

REGULARITY OF ALMOST-SURELY INJECTIVE PROJECTIONS IN EUCLIDEAN SPACES

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ABSTRACT. In a previous work we proved that if a finite Borel measure μ in a Euclidean space has Hausdorff dimension smaller than a positive integer k , then the orthogonal projection onto almost every k -dimensional linear subspace is injective on a set of full μ -measure. In this paper we study the regularity of the inverses of these projections and prove that if μ has a compact support X such that (respectively) the Hausdorff, upper box-counting or Assouad dimension of X is smaller than k , then the inverse is (respectively) continuous, pointwise α -Hölder for some $\alpha \in (0, 1)$ or pointwise α -Hölder for every $\alpha \in (0, 1)$. The results generalize to the case of typical linear perturbations of Lipschitz maps and strengthen previously known ones in the lossless analog compression literature. We provide examples showing the sharpness of the statements. Additionally, we construct a non-trivial measure on the plane which admits almost-surely injective projections in every direction, and show that no homogeneous self-similar measure has this property.

1. INTRODUCTION AND MAIN RESULTS

1.1. Projections of sets and measures in Euclidean spaces. The study of geometric and dimensional properties of the images of a set $X \subset \mathbb{R}^N$, $N \in \mathbb{N}$, under orthogonal projections

$$P_V: \mathbb{R}^N \rightarrow V$$

onto k -dimensional linear spaces $V \subset \mathbb{R}^N$, is a well-known subject of interest in geometric measure theory. The space of all k -dimensional linear subspaces of \mathbb{R}^N (or, equivalently, the space of corresponding orthogonal projections) forms the Grassmannian $\text{Gr}(k, N)$, which has a structure of a $k(N - k)$ -dimensional compact manifold, equipped with the standard rotation-invariant (Haar) measure. Throughout the paper, the terms ‘almost every linear space’ or ‘almost every projection’ will be used in relation to this measure.

A classical result in this area is the celebrated Marstrand–Mattila theorem, proved in [Mar54, Mat75].

Theorem 1.1 (Marstrand–Mattila projection theorem). *Let X be a Borel set in \mathbb{R}^N . Then the following hold.*

- (i) $\dim_H P_V(X) = \min\{k, \dim_H X\}$ for almost every k -dimensional linear subspace V of \mathbb{R}^N .
- (ii) If $\dim_H X > k$, then $P_V(X)$ has positive k -dimensional Hausdorff measure for almost every k -dimensional linear subspace V of \mathbb{R}^N .

Here and in the sequel $\dim_H X$ denotes the Hausdorff dimension of the set X , while \mathcal{H}^k denotes the k -dimensional Hausdorff measure.

A number of further results related to Marstrand–Mattila projection theorem have been obtained, including versions valid for various types of dimension, and estimates on the size of the set of exceptional projections, see e.g. [FH97, PS00, Mat04, Bou10, FO14, FO17, Orp21, DOV22] and the references therein, as well as [FFJ15] for a comprehensive survey. In particular, a version of Marstrand–Mattila’s projection theorem for measures has been established (see [HT94, HK97]).

Theorem 1.2 (Marstrand–Mattila projection theorem for measures). *Let μ be a finite Borel measure in \mathbb{R}^N . Then the following hold.*

- (i) $\dim_H P_V \mu = \min\{k, \dim_H \mu\}$ and $\underline{\dim}_H P_V \mu = \min\{k, \underline{\dim}_H \mu\}$ for almost every k -dimensional linear subspace V of \mathbb{R}^N .

(ii) If $\underline{\dim}_H \mu > k$, then $P_V \mu$ is absolutely continuous with respect to k -dimensional Hausdorff measure for almost every k -dimensional linear subspace V of \mathbb{R}^N .

Here $P_V \mu$ denotes the image of μ under P_V , while $\dim_H \mu$ and $\underline{\dim}_H \mu$ denote, respectively, the upper and lower Hausdorff dimensions of a measure μ (see Section 2.1 for the definitions).

1.2. Injective and almost-surely injective projections. Apart from considering the dimension of the images of X under orthogonal projections P_V , another line of research is to study under which conditions the projections P_V are injective on X , at least for typical V .¹ Note that if this occurs, then P_V provides a *topological embedding* of X into a k -dimensional linear space V , and X can be considered as the graph of a function from $P_V(X) \subset V \simeq \mathbb{R}^k$ to $V^\perp \simeq \mathbb{R}^{N-k}$.

It is known that if X is a compact subset of \mathbb{R}^N and $2\overline{\dim}_B X < k$, where $\overline{\dim}_B$ denotes the upper box-counting (Minkowski) dimension, then P_V is injective for a typical k -dimensional linear space $V \subset \mathbb{R}^N$. This fact is commonly referred to as the *Mañé projection theorem*. Indeed, Mañé proved this result for topologically generic projections in [Mn81], while a version valid for almost every projection (or, equivalently, for almost every linear map $L: \mathbb{R}^N \rightarrow \mathbb{R}^k$) was presented in [SYC91, Rob11]. Mañé theorem's statement is in the spirit of the well-known Menger–Nöbeling embedding theorem (see e.g. [HW41, Theorem 5.2], which states that for a compact metric space X with Lebesgue covering dimension at most k , a generic continuous transformation $\phi: X \rightarrow \mathbb{R}^{2k+1}$ is injective).

Given a projection P_V which is injective on a set X , it is natural to ask what is the regularity of its *inverse*

$$(P_V|_X)^{-1}: P_V(X) \rightarrow X.$$

In [HK99, Theorem 3.1], Hunt and Kaloshin proved that if $X \subset \mathbb{R}^N$ is compact and $2\overline{\dim}_B X < k$, then for almost every k -dimensional linear space $V \subset \mathbb{R}^N$, the projection P_V restricted to X has an α -Hölder inverse for $0 < \alpha < 1 - \frac{2\overline{\dim}_B X}{k}$ (see also [BAEFN93, EFNT94] for earlier results in this direction and [Rob11] for a detailed exposition). In [RS19, Theorem 2.1], Rossi and Shmerkin gave upper bounds on the Hausdorff dimension of the set of exceptional projections. Furthermore, the regularity of the inverses can be improved, if the assumption $2\overline{\dim}_B X < k$ is replaced by $2\dim_A X < k$, where \dim_A is the Assouad dimension (see Definition 2.2). More precisely, in this case almost all projections onto k -dimensional linear subspaces of \mathbb{R}^N have inverses which are α -Hölder for any $\alpha \in (0, 1)$, see [Ols02, Theorem 5.2] and [Rob11, Theorem 9.18]. The problem of the existence of linear embeddings and the regularity of their inverses was also studied for finite-dimensional subsets of Banach spaces, see [Rob11, Chapters 5–9] and references therein.

It is known that in general, the bound $2\overline{\dim}_B X < k$ in the Mañé projection theorem cannot be diminished (see [HW41, Example V.3]), and $\overline{\dim}_B$ cannot be replaced by \dim_H (see [SYC91, Appendix])². However, the situation changes if instead of the injectivity of P_V on X , one is interested in *almost sure injectivity* of P_V , i.e. the injectivity of P_V on a full μ -measure Borel subset of X , according to a given Borel measure μ on X . In our previous paper [BGŚ20, Corollary 3.4], strengthening a result by Alberti, Bölcskei, De Lellis, Koliander and Riegler [ABDL⁺19], we showed the following.

Theorem 1.3 (Probabilistic injective projection theorem). *Let X be a Borel subset of \mathbb{R}^N equipped with a finite or σ -finite Borel measure μ and assume $\mathcal{H}^k(X) = 0$ for some positive integer $k \leq N$ (in particular, it is enough to assume $\dim_H X < k$). Then for almost every k -dimensional linear subspace V of \mathbb{R}^N , the orthogonal projection P_V is injective on a full μ -measure Borel set $X_V \subset X$.*

See also Theorem 1.11 for a more general statement. Consequently, for a given set X , the minimal dimension k sufficient for a *typical* projection to be injective may be reduced by half when injectivity is considered in a ‘probabilistic’ setup compared to a ‘deterministic’ one.

¹Another question one may consider is when a projection P_V is injective on X for *some* V . In [ABDL⁺19, Example III.4] one finds a construction of a compact set X in \mathbb{R}^3 with $\dim_H X = 2$, such that there is a projection $P: \mathbb{R}^3 \rightarrow \mathbb{R}$ injective on X .

²In fact, these examples show that even the existence of an orthogonal projection injective on X does not hold under weaker assumptions.

We note that the condition $\dim_H \mu \leq k$ is necessary for a typical almost sure injectivity of P_V . Indeed, we have the following.

Proposition 1.4. *Let μ be a finite Borel measure in \mathbb{R}^N with $\dim_H \mu > k$. Then for almost every k -dimensional linear space $V \subset \mathbb{R}^N$ and every Borel set $Y \subset \mathbb{R}^N$ of full μ -measure, the orthogonal projection P_V is not injective on Y .*

The proof of Proposition 1.4 is presented in Section 3.

Remark 1.5. Note that one cannot extend Proposition 1.4 to projections onto all k -dimensional linear subspaces $V \subset \mathbb{R}^N$. An obvious counterexample is the lift of 1-dimensional Lebesgue measure on the interval $[0, 1]$ in the x -axis on the plane to a graph of a Borel function $[0, 1] \rightarrow \mathbb{R}$ with Hausdorff dimension greater than 1 (for instance, the graph of a continuous nowhere-differentiable Weierstrass-type function, see e.g. [BBR14, She18, RS21]), which projects injectively onto the x -axis. On the other hand, if μ is s -analytic for $s > k$, then no projection onto a k -dimensional linear space $V \subset \mathbb{R}^N$ is injective on a set of positive μ -measure, see [ABDL⁺19, Corollary IV.2].

In the border case $\dim_H \mu = k$, if μ is not singular with respect to the k -dimensional Hausdorff measure, then different types of behaviour may occur, e.g. for $k = 1$ and 1-dimensional Hausdorff measure on an interval in \mathbb{R}^2 , a projection onto a typical line is injective on its support, while for 1-dimensional Hausdorff measure on a circle in \mathbb{R}^2 , no Lipschitz map $\phi: \mathbb{R}^2 \rightarrow \mathbb{R}$ is injective on a set of full measure, see [BGŚ20, Example 3.5]. Understanding the border case is closely related to the following open problem in geometric measure theory: if $X \subset \mathbb{R}^2$ is a \mathcal{H}^1 -measurable, purely unrectifiable set with $\mathcal{H}^1(X) < \infty$, is it true that for \mathcal{H}^1 -almost every $x \in X$, almost every line passing through x meets X only at x ? Refer to [Mat04, Problem 12] for the original formulation and to [Mat23, Conjecture 4.15] for a more recent account. While the problem remains unsolved in the general case, it has been answered affirmatively for some classes of sets, e.g. for self-similar 1-sets satisfying the Open Set Condition, see [SS07].

In this paper, assuming suitable bounds on the dimension of a compact set $X \subset \mathbb{R}^N$, we study the question of the regularity of the *almost sure inverse map*

$$(P_V|_{X_V})^{-1}: P_V(X_V) \rightarrow X_V$$

for a typical k -dimensional linear space $V \subset \mathbb{R}^N$, where P_V is injective on a full μ -measure Borel set $X_V \subset X$.

Definition 1.6. A map $\phi: A \rightarrow \mathbb{R}^N$, where $A \subset \mathbb{R}^k$, is **pointwise α -Hölder**, if for every $x \in A$ there exists $c_x > 0$ such that

$$\|\phi(x) - \phi(y)\| \leq c_x \|x - y\|^\alpha$$

for every $y \in A$.

The basic result of this paper is the following.

Theorem 1.7 (Regularity of the inverse of almost-surely injective projections). *Let X be a compact subset of \mathbb{R}^N and let μ be a finite Borel measure supported on X . Consider orthogonal projections $P_V: \mathbb{R}^N \rightarrow V$ onto k -dimensional linear spaces $V \subset \mathbb{R}^N$. Then the following hold.*

- (i) *If $\dim_H X < k$, then for almost every V there exists a full μ -measure Borel set $X_V \subset X$ such that the restriction of P_V to X_V is injective with continuous inverse.*
- (ii) *If $\overline{\dim}_B X < k$, then for almost every V there exists a full μ -measure Borel set $X_V \subset X$ such that the restriction of P_V to X_V is injective with inverse which is pointwise α -Hölder for every $\alpha \in (0, 1 - \frac{\overline{\dim}_B X}{k})$.*
- (iii) *If $\dim_A X < k$, then for almost every V there exists a full μ -measure Borel set $X_V \subset X$ such that the restriction of P_V to X_V is injective with inverse which is pointwise α -Hölder for every $\alpha \in (0, 1)$.*

Theorem 1.7 follows from a more general Theorem 1.12, presented in Subsection 1.4. In Section 6 we provide examples showing that the result is sharp in several ways. In particular, the bound $1 - \frac{\overline{\dim}_B X}{k}$ for the Hölder exponent in assertion (ii) cannot be improved in terms of $\overline{\dim}_B X$.

Remark 1.8. It is important to note that, in general, we cannot obtain Hölder continuity (instead of pointwise Hölder continuity) of the inverse of $P_V|_{X_V}$ in Theorem 1.7, even under the assumption $\dim_A X < k$. Indeed, if $X \subset \mathbb{R}^N$ is a compact set with $\dim_A X < k$, which does not embed topologically into \mathbb{R}^k (for example, there exist simplicial complexes of dimension n which do not embed topologically into \mathbb{R}^{2n} , see [HW41, Example V.3]) and μ is any measure with $\text{supp } \mu = X$, then a projection P_V onto a k -dimensional linear space $V \subset \mathbb{R}^N$, which is injective on a set of full μ -measure with a Hölder inverse is actually a homeomorphism on X , which is impossible, as X does not embed topologically into \mathbb{R}^k . See [BGŚ20, Remark 3.10] for the details of this argument.

1.3. Measures with almost-surely injective projections in every direction. An important question in the geometric measure theory, which has gained increasing interest in recent years, is finding conditions under which suitable projection theorems hold for *every* (rather than almost every) projection. This question is particularly interesting in the context of *self-similar measures* for iterated function systems, i.e. Borel probability measures μ in \mathbb{R}^N satisfying

$$\mu = \sum_{i \in I} p_i \varphi_i \mu,$$

where I is a finite set, $(p_i)_{i \in I}$ is a strictly positive probability vector and $\{\varphi_i : i \in I\}$ is an iterated function system (IFS) consisting of contracting similarities

$$\varphi_i : \mathbb{R}^N \rightarrow \mathbb{R}^N, \quad \varphi_i(x) = r_i O_i(x) + t_i$$

with scales $r_i \in (0, 1)$, orthogonal matrices $O_i \in \mathbb{R}^{N \times N}$ and translation vectors $t_i \in \mathbb{R}^N$.

In [HS12, Theorem 1.6], Hochman and Shmerkin proved that if the system $\{\varphi_i : i \in I\}$ satisfies the Strong Separation Condition and the semigroup generated by $\{O_i : i \in I\}$ acts minimally on $\text{Gr}(k, N)$, then the first assertion of Marstrand–Mattila’s projection theorem for measures (Theorem 1.2) holds for all orthogonal projections onto k -dimensional linear subspaces of \mathbb{R}^N . The result was extended by Farkas [Far16, Theorem 1.6] for systems without any separation conditions. On the other hand, Rapaport [Rap17] constructed a self-similar measure in the same class, which does not satisfy the absolute continuity part of Theorem 1.2 for a Baire-residual set of projections. Similarly, he showed that for such measures, the slicing theorem (Theorem 3.3) can fail for a residual set of projections. Nevertheless, in [Rap20] he proved that typical *homogeneous* self-similar measures (i.e. the ones satisfying $r_i O_i = r_j O_j$ for all i, j) in the plane with the Strong Separation Condition satisfy both assertions of Theorem 1.2 for every projection. In the case of random constructions, Simon and Rams [RS14, RS15] showed that almost every fractal percolation satisfies the assertions of Theorem 1.1 for every projection.

Considering the question of the injectivity of projections, we note that, obviously, a non-singleton set in \mathbb{R}^N cannot be projected injectively onto all k -dimensional linear subspaces of \mathbb{R}^N , for any $1 \leq k \leq N - 1$. This naturally leads us to the setup of almost-surely injective projections. The first question appearing in this context is whether there exist non-trivial (with non-singleton support) measures which project almost-surely injectively in every direction. The following result shows that in fact such measures exist.

Theorem 1.9 (Existence of a measure with almost-surely injective projections in every direction). *There exists a compactly supported Borel probability measure μ in \mathbb{R}^2 with positive Hausdorff dimension (and hence with non-singleton support), such that every orthogonal projection $P_V : \mathbb{R}^2 \rightarrow V$ onto a line $V \subset \mathbb{R}^2$ is injective on a Borel set X_V of full μ -measure.*

The proof of Theorem 1.9 is presented in Section 5.

Furthermore, we show that, unlike for the Marstrand–Mattila and slicing theorems, non-degenerated homogeneous self-similar measures cannot satisfy this property, even generically.

Proposition 1.10. *Let μ be a self-similar measure in \mathbb{R}^N corresponding to a homogeneous IFS $\varphi_i(x) = rO(x) + t_i$, $i \in I$, such that φ_i are not all equal. Then for every $k \in \{1, \dots, N - 1\}$ there exists a k -dimensional linear space $V \subset \mathbb{R}^N$ such that for every Borel set $Y \subset \mathbb{R}^N$ of full μ -measure, the orthogonal projection P_V onto V is not injective on Y .*

Proposition 1.10 follows from a more general fact (Proposition 5.4), which is proved in Section 5.

1.4. Regularity of the inverse for typical linear perturbations of Lipschitz maps. The result described in Theorem 1.7 can be generalized to the setup of almost-surely injective linear perturbations of Lipschitz maps on compact sets in Euclidean spaces. In a previous paper, we proved the following.

Theorem 1.11 ([BGS²⁰, Theorem 3.1]). *Let μ be a finite or σ -finite Borel measure in \mathbb{R}^N supported on a set X , such that μ is singular with respect to the k -dimensional Hausdorff measure for some $k \in \mathbb{N}$ (in particular, it is enough to assume $\dim_H \mu < k$), and let $\phi: X \rightarrow \mathbb{R}^k$ be a Lipschitz map. Then for almost every linear transformation $L: \mathbb{R}^N \rightarrow \mathbb{R}^k$ there exists a Borel set $X_L \subset X$ of full μ -measure, such that the map $\phi_L = \phi + L$ is injective on X_L .*

Here and in the sequel, ‘almost every linear map’ refers to the Lebesgue measure in the space $\text{Lin}(\mathbb{R}^N, \mathbb{R}^k) \simeq \mathbb{R}^{Nk}$. The result can be generalized to the case of Hölder maps $\phi: X \rightarrow \mathbb{R}^k$, see [BGS²⁰] for details. The conclusion of Theorem 1.11 holds also for prevalent sets in the spaces of Lipschitz and C^r -maps $\phi: X \rightarrow \mathbb{R}^k$ (see Definition 2.3 and Remark 2.4).

Extending the setup of Subsection 1.2, we study regularity properties of the inverse maps

$$(\phi_L|_{X_L})^{-1}: \phi_L(X_L) \rightarrow X_L.$$

The following is our main result.

Theorem 1.12 (Regularity of the inverse for almost-surely injective linear perturbations of Lipschitz maps). *Let μ be a finite Borel measure in \mathbb{R}^N with a compact support X and let $\phi: X \rightarrow \mathbb{R}^k$ be a Lipschitz map. We write $\phi_L = \phi + L$ for linear maps $L \in \text{Lin}(\mathbb{R}^N, \mathbb{R}^k)$. Then the following hold.*

- (i) *If $\dim_H X < k$, then for almost every linear map L there exists a Borel set $X_L \subset X$ of full μ -measure such that ϕ_L is injective on X_L with continuous inverse.*
- (ii) *If $\overline{\dim}_B X < k$, then for almost every linear map L there exists a Borel set $X_L \subset X$ of full μ -measure such that ϕ_L is injective on X_L with inverse which is pointwise α -Hölder for every $\alpha \in (0, 1 - \frac{\overline{\dim}_B X}{k})$.*
- (iii) *If $\dim_A X < k$, then for almost every linear map L there exists a Borel set $X_L \subset X$ of full μ -measure on which ϕ_L is injective with inverse which is pointwise α -Hölder for every $\alpha \in (0, 1)$.*

Instead of linear maps from $\text{Lin}(\mathbb{R}^N, \mathbb{R}^k) \simeq \mathbb{R}^{Nk}$, $k \leq N$, with the Lebesgue measure, one can equivalently consider orthogonal projections $P_V: \mathbb{R}^N \rightarrow V$ for $V \in \text{Gr}(k, N)$ with the rotation-invariant measure (see Remark 3.1). Therefore, Theorem 1.12 implies immediately Theorem 1.7 by setting $\phi = 0$. On the other hand, considering ϕ within the spaces of Lipschitz or C^r -maps, $r = 1, 2, \dots, \infty$, one obtains a result for almost every map in these spaces in the sense of prevalence, as defined in [HSY92]. See Definition 2.3 and Remark 2.4 for more details. Examples presented in Section 6 show that Theorem 1.12 cannot be improved in the setting of prevalence within the above spaces.

1.5. Relation to the theory of compressed sensing. The field of *compressed sensing* grew out of the work of Candès, Donoho, Romberg and Tao ([Can06, CRT06b, Don06, FR13]). The fundamental problem of the theory is to find conditions enabling to recover an input vector $x \in \mathbb{R}^N$ from its linear measurement $y = Ax \in \mathbb{R}^m$, where $A \in \mathbb{R}^{m \times N}$, even though $m \ll N$. A key theorem ([FR13, Theorem 9.12], see also [CT06, CRT06a]) states that with high probability one may recover x with $\|x\|_0 := |\{j : x_j \neq 0\}| \leq s$ (the *s-sparsity* condition) from y when $A \in \mathbb{R}^{m \times N}$ is a random Gaussian matrix with $m \approx s \ln \frac{N}{s}$ via an ℓ_1 -minimization *basis pursuit* algorithm [Mal98, §1.4.3], see also [CDS01]. Capitalizing on sparsity, compressed sensing has found many applications (see e.g. [LDP07, DDT⁺08, BS07, HS09]).

In [ABDL⁺19], Alberti, Bölcskei, De Lellis, Koliander and Riegler studied the above-mentioned problem in a setting where both the input vector x and the sensing matrix A are random: $x \in \mathbb{R}^N$ is given according to a probability measure μ in \mathbb{R}^n , $A \in \mathbb{R}^{m \times N}$ is given according to the Lebesgue

measure in $\mathbb{R}^{m \times N}$ and one seeks to recover x from $y = Ax$ μ -almost surely. In particular, they proved a version of Theorem 1.11 for probability measures, with $\phi = 0$ and the Hausdorff dimension replaced with the lower modified Minkowski dimension, see [ABDL⁺19, Theorem II.1] (for related earlier results see [WV10, Theorem 18 and Corollary 1]).

In recent years there has been a surge of interest in a compressed sensing framework for analog signals modelled by continuous-alphabet discrete-time stochastic processes³ with general (not necessarily sparse) distributions ([WV10, DT10, DMM11, JP17, RJEP17, GK19, GŚ19, GŚ20]). In [WV10], Wu and Verdú emphasized that the regularity of both of the encoder and decoder is crucial, as it introduces resilience to noise. Translated to the setting of [ABDL⁺19] and Theorem 1.11, as the encoder is already assumed to be linear, this corresponds to investigating the regularity of the inverse map $L^{-1}: L(X_L) \rightarrow X_L$. Thus, one may interpret our main result as giving almost sure regularity guarantees for decompression under various dimension assumptions on the measure generating the input vector. For example, Theorem 1.12 allows us to improve [WV10, Corollary 1]. According to this result, denoting by Σ_s the set of all s -sparse vectors in \mathbb{R}^N , i.e.

$$\Sigma_s = \{x \in \mathbb{R}^N : x = (x_1, \dots, x_N) \text{ with at most } s \text{ of } x_i \text{ being non-zero}\},$$

if μ is a σ -finite Borel measure on Σ_s , $k > s$, $\varepsilon > 0$ and $0 < \alpha < 1 - \frac{s}{k}$, then there exists a linear map $L: \mathbb{R}^N \rightarrow \mathbb{R}^k$ and an α -Hölder map $g: \mathbb{R}^k \rightarrow \mathbb{R}^N$ such that $\mu(\{x \in \mathbb{R}^N : g(Lx) \neq x\}) \leq \varepsilon$.

As Σ_s is a finite union of s -linear subspaces of \mathbb{R}^N , one has $\dim_A \Sigma_s = s$ and thus Theorem 1.12(iii) implies in a straightforward manner the following result, where the Hölder exponent α can be taken arbitrarily close to 1.

Corollary 1.13. *Let μ be any compactly supported finite Borel measure on Σ_s and fix $k > s$. Then almost every linear map $L: \mathbb{R}^N \rightarrow \mathbb{R}^k$ is injective on a set of full μ -measure with an inverse which is pointwise α -Hölder for every $\alpha \in (0, 1)$. Consequently, for almost every linear map $L: \mathbb{R}^N \rightarrow \mathbb{R}^k$, every $\varepsilon > 0$ and every $\alpha \in (0, 1)$, there exists an α -Hölder map $g: \mathbb{R}^k \rightarrow \mathbb{R}^N$ such that $\mu(\{x \in \mathbb{R}^N : g(Lx) \neq x\}) \leq \varepsilon$.*

In fact, we can guarantee g to be Lipschitz up to a logarithmic factor, see Theorem 4.4. The result extends to the setup of compactly supported measures on a finite union of s -dimensional C^1 manifolds. For related results see [SRAB17].

The article [ABDL⁺19] emphasizes the importance of proving *converse statements* for compression results, i.e. the ones demonstrating that if the dimension of the image space is too small, then compression is not possible, and provides such a result for *s-analytic* measures, see [ABDL⁺19, Corollary IV.2]. Proposition 1.4 provides a converse statement to arbitrary finite Borel measures in \mathbb{R}^N , in terms of their Hausdorff dimension.

1.6. Further related topics. Let us mention that almost sure injectivity plays an important role in some other nonlinear projection schemes. For instance, in the context of natural projection maps from the symbolic space for iterated function systems, this property is called *weak almost unique coding* (see [KS19, Definition 1.9]) and for self-similar systems it is known to be equivalent to the *no dimension drop* condition (i.e. the equality of the dimension of a given ergodic invariant measure to the ratio of its entropy and Lyapunov exponent), see [KS19, Appendix] and [Fen23, Corollary 4.7]. This observation can be successfully utilized for obtaining dimension results for certain classes of fractal attractors, see e.g. [KS19]. Moreover, basic techniques developed for studying typical properties of orthogonal projections can be often transferred to parametrized families of iterated functions systems satisfying the *transversality condition* (an analogue of Lemma 4.1). It was first used by Pollicott and Simon [PS95] and led, for example, to results analogous to Marstrand's projection theorem ([SSU01a, SSU01b], see also [BSSŚ22] for a more detailed overview and [Sol23] for a recent survey).

Finally, techniques originating from the study of orthogonal projections can also be used to analyse so-called *delay-coordinate maps*, i.e. maps of the form $\phi(x) = (h(x), h(Tx), \dots, h(T^{k-1}x))$,

³The rigorous passage between continuous-time signals and discrete-time signals is justified by the Shannon sampling theorem ([Hig96, Chapter 1]).

where $T: X \rightarrow X$ is a discrete-time dynamical system and $h: X \rightarrow \mathbb{R}$ is a real function (observable). The injectivity or almost sure injectivity of ϕ implies that the original dynamics can be faithfully modelled based only on the values of the observable. Related embedding results, known as Takens-type theorems, serve as a framework for applications in natural sciences, see e.g. [Tak81, SM90, SYC91, Aba96]. Recently, a probabilistic counterpart of this theory has been developed [SSOY98, BGŚ20, BGŚ22, BGS24], where the regularity of almost-surely defined mappings related to ϕ plays a crucial role. See [BGS24] for a more detailed discussion and further references.

Structure of the paper. Section 2 contains basic definitions and a description of technical tools used in subsequent parts of the paper. In Section 3 we prove Proposition 1.4, while Section 4 contains the proof of Theorem 1.12. Theorem 1.9 and Proposition 1.10 are proved in Section 5. The last Section 6 provides examples showing the sharpness of the obtained results.

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2. PRELIMINARIES

We consider the Euclidean space \mathbb{R}^N , $N \geq 1$, with the standard inner product $\langle \cdot, \cdot \rangle$. The corresponding norm and diameter are denoted, respectively, by $\|\cdot\|$ and $|\cdot|$. The symbol $B_N(x, r)$ denotes the r -ball centred at x in the Euclidean norm in \mathbb{R}^N . We often write $B(x, r)$ when the dimension is clear from the context. By $\#A$ we denote the cardinality of a set A .

2.1. Dimensions.

Definition 2.1. For $s > 0$, the **s -dimensional (outer) Hausdorff measure** of a set $X \subset \mathbb{R}^N$ is defined as

$$\mathcal{H}^s(X) = \lim_{\delta \rightarrow 0} \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : X \subset \bigcup_{i=1}^{\infty} U_i, |U_i| \leq \delta \right\}.$$

The **Hausdorff dimension** of X is given as

$$\dim_H X = \inf\{s > 0 : \mathcal{H}^s(X) = 0\} = \sup\{s > 0 : \mathcal{H}^s(X) = \infty\}.$$

For a bounded set $X \subset \mathbb{R}^N$ and $\delta > 0$, let $N(X, \delta)$ denote the minimal number of balls of radius δ required to cover X . The **lower** and **upper box-counting (Minkowski) dimensions** of X are defined, respectively, as

$$\underline{\dim}_B X = \liminf_{\delta \rightarrow 0} \frac{\log N(X, \delta)}{-\log \delta}, \quad \overline{\dim}_B X = \limsup_{\delta \rightarrow 0} \frac{\log N(X, \delta)}{-\log \delta}.$$

For a finite Borel measure μ in \mathbb{R}^N , we define its (upper) Hausdorff dimension as

$$\dim_H \mu = \inf \left\{ \dim_H X : X \subset \mathbb{R}^N \text{ Borel with } \mu(\mathbb{R}^N \setminus X) = 0 \right\}$$

and the lower Hausdorff dimension as

$$\underline{\dim}_H \mu = \inf \left\{ \dim_H X : X \subset \mathbb{R}^N \text{ Borel with } \mu(X) > 0 \right\}.$$

Definition 2.2. A bounded set $X \subset \mathbb{R}^N$ is said to be **(M, s) -homogeneous** if $N(X \cap B(x, r), \rho) \leq M(r/\rho)^s$ for every $x \in X$, $0 < \rho < r$, i.e. the intersection $B(x, r) \cap X$ can be covered by at most $M(r/\rho)^s$ balls of radius ρ . The **Assouad dimension** of X is defined as

$$\dim_A X = \inf\{s > 0 : X \text{ is } (M, s)\text{-homogeneous for some } M > 0\}.$$

It is easy to see that in the definitions of box-counting and Assouad dimensions, it is enough to consider covers by balls centred in X . For a bounded set $X \subset \mathbb{R}^N$ and a finite Borel measure μ on X , we have the following inequalities (see e.g. [Rob11, (9.1)]).

$$\underline{\dim}_H \mu \leq \dim_H \mu \leq \dim_H X \leq \underline{\dim}_B X \leq \overline{\dim}_B X \leq \dim_A X.$$

2.2. Prevalence. A notion of *prevalence* was introduced by Hunt, Shroer and Yorke in [HSY92] and is regarded to be an analogue of ‘Lebesgue almost sure’ condition in infinite dimensional linear spaces.

Definition 2.3. Let \mathcal{V} be a complete linear metric space (i.e. a linear space with a complete metric which makes addition and scalar multiplication continuous). A Borel set $\mathcal{S} \subset \mathcal{V}$ is called **prevalent** if there exists a Borel measure ν in \mathcal{V} , which is positive and finite on some compact set in \mathcal{V} , such that for every $v \in \mathcal{V}$, we have $v + e \in \mathcal{S}$ for ν -almost every $e \in \mathcal{V}$. A non-Borel subset of \mathcal{V} is prevalent if it contains a prevalent Borel subset.

We focus mainly on the prevalence in the space $\text{Lip}(X, \mathbb{R}^k)$ of all Lipschitz functions $h: X \rightarrow \mathbb{R}^k$ on a compact set $X \subset \mathbb{R}^N$, endowed with the Lipschitz norm

$$(1) \quad \|h\|_{\text{Lip}} = \|h\|_{\infty} + \text{Lip}(h),$$

where $\|h\|_{\infty}$ is the supremum norm and $\text{Lip}(h)$ is the Lipschitz constant of h . Note, however, that in Theorem 1.12 we can consider prevalence in other spaces, as explained in the following remark.

Remark 2.4. Let \mathcal{V} be any of the spaces of Lipschitz or C^r , $r = 1, 2, \dots, \infty$, maps from (a bounded open neighbourhood of) a compact set $X \subset \mathbb{R}^N$ into \mathbb{R}^k , endowed with the natural complete linear metric. Note that in order to show prevalence of a set $\mathcal{S} \subset \mathcal{V}$ via linear maps, it is enough to prove that for every $\phi \in \mathcal{V}$ and Lebesgue-almost every $L \in \text{Lin}(\mathbb{R}^N, \mathbb{R}^k) \simeq \mathbb{R}^{Nk}$ we have $\phi + L \in \mathcal{S}$. As all the C^r -spaces listed above are contained in the space of Lipschitz maps, it is enough to consider Lipschitz maps ϕ .

2.3. Conditional measures. It will be useful to work with a system of conditional measures for considered projections. In the sequel, we will denote Dirac’s measure at a point x by δ_x . The symbol $\phi\mu$ denotes the image of a measure μ under a map ϕ .

Definition 2.5. Given a continuous map $\phi: X \rightarrow \mathbb{R}^k$ on a compact set $X \subset \mathbb{R}^N$ there exists a **system of conditional measures** of a probability measure μ on X with respect to ϕ , i.e. a family $\{\mu_y : y \in \mathbb{R}^k\}$, such that

- (i) for every $y \in \mathbb{R}^k$, μ_y is a (possibly zero) Borel measure on $\phi^{-1}(\{y\})$,
- (ii) for $\phi\mu$ -almost every $y \in \mathbb{R}^k$, μ_y is a Borel probability measure,
- (iii) for every μ -measurable set $A \subset X$, the function $\mathbb{R}^k \ni y \mapsto \mu_y(A)$ is $\phi\mu$ -measurable and

$$\mu(A) = \int_{\mathbb{R}^k} \mu_y(A) d\phi\mu(y).$$

The existence and $\phi\mu$ -almost sure uniqueness of the system of conditional measures follows from the Rokhlin’s Disintegration Theorem [Roh52]. See also [Sim12] for a more direct approach.

The following lemma characterizes almost sure injectivity in terms of conditional measures.

Lemma 2.6. *Let $\phi: X \rightarrow \mathbb{R}^k$ be a continuous map on a compact set $X \subset \mathbb{R}^N$ and let μ be a Borel probability measure on X . Then ϕ is injective on a Borel set $X_\phi \subset X$ of full μ -measure if and only if the system $\{\mu_{\phi(x)} : x \in X\}$ of conditional measures of μ with respect to ϕ satisfies $\mu_{\phi(x)} = \delta_x$ for μ -almost every $x \in X$.*

Proof. If ϕ is injective on X_ϕ , then setting $\mu_{\phi(x)} = \delta_x$ for $x \in X_\phi$ and $\mu_y = 0$ for $y \notin \phi(X_\phi)$ gives a system of conditional measures of μ with respect to ϕ (see [BGŚ22, p. 620] for a detailed argument). Hence, the first implication follows by the almost sure uniqueness of the system of conditional measures. For the other implication, assume $\mu_{\phi(x)} = \delta_x$ for μ -almost every $x \in X$. Then $X_\phi = \{x \in X : \mu_{\phi(x)} = \delta_x\}$ is the required set of injectivity. Indeed, $\mu(X_\phi) = 1$ by assumption, and if $x, y \in X_\phi$ and $\phi(x) = \phi(y)$, then $\delta_x = \delta_y$, so $x = y$. \square

3. ORTHOGONAL PROJECTIONS AND SLICES

As noted in the introduction, we consider the Grassmannian $\text{Gr}(k, N)$ of k -dimensional linear subspaces of \mathbb{R}^N . We denote by $\gamma_{k,N}$ the unique rotation-invariant measure on $\text{Gr}(k, N)$ (see [Mat95, Section 3.9] for details and [FR02] for an alternative construction). Recall that for $V \in \text{Gr}(k, N)$ we denote by $P_V: \mathbb{R}^N \rightarrow V \simeq \mathbb{R}^k$ the orthogonal projection onto V .

Remark 3.1. As we switch between Lebesgue-almost sure statements for linear transformations $L \in \text{Lin}(\mathbb{R}^N, \mathbb{R}^k) \simeq \mathbb{R}^{Nk}$ and $\gamma_{k,N}$ -almost sure statements for orthogonal projections P_V , $V \in \text{Gr}(k, N)$, it is useful to note that the two kinds of statements are equivalent if one is interested in the injectivity of L and P_V on a full-measure set. Namely, Lebesgue-almost every linear map $L: \mathbb{R}^N \rightarrow \mathbb{R}^k$, $k \leq N$, has full rank and can be represented uniquely as $L = \Psi \circ P_V$, where V is the k -dimensional orthogonal complement of $\text{Ker } L$, P_V is the orthogonal projection onto V and $\Psi: V \rightarrow \mathbb{R}^k$ is a linear isomorphism depending continuously on L . It is easy to see that under this identification, full Lebesgue-measure sets in $\text{Lin}(\mathbb{R}^N, \mathbb{R}^k)$ correspond to full $\gamma_{k,N}$ -measure sets in $\text{Gr}(k, N)$. Moreover, the injectivity of L on a set $Y \subset \mathbb{R}^N$ is equivalent to the injectivity of P_V on Y . Similarly, an equivalence holds between the continuity or pointwise Hölder continuity of the inverses of $L|_Y$ and $P_V|_Y$.

Extending the notation from the previous section, for a compactly supported finite measure μ in \mathbb{R}^N and a linear space $V \in \text{Gr}(k, N)$, we denote by $\{\mu_a^V : a \in V\}$ the system of conditional measures of μ with respect to P_V . The measures μ_a^V are concentrated on $P_V^{-1}(a) = V^\perp + a$. By Lemma 2.6 and Remark 3.1, Theorem 1.11 implies the following corollary.

Corollary 3.2. *Let μ be a compactly supported finite Borel measure in \mathbb{R}^N with $\dim_H \mu < k$. Then for $\gamma_{k,N}$ -almost every $V \in \text{Gr}(k, N)$, the sliced measure μ_a^V is a point mass for $P_V \mu$ -almost every $a \in V$.*

Järvenpää and Mattila [JM98, Theorem 3.3] proved a general ‘slicing’ theorem for measures⁴, which in our notation reads as follows.

Theorem 3.3 (Slicing Theorem). *Let μ be a compactly supported finite Borel measure in \mathbb{R}^N . Then for $\gamma_{k,N}$ -almost every $V \in \text{Gr}(k, N)$,*

$$\underline{\dim}_H \mu_a^V \geq \underline{\dim}_H \mu - k \quad \text{for } P_V \mu\text{-almost every } a \in V.$$

Using the above result we can give a proof of Proposition 1.4.

Proof of Proposition 1.4. We claim that under the assumptions of the proposition, there exists a compact set $X \subset \mathbb{R}^N$ such that $\mu(X) > 0$ and $\nu = \mu|_X$ satisfies $\underline{\dim}_H \nu > k$. Indeed, recall (see e.g. [Fal97, Proposition 10.3]) that

$$\dim_H \mu = \text{esssup}_{x \sim \mu} \liminf_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r}.$$

Now choose X to be a compact subset of positive μ -measure of the set

$$\left\{ x \in \mathbb{R}^N : \liminf_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} \geq s \right\}$$

for fixed $s > 0$ with $k < s < \dim_H \mu$. Applying Frostman’s lemma (see e.g. [PU10, Theorem 8.6.3]) we obtain $\underline{\dim}_H \nu \geq s > k$. By Theorem 3.3, for $\gamma_{k,N}$ -almost every $V \in \text{Gr}(k, N)$,

$$\underline{\dim}_H \nu_a^V \geq \underline{\dim}_H \nu - k > 0 \text{ for } P_V \nu\text{-almost every } a \in V.$$

⁴Järvenpää and Mattila define ‘sliced measures’ in [JM98] as weak- $*$ limits of $\delta^{-k} \mu|_{P_V^{-1}(B(a, \delta))}$ as $\delta \searrow 0$, which exist for \mathcal{H}^k -almost every $a \in V$. This is a different notion than conditional measures defined in Definition 2.5, which satisfy $\mu_a^V = \lim_{\delta \rightarrow 0} (\mu(P_V^{-1}(B(a, \delta)))^{-1} \mu|_{P_V^{-1}(B(a, \delta))})$ (see [Sim12]). However, if $\underline{\dim}_H \mu > k$ (which is the only non-trivial case in Theorem 3.3), then $P_V \mu$ is absolutely continuous with respect to \mathcal{H}^k for $\gamma_{k,N}$ -almost every $V \in \text{Gr}(k, N)$ by Theorem 1.2, so the two definitions are equal up to a constant for almost every $V \in \text{Gr}(k, N)$ and $P_V \mu$ -almost every $a \in V$. Moreover, Theorem 3.3 in [JM98, Theorem 3.3] is valid for \mathcal{H}^k -almost every $a \in V$ with $\mu_a^V(\mathbb{R}^N) > 0$, which is equivalent to the statement for $P_V \mu$ -almost every $a \in V$ (by the same reason).

As Dirac's measure has dimension zero, this means that for $\gamma_{k,N}$ -almost every $V \in \text{Gr}(k, N)$, almost every conditional measure of ν with respect to P_V is not a Dirac's measure, hence by Lemma 2.6 there cannot exist a set of full ν -measure on which P_V is injective. As ν is absolutely continuous with respect to μ , the same is true for μ . By Remark 3.1, this holds also for almost every linear map $L: \mathbb{R}^N \rightarrow \mathbb{R}^k$. \square

4. PROOF OF THEOREM 1.12

Let $E = E(N, k)$ denote the set of linear maps $L: \mathbb{R}^N \rightarrow \mathbb{R}^k$ of the form

$$Lx = (\langle l_1, x \rangle, \dots, \langle l_k, x \rangle),$$

where $l_1, \dots, l_k \in \mathbb{R}^N$ satisfy $\|l_1\|, \dots, \|l_k\| \leq 1$. As E may be identified with $(B_N(0, 1))^k$, we will denote by Leb the normalized k -fold product of Lebesgue measures on $B_N(0, 1)$, considered as a probability measure on E . Note that it is enough to prove the assertion of Theorem 1.12 for Lebesgue-almost every $L \in E$, as then a rescaling gives the result for almost every linear mapping $L: \mathbb{R}^N \rightarrow \mathbb{R}^k$. By the Cauchy–Schwarz inequality, for all $x \in \mathbb{R}^N$,

$$(2) \quad \|Lx\| \leq \sqrt{N} \|x\|.$$

The following lemma is the key technical ingredient of the proof.

Lemma 4.1. *For every $x \in \mathbb{R}^N \setminus \{0\}$, $z \in \mathbb{R}^k$ and $\varepsilon > 0$,*

$$\text{Leb}(\{L \in E : \|Lx + z\| \leq \varepsilon\}) \leq C \frac{\varepsilon^k}{\|x\|^k},$$

where $C > 0$ depends only on N and k .

For the proof see [Rob11, Lemma 4.1]. We will prove each of the assertions of Theorem 1.12 separately.

4.1. Proof of assertion (i) of Theorem 1.12. We actually prove the following, slightly stronger version of the statement.

Theorem 4.2. *Let μ be a finite Borel measure in \mathbb{R}^N with a compact support X satisfying $\mathcal{H}^k(X) = 0$ and let $\phi: X \rightarrow \mathbb{R}^k$ be a Lipschitz map. Then for almost every linear map $L: \mathbb{R}^N \rightarrow \mathbb{R}^k$ there exists a Borel set $X_L \subset X$ of full μ -measure such that for every $x \in X_L$ and every $\varepsilon > 0$, there exists $\delta > 0$ for which the map $\phi_L = \phi + L$ satisfies*

$$(3) \quad \text{for every } y \in X, \text{ if } \|\phi_L(x) - \phi_L(y)\| \leq \delta, \text{ then } \|x - y\| \leq \varepsilon.$$

Proof. The first part of the argument is obtained directly from the proof of [BGŚ20, Theorem 3.1]. We include the arguments for the convenience of the reader. First, we will prove that for every $x \in X$ we have

$$(4) \quad \text{Leb}(\{L \in E : \exists_{y \in X \setminus \{x\}} \phi_L(x) = \phi_L(y)\}) = 0.$$

Note that the above set, as well as all similar sets we consider in this section, are Borel measurable as a consequence of standard considerations (see [BGŚ20, Lemma 2.4]). As $\phi: X \rightarrow \mathbb{R}^k$ is a Lipschitz map, there exists $H > 0$ so that for all $x, y \in X$,

$$(5) \quad \|\phi(x) - \phi(y)\| \leq H \|x - y\|.$$

Fix $x \in X$, $\varepsilon > 0$ and let

$$K_n = \left\{ y \in X : \|x - y\| \geq \frac{1}{n} \right\}$$

for $n \in \mathbb{N}$. Define

$$B_n = \{L \in E : \exists_{y \in K_n} \phi_L(x) = \phi_L(y)\}.$$

and note that for (4) it suffices to prove $\text{Leb}(B_n) = 0$ for each n . As $\mathcal{H}^k(K_n) \leq \mathcal{H}^k(X) = 0$, there exists a collection of balls $B_N(y_i, \varepsilon_i)$, $i \in \mathbb{N}$, for some $y_i \in K_n$ and $\varepsilon_i > 0$, such that

$$K_n \subset \bigcup_{i \in \mathbb{N}} B_N(y_i, \varepsilon_i) \quad \text{and} \quad \sum_{i=1}^{\infty} \varepsilon_i^k \leq \varepsilon.$$

Let $L \in B_n$. Then there is $y \in K_n$ such that $\phi_L(x) = \phi_L(y)$. Clearly, $y \in B_N(y_i, \varepsilon_i)$ for some $i \in \mathbb{N}$. We calculate

$$\begin{aligned} \|\phi_L(x) - \phi_L(y_i)\| &\leq \|\phi_L(x) - \phi_L(y)\| + \|\phi_L(y) - \phi_L(y_i)\| \\ &= \|\phi_L(y) - \phi_L(y_i)\| \\ &\leq \|\phi(y_i) - \phi(y)\| + \|L(y_i - y)\| \\ &\leq H\|y_i - y\| + \sqrt{N}\|y_i - y\| \\ &\leq M\varepsilon_i \end{aligned}$$

for $M = H + \sqrt{N}$, by (2) and (5). This shows

$$B_n \subset \bigcup_{i \in \mathbb{N}} \{L \in E : \|\phi_L(x) - \phi_L(y_i)\| \leq M\varepsilon_i\}.$$

Thus, using Lemma 4.1 and the fact $\|x - y_i\| \geq \frac{1}{n}$, we obtain

$$\begin{aligned} \text{Leb}(B_n) &\leq \sum_{i=1}^{\infty} \text{Leb}(\{L \in E : \|L(x - y_i) + \phi(x) - \phi(y_i)\| \leq M\varepsilon_i\}) \\ &\leq \frac{CM^k}{1/n^k} \sum_{i=1}^{\infty} \varepsilon_i^k \leq CM^k n^k \varepsilon. \end{aligned}$$

As $\varepsilon > 0$ is arbitrary, we obtain $\text{Leb}(B_n) = 0$, and thus (4) is established. Combining (4) with Fubini's theorem (see e.g. [Rud87, Theorem 8.8]), we obtain

$$(6) \quad \mu(\{x \in X : \exists_{y \in X \setminus \{x\}} \phi_L(x) = \phi_L(y)\}) = 0$$

for Lebesgue-almost every $L \in E$. Hence, the set

$$X_L = X \setminus \{x \in X : \exists_{y \in X \setminus \{x\}} \phi_L(x) = \phi_L(y)\}$$

is a full μ -measure set on which ϕ_L is injective (which proves Theorem 1.11). To obtain additionally the continuity of ϕ_L^{-1} on X_L , fix $L \in E$ satisfying (6). We claim that every $x \in X_L$ satisfies (3). If not, then there exists $\varepsilon > 0$ such that for every $n \geq 1$ there exists $y_n \in X$ satisfying

$$\|\phi_L(x) - \phi_L(y_n)\| \leq \frac{1}{n} \quad \text{and} \quad \|x - y_n\| > \varepsilon.$$

As X is compact, there is a converging subsequence $y_{n_k} \rightarrow y$ for some $y = y(L, x) \in X$. By the continuity of ϕ_L , we have $\phi_L(x) = \phi_L(y)$ and $\|x - y\| \geq \varepsilon$, in particular $x \neq y$, contradicting $x \in X_L$. \square

4.2. Proof of assertion (ii) of Theorem 1.12. The proof combines the techniques of [HK99, Theorem 3.1] and [BGŚ20, Theorem 3.1].

Lemma 4.3. *Let X be a compact subset of \mathbb{R}^N with $\overline{\dim}_B X < k$. Fix $\theta \in (0, k - \overline{\dim}_B X)$ and a Lipschitz map $\phi: X \rightarrow \mathbb{R}^k$. Then there exists a constant $D > 0$ such that*

$$\text{Leb}(\{L \in E : \exists_{y \in X} \|\phi_L(x) - \phi_L(y)\| \leq \varepsilon \text{ and } \|x - y\| \geq \delta\}) \leq D\delta^{-k} \varepsilon^{k - \overline{\dim}_B X - \theta}$$

for every $x \in X$ and $0 < 2\varepsilon \leq \delta$, where $\phi_L = \phi + L$.

Proof. Set $d = \overline{\dim}_B X$. By the definition of $\overline{\dim}_B$, there exists a constant $D = D(X, \theta)$ such that for every $\varepsilon > 0$ there exists a cover

$$(7) \quad X \subset \bigcup_{i=1}^{N_\varepsilon} B(y_i, \varepsilon) \quad \text{with} \quad N_\varepsilon \leq D\varepsilon^{-(d+\theta)}.$$

Consider $x, y \in X$ and $L \in E$ such that $\|\phi_L(x) - \phi_L(y)\| \leq \varepsilon$ and $\|x - y\| \geq \delta$. Let y_i be such that $y \in B(y_i, \varepsilon)$. Then, recalling that we assume $2\varepsilon \leq \delta$,

$$\begin{aligned} \|L(x - y_i) - (\phi(y_i) - \phi(x))\| &= \|\phi_L(x) - \phi_L(y_i)\| \leq \|\phi_L(x) - \phi_L(y)\| + \|\phi_L(y) - \phi_L(y_i)\| \\ &\leq \varepsilon + (\text{Lip}(\phi) + \sup_{L \in E} \|L\|)\varepsilon \leq M\varepsilon, \end{aligned}$$

where $M = 1 + \text{Lip}(\phi) + \sup_{L \in E} \|L\| < \infty$. Moreover,

$$\|x - y_i\| \geq \|x - y\| - \|y - y_i\| \geq \delta - \varepsilon \geq \frac{\delta}{2}.$$

Therefore,

$$\begin{aligned} &\{L \in E : \exists_{y \in X} \|\phi_L(x) - \phi_L(y)\| \leq \varepsilon \text{ and } \|x - y\| \geq \delta\} \\ &\subset \bigcup_{i=1}^{N_\varepsilon} \{L \in E : \|L(x - y_i) - (\phi(y_i) - \phi(x))\| \leq M\varepsilon \text{ and } \|x - y_i\| \geq \delta/2\}. \end{aligned}$$

Hence, by Lemma 4.1 and (7),

$$\begin{aligned} &\text{Leb}(\{L \in E : \exists_{y \in X} \|\phi_L(x) - \phi_L(y)\| \leq \varepsilon \text{ and } \|x - y\| \geq \delta\}) \\ &\leq C2^k N_\varepsilon \delta^{-k} M^k \varepsilon^k \leq C2^k D M^k \delta^{-k} \varepsilon^{k-d-\theta}. \end{aligned}$$

□

Proof of Theorem 1.12ii. Set $d = \overline{\dim}_B(\text{supp } \mu)$. Fix $\alpha \in (0, 1 - \frac{d}{k})$ and let $\theta \in (0, k - d)$ be such that $\alpha < 1 - \frac{d+\theta}{k}$. Let $H = \sup_{L \in E} \text{diam}(\phi_L(X))$. For a fixed $x \in X$, by (4) (which we can apply as $\dim_H \mu \leq d < k$) and Lemma 4.3,

$$\begin{aligned} &\text{Leb}(\{L \in E : \forall_{M>0} \exists_{y \in X} \|x - y\| > M \|\phi_L(x) - \phi_L(y)\|^\alpha\}) \\ &= \lim_{M \rightarrow \infty} \text{Leb}(\{L \in E : \exists_{y \in X} \|x - y\| > M \|\phi_L(x) - \phi_L(y)\|^\alpha\}) \\ &= \lim_{M \rightarrow \infty} \sum_{m=0}^{\infty} \text{Leb}(\{L \in E : \exists_{y \in X} 2^{-(m+1)}H < \|\phi_L(x) - \phi_L(y)\| \leq 2^{-m}H \\ &\quad \text{and } \|x - y\| > M \|\phi_L(x) - \phi_L(y)\|^\alpha\}) \\ &\quad + \text{Leb}(\{L \in E : \exists_{y \in X} \phi_L(x) = \phi_L(y) \text{ and } \|x - y\| > 0\}) \\ &\stackrel{(4)}{\leq} \lim_{M \rightarrow \infty} \sum_{m=0}^{\infty} \text{Leb}(\{L \in E : \exists_{y \in X} \|\phi_L(x) - \phi_L(y)\| \leq 2^{-m}H \text{ and } \|x - y\| > MH^\alpha 2^{-\alpha(m+1)}\}) \\ &\stackrel{\text{Lemma 4.3}}{\leq} \lim_{M \rightarrow \infty} \sum_{m=0}^{\infty} DM^{-k} H^{-\alpha k} 2^{\alpha k(m+1)} 2^{-m(k-d-\theta)} H^{k-d-\theta} \\ &= \lim_{M \rightarrow \infty} DM^{-k} H^{k(1-\alpha)-d-\theta} 2^{\alpha k} \sum_{m=0}^{\infty} 2^{m(\alpha k - k + d + \theta)} = 0, \end{aligned}$$

as the series $\sum_{m=0}^{\infty} 2^{m(\alpha k - k + d + \theta)}$ converges since we assume $\alpha < 1 - \frac{d+\theta}{k}$. This proves that for every $x \in X$, the condition

$$(8) \quad \|x - y\| \leq M \|\phi_L(x) - \phi_L(y)\|^\alpha \quad \text{for some } M = M(x, L) \text{ and every } y \in X$$

is satisfied for almost every $L \in E$. Therefore, by Fubini's theorem, for almost every $L \in E$, the condition (8) holds for μ -almost every $x \in X$. Finally, note that by taking a countable intersection of full Lebesgue measure sets, we can assume that for almost every $L \in E$, the condition (8) holds for every $\alpha < 1 - \frac{d}{k}$. □

4.3. Proof of assertion (iii) of Theorem 1.12. Again, we prove a stronger result.

Theorem 4.4. *Let $R > 0$ and $\eta > 1$. Let μ be a probability measure in \mathbb{R}^N with a compact support X satisfying $|X| \leq R$ and $\dim_A X < k$. Let $\phi: X \rightarrow \mathbb{R}^k$ be a Lipschitz map and fix $\theta \in (0, k - \dim_A X)$. Then for almost every linear mapping $L: \mathbb{R}^N \rightarrow \mathbb{R}^k$ there exists a Borel set $X_L \subset X$ of full μ -measure such that for each point $x \in X_L$ there exists $C > 0$ for which the map $\phi_L = \phi + L$ satisfies*

$$(9) \quad \|\phi_L(x) - \phi_L(y)\| \geq Cf(\|x - y\|) \quad \text{for every } y \in X,$$

where

$$f(x) = \frac{x}{(\log_2(2R/x))^{\eta/\theta}}.$$

Proof. Once more, by Fubini's theorem it is enough to prove that for every $x \in X$, the condition (9) holds for almost every $L \in E$. The rest of the proof is a combination of the methods set forth in [Ols02, Theorem 5.2] and [BGŚ20, Theorem 3.1]. As $\phi: X \rightarrow \mathbb{R}^k$ is a Lipschitz map, there exists $H > 0$ so that for all $x, y \in X$,

$$(10) \quad \|\phi(x) - \phi(y)\| \leq H \|x - y\|.$$

Fix $x \in X$. Define $r_n = \frac{R}{2^n}$ and $\rho_n = f(r_{n-1}) > 0$. Note that for $0 < x \leq R$, we have $\frac{f(x)}{x} \leq 1$ as $\log_2 \frac{2R}{x} \geq 1$. Moreover, a simple calculation shows that the function f is monotone increasing on $(0, R]$. For $n \geq 1$, define

$$Z_n = \{y \in X : r_n \leq \|y - x\| \leq r_{n-1}\}.$$

By the definition of the Assouad dimension, there exists $K > 0$ such that for every $0 < s < r_{n-1}$ and a ball B of radius r_{n-1} , the set $X \cap B$ may be covered by $K(\frac{r_{n-1}}{s})^{k-\theta}$ balls of radius s . Let $c > 2$ satisfy $c^\theta > K$. We conclude that the set Z_n , which is contained in a ball B of radius r_{n-1} around x , may be covered by at most $\ell_{n,i} \leq Kc^{i(k-\theta)}(\frac{r_{n-1}}{\rho_n})^{k-\theta}$ balls $\{B(a_{n,i,j}, \frac{\rho_n}{c^i})\}_{j=1}^{\ell_{n,i}}$ of radius $\frac{\rho_n}{c^i}$ (with centers in Z_n) for $i \geq 1$ (recall that $\rho_n \leq r_{n-1}$). Thus, $Z_n \subset \bigcup_{j=1}^{\ell_{n,i}} B(a_{n,i,j}, \frac{\rho_n}{c^i})$.

For $i \geq 2$, define

$$U_i = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{\ell_{n,i}} B\left(a_{n,i,j}, \frac{2\rho_n}{c^i}\right).$$

Every center $a_{n,i,j}$ satisfies $\|x - a_{n,i,j}\| \geq r_n$, so the ball centred at $a_{n,i,j}$ of radius $\frac{2\rho_n}{c^i} < \frac{\rho_n}{2} \leq \frac{r_{n-1}}{2} \leq r_n$ does not contain x . Thus, for $i \geq 2$,

$$X \setminus \{x\} \subset U_i.$$

In order to establish the condition (9) for a fixed $x \in X$ and Lebesgue-almost every $L \in E$, it is enough to show

$$\lim_{i \rightarrow \infty} \text{Leb} \left\{ L \in E : \exists_{y \in X} \exists_{n \geq 1} \|\phi_L(x) - \phi_L(y)\| < \frac{\rho_n}{c^i} \text{ and } \|x - y\| \geq r_n \right\} = 0$$

Indeed, this implies that for almost every $L \in E$ there exists $i = i(L) \geq 2$, such that for every $y \in X$, $\|x - y\| \geq r_n$ implies $\|\phi_L(x) - \phi_L(y)\| \geq \frac{\rho_n}{c^i}$. As every $y \in X \setminus \{x\}$ is contained in some Z_n (recall that $|X| \leq R$), this implies that for every $y \in X \setminus \{x\}$ (using monotonicity of f on $(0, R]$), we obtain

$$\|\phi_L(x) - \phi_L(y)\| \geq \frac{\rho_n}{c^i} = \frac{f(r_{n-1})}{c^i} \geq \frac{1}{c^i} f(\|x - y\|).$$

Let

$$A_i = \left\{ L \in E : \exists_{y \in X} \exists_{n \geq 1} \|\phi_L(x) - \phi_L(y)\| < \frac{\rho_n}{c^i} \text{ and } \|x - y\| \geq r_n \right\}.$$

Clearly, A_i is a Borel set. For $L \in A_i$ one may find $y \in X$ and $n \geq 1$ such that $\|\phi_L(x) - \phi_L(y)\| < \frac{\rho_n}{c^i}$ and $\|x - y\| \geq r_n$. Consequently, $y \in \bigcup_{m=1}^n Z_m$. Therefore, one may find a center $a_{m,i,j}$ such that $y \in B(a_{m,i,j}, \frac{\rho_m}{c^i})$. Note that

$$\|\phi_L(x) - \phi_L(a_{m,i,j})\| \leq \|\phi_L(x) - \phi_L(y)\| + \|\phi_L(y) - \phi_L(a_{m,i,j})\|.$$

By (2) and (10),

$$\|\phi_L(y) - \phi_L(a_{m,i,j})\| \leq \|Ly - La_{m,i,j}\| + \|\phi(y) - \phi(a_{m,i,j})\| \leq \sqrt{N} \frac{\rho_m}{c^i} + H \frac{\rho_m}{c^i}.$$

Thus,

$$\|\phi_L(x) - \phi_L(a_{m,i,j})\| \leq \frac{\rho_n}{c^i} + (\sqrt{N} + H) \frac{\rho_m}{c^i} \leq Q \frac{\rho_m}{c^i},$$

where $Q = 1 + \sqrt{N} + H$. We conclude

$$A_i \subset \bigcup_{m=1}^{\infty} \bigcup_{j=1}^{\ell_{m,i}} A_{m,i,j}$$

for

$$A_{m,i,j} = \left\{ L \in E : \|\phi_L(x) - \phi_L(a_{m,i,j})\| \leq Q \frac{\rho_m}{c^i} \right\}.$$

By Lemma 4.1 (recall that $a_{m,i,j} \in Z_m$),

$$\text{Leb}(A_{m,i,j}) \leq C \frac{(Q \frac{\rho_m}{c^i})^k}{r_m^k}.$$

Thus,

$$\text{Leb}(A_i) \leq \sum_{m=1}^{\infty} \sum_{j=1}^{\ell_{m,i}} \text{Leb}(A_{m,i,j}) \leq \sum_{m=1}^{\infty} K c^{i(k-\theta)} \left(\frac{r_{m-1}}{\rho_m} \right)^{k-\theta} C \frac{(Q \frac{\rho_m}{c^i})^k}{r_m^k} \leq \frac{KC(2Q)^k}{c^{i\theta}} \sum_{m=1}^{\infty} \left(\frac{f(r_{m-1})}{r_{m-1}} \right)^{\theta}.$$

We notice

$$\left(\frac{f(r_{m-1})}{r_{m-1}} \right)^{\theta} = \frac{1}{(\log_2(2^m))^{\eta}} = \frac{1}{m^{\eta}}.$$

Thus,

$$\text{Leb}(A_i) \leq \frac{KC(2Q)^k}{c^{i\theta}} \sum_{m=1}^{\infty} \frac{1}{m^{\eta}}.$$

As $\sum_{m=1}^{\infty} \frac{1}{m^{\eta}} < \infty$, this implies $\lim_{i \rightarrow \infty} \text{Leb}(A_i) = 0$, which ends the proof. \square

5. MEASURES WITH ALL ALMOST-SURELY INJECTIVE PROJECTIONS

5.1. Proof of Theorem 1.9. The measure μ will be defined in two steps. First, we define a measure ν on the unit interval by randomizing digits in dyadic expansions and then push ν to the graph of the function $x \mapsto x^2$. For the first step, it is convenient to work in the symbolic space $\{0, 1\}^{\mathbb{N}}$.

Partition $\mathbb{N} = \{1, 2, \dots\}$ into blocks $B_n = \{2n-1, 2n\}$, $n \geq 1$ and denote $L_n = 2n-1$, $R_n = 2n$, so that the block B_n consists of the left bit L_n and the right bit R_n . For $\omega \in \{0, 1\}^{\mathbb{N}}$ we use the notation $\omega = (\omega_1, \omega_2, \dots)$. Set

$$\Sigma = \{\omega \in \{0, 1\}^{\mathbb{N}} : \omega_{L_n} = 0 \text{ for every } n \geq 1\}.$$

Fix $p \in (0, 1/2)$. Define a probability measure \mathbf{p} on $\{0, 1\}^2$ as

$$\mathbf{p} = p\delta_{(0,1)} + (1-p)\delta_{(0,0)}$$

and let \mathbb{P} be a probability measure on $\{0, 1\}^{\mathbb{N}}$ given by

$$\mathbb{P} = \mathbf{p}^{\otimes \mathbb{N}}$$

(we use above the identification $\{0, 1\}^{\mathbb{N}} = (\{0, 1\}^2)^{\mathbb{N}}$). Clearly, $\mathbb{P}(\Sigma) = 1$. Now we transport \mathbb{P} to the unit interval by setting

$$\nu = \Pi\mathbb{P}, \quad X = \Pi(\Sigma),$$

where Π is given by

$$\Pi(\omega) = \sum_{j=1}^{\infty} \omega_j 2^{-j}.$$

The last step is to lift ν to the graph of a non-linear function. Let $f: [0, 1] \rightarrow [0, 1]$ be given by $f(x) = x^2$ and $\Psi: [0, 1] \rightarrow [0, 1]^2$ by $\Psi(x) = (x, f(x))$. Finally, we set

$$\mu = \Psi\nu$$

and claim that the measure μ satisfies the properties from Theorem 1.9. First, note that $\dim_H \nu = \frac{-p \log p - (1-p) \log(1-p)}{\log 4}$ (see [BP17, Example 1.5.6]). As Ψ is bi-Lipschitz, we have $\dim_H \mu = \dim_H \nu > 0$. For the injectivity part of Theorem 1.9, fix a non-zero linear map $L: \mathbb{R}^2 \rightarrow \mathbb{R}$, $L(x, y) = \alpha x + \beta y$. As f is a bijection on $[0, 1]$, we can assume that $\alpha \neq 0$ and $\beta \neq 0$, as otherwise L is injective on the whole graph $\Psi([0, 1])$ and the claim of Theorem 1.9 follows trivially. Note that for $x, y \in [0, 1]$ with $x \neq y$ we have

$$(11) \quad L(x, f(x)) = L(y, f(y)) \iff \frac{f(y) - f(x)}{x - y} = \frac{\alpha}{\beta} \iff x + y = -\frac{\alpha}{\beta}.$$

Therefore, in order to show the injectivity of L , we need to study solutions of the equation $x + y = z$ for fixed z and x, y taken from X . We will do so in terms of the dyadic expansions.

Note that every $z \in [0, 1)$ has a unique dyadic expansion $z = \sum_{j=1}^{\infty} z_j 2^{-j}$ such that the sequence $(z_j)_{j=1}^{\infty} \in \{0, 1\}^{\mathbb{N}}$ is not eventually equal to 1 (we say that the dyadic expansion does not terminate with 1's). Moreover, the only points $z \in [0, 1)$ which have a non-unique dyadic expansion are the dyadic rationals from $(0, 1)$. For them, there are exactly two expansions, one terminating with 1's and one terminating with 0's. As Σ does not contain sequences terminating with 1's, we see that Π is injective on Σ . Therefore, for $x, y \in X$ we will write $x = \sum_{j=1}^{\infty} x_j 2^{-j}$, $y = \sum_{j=1}^{\infty} y_j 2^{-j}$ for its unique dyadic expansion which does not terminate with 1's, so that $\Pi((x_1, x_2, \dots)) = x$, $\Pi((y_1, y_2, \dots)) = y$ and $(x_1, x_2, \dots), (y_1, y_2, \dots) \in \Sigma$. First, we need a technical lemma.

Lemma 5.1. *Let $x, y, z \in [0, 1)$ have dyadic expansions*

$$x = \sum_{j=1}^{\infty} x_j 2^{-j}, \quad y = \sum_{j=1}^{\infty} y_j 2^{-j}, \quad z = \sum_{j=1}^{\infty} z_j 2^{-j}$$

which do not terminate with 1's. Assume $x + y = z$. Then for every $k \in \mathbb{N}$, the condition $x_k = y_k = 0$ implies $\sum_{j < k} z_j 2^{-j} = \sum_{j < k} (x_j + y_j) 2^{-j}$. Consequently, for every $k, m \in \mathbb{N}$ with $k < m$ such that $x_k = y_k = x_m = y_m = 0$ we have

$$\sum_{k \leq j < m} z_j 2^{-j} = \sum_{k \leq j < m} (x_j + y_j) 2^{-j}.$$

Proof. We begin by proving the first assertion. Assume $x_k = y_k = 0$. Then

$$\sum_{j < k} z_j 2^{-j} = a 2^{-(k-1)}, \quad \sum_{j < k} (x_j + y_j) 2^{-j} = b 2^{-(k-1)}$$

for some $a, b \in \mathbb{N} \cup \{0\}$. As $x + y = z$, we have

$$(12) \quad a 2^{-(k-1)} + \sum_{j \geq k} z_j 2^{-j} = b 2^{-(k-1)} + \sum_{j \geq k} (x_j + y_j) 2^{-j}.$$

Since x_j, y_j, z_j are not eventually equal to 1 and $x_k = y_k = 0$, we obtain

$$0 \leq \sum_{j \geq k} z_j 2^{-j} < 2^{-(k-1)}, \quad 0 \leq \sum_{j \geq k} (x_j + y_j) 2^{-j} < 2^{-(k-1)}.$$

Combining this with (12) yields

$$|a - b| 2^{-(k-1)} = \left| \sum_{j \geq k} z_j 2^{-j} - \sum_{j \geq k} (x_j + y_j) 2^{-j} \right| < 2^{-(k-1)},$$

so $a = b$ and, consequently, $\sum_{j < k} z_j 2^{-j} = \sum_{j < k} (x_j + y_j) 2^{-j}$.

Now, if $x_k = y_k = x_m = y_m = 0$, then by the first assertion,

$$\begin{aligned} \sum_{k \leq j < m} z_j 2^{-j} &= \sum_{j < m} z_j 2^{-j} - \sum_{j < k} z_j 2^{-j} = \sum_{j < m} (x_j + y_j) 2^{-j} - \sum_{j < k} (x_j + y_j) 2^{-j} \\ &= \sum_{k \leq j < m} (x_j + y_j) 2^{-j} = \sum_{k < j < m} (x_j + y_j) 2^{-j}. \end{aligned}$$

□

The above lemma and the structure of Σ provide the following consequence.

Lemma 5.2. *Let $z = x + y$ for some $x, y \in X$, such that z has a dyadic expansion $z = \sum_{j=1}^{\infty} z_j 2^{-j}$ which does not terminate with 1's. Then, denoting $z|_{B_n} = (z_{L_n}, z_{R_n})$, for every $n \geq 1$ we have the following.*

- (i) $z|_{B_n} \in \{(0, 0), (0, 1), (1, 0)\}$.
- (ii) If $z|_{B_n} = (0, 0)$, then $x_{R_n} = y_{R_n} = 0$.
- (iii) If $z|_{B_n} = (1, 0)$, then $x_{R_n} = y_{R_n} = 1$.
- (iv) If $z|_{B_n} = (0, 1)$, then exactly one of bits x_{R_n}, y_{R_n} is equal to 1.

Proof. As $x_{L_n} = y_{L_n} = 0$ for $n \geq 1$, we see from Lemma 5.1 that

$$\sum_{j \in B_n} z_j 2^{-j} = (x_{R_n} + y_{R_n}) 2^{-R_n}.$$

The assertions of the lemma follow from this equality by recalling that $B_n = \{R_n - 1, R_n\} = \{L_n, R_n\}$. □

Theorem 1.9 is an immediate consequence of the following proposition.

Proposition 5.3. *Let $L: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a linear map of the form $L(x, y) = \alpha x + \beta y$ with $\alpha \neq 0$ and $\beta \neq 0$. Then there exists a Borel set $\Sigma_L \subset \Sigma$ with $\mathbb{P}(\Sigma_L) = 0$, such that if $x, y \in X$, $x \neq y$ have dyadic expansions $x = \sum_{j=1}^{\infty} x_j 2^{-j}$, $y = \sum_{j=1}^{\infty} y_j 2^{-j}$ which do not terminate with 1's and satisfy $L(x, f(x)) = L(y, f(y))$, then at least one of the sequences $(x_1, x_2, \dots), (y_1, y_2, \dots)$ belongs to Σ_L .*

Indeed, if we set $X_L = \Psi(X \setminus \Pi(\Sigma_L))$, then X_L is Borel (by [Kec95, Theorem 5.1]), $\mu(X_L) = 1$ and L is injective on X_L .

Proof of Proposition 5.3. Assume that there exist $x, y \in X$ such that $L(x, f(x)) = L(y, f(y))$, as otherwise there is nothing to prove. Set $z = z(L) = -\frac{\alpha}{\beta}$. Note that $z \in [0, 1)$ as $x_1 = y_1 = 0$ and the expansions of x and y do not terminate with 1's. Therefore, z has a unique dyadic expansion $z = \sum_{j=1}^{\infty} z_j 2^{-j}$ which does not terminate with 1's. We define the set Σ_L in terms of the sequence $(z_j)_{j=1}^{\infty}$.

First, assume that $z|_{B_n} \in \{(0, 0), (1, 0)\}$ for infinitely many $n \geq 1$. In this case we set

$$\Sigma_L = \{\omega \in \{0, 1\}^{\mathbb{N}} : \text{for each } n \geq 1, \omega_{R_n} = 0 \text{ if } z|_{B_n} = (0, 0) \text{ and } \omega_{R_n} = 1 \text{ if } z|_{B_n} = (1, 0)\}.$$

Clearly, $\mathbb{P}(\Sigma_L) = 0$. If $x, y \in X$, $x \neq y$ with dyadic expansions $x = \Pi(x_1, x_2, \dots)$, $y = \Pi(y_1, y_2, \dots)$ satisfy $L(x, f(x)) = L(y, f(y))$, then by (11), we have $x + y = z$, hence Lemma 5.2 implies $(x_1, x_2, \dots) \in \Sigma_L$ and $(y_1, y_2, \dots) \in \Sigma_L$.

The remaining case is the one with $z|_{B_n} \in \{(0, 0), (1, 0)\}$ only for finitely many $n \geq 1$. Then, by Lemma 5.2 we must have $z|_{B_n} = (0, 1)$ for all n large enough. Set

$$\Sigma_L = \left\{ \omega \in \{0, 1\}^{\mathbb{N}} : \limsup_{N \rightarrow \infty} \frac{1}{N} \#\{1 \leq n \leq N : \omega_{R_n} = 1\} \geq \frac{1}{2} \right\}.$$

As $p < 1/2$, the Birkhoff ergodic theorem gives $\mathbb{P}(\Sigma_L) = 0$. Again, if $x, y \in X$, $x \neq y$ satisfy $L(x, f(x)) = L(y, f(y))$, then $x + y = z$, so by Lemma 5.2 and the assumption on z , we have

$$\begin{aligned} 1 &= \limsup_{N \rightarrow \infty} \frac{1}{N} \# \{1 \leq n \leq N : z_{B_n} = (0, 1)\} \\ &\leq \limsup_{N \rightarrow \infty} \frac{1}{N} \# \{1 \leq n \leq N : x_{R_n} = 1 \text{ or } y_{R_n} = 1\} \\ &\leq \limsup_{N \rightarrow \infty} \frac{1}{N} \# \{1 \leq n \leq N : x_{R_n} = 1\} + \limsup_{N \rightarrow \infty} \frac{1}{N} \# \{1 \leq n \leq N : y_{R_n} = 1\}. \end{aligned}$$

Therefore, $(x_1, x_2, \dots) \in \Sigma_L$ or $(y_1, y_2, \dots) \in \Sigma_L$. This concludes the proof of the proposition. \square

5.2. Self-similar measures. In this subsection we will prove Proposition 1.10. We actually obtain a stronger statement. Let $\varphi_i: \mathbb{R}^N \rightarrow \mathbb{R}^N$, $i \in I$, be a finite collection of contractions and let $(p_i)_{i \in I}$ be a strictly positive probability vector. It is well-known (see [Hut81]) that there exists a unique Borel probability measure in \mathbb{R}^N which is stationary for this system, i.e. satisfies

$$\mu = \sum_{i \in I} p_i \varphi_i \mu.$$

Proposition 5.4. *Assume that the iterated function system given by $\{\varphi_i : i \in I\}$ satisfies $\varphi_{i_2} = \varphi_{i_1} + t$ for $i_1, i_2 \in I$, $i_1 \neq i_2$ and some non-zero vector $t \in \mathbb{R}^N$. Let μ be the stationary measure for a probability vector $(p_i)_{i \in I}$. Consider $V \in \text{Gr}(k, N)$ contained in the orthogonal complement of t . Then P_V is not injective on any set of full μ -measure.*

Proof. We can assume $i_1 = 1$, $i_2 = 2$. Let $\nu = p_1 \varphi_1 \mu + p_2 \varphi_2 \mu$. As ν is absolutely continuous with respect to μ , it is enough to prove the statement for the measure ν . Consider the system of conditional measures $\{\nu_a : a \in V\}$ of ν with respect to the map P_V . Let $\nu^{(1)} = \varphi_1 \mu$, $\nu^{(2)} = \varphi_2 \mu$, so that

$$(13) \quad \nu = p_1 \nu^{(1)} + p_2 \nu^{(2)}$$

and let $\{\nu_a^{(i)} : a \in V\}$, $i = 1, 2$ be the system of conditional measures of $\nu^{(i)}$ with respect to P_V . Let

$$S_t: \mathbb{R}^N \rightarrow \mathbb{R}^N, \quad S_t(x) = x + t.$$

A crucial observation is that

$$(14) \quad \nu_a^{(2)} = S_t \nu_a^{(1)} \quad \text{for almost every } a \in V \text{ with respect to the measure } P_V \nu^{(1)} = P_V \nu^{(2)}.$$

To verify that, note first that as $\varphi_2(x) = \varphi_1(x) + t$ for all $x \in \mathbb{R}^N$, we have $\nu^{(2)} = S_t \nu^{(1)}$. As V is contained in the orthogonal complement of t , we have $P_V \circ S_t = P_V$, so

$$(15) \quad P_V \nu^{(2)} = P_V S_t \nu^{(1)} = P_V \nu^{(1)}.$$

Therefore, to obtain (14), it is enough to check that $\{S_t \nu_a^{(1)} : a \in V\}$ is a system of conditional measures of $\nu^{(2)}$ with respect to P_V . As $\nu_a^{(1)}(P_V^{-1}(a)) = 1$, we have

$$S_t \nu_a^{(1)}(P_V^{-1}(a)) = \nu_a^{(1)}(S_t^{-1} P_V^{-1}(a)) = \nu_a^{(1)}(P_V^{-1}(a)) = 1,$$

hence we see from (15) that the conditions (i)–(ii) of Definition 2.5 are satisfied. For the condition (iii), we use (15) once more to obtain

$$\nu^{(2)}(A) = \nu^{(1)}(S_t^{-1} A) = \int_V \nu_a^{(1)}(S_t^{-1} A) dP_V \nu^{(1)}(a) = \int_V S_t \nu_a^{(1)}(A) dP_V \nu^{(2)}(a).$$

This shows that indeed $\{S_t \nu_a^{(1)} : a \in V\}$ is a system of conditional measures for $\nu^{(2)}$ with respect to P_V , hence (14) is established by its uniqueness. By (13) and (15) we have

$$(16) \quad \nu_a = p_1 \nu_a^{(1)} + p_2 \nu_a^{(2)} \quad \text{for } P_V \nu\text{-almost every } a \in V.$$

Now, if P_V is injective on a set of full ν -measure, then by Lemma 2.6, we could conclude that ν_a is a Dirac's measure for $P_V \nu$ -almost every $a \in V$. By (16), this would imply that $\nu_a^{(1)} = \nu_a^{(2)}$

for $P_V\nu$ -almost every $a \in V$. This would make a contradiction with (14), as t is non-zero and (15) gives $P_V\nu = P_V\nu^{(1)} = P_V\nu^{(2)}$. \square

Proposition 1.10 is an immediate consequence of Proposition 5.4, as any homogeneous iterated function system consisting of similarities which are not all equal must contain two distinct translation vectors t_i .

6. EXAMPLES

In this section we provide examples showing that Theorems 1.7 and 1.12 cannot be improved in the following directions:

- (a) in assertion (i), $\overline{\dim}_H X$ cannot be replaced by $\dim_H \mu$,
- (b) in assertion (ii), $\overline{\dim}_B X$ cannot be replaced by $\dim_H X$ and the range of α cannot be extended (in terms of $\overline{\dim}_B X$) to any interval strictly larger than $(0, 1 - \overline{\dim}_B X/k]$,
- (c) in assertion (iii), $\dim_A X$ cannot be replaced by $\overline{\dim}_B X$.

Considering the setup of Theorem 1.12, we construct suitable examples for a measure μ of compact support $X \subset \mathbb{R}^N$ and an open set of maps $\mathcal{U} \subset \text{Lip}(X, \mathbb{R}^k)$, $k < N$, containing all full-rank linear maps $L \in \text{Lin}(\mathbb{R}^N, \mathbb{R}^k)$ restricted to X . As every prevalent set is dense (see [HSY92, Fact 2]), the examples show that Theorem 1.12 is sharp (in the sense of the items (a)–(c)) also as a result on embeddings via prevalent maps in the space of Lipschitz or C^r , $r = 1, 2, \dots, \infty$, maps into \mathbb{R}^k (see Remark 2.4).

As described in Remark 3.1, every full-rank linear map $L \in \text{Lin}(\mathbb{R}^N, \mathbb{R}^k)$, $k \leq N$, can be uniquely represented as $L = \Psi \circ P_V$, where $V = (\text{Ker } L)^\perp$ and $\Psi: V \rightarrow \mathbb{R}^k$ is a linear isomorphism, such that Lebesgue-almost all linear maps $L \in \text{Lin}(\mathbb{R}^N, \mathbb{R}^k)$ correspond to $\gamma_{k,N}$ -almost all orthogonal projections P_V , $V \in \text{Gr}(k, N)$, for the rotation-invariant measure $\gamma_{k,N}$, and open sets in $\text{Lin}(\mathbb{R}^N, \mathbb{R}^k)$ correspond to open sets in $\text{Gr}(k, N)$. It follows that the examples are valid also within the setup of Theorem 1.7.

The following proposition provides examples corresponding to the item (a).

Proposition 6.1. *For every $1 \leq k < N$ there exists a finite Borel measure in \mathbb{R}^N with a compact support X and an open set $\mathcal{U} \subset \text{Lip}(X, \mathbb{R}^k)$, containing all full-rank linear maps $L \in \text{Lin}(\mathbb{R}^N, \mathbb{R}^k)$ restricted to X , such that $\dim_H \mu = 0$ and no map $\phi \in \mathcal{U}$ is injective on a full μ -measure set with continuous inverse.*

Proof. Let $X = \overline{B_N(0, 1)} \subset \mathbb{R}^N$ and let Y be a countable dense subset of X . Consider a finite Borel measure μ on Y such that $\mu(\{x\}) > 0$ for every $x \in Y$. Then the (topological) support of μ is equal to X and $\dim_H \mu = 0$. Let $L \in \text{Lin}(\mathbb{R}^N, \mathbb{R}^k)$ be a full-rank linear map. As noted above, we can write $L = \Psi \circ P_V$, where $V = (\text{Ker } L)^\perp$ and $\Psi: V \rightarrow \mathbb{R}^k$ is a linear isomorphism.

Choose $w \in V^\perp$ such that $\|w\| = \frac{1}{2}$ and let

$$A_+ = (V + w) \cap X, \quad A_- = (V - w) \cap X.$$

Then $\text{dist}(A_+, A_-) = 1$ and $B = P_V(A_+) = P_V(A_-)$ is a closed k -dimensional ball in V . Take $f \in \text{Lip}(X, \mathbb{R}^k)$ such that $\|f\|_{\text{Lip}} < \varepsilon$ for a small $\varepsilon > 0$, where the Lipschitz norm $\|\cdot\|_{\text{Lip}}$ is defined in (1). Then $g = \Psi^{-1} \circ f \in \text{Lip}(X, V) \simeq \text{Lip}(X, \mathbb{R}^k)$ and $\|g\|_{\text{Lip}}$ is arbitrarily small for sufficiently small ε . Set $\psi = P_V + g: \mathbb{R}^N \rightarrow V$. As $P_V|_{A_\pm}$ is a translation onto B , the map $\psi|_{A_\pm}$ is bi-Lipschitz and

$$(17) \quad \psi(A_+) \cap \psi(A_-) \supset U \cap V \neq \emptyset$$

for an open set $U \subset \mathbb{R}^N$, provided ε is chosen sufficiently small. Let \tilde{A}_+, \tilde{A}_- be disjoint compact sets containing, respectively, some neighbourhoods of A_+, A_- . By (17) and the density of Y , there exists $x_+ \in \tilde{A}_+ \cap Y$ such that $\psi(x_+) \in U \cap V$. Then, again by (17) and the density of Y , there exists $x_- \in \tilde{A}_-$ such that $\psi(x_-) = \psi(x_+)$, and a sequence of points $x_n \in \tilde{A}_- \cap Y$ with $x_n \rightarrow x_-$.

Suppose ψ is injective on a full μ -measure subset of X . Then ψ is injective on Y . However, we have $(\psi|_Y)^{-1}(\psi(x_+)) = x_+ \in \tilde{A}_+$, $\psi(x_n) \rightarrow \psi(x_-) = \psi(x_+)$ and $(\psi|_Y)^{-1}(\psi(x_n)) = x_n \in \tilde{A}_-$. As \tilde{A}_+, \tilde{A}_- are disjoint compact sets, $(\psi|_Y)^{-1}$ cannot be continuous at $\psi(x_+)$.

Concluding, for every full-rank linear map $L \in \text{Lin}(\mathbb{R}^N, \mathbb{R}^k)$ we have found $\varepsilon = \varepsilon(L) > 0$ such that $\psi = P_V + g$ (where $L = \Psi \circ P_V$ and $g = \Psi^{-1} \circ f$) is not injective on a full μ -measure set with continuous inverse for every $f \in \text{Lip}(X, \mathbb{R}^k)$ with $\|f\|_{\text{Lip}} < \varepsilon$. Consequently, $\phi = L + f = \Psi \circ (P_V + g)$ is not injective on a full μ -measure set with continuous inverse. This proves the proposition, as \mathcal{U} can be taken as the union over L of balls in $\text{Lip}(X, \mathbb{R}^k)$ of radius $\varepsilon(L)$, centred at L . \square

Examples corresponding to the items (b)–(c) are inspired by [FH96, Example 1] and [SY97, Example 3.7], but differ in details and require different analysis than the ones presented in the cited works. The common construction in both cases (b)–(c) is as follows. Consider a pair of sequences $(r_i)_{i=1}^\infty, (\ell_i)_{i=1}^\infty$ of positive real numbers, which decrease to 0 with $\ell_i < r_i$ (in fact, ℓ_i will be chosen to decay much faster than r_i). Fix $N \in \mathbb{N}$, $1 \leq k < N$ and let e_1, \dots, e_N be the standard orthonormal basis of \mathbb{R}^N . Denote

$$\mathcal{J} = \{J \subset \{1, 2, \dots, N\} : \#J = k + 1\},$$

and for $J \in \mathcal{J}$ let $S_J \subset \mathbb{R}^N$ be the k -dimensional unit sphere defined as

$$S_J = \{(x_1, \dots, x_N) \in \mathbb{R}^N : \sum_{j \in J} x_j^2 = 1 \text{ and } x_j = 0 \text{ for } j \notin J\}.$$

For each $J \in \mathcal{J}$ and $i \geq 1$, let $X_{J,i} \subset \mathbb{R}^N$ be an ℓ_i -separated set of maximal cardinality in

$$r_i S_J = \{r_i x : x \in S_J\}.$$

Then $X_{J,i}$ is an ℓ_i -net in $r_i S_J$ and

$$C_2 \left(\frac{r_i}{\ell_i}\right)^k \leq \#X_{J,i} \leq C_1 \left(\frac{r_i}{\ell_i}\right)^k$$

for some constants $C_1, C_2 > 0$ independent of i and J . Hence, $X_{J,i}$ can be seen as (approximately) uniformly distributed in $r_i S_J$. For $J \in \mathcal{J}$ we define a compact set $X_J \subset \mathbb{R}^N$ by

$$X_J = \bigcup_{i=1}^{\infty} X_{J,i} \cup \{0\}$$

and set

$$X = \bigcup_{J \in \mathcal{J}} X_J.$$

To show the items (b)–(c), we prove the following proposition.

Proposition 6.2. *For every $1 \leq k < N$ and the set X defined above, there exists an open set $\mathcal{U} \subset \text{Lip}(X, \mathbb{R}^k)$ containing all full-rank linear maps $L \in \text{Lin}(\mathbb{R}^N, \mathbb{R}^k)$ restricted to X , such that the following hold.*

- (i) *For $r_i = 2^{-i}$ and $\ell_i = 2^{-i^2}$, we have $\dim_H X = 0$ and no map $\phi \in \mathcal{U}$ is injective on X with a pointwise α -Hölder inverse for any $\alpha > 0$.*
- (ii) *For every $s \in (0, k)$ and $t = \frac{k}{k-s}$, $r_i = 2^{-i}$, $\ell_i = 2^{-it}$, we have $\underline{\dim}_B X = \overline{\dim}_B X = s$ and no map $\phi \in \mathcal{U}$ is injective on X with a pointwise α -Hölder inverse for $\alpha > 1 - \frac{\overline{\dim}_B X}{k}$.*

In particular, if μ is a finite Borel measure supported on X with $\mu(\{x\}) > 0$ for every $x \in X$, in both cases we conclude that there is no full μ -measure set on which ϕ is injective with a pointwise α -Hölder inverse for α in the given range.

Note that as Proposition 6.2ii provides a compact set X with $\overline{\dim}_B X < k$ and an open set of Lipschitz maps (containing all full-rank linear maps) without a pointwise α -Hölder inverse on X for some $\alpha \in (0, 1)$, we conclude that in the assertion (iii) of Theorems 1.7 and 1.12, the Assouad dimension cannot be replaced by the upper box-counting dimension.

The proof of Proposition 6.2 is preceded by two lemmas.

Lemma 6.3. *For any choice of the sequences $r_i \searrow 0$ and $\ell_i < r_i$, the set X satisfies $\dim_H X = 0$. For $r_i = 2^{-i}$ and $\ell_i = 2^{-ti}$, $t > 1$, we have $\underline{\dim}_B X = \overline{\dim}_B X = k \frac{t-1}{t}$.*

Proof. The first statement is immediate as X is countable. For the second one, take $J \in \mathcal{J}$, $r_i = 2^{-i}$, $\ell_i = 2^{-ti}$, $t > 1$, and note that for a small $r > 0$ one has the following bounds on the covering number $N(X_{J,i}, r)$, coming, respectively, from covering each point of $X_{J,i}$ separately, and covering the whole sphere $r_i S_J$:

$$(18) \quad N(X_{J,i}, r) \leq \#X_{J,i} \leq C_1 \left(\frac{r_i}{\ell_i}\right)^k = C_1 2^{k(t-1)i},$$

$$(19) \quad N(X_{J,i}, r) \leq C \left(\frac{r_i}{r}\right)^k = \frac{C}{r^k 2^{ki}},$$

with a constant $C > 0$ independent of i and J .

Let $m = m(r)$ be such that $2^{-m} \leq r < 2^{-m+1}$. Note that $X_{J,i} \subset B(0, r)$ for $i > m$. Hence, applying the bounds (18) for $i \leq \lceil m/t \rceil$ and (19) for $i > \lceil m/t \rceil$ we obtain

$$N(X_J, r) \leq 1 + C_1 \sum_{i=1}^{\lceil m/t \rceil} 2^{k(t-1)i} + C \sum_{i=\lceil m/t \rceil+1}^m 2^{k(m-i)} \leq C'_1 2^{km(t-1)/t}$$

for some constant $C'_1 > 0$ and every sufficiently small $r > 0$. This gives

$$\overline{\dim}_B X_J \leq \lim_{m \rightarrow \infty} \frac{\log C'_1 + (km(t-1)/t) \log 2}{(m-1) \log 2} = k \frac{t-1}{t}.$$

On the other hand, for $i = \lfloor (m-2)/t \rfloor$ we have $2r \leq \ell_i$, so the set $X_{J, \lfloor (m-2)/t \rfloor}$ is $2r$ -separated, and hence

$$\begin{aligned} N(X_J, r) &\geq N(X_{J, \lfloor (m-2)/t \rfloor}, r) \geq \#X_{J, \lfloor (m-2)/t \rfloor} \geq C_2 \left(\frac{r_{\lfloor (m-2)/t \rfloor}}{\ell_{\lfloor (m-2)/t \rfloor}}\right)^k \\ &= C_2 2^{k(t-1)\lfloor (m-2)/t \rfloor} \geq C'_2 2^{km(t-1)/t} \end{aligned}$$

for some constant $C'_2 > 0$ and sufficiently large m . Thus,

$$\underline{\dim}_B X_J \geq \lim_{m \rightarrow \infty} \frac{\log C'_2 + (km(t-1)/t) \log 2}{(m-1) \log 2} = k \frac{t-1}{t}.$$

As X is a finite union of X_J , $J \in \mathcal{J}$, we have $\underline{\dim}_B X = \overline{\dim}_B X = k \frac{t-1}{t}$. \square

Lemma 6.4. *For every full-rank linear map $L \in \text{Lin}(\mathbb{R}^N, \mathbb{R}^k)$, there exist $J = J(L) \in \mathcal{J}$ and $\varepsilon = \varepsilon(L) > 0$ such that for every $f \in \text{Lip}(X, \mathbb{R}^k)$ with $\|f\|_{\text{Lip}} < \varepsilon$ and every $0 < r \leq 1$ there exists $y \in rS_J$ with $(L+f)(y) = (L+f)(0)$.*

Proof. Fix a full-rank linear map $L \in \text{Lin}(\mathbb{R}^N, \mathbb{R}^k)$. As previously, write $L = \Psi \circ P_V$, where $V = (\text{Ker } L)^\perp$ and $\Psi: V \rightarrow \mathbb{R}^k$ is a linear isomorphism. Since $\{e_1, e_2, \dots, e_N\}$ is a linear basis of \mathbb{R}^N and $\dim V^\perp = N - k$, there exists $J \in \mathcal{J}$ such that $\text{span}(V^\perp, \{e_j : j \in J\}) = \mathbb{R}^N$ (actually, we can obtain this with a subset of J of cardinality k , but it is crucial for the construction that $\#J = k + 1$). Note that $\text{span}(S_J) = \text{span}(\{e_j : j \in J\})$ and take any $y_0 \in V^\perp \cap S_J$. Such y_0 exists, as otherwise $V^\perp \cap \text{span}(S_J) = \{0\}$, but this is impossible since $\dim V^\perp = N - k$ and $\dim(\text{span}(S_J)) = k + 1$. Let now W be the orthogonal complement of y_0 in $\text{span}(S_J)$, so that $W \subset \text{span}(S_J)$ and $\dim W = k$. Note that $W \cap V^\perp = \{0\}$, as $\dim W + \dim V^\perp = N$ and $\text{span}(W, V^\perp) = \text{span}(W, y_0, V^\perp) = \text{span}(S_J, V^\perp) = \mathbb{R}^N$. As $y_0 \in V^\perp = \text{Ker } P_V$, we have $P_V(y_0) = 0$. Since $W \cap \text{Ker } P_V = W \cap V^\perp = \{0\}$, we see that P_V is injective on W . As $\dim W = k = \dim V$, we conclude that P_V is a linear isomorphism between W and V and hence it is a bi-Lipschitz homeomorphism.

Now we show that

$$(20) \quad P_V \text{ is a bi-Lipschitz homeomorphism from } S_J \cap B(y_0, \delta) \text{ onto its image for some } \delta > 0.$$

To prove (20), we first note that W is the tangent space to S_J at y_0 , as W is the orthogonal complement of y_0 in $\text{span}(S_J)$. Therefore, for any sufficiently small $\eta > 0$ there exists $\delta > 0$ such that the map $S_J \cap B(y_0, \delta) \ni x \mapsto P_W(x - y_0) \in W$ is a bi-Lipschitz homeomorphism onto

its image and such that $h: S_J \cap B(y_0, \delta) \rightarrow \mathbb{R}^N$ given by $h(x) = x - y_0 - P_W(x - y_0)$ satisfies $\|h\|_{\text{Lip}} < \eta$. As $x = y_0 + P_W(x - y_0) + h(x)$, we have

$$(21) \quad P_V(x) = P_V(P_W(x - y_0)) + P_V(h(x)).$$

for $x \in S_J \cap B(y_0, \delta)$. As $x \mapsto P_W(x - y_0)$ is bi-Lipschitz as a map from $S_J \cap B(y_0, \delta)$ to W and P_V is bi-Lipschitz on W , we have that $x \mapsto P_V(P_W(x - y_0))$ is bi-Lipschitz on $S_J \cap B(y_0, \delta)$. On the other hand, $\|P_V \circ h\|_{\text{Lip}} < \eta$, so we see from (21) that P_V is bi-Lipschitz on $S_J \cap B(y_0, \delta)$ if η is small enough (compared to the Lipschitz constants of $x \mapsto P_V(P_W(x - y_0))$ and its inverse on $S_J \cap B(y_0, \delta)$, which are uniformly bounded away from 0 and ∞ for all small δ). Such η exists for δ small enough, which shows (20).

Take $y_0 \in S_J$ and $\delta > 0$ as in (20). Then, by a homothetic change of coordinates, P_V is a bi-Lipschitz homeomorphism from $rS_J \cap B(ry_0, r\delta)$ onto its image for every $0 < r \leq 1$, with Lipschitz constants independent of r . As previously, note that if $f \in \text{Lip}(X, \mathbb{R}^k)$ satisfies $\|f\|_{\text{Lip}} < \varepsilon$ for a small $\varepsilon > 0$, then $g = \Psi^{-1} \circ f \in \text{Lip}(X, V) \simeq \text{Lip}(X, \mathbb{R}^k)$ and $\|g\|_{\text{Lip}}$ is arbitrarily small for sufficiently small ε . Consequently, setting $\psi = P_V + g$, we see that there exists $\varepsilon = \varepsilon(L) > 0$ such that if $\|f\|_{\text{Lip}} < \varepsilon$, then ψ is a bi-Lipschitz homeomorphism from $rS_J \cap B(ry_0, r\delta)$ onto its image with uniform Lipschitz constants, i.e. there exist $c, C > 0$ such that

$$(22) \quad c\|x - y\| \leq \|\psi(x) - \psi(y)\| \leq C\|x - y\|$$

for every $0 < r \leq 1$ and $x, y \in rS_J \cap B(ry_0, r\delta)$. Our goal is to show that $\psi(0) = g(0)$ belongs to the image $\psi(rS_J \cap B(ry_0, r\delta))$. To see this, note that as ψ is a bi-Lipschitz homeomorphism of the k -dimensional ball $rS_J \cap B(ry_0, r\delta)$, (22) implies that $\psi(rS_J \cap B(ry_0, r\delta))$ contains a ball in \mathbb{R}^k of radius $cr\delta$ centred at $\psi(ry_0) = g(ry_0)$. As $\|g(ry_0) - g(0)\| \leq \varepsilon r$, we see that $g(0) \in \psi(rS_J \cap B(ry_0, r\delta))$ provided that ε is small enough to satisfy $\varepsilon < c\delta$. Hence, $\psi(y) = \psi(0)$ for some $y \in rS_J$. As $\psi = P_V + g$, $L = \Psi \circ P_V$ and $f = \Psi \circ g$, this implies $(L + f)(y) = (L + f)(0)$, which ends the proof of the lemma. \square

Proof of Proposition 6.2. As previously, we set \mathcal{U} to be the union over all full-rank linear maps $L \in \text{Lin}(\mathbb{R}^N, \mathbb{R}^k)$ of balls in $\text{Lip}(X, \mathbb{R}^k)$ with radius $\varepsilon(L)$, centred at L , where $\varepsilon(L)$ is as in Lemma 6.4. Consider $\phi = L + f \in \mathcal{U}$, where $L \in \text{Lin}(\mathbb{R}^N, \mathbb{R}^k)$ and $\|f\|_{\text{Lip}} < \varepsilon(L)$. By Lemma 6.4, for $i \geq 1$ and $J = J(L)$, we can find $y_i \in r_i S_J$ with $\psi(y_i) = \psi(0)$. Let $x_i \in X_{J,i}$ be such that $\|x_i - y_i\| \leq \ell_i$ (recall that $X_{J,i}$ is an ℓ_i -net in $r_i S_J$). Suppose that ϕ is injective on X with a pointwise α -Hölder inverse. Then there exists $M > 0$ such that for every $i \geq 1$,

$$\begin{aligned} r_i &= \|x_i\| \leq M\|\phi(x_i) - \phi(0)\|^\alpha = M\|\phi(x_i) - \phi(y_i)\|^\alpha \\ &\leq M(\|L(x_i) - L(y_i)\| + \|f(x_i) - f(y_i)\|)^\alpha \\ &\leq M(\|L\| + \varepsilon)^\alpha \|x_i - y_i\|^\alpha \leq M(\|L\| + \varepsilon)^\alpha \ell_i^\alpha, \end{aligned}$$

so

$$\frac{r_i}{\ell_i^\alpha} \leq M(\|L\| + \varepsilon)^\alpha.$$

This however cannot be the case if $\lim_{i \rightarrow \infty} \frac{r_i}{\ell_i^\alpha} = \infty$. Therefore, for $r_i = 2^{-i}$, $\ell_i = 2^{-i^2}$ we conclude that ϕ is not injective on X with a pointwise α -Hölder inverse for any $\alpha > 0$. This proves the first assertion of the proposition. For the choice $r_i = 2^{-i}$, $\ell_i = 2^{-ti}$, $t > 1$, we obtain that ϕ is not injective on X with a pointwise α -Hölder inverse for any $\alpha > 1/t$. Setting $t = \frac{k}{k-s}$ for $s \in (0, k)$, we obtain $\frac{1}{t} = 1 - \frac{s}{k}$ and $\underline{\dim}_B X = \overline{\dim}_B X = k \frac{t-1}{t} = s$ by Lemma 6.3. This finishes the proof. \square

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