

A generalisation of Kani-Rosen decomposition theorem for Jacobian varieties

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Abstract. In this short paper we generalise a theorem due to Kani and Rosen on decomposition of Jacobian varieties of Riemann surfaces with group action. This generalisation extends the set of Jacobians for which it is possible to obtain an isogeny decomposition where all the factors are Jacobians.

Mathematics Subject Classification (2010): 14H37 (primary); 14H40, 14L30 (secondary).

1. Introduction and statement of the results

Let C be a compact Riemann surface of genus $g_C \geq 2$ and JC its Jacobian variety. Given a group H of automorphisms of C , we denote by $|H|$ its order and by C_H the underlying Riemann surface structure of the quotient C/H .

Kani and Rosen in [16] studied relations among idempotents in the algebra of rational endomorphisms of an arbitrary Abelian variety. By means of these relations, in the case of the Jacobian variety of a Riemann surface C with group action, they succeeded in proving a decomposition theorem for JC in which each factor is isogenous to the Jacobian of a quotient of C .

Applications of Kani-Rosen theorem can be found, for example, in [3,5,9,11,13,15,19]. For the sake of explicitness, we exhibit here this result:

Theorem C (Kani and Rosen). *Let H_1, \dots, H_t be groups of automorphisms of a Riemann surface C . If:*

- (1) $H_i H_j = H_j H_i$ for all $i \neq j$;
- (2) The genus of $C_{H_i H_j}$ is zero for all $i \neq j$;
- (3) $g_C = \sum_{i=1}^t g_{C_{H_i}}$;

Partially supported by Postdoctoral Fondecyt Grant 3160002, Fondecyt Grant 1141099 and Anillo ACT 1415 PIA-CONICYT Grant.

Received May 8, 2017; accepted in revised form September 11, 2017.

Published online June 2019.

then

$$JC \sim JC_{H_1} \times \cdots \times JC_{H_t}.$$

The aim of this article is to provide a generalisation of the aforementioned theorem. More precisely, we prove the following result:

Theorem 1.1. *Let $\{H_1, \dots, H_t\}$ be an admissible collection of groups of automorphisms of a Riemann surface C . Then*

$$JC \sim JC_{H_1} \times \cdots \times JC_{H_t} \times P$$

for some Abelian subvariety P of JC .

Furthermore, if $\{H_1, \dots, H_t\}$ is admissible and $g_C = \sum_{i=1}^t g_{C_{H_i}}$, then

$$JC \sim JC_{H_1} \times \cdots \times JC_{H_t}.$$

A precise definition of what we call *admissible* collection of groups of automorphisms will be provided in Subsection 2.3.

Theorem 1.1 enlarges the collection of Jacobians JC for which it is possible to obtain an isogeny decomposition in terms of Jacobians of quotients of C . In fact, at the end of this article, we provide examples of admissible collections which do not satisfy each one of the hypothesis of Theorem C.

We anticipate the fact that the hypothesis of Theorem 1.1 can be rephrased in a purely algebraic way; see Proposition 3.1.

Let H be a group of automorphisms of C . We denote by $\pi_H : C \rightarrow C_H$ the associated regular covering map, and by

$$\pi_H^* : JC_H \rightarrow JC$$

the induced homomorphism between the respective Jacobian varieties. As the set $\pi_H^*(JC_H)$ is an Abelian subvariety of JC which is isogenous to JC_H , there exists an Abelian subvariety $P(C \rightarrow C_H)$ of JC such that

$$JC \sim JC_H \times P(C \rightarrow C_H). \quad (1.1)$$

The factor $P(C \rightarrow C_H)$ is known as the *Prym variety* associated to the covering map induced by H .

As we shall see later, the collection $\{H_1\}$ is admissible for each group of automorphisms H_1 of C . Hence, the trivial case $t = 1$ corresponds to the decomposition (1.1) with $H = H_1$.

The case $t = 2$ is slightly more challenging. In fact, it is not a difficult task to produce (or find in the literature; see for example the Klein group case in [21]) examples of C and pairs of groups of automorphisms H_1 and H_2 of C such that the dimension of $JC_{H_1} \times JC_{H_2}$ exceeds the genus of C . Nevertheless, roughly speaking, part of such an *excess* can be geometrically identified. More precisely:

Proposition 1.2. *Let H_1 and H_2 be groups of automorphisms of a Riemann surface C . Then*

$$JC \times JC_{\langle H_1, H_2 \rangle} \sim JC_{H_1} \times JC_{H_2} \times P$$

for some Abelian subvariety P of JC .

In particular, if the genus of $C_{\langle H_1, H_2 \rangle}$ is zero and $g_C = g_{C_{H_1}} + g_{C_{H_2}}$, then

$$JC \sim JC_{H_1} \times JC_{H_2}.$$

Note that, in the previous proposition, the collection $\{H_1, H_2\}$ is not asked to be admissible. The key point in the proof of this result is the classically known formula to compute the dimension of the sum of two vector subspaces. The same argument cannot be generalized for $t \geq 3$ since, as a matter of fact, there does not exist such a formula for more than two summands.

As a direct consequence of Theorem 1.1 we obtain:

Corollary 1.3. *Let $\{H_1, \dots, H_t\}$ be an admissible collection of groups of automorphism of a Riemann surface C .*

Then, for each fixed $k \in \{1, \dots, t\}$, the product

$$JC_{H_1} \times \dots \times JC_{H_{k-1}} \times JC_{H_{k+1}} \times \dots \times JC_{H_t}$$

is isogenous to an Abelian subvariety of $P(C \rightarrow C_{H_k})$.

Furthermore, if $\{H_1, \dots, H_t\}$ is admissible and $g_C = \sum_{i=1}^t g_{C_{H_i}}$, then

$$JC_{H_1} \times \dots \times JC_{H_{k-1}} \times JC_{H_{k+1}} \times \dots \times JC_{H_t} \sim P(C \rightarrow C_{H_k}).$$

The previous result can be employed to obtain, in a very straightforward way, Jacobians which are isogenous to Prym varieties (by considering $t = 2$).

As a further consequence of Theorem 1.1, we prove the following corollary regarding the existence of Riemann surfaces with prescribed data on their Jacobian varieties.

Corollary 1.4. *Let $t \geq 2$ and $g_1, \dots, g_t \geq 1$ be integers. Given t hyperelliptic Riemann surfaces C_1, \dots, C_t of genus g_1, \dots, g_t respectively, there exists a Riemann surface C of genus*

$$g = 1 - 2^t + 2^{t-1}(t + \sum_{i=1}^t g_i)$$

such that

$$JC \sim JC_1 \times \dots \times JC_t \times P,$$

for a suitable Abelian subvariety P of JC of dimension

$$1 + 2^{t-1}t - 2^t + (2^{t-1} - 1)\sum_{i=1}^t g_i.$$

In [7] Ekedahl and Serre introduced the problem of determining all genera for which there is a Riemann surface whose Jacobian can be decomposed, up to isogeny, as a product of elliptic curves (for recent progress on this topic we refer to [20]). Since their paper, there has been much interest in this sort of Riemann surfaces, particularly in their applications to number theory.

More restrictive approaches to this problem were developed by Paulhus in [19] and Hidalgo in [11]. More precisely, Paulhus dealt with the following problem: given an integer g , she asked for the largest integer $t = t(g)$ such that E^t is isogenous to an Abelian subvariety of JC , for some Riemann surface C of genus g and for some elliptic curve E . She answered this question for small genera. Meanwhile, a similar problem was considered by Hidalgo: given an integer t , he asked for the existence of a Riemann surface C of minimum genus $g = g(t)$, such that its Jacobian is isogenous to the product of at least t elliptic curves and other Jacobians. By means of very explicit constructions, he determined bounds for g in terms of t .

Let E_1, \dots, E_t be $t \geq 2$ elliptic curves. We remark that, as a direct application of the previous corollary, we can ensure the existence of a Riemann surface of genus $g_t = 1 + 2^t(t - 1)$ so that its Jacobian is isogenous to the product $E_1 \times \dots \times E_t \times P$ for a suitable Abelian subvariety P of it. Furthermore, using our theorem we are in position to guarantee the existence of a Riemann surface of smaller genus than g_t and satisfying the same property, as shown in the next corollary:

Corollary 1.5. *Let $t \geq 2$ be an integer and let E_1, \dots, E_t be elliptic curves. Then there exists a compact Riemann surface C of genus*

$$g = \begin{cases} 1 - 2^{t/2} + (3t)2^{(t/2)-2} & \text{if } t \text{ is even} \\ 1 - 2^{(t+1)/2} + (3t + 1)2^{(t-3)/2} & \text{if } t \text{ is odd} \end{cases}$$

so that

$$JC \sim E_1 \times \dots \times E_t \times P$$

for a suitable Abelian subvariety P of JC of dimension $g - t$.

In particular, if the elliptic curves are pairwise isogenous, then

$$JC \sim E_1^t \times P.$$

The next theorem was also proved in [16]; we give an alternative proof, substantially simpler than the original one.

Theorem B (Kani and Rosen). *Let G be a finite group of automorphisms of a Riemann surface C such that $G = H_1 \cup \dots \cup H_t$ where the subgroups H_i of G satisfy $H_i \cap H_j = \{1\}$ for $i \neq j$. Then*

$$JC^{t-1} \times JC_G^{|G|} \sim JC_{H_1}^{|H_1|} \times \dots \times JC_{H_t}^{|H_t|}.$$

This paper is organized as follows. In Section 2 we shall briefly review the preliminaries: representations of groups and the group algebra decomposition theorem for

Jacobians varieties (on which the proofs are based). Section 3 is devoted to proving the results. In the last section we exhibit an explicit example in order to illustrate how our results can be applied.

ACKNOWLEDGEMENTS. The authors are grateful to their colleague Angel Carocca for his helpful suggestions throughout the preparation of this manuscript.

2. Preliminaries

2.1. Representations of groups

Let G be a finite group and let $\rho : G \rightarrow \text{GL}(V)$ be a complex representation of G . By abuse of notation, we shall also write V to refer to the representation ρ . The *degree* d_V of V is the dimension of V as a complex vector space, and the *character* of V is the map obtained by associating to each $g \in G$ the trace of the matrix $\rho(g)$. Two representations V_1 and V_2 are *equivalent* if and only if their characters agree; we write $V_1 \cong V_2$. The *character field* K_V of V is the field obtained by extending the rational numbers by the character values. The *Schur index* s_V of V is the smallest positive integer such that there exists a field extension of K_V of degree s_V over which V can be defined.

It is known that for each rational irreducible representation W of G there is a complex irreducible representation V of G such that

$$W \otimes_{\mathbb{Q}} \mathbb{C} \cong (\oplus_{\sigma} V^{\sigma}) \oplus \cdot^{s_V} \cdot \oplus (\oplus_{\sigma} V^{\sigma}) = s_V (\oplus_{\sigma} V^{\sigma}), \tag{2.1}$$

where the sum $\oplus_{\sigma} V^{\sigma}$ is taken over the Galois group associated to $\mathbb{Q} \leq K_V$. The representation V is said to be *associated* to W . If V' is another complex irreducible representation associated to W then V and V' are said to be *Galois associated*.

Let H be a subgroup of G . We denote by V^H the vector subspace of V consisting of those elements which are fixed under H . By Frobenius reciprocity theorem, its dimension – denoted by d_V^H – agrees with $\langle \rho_H, V \rangle_G$, where ρ_H stands for the representation of G induced by the trivial one of H , and the brackets for the usual inner product of characters.

We refer to [24] for further basic facts related to representations of groups.

2.2. Complex tori and Abelian varieties

A g -dimensional *complex torus* $X = V/\Lambda$ is the quotient between a g -dimensional complex vector space V by a maximal rank discrete subgroup Λ . Each complex torus is an Abelian group and a g -dimensional compact connected complex analytic manifold. *Homomorphisms* between complex tori are holomorphic maps which are homomorphisms of groups; we shall denote by $\text{End}(X)$ the ring of endomorphisms of X . An *isogeny* between two complex tori is a surjective homomorphism with

finite kernel; isogenous tori are denoted by $X_1 \sim X_2$. The isogenies of a complex torus X into itself are the invertible elements of the ring of rational endomorphisms of X

$$\text{End}_{\mathbb{Q}}(X) := \text{End}(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

An *Abelian variety* is by definition a complex torus which is also a complex projective algebraic variety. The Jacobian variety JC of a Riemann surface C of genus g is an Abelian variety of dimension g .

We refer to [1] for basic material on this topic.

2.3. Group algebra decomposition

We consider a finite group G and its rational irreducible representations W_1, \dots, W_r . It is classically known that if G acts on C then this action induces a \mathbb{Q} -algebra homomorphism

$$\Phi : \mathbb{Q}[G] \rightarrow \text{End}_{\mathbb{Q}}(JC).$$

For each $\alpha \in \mathbb{Q}[G]$ we define the Abelian subvariety

$$A_{\alpha} := \text{Im}(\alpha) = \Phi(l\alpha)(JC) \subset JC$$

where l is some positive integer chosen in such a way that $l\alpha \in \mathbb{Z}[G]$.

The decomposition of $1 = e_1 + \dots + e_r \in \mathbb{Q}[G]$, where each e_l is a uniquely determined central idempotent (computed from W_l), yields an isogeny

$$JC \sim A_{e_1} \times \dots \times A_{e_r}$$

which is G -equivariant. See [17].

Additionally, there are idempotents f_{l1}, \dots, f_{ln_l} such that $e_l = f_{l1} + \dots + f_{ln_l}$ where $n_l = d_{V_l}/s_{V_l}$ and V_l is a complex irreducible representation of G associated to W_l . These idempotents provide n_l subvarieties of JC which are isogenous between them; let B_l be one of them, for every l . Thus

$$JC \sim_G B_1^{n_1} \times \dots \times B_r^{n_r}, \tag{2.2}$$

called the *group algebra decomposition* of JC with respect to G . See [4].

If the representations are labelled in such a way that $W_1 (= V_1)$ denotes the trivial one (as we will do in this paper) then $n_1 = 1$ and $B_1 \sim JC_G$.

Notation. Throughout this paper we shall reserve the notation \sim_G to refer to the group algebra decomposition (2.2) of JC with respect to G . Observe that each product $B_i^{n_i}$ admits a G -action (by an appropriate multiple of W_i), but, in general, each B_i does not.

Let H be a subgroup of G . It was also proved in [4] that the group algebra decomposition of JC with respect to G induces a isogeny decomposition

$$JC_H \sim B_1^{n_1^H} \times \dots \times B_r^{n_r^H} \tag{2.3}$$

of JC_H , where $n_l^H = d_{V_l^H}/s_{V_l^H}$.

Note that, in general, the isogeny (2.3) is not a group algebra decomposition because the quotient C_H does not necessarily have group action. We also mention that the dimension of each factor in (2.2) is explicitly computable in terms of the monodromy of the action of G on C . See [23].

Now, once the basic preliminaries have been introduced, we are in position to bring in the precise definition of *admissible* collection of automorphisms.

Definition 2.1. Let C be a compact Riemann surface and let G, H_1, \dots, H_t be subgroups of automorphisms of C such that G contains H_i for each $1 \leq i \leq t$. Consider the group algebra decomposition (2.2) with respect to G .

The collection $\{H_1, \dots, H_t\}$ will be called *G-admissible* if

$$d_{V_l}^{H_1} + \dots + d_{V_l}^{H_t} \leq d_{V_l}$$

for every complex irreducible representation V_l of G such that $B_l \neq 0$. The collection will be called *admissible* if it is *G-admissible* for some group G .

We emphasize the fact that our definition of admissibility is based on the dimensions of the vector subspaces fixed under the corresponding subgroups; consequently, it is based on the induced isogenies (2.3) with $H = H_i$. As the reader will note in the next section, these isogenies will play a key role in our proofs; indeed, this is the new ingredient that was not available when the Kani-Rosen decomposition theorem was originally proved.

3. Proofs

3.1. Proof of Theorem 1.1

Let us assume that the collection $\{H_1, \dots, H_t\}$ is *G-admissible*, and consider the group algebra decomposition of JC with respect to G given by

$$JC \sim_G B_1^{n_1} \times \dots \times B_r^{n_r} \tag{3.1}$$

and

$$JC_{H_i} \sim B_1^{n_1 H_i} \times \dots \times B_r^{n_r H_i} \tag{3.2}$$

the corresponding induced isogeny decomposition of JC_{H_i} .

Suppose that the factor B_l is associated to the rational irreducible representation W_l , and that this is, in turn, associated to the complex irreducible representation V_l of G .

If $t = 1$, the decomposition (1.1) with $H = H_1$ proves the result. Thus, from now on we assume $t > 1$.

Claim. The genus of S_G is zero.

Assume that the genus of S_G is positive or, equivalently, that the factor B_1 has positive dimension. Note that

$$d_{V_1}^{H_1} + \dots + d_{V_1}^{H_t} = t$$

and therefore, as the collection $\{H_1, \dots, H_t\}$ is G -admissible, we must have

$$t = d_{V_1}^{H_1} + \dots + d_{V_1}^{H_t} \leq d_{V_1} = 1;$$

a contradiction. This proves the claim.

It follows that the factor B_1 equals zero. Without loss of generality, we may assume that $B_l \neq 0$ for each $l > 1$. Now, as the collection $\{H_1, \dots, H_t\}$ is supposed to be G -admissible, we have that

$$d_{V_l}^{H_1} + \dots + d_{V_l}^{H_t} + \delta_l = d_{V_l}$$

for some $\delta_l \geq 0$, for each $l \neq 1$. The last equality can be also written as

$$n_l^{H_1} + \dots + n_l^{H_t} + \tilde{\delta}_l = n_l,$$

where $\tilde{\delta}_l = \delta_l/s_{V_l}$ for each $l \neq 1$. We remark that all $\tilde{\delta}_l$ are integers.

In this way, we obtain that

$$B_l^{n_l^{H_1}} \times \dots \times B_l^{n_l^{H_t}} \times B_l^{\tilde{\delta}_l} = B_l^{n_l}$$

for all $l \neq 1$.

Thereby

$$\prod_{l=1}^r B_l^{\tilde{\delta}_l} \times \prod_{l=1}^r \left(B_l^{n_l^{H_1}} \times \dots \times B_l^{n_l^{H_t}} \right) = \prod_{l=1}^r B_l^{n_l}$$

or, equivalently, if we reorder the products, we see that

$$\prod_{l=1}^r B_l^{\tilde{\delta}_l} \times \prod_{i=1}^t \prod_{l=1}^r B_l^{n_l^{H_i}} = \prod_{l=1}^r B_l^{n_l}.$$

Now, by considering the isogenies (3.1) and (3.2), it follows that

$$P \times JC_{H_1} \times \dots \times JC_{H_t} \sim JC, \tag{3.3}$$

where $P := \prod_{l=1}^r B_l^{\tilde{\delta}_l}$. This proves the first result.

The second one is now straightforward. Indeed, if we suppose $\{H_1, \dots, H_t\}$ to be G -admissible and g_C to be equal to $\sum_{i=1}^t g_{C_{H_i}}$, then by comparing dimensions in both sides of (3.3), the factor P must clearly be zero. Thereby

$$JC \sim JC_{H_1} \times \dots \times JC_{H_t}$$

in this case, and the proof is complete.

3.2. Restatement of the hypothesis of Theorem 1.1

Let $H_1(C, \mathbb{Z})$ denote the integral first homology group of C . It is classically known that the action of G on C induces a representation of degree $2g$

$$\rho_{\text{rac}} : G \rightarrow \text{GL}(H_1(C, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q})$$

of G , known as the *rational* representation of G . By abuse of notation, we shall also write ρ_{rac} to refer to its complexification $\rho_{\text{rac}} \otimes \mathbb{C}$.

The next result exhibits algebraic restatements of the hypothesis of our main theorem:

Proposition 3.1. *Let C be a compact Riemann surface and let G, H_1, \dots, H_t be subgroups of automorphisms of C such that G contains H_i for each $1 \leq i \leq t$. Consider the group algebra decomposition (2.2) with respect to G .*

The following statements are equivalent:

- (1) *The collection $\{H_1, \dots, H_t\}$ is G -admissible and $g_C = \sum_{i=1}^t g_{C_{H_i}}$;*
- (2) *$\sum_{i=1}^t d_{V_i}^{H_i} = d_{V_i}$ for every l such that $B_l \neq 0$;*
- (3) *There are non-negative integers a_l such that*

$$\oplus_{i=1}^t \rho_{H_i} \cong \sum_{\langle \rho_{\text{rac}}, W_l \rangle_G \neq 0} n_l W_l \oplus \sum_{\langle \rho_{\text{rac}}, W_l \rangle_G = 0} a_l W_l.$$

Proof. The equivalence between statements (1) and (2) follows directly from the proof of Theorem 1.1.

We proceed to verify the one between statements (2) and (3). It is a known fact that

$$\rho_{H_i} \cong \oplus_{l=1}^r n_l^{H_i} W_l$$

for each $1 \leq i \leq t$ (see, for example [4, Lemma 4.3]). Thus,

$$\oplus_{i=1}^t \rho_{H_i} \cong \oplus_{l=1}^r \left(n_l^{H_1} + \dots + n_l^{H_t} \right) W_l.$$

Now, as

$$\dim B_l = 0 \quad \text{if and only if} \quad \langle \rho_{\text{rac}}, W_l \rangle_G = 0,$$

we see that if statement (2) is assumed, then

$$\oplus_{i=1}^t \rho_{H_i} \cong \sum_{\langle \rho_{\text{rac}}, W_l \rangle_G \neq 0} n_l W_l \oplus \sum_{\langle \rho_{\text{rac}}, W_l \rangle_G = 0} \left(n_l^{H_1} + \dots + n_l^{H_t} \right) W_l$$

proving statement (3) with $a_l = n_l^{H_1} + \dots + n_l^{H_t}$. The converse is direct. □

As a particular case, we consider the situation when the dimension of B_l equals zero if and only if $l = 1$. Then, the proposition above ensures that in this case the collection $\{H_1, \dots, H_t\}$ is G -admissible and $g_C = \sum_{i=1}^t g_{C_{H_i}}$ if and only if

$$\bigoplus_{i=1}^t \rho_{H_i} \cong \sum_{l \neq 1} n_l W_l \oplus a_1 W_1 \cong \rho_{\text{reg}} \oplus (a_1 - 1)W_1 \tag{3.4}$$

for some $a_1 \geq 1$, where ρ_{reg} stands for the regular representation of G .

The value of a_1 can easily be determined by comparing in (3.4) either the character at the identity or the multiplicity of the trivial representation; namely

$$a_1 = 1 + |G| \cdot \left(-1 + \sum_{i=1}^t \frac{1}{|H_i|} \right) = t.$$

Hence, at the end, the hypothesis of Theorem 1.1 are equivalent to

$$\bigoplus_{i=1}^t \rho_{H_i} \cong \rho_{\text{reg}} \oplus (t - 1)W_1$$

in this case.

3.3. Proof of Proposition 1.2

Let H_1 and H_2 be two subgroups of automorphisms of C . Clearly, there is an integer $\delta_l \geq 0$ such that

$$\dim \left(V_l^{H_1} + V_l^{H_2} \right) + \delta_l = \dim V_l$$

for each l . By considering the dimension formula for the sum of two vector subspaces, we can assert that

$$\dim V_l^{H_1} + \dim V_l^{H_2} - \dim \left(V_l^{H_1} \cap V_l^{H_2} \right) + \delta_l = \dim V_l.$$

Now, as

$$V_l^{H_1} \cap V_l^{H_2} = V_l^{\langle H_1, H_2 \rangle},$$

the following equality is obtained:

$$n_l^{H_1} + n_l^{H_2} + \tilde{\delta}_l = n_l + n_l^{\langle H_1, H_2 \rangle},$$

where $\tilde{\delta}_l = \delta_l/s_{V_l}$ is a non negative integer.

The remaining part of the proof follows analogously as done in the proof of Theorem 1.1.

3.4. Proof of Corollary 1.3

The isogeny decomposition (1.1) of J_C associated to the regular covering map $\pi_{H_k} : C \rightarrow C_{H_k}$ together with Theorem 1.1 ensure that

$$J_{C_{H_1}} \times \dots \times J_{C_{H_{k-1}}} \times J_{C_{H_{k+1}}} \times \dots \times J_{C_{H_t}} \times P \sim P(C \rightarrow C_{H_k}),$$

for some Abelian subvariety P of J_C , and the proof is done.

3.5. Proof of Corollary 1.4

We start by recalling the well-known fact that for each $1 \leq i \leq t$ there exists a two-fold regular covering map over the Riemann sphere

$$f_i : C_i \rightarrow \mathbb{P}^1$$

ramified over $2g_i + 2$ values. Furthermore, if $B_i \subset \mathbb{P}^1$ denotes the set of branch values of f_i , then we choose, without loss of generality, the f_i such that the intersection $B_i \cap B_j$ is empty for $i \neq j$. We denote by $H_i \cong \mathbb{Z}_2$ the deck group of f_i ; that is, $C_i/H_i \cong \mathbb{P}^1$.

Let

$$C = \{(z_1, \dots, z_t) \in C_1 \times \dots \times C_t : f_1(z_1) = \dots = f_t(z_t)\}$$

be the fiber product of the coverings f_1, \dots, f_t . Clearly, C is endowed with canonical projections

$$\pi_i : C \rightarrow C_i \quad (z_1, \dots, z_t) \mapsto z_i.$$

We recall that C has the structure of a compact Riemann surface (or, equivalently, a connected irreducible smooth complex projective algebraic curve; see [8, 12, 14]), and that the direct product

$$H = H_1 \times \dots \times H_t \cong \mathbb{Z}_2^t$$

acts canonically on C . It is not difficult to see that the correspondence

$$f : C \rightarrow \mathbb{P}^1, \quad z = (z_1, \dots, z_t) \mapsto f(z) := f_1(z_1) = \dots = f_t(z_t)$$

is a branched regular covering map of degree 2^t admitting H as deck group. Note that, as the coverings f_i have been chosen with disjoint ramification, the set of branch values B of f agrees with $B_1 \cup \dots \cup B_t$.

The projection $\pi_i : C \rightarrow C_i$ is a regular covering map of degree 2^{t-1} admitting

$$K_i \cong H/H_i \cong \prod_{k=1, k \neq i}^t H_k$$

as deck group; then $C_i \cong C/K_i$.

Claim. The collection $\{K_1, \dots, K_t\}$ is H -admissible.

The complex irreducible representations of $H \cong \langle K_1, \dots, K_t \rangle$ are of the form

$$V := V_1 \otimes \dots \otimes V_t,$$

where V_j is a complex irreducible representation of H_j . Note that

$$V^{K_i} \cong V_1^{H_1} \otimes \dots \otimes V_{i-1}^{H_{i-1}} \otimes V_i \otimes V_{i+1}^{H_{i+1}} \otimes \dots \otimes V_t^{H_t}.$$

We remark the following obvious observation: if V_j is the non-trivial representation of $H_j \cong \mathbb{Z}_2$ then

$$V_j^{H_j} = 0.$$

Thus, it follows that:

- (1) If the representation V_j is non-trivial for some $j \neq i$, then

$$d_V^{K_i} = 0;$$

- (2) If the representations V_j are trivial for all $j \neq i$, then

$$d_V^{K_i} = d_{V_i} = 1.$$

Thereby

$$d_V^{K_1} + \dots + d_V^{K_t} \leq 1 = d_V$$

and the proof of the claim is done.

It now follows from Theorem 1.1 that

$$JC \sim JC_1 \times \dots \times JC_t \times P$$

for an Abelian subvariety P of JC . Clearly, the dimension of P is $g_C - \sum_{i=1}^t g_i$.

It only remains to compute the genus of C . This task can be accomplished by applying the Riemann-Hurwitz formula (see, for example [18, page 80]) to the involved coverings. More precisely, this formula applied to f_i says that

$$R_i = g_i + 1 \tag{3.5}$$

where R_i is the branch number of f_i . We recall that $R_i = \sum(1 - 1/m_l)$ where the sum is taken over a maximal collection of non-equivalent branch points $\{p_1^i, \dots, p_s^i\}$ of f_i and with m_l denoting the order of the H_i -stabilizer subgroup of p_l . Similarly, the Riemann-Hurwitz formula applied to f says that

$$2g_C - 2 = 2^t(-2) + 2^t R. \tag{3.6}$$

Again, as the ramification of the coverings f_i have been chosen to be disjoint, the branch number R of f agrees with $R_1 + \dots + R_t$. Finally, the desired expression for g_C is obtained after replacing (3.5) in (3.6).

The proof is done. □

3.6. Proof of Corollary 1.5

It is a known fact that, given two elliptic curves E_1 and E_2 , there exists a compact Riemann surface C of genus two such that JC is isogenous to the product $E_1 \times E_2$ (see, for example [6, 10, 22]; see also [11, Theorem 1] for an explicit construction).

Assume t to be even; say $t = 2s$ for some $s \geq 1$. Let C_j be a Riemann surface such that

$$JC_j \sim E_j \times E_{j+s}$$

for each $1 \leq j \leq s$. Now, we apply Corollary 1.4 to the Riemann surfaces C_1, \dots, C_s to guarantee the existence of a Riemann surface C such that

$$JC \sim \prod_{j=1}^s JC_j \times P \sim \prod_{i=1}^t E_i \times P$$

for a suitable Abelian subvariety P of JC .

Assume t to be odd; say $t = 2s + 1$ for some $s \geq 1$. Let C_j be a Riemann surface such that

$$JC_j \sim E_j \times E_{j+s}$$

for each $1 \leq j \leq s$. Now, we apply Corollary 1.4 to the Riemann surfaces C_1, \dots, C_s, E_t to guarantee the existence of a Riemann surface C such that

$$JC \sim \prod_{j=1}^s JC_j \times E_t \times P \sim \prod_{i=1}^t E_i \times P$$

for a suitable Abelian subvariety P of JC .

The computation of the genera is straightforward, and the proof is done. □

3.7. Proof of Theorem B

Lemma 3.2. *Let G be a finite group such that $G = H_1 \cup \dots \cup H_t$ where the subgroups H_i of G satisfy $H_i \cap H_j = \{1\}$ for $i \neq j$. Then*

$$\bigoplus_{i=1}^t |H_i| \rho_{H_i} \cong (t - 1) \rho_{\text{reg}} \oplus |G| W_1.$$

Proof. Following [24, page 30], the character of the representation $|H_i| \rho_{H_i}$ at g in G is

$$\begin{cases} |G| & \text{for } g = 1 \\ \#\mathcal{C}_{g,i} & \text{for } g \neq 1 \end{cases}$$

where $\mathcal{C}_{g,i} = \{s \in G : s^{-1}gs \in H_i\}$. Now, as G is partitioned into its subgroups H_i , we see that for each $g \neq 1$ the sets $\mathcal{C}_{g,i}$ and $\mathcal{C}_{g,j}$ are disjoint for $i \neq j$. It follows that the character of $\bigoplus_{i=1}^t |H_i| \rho_{H_i}$ is

$$\begin{cases} t|G| & \text{for } g = 1 \\ |G| & \text{for } g \neq 1 \end{cases}$$

and, as the character of the regular representation is $|G|$ if $g = 1$ and zero otherwise, the result follows directly by comparison of characters. □

Proof of Theorem B. Let us assume

$$JC \sim_G JC_G \times B_2^{n_2} \times \cdots \times B_r^{n_r} \tag{3.7}$$

to be the group algebra decomposition of JC with respect to G , and

$$JC_{H_i} \sim JC_G \times B_2^{n_2^{H_i}} \times \cdots \times B_r^{n_r^{H_i}},$$

the induced isogeny decomposition of JC_{H_i} where $n_l^{H_i} = d_{V_l}^{H_i} / s_{V_l}$. Thus

$$JC_{H_1}^{|H_1|} \times \cdots \times JC_{H_t}^{|H_t|} \sim JC_G^{\sum_{i=1}^t |H_i|} \times B_2^{\sum_{i=1}^t n_2^{|H_i|}} \times \cdots \times B_r^{\sum_{i=1}^t n_r^{|H_i|}}.$$

The fact that $|G| + t - 1 = \sum_{i=1}^t |H_i|$ implies that

$$JC_{H_1}^{|H_1|} \times \cdots \times JC_{H_t}^{|H_t|} \sim JC_G^{|G|} \times \left(JC_G^{t-1} \times B_2^{\sum_{i=1}^t n_2^{|H_i|}} \times \cdots \times B_r^{\sum_{i=1}^t n_r^{|H_i|}} \right)$$

and therefore we only need to prove that

$$JC^{t-1} \sim JC_G^{t-1} \times B_2^{\sum_{i=1}^t n_2^{|H_i|}} \times \cdots \times B_r^{\sum_{i=1}^t n_r^{|H_i|}}.$$

Again, after considering (3.7), it is enough to prove that

$$B_2^{(t-1)n_2} \times \cdots \times B_r^{(t-1)n_r} = B_2^{\sum_{i=1}^t n_2^{|H_i|}} \times \cdots \times B_r^{\sum_{i=1}^t n_r^{|H_i|}}.$$

This equality is obtained if

$$\sum_{i=1}^t \left[n_l^{H_i} |H_i| - (t-1)n_l \right] = \frac{1}{s_{V_l}} \left[\sum_{i=1}^t d_{V_l}^{H_i} |H_i| - (t-1)d_{V_l} \right]$$

equals zero, for every $2 \leq l \leq r$.

Finally, by Frobenius reciprocity theorem, the previous expression can be rewritten as

$$\frac{1}{s_{V_l}} (\oplus_{i=1}^t |H_i| \rho_{H_i} - (t-1)\rho_{\text{reg}}, V_l)_G$$

and the result follows from Lemma 3.2. The proof of Theorem B is done. □

4. Example

Let q be an odd prime number. We consider a three-dimensional family \mathcal{F} of compact Riemann surfaces C of genus $g_C = 4q - 1$ admitting the action of a group of automorphisms G isomorphic to the dihedral group

$$\mathbf{D}_{2q} = \langle r, s : r^{2q} = s^2 = (sr)^2 = 1 \rangle$$

of order $4q$, in such a way that:

- (1) The quotient C_G has genus zero;
- (2) The associated $4q$ -fold covering map $C \rightarrow C_G$ ramifies over six values; two values having $2q$ preimages (each with stabilizer a conjugate of the subgroup generated by s), two values having $2q$ preimages (each with stabilizer a conjugate of the subgroup generated by sr), and two values having 2 preimages (each with stabilizer the subgroup generated by r).

The existence of the family \mathcal{F} is guaranteed by Riemann’s existence theorem (see, for example, [2, Proposition 2.1]).

Consider the subgroups

$$H_1 = \langle s \rangle, \quad H_2 = \langle sr \rangle \quad \text{and} \quad H_3 = \langle r \rangle$$

of G . We remark that Theorem C cannot be applied in this case, since the hypothesis (1) is not satisfied (H_1 and H_2 do not permute).

Claim. The collection $\{H_1, H_2, H_3\}$ is G -admissible.

To prove the claim it is enough to consider a maximal collection of non-Galois associated (and non-trivial) complex irreducible representations of G . Namely, three representations of degree one

$$V_2 : r \mapsto 1, s \mapsto -1, \quad V_3 : r \mapsto -1, s \mapsto 1, \quad V_4 : r \mapsto -1, s \mapsto -1$$

and two of degree two

$$V_5 : r \mapsto \begin{pmatrix} \omega_{2q} & 0 \\ 0 & \overline{\omega_{2q}} \end{pmatrix} \quad s \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$V_6 : r \mapsto \begin{pmatrix} \omega_q & 0 \\ 0 & \overline{\omega_q} \end{pmatrix} \quad s \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $\omega_t := \exp(2\pi i/t)$.

The claim follows after considering the following table, which summarizes the vector subspaces of V_2, \dots, V_6 fixed under each of the subgroups H_1, H_2, H_3 .

	V_2	V_3	V_4	V_5	V_6
H_1	$\{0\}$	$\langle 1 \rangle$	$\{0\}$	$\langle (1, 1) \rangle$	$\langle (1, 1) \rangle$
H_2	$\{0\}$	$\{0\}$	$\langle 1 \rangle$	$\langle (1, \omega_{2q}) \rangle$	$\langle (1, \omega_q) \rangle$
H_3	$\langle 1 \rangle$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$

Hence, by Theorem 1.1, the claim above implies that

$$JC \sim JC_{H_1} \times JC_{H_2} \times JC_{H_3} \times P$$

for a suitable Abelian subvariety P of JC .

Furthermore, it is not difficult to compute that

$$g_{C_{H_1}} = 2q - 1, \quad g_{C_{H_2}} = 2q - 1 \quad \text{and} \quad g_{C_{H_3}} = 1,$$

showing that their sum agrees with g_C . Thereby $P = 0$ and the decomposition

$$JC \sim JC_{H_1} \times JC_{H_2} \times JC_{H_3} \tag{4.1}$$

is obtained.

We also mention some facts related to this example:

- (1) By Proposition 3.1, we have that $\bigoplus_{i=1}^3 \rho_{H_i} \cong \rho_{\text{reg}} \oplus 2W_1$;
- (2) By Corollary 1.3, the Prym varieties associated to the subgroups H_1 and H_2 –whose dimension is $2q$ – contain an elliptic curve, for all q ;
- (3) The group algebra decomposition of JC with respect to G is

$$JC \sim_G B_1 \times B_2 \times B_3 \times B_4 \times B_5^2 \times B_6^2, \tag{4.2}$$

by [23, Theorem 5.12], the dimensions of the factors are

$$\dim(B_l) = \begin{cases} 0 & \text{if } l = 1 \\ 1 & \text{if } l = 2, 3, 4 \\ q - 1 & \text{if } l = 5, 6 \end{cases}$$

and the decompositions (4.1) and (4.2) are related by the isogenies

$$JC_{H_1} \sim B_3 \times B_5 \times B_6, \quad JC_{H_2} \sim B_4 \times B_5 \times B_6, \quad JC_{H_3} \sim B_2.$$

- (4) The collection $\{H_1, H_3\}$ satisfies hypotheses (1) and (2) of Theorem C, but it does not satisfy hypothesis (3). Indeed,

$$g_{C_{H_1}} + g_{C_{H_3}} = 2q \neq 4q - 1 = g_C.$$

Thus, Theorem C cannot be applied. In contrast, it is straightforward to see that $\{H_1, H_3\}$ is, in fact, a G -admissible collection. Then Theorem 1.1 asserts that

$$JC \sim JC_{H_1} \times JC_{H_3} \times Q$$

for some Abelian subvariety Q of JC of dimension $2q - 1$;

- (5) The subgroup H_1 and $H_4 = \langle r^q \rangle$ permute, but the genus of $C_{H_1 H_4}$ is positive. Then hypothesis (2) in Theorem C is not satisfied, and this result cannot be applied. On the other hand, the vector subspaces of V_2, \dots, V_6 fixed under H_1 and H_4 are

	V_2	V_3	V_4	V_5	V_6
H_1	{0}	{1}	{0}	$\langle(1, 1)\rangle$	$\langle(1, 1)\rangle$
H_4	{1}	{0}	{0}	{0}	$\langle(1, 1)\rangle$

showing that the collection $\{H_1, H_4\}$ is G -admissible. Hence, by Theorem 1.1, we have that

$$JC \sim JC_{H_1} \times JC_{H_4} \times E$$

where E is an elliptic curve;

- (6) Note that the collection $\{H_1, H_4\}$ is G -admissible but it is not $\langle H_1, H_4 \rangle$ -admissible.

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