FINE PROPERTIES OF METRIC SPACE-VALUED MAPPINGS OF BOUNDED VARIATION IN METRIC MEASURE SPACES

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Abstract. Here we consider two notions of mappings of bounded variation (BV) from a metric measure space into a metric space; one based on relaxations of Newton-Sobolev functions, and the other based on a notion of AM-upper gradients. We show that when the target metric space is a Banach space, these two notions coincide with comparable energies, but for more general target metric spaces, the two notions can give different function-classes. We then consider the fine properties of BV mappings (based on the AM-upper gradient property), and show that when the target space is a proper metric space, then for a BV mapping into the target space, co-dimension 1-almost every point in the jump set of a BV mapping into the proper space has at least two, and at most k_0, number of jump values associated with it, and that the preimage of balls around these jump values have lower density at least γ at that point. Here k_0 and γ depend solely on the structural constants associated with the metric measure space, and jump points are points at which the map is not approximately continuous.

Key words and phrases: Bounded variation, metric measure space, Poincaré inequality, doubling measure, vector-valued maps, approximate continuity, jump points, jump values

Acknowledgement: The research of the authors J.K. and N.S. were partially supported by grant DMS #2054960 from the National Science Foundation (U.S.A.). J.K. was also partially supported by the University of Cincinnati's University Research Council summer grant. I.C. was supported by the grant PID2022-138758NB-I00 (Spain). The authors thank Ryan Gibara and Panu Lahti for valuable feedback on early versions of the manuscript. They also thank the anonymous referee for the comments that helped improve the exposition of the paper.

Mathematics Subject Classification (2020): Primary: 26A45, 46E36; Secondary: 30L99, 26B30, 30L05, 54E40

1. Introduction

The theory of functions of bounded variation were first developed in order to study existence and regularity properties of minimal surfaces, and a nice overview can be obtained from the book [5]; much of the original work on regularity theory can be found in the collection [10], and the discussion in [16] gives a nice discussion on fine properties of BV functions in Euclidean spaces. Since then the theory has found applications in other areas as well, including image processing [4, 8], plasma physics [29, 19], and quasiconformal mappings [18, 32], and the references contained in these papers provide further valuable information. In image processing or in plasma-blistering in media that are not uniform and might even exhibit non-smoothness, a theory of functions of bounded variation in metric spaces is useful. Recent research on mappings of finite distortion and quasisymmetric mappings indicate a need to understand metric space-valued mappings of bounded variation on metric measure spaces, see for instance [2, 34, 24, 41, 9, 7] for a sampling of the currently extant literature on this topic. In this paper we seek to study mappings of bounded variation in non-smooth metric measure spaces of controlled geometry, that is, spaces where the measure is doubling and supports a 1-Poincaré inequality (see Section 3).

Unlike Sobolev functions, functions of bounded variation exhibit less regularity; classic examples include the Cantor staircase function on the Euclidean unit interval and characteristic functions of smooth Euclidean sets. However, a Euclidean set whose characteristic function is of bounded variation can have non-smooth boundary. As with characteristic functions of such sets, more general functions of bounded variation in Euclidean domains exhibit discontinuous behavior along certain subsets, called jump sets. The situation gets more complicated when the function of bounded variation is not real-valued but a map from a Euclidean domain into a metric space, as in [2]. Yet another layer of complication comes from considering functions of bounded variation from a metric measure space into a metric space. The goal of the present paper is to explore regularity properties of such maps. In this case, the lack of smoothness implies that we have no notion of inward normal direction in the sense analogous to [17] or [42, (1.2)], and hence the classical definition of jump points as in [17, 42] (see also [5, Definition 3.67]) is not suitable here.

To do so, the first question to address is what is a reasonable notion of mappings of bounded variation from a metric measure space into a metric space. First proposed by Miranda Jr. in [40], the notion of real-valued functions of bounded variation on metric measure spaces equipped with a doubling measure supporting a 1-Poincaré inequality has been extensively studied, and the papers [3, 6, 13, 14, 25, 26, 27, 28, 33, 36, 37] contain a small sample of the outcomes from such a study. The papers [40, 3, 6, 14, 36] consider the definition of functions of bounded variation in the metric setting via relaxation of Sobolev functions, while the papers [25, 26, 27, 28] consider functions of bounded variation as those whose local behavior is controlled by a sequence of non-negative Borel functions that serve as a substitute for upper gradients [39]. In [13] it was shown that these two approaches yield the same class of real-valued functions of bounded variation.

In the present paper we consider two definitions of mappings of bounded variation from a metric measure space into a metric space or, in particular, a Banach space, by adopting the two approaches described above. We show that when the domain metric measure space is complete, doubling, and supports a 1-Poincaré inequality and the target metric space is a Banach space, both notions yield the same class of maps. However, when the target metric space is not a Banach space, the two approaches do not in general yield the same function class, with the notion of relaxation of Sobolev functions yielding a *strictly smaller* subclass of maps. Thus, in the setting of general metric space targets, it is more appropriate to study mappings of bounded variation based on the sequence of upper gradients as first proposed by Martio in [39]. Other alternate notions of metric-valued BV mappings defined via relaxation with simple maps and test plan-based BV mappings were studied in [7], where they were shown to be equivalent to definitions given by test plans and post-composition with Lipschitz functions. However, the fine properties of those mappings are not considered in [7].

Having made the choice of the definition of mappings of bounded variation, in the second part of the paper we explore the fine properties of mappings of bounded variation, from a complete doubling metric measure space supporting a 1-Poincaré inequality, into a proper metric space. We determine Hausdorff co-dimensional measure properties of sets of jump discontinuity points of such mappings. While the study of fine properties of *real-valued* BV functions on Euclidean domains is now well-established (see for instance [42, 17, 5]), the corresponding study of *metric space-valued* mappings on Euclidean domains has a much shorter history. Moreover, real-valued BV functions in Euclidean domains can exhibit more than two jump values at a jump point, as demonstrated by the function f

on the complex plane given by $f = 2\chi_A + \chi_E - \chi_F$ with $A = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x^2+y^2 \leq 1\}$, $E = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y < 0, x^2 + y^2 \leq 1\}$, and $F = \{(x, y) \in \mathbb{R}^2 : x < 0, y < 0, x^2 + y^2 \leq 1\}$ having four jump values at the point $(0, 0)$. However, at $\mathcal{H}^{n-1}$-a.e. point in $\mathbb{R}^n$ we see only two jump values. In such a Euclidean setting, similar results hold true once we know that there *are* jump values for metric space-valued BV maps. The proof of Theorem 1.3 below gives a more complete and alternate verification of the corresponding result stated in [2] for *metric space-valued* BV mappings from Euclidean domains, as the existence of jump values as claimed in [1, 2], lacks some details. One key part of the verification requires the directions ν_φ in [2, Theorem 2.3] to be independent of the functions φ, for which [2] points to the unproven statement in [2, Remark 1.5]. Note also that the results of this paper also covers mappings from subsets of $\mathbb{R}^n$ even when the subset is not open, provided it carries a doubling measure supporting a suitable Poincaré inequality. A few words of caution are appropriate here. The definition of *jump set* as considered here follows the convention of [16, Definition 5.9]. In [5] the term *approximate discontinuity set* is used instead (with the notation S_u used for the set corresponding to the BV function u in [5, 6]), in order to distinguish the points where one can obtain two jump values, see [5, Definition 3.67]. As we do not have access to the notion of inner normal vectors for level surfaces of BV functions, we follow the simpler categorization of [16] instead.

The following are the two main results of this note. The first result focuses on comparing the two notions of mappings of bounded variation. The space $BV(X : V)$ is defined using the Miranda Jr. [40] approach of relaxation of Sobolev function class, while the space $BV_{AM}(X : V)$ is obtained by using sequences of non-negative Borel functions that act as upper gradients as in [39].

Theorem 1.1. *Let (X, d, μ) be a complete doubling metric measure space supporting a 1-Poincaré inequality, and let V be a Banach space. Suppose also that for μ-a.e. $x \in X$ we have that $\liminf_{r\to 0^+} \mu(B(x, r))/r = 0$. Then $BV(X : V) = BV_{AM}(X : V)$, with comparable BV energy seminorms.*

Theorem 1.1 will be proved in Section 4. Before doing so, in Section 3 we adapt the notion of Semmes pencil of curves and Poincaré inequality to the setting of Banach space-valued BV functions.

In the next main theorem, we determine the fine properties of a metric space-valued BV_{AM}-map, when the metric space target is proper (that is, closed and bounded subsets of Y are compact).

In what follows, we consider maps $u \in BV_{AM}(X : Y)$ with (Y, d_Y) a metric space.

Definition 1.2. A point $x \in X$ is said to be a *point of approximate continuity* of u if there is a point $y_x \in Y$ such that for each $\varepsilon > 0$ we have

$$\limsup_{r\to 0^+} \frac{\mu(B(x, r) \setminus u^{-1}(B(y_x, \varepsilon)))}{\mu(B(x, r))} = 0.$$

We say that x is a *jump point* of u, that is, $x \in \mathcal{J}(u)$, if it is not a point of approximate continuity of u. If x is a jump point of u, we say that a point $y \in Y$ is a *jump value* of u at x if for all $\varepsilon > 0$, we have

$$\limsup_{r\to 0^+} \frac{\mu(B(x, r) \cap u^{-1}(B(y, \varepsilon)))}{\mu(B(x, r))} > 0.$$

Theorem 1.3. *Let (X, d, μ) be a complete doubling metric measure space supporting a 1-Poincaré inequality, and let (Y, d_Y) be a proper metric space. Then for each $u \in BV_{AM}(X : Y)$ we have that $\mathcal{J}(u)$ is σ-finite with respect to the codimension 1 Hausdorff measure $\mathcal{H}^{-1}$ on X and there exists a set $N \subset \mathcal{J}(u)$ with $\mathcal{H}^{-1}(N) = 0$ such that the following hold:*

(a) *A point* $x_0 \in X$ *belongs to* $\mathcal{J}(u)$ *if and only if there exist sets* $E_1, E_2 \subset X$ *such that*

$$\limsup_{r\to 0} \frac{\mu(B(x_0,r)\cap E_i)}{\mu(B(x_0,r)} > 0 \quad \text{for} \quad i=1,2,$$

and there exist balls $B_1, B_2 \subset Y$ *with* $\operatorname{dist}(B_1,B_2) \geq \operatorname{rad}(B_1)$ *such that* $u(E_i) \subset B_i$ *for* $i=1,2$.

(b) *For each* $x_0 \in \mathcal{J}(u)\setminus N$ *there are at least two, and at most* k_0*, number of jump values, that is, points* $y_1, y_2, \cdots, y_k \in Y$ *such that for each* $\varepsilon > 0$ *and* $i = 1,2,\cdots,k$ *we have*

$$\liminf_{r\to 0^+} \frac{\mu(B(x,r)\cap u^{-1}(B(y_i,\varepsilon)))}{\mu(B(x,r))} \geq \gamma,$$

and

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r)\setminus \bigcup_{i=1}^k u^{-1}(B(y_i,\varepsilon)))}{\mu(B(x,r))} = 0.$$

In the above, both k_0 *and* γ *are constants that depend solely on the doubling and Poincaré constants of the space* X*, and in particular are independent of* Y*,* u *and* ε*.*

Theorem 1.3 will be proved in Section 5. The set $\mathcal{J}(u)$ is called the *jump set* of u, and is defined in the initial discussion of Section 5 as the complement of the set of points of approximate continuity of u, and so (a) is immediate from the construction. The σ-finiteness of the jump set is proved as Corollary 5.10. Subsequently, (b) is proved via Proposition 5.12, completing the proof. We point out here that, with our definition of the jump set $\mathcal{J}(u)$, we cannot be guaranteed that the set N referred to in the above theorem will be empty. Indeed the argument function $f(z) = \operatorname{Arg}(z)$ in the complex unit disk has the complex number 0 as part of its jump set, but at 0 the function f takes on infinitely many jump values. From the expository monograph [5] we know that for a real-valued BV function on Euclidean domains, such points form a very small set; the above theorem extends this to metric space-valued mappings in metric measure spaces.

2. Background notions

In this note, (X,d,μ) will denote a metric measure space, where (X,d) is a complete metric space and μ a Borel measure, and V is a general Banach space. The ball centered at $x \in X$ with radius $r > 0$ will be denoted $B(x,r) = \{y \in X : d(x,y) < r\}$. A ball in X may have more than one center and more than one radius. Hence, by a ball, we understand that it comes with a pre-selected center and radius. The radius of a ball B will be denoted by $\operatorname{rad}(B)$. The *closed* ball centered at x with radius $r > 0$ is the set $\overline{B}(x,r) := \{z \in X : d(x,z) \leq r\}$, and is in general potentially larger than the topological closure of the open ball $B(x,r)$. Moreover, given two sets $E, F \subset X$, the distance between them is denoted $\operatorname{dist}(E,F) := \inf\{d(x,y) : x \in E, y \in F\}$.

We will assume throughout that the measure μ is *doubling*, that is, there is some constant $C_d \geq 1$ such that whenever $x \in X$ and $r > 0$, we have

$$0 < \mu(B(x,2r)) \leq C_d\, \mu(B(x,r)) < \infty.$$

Given such a measure μ, and a set $A \subset X$, the *co-dimension* 1 Hausdorff measure of A is given by

$$\mathcal{H}^{-1}(A) := \lim_{\delta\to 0^+} \inf\left\{ \sum_{i\in I\subset\mathbb{N}} \frac{\mu(B_i)}{\operatorname{rad}(B_i)} \,:\, A \subset \bigcup_{i\in I} B_i,\, \operatorname{rad}(B_i) \leq \delta\right\}.$$

Next, we introduce the definitions of two notions of mappings of bounded variation. The first one, $BV(X:V)$, was widely studied in [40], while the other one, $BV_{AM}(X:V)$, was first introduced in [39] and was proven to be equal to $BV(X : V)$ in [13], when $V = \mathbb{R}$ and the measure on X is doubling and supports a 1-Poincaré inequality. As a natural question, we will study here the equality of both spaces when V is a general Banach space.

Since the approach in [40] is based on approximation via Newtonian mappings, we recall here the definition of upper gradients and the Newton-Sobolev space. An *upper gradient* of a mapping $f : X \to V$ is a non-negative Borel function $g : X \to [0, \infty]$ such that for each nonconstant compact rectifiable curve $\gamma : [a, b] \to X$ (with $a, b \in \mathbb{R}$ and $a < b$), we have

$$\|f(\gamma(b)) - f(\gamma(a))\| \le \int_\gamma g\, ds. \tag{2.1}$$

The function g is called a 1-*weak upper gradient* of f if (2.1) holds for 1-almost every curve (see Section 2.2 below for the concept of 1-modulus). It turns out that if f has a 1-weak upper gradient in the class $L^1(X)$, then there is a unique 1-weak upper gradient of f with the smallest L^1-norm; moreover, every 1-weak upper gradient of f that belongs to $L^1(X)$ is μ-a.e. bounded below by this unique weak upper gradient; we denote by g_u the minimal 1-weak upper gradient of u. Note that modifying a map f, that has an upper gradient belonging to $L^1(X)$, on a set of measure zero can result in a function with no upper gradient in $L^1(X)$. Thus the above notion is at the level of *maps*, not equivalence classes of maps. However, we say that two maps f_1, f_2 are equivalent if $f_1 = f_2$ μ-a.e. in X and in addition, for each $\varepsilon > 0$ there is an upper gradient g_ε of $f_1 - f_2$ such that $\int_X g_\varepsilon\, d\mu < \varepsilon$. The *Newton Sobolev space* $N^{1,1}(X : V)$ is the collection of all equivalence classes of mappings $f : X \to V$ with $f \in L^1(X : V)$ and each map in the equivalence class having an upper gradient in $L^1(X)$. We refer the interested reader to [23] for more details on $N^{1,1}(X : V)$ and upper gradients.

2.1. **Vector-valued mappings of bounded variation via relaxation of Newton-Sobolev mappings.** Let $u \in L^1(X : V)$ with $L^1(X : V)$ in the sense of Bochner integrals, and define

$$\|Du\|(X) := \inf\left\{\liminf_{i\to\infty} \int_X g_{u_i}\, d\mu : (u_i)_{i\in\mathbb{N}} \in N^{1,1}(X : V), u_i \xrightarrow{L^1} u\right\}.$$

Definition 2.2. Let (X, d, μ) be a metric measure space and V a Banach space. Following Miranda [40], we define $BV(X : V)$ to be the class of mappings $u \in L^1(X : V)$ such that $\|Du\|(X) < \infty$. We denote $BV(X) := BV(X : \mathbb{R})$.

It was shown in [40] that the map $U \mapsto \|Du\|(U)$ for open sets $U \subset X$ can be extended via a Carathéodory construction to a Radon outer measure on X, which is also denoted by $\|Du\|$; in particular, for Borel sets $A \subset X$ we set

$$\|Du\|(A) := \inf\{\|Du\|(U) : U \text{ is open in } X \text{ and } A \subset U\}.$$

If $E \subset X$ is a measurable set, we say that E is of finite perimeter if $\chi_E \in BV(X)$ and we denote the perimeter measure by $P(E, \cdot) := \|D\chi_E\|(\cdot)$. For functions in the class $BV(X)$ the following co-area formula is known.

Lemma 2.3. *(coarea formula,* [40, Proposition 4.2]*) Let $E \subset X$ be a Borel set and $u \in BV(X)$. Then*

$$\|Du\|(E) = \int_{-\infty}^{\infty} P(\{u > t\}, E)dt.$$

Thanks to the work of Ambrosio [3], we know the structure of sets of finite perimeter. To describe these results we first describe the measure-theoretic and reduced boundaries of subsets of X. For $E \subset X$ we say that a point $x \in X$ belongs to the measure-theoretic boundary $\partial_* E$ of E if

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r)\cap E)}{\mu(B(x,r))} > 0 \quad \text{and} \quad \limsup_{r\to 0^+} \frac{\mu(B(x,r)\setminus E)}{\mu(B(x,r))} > 0. \tag{2.4}$$

For a real number $\beta > 0$ we say that $x \in X$ belongs to the reduced boundary $\Sigma_\beta E$ of E if

$$\liminf_{r\to 0^+} \frac{\mu(B(x,r)\cap E)}{\mu(B(x,r))} \ge \beta \quad \text{and} \quad \liminf_{r\to 0^+} \frac{\mu(B(x,r)\setminus E)}{\mu(B(x,r))} \ge \beta. \tag{2.5}$$

Lemma 2.6. *Suppose that X is complete and that μ is doubling and supports a 1-Poincaré inequality. Then there is a positive real number $\gamma \le 1/2$, depending only on the doubling constant and constants associated with the Poincaré inequality, such that for each set E of finite perimeter,*

$$\mathcal{H}^{-1}(\partial_* E \setminus \Sigma_\gamma E) = 0 \quad \textit{and} \quad P(E,X) \approx \mathcal{H}^{-1}(\Sigma_\gamma E).$$

We also point out that in fact, the property of a measurable set being of finite perimeter is characterized by the property that $\mathcal{H}^{-1}(\Sigma_\gamma E)$ is finite; this result was first proved by Lahti [33], and is new even in the Euclidean setting, refining Federer's characterization of Euclidean sets of finite perimeter.

2.2. **1-modulus and AM-modulus.** Recall the definition of 1-modulus of a family of nonconstant, compact and rectifiable curves Γ:

$$\mathrm{Mod}_1(\Gamma) := \inf_\rho \int_X \rho \, d\mu$$

where the infimum is taken over all non negative Borel functions $\rho : X \to [0, +\infty]$ s.t. $\int_\gamma \rho ds \ge 1$ for each $\gamma \in \Gamma$. It turns out that there is another notion of modulus that is better suited to the study of BV functions. This notion, called AM-modulus, was first proposed by Martio in [39]. Following [39] we define the AM-modulus to be

$$\mathrm{AM}(\Gamma) := \inf_{(\rho_i)_{i\in\mathbb{N}}} \liminf_{i\to\infty} \int_X \rho_i d\mu,$$

where the infimum is taken over all sequences of AM-admissible functions, that is, sequences $(\rho_i)_i$ of non negative Borel functions such that for each $\gamma \in \Gamma$ we have

$$\liminf_{i\to\infty} \int_\gamma \rho_i ds \ge 1.$$

We say that a property holds for 1-almost every curve (respectively AM-almost every curve) on X if it holds outside a family of curves of zero 1-modulus (resp. AM-modulus).
For a curve family Γ we always have $\mathrm{AM}(\Gamma) \le \mathrm{Mod}_1(\Gamma)$.

Lemma 2.7. *Let Γ be a family of curves in X. Then*

(a) *$Mod_1(\Gamma) = 0$ if and only if there is a non-negative Borel function $\rho \in L^1(X)$ such that for each $\gamma \in \Gamma$ we have $\int_\gamma \rho \, ds = \infty$.*
(b) *$AM(\Gamma) = 0$ if and only if there is a sequence $(\rho_i)_{i\in\mathbb{N}}$ of non-negative Borel functions with $\sup_i \int_X \rho_i \, d\mu < \infty$ such that for each $\gamma \in \Gamma$ we have*

$$\liminf_{i\to\infty} \int_\gamma \rho_i \, ds = \infty.$$

Proof. A proof of (a) can be found in [23, Lemma 5.2.8]. To prove (b) we argue as follows. Suppose first that $\mathrm{AM}(\Gamma) = 0$. Then for each positive integer k we can find a sequence $(\rho_{k,i})_{i\in\mathbb{N}}$ of non-negative Borel functions on X with $\liminf_{i\to\infty}\int_\gamma \rho_{k,i}\,ds \ge 1$ for each $\gamma \in \Gamma$, such that $\sup_i \int_X \rho_{k,i}\,d\mu < 2^{-k}$. For each positive integer i we set $\rho_i = \sum_{k=1}^\infty \rho_{k,i}$. By the monotone convergence theorem we know that for each $\gamma \in \Gamma$,

$$\int_\gamma \rho_i\,ds = \sum_{k=1}^\infty \int_\gamma \rho_{k,i}\,ds,$$

and so

$$\liminf_i \int_\gamma \rho_i\,ds \ge \sum_{k=1}^\infty \liminf_i \int_\gamma \rho_{k,i}\,ds = \infty,$$

and at the same time, for each positive integer i we have

$$\int_X \rho_i\,d\mu = \sum_{k=1}^\infty \int_X \rho_{k,i}\,d\mu \le \sum_{k=1}^\infty 2^{-k} = 1.$$

The desired conclusion follows.

Now suppose that Γ is such that there is a sequence $(\rho_i)_{i\in\mathbb{N}}$ of non-negative Borel functions on X such that $\sup_i \int_X \rho_i\,d\mu =: \alpha < \infty$ and for each $\gamma \in \Gamma$ we have $\liminf_{i\to\infty}\int_\gamma \rho_i\,ds = \infty$. Then for each $\varepsilon > 0$ the sequence $(\varepsilon\rho_i)_{i\in\mathbb{N}}$ is a sequence of AM-admissible functions for Γ, with $\limsup_i \int_X \varepsilon\rho_i\,d\mu = \varepsilon\alpha$. Thus $\mathrm{AM}(\Gamma) \le \varepsilon\alpha$ for each $\varepsilon > 0$. Thus we have that $\mathrm{AM}(\Gamma) = 0$. □

From the above lemma, it follows that if Γ is a family of curves with $\mathrm{AM}(\Gamma) = 0$, then for each $\varepsilon > 0$ there is a sequence $(\rho_i)_i$ such that $\sup_i \int_X \rho_i\,d\mu < \varepsilon$ and for each $\gamma \in \Gamma$ we have $\liminf_{i\to\infty}\int_\gamma \rho_i\,ds = \infty$.

2.3. **The notion of $BV_{AM}(X:V)$.** Now we turn our attention to the definition of $BV_{AM}(X:V)$. The notion of $BV_{AM}(X:\mathbb{R})$ was first proposed by Honzlová-Exnerová, Malý, and Martio in a series of papers [26, 27, 28] using the notion of AM-modulus, see also [25]. This notion was adopted by Lahti in [34] to study metric space-valued BV mappings. In this section we focus on this notion of BV maps.

Definition 2.8. Let (X, d, μ) be a metric measure space and V a Banach space. Let $(\rho_i)_{i\in\mathbb{N}}$ be a sequence of non-negative Borel functions on X. We say that this sequence is an *AM-bounding sequence* for a function $u : X \to V$ if for AM-a.e. curve $\gamma : [a, b] \to X$ there is a null set $N_\gamma \subset [a, b]$ (that is, $\mathcal{H}^1(N_\gamma) = 0$) such that for every $\tau, t \in [a, b] \setminus N_\gamma$ with $\tau < t$, we have

$$\|u(\gamma(\tau)) - u(\gamma(t))\| \le \liminf_{i\to\infty} \int_{\gamma|_{[\tau,t]}} \rho_i ds. \tag{2.9}$$

We say that a mapping $u \in L^1(X : V)$ is in the class $BV_{AM}(X : V)$ if there is an AM-bounding sequence $(\rho_i)_{i\in\mathbb{N}}$ for u such that

$$\liminf_{i\to\infty} \int_X \rho_i\,d\mu < \infty.$$

We set

$$\|D_{AM}u\|(X) := \inf_{(\rho_i)_i} \liminf_{i\to\infty} \int_X \rho_i\,d\mu,$$

where the infimum is taken over all AM-bounding sequences $(\rho_i)_{i\in\mathbb{N}}$ of u.

As in the case of the object $\|Du\|$, the above $\|D_{AM}u\|$ can be extended to be a Radon measure on X via a Carathéodory construction, see for example [39] and Section 6.1.

Note that in considering an AM-bounding sequence for u, we discount a family Γ of curves in X such that $\mathrm{AM}(\Gamma) = 0$. If the AM-bounding sequence $(\rho_i)_{i\in\mathbb{N}}$ is such that the exceptional family Γ is empty, then we say that $(\rho_i)_{i\in\mathbb{N}}$ is a *strong bounding sequence* for u.

Lemma 2.10. *Let $u \in BV_{AM}(X;V)$ and $v : X \to V$. Suppose that there is a set $N \subset X$ with $\mu(N) = 0$ such that for each $x \in X \setminus N$ we have $u(x) = v(x)$. Then a sequence $(\rho_i)_{i\in\mathbb{N}}$ is an AM-bounding sequence for u if and only if it is an AM-bounding sequence for v; hence $v \in BV_{AM}(X;V)$ with $\|D_{AM}u\|(X) = \|D_{AM}v\|(X)$ and $\|D_{AM}(u-v)\|(X) = 0$.*

Proof. Since N is a null-set, by enlarging it if need be (recall that μ is a Borel measure), we can also assume that it is a Borel set as well. It follows that with Γ_N^+ the collection of all nonconstant compact rectifiable curves $\gamma : [a,b] \to X$ for which $\mathcal{H}^1(\gamma^{-1}(N)) > 0$, we have $\mathrm{AM}(\Gamma_N^+) \le \mathrm{Mod}_1(\Gamma_N^+) = 0$. Thus, for each AM-bounding sequence $(\rho_i)_{i\in\mathbb{N}}$ of the original function u and for each $\gamma \notin \Gamma_0 \cup \Gamma_N^+$, we can replace N_γ with $N_\gamma \cup \gamma^{-1}(N)$ to see that this is an AM-bounding sequence for v as well. Here Γ_0 is the exceptional family associated with the bounding sequence; so $\mathrm{AM}(\Gamma_0) = 0$.

Since $u - v = 0$ μ-a.e. in X, the final claims follows from noting that $\mathrm{AM}(\Gamma_N^+) = 0$ and so the sequence $(g_i)_{i\in\mathbb{N}}$, with each g_i the zero function, is an AM-bounding sequence for $u - v$. □

Lemma 2.11. *Suppose that $(\rho_i)_{i\in\mathbb{N}}$ is an AM-bounding sequence for a map u from X to V such that $\sup_i \int_X \rho_i \, d\mu < \infty$. Then for each $\varepsilon > 0$ we can find a strong bounding sequence $(g_i)_{i\in\mathbb{N}}$ of u such that for each $i \in \mathbb{N}$ we have $\int_X |g_i - \rho_i| \, d\mu < \varepsilon$.*

Proof. Let $u \in BV_{AM}(X : V)$ and $(\rho_i)_{i\in\mathbb{N}}$. Then there exists Γ with $\mathrm{AM}(\Gamma) = 0$ such that for each non constant compact rectifiable curve $\gamma \notin \Gamma$, the relation (2.9) holds for $s, t \in \mathrm{dom}(\gamma) \setminus \gamma^{-1}(N_\gamma)$ where $\mathcal{H}^1(N_\gamma) = 0$. Since $\mathrm{AM}(\Gamma) = 0$, by Lemma 2.7 there exists a sequence of non-negative Borel functions $(g_i)_i$ such that

$$\liminf_{i\to\infty} \int_X g_i \, d\mu < \infty \quad \text{and} \quad \liminf_{i\to\infty} \int_\gamma g_i \, ds = \infty \ \ \forall \gamma \in \Gamma.$$

Now let Γ_0 be the family of all nonconstant compact rectifiable curves γ in X for which we have $\liminf_{i\to\infty} \int_\gamma g_i ds = \infty$. Then $\mathrm{AM}(\Gamma_0) = 0$ and each subcurve of a curve that is not in Γ_0 is also not in Γ_0. For each $\varepsilon > 0$, since $\Gamma \subset \Gamma_0$, we have that for each $\gamma \notin \Gamma_0$,

$$\|u(\gamma(\tau)) - u(\gamma(t))\| \le \liminf_{i\to\infty} \int_{\gamma|_{[\tau,t]}} \rho_i ds \le \liminf_{i\to\infty} \int_{\gamma|_{[\tau,t]}} \rho_i + \varepsilon g_i ds \tag{2.12}$$

for every $\tau, t \notin \gamma^{-1}(N_\gamma)$.

If $\gamma \in \Gamma_0$ is such that every subcurve of γ also belongs to Γ_0, then for each $\tau, t \in [a,b]$ with $\tau < t$ we have $\liminf_{i\to\infty} \int_{\gamma|_{[\tau,t]}} g_i \, ds = \infty$, and so the choice of $N_\gamma = \emptyset$ works. If it is not the case that every subcurve of γ also belongs to Γ_0, then let $\mathcal{C}_0(\gamma)$ be the collection of all non-degenerate (that is, containing more than one point) intervals $I \subset [a,b]$ for which, whenever J is a compact subinterval of I we must have $\gamma|_J \notin \Gamma_0$, and whenever J is a compact subinterval of $[a,b]$ containing I in its interior, we must have $\gamma|_J \in \Gamma_0$. By the maximality of the intervals in the collection $\mathcal{C}_0(\gamma)$, two intervals in this collection are either disjoint or are equal as intervals. Moreover, these intervals have non-empty interior. It follows that as $\mathbb{Q}$ is dense in $\mathbb{R}$, the collection $\mathcal{C}_0(\gamma)$ is countable.

With $\gamma : [a,b] \to X$, consider all $a_0, b_0 \in [a,b] \cap \mathbb{Q}$ with $a_0 < b_0$ for which $\gamma|_{[a_0,b_0]} \notin \Gamma_0$, that is, $\liminf_{i\to\infty} \int_{\gamma|_{[a_0,b_0]}} g_i \, ds < \infty$; hence there is a corresponding null set $N[a_0,b_0] \subset [a_0,b_0]$ with

$\mathcal{H}^1(N[a_0,b_0])=0$, so that for each $\tau,t\in[a_0,b_0]\setminus N[a_0,b_0]$ we have (2.12) holding true. Let $\mathcal{C}(\gamma)$ be the collection of all such $[a_0,b_0]\subset[a,b]$, and set

$$N_\gamma:=\bigcup_{[a_0,b_0]\in\mathcal{C}(\gamma)}N[a_0,b_0]\cup\bigcup_{J\in\mathcal{C}_0(\gamma)}\{\inf J,\sup J\}.$$

Note that as $a_0,b_0\in\mathbb{Q}$, the collection $\mathcal{C}(\gamma)$ is a countable collection. Hence $\mathcal{H}^1(N_\gamma)=0$ by the subadditivity of $\mathcal{H}^1$ on $[a,b]$. Now let $\tau,t\in[a,b]\setminus N_\gamma$ with $\tau<t$. If $[\tau,t]\subset[a_0,b_0]$ for some $[a_0,b_0]\in\mathcal{C}(\gamma)$, then (2.12) holds. If there is no $[a_0,b_0]\in\mathcal{C}(\gamma)$ for which $[\tau,t]\subset[a_0,b_0]$, then we must have that $\liminf_{i\to\infty}\int_{\gamma|_{[\tau,t]}}g_i\,ds=\infty$, and so we now have

$$\|u(\gamma(\tau))-u(\gamma(t))\|\le\liminf_{i\to\infty}\int_{\gamma|_{[\tau,t]}}(\rho_i+\varepsilon g_i)\,ds=\infty.$$

Therefore $(\rho_i+\varepsilon g_i)_i$ satisfies (2.9) for every non constant compact rectifiable curve. Moreover, since ε is arbitrary, one can approach the energy $\|D_{AM}u\|(X)$ just by taking the infimum over the upper bounds of u that verify (2.9) for every non constant compact rectifiable curve. □

Lemma 2.13. *Suppose that $(\rho_i)_i$ is an AM–bounding sequence for u and that η is a non-negative L-Lipschitz function with support in a bounded set U; moreover, suppose that η is constant on an open set $V\Subset U$. Then $(\eta\rho_i+L\,|u|\,\chi_{U\setminus V})_i$ is an AM–bounding sequence for ηu.*

Proof. For each $i\in\mathbb{N}$ we set $g_i:=\eta\rho_i+L\,|u|\,\chi_{U\setminus V}$.

Let Γ be the exceptional family for the AM–bounding sequence $(\rho_i)_i$ with respect to u; so $\mathrm{AM}(\Gamma)=0$, and for each nonconstant compact rectifiable curve $\gamma:[a,b]\to X$ with $\gamma\notin\Gamma$, we have a null set $N_\gamma\subset[a,b]$ with $\mathcal{H}^1(N_\gamma)=0$ such that whenever $t,\tau\in[a,b]\setminus N_\gamma$ with $t<\tau$, we have

$$|u(\gamma(t))-u(\gamma(\tau))|\le\liminf_{i\to\infty}\int_{\gamma|_{[t,\tau]}}\rho_i\,ds.$$

Fix such $\tau,t\in[a,b]$. Consider a partition $t=t_0<t_1<\cdots<t_k=\tau$ of the interval $[t,\tau]$ so that $t_1,\cdots,t_{k-1}\notin N_\gamma$. Then

$$\begin{aligned}|u(\gamma(t))\eta(\gamma(t))-u(\gamma(\tau))\eta(\gamma(\tau))|&\le\sum_{j=1}^k|u(\gamma(t_{j-1}))\eta(\gamma(t_{j-1}))-u(\gamma(t_j))\eta(\gamma(t_j))|\\&\le\sum_{j=1}^k|u(\gamma(t_{j-1}))\eta(\gamma(t_{j-1}))-u(\gamma(t_j))\eta(\gamma(t_{j-1}))|\\&\qquad+\sum_{j=1}^k|u(\gamma(t_j))\eta(\gamma(t_{j-1}))-u(\gamma(t_j))\eta(\gamma(t_j))|\\&\le\liminf_{i\to\infty}\sum_{j=1}^k\int_{\gamma|_{[t_{j-1},t_j]}}[\eta(\gamma(t_{j-1}))\rho_i+|u(\gamma(t_j))|\mathrm{Lip}\,\eta]\,ds.\end{aligned}$$

The above must be true for all such partitions of the interval $[t,\tau]$. Since $u\circ\gamma$ and $\mathrm{Lip}\,\eta\circ\gamma$ are Borel functions on $[a,b]$, it follows that

$$|u(\gamma(t))\eta(\gamma(t))-u(\gamma(\tau))\eta(\gamma(\tau))|\le\liminf_{i\to\infty}\int_{\gamma|_{[t,\tau]}}g_i\,ds.$$ □

Lemma 2.14. *Let $u \in BV(X:V)$. Then the upper gradients of an approximating sequence form an AM-bounding sequence for u. In particular $u \in BV_{AM}(X:V)$ with $\|D_{AM}u\|(X) \leq \|Du\|(X)$.*

Proof. If $u \in BV(X:V)$, then there exists a sequence $(u_i)_{i\in\mathbb{N}} \in N^{1,1}(X:V)$ with upper gradients g_i such that

$$\limsup_{i\to\infty} \int_X g_i d\mu < \infty,$$

and such that $u_i \to u$ in $L^1(X:V)$. In particular, (passing to a subsequence if necessary) $u_i \to u$ pointwise almost everywhere in X. Then there exists a null set N such that $\lim_{i\to\infty} u_i(x) = u(x)$ for every $x \in X \setminus N$. By enlarging N if necessary, we may also assume that N is a Borel set (recall that μ is Borel regular). Hence, by considering the non-negative Borel measurable function $\rho = \infty\chi_N$ on X, we know that if Γ_N^+ is the collection of all compact nonconstant rectifiable curves γ in X with $\mathcal{H}^1(\gamma^{-1}(N)) > 0$, then $\mathrm{AM}(\Gamma_N^+) \leq \mathrm{Mod}_1(\Gamma_N^+) = 0$.

Let $\gamma : [a,b] \to X$ be a nonconstant compact rectifiable curve such that $\gamma \notin \Gamma_N^+$. We set $N_\gamma := \gamma^{-1}(N)$. Then for $\tau, t \in [a,b] \setminus N_\gamma$ with $\tau < t$, we have that

$$\lim_{i\to\infty} u_i(\gamma(t)) - u_i(\gamma(\tau)) = u(\gamma(t)) - u(\gamma(\tau)).$$

Moreover, since g_i is an upper gradient of u_i, it follows that

$$\|u_i(\gamma(t)) - u_i(\gamma(\tau))\| \leq \int_{\gamma|_{[\tau,t]}} g_i\, ds.$$

Combining the above two, we see that for $\tau, t \in [a,b] \setminus N_\gamma$ with $\tau < t$, we have

$$\|u(\gamma(t)) - u(\gamma(\tau))\| \leq \liminf_{i\to\infty} \int_{\gamma|_{[\tau,t]}} g_i\, ds.$$

Thus we have shown that $(g_i)_{i\in\mathbb{N}}$ is an AM-bounding sequence for u. Now fix $\varepsilon > 0$ and choose $(u_i)_{i\in\mathbb{N}}$ and $(g_i)_{i\in\mathbb{N}}$ such that

$$\liminf_{i\to\infty} \int_X g_i d\mu \leq \|Du\|(X) + \varepsilon.$$

Since the previous argument holds for any choice of $(u_i)_{i\in\mathbb{N}}$ and $(g_i)_{i\in\mathbb{N}}$, we have that $(g_i)_{i\in\mathbb{N}}$ is an AM-bounding sequence for u, and hence,

$$\|D_{\mathrm{AM}}u\|(X) \leq \liminf_{i\to\infty} \int_X g_i d\mu \leq \|Du\|(X) + \varepsilon,$$

and then taking $\varepsilon \to 0$ completes the proof. □

Remark 2.15. Notice that condition (2.9) holds outside a null set N_γ, which depends on the choice of γ, but as seen in the previous proof, whenever $u \in BV(X:V)$, we can choose a null set $N \subset X$ to be independent of the curve, such that $N_\gamma = \gamma^{-1}(N)$. In the following sections we will prove that $BV(X:V) = BV_{AM}(X:V)$. However the construction of the approximation by Newtonian mappings of a BV_{AM}-mapping will yield a sequence of upper gradients different to the original AM-bounding sequence for u, and so in general we cannot assume that *every* AM-bounding sequence of u comes with a universal null set $N \subset X$ as above.

2.4. **Metric space-valued functions of bounded variation.** In the prior two sections we considered mappings from the metric space X into a Banach space V; in this section we consider the case of mappings into a metric space (Y, d_Y). We will assume here that Y is complete and separable. Given $\Omega \subset X$ and $y_0 \in Y$, we mean by $f \in L^1(\Omega : Y, y_0)$ that

$$\int_\Omega d_Y(f(\cdot), y_0)d\mu < \infty.$$

We say that $f \in L^1_{loc}(X : Y)$ if for each $y_0 \in Y$ (or, equivalently, for some $y_0 \in Y$) we have that the real-valued function $x \mapsto d_Y(f(x), y_0)$ belongs to $L^1_{loc}(X)$.

We begin with the intrinsic definitions, analogous to the definitions of $BV(X : V)$ and $BV_{AM}(X : V)$ given above.

Lemma 2.16. *Let $u : X \to Y$ be a measurable function. Then the following are equivalent:*

(a) *There is a family Γ_0 of curves in X with $AM(\Gamma_0) = 0$ and a sequence $(\rho_i)_{i\in\mathbb{N}}$ of non-negative Borel measurable functions on X such that for each nonconstant compact rectifiable curve $\gamma : [a, b] \to X$ with $\gamma \notin \Gamma_0$ there is a set $N_\gamma \subset [a, b]$ with $\mathcal{H}^{-1}(N_\gamma) = 0$ such that for each $\tau, t \in [a, b] \setminus N_\gamma$ with $\tau < t$ we have*

$$d_Y(u(\gamma(\tau)), u(\gamma(t))) \le \liminf_{i\to\infty} \int_{\gamma|_{[\tau,t]}} \rho_i \, ds,$$

with

$$\sup_i \int_X \rho_i \, d\mu < \infty.$$

(b) *For every Banach space V and isometric embedding $\Phi : Y \to V$ we have that $\Phi \circ u \in BV_{AM}(X; V)$.*

(c) *There is a Banach space V and an isometric embedding $\Phi : Y \to V$ such that $\Phi \circ u \in BV_{AM}(X; V)$.*

In any (and hence all) of the cases above, we have that

$$\|D_{AM}\Phi \circ u\|(X) = \inf_{(\rho_i)_{i\in\mathbb{N}}} \liminf_{i\to\infty} \int_X \rho_i \, d\mu$$

where the infimum is over all sequences $(\rho_i)_{i\in\mathbb{N}}$ satisfying (a) above.

Proof. If V is a Banach space and Φ is an isometric embedding of Y into V, then for each $x, z \in X$ we have that $d_Y(u(x), u(z)) = \|\Phi(u(x)) - \Phi(u(z))\|$, and so we know that (a) implies (b) and that (b) implies (c). Indeed, every complete separable metric space can be isometrically embedded in the Banach space ℓ^∞ by the Kuratowski embedding theorem (see, e.g., [23, page 100]).

Thus it only remains to show that (c) implies (a). To this end, suppose that V is a Banach space, Φ an isometric embedding of Y into V, and that $\Phi \circ u \in BV_{AM}(X; V)$. Then there is a sequence $(\rho_i)_{i\in\mathbb{N}}$ that is an AM-bounding sequence for $\Phi \circ u$, and a family Γ_0 with $\mathrm{AM}(\Gamma_0) = 0$ such that whenever $\gamma : [a, b] \to X$ with $\gamma \notin \Gamma_0$ there is a set $N_\gamma \subset [a, b]$ with $\mathcal{H}^{-1}(N_\gamma) = 0$ such that for each $\tau, t \in [a, b] \setminus N_\gamma$ with $\tau < t$ we have

$$\|\Phi \circ u(\gamma(\tau)) - \Phi \circ u(\gamma(t))\| \le \liminf_{i\to\infty} \int_{\gamma|_{[\tau,t]}} \rho_i \, ds,$$

with

$$\sup_i \int_X \rho_i \, d\mu < \infty.$$

As Φ is an isometric embedding of Y into V, it follows that $d_Y(u(\gamma(\tau)), u(\gamma(t))) = \|\Phi \circ u(\gamma(\tau)) - \Phi \circ u(\gamma(t))\|$. The condition (a) follows.

The above argument shows that the sequence $(\rho_i)_{i\in\mathbb{N}}$ satisfies (a) if and only if it is an AM-bounding sequence for $\Phi \circ u$. The final claim of the above lemma now follows. $\square$

Definition 2.17. We say that a map $u : X \to Y$ is in $BV_{AM}(X; Y)$ if $u \in L^1(X : Y)$ and there is a sequence $(\rho_i)_{i\in\mathbb{N}}$ satisfying the hypothesis of Lemma 2.16 (a).

Lemma 2.18. *Let $\Phi : Y \to V$ be an isometric embedding of the metric space Y into a Banach space V. Suppose that*

(a) *$u \in BV_{AM}(X; V)$, and*
(b) *there is a set $N \subset X$ with $\mu(N) = 0$ such that for each $x \in X \setminus N$ we have that $u(x) \in \Phi(Y)$.*

Then $u \circ \Phi^{-1} \in BV_{AM}(X; Y)$. Here Φ^{-1} stands in for the inverse map of the bijective map $\Phi : Y \to \Phi(Y)$.

Proof. Since any modification of u on a set of μ-measure zero results in the same equivalence class of u in $BV_{AM}(X; V)$ (see Lemma 2.10), we can modify u on N by setting $u(x)$ to be some fixed point in $\Phi(Y)$ if $x \in N$. The conclusion now follows from the previous lemma. $\square$

Unlike $BV_{AM}(X; Y)$, the situation for $BV(X; Y)$ is more complicated.

Definition 2.19. We say that a map $u : X \to Y$ is in $BV(X; Y)$ if there is a sequence $(u_k)_{k\in\mathbb{N}}$ from $N^{1,1}(X; Y)$ such that $u_k \to u$ in $L^1(X; Y)$ and $\sup_{k\in\mathbb{N}} \int_X g_{u_k}\, d\mu < \infty$. Here g_{u_k} is the minimal 1-weak upper gradient of u_k in the sense of [23, page 161].

We will see that $BV_{AM}(X; V) = BV(X; V)$ whenever V is a Banach space (see Theorem 1.1), and as in the proof of Lemma 2.14, we can see that $BV(X; Y) \subset BV_{AM}(X; Y)$. However, $BV(X; Y)$ is in general a strictly smaller subset of $BV_{AM}(X; Y)$. This supports the choice of $BV_{AM}(X; Y)$ as the space of mappings of bounded variation in [34, 35].

Example 2.20. Consider $X = [-1, 1]$ and $Y = \{0, 1\}$. Let $u := \chi_{[0,1]}$ and $\rho_i := i\chi_{[-1/i,0]}$. Then for each $x, y \in [-1, 1]$, if $x, y < 0$ or $x, y \geq 0$ then $|u(x) - u(y)| = 0$ so it is immediate that $(\rho_i)_i$ satisfies the upper bound inequality. If $x < 0$ and $y \geq 0$ then

$$\liminf_{i\to\infty} \int_x^y i\chi_{[-1/i,0]} d\mathcal{L}^1 = 1 = |u(x) - u(y)|.$$

Thus the sequence $(\rho_i)_{i\in\mathbb{N}}$ is an AM-bounding sequence for u (and indeed, it is a strong bounding sequence for u. Therefore we know that $u \in BV_{AM}(X : Y)$. However Newtonian mappings are absolutely continuous on $[-1, 1]$, and so, since $Y = \{0, 1\}$, they must be constant; hence it is not possible to approximate (in L^1 norm) u by Newtonian mappings, proving that $u \notin BV(X : Y)$.

While the above example shows how topological obstructions can prevent approximations by $N^{1,1}$-maps, the next example provides a more analytical obstruction.

Example 2.21. Let $X = [-1, 1]$ and $Y = (\{0\} \times [0, 1]) \cup ([0, 1] \times \{1\}) \cup (\{1\} \times [0, 1])$ be both equipped with the Euclidean metric, and X be also equipped with the Lebesgue measure $\mathcal{L}^1$. Let $u \in BV_{AM}(X : Y)$ be given by $u(x) = (0, 0)$ when $-1 \leq x \leq 0$ and $u(x) = (1, 0)$ when $0 < x \leq 1$. If $(u_k)_k$ is a sequence of functions from $N^{1,1}(X : Y)$ such that $u_k \to u$ in $L^1(X)$, then for sufficiently large k we know that are points $x_k, y_k \in X$ with $|x_k + 1| < \frac{1}{10}$ and $|y_k - 1| < \frac{1}{10}$ such that $|u(x_k) - (0, 0)| < \frac{1}{10}$ and $|u(y_k) - (1, 0)| < \frac{1}{10}$. On the other hand, by the absolute continuity on $[-1, 1]$ of functions in $N^{1,1}(X : Y)$, we have that $\int_X g_{u_k}\, d\mathcal{L}^1 = \text{length}(u_k) \geq 1 + 2 \times \frac{9}{10} = \frac{14}{5}$. In inferring the above, we used the fact that u_k is also a path in Y. On the other hand, $\|D_{AM}u\|(X) = 1$, and so it is not possible to have $\|D_{AM}u\|(X) = \inf_{(u_k)\subset N^{1,1}(X:Y)} \liminf_{k\to\infty} \int_X g_{u_k}\, d\mathcal{L}^1$.

The above example does not preclude an L^1-approximation of the AM–BV function by a sequence of Newton-Sobolev functions. The next example gives a metric obstruction to the existence of even such an approximation.

Example 2.22. Let $X = [-1, 1]$ and $Y = (\{0\} \times [-1, 1]) \cup \{(x, \sin(1/x)) : 0 < x \leq 1\}$ be both equipped with the Euclidean metrics, and X also be equipped with the 1-dimensional Lebesgue measure. Let $u : X \to Y$ be given by $u(x) = (0, 0)$ if $-1 \leq x \leq 0$, and $u(x) = (1, \sin(1))$ when $0 < x \leq 1$. Then $\|D_{AM}u\|(X) = \sqrt{1 + \sin^2(1)}$, but due to the absolute continuity of functions in $N^{1,1}(X : Y)$ and the lack of paths in Y connecting $u(-1)$ to $u(1)$, there can be no sequence of functions in $N^{1,1}(X : Y)$ that gives an L^1-approximation of u.

By the ACL (absolute continuity on lines) property of functions in $N^{1,1}(X : Y)$, the above examples have higher dimensional analogs, but we will not go into details here.

Remark 2.23. While $BV(X : Y)$ need not equal $BV_{AM}(X : Y)$ in general, we do have a relationship between the two notions. Thanks to the Kuratowski embedding theorem ([23, page 100]), we can isometrically embed any separable metric space (Y, d_Y) into the Banach space ℓ^∞. Thanks to Theorem 1.1 and Lemma 2.18, we know that with V any Banach space and $\Phi : Y \to V$ an isometric embedding, whenever Y is complete, we have $BV_{AM}(X : Y)$ is the same as the class

$$\left\{u \in BV(X : V) : \mu(\{x \in X : u(x) \notin \Phi(Y)\}) = 0\right\}.$$

3. Poincaré inequalities and Semmes Pencil.

We say that the metric measure space (X, d, μ) supports a 1-Poincaré inequality if there are constants $C > 0, \lambda \geq 1$ such that for each $u, g \in L^1_{loc}(X)$, with g an upper gradient of u, we have

$$\fint_B \|u - u_B\| d\mu \leq C\text{rad}(B) \fint_{\lambda B} g d\mu$$

for each ball $B \subset X$. The metric measure space X supports an AM–Poincaré inequality if there are $C > 0$, $\lambda \geq 1$ so that for each $u \in BV_{AM}(X : V)$ and any AM-upper bound $(\rho_i)_i$ of u we have

$$\fint_B \|u - u_B\| d\mu \leq C\text{rad}(B) \liminf_{i\to\infty} \fint_{\lambda B} \rho_i d\mu$$

for each ball $B \subset X$. As with the 1-Poincaré inequality, the AM-Poincaré inequality implies the following version, which involves the AM-BV energy:

Lemma 3.1. *If X supports an AM–Poincaré inequality, then for $u \in BV_{AM}(X : V)$, we have that*

$$\fint_B \|u - u_B\| d\mu \leq C\, \text{rad}(B) \frac{\|D_{AM}u\|(\lambda B)}{\mu(\lambda B)}$$

for each ball $B \subset X$.

Proof. Let $\varepsilon > 0$ and let $\{\rho_k\}_k$ be an AM-upper bound for u in λB such that

$$\liminf_{k\to\infty} \int_{\lambda B} \rho_k d\mu < \|D_{AM}u\|(\lambda B) + \varepsilon.$$

Let $0<\delta<1/2$, and let η be a $1/\delta$-Lipschitz function such that $\eta=1$ on $\lambda(1-\delta)B$ and $\eta=0$ on $X\setminus\lambda B$, with $0\le\eta\le 1$ on X. We then have that $\{\eta\rho_k+\delta^{-1}|u|\,\chi_{\lambda B\setminus\lambda(1-\delta)B}\}_k$ is an AM-upper bound for ηu in X, see for example Lemma 2.13 above. Thus, by the AM–Poincaré inequality and by the doubling property of μ, we have that

$$\begin{aligned}\fint_{(1-\delta)B}\fint_{(1-\delta)B}\|u(y)-u(x)\|\,d\mu(y)\,d\mu(x)&\le 2\fint_{(1-\delta)B}\|u-u_{(1-\delta)B}\|d\mu\\ &\le C(1-\delta)\,\mathrm{rad}(B)\liminf_{k\to\infty}\fint_{\lambda(1-\delta)B}\left(\eta\rho_k+\frac{|u|}{\delta}\chi_{\lambda B\setminus\lambda(1-\delta)B}\right)d\mu\\ &\le C\,\mathrm{rad}(B)\liminf_{k\to\infty}\fint_{\lambda B}\rho_k d\mu\\ &< C\,\mathrm{rad}(B)\left(\frac{\|D_{AM}u\|(\lambda B)+\varepsilon}{\mu(\lambda B)}\right).\end{aligned}$$

Now letting $\delta\to 0$ and taking $\varepsilon\to 0$, we have that

$$\fint_B\|u-u_B\|d\mu\le C\,\mathrm{rad}(B)\frac{\|D_{AM}u\|(\lambda B)}{\mu(\lambda B)}.$$

Here u_B is the Bochner integral average of u over the ball B. □

The goal of this section is to see that if X supports a 1-Poincaré inequality, then it supports an AM–Poincaré inequality. For that we will use the fact that spaces supporting a 1-Poincaré inequality have a Semmes Pencil of curves (see [13, Theorem 3.10]).

Definition 3.2. We say that the metric measure space (X,d,μ) supports a Semmes pencil of curves if there exists $C>0$ so that for each $x,y\in X$ there exists a family $\Gamma_{x,y}$ of nonconstant compact rectifiable curves equipped with a probability measure $\sigma_{x,y}$ such that each $\gamma\in\Gamma_{x,y}$ connects x to y, $\ell(\gamma)\le Cd(x,y)$, and for each Borel set $A\subset X$ the map $\gamma\mapsto\ell(\gamma\cap A)$ is $\sigma_{x,y}$-measurable with

$$\int_{\Gamma_{x,y}}\ell(\gamma\cap A)d\sigma_{x,y}(\gamma)\le C\int_{A\cap CB_{x,y}}R_{x,y}(z)d\mu(z),$$

where $CB_{x,y}:=B(x,Cd(x,y))\cup B(y,Cd(x,y))$ and

$$R_{x,y}(z):=\frac{d(x,z)}{\mu(B(x,d(x,z)))}+\frac{d(y,z)}{\mu(B(y,d(y,z)))}.$$

Theorem 3.3. *Suppose that μ is doubling, X has a Semmes pencil of curves, and that for μ-a.e. $x\in X$ we have $\liminf_{r\to 0^+}\mu(B(x,r))/r=0$. Then X supports an AM–Poincaré inequality for every Banach space V.*

The proof follows along the lines of [13, Proposition 3.9], but as we are now dealing with a vector-valued map, we provide the complete proof here, especially since there seems to be a gap in the details of the proof in [13] which we fixed here. In doing so, we saw that we needed the additional condition that $\liminf_{r\to 0^+}\frac{\mu(B(x,r))}{r}=0$ for almost every $x\in X$. This condition fails for example when $X=\mathbb{R}$, but in spaces that are not one-dimensional in nature this is automatically satisfied via the upper mass bound estimates for the doubling measure μ on the connected space X, when the upper mass bound exponent can be taken to be larger than 1. When $X=\mathbb{R}^2$ is equipped with the measure $d\mu(x)=|x|^{-1}d\mathcal{L}^2$, the point $x=0$ fails the condition $\liminf_{r\to 0^+}\frac{\mu(B(x,r))}{r}=0$, but this condition holds at all other $x\in\mathbb{R}^2$.

Proof. Fix a Banach space V and let $u \in BV_{AM}(X:V)$. Let B be a ball in X and $(\rho_k)_{k=1}^{\infty}$ be a strong bounding sequence for u in $4CB$ with C the constant associated with the Semmes pencil of curves, that is, condition (2.9) holds for all curves in $4CB$. By the Lebesgue differentiation theorem for vector-valued functions (see for instance [23, page 77]), we know that $\mu(N)=0$, where N is the set of points $x \in X$ for which either $\liminf_{r\to 0^+} \mu(B(x,r))/r > 0$ or

$$\limsup_{r\to 0^+} \fint_{B(x,r)} \|u(x)-u(z)\| d\mu(z) > 0.$$

Let $x, y \in X \setminus N$ be two distinct points, and for each $\varepsilon > 0$ consider the sets

$$E_\varepsilon(x) := \{z \in X : \|u(x)-u(z)\| > \varepsilon\}, \quad E_\varepsilon(y) := \{z \in X : \|u(y)-u(z)\| > \varepsilon\}.$$

Now, since x and y are Lebesgue points of u, we know that

$$\limsup_{r\to 0} \frac{\mu(B(x,r)\cap E_\varepsilon(x))}{\mu(B(x,r))} = 0 \quad \text{and} \quad \limsup_{r\to 0} \frac{\mu(B(y,r)\cap E_\varepsilon(y))}{\mu(B(y,r))} = 0.$$

Let $\{r_i\}_i$ be a decreasing sequence of radii so that $r_1 \le \frac{1}{4}d(x,y)$, $r_{i+1} \le \frac{1}{4}r_i$, $\frac{\mu(B(x,r_i))}{r_i} \le 2^{-i}$, $\frac{\mu(B(y,r_i))}{r_i} \le 2^{-i}$, and in addition

$$\frac{\mu(B(x,r_i)\cap E_\varepsilon(x))}{\mu(B(x,r_i))} < 2^{-i} \quad \text{and} \quad \frac{\mu(B(y,r_i)\cap E_\varepsilon(y))}{\mu(B(y,r_i))} < 2^{-i}.$$

For each i let $\mathrm{R}_i(x) := B(x,r_i)\setminus B(x,r_i/2)$ and denote by $\Gamma_i(x)$ the collection of all curves $\gamma \in \Gamma_{x,y}$ such that

$$\mathcal{H}^1\big(\gamma^{-1}(\mathrm{R}_i(x)\setminus E_\varepsilon(x))\big) = 0$$

and define $\Gamma_i(y)$ analogously, replacing x by y. Now use the fact that μ is doubling and $\Gamma_{x,y}$ is a Semmes family of curves to obtain

$$\begin{aligned}
\frac{r_i}{2}\sigma_{x,y}(\Gamma_i(x)) &\le \int_{\Gamma_{x,y}} \ell(\gamma \cap \mathrm{R}_i(x)\cap E_\varepsilon(x))\sigma_{x,y}(\gamma) \\
&\le \int_{CB_{x,y}\cap E_\varepsilon(x)\cap \mathrm{R}_i(x)} R_{x,y}(z)d\mu(z) \\
&\le \int_{CB_{x,y}\cap E_\varepsilon(x)\cap \mathrm{R}_i(x)} \left(\frac{d(x,z)}{\mu(B(x,d(x,z)))} + \frac{d(y,z)}{\mu(B(y,d(y,z)))}\right) d\mu(z) \\
&\le \int_{CB_{x,y}\cap E_\varepsilon(x)\cap \mathrm{R}_i(x)} \left(\frac{r_i}{\mu(B(x,r_i/2))} + \frac{2C_d d(x,y)}{\mu(B(x,d(x,y)/2))}\right) d\mu(z) \\
&\le \frac{r_i}{\mu(B(x,r_i/2))}\mu(E_\varepsilon(x)\cap B(x,r_i)) + \frac{2C_d d(x,y)}{\mu(B(x,d(x,y)/2))}\mu(B(x,r_i)) \\
&\le r_i C_d 2^{-i} + \frac{2C_d d(x,y)}{\mu(B(x,d(x,y)/2))}\mu(B(x,r_i)).
\end{aligned}$$

We note that the above estimate also fills in the gap found in the proof for the real-valued case in [13, page 243]. Set $C_{x,y} := C_d + \frac{2C_d d(x,y)}{\mu(B(x,d(x,y)/2))}$. Then from the above argument we see that

$$\sigma_{x,y}(\Gamma_i(x)) \le 2C_d\, 2^{-i} + \frac{4C_d d(x,y)}{\mu(B(x,d(x,y)/2))}\frac{\mu(B(x,r_i))}{r_i} \le 2C_{x,y}\, 2^{-i}.$$

Then for each positive integer n we have

$$\sigma_{x,y}\left(\bigcup_{i=n}^{\infty}\Gamma_i(x)\right)\le C_{x,y}\,2^{1-n}.$$

Define

$$\Gamma(x):=\bigcap_{n\in\mathbb{N}}\bigcup_{i=n}^{\infty}\Gamma_i(x).$$

It follows that $\sigma_{x,y}(\Gamma(x))=0$. When $\gamma\in\Gamma_{x,y}\backslash\Gamma(x)$, there exists a positive integer n_0 such that $\gamma\notin\Gamma_i(x)$ for every $i\ge n_0$. It suffices to have $\gamma\notin\Gamma_i(x)$ for some $i\ge n_0$ to get that there exists $\hat{x}\in\gamma\backslash(E_\varepsilon(x)\cup N_\gamma)$ for any $\mathcal{H}^1$-null set N_γ in γ. Now consider the same argument replacing x by y in order to construct $\Gamma(y)$. Note that $\sigma_{x,y}(\Gamma(x)\cup\Gamma(y))=0$.

Recall that condition (2.9) holds for every nonconstant, compact, rectifiable curve because $(\rho_i)_i$ is a strong AM-bounding sequence. Therefore, for every curve $\gamma\in\Gamma_{x,y}\backslash(\Gamma(x)\cup\Gamma(y))$ there exists an $\mathcal{H}^1$-null set N_γ such that

$$\|u(\gamma(\tau))-u(\gamma(t))\|\le\liminf_{k\to\infty}\int_\gamma\rho_k\,ds$$

whenever $\tau,t\in\mathrm{dom}(\gamma)\backslash\gamma^{-1}(N_\gamma)$. Since $\gamma\notin\Gamma(x)\cup\Gamma(y)$, there exist $\hat{x}\in\gamma\backslash(E_\varepsilon(x)\cup N_\gamma)$ and $\hat{y}\in\gamma\backslash(E_\varepsilon(y)\cup N_\gamma)$ such that

$$\|u(x)-u(y)\|\le\|u(\hat{x})-u(\hat{y})\|+2\varepsilon\le\liminf_{k\to\infty}\int_\gamma\rho_k ds+2\varepsilon. \tag{3.4}$$

(Notice that we can actually get not only such $\hat{x}$ and $\hat{y}$ but two sequences of points $x_i\notin E_\varepsilon(x)\cup N_\gamma$ and $y_i\notin E_\varepsilon(y)\cup N_\gamma$ converging to x and y respectively, but we do not need that here). By the Semmes family inequality, we have

$$\int_{\Gamma_{x,y}}\int_\gamma\rho_k\,ds\,d\sigma_{x,y}(\gamma)\le C\int_{CB_{x,y}}\rho_k(z)R_{x,y}(z)d\mu(z)$$

for each $x,y\in X\setminus N$, $k\in\mathbb{N}$. Therefore, by $\sigma_{x,y}(\Gamma(x)\cup\Gamma(y))=0$ and by (3.4), we see that

$$\|u(x)-u(y)\|=\int_{\Gamma_{x,y}}\|u(x)-u(y)\|d\sigma_{x,y}(\gamma)\le C\,\liminf_{k\to\infty}\int_{CB_{x,y}}\rho_k(z)R_{x,y}(z)d\mu(z)+2\varepsilon.$$

Recall that $\mu(N)=0$. Now, for each ball $B\subset X$,

$$\begin{aligned}
\fint_B\|u-u_B\|d\mu &\le\fint_B\fint_B\|u(x)-u(y)\|d\mu(y)d\mu(x)\\
&\le C\fint_B\fint_B\liminf_{k\to\infty}\int_{CB_{x,y}}\rho_k(z)R_{x,y}(z)d\mu(z)d\mu(y)d\mu(x)+2\varepsilon\\
&\le\frac{C}{\mu(B)^2}\liminf_{k\to\infty}\int_B\int_B\int_{4CB}\rho_k(z)R_{x,y}(z)d\mu(z)d\mu(y)d\mu(x)+2\varepsilon\\
&=\frac{C}{\mu(B)^2}\liminf_{k\to\infty}\int_{4CB}\rho_k(z)\int_B\int_B R_{x,y}(z)d\mu(y)d\mu(x)d\mu(z)+2\varepsilon,
\end{aligned}\tag{3.5}$$

where we have used Tonelli's theorem in the last equality.

Now, to obtain an estimate for the inner two integrals above, we fix $z \in 4CB$. Let $R := \mathrm{rad}(B)$. By the doubling property of μ, we have with $B_i = B(z, 5C\,2^{-i}R)$ for $i = 0, 1, \cdots$,

$$\begin{aligned}\int_B \int_B \frac{d(x,z)}{\mu(B(x,d(x,z)))}\,d\mu(y)\,d\mu(x) &= \mu(B) \int_B \frac{d(x,z)}{\mu(B(x,d(x,z)))}\,d\mu(x) \\ &\le \mu(B) \int_{B(z,5CR)} \frac{d(x,z)}{\mu(B(x,d(x,z)))}\,d\mu(x) \\ &\lesssim \mu(B) \sum_{i=0}^{\infty} \int_{B_i \setminus B_{i+1}} \frac{2^{-i}R}{\mu(B(z,5C\,2^{-i}R))}\,d\mu(x) \\ &\lesssim \mu(B) \sum_{i=0}^{\infty} 2^{-i}R \\ &\lesssim \mu(B)\,R,\end{aligned}$$

where we have implicitly used the fact that $\mu(\{w\}) = 0$ for each $w \in X$. The comparison constants above depend solely on the doubling constant of μ and the constant C. A similar estimate also gives

$$\int_B \int_B \frac{d(y,z)}{\mu(B(y,d(y,z)))}\,d\mu(y)\,d\mu(x) \lesssim \mu(B)\,R.$$

Now from (3.5) we see that

$$\fint_B \|u - u_B\|\,d\mu \lesssim \frac{C\ \mathrm{rad}(B)}{\mu(B)}\ \liminf_{k\to\infty} \int_{4CB} \rho_k(z)\,d\mu(z)\ +\ 2\varepsilon.$$

Taking $\varepsilon \to 0$, we have that

$$\fint_B \|u - u_B\| d\mu \le C\,\mathrm{rad}(B) \liminf_{k\to\infty} \fint_{4CB} \rho_k d\mu.$$

Now, let $\{\rho_k\}_{k=1}^{\infty}$ be an AM-upper bound for u. That is, (2.9) holds for AM-a.e. curve. Then by Lemma 2.11, we know that there exists a sequence of non-negative Borel functions $\{g_k\}_{k=1}^{\infty}$ with $\limsup_{k\to\infty} \int_X g_k\,d\mu < \infty$ such that for all $\varepsilon > 0$, $\{\rho_k + \varepsilon g_k\}_{k=1}^{\infty}$ is a strong bounding sequence for u, that is, (2.9) holds for all curves. Applying the above result, we obtain

$$\begin{aligned}\fint_B \|u - u_B\| d\mu &\le C\,\mathrm{rad}(B) \liminf_{k\to\infty} \fint_{4CB} (\rho_k + \varepsilon g_k) d\mu \\ &\le C\,\mathrm{rad}(B) \left(\liminf_{k\to\infty} \fint_{4CB} \rho_k d\mu + \varepsilon \limsup_{k\to\infty} \fint_{4CB} g_k d\mu \right) \\ &\le C\,\mathrm{rad}(B) \left(\liminf_{k\to\infty} \fint_{4CB} \rho_k d\mu + \frac{\varepsilon}{\mu(4CB)} \limsup_{k\to\infty} \int_X g_k d\mu \right).\end{aligned}$$

Taking $\varepsilon \to 0^+$ yields the desired result. □

Corollary 3.6. *The following are equivalent whenever (X, d, μ) is a metric measure space with μ a doubling measure satisfying $\limsup_{r\to 0^+} \mu(B(x,r))/r = 0$ for each $x \in X$.*

- *(i) X supports an AM–Poincaré inequality for every Banach space V target.*
- *(ii) X supports an AM–Poincaré inequality for some Banach space V target.*
- *(iii) X supports an AM–Poincaré inequality for real-valued functions.*
- *(iv) X supports a Semmes pencil of curves.*
- *(v) X supports a 1-Poincaré inequality.*

Proof. $(i) \Rightarrow (ii)$ is immediate. If (ii) holds, then in particular X supports a 1-Poincaré inequality for Banach space-valued functions, and hence supports a 1-Poincaré inequality for real-valued functions, as $\mathbb{R}$ can be isometrically embedded into that Banach space, that is, $(ii) \Rightarrow (v)$. From [13, Theorem 3.10] we know that (v), (iv), and (iii) are equivalent. Finally, $(iv) \Rightarrow (i)$ follows from Theorem 3.3. □

The condition $\liminf_{r\to 0^+} \mu(B(x,r))/r = 0$ for μ-a.e. $x \in X$ precludes us from considering spaces that have components that are one-dimensional in nature, as for example in $\mathbb{R}$ and graphs. It is perhaps possible to handle this situation separately, as was shown for real-valued BV functions in [37]. We do not do so here.

4. Proof of theorem 1.1

The focus of this section is to complete the proof of the first main theorem of the paper, Theorem 1.1. By Lemma 2.14 we have seen that $BV(X : V) \subset BV_{AM}(X : V)$ with the energy seminorm control $\|D_{\mathrm{AM}}u\|(X) \le \|Du\|(X)$. Thus it only remains to show the reverse inequality. To this end, let $u \in BV_{AM}(X : V)$. We will make use of the version of Poincaré inequality identified in Lemma 3.1 above.

Since the measure μ is doubling, for each $\varepsilon > 0$ there is a countable covering $\{B_i\}_i$ of X by balls of radius ε such that for each $T \ge 1$ there is a constant $C_T > 0$, depending solely on T and the doubling constant associated with μ, such that $\sum_i \chi_{TB_i} \le C_T$ on X. Moreover, for each i there is a non-negative C/ε-Lipschitz function φ_i, with support in $2B_i$, so that $\sum_i \varphi_i = 1$ on X; see for example the discussion at the beginning of [23, Section 9.2]. Such a collection of functions $\{\varphi_i\}_i$ is called a Lipschitz partition of unity in X. Using this Lipschitz partition of unity, we now construct a locally Lipschitz continuous approximation of u as follows:

$$u_\varepsilon := \sum_i u_{B_i}\, \varphi_i, \qquad \text{where} \qquad u_{B_i} := \fint_{B_i} u \, d\mu.$$

Let $x \in X$ and fix an index j such that $x \in B_j$. Then it follows that whenever $\varphi_i(x) \neq 0$, necessarily $x \in 2B_i$ and so $2B_i \cap B_j$ is non-empty; in this case, $2B_i \subset 5B_j$. Hence, using also the fact that $u(x) = \sum_i u(x)\varphi_i(x)$, we obtain

$$\begin{aligned} u_\varepsilon(x) - u(x) = \sum_i [u_{B_i} - u(x)]\varphi_i(x) &= \sum_{i;2B_i\cap B_j\neq\emptyset} [u_{B_i} - u(x)]\varphi_i(x) \\ &= \sum_{i;2B_i\cap B_j\neq\emptyset} \varphi_i(x) \fint_{B_i} [u - u(x)]\, d\mu. \end{aligned}$$

Thus by the doubling property of μ and the bounded overlap property of the balls $\{5B_j\}_j$, we obtain

$$\begin{aligned} \|u_\varepsilon(x) - u(x)\| &\le \sum_{i;2B_i\cap B_j\neq\emptyset} \fint_{B_i} \|u - u(x)\|\, d\mu \\ &\le \sum_{i;2B_i\cap B_j\neq\emptyset} \fint_{B_i} \|u - u_{5B_j}\|\, d\mu + \|u_{5B_j} - u(x)\| \\ &\lesssim \fint_{5B_j} \|u - u_{5B_j}\|\, d\mu + \|u_{5B_j} - u(x)\|, \end{aligned}$$

and integrating over B_j and summing up over j, and using the fact that $\{B_j\}_j$ is a cover of X, we obtain

$$\begin{aligned}\int_X \|u_\varepsilon(x) - u(x)\| \, d\mu(x) &\le \sum_j \int_{B_j} \|u_\varepsilon(x) - u(x)\| \, d\mu(x) \\ &\lesssim \sum_j \int_{B_j} \left(\fint_{5B_j} \|u - u_{5B_j}\| \, d\mu + \|u_{5B_j} - u(x)\| \right) d\mu(x) \\ &\lesssim \sum_j \left(\mu(B_j) \fint_{5B_j} \|u - u_{5B_j}\| \, d\mu + \mu(B_j) \fint_{5B_j} \|u_{5B_j} - u(x)\| \, d\mu(x) \right) \\ &\lesssim \sum_j \mu(B_j) \fint_{5B_j} \|u - u_{5B_j}\| \, d\mu \\ &\lesssim \sum_j \varepsilon \, \|D_{AM} u\|(5\lambda B_j) \\ &\lesssim \varepsilon \, \|D_{AM} u\|(X).\end{aligned}$$

In obtaining the penultimate inequality above, we used the AM-Poincaré inequality, and in obtaining the last inequality above, we relied on the bounded overlap of the collection $\{5\lambda B_j\}_j$. Thus $u_\varepsilon \to u$ in $L^1(X : V)$ as $\varepsilon \to 0^+$. As u_ε is locally Lipschitz continuous (as we will show next) on the separable metric space X, it follows that u_ε is Bochner measurable, and so the convergence holds in $L^1(X : V)$.

To show that $u_\varepsilon \in N^{1,1}(X : V)$, it suffices to show that u_ε is locally Lipschitz continuous on X with its local Lipschitz constant function $\operatorname{Lip} u_\varepsilon \in L^1(X)$. Here,

$$\operatorname{Lip} u_\varepsilon(x) := \limsup_{y \to x} \frac{\|u(y) - u(x)\|}{d(x,y)}.$$

To do so, we fix $x \in X$ and choose an index j such that $x \in B_j$. Considering $y \in B_j$ as well, we see that

$$u_\varepsilon(y) - u_\varepsilon(x) = \sum_{i; 2B_i \cap B_j \neq \emptyset} u_{B_i} \, (\varphi_i(x) - \varphi_i(y)) = \sum_{i; 2B_i \cap B_j \neq \emptyset} \left(u_{B_i} - u_{5B_j} \right) \, (\varphi_i(x) - \varphi_i(y)) \, .$$

Using the Lipschitz property of the functions φ_i, we now see by the Poincaré inequality that

$$\begin{aligned}\|u_\varepsilon(y) - u_\varepsilon(x)\| \lesssim \frac{d(x,y)}{\varepsilon} \sum_{i; 2B_i \cap B_j \neq \emptyset} \|u_{B_i} - u_{5B_j}\| &\lesssim \frac{d(x,y)}{\varepsilon} \sum_{i; 2B_i \cap B_j \neq \emptyset} \fint_{B_i} \|u - u_{5B_j}\| \, d\mu \\ &\lesssim \frac{d(x,y)}{\varepsilon} \fint_{5B_j} \|u - u_{5B_j}\| \, d\mu \\ &\lesssim d(x,y) \, \frac{\|D_{AM} u\|(5\lambda B_j)}{\mu(B_j)}.\end{aligned}$$

It follows that

$$\operatorname{Lip} u_\varepsilon(x) \lesssim \inf_{j \, : x \in B_j} \frac{\|D_{AM} u\|(5\lambda B_j)}{\mu(B_j)}.$$

Thus u_ε is locally Lipschtiz continuous on X, and it only remains to show that $\operatorname{Lip} u_\varepsilon \in L^1(X)$. Using the fact that $\{B_j\}_j$ covers X, we see that

$$\int_X \operatorname{Lip} u_\varepsilon \, d\mu \lesssim \sum_j \int_{B_j} \frac{\|D_{AM}u\|(5\lambda B_j)}{\mu(B_j)} \, d\mu = \sum_j \|D_{AM}u\|(5\lambda B_j) \lesssim \|D_{AM}u\|(X) < \infty,$$

and this completes the proof that $u_\varepsilon \in N^{1,1}(X)$. As $u_\varepsilon \to u$ in $L^1(X : V)$ and as $\sup_\varepsilon \int_X \operatorname{Lip} u_\varepsilon \, d\mu \lesssim \|D_{AM}u\|(X) < \infty$, it follows that $u \in BV(X : V)$ with $\|Du\|(X) \lesssim \|D_{AM}u\|(X)$, completing the proof of Theorem 1.1. Note that the comparison constant in the above inequality depends solely on the doubling constant of the measure μ and the constants from the Poincaré inequality.

5. Approximate continuity and jump sets; proof of Theorem 1.3

Throughout this section, in addition to the measure μ being doubling and supporting a 1-Poincaré inequality, we will also assume that X is complete. In this section we consider the regularity properties of functions in the class $BV_{AM}(X : Y)$, with (Y, d_Y) a proper metric space (that is, closed and bounded subsets of Y are compact). As seen from the examples in Subsection 2.4, when Y is not a Banach space, it is more natural to consider the class $BV_{AM}(X : Y)$ rather than $BV(X : Y)$.

For functions u in the class $L^1(X : Y)$, by isometrically embedding Y into a Banach space if necessary, we know that for μ-almost every $x \in X$, the Lebesgue point property holds at x:

$$\limsup_{r \to 0^+} \fint_{B(x,r)} d_Y(u(y), u(x)) \, d\mu(y) = 0,$$

and we refer the interested reader for more on this topic to [23, Page 77]. At such points x, as in the proof of Theorem 3.3, if we set $E_\varepsilon(x) := \{y \in X : d_Y(u(y), u(x)) > \varepsilon\}$, then we have that

$$\limsup_{r \to 0^+} \frac{\mu(B(x,r) \cap E_\varepsilon(x))}{\mu(B(x,r))} = 0.$$

For the convenience of the reader, we rephrase the definition of approximate continuity from Definition 1.2 now.

Definition 5.1. We say that a point $x \in X$ is a point of approximate continuity of u if for every $\varepsilon > 0$ we have

$$\limsup_{r \to 0^+} \frac{\mu(\{y \in B(x,r) : d_Y(u(x), u(y)) \geq \varepsilon\})}{\mu(B(x,r))} = 0.$$

The discussion from the previous paragraph tells us that μ-almost every point in X is a point of approximate continuity of $u \in L^1(X : Y)$. For functions $u \in BV_{AM}(X : Y)$ we would like a better control. We may broaden the definition of approximate continuity by saying that u is approximately continuous at x if there is some $y_0 \in Y$ such that for every $\varepsilon > 0$ we have

$$\limsup_{r \to 0^+} \frac{\mu(\{y \in B(x,r) : d_Y(y_0, u(y)) \geq \varepsilon\})}{\mu(B(x,r))} = 0.$$

Since μ-almost every point in X is a point of approximate continuity of u, if $x \in X$ such that there is some y_0 satisfying the above density condition, then we can re-define u at x by setting $u(x) := y_0$; such a modification is a better representative of u; moreover, such a modification needs to be done only on a set of μ-measure zero, thanks to the Lebesgue differentiation theorem mentioned above.

Note that if x is a point of approximate continuity in the above sense with y_0 the corresponding value, and if Y is bounded, then for each $\varepsilon > 0$ we have

$$\limsup_{r\to 0^+} \fint_{B(x,r)} d_Y(u(z), y_0)\, d\mu(z) \le \varepsilon + \mathrm{diam}(Y) \limsup_{r\to 0^+} \frac{\mu(\{y \in B(x,r) \,:\, d_Y(y_0, u(y)) \ge \varepsilon\})}{\mu(B(x,r))} = \varepsilon,$$

and so necessarily x is a Lebesgue point of u as well.

The notions of approximate continuity and jump values as considered in [1, page 294] are somewhat different than ours in that it is required there that for every continuous function $g : Y \to \mathbb{R}$, the map $g \circ u$ is approximately continuous at x with the approximate limit being $g(y_0)$. Such an indirect definition seems to be not needed here, and we take the definition of approximate continuity proposed by Ambrosio in [2, Definition 1.1].

Let us consider points $x \in X$ that are not points of approximate continuity in the above, more expanded, sense. We would like to know that for $x \in \mathcal{J}(u)$ there are only finitely many points $y_1, y_2, \cdots, y_k$, with $k \le k_0$ where k_0 is independent of u and x, that act as jump values of u at x. This may not be possible at all $x \in \mathcal{J}(u)$, but we would like to ensure that this is possible for $\mathcal{H}^{-1}$-a.e. $x \in \mathcal{J}(u)$.

Remark 5.2. A drawback in [2] is that, as pointed out in the introduction, the discussion regarding jump sets is incomplete; if we know that there is a set F, of positive density at $x \in \mathcal{J}(u)$, for which $g \circ u$ takes on an approximate limit at x along F for *each* continuous $g : Y \to \mathbb{R}$, then from [1, Proposition 1.1(v)] we know that u has an approximate continuity value *along* F at x. The *proof of the existence of* such F is not provided in [2, Proposition 1.1] nor in [2, Remark 1.5]. Indeed, in proving [2, Theorem 2.3], the family $\mathcal{F}$ considered in [2] is a countable collection of distance functions, and in the case of more general metric space targets, $\mathcal{F}$ is too small. Indeed, in order to detect the distinct possible jump values without knowing ahead of time that there are *only* two jump values, we would need to expand $\mathcal{F}$ to include 1-Lipschitz functions on the target metric space such that whenever $y_1, \cdots, y_j$ are distinct points in that target space, there is a function $\psi \in \mathcal{F}$ such that $\{\psi(y_k) \,:\, k = 1, \cdots, j\}$ is of cardinality j. It is not clear to us that such $\mathcal{F}$ always exists. For instance, with the target metric space $\mathbb{R}^2$ equipped with the ℓ_∞-metric, we are unable to guarantee such a separation when $j \ge 6$. The proof of [2, Theorem 2.3] might perhaps be completed by considering instead the directions ν_φ of the normal to the jump sets of $\varphi \circ u$, but we work directly with the maps u themselves. In order to be able to locate jump values, we need an alternate characterization of points in $\mathcal{J}(u)$, as in the claim of Theorem 1.3(a). This is the focus of the proof below.

Proof of Theorem 1.3(a). We first make the simplifying reduction that Y is a compact metric space. We refer the interested reader to the final section of the paper, the appendix, for the final step that allows us to extend the result to non-compact proper metric space Y.

We fix $u \in \mathrm{BV}_{AM}(X : Y)$. Now, if $x \in \mathcal{J}(u)$, then for every $y \in Y$ there is some $\varepsilon_y > 0$ such that

$$\limsup_{r\to 0^+} \frac{\mu(\{z \in B(x,r) \,:\, d_Y(y, u(z)) \ge \varepsilon_y\})}{\mu(B(x,r))} > 0.$$

For each $y \in Y$ and $\varepsilon > 0$ set

$$F(y,\varepsilon) := \{z \in X \,:\, d_Y(u(z), y) \ge \varepsilon\}. \tag{5.3}$$

Note that then for every $0 < \varepsilon \le \varepsilon_y$, we have that

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r) \cap F(y,\varepsilon))}{\mu(B(x,r))} > 0.$$

We fix $\varepsilon > 0$, and cover the compact set Y by finitely many balls $B(y_i, \varepsilon)$, $i = 1, \cdots, N_\varepsilon$. Note that as $B(x,r) = \bigcup_{i=1}^{N_\varepsilon} u^{-1}(B(y_i,\varepsilon)$, necessarily there is some $y_1 \in Y$, relabeled if necessary, such that

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r)\cap u^{-1}(B(y_1,\varepsilon)))}{\mu(B(x,r))} > 0.$$

If we also have that

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r)\cap u^{-1}(Y\setminus B(y_1,3\varepsilon)))}{\mu(B(x,r))} > 0, \tag{5.4}$$

then we have two sets $E_1, E_2 \subset X$ with $E_1 = u^{-1}(B(y_1,\varepsilon))$ and $E_2 = u^{-1}(B(w_1,\varepsilon))$, $d_Y(y_1,w_1) \geq 3\varepsilon$, such that

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r)\cap E_1)}{\mu(B(x,r))} > 0, \text{ and } \limsup_{r\to 0^+} \frac{\mu(B(x,r)\cap E_2)}{\mu(B(x,r))} > 0. \tag{5.5}$$

If (5.4) fails, then we know that

$$\lim_{r\to 0^+} \frac{\mu(B(x,r)\setminus u^{-1}(B(y_1,3\varepsilon)))}{\mu(B(x,r))} = 0,$$

and so

$$\lim_{r\to 0^+} \frac{\mu(B(x,r)\cap u^{-1}(\overline{B}(y_1,3\varepsilon)))}{\mu(B(x,r))} = 1.$$

In this case, we can cover the compact set $\overline{B}(y_1, 3\varepsilon)$ by balls of radii $\varepsilon/6^2$, and obtain a point $y_2 \in \overline{B}(y_1, 3\varepsilon)$ so that

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r)\cap u^{-1}(B(y_2,\varepsilon/6^2)))}{\mu(B(x,r))} > 0.$$

If we know that

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r)\setminus u^{-1}(B(y_2,3\varepsilon/6^2)))}{\mu(B(x,r))} > 0,$$

then we can set $E_1 = u^{-1}(B(y_2,\varepsilon/6^2))$ and $E_2 = u^{-1}(B(w_2,\varepsilon/6^2))$, with $6\varepsilon \geq d_Y(y_2,w_2) \geq 3\varepsilon/6^2$ such that (5.5) holds. If the above analog of (5.4) fails, then we know that

$$\lim_{r\to 0^+} \frac{\mu(B(x,r)\cap u^{-1}(\overline{B}(y_2,3\varepsilon/6^2)))}{\mu(B(x,r))} = 1,$$

and the process inductively continues. Thus we obtain a sequence of points $y_1, y_2, \cdots$ with $d_Y(y_i, y_{i+1}) \leq 3\varepsilon/6^i$ and so that

$$\lim_{r\to 0^+} \frac{\mu(B(x,r)\cap u^{-1}(\overline{B}(y_i,3\varepsilon/6^i)))}{\mu(B(x,r))} = 1.$$

If this process continues ad infinitum, then we obtain a Cauchy sequence $\{y_i\}_i$ in Y which, by the completeness of Y, must converge to a point y_∞ for which we would have that for each $\tau > 0$,

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r)\cap u^{-1}(B(y_\infty,\tau)))}{\mu(B(x,r))} = 1,$$

and so re-setting $u(x) = y_\infty$ would show that $x \notin \mathcal{J}(u)$. Therefore the inductive process above must terminate at some index k, and so we know that there is some $y_k, w_k \in Y$ such that $d_Y(y_k, w_k) \geq 3\varepsilon/6^k$, and with $E_1 = u^{-1}(B(y_k,\varepsilon/6^k))$ and $E_2 = u^{-1}(B(w_k,\varepsilon/6^k))$, condition (5.5) holds. Note that $\text{dist}(B(y_k,\varepsilon/6^k), B(w_k,\varepsilon/6^k)) \geq \varepsilon/6^k > 0$. Hence, the condition described in Theorem 1.3(a) is a characterization of a jump point of $u \in BV_{AM}(X:Y)$. $\square$

Given the above proof, we re-cast the definition of jump points next, as this version is most useful in the proof of Theorem 1.3(b), see Definition 1.2 for the original construction of $\mathcal{J}(u)$.

Definition 5.6. Let $u : X \to Y$. We say that $x_0 \in X$ is a jump point of u if there exist sets $E_1, E_2 \subset X$ such that

$$\limsup_{r\to 0} \frac{\mu(B(x_0,r)\cap E_i)}{\mu(B(x_0,r)} > 0 \quad \text{for} \quad i = 1,2, \tag{5.7}$$

and there exist balls $B_1, B_2 \subset Y$ with $\operatorname{dist}(B_1, B_2) \geq \operatorname{rad}(B_1)$ such that $u(E_i) \subset B_i$ for $i = 1, 2$.

From the discussion preceding the above definition, we know that $x \in \mathcal{J}(u)$ if and only if x is a jump point in the sense of Definition 5.6 above.

Since $Y \ni w \mapsto d_Y(w, y_0)$ is a 1-Lipschitz map for each $y_0 \in Y$, the next lemma follows by an easy verification of (2.9) with the aid of triangle inequality.

Lemma 5.8. *Let $u \in BV_{AM}(X : Y)$, and $y_0 \in Y$. Then $v : X \to \mathbb{R}$ given by $v(x) = d_Y(u(x), y_0)$ belongs to the class $BV_{AM}(X) = BV(X)$.*

As a corollary to the above lemma, the co-area formula from Lemma 2.3 yields the following, from which we obtain σ-finiteness of the jump set with respect to $\mathcal{H}^{-1}$.

Corollary 5.9. *Let $u \in BV_{AM}(X : Y)$. For each $y \in Y$ and $\rho > 0$ set $E(y,\rho) := u^{-1}(B(y,\rho))$. Then for each $y \in Y$ there is a set $D_y \subset [0,\infty)$ with $\mathcal{L}^1(D_y) = 0$ such that for each $\rho \in (0,\infty) \setminus D_y$ we have that $E(y,\rho)$ is of finite perimeter in X.*

The next result proves that the set $\mathcal{J}(u)$, as constructed above, satisfies its σ-finiteness with respect to the co-dimensional measure $\mathcal{H}^{-1}$ claimed in the statement of Theorem 1.3.

Corollary 5.10. *For each $u \in BV_{AM}(X : Y)$, the jump set $\mathcal{J}(u)$ is σ-finite with respect to the co-dimension 1 Hausdorff measure $\mathcal{H}^{-1}$ on X.*

Proof. As Y is separable, there exists a countable dense subset Y_0 of Y, and for each $y \in Y_0$, let

$$\mathcal{R}(y) := \{\rho > 0 : P(E(y,\rho), X) < \infty\}.$$

By Corollary 5.9, we have that $\mathcal{L}^1((0,\infty) \setminus \mathcal{R}(y)) = 0$, and so there exists a countable subset $\mathcal{R}_0(y) \subset \mathcal{R}(y)$ dense in $(0,\infty)$. By Lemma 2.6, it follows that $\mathcal{H}^{-1}(\partial_*(E(y,\rho)) < \infty$ for each $\rho \in \mathcal{R}(y)$, where $\partial_* E(y,\rho)$ is the measure-theoretic boundary of $E(y,\rho)$, as given by (2.4).

Now, for each $x \in \mathcal{J}(u)$, we have by Definition 5.6 and the density of Y_0 in Y, that there exists $y_1, y_2 \in Y_0$, $\rho_1 \in \mathcal{R}_0(y_1)$, and $\rho_2 \in \mathcal{R}_0(y_2)$ such that $u(E_1) \subset B_1 \subset B(y_1,\rho_1)$, $u(E_2) \subset B_2 \subset B(y_2,\rho_2)$, and $\operatorname{dist}(B(y_1,\rho_1), B(y_2,\rho_2)) > 0$. Here E_1, E_2, B_1, and B_2 are as given in Definition 5.6. Then, we have that $x \in \partial_* E(y_1,\rho_1)$, and so it follows that

$$\mathcal{J}(u) \subset \bigcup_{y\in Y_0} \bigcup_{\rho\in\mathcal{R}_0(y)} \partial_* E(y,\rho). \qquad \square$$

The above notion of jump sets agrees with the notion of jump sets for real-valued BV functions, see for example [36, 14] for real-valued BV functions in the metric setting, and [16] for the Euclidean setting (see, however, the discussion in Section 1 on alternate nomenclature used in literature on Euclidean BV functions). The discussion towards the end of this section gives a brief overview of why these notions agree. However, as pointed out in [36], a BV function can take on infinitely many values near the jump point, but such a bad behavior cannot happen on a large set. To demonstrate a similar behavior of metric space-valued BV functions, we first consider what it means for a point in the target metric space to be a jump value near a jump point of the BV function.

Definition 5.11. With $u \in BV_{AM}(X:Y)$ and $x \in \mathcal{J}(u)$, we say that a point $y_0 \in Y$ is a jump value of u at x if for every $\varepsilon > 0$ we have that

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r)\cap u^{-1}(B(y_0,\varepsilon)))}{\mu(B(x,r))} > 0.$$

The next proposition verifies the claim (b) of Theorem 1.3 whenever Y is compact. We refer the reader to Appendix 6.2 for the case where Y is unbounded and hence is noncompact.

Proposition 5.12. *There exists $k_0 \in \mathbb{N}$ so that for every $u \in BV_{AM}(X:Y)$ there is a set $N \subset X$ with $\mathcal{H}^{-1}(N) = 0$ such that for each $x \in \mathcal{J}(u) \setminus N$ there are at least two and at most k_0 jump values $y_1, \cdots, y_k \in Y$ of u at x. Furthermore, for every $\varepsilon > 0$ and $i = 1, 2, \cdots, k$, we have*

$$\liminf_{r\to 0^+} \frac{\mu(B(x,r)\cap u^{-1}(B(y_i,\varepsilon)))}{\mu(B(x,r))} \ge \gamma,$$

and

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r)\setminus \bigcup_{i=1}^k u^{-1}(B(y_i,\varepsilon)))}{\mu(B(x,r))} = 0.$$

Here k_0 and γ are constants depending only on the doubling constant and the Poincaré constants of X, and in particular are independent of Y, u, and ε.

Proof. Since Y is compact, it is separable. As above, let Y_0 be a countable dense subset of Y, and for each $y \in Y_0$ let

$$\mathcal{R}(y) = \{\rho > 0 : P(E(y,\rho), X) < \infty\}.$$

Note from Corollary 5.9 that $\mathcal{L}^1((0,\infty)\setminus \mathcal{R}(y)) = 0$. Let $\mathcal{R}_0(y)$ be a countable dense subset of $\mathcal{R}(y)$. For each $y \in Y_0$ and $\rho \in \mathcal{R}(y)$ we know that $\mathcal{H}^{-1}(\partial_* E(y,\rho)\setminus \Sigma_\gamma(E(y,\rho))) = 0$, where $\partial_* E(y,\rho)$ is the measure-theoretic boundary of $E(y,\rho)$, as given by (2.4), and $\Sigma_\gamma(E(y,\rho))$ is the reduced boundary of $E(y,\rho)$, as given by (2.5). Here $0 < \gamma \le \frac{1}{2}$ is a number that depends solely on the constants associated with the doubling property of μ and the Poincaré inequality; see for example [3, Theorem 5.3]. Let

$$N := \bigcup_{y\in Y_0} \bigcup_{\rho\in\mathcal{R}_0(y)} \partial_*(E(y,\rho))\setminus \Sigma_\gamma(E(y,\rho)).$$

Then, by the countability of the collections, we have that $\mathcal{H}^{-1}(N) = 0$. We now fix $x \in \mathcal{J}(u)\setminus N$. We proceed in an inductive fashion hinted at in the discussion preceding Definition 5.6.

Let E_1 be one of the two sets identified in Definition 5.6, associated with the jump point x, and let B_1 be the corresponding ball in Y such that $u(E_1) \subset B_1$ and

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r)\cap E_1)}{\mu(B(x,r))} > 0 \text{ and } \limsup_{r\to 0^+} \frac{\mu(B(x,r)\setminus E_1)}{\mu(B(x,r))} > 0.$$

Since the distance between the balls B_1 and B_2 in Definition 5.6 is positive, we are free to choose the center y_1 of B_1 to be in Y_0 and then, by increasing the radius slightly if necessary, have the radius of B_1 be in the set $\mathcal{R}_0(y_1)$. A similar modification can be made to the ball B_2. We can now replace E_1 with $u^{-1}(B_1)$ and E_2 with $u^{-1}(B_2)$; hence from now on, $E_1 = u^{-1}(B_1)$. Thus we have that $x \in \partial_* E_1$ because of the existence of B_2, and as $x \notin N$, we see that

$$\liminf_{r\to 0^+} \frac{\mu(B(x,r)\cap E_1)}{\mu(B(x,r))} \ge \gamma \text{ and } \liminf_{r\to 0^+} \frac{\mu(B(x,r)\setminus E_1)}{\mu(B(x,r))} \ge \gamma.$$

Let $\rho > 0$ be the radius of the ball B_1, and note that the distance between B_1 and B_2 is at least ρ. Covering the closed ball $\overline{B_1}$ by balls $B(y_{2,1}, \rho/12), \cdots, B(y_{2,N_2}, \rho/12)$, with $y_{2,i} \in Y_0$ for $i = 1, \cdots, N_2$, and $B(y_{2,i}, \rho/12)$ intersects $\overline{B_1}$. By doing so, we can find a point $y_2 \in \frac{13}{12} B_1$ such that

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r) \cap u^{-1}(B(y_2, \rho/12)))}{\mu(B(x,r))} > 0.$$

We can then find $\rho_2 \in \mathcal{R}_0(y_2)$ such that $\rho/12 \le \rho_2 < \rho/11$, so that with $E_{2,1} = u^{-1}(B(y_2, \rho_2))$, we have by the fact that $x \notin N$,

$$\liminf_{r\to 0^+} \frac{\mu(B(x,r) \cap E_{2,1})}{\mu(B(x,r))} \ge \gamma.$$

In the above, we have used the fact that B_2 does not intersect $\overline{B_1} \cup \overline{B}(y_2, \rho_2)$ to know that $x \in \partial_* E_{2,1}$. We proceed inductively as follows. Once $y_i \in Y_0$ and $\rho_i \in \mathcal{R}_0(y_i)$, $i = 1, \cdots, k$, are selected such that $d_Y(y_i, y_{i+1}) < 2\rho_i$ and $\rho_{i+1} < \rho_i/11$, and with $E_{i,1} = u^{-1}(B(y_i, \rho_i))$, we have

$$\liminf_{r\to 0^+} \frac{\mu(B(x,r) \cap E_{i,1})}{\mu(B(x,r))} \ge \gamma$$

for $i = 2, \cdots, k$. We cover $\overline{B}(y_k, \rho_k)$ by balls $B(y_{k+1,1}, \rho_k/12), \cdots, B(y_{k+1,N_{k+1}}, \rho_k/12)$, each intersecting $\overline{B}(y_k, \rho_k)$ with $y_{k+1,i} \in Y_0$ for $i = 1, \cdots, N_{k+1}$, and hence find $y_{k+1} \in Y_0$ so that $d(y_k, y_{k+1}) < 2\rho_k$, and

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r) \cap u^{-1}(B(y_{k+1}, \rho_k/12)))}{\mu(B(x,r))} > 0.$$

We then find $\rho_{k+1} \in \mathcal{R}_0(y_{k+1})$ such that $\rho_k/12 \le \rho_{k+1} < \rho_k/11$, and hence, with $E_{k+1,1} = u^{-1}(B(y_{k+1}, \rho_{k+1}))$, we have that

$$\liminf_{r\to 0^+} \frac{\mu(B(x,r) \cap E_{k+1,1})}{\mu(B(x,r))} \ge \gamma.$$

Note that as for each j we have $\rho_j < (11)^{-j}\, \rho$, and as $\operatorname{dist}(B_1, B_2) > \rho$, necessarily $B(y_k, \rho_k) \cap B_2 = \varnothing$. Moreover, as $d_Y(y_k, y_{k+1}) < (11)^{k-1}\rho$, we also have that the sequence $\{y_j\}_j$ is a Cauchy sequence in Y, and as Y is complete, converges to some $y_\infty \in Y$. We now show that y_∞ is a jump value of u at x. Let $\varepsilon > 0$; then there is some positive integer k so that $B(y_k, \rho_k) \subset B(y_\infty, \varepsilon)$. It follows that

$$\liminf_{r\to 0} \frac{\mu(B(x,r) \cap u^{-1}(B(y_\infty, \varepsilon)))}{\mu(B(x,r))} \ge \liminf_{r\to 0^+} \frac{\mu(B(x,r) \cap E_{k,1})}{\mu(B(x,r))} \ge \gamma > 0.$$

Thus u has at least one jump value at x, and moreover, we also have that for each $\varepsilon > 0$,

$$\liminf_{r\to 0} \frac{\mu(B(x,r) \cap u^{-1}(B(y_\infty, \varepsilon)))}{\mu(B(x,r))} \ge \gamma.$$

Note also, from switching the roles of the sets E_1 and E_2, we obtain a second jump value of u at x.

Now, if $z \in Y$ is any other jump value of u at x, then for each $\varepsilon > 0$ with $\varepsilon < d_Y(z, y_\infty)/20$, we can find $z_1 \in B(z, \varepsilon/2) \cap Y_0$ and $0 < \tau < \varepsilon/4$ such that $\tau \in \mathcal{R}_0(z_1)$ and note that $u^{-1}(B(z_1, \tau)) \subset u^{-1}(B(z, \varepsilon))$ with

$$\liminf_{r\to 0} \frac{\mu(B(x,r) \cap u^{-1}(B(z, \varepsilon)))}{\mu(B(x,r))} \ge \liminf_{r\to 0} \frac{\mu(B(x,r) \cap u^{-1}(B(z_1, \tau)))}{\mu(B(x,r))} \ge \gamma; \tag{5.13}$$

that is, *each* jump value of u at x satisfies the above lower density at least γ at x_0. As $\gamma > 0$, there are at most $k_0 := \lceil 1/\gamma \rceil$ number of such jump values for x.

Now suppose that we have identified all the jump values $y_1, \cdots, y_k$ of u at x, with $2 \le k \le k_0$. We claim that for each $\tau > 0$, the set $E(\tau) := \bigcup_{j=1}^k E(y_j, \tau_j)$ has density 1 at x, that is,

$$\liminf_{r\to 0^+} \frac{\mu(B(x,r)\cap E(\tau))}{\mu(B(x,r))} = 1.$$

Here $\tau_i \in \mathcal{R}_0(y_i)$ is such that $\frac{2}{3}\tau < \tau_i \le \tau$. It suffices to know this for all sufficiently small $\tau > 0$, and so we consider $\tau > 0$ for which the closed balls $\overline{B}(y_i, \tau_i)$ are pairwise disjoint. If the claim does not hold, then we would have that

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r)\setminus E(\tau))}{\mu(B(x,r))} > 0.$$

Then let $0 < \varepsilon < \tau/20$ such that for each i, j with $i \neq j$ we have that $\operatorname{dist}(\overline{B}(y_i,\tau_i), \overline{B}(y_j,\tau_j)) > 20\varepsilon$. Now setting $K(\tau) = X \setminus E(\tau)$, we have from (5.13) that

$$\liminf_{r\to 0^+} \frac{\mu(B(x,r)\setminus K(\tau))}{\mu(B(x,r))} \ge \gamma > 0 \text{ and simultaneously, } \limsup_{r\to 0^+} \frac{\mu(B(x,r)\cap K(\tau))}{\mu(B(x,r))} > 0.$$

Now, by a repeat of the covering argument employed in the first part of this proof, we cover $Y \setminus \bigcup_{j=1}^k B(y_j, \tau_j)$ by finitely many balls of radii ε, and so find a ball B_1, centered at $w_1 \in Y \setminus \bigcup_{j=1}^k B(y_j, \tau_j)$, such that

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r)\cap K(\tau)\cap E(w_1,\varepsilon))}{\mu(B(x,r))} > 0.$$

Then by modifying w_1 if necessary, we can ensure that $w_1 \in Y_0$, and then find $\rho_1 \in \mathcal{R}_0(w_1)$ so that $\varepsilon \le \rho_1 < \frac{13}{12}\varepsilon$. Note that $B(w_1, \rho_1)$ is necessarily disjoint from $u(E(\tau/2))$ by this choice. Therefore we must have $E(w_1, \rho_1) \subset K(\tau/2)$, and so

$$\liminf_{r\to 0^+} \frac{\mu(B(x,r)\cap K(\tau/2)\cap E(w_1,\rho_1))}{\mu(B(x,r))} = \liminf_{r\to 0^+} \frac{\mu(B(x,r)\cap E(w_1,\rho_1))}{\mu(B(x,r))} \ge \gamma.$$

At this point, we can repeat the preceding argument that established the existence of the jump values to conclude that there must be a jump value in Y attained by u along $K(\tau)$, violating the maximality of the collection of jump values considered above. It follows that $K(\tau)$ has density 0 at x. □

As pointed out in the early sections of this paper, the theory of real-valued functions of bounded variation on complete doubling metric measure spaces supporting a 1-Poincaré inequality is reasonably well-established. The notion of real-valued functions of bounded variation in metric measure spaces was first proposed by Miranda Jr. in [40], and its fine properties were studied in [6, 33]; an elegant account of real-valued functions of bounded variation and their fine properties in the Euclidean setting can be found in [16, Definition 5.9 and Theorem 5.17]. The fine properties of real-valued BV functions studied there includes approximate continuity and jump points. The notion of approximate continuity, as proposed in Section 5, is the same as that found in real analysis texts and in [3, 6]. We now verify that the notion of jump values, as given in Section 5, agrees with the corresponding notion as given in [5].

As considered in [3], given $u : X \to \mathbb{R}$ $x_0 \in X$ is a jump point of u if $u^\wedge(x_0) < u^\vee(x_0)$, where

$$u^\wedge(x_0) = \sup\left\{t \in \mathbb{R} : \lim_{r\to 0} \frac{\mu(B(x_0,r)\cap\{u \le t\})}{\mu(B(x_0,r))} = 0\right\},$$
$$u^\vee(x_0) = \inf\left\{t \in \mathbb{R} : \lim_{r\to 0} \frac{\mu(B(x_0,r)\cap\{u \ge t\})}{\mu(B(x_0,r))} = 0\right\}.$$

Lemma 5.14. *Let $u : X \to \mathbb{R}$. Then $x_0 \in \mathcal{J}(u)$ if and only if $u^\wedge(x_0) < u^\vee(x_0)$.*

Proof. Suppose first that $u^\wedge(x_0) = u^\vee(x_0) =: \beta$. It then follows that for each $\varepsilon > 0$,

$$\lim_{r\to 0^+} \frac{\mu(B(x_0,r)\cap\{|u-\beta| \ge \varepsilon\})}{\mu(B(x_0,r))} = 0,$$

and so by Definition 5.1 the point x_0 is a point of approximate continuity of u, that is, $x_0 \notin \mathcal{J}(u)$.

For the converse suppose that $-\infty < u^\wedge(x_0) < u^\vee(x_0) < \infty$, and choose t_1^-, t_1^+, t_2^-, and t_2^+ such that $t_1^- < u^\wedge(x_0) < t_1^+ < t_2^- < u^\vee(x_0) < t_2^+$. Then

$$\lim_{r\to 0} \frac{\mu(B(x_0,r)\cap\{u \le t_1^-\})}{\mu(B(x_0,r))} = 0 \quad\text{and}\quad \lim_{r\to 0} \frac{\mu(B(x_0,r)\cap\{u \ge t_2^+\})}{\mu(B(x_0,r))} = 0.$$

Since $u^\wedge(x_0) < t_1^+$ and $u^\vee(x_0) > t_2^-$ we also have

$$\limsup_{r\to 0} \frac{\mu(B(x_0,r)\cap\{u \le t_1^+\})}{\mu(B(x_0,r))} > 0 \quad\text{and}\quad \limsup_{r\to 0} \frac{\mu(B(x_0,r)\cap\{u \ge t_2^-\})}{\mu(B(x_0,r))} > 0,$$

and so if we set $E_i = \{t_i^- \le u \le t_i^+\}$ then

$$\lim_{r\to 0} \frac{\mu(B(x_0,r)\cap E_i)}{\mu(B(x_0,r))} > 0 \quad\text{for}\quad i = 1, 2,$$

with $u(E_i) \subset B_i := (t_i^-, t_i^+)$. Since $t_1^+ < t_2^-$, $\operatorname{dist}(B_1, B_2) > 0$. Thus x_0 is not a point of approximate continuity of u, that is, $x_0 \in \mathcal{J}(u)$. If $u^\wedge(x_0) = -\infty$ or if $u^\vee(x_0) = \infty$, then we replace u with $\chi_{K_n} \cdot u$ and proceed as above, with $K_n = \{|u| \le n\}$. $\square$

6. Appendix

6.1. **The outer measure property of $\|D_{AM}u\|$.** In [40, Theorem 3.4], Miranda Jr. proves the outer measure property of $\|Du\|$ for a function in $u \in BV(X)$ using the criterion given in DeGiorgi–Letta [11, Theorem 5.1]. To do so, he relies upon a delicate construction by which approximating sequences of locally Lipschitz functions defined on two open sets are stitched together to obtain an approximating sequence defined on the union of the open sets. By the nature of Definition 2.2, this must be done in a manner so that both the L^1-convergence and energies of the new sequence of functions are controlled. In [39, Theorem 4.1], Martio proves the outer measure property of $\|D_{AM}u\|$ for $u \in BV_{AM}(X)$ using [11, Theorem 5.1] in a similar manner. However, the stitching argument employed there is much simpler due to definition of $BV_{AM}(X)$, since one only needs to stitch together the AM-upper bounds. We include a detailed proof of this stitching lemma for the convenience of the reader.

Lemma 6.1. [39, Lemma 2.4] *Let $\Omega_1, \Omega_2 \subset X$ be open sets, and let $\{g_i^1\}_i$ and $\{g_i^2\}_i$ be AM-upper bounds for a function $u \in L^1(\Omega_1 \cup \Omega_2 : V)$ in Ω_1 and Ω_2 respectively. Then*

$$g_i(x) = \begin{cases} g_i^1(x), & x \in \Omega_1 \setminus \Omega_2 \\ \max\{g_i^1(x), g_i^2(x)\}, & x \in \Omega_1 \cap \Omega_2 \\ g_i^2(x), & x \in \Omega_2 \setminus \Omega_1 \end{cases}$$

is an AM-upper bound for u in $\Omega_1 \cup \Omega_2$.

Proof. For $j = 1, 2$, let Γ_j denote the collection of curves in Ω_j for which (2.9) fails for the AM-upper bound $\{g_i^j\}_i$. Let Γ denote the collection of curves in $\Omega_1 \cup \Omega_2$ which have a subcurve in $\Gamma_1 \cup \Gamma_2$. Then it follows that $AM(\Gamma) \le AM(\Gamma_1 \cup \Gamma_2) = 0$.

Let γ be a curve in $\Omega_1 \cup \Omega_2$ such that $\gamma \notin \Gamma$. By compactness of $\gamma([0, l(\gamma)])$, there exists $\delta > 0$ such that γ' lies in Ω_1 or Ω_2 whenever γ' is a subcurve of γ with $l(\gamma') < \delta$. Choose a partition $0 = t_0 < t_1 < \cdots < t_n = l(\gamma)$ such that $t_k - t_{k-1} < \delta/2$ for $1 \le k \le n$. Since $\gamma \notin \Gamma$, it follows that $\gamma|_{[t_k, t_{k+2}]} \notin \Gamma_1 \cup \Gamma_2$ for $0 \le k \le n-2$. Therefore, for each such k, there exists a subset $N_k \subset [t_k, t_{k+2}]$ with $\mathcal{H}^1(N_k) = 0$ and a such that for all $s, t \in [t_k, t_{k+2}] \setminus N_k$, we have

$$\|u(\gamma(s)) - u(\gamma(t))\| \le \liminf_{k\to\infty} \int_{\gamma|_{[s,t]}} g_i ds. \tag{6.2}$$

Let $N = \bigcup_k N_k$, and let $\tau, t \in [0, l(\gamma)] \setminus N$, with $\tau < t$. Then there exists $0 \le k_1 \le k_2 \le n-1$ such that $s \in [t_{k_1}, t_{k_1+1}]$ and $t \in [t_{k_2}, t_{k_2+1}]$. Let $s = s_{k_1}$, $t =: s_{k_2}$, and for each $k_1 < k < k_2$ choose $s_k \in [t_k, t_{k+1}] \setminus N$. By the triangle inequality and (6.2), it follows that

$$\begin{aligned} \|u(\gamma(\tau)) - u(\gamma(t))\| \le \sum_{k=k_1}^{k_2-1} \|u(\gamma(s_k)) - u(\gamma(s_{k+1}))\| &\le \sum_{k=k_1}^{k_2-1} \liminf_{i\to\infty} \int_{\gamma|_{[s_k, s_{k+1}]}} g_i\, ds \\ &\le \liminf_{i\to\infty} \int_{\gamma|_{[\tau,t]}} g_i\, ds. \end{aligned}$$

□

Using Lemma 6.1, Martio obtains the following using an argument analogous to the proof of [40, Theorem 3.4]:

Theorem 6.3. [39, Theorem 4.1] *If $u \in BV_{AM}(X : V)$, then $\|D_{AM}u\|(\cdot)$ (defined on open sets) defines a Borel outer measure in X.*

6.2. **Dealing with a non-compact proper Y.** We now consider the case that the metric space (Y, d_Y) is a proper metric space that is not compact. Recall that the proofs and discussions in Section 5 dealt with the case that Y is compact as then we can focus on covering Y by *finitely many* balls of radius $\varepsilon > 0$ and hence find a ball whose pre-image has positive density at a point $x \in \mathcal{J}(u)$. If we had instead a countably infinite many balls needed to cover Y, then we do not know that there must be one ball whose pre-image has positive density at x. When Y is not compact, this is because Y is not bounded; hence we cannot cover Y by finitely many balls of fixed radius $\varepsilon > 0$. In this subsection we point out how to deal with this situation.

As in the proof of Proposition 5.12, let Y_0 be a countable dense subset of Y, and for each $y \in Y_0$ let $\mathcal{R}_0(y)$ be a countable dense subset of $\mathcal{R}(y)$. If there is some $R > 0$ and $a \in Y$ such that $\mu(u^{-1}(Y \setminus B(a, R))) = 0$, then we can replace Y with $\overline{B}(a, R)$ and the proof of Proposition 5.12 identifies the jump values of u at points in $\mathcal{J}(u) \setminus N$. Hence we may assume without loss of generality that no such a, R, exists. In this case, we fix a point $a \in Y_0$ and note by the co-area formula Lemma 2.3 applied to the real-valued function $d_a \circ u$ of bounded variation given by $x \mapsto d_Y(a, u(x))$, that

$$\int_0^\infty P(u^{-1}(B(a,t)), X)\, dt = \|D\, d_a \circ u\|(X) \le \|D_{AM} u\|(X) < \infty.$$

It follows that for each positive integer n we can find $R_n > n$ such that $P(u^{-1}(B(a,R_n)), X) < 1/n$. We now enlarge the null set N, chosen in the proof of Proposition 5.12, by replacing N with

$$N \cup \bigcup_{k\in\mathbb{N}} \partial_* u^{-1}(B(a,R_k)) \setminus \Sigma_\gamma u^{-1}(B(a,R_k)).$$

We now fix $x \in \mathcal{J}(u) \setminus N$. Then, with $x \in \mathcal{J}(u) \setminus N$ as in the proof of Proposition 5.12, we have one of two cases:

(a) For each positive integer n we have that

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r) \cap u^{-1}(\overline{B}(a,R_n)))}{\mu(B(x,r))} = 0.$$

(b) There is some positive integer n_0 such that for each $n \ge n_0$ we have

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r) \cap u^{-1}(\overline{B}(a,R_n)))}{\mu(B(x,r))} > 0.$$

Should Case (a) happen, we say that u is approximately continuous at x with approximate limit ∞. Such points form a μ-measure null subset of X because, by embedding Y into a Banach space and using Bochner integrals, we know that μ-a.e. point in X is a Lebesgue point of u as $u \in L^1(X:V)$; note that the value of the function at a Lebesgue point must necessarily be a point in the Banach space and hence cannot be infinite in nature. We can include them in the set of approximately continuous points of u. Thus it suffices to take care of Case (b). In this case, we focus on covering the compact set $\overline{B}(a,R_n)$ for some fixed $n \ge n_0$ by balls $B(y_i,\varepsilon)$, $i = 1, \cdots, N_\varepsilon$, where implicitly N_ε now depends on the choice of R_n as well, but as n is fixed, this dependence is suppressed. Here we ensure that $0 < \varepsilon < R_n/10$. In so doing, we find one point, say y_1, such that

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r) \cap u^{-1}(B(y_1,\varepsilon)))}{\mu(B(x,r))} > 0.$$

Thus we can choose $E_1 = u^{-1}(B(y_1,\varepsilon))$, and as x is not a point of approximate continuity of u, we also know that

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r) \setminus u^{-1}(B(y_1,\varepsilon)))}{\mu(B(x,r))} > 0.$$

If we also have

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r) \setminus u^{-1}(B(a,R_{2n})))}{\mu(B(x,r))} > 0,$$

then necessarily $x \in \partial_* u^{-1} B(a,R_{2n})$ and so as $x \notin N$, we must have that

$$\liminf_{r\to 0^+} \frac{\mu(B(x,r) \setminus u^{-1}(B(a,R_{2n})))}{\mu(B(x,r))} \ge \gamma.$$

If for all positive integers n the above density property holds for $u^{-1}(B(a,R_{2n}))$, then we can consider ∞ to be one of the jump values of u at x. Continuing the argument found in the proof of Proposition 5.12 by covering $\overline{B}(a, \frac{11}{10}R_n)$ by balls of radius $\varepsilon/6^2$ to find y_2, and proceeding from there to find a sequence $y_i \in Y$ that converges to $y_\infty \in Y$, we see that y_∞ must also be a jump value of u at x. The rest of the argument as found in the proof of Proposition 5.12 holds, as long as we consider ∞ to be one of the jump values if necessary.

If ∞ is a jump value of u at x, then we must necessarily have that $x \in \Sigma_\gamma u^{-1}(B(a, R_k))$ for each k. As

$$1/k > P(u^{-1}(B(a, R_k)), X) \approx \mathcal{H}^{-1}(\Sigma_\gamma u^{-1}(B(a, R_k))),$$

we must have that

$$\mathcal{H}^{-1}(\bigcap_k \Sigma_\gamma u^{-1}(B(a, R_k))) = 0.$$

That is, the collection of all points $x \in \mathcal{J}(u) \setminus N$ that have ∞ as a jump value must be of $\mathcal{H}^{-1}$-measure zero as well. All other points in $\mathcal{J}(u)$ can be handled by the proof of Proposition 5.12 by using covering arguments only for the compact set $\overline{B}(a, R_j)$ for sufficiently large j.

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