SYMMETRY FOR A FULLY NONLINEAR FREE BOUNDARY PROBLEM WITH HIGHLY SINGULAR TERM

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ABSTRACT. In this paper, we establish the radial symmetry of solutions to a free boundary problem, characterized by a singular right-hand side, in both the elliptic and parabolic regimes. Specifically, we focus on the unit ball B_1, where we examine a solution to the fully nonlinear elliptic problem given by:

$$\begin{cases} F(D^2u) = f(u) & \text{in } B_1 \cap \{u > 0\}, \\ u = M & \text{on } \partial B_1, \\ 0 \le u < M & \text{in } B_1. \end{cases}$$

The right-hand side $f(u)$ behaves near the free boundary $\partial\{u > 0\}$ like u^a, with negative values for $a \in (-1, 0)$. It is important to note that due to the lack of C^2-smoothness of both u and the free boundary $\partial\{u > 0\}$, the well-known Serrin-type boundary point lemma cannot be directly applied. To overcome this challenge, we devise an approach based on an exact assumption concerning the first-order expansion and the decay on the second order, complemented by an ad-hoc comparison principle. Furthermore, we extend our analysis to encompass the parabolic case of the problem, and present a corresponding result.

CONTENTS

1. INTRODUCTION

1.1. **Background.** In this paper, we tackle the issue of rigidity concerning a fully nonlinear Dirichlet problem within the confines of the unit ball, featuring constant boundary values. Our problem presents a unique challenge due to the presence of a discontinuous right-hand side, resulting in the emergence of a free boundary.

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The right-hand side in our problem is described by $f(u) \approx u^a$ with $-1 < a < 0$, particularly when $0 < u \approx 0$.

These singularities introduce significantly reduced regularity of the solutions up to the free boundary, as commonly assumed in rigidity theory. Moreover, our problem faces an additional difficulty since the moving plane technique, which serves as our primary tool, typically relies on Lipschitz right-hand sides. However, this necessary condition is naturally absent in our case, owing to the highly singular nature of the right-hand side.

The fully nonlinear case, as dealt with here, introduces yet another technical problem, which has to do with the equation itself and what definition should be appropriate. An obvious choice of definition would be viscosity (which we also adopt in the bulk) but the problem is the regularity issue for such solutions at the free boundary.

In the context of the Laplacian case, previous studies by Alt-Phillips [2] and Teixeira [21] have examined problems with highly singular right-hand sides in a more general setting. Alt-Phillips considered minimizers, while Teixeira treated viscosity solutions. A recent work by De silva-Savin [12] explored the case where $-3 < a < -1$, defining viscosity solutions based on the asymptotic behavior of solutions near the free boundary.

In a related study, two of the authors [13] dealt with the symmetry problem for the Laplacian case. They employed a stronger form of asymptotic expansion for the solution to overcome the challenges posed by the low-regularity nature of the solutions and the free boundary. This current paper can be viewed as an extension of [13] from the Laplacian to the fully nonlinear setting, as well as from the elliptic to the parabolic framework. We build upon the original concept from [13], but it is worth noting that the technical aspects form the central and most demanding complexities.

The fully nonlinear case with singular terms has been considered earlier; see e.g., [4], [15] in the elliptic and [3] in the parabolic contexts. However, no results has been found in the literature where symmetry aspects have been considered, for singular right hand side.

It would be an interesting project to consider analysis of the free boundary problem in the nonlinear setting, in the way De silva-Savin [11] have done for the Laplacian case. Also the weaker version of asymptotic expansion of De Silva-Savin is far from sufficient for treating any symmetry problems. To circumvent this obstacle, we shall assume a second order asymptotic expansion, which is justified[1] from a regularity point of view; see Asymptotic Property (1.4), (1.5).

We will now define our problem more specifically, in both elliptic and parabolic cases. The elliptic case of our problem is given by the equation (with $\partial\{u > 0\}$ a priori unknown)

$$
\begin{cases} F(D^2u) = f(u) & \text{in } B_1 \cap \{u > 0\}, \\ u = M & \text{on } \partial B_1, \\ 0 \le u < M & \text{in } B_1, \end{cases} \tag{1.1}
$$

where $B_1 \subset \mathbb{R}^n$ ($n \ge 2$) is the unit ball, $M > 0$ is a given fixed constant, and the functions f and F are defined below.

[1] Although this is a challenging problem even in the Laplacian case, it is reasonable to expect such an expansion.

The parabolic counterpart of our problem is expressed through the equation

$$\begin{cases} F(D^2u) - \partial_t u = f(u) & \text{in } Q_1 \cap \{u > 0\}, \\ u(x,t) = M_t & \text{on } \partial_p Q_1, \\ 0 \le u(x,t) < M_t & \text{in } Q_1. \end{cases} \tag{1.2}$$

Here, $Q_1 = B_1 \times (0,1]$ is the parabolic unit cylinder and $\partial_p Q_1 = (\partial B_1 \times [0,1]) \cup (B_1 \times \{0\})$ is its parabolic boundary. M_t is a positive continuous function of $t \in [0,1]$.

In both problems, we shall have an additional assumption regarding the behavior of the solution close to the (unknown) free boundary $\partial\{u > 0\}$. This assumption is expressed in terms of an asymptotic expansion, postulating that the solution close to C^1 free boundaries behaves like a one-dimensional solution, and the second term in the expansion decays much faster concerning the distance to the free boundary.

For the Laplacian case, Alt-Phillips considered the existence and regularity of solutions and the free boundary in their work [2] through a minimization approach, which, unfortunately, does not work for non-divergence equations.

For the fully nonlinear case, one may consider a singular perturbation technique, as shown in [4]. The latter approach seems plausible for the parabolic case.

1.2. **Main Results.** To specify the function f in (1.1) and (1.2), we fix $-1 < a < 0$, $\kappa_0 > 0$ and $\varepsilon_1 > 0$ small. We assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies the following properties

$$\begin{cases} (a)\ \sup_{\varepsilon>0} \frac{f(\rho+\varepsilon)-f(\rho)}{\varepsilon} \le \kappa_0 \rho^{a-1}, \quad \rho > 0, \\ (b)\ f(\rho) = 0, \quad \rho \le 0, \\ (c)\ f(\rho) = \rho^a, \quad 0 < \rho < \varepsilon_1, \\ (d)\ |f(\rho)| \le C_0, \quad \rho \ge \varepsilon_1. \end{cases} \tag{1.3}$$

To describe the operator F, let $S = S(n)$ be the space of $n \times n$ symmetric matrices, $\Lambda \ge \lambda > 0$ be constants, and $P^-_{\lambda,\Lambda}$, $P^+_{\lambda,\Lambda}$ be the extremal Pucci operators defined by

$$P^-_{\lambda,\Lambda}(N) = \lambda \Sigma_{e_i>0} e_i + \Lambda \Sigma_{e_i<0} e_i, \quad P^+_{\lambda,\Lambda}(N) = \lambda \Sigma_{e_i<0} e_i + \Lambda \Sigma_{e_i>0} e_i,$$

where each e_i is an eigenvalue of $N \in S$. Our assumption on $F : S \to \mathbb{R}$ is the following:

$$\begin{cases} \bullet\ F \text{ is convex.} \\ \bullet\ F(0) = 0. \\ \bullet\ F \text{ is homogeneous of degree 1, i.e., } F(rN) = rF(N) \text{ for any } r > 0,\ N \in S. \\ \bullet\ \text{There are constants } \Lambda \ge \lambda > 0 \text{ such that for any } N_1, N_2 \in S, \text{ we have} \\ \quad P^-_{\lambda,\Lambda}(N_1 - N_2) \le F(N_1) - F(N_2) \le P^+_{\lambda,\Lambda}(N_1 - N_2). \\ \bullet\ 1 \le \Lambda/\lambda \le \eta_0 \text{ for some } \eta_0 > 1, \text{ defined in Appendix B.} \\ \bullet\ F \text{ is Hessian, i.e., } F(R^T N R) = F(N) \text{ for each orthogonal matrix } R \text{ and } N \in S. \end{cases}$$

Asymptotic Property

For $-1 < a < 0$ we fix the following constant throughout the paper:

$$\beta = \frac{2}{1-a}.$$

Elliptic case: We say that a function $u : B_1 \to \mathbb{R}$ satisfies *Asymptotic Property* if at any free boundary point $x^0 \in \partial\{u > 0\}$ the asymptotic expansion holds

$$|u(x) - A_{x^0}((x - x^0) \cdot \nu^{x^0})^\beta_+| \le C_0|x - x^0|^{2+\delta_\beta}, \quad x \in B_{d_{x_0}}(x^0). \tag{1.4}$$

Here, $C_0 > 0$ and $0 < \delta_\beta < \beta - 1$ are fixed constants independent of x^0, while a unit vector $\nu^{x^0} \in \partial B_1$, a radius $d_{x_0} \ge c_0 > 0$ and a constant $A_{x^0} > 0$ depend on x^0.

Parabolic case: The parabolic case needs a similar type of asymptotic property. Let $u : Q_1 \to \mathbb{R}$ be a function, with $0 < T_1 < 1$ the first time we encounter a free boundary point $\partial\{u > 0\}$. We say that u satisfies *(Parabolic) Asymptotic Property* if at every free boundary point $z^0 = (x^0, t^0) \in \partial\{u > 0\}$ with $t^0 > T_1$ the asymptotic expansion holds

$$\left|u(x,t) - A_{z^0}((x - x^0) \cdot \mu^{z^0})^\beta_+\right| \le C_0\left(|x - x^0| + \sqrt{|t - t^0|}\right)^{2+\delta_\beta}, \quad (x,t) \in \tilde{Q}_{d_{z^0}}(z^0). \tag{1.5}$$

Here, $C_0 > 0$ and $0 < \delta_\beta < \beta - 1$ are fixed constants independent of z^0, while a spatial unit vector $\mu^{z^0} \in \partial B_1$ in $\mathbb{R}^n$, a constant $A_{z^0} > 0$ and a radius

$$d_{z^0} := \min\{|1 - t^0|^{1/2}, |t^0 - T_1|^{1/2}\}$$

depend on z^0.

Remark 1. *It is noteworthy that the above asymptotic property* (1.4) *implies that the free boundary is uniformly* $C^{1,|a|}$ *in the elliptic case.*

On the other hand, in the parabolic case the asymptotic property (1.5) *implies* $C^{1/2+|a|/2}$ *in time direction and* $C^{1,|a|}$ *in space directions for each point on* $\partial\{u > 0\} \cap \{t > T_1\}$, *with uniform norm, depending on the distance of the free boundary point to the time slice* $\{t = T_1\}$.

The asymptotic expansions (1.4) and (1.5) guarantee that the homogeneous rescalings $u_r(x) := \frac{u(rx+x^0)}{r^\beta}$ converge (up to a subsequence) to $q_{x^0}(x) := A_{x^0}(x \cdot \nu^{x^0})^\beta_+$, and similarly $u_r(x,t) := \frac{u(rx+x^0, r^2t+t^0)}{r^\beta}$ converge to (a time-independent version) $q_{z^0}(x,t) := A_{z^0}(x \cdot \mu^{z^0})^\beta_+$. Since $f(u) = u^a$ near the free boundary $\partial\{u > 0\}$, it is reasonable to expect that they solve $F(D^2 q_{x^0}) = q^a_{x^0}$ and $F(D^2 q_{z^0}) - \partial_t q_{z^0} = q^a_{z^0}$, which will determine A_{x^0} and A_{z^0}, respectively. See Appendix A for their exact values and properties.

By the result of Remark 1 on the regularity of the free boundary $\partial\{u > 0\}$, it is easy to see that the vector ν^{x^0} in (1.4) should be the unit normal to the free boundary at x^0, which points toward $\{u > 0\}$. Similarly, in (1.5), if $\nu^{z^0} = (\nu^{z^0}_x, \nu^{z^0}_t)$ is the unit normal to $\partial\{u > 0\}$ pointing toward $\{u > 0\}$ then $\mu^{z^0} = \frac{\nu^{z^0}_x}{|\nu^{z^0}_x|}$.

Our results are the following theorems.

Theorem 1. *For* $-1 < a < 0$, *let* u *be a viscosity solution of* (1.1). *Suppose that* f *satisfies* (1.3) *and* $f(\rho) \ge 0$ *for* $\rho \ge M - \varepsilon_1$, *and that the asymptotic property* (1.4) *holds. Then* u *is spherically symmetric in* B_1 *and* $\partial_{|x|}u \ge 0$.

Our second main result concerning the parabolic equation is as follows.

Theorem 2. *For* $-1 < a < 0$, *let* u *be a viscosity solution of* (1.2). *Suppose that* f *satisfies* (1.3), $f(\rho) \ge 0$ *for* $\rho \ge \inf_{0 \le t \le 1} M_t - \varepsilon_1$ *and* $\inf_{\varepsilon > 0} \frac{f(\rho+\varepsilon) - f(\rho)}{\varepsilon} \ge -\kappa_0 \rho^{a-1}$ *for* $\rho > 0$. *We assume that the asymptotic property* (1.5) *holds. Then, for each* $0 < t < 1$, $u(\cdot, t)$ *is spherically symmetric around the origin, and* $\partial_{|x|}u(x,t) \ge 0$.

The assumption in the parabolic case, that the AP property holds close to every free boundary point, is somewhat limiting, and we firmly believe that the theorem should hold true without this constraint. The challenge arises from points that fail to satisfy (1.5) after time T_1. Dealing with such points becomes considerably more difficult when applying standard comparison and maximum principles, as several possibilities emerge. While we managed to handle some of these cases, others resisted all our attempts, requiring further tools that remain unknown to us at this stage.

It is worth noting that for $a \geq 0$, the symmetry results above can be easily derived from the standard moving plane technique, thanks to the monotonicity of the right-hand side $f(u) = u^a$ near $u = 0$. As a result, we excluded this particular case from our study since a similar moving plane machinery would be necessary, but the intricate technical approach employed in this paper is no longer required for this scenario.

1.3. **Notation.** For $z^0 = (x^0, t^0) \in \mathbb{R}^{n+1}$, where $x^0 \in \mathbb{R}^n$ and $t \in \mathbb{R}$, and $r > 0$, we let

$$\begin{aligned} B_r(x^0) &= \{x \in \mathbb{R}^n : |x - x^0| < r\} : \text{ Euclidean ball},\\ Q_r(z^0) &= B_r(x^0) \times (t^0, t^0 + r^2] : \text{ parabolic cylinder},\\ \tilde{Q}_r(z^0) &= B_r(x^0) \times (t^0 - r^2, t^0 + r^2) : \text{ full parabolic cylinder},\\ \partial_p Q_r(z^0) &= \overline{Q_r(z^0)} \setminus Q_r(z^0) : \text{ parabolic boundary}. \end{aligned}$$

For a general bounded open set $\Omega \subset \mathbb{R}^{n+1}$, its parabolic boundary $\partial_p\Omega$ is defined as the set of all points $z \in \partial\Omega$ such that for any $\varepsilon > 0$ the interior of the parabolic cylinder $Q_\varepsilon(z)$ contains at least one point in Ω^c.

2. Proof of Theorem 1

For $0 \leq \tau < 1$ and $x = (x_1, \cdots, x_n)$, we define

$$\begin{cases} \Pi_\tau = \{x \in \mathbb{R}^n : x_1 = \tau\} : \text{ the hyperplane},\\ x^\tau = (2\tau - x_1, x_2, x_3, \cdots x_n) : \text{ the reflection of } x \text{ with respect to } \Pi_\tau,\\ \Sigma_\tau = \{x : x^\tau \in B_1,\ x_1 < \tau\} : \text{ the reflection of the right side of } B_1,\\ u_\tau(x) = u(x^\tau) : \text{ the reflection of } u \text{ with respect to } \Pi_\tau. \end{cases}$$

By Hessian property of F it is enough to prove symmetry only in x_1-direction. It further suffices to show that $u_\tau \geq u$ in Σ_τ for any $0 < \tau < 1$. The theorem will then follow from this.

Before proving $u_\tau \geq u$ in Σ_τ, $0 < \tau < 1$, we observe that (by Hessian property) u_τ satisfies the equation

$$F(D^2 u_\tau) = f(u_\tau) \quad \text{in } \Sigma_\tau \cap \{u_\tau > 0\}, \tag{2.1}$$

and also the asymptotic property at every $y \in \partial\{u_\tau > 0\}$

$$|u_\tau(x) - A_{y^\tau}\left((x - y) \cdot \tilde{\nu}^{y^\tau}\right)_+^\beta| \leq C_0|x - y|^{2+\delta_\beta}, \quad x \in B_{d_{y^\tau}}(y). \tag{2.2}$$

Here, $\tilde{\nu}^{y^\tau}$ is the reflection of the unit normal ν^{y^τ} of u at y^τ through the plane $\{x_1 = 0\}$ (i.e., if $\nu^{y^\tau} = (\nu_1, \cdots, \nu_n)$ then $\tilde{\nu}^{y^\tau} = (-\nu_1, \nu_2, \cdots, \nu_n)$). Notice that $\tilde{\nu}^{y^\tau}$ is equal to the unit normal of u_τ at y pointing into $\{u_\tau > 0\}$. Indeed, for (2.1), one can compute, using $u_\tau(x) = u(x^\tau) = u(2\tau - x_1, x_2, \cdots, x_n)$, $D^2u_\tau(x) = R^T D^2 u(x^\tau) R$ for a diagonal matrix R with entries $-1, 1, \cdots, 1$. Then

$$F(D^2u_\tau(x)) = F(R^T D^2 u(x^\tau) R) = F(D^2 u(x^\tau)) = f(u(x^\tau)) = f(u_\tau(x)) \quad \text{in } \Sigma_\tau \cap \{u_\tau > 0\}.$$

To prove (2.2), we employ the asymptotic property of u at $y^\tau \in \partial\{u > 0\}$ to deduce that for $x \in B_{d_{y^\tau}}(y)$

$$\begin{aligned}\left|u_\tau(x) - A_{y^\tau}\left((x-y)\cdot\tilde{\nu}^{y^\tau}\right)_+^\beta\right| &= \left|u(x^\tau) - A_{y^\tau}\left((x^\tau - y^\tau)\cdot\nu^{y^\tau}\right)_+^\beta\right| \\ &\le C_0|x^\tau - y^\tau|^{2+\delta_\beta} = C_0|x-y|^{2+\delta_\beta}.\end{aligned}$$

Using $P^+_{\lambda,\Lambda}(D^2u) \ge F(D^2u) - F(0) = f(u) \ge 0$ near ∂B_1, we have by Hopf's Lemma (Proposition 2.6 in [10]) that

$$\partial_\nu u < 0 \quad \text{on } \partial B_1, \tag{2.3}$$

for any unit vector ν on ∂B_1 pointing into B_1. Thus $\partial_{x_1} u > 0$ in a small neighborhood of $e_1 = (1, 0, 0, \cdots, 0)$. This gives that for any $\tau \in (0, 1)$ close to 1, $u_\tau > u$ in Σ_τ. Hence we can start our moving plane and we define τ_0 as the smallest value such that $u_\tau \ge u$ in Σ_τ for all $\tau_0 < \tau < 1$.

Towards a contradiction, we assume that $\tau_0 > 0$, and take sequences $\varepsilon_i \searrow 0$, $\tau_i = \tau_0 - \varepsilon_i > 0$ satisfying $D_i := \{w_{\tau_i} < 0\} \cap \Sigma_{\tau_i} \neq \emptyset$, where $w_{\tau_i} := u_{\tau_i} - u$. For such D_i, we define

$$D_0 := \{x^0 \in \overline{B_1} : x^{i_k} \to x^0 \text{ for some subsequence } x^{i_k} \in D_{i_k}\} \subset \overline{\Sigma_{\tau_0}}.$$

From $D_i \neq \emptyset$, we have $D_0 \neq \emptyset$. For $w_{\tau_0} := \lim_{i\to\infty} w_{\tau_i} = u_{\tau_0} - u$, we claim

$$w_{\tau_0} = |\nabla w_{\tau_0}| = 0 \quad \text{in } D_0.$$

Indeed, the fact that $w_{\tau_i} < 0$ in D_i yields $w_{\tau_0} \le 0$ on D_0, while the definition of τ_0 implies $w_{\tau_0} \ge 0$ in $\overline{\Sigma_{\tau_0}} \supset D_0$. Thus $w_{\tau_0} = 0$ on D_0. To see that $\nabla w_{\tau_0} = 0$ on D_0, let $x^0 \in D_0$ and $x^i \in D_i$ with $x^i \to x^0$ over a subsequence. We also let $e \in \partial B_1$ be an arbitrary direction. Since D_i is open, there is a line segment $(z_1^i, z_2^i) \subset D_i$ such that $z_1^i, z_2^i \in \partial D_i$, $z_1^i = z_2^i + re$ for some $r > 0$, and x^i lies on (z_1^i, z_2^i). Since $w_{\tau_i} \ge 0$ on $\partial\Sigma_{\tau_i}$, we have $w_{\tau_i} = 0$ on ∂D_i. In particular, $w_{\tau_i}(z_1^i) = w_{\tau_i}(z_2^i) = 0$. Combining this with the C^1-regularity of w_{τ_i} in $\overline{\Sigma_{\tau_i}}$, we deduce that if $z_3^i \in (z_1^i, z_2^i)$ is a local minimum point of w_{τ_i} on (z_1^i, z_2^i) then $\partial_e w_{\tau_i}(z_3^i) = 0$. Over a subsequence $[x^i, z_3^i]$ converges either to a point $\{x^0\}$ or to a line segment $[x^0, z^0] \subset D_0$. In either case we have $\partial_e w_{\tau_0}(x^0) = 0$. Since e and x^0 are arbitrary, we conclude that $\nabla w_{\tau_0} = 0$ on D_0.

Now, we decompose D_0 into two parts

$$D_0 = (D_0 \cap \{u > 0\}) \cup (D_0 \cap \{u = 0\}),$$

and will prove that these sets are empty, contradicting that $D_0 \neq \emptyset$.

Claim A: $D_0 \cap \{u > 0\} = \emptyset$.

Claim A will follow once we show the following:
(A1) $D_0 \cap \Sigma_{\tau_0} \cap \{u > 0\} = \emptyset$,
(A2) $D_0 \cap \Pi_{\tau_0} \cap \{u > 0\} = \emptyset$.

We first prove (A1). From (2.1) and (a) in (1.3), we can see that w_{τ_0} satisfies in $\Sigma_{\tau_0} \cap \{u > 0\}$

$$P^-_{\lambda,\Lambda}(D^2 w_{\tau_0}) \le F(D^2 u_{\tau_0}) - F(D^2 u) = f(u_{\tau_0}) - f(u) \le \kappa_0 u^{a-1} w_{\tau_0}. \tag{2.4}$$

We also note that $w_{\tau_0} \ge 0$ in $\overline{\Sigma_{\tau_0}}$ and u^{a-1} is bounded in every compact subset of $\Sigma_{\tau_0} \cap \{u > 0\}$. By the strong minimum principle (Proposition 2.6 in [10]), for every connected component C of $\Sigma_{\tau_0} \cap \{u > 0\}$, either $w_{\tau_0} = 0$ in C or $w_{\tau_0} > 0$ in C. We claim that

$$w_{\tau_0} > 0 \quad \text{in } \Sigma_{\tau_0} \cap \{u > 0\}, \tag{2.5}$$

which readily implies (A1). To this end, we assume to the contrary that for some component C, $w_{\tau_0} = 0$ in C. Since C is an open set and $\partial C \cap (\partial\Sigma_{\tau_0} \setminus \Pi_{\tau_0}) = \emptyset$, we can take two points y, z with $y \in \partial C \cap \Sigma_{\tau_0}$ and $z \in C$ satisfying $u(y) = u_{\tau_0}(y) = 0$, $u(z) = u_{\tau_0}(z) > 0$ and $z = y + re_1$ for some $r > 0$. Then the reflected points y^{τ_0}, $z^{\tau_0} \in B_1 \cap \{x_1 > \tau_0\}$ satisfy $u(y^{\tau_0}) = 0$, $u(z^{\tau_0}) > 0$ and $y = z + re_1$. This is a contradiction, since u is nondecreasing in x_1-direction in $B_1 \cap \{x_1 > \tau_0\}$ by the definition of τ_0; we could also have used the fact that $u_\tau \geq u$ for $\tau = \frac{y_1^{\tau_0} + z_1^{\tau_0}}{2} > \tau_0$.

To prove (A2), suppose there is a point $x^0 \in D_0 \cap \Pi_{\tau_0} \cap \{u > 0\}$. If $x^0 \in \partial B_1$, then $\partial_{x_1} u(x^0) = -1/2\partial_{x_1} w_{\tau_0}(x^0) = 0$, which contradicts (2.3). Thus we may assume that $x^0 \in B_1$. Then we can take a small ball in $\Sigma_{\tau_0} \cap \{u > 0\}$ which touches Π_{τ_0} at x^0. From (2.5), $w_{\tau_0}(x^0) = 0$ and $w_{\tau_0} > 0$ in that ball. By using (2.4) and Hopf's lemma, we get

$$\partial_{x_1} w_{\tau_0}(x^0) < 0.$$

This is a contradiction since $\nabla w_{\tau_0} = 0$ on D_0, and the proof of *Claim A* is completed.

Claim B: $D_0 \cap \{u = 0\} = \emptyset$.

For each $D_i = \{w_{\tau_i} < 0\} \cap \Sigma_{\tau_i}$ defined in the beginning of the proof, we take a minimum point x^i of w_{τ_i} in D_i (or, equivalently, in Σ_{τ_i}), so that $\nabla w_{\tau_i}(x^i) = 0$. This is possible since $w_{\tau_i} < 0$ in D_i and $w_{\tau_i} \geq 0$ on ∂D_i. Then, over a subsequence $x^i \to x^0 \in D_0$. By the result of *Claim A*, $x^0 \in D_0 \cap \{u = 0\}$. Note that we have from $w_{\tau_0} = u_{\tau_0} - u = M - u > 0$ on $\partial\Sigma_{\tau_0} \setminus \Pi_{\tau_0}$ and $w_{\tau_0} = 0$ on D_0 that $D_0 \subset \Sigma_{\tau_0} \cup \Pi_{\tau_0}$. We then have the following three possibilities:

(i) $x^0 \in D_0 \cap \Sigma_{\tau_0} \cap \{u = 0\}$.
(ii) $x^0 \in D_0 \cap \Pi_{\tau_0} \cap \{u = 0\}$ and Π_{τ_0} is orthogonal to $\partial\{u > 0\}$ at x^0.
(iii) $x^0 \in D_0 \cap \Pi_{\tau_0} \cap \{u = 0\}$ and Π_{τ_0} is non-orthogonal[2] to $\partial\{u > 0\}$ at x^0.

Before discussing the above three cases, we prove that $x^0 \in \partial\{u > 0\} \cap \partial\{u_{\tau_0} > 0\}$.

Indeed, it follows from $u(x^i) > u_{\tau_i}(x^i) \geq 0$ and $u(x^0) = 0$ that $x^0 \in \partial\{u > 0\}$. In addition, from $u > 0$ on ∂B_1 we see $D_0 \cap \{u = 0\} \subset \Sigma_{\tau_0} \cup (\Pi_{\tau_0} \setminus \partial B_1)$. This implies $x^i \in \Sigma_{\tau_0}$ for large i. Using $u_{\tau_0} \geq u$ in Σ_{τ_0}, we have $u_{\tau_0}(x^i) \geq u(x^i) > 0$. Combining those with $u_{\tau_0}(x^0) = w_{\tau_0}(x^0) + u(x^0) = 0$ yields $x^0 \in \partial\{u_{\tau_0} > 0\}$.

We thoroughly examine both cases (i) and (ii) in parallel, as we expect to reach a contradiction in each scenario by carefully utilizing the comparison principle with barriers. It is important to note that while constructing the barrier in the Laplacian case [13] was relatively straightforward, in the fully nonlinear setting, the process becomes considerably more intricate. Nevertheless, we successfully address this complexity in Appendix B.

We claim that the asymptotic property gives

$$w_{\tau_0}(x) = O\left(|x - x^0|^{2+\delta_\beta}\right). \tag{2.6}$$

Indeed, (2.6) is trivial for the case (ii) due to its orthogonality assumption. For the case (i), we use the asymptotic expansions (1.4) of u at x^0 and (2.2) of u_{τ_0} at x^0,

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[2]The reader may ask why this case is handled in our problem, despite it being ignored in most (if not all) other similar problems. This depends on the use of the maximum principle in small domains, that does not work (at least we do not know how) in our case due to the singularity in the r.h.s.

respectively, to have

$$u(x) = A_{x^0}\left((x-x^0)\cdot \nu^{x^0}\right)_+^\beta + O\left(|x-x^0|^{2+\delta_\beta}\right),$$
$$u_{\tau_0}(x) = A_{(x^0)^{\tau_0}}\left((x-x^0)\cdot \tilde{\nu}^{(x^0)^{\tau_0}}\right)_+^\beta + O\left(|x-x^0|^{2+\delta_\beta}\right).$$

The fact that $u_{\tau_0} \geq u$ in Σ_{τ_0} and $x^0 \in \Sigma_{\tau_0}$ implies $u_{\tau_0} \geq u$ in a neighborhood of x^0, which yields $\nu^{x^0} = \tilde{\nu}^{(x^0)^{\tau_0}}$. This in turn implies, employing the equality $A_y = [\beta(\beta-1)F(\nu^y\otimes\nu^y)]^{-\beta/2}$, $y \in \partial\{u>0\}$, in Appendix A, that $A_{x^0} = A_{(x^0)^{\tau_0}}$. Thus, (2.6) holds in Case (i) as well.

Next, we observe that for small $r > 0$

$$P^-_{\lambda,\Lambda}(D^2 w_{\tau_0}) \leq f(u_{\tau_0}) - f(u) = u^a_{\tau_0} - u^a \leq 0 \quad \text{in } A_r := \Sigma_{\tau_0}\cap\{u>0\}\cap B_r(x^0).$$

To use the result in Appendix B.1, let the n-dimensional cone K^ε and the function H^ε be as in Appendix B.1, with small $\varepsilon > 0$ satisfying (B.2). Thanks to the $C^{1,|\alpha|}$-regularity of $\partial\{u>0\}$, we can take possible rotations and translations on K^ε and H^ε to obtain a cone K and a function H satisfying for small $r>0$

$$K\cap B_r(x^0) \subset A_r,$$
$$P^-_{\lambda,\Lambda}(D^2H) = 0 \quad \text{in } K\cap B_r(x^0), \quad H = 0 \quad \text{on } \partial K \cap B_r(x^0),$$
$$H(x^j) \geq |x^j - x^0|^{2+\delta_\beta/2} \quad \text{for a sequence } x^j \in K\cap B_r(x^0) \text{ with } |x^j-x^0| \searrow 0. \tag{2.7}$$

Since $K\cap\partial B_r(x^0)$ is compactly supported in $\{w_{\tau_0} > 0\}$, the minimum of w_{τ_0} on $K\cap\partial B_r(x^0)$ is strictly positive. Thus we have for some constant $c_* > 0$

$$c_* H \leq w_{\tau_0} \quad \text{on } K\cap\partial B_r(x^0).$$

This, together with the fact that $c_*H = 0 \leq w_{\tau_0}$ on $\partial K\cap B_r(x^0)$, gives

$$w_{\tau_0} - c_*H \geq 0 \quad \text{on } \partial(K\cap B_r(x^0)).$$

Moreover,

$$P^-_{\lambda,\Lambda}(D^2(w_{\tau_0} - c_*H)) \leq P^-_{\lambda,\Lambda}(D^2 w_{\tau_0}) - c_* P^-_{\lambda,\Lambda}(D^2H) \leq 0 \quad \text{in } K\cap B_r(x^0).$$

Therefore, we have by the maximum principle $w_{\tau_0} \geq c_*H$ in $K\cap B_r(x^0)$. This contradicts (2.6)–(2.7).

Now we consider case (iii). The proof in this case heavily relies on the asymptotic property (1.4) and the blow-ups of homogeneous rescalings (2.8), just as in the Laplacian case [13]. Consequently, the two proofs are expected to be nearly identical. However, during our analysis, we recognized the need for adjustments in the construction of the rescalings from [13]. As a result, we present a refined proof in this paper, utilizing more intricate techniques to address these necessary modifications.

Let $\nu^0 = (\nu^0_1, \cdots, \nu^0_n)$ be the unit normal vector of $\partial\{u>0\}$ at x^0 pointing toward $\{u>0\}$, i.e., $\nu^0 = \nu^{x_0}$. Since the non-orthogonality assumption in (iii) gives $\nu^0_1 \neq 0$ and the fact that $w_{\tau_0} \geq 0$ in Σ_{τ_0} implies $\nu^0_1 \geq 0$, we have $\nu^0_1 > 0$.

For each i we take a free boundary point $y^i \in \partial\{u>0\}$ closest to x^i. Then, for $\rho_i := |x^i - y^i|$, $\nu^i := \frac{x^i-y^i}{\rho_i}$ is the unit normal to $\partial\{u>0\}$ at y^i. Due to the C^1-smoothness of $\partial\{u>0\}$, $\nu^i \to \nu^0$, and thus $\nu^i_1 \geq \nu^0_1/2 > 0$ for large i. Similarly, for each i we take a point $\bar{y}^i \in \partial\{u_{\tau_i} > 0\}$ closest to x^i. Then, for $\bar{\rho}_i := |x^i - \bar{y}^i|$, $\bar{\nu}^i := \frac{x^i-\bar{y}^i}{\bar{\rho}_i}$ is the unit normal to $\partial\{u_{\tau_i}>0\}$ at $\bar{y}^i$. $\nu^0_1 > 0$ implies that the first components y^i_1, $\bar{y}^i_1$ of y^i, $\bar{y}^i$, respectively, satisfy $y^i_1 < \tau_i < \bar{y}^i_1$. This, combined with $x^i \in \Sigma_{\tau_i} \subset \{x_1 < \tau_i\}$, yields $\bar{\rho}_i \geq \rho_i$ and $\bar{\nu}^i_1 \leq 0$. Notice that x^i, y^i, $\bar{y}^i$ and $(\bar{y}^i)^{\tau_i}$ all converge to x^0.

We claim that $\frac{\bar\rho_i}{\rho_i} \le 2$ for large i. Otherwise, $\frac{\bar\rho_i}{\rho_i} > 2$ over a subsequence. Using $\rho_i = \operatorname{dist}(x^i, \partial\{u > 0\})$ and applying asymptotic property (1.4) of u at y^i gives for large i

$$u(x^i) \le A_{x^i}\rho_i^\beta + C_0\rho_i^{2+\delta_\beta} \le \sqrt{3/2}A_{x^0}\rho_i^\beta + C_0\rho_i^{2+\delta_\beta} \le \sqrt{2}A_{x^0}\rho_i^\beta.$$

Similarly, using $\bar\rho_i = \operatorname{dist}(x^i, \partial\{u_{\tau_i} > 0\})$ and applying asymptotic property (2.2) of u_{τ_i} at $\bar y^i$, we obtain for large i

$$u_{\tau_i}(x^i) \ge A_{(\bar y^i)^{\tau_i}}\left((x^i - \bar y^i)\cdot \tilde\nu^{(\bar y^i)^{\tau_i}}\right)_+^\beta - C_0|x^i - \bar y^i|^{2+\delta_\beta} = A_{(\bar y^i)^{\tau_i}}\bar\rho_i^\beta - C_0\bar\rho_i^{2+\delta_\beta} \ge \frac{A_{x_0}}{\sqrt2}\bar\rho_i^\beta.$$

Here, in the second step we have used the fact that $\tilde\nu^{(\bar y^i)^{\tau_i}}$ is the unit normal $\bar\nu^i$ of u_{τ_i} at $\bar y^i$. If follows that

$$w_{\tau_i}(x^i) = u_{\tau_i}(x^i) - u(x^i) \ge \frac{A_{x^0}}{\sqrt2}\bar\rho_i^\beta - \sqrt2 A_{x^0}\rho_i^\beta > 0,$$

which contradicts $x^i \in D_i = \{w_{\tau_i} < 0\} \cap \Sigma_{\tau_i}$.

Next, we consider the rescalings

$$q_i(x) := \frac{u(\rho_i x + y^i)}{\rho_i^\beta}, \qquad \bar q_i(x) := \frac{u_{\tau_i}(\rho_i x + y^i)}{\rho_i^\beta}. \tag{2.8}$$

Using the asymptotic property of u at y^i and u_{τ_i} at $\bar y^i$ again,

$$|q_i(x) - A_{y^i}(x\cdot\nu^i)_+^\beta| \le C_0\rho_i^{2+\delta_\beta-\beta}|x|^{2+\delta_\beta},$$
$$\left|\bar q_i(x) - A_{(\bar y^i)^{\tau_i}}\left(\left(x + \frac{y^i - \bar y^i}{\rho_i}\right)\cdot\bar\nu^i\right)_+^\beta\right| \le C_0\rho_i^{2+\delta_\beta-\beta}\left|x + \frac{y^i - \bar y^i}{\rho_i}\right|^{2+\delta_\beta}.$$

For large i these estimates hold in $B_{1/2}(\nu^i)$, and thus in $B_{1/4}(\nu^0)$ as well. From $\frac{\bar\rho_i}{\rho_i} \le 2$, we see that

$$\frac{|y^i - \bar y^i|}{\rho_i} \le \frac{|y^i - x^i| + |x^i - \bar y^i|}{\rho_i} = \frac{\rho_i + \bar\rho_i}{\rho_i} \le 3.$$

Thus, over a subsequence,

$$q_i \to q_0(x) := A_{x^0}(x\cdot\nu^0)_+^\beta \quad \text{uniformly in } B_{1/4}(\nu^0),$$
$$\bar q_i \to \bar q_0(x) := A_{x^0}((x + z^0)\cdot\bar\nu^0)_+^\beta \quad \text{uniformly in } B_{1/4}(\nu^0),$$

for some $z^0 \in \overline{B_3}$ and $\bar\nu^0 \in \partial B_1$. Note that from $\bar\nu_1^i \le 0$, $\bar\nu_1^0 \le 0$.

In fact, we have the above convergence in the C^1-sense as well. For this purpose, we observe that by the asymptotic property, there exist constants $c_1 > 0$ and $C_1 > 0$, independent of i, such that

$$c_1 \le \frac{u}{\rho_i^\beta} \le C_1, \quad c_1 \le \frac{u_{\tau_i}}{\bar\rho_i^\beta} \le C_1 \quad \text{in } B_{\rho_i/2}(x^i).$$

This is equivalent to

$$c_1 \le q_i \le C_1, \quad c_1 \le \left(\frac{\rho_i}{\bar\rho_i}\right)^\beta \bar q_i \le C_1 \quad \text{in } B_{1/2}(\nu^i).$$

From $1 \le \frac{\bar\rho_i}{\rho_i} \le 2$, it follows that q_i and $\bar q_i$ are uniformly bounded below and above by positive constants (independent of i) in $B_{1/3}(\nu^0)$. In addition, since $F(D^2u) = u^a$ in $\{u > 0\}$ near the free boundary, we have $F(D^2q_i) = q_i^a$ and $F(D^2\bar q_i) = \bar q_i^a$ in $B_{1/3}(\nu^0)$. Thus, for any $\alpha \in (0,1)$,

$$\|q_i\|_{C^{1,\alpha}(B_{1/4}(\nu^0))} \le C(n,\lambda,\Lambda,\alpha)\left(\|q_i\|_{L^\infty(B_{1/3}(\nu^0))} + \|q_i^a\|_{L^\infty(B_{1/3}(\nu^0))}\right) \le C,$$

and similarly,

$$\|\bar{q}_i\|_{C^{1,\alpha}(B_{1/4}(\nu^0))} \le C.$$

Therefore,

$$\begin{aligned} q_i &\to q_0 \quad \text{in } C^1_{\text{loc}}(B_{1/4}(\nu^0)),\\ \bar{q}_i &\to \bar{q}_0 \quad \text{in } C^1_{\text{loc}}(B_{1/4}(\nu^0)). \end{aligned}$$

Now, to reach a contradiction (and complete the proof), we recall that $\nabla w_{\tau_i}(x^i) = 0$, which is equivalent to $\nabla(q_i - \bar{q}_i)(\nu^i) = 0$. By the C^1-convergence we have $\nabla q_0(\nu^0) = \nabla\bar{q}_0(\nu^0)$. Comparing their first components, we get (by using $q_0(x) = A_0(x\cdot\nu^0)^\beta_+$ and $\bar{q}_0(x) = A_0((x+z^0)\cdot\bar{\nu}^0)^\beta_+$)

$$\nu^0_1(\nu^0\cdot\nu^0)^{\beta-1}_+ = \bar{\nu}^0_1((\nu^0+z^0)\cdot\bar{\nu}^0)^{\beta-1}_+.$$

Since $\nu^0_1 > 0$ and $\bar{\nu}^0_1 \le 0$, we see that the left-hand side of the equation is strictly positive, while the right-hand side is nonpositive. This is a contradiction.

3. Proof of Theorem 2

In analogy with the elliptic case, we use the following notations in this proof: for $0 \le \tau < 1$ and $z = (x,t) = (x_1,\cdots,x_n,t)$,

$$\begin{cases} \Pi_\tau = \{z\in\mathbb{R}^{n+1} : x_1 = \tau\} : \text{the hyperplane},\\ z^\tau = (2\tau - x_1, x_2, x_3, \cdots x_n, t) : \text{the reflection of } z \text{ with respect to } \Pi_\tau,\\ \Sigma_\tau = \{z : z^\tau \in Q_1,\, x_1 < \tau\} : \text{the reflection of the right side of } Q_1,\\ u_\tau(z) = u(z^\tau) : \text{the reflection of } u \text{ with respect to } \Pi_\tau. \end{cases}$$

We observe that u_τ, $0<\tau<1$, satisfies the equation

$$F(D^2u_\tau) - \partial_t u_\tau = f(u_\tau) \quad \text{in } \Sigma_\tau \cap \{u_\tau > 0\}, \tag{3.1}$$

and the asymptotic property at each $z^0 = (x^0,t^0) \in \partial\{u_\tau > 0\}$ with $t^0 > T_1$

(3.2)

$$\left|u_\tau(x,t) - A_{(z^0)^\tau}\left((x-x^0)\cdot\tilde{\mu}^{(z^0)^\tau}\right)^\beta_+\right| \le C_0\left(|x-x^0| + \sqrt{|t-t^0|}\right)^{2+\delta_\beta}, \quad (x,t)\in\tilde{Q}_{d_{(z^0)^\tau}}(z^0).$$

They follow from Hessian property of F and the asymptotic property (1.5) of u at $(z^0)^\tau \in \partial\{u>0\}$, respectively, as in their elliptic counterparts (2.1) and (2.2).

We first prove the symmetry and monotonicity results in Theorem 2 for $0 < t < T_1$, where there is no free boundary points. As in the elliptic case, by Hessian property of F, it is enough to show that $u_\tau \ge u$ in Σ_τ for every $0<\tau<1$. To prove it by contradiction, we assume that the set $D^T_\tau := \{u_\tau < u\}\cap\Sigma_\tau\cap\{0<t<T\}$ is nonempty for some $0<\tau<1$ and $0<T<T_1$.

By using the additional assumption on f in Theorem 2, (3.1) and $\inf_{B_1\times[0,T]} u > 0$, $w_\tau := u_\tau - u$ satisfies in $D^T_\tau = \{w_\tau < 0\}$

$$\begin{aligned} P^-_{\lambda,\Lambda}(D^2w_\tau) - \partial_t w_\tau &\le (F(D^2u_\tau) - \partial_t u_\tau) - (F(D^2u) - \partial_t u)\\ &= f(u_\tau) - f(u) = -(f(u) - f(u_\tau))\\ &\le \kappa_0 u_\tau^{a-1}(u - u_\tau) \le -\kappa_0\left(\inf_{B_1\times[0,T]} u\right)^{a-1} w_\tau. \end{aligned}$$

Moreover, (1.2) implies that $w_\tau \ge 0$ on $\partial_p\Sigma_\tau$, thus $w_\tau = 0$ on $\partial_p D^T_\tau$. For $m := \kappa_0\left(\inf_{B_1\times[0,T]} u\right)^{a-1}$, we define $W_\tau(x,t) := e^{-m(t-t^0)}w_\tau(x,t)$. Then

$$W_\tau = 0 \quad \text{on } \partial_p D^T_\tau, \qquad W_\tau < 0 \quad \text{in } D^T_\tau,$$

and

$$P^-_{\lambda,\Lambda}(D^2W_\tau) - \partial_t W_\tau = e^{-m(t-t^0)}\left(P^-_{\lambda,\Lambda}(D^2w_\tau) - \partial_t w_\tau\right) + me^{-m(t-t^0)}w_\tau \le 0 \quad \text{in } D^T_\tau.$$

Note that in D^T_τ, W_τ takes its minimum at an interior point $z^0 = (x^0, t^0)$, say. We then take $r > 0$ so that the cylinder $Q := B_r(x^0) \times (t^0 - r^2, t^0)$ is inside D^T_τ and $\partial_p Q$ touches $\partial_p D^T_\tau$. Then the minimum principle implies that $W_\tau \equiv \inf_{D^T_\tau} W_\tau < 0$ in Q. This is a contradiction since $W_\tau = 0$ on $\partial_p D^T_\tau$.

By continuity $u(\cdot, T_1)$ is radial symmetric and $\partial_{|x|}u(\cdot, T_1) \ge 0$ in B_1, which implies $\{u(\cdot, T_1) = 0\}$ is either a one-point set $\{\mathbf{0}\}$ or a ball B_{r_0} for some $0 < r_0 < 1$. We want to show $\{u(\cdot, T_1) = 0\} = \{\mathbf{0}\}$. For this purpose, we assume towards a contraction $\{u(\cdot, T_1) = 0\} = B_{r_0}$. From $(r_0e_1, T_1) \in \partial\{u > 0\}$, we can find a small constant $0 < \rho \ll r_0$ such that $u < \varepsilon_1$ in a cylinder $B_{3\rho}(r_0e_1) \times (T_1 - \rho^2, T_1]$, where ε_1 is as in (1.3). We consider a smaller cylinder $A := B_\rho((r_0 - 2\rho)e_1) \times (T_1 - \rho^2, T_1]$ and let $\tau := r_0 - \rho \in (0, 1)$. Then $w_\tau = u_\tau - u \ge 0$ in $A \subset B_1 \times (0, T_1]$ and

$$P^-_{\lambda,\Lambda}(D^2w_\tau) - \partial_t w_\tau \le f(u_\tau) - f(u) = u^a_\tau - u^a \le 0 \quad \text{in } A.$$

Since $w_\tau((r_0 - 2\rho)e_1, T_1) = u(r_0e_1, T_1) - u((r_0 - 2\rho)e_1, T_1) = 0 - 0 = 0$ and $((r_0 - 2\rho)e_1, T_1) \in A \setminus \partial_p A$, the minimum principle gives $w_\tau \equiv 0$ in A. However, using $\{u(\cdot, T_1) = 0\} = B_{r_0}$, we have $w_\tau((r_0-3\rho)e_1, T_1) = u((r_0+\rho)e_1, T_1)-u((r_0-3\rho)e_1, T_1) > 0$ for a point $((r_0 - 3\rho)e_1, T_1) \in \overline{A}$. This is a contradiction.

We now prove Theorem 2 for every $0 < t < 1$. As before, it suffices to show that $u_\tau \ge u$ in Σ_τ for each $0 < \tau < 1$. Since $F(D^2u) - \partial_t u = f(u) \ge 0$ near $\partial_p Q_1$, we have by parabolic Hopf Lemma (Theorem 4.2 in [8]) that

$$\partial_\nu u < 0 \quad \text{on } \partial B_1 \times (0, 1], \tag{3.3}$$

for any spatial unit vector ν on $\partial B_1 \times (0, 1]$ which points into Q_1. This implies that $\partial_{x_1} u > 0$ in a neighborhood of $\{e_1\}\times[T_1/2, 1]$. Combining this with the monotonicity of $u(\cdot, t)$ for $t < T_1$ gives that for any $\tau \in (0, 1)$ close to 1, we have $u_\tau \ge u$ in Σ_τ. Hence we can start our moving plane and we let τ_0 to be the smallest value such that $u_\tau \ge u$ in Σ_τ for all $\tau_0 < \tau < 1$.

Towards a contradiction, we assume $\tau_0 > 0$, and take sequences $\varepsilon_i \searrow 0$ and $\tau_i := \tau_0 - \varepsilon_i > 0$ such that $D_i := \{w_{\tau_i} < 0\} \cap \Sigma_{\tau_i} \ne \emptyset$, where $w_{\tau_i} := u_{\tau_i} - u$. For such D_i's, we define

$$D_0 := \{z^0 \in \overline{Q_1} : z^{i_k} \to z^0 \text{ for some subsequence } z^{i_k} \in D_{i_k}\}.$$

Note that $D_i \ne \emptyset$ implies $D_0 \ne \emptyset$. For $w_{\tau_0} := \lim_{i\to\infty} w_{\tau_i} = u_{\tau_0} - u$, we claim that

$$w_{\tau_0} = |\nabla w_{\tau_0}| = 0 \quad \text{in } D_0.$$

Indeed, from $w_{\tau_i} < 0$ in D_i we have $w_{\tau_0} \le 0$ on D_0. Moreover, by the definition of τ_0, we also have $w_{\tau_0} \ge 0$ in $\overline{\Sigma_{\tau_0}} \supset D_0$, thus $w_{\tau_0} = 0$ on D_0. To see that $\nabla w_{\tau_0} = 0$ on D_0, let $z^0 \in D_0$ and $z^i \in D_i$ with $z^i \to z^0$ over a subsequence. Fix a spatial unit vector $e \in \partial B_1$. Since each D_i is open relative to Q_1, there is a line segment $(y^i_1, y^i_2) \subset D_i$ such that y^i_1, $y^i_2 \in \partial_p D_i$, $y^i_1 = y^i_2 + se$ for some $s > 0$, and z^i lies on (y^i_1, y^i_2). Since $w_{\tau_i} \ge 0$ on $\partial_p \Sigma_{\tau_i}$, we have $w_{\tau_i} = 0$ on $\partial_p D_i$. In particular, $w_{\tau_i}(y^i_1) = w_{\tau_i}(y^i_2) = 0$. Since w_{τ_i} is at least pointwise C^1_x in $\overline{\Sigma_{\tau_i}}$, if $y^i_3 \in (y^i_1, y^i_2)$ is a local minimum point of w_{τ_i} on (y^i_1, y^i_2), then $\partial_e w_{\tau_i}(y^i_3) = 0$. Then, over a subsequence $[z^i, y^i_3]$ converges either to a point $\{z^0\}$ or to a line segment $[z^0, y^0] \subset \overline{D_0}$. In either case we have $\partial_e w_{\tau_0}(z^0) = 0$. Since e and z^0 are arbitrary, we see that $\nabla w_{\tau_0} = 0$ on D_0.

We now decompose D_0 into two parts

$$D_0 = (D_0 \cap \{u > 0\}) \cup (D_0 \cap \{u = 0\})$$

and will prove that these sets are empty, contradicting $D_0 \neq \emptyset$.

Claim A: $D_0 \cap \{u > 0\} = \emptyset$.

Note that *Claim A* follows once we show
(A1) $D_0 \cap \Sigma_{\tau_0} \cap \{u > 0\} = \emptyset$,
(A2) $D_0 \cap \Pi_{\tau_0} \cap \{u > 0\} = \emptyset$.

Since $w_{\tau_0} = 0$ in D_0, to prove (A1) it is enough to show that

$$w_{\tau_0} > 0 \quad \text{in } \Sigma_{\tau_0} \cap \{u > 0\}. \tag{3.4}$$

To this aim, we recall that $u_{\tau_0} \geq u$ in Σ_{τ_0} and observe that in $\Sigma_{\tau_0} \cap \{u > 0\}$

$$\begin{aligned} P^-_{\lambda,\Lambda}(D^2 w_{\tau_0}) - \partial_t w_{\tau_0} &\leq (F(D^2 u_{\tau_0}) - \partial_t u_{\tau_0}) - (F(D^2 u) - \partial_t u) \\ &= f(u_{\tau_0}) - f(u) \leq \kappa_0 u^{a-1} w_{\tau_0}. \end{aligned} \tag{3.5}$$

Let C be a connected component of $\Sigma_{\tau_0} \cap \{u > 0\}$. Then, using (3.5) and the fact that u^{a-1} is bounded in every compact subset of $\Sigma_{\tau_0} \cap \{u > 0\}$, we can deduce by parabolic Hopf Lemma that w_{τ_0} cannot attain its minimum in the interior of C unless it is a constant in C. In particular, we have either $w_{\tau_0} \equiv 0$ in C or $w_{\tau_0} > 0$ in C. To prove (3.4) by contradiction, we assume that $w_{\tau_0} \equiv 0$ in some component C. From the fact that $\Sigma_{\tau_0} \cap \{u > 0\}$ is relatively open in Q_1 and $w_{\tau_0} > 0$ on $(\partial_p \Sigma_{\tau_0} \setminus \Pi_{\tau_0}) \cap Q_1$, it follows that C is relatively open and $\partial_p C \cap (\partial_p \Sigma_{\tau_0} \setminus \Pi_{\tau_0}) \cap Q_1 = \emptyset$. This enables us to take two points $y \in \partial_p C \cap \Sigma_{\tau_0}$ and $z \in C$ such that $u(y) = u_{\tau_0}(y) = 0$, $u(z) = u_{\tau_0}(z) > 0$, and $z - y = (re_1, 0)$ for some $r > 0$. Then, the reflected points $y^{\tau_0}, z^{\tau_0} \in Q_1 \cap \{x_1 > \tau_0\}$ satisfy that $u(y^{\tau_0}) = 0$, $u(z^{\tau_0}) > 0$, and $y^{\tau_0} - z^{\tau_0} = (re_1, 0)$. This is a contradiction, since u is nondecreasing in x_1-direction in $Q_1 \cap \{x_1 > \tau_0\}$ by the definition of τ_0.

To prove (A2) by contradiction, we suppose that there exists a point $z^0 = (x^0, t^0) \in D_0 \cap \Pi_{\tau_0} \cap \{u > 0\}$. Note that $t^0 \geq T_1$. If $x^0 \in \partial B_1$, then $\partial_{x_1} u(z^0) = -\frac{1}{2}\partial_{x_1} w_{\tau_0}(z^0) = 0$, which contradicts (3.3). Thus we may assume $x^0 \in B_1$. Then, we can find a small $r > 0$ such that a cylinder $B_r(x^0 - re_1) \times (t^0 - r^2, t^0)$ is contained in $\Sigma_{\tau_0} \cap \{u > 0\}$. Notice that the cylinder touches Π_{τ_0} on $\{x^0\} \times (t^0 - r^2, t^0)$. By taking smaller r if necessary, we may assume that $u > c_0$ in the cylinder for some positive constant $c_0 > 0$. From (3.4) and (3.5), together with the fact that $w_{\tau_0} = 0$ on Π_{τ_0}, we have $\partial_{x_1} w_{\tau_0}(z^0) < 0$ by Hopf Lemma. This contradicts that $\nabla w_{\tau_0} = 0$ on D_0, and completes the the proof of *Claim A*.

Claim B: $D_0 \cap \Sigma_{\tau_0} \cap \{u = 0\} = \emptyset$.

For each open set $D_i = \{w_{\tau_i} < 0\} \cap \Sigma_{\tau_i}$ defined in the beginning of the proof, since $w_{\tau_i} < 0$ in D_i and $w_{\tau_i} \geq 0$ on $\partial_p D_i$, we can take a point $z^i = (x^i, t^i) \in D_i$ such that $\nabla w_{\tau_i}(z^i) = 0$. Then, over a subsequence $z^i \to z^0 = (x^0, t^0) \in D_0$. From *Claim A*, $z^0 \in D_0 \cap \{u = 0\}$. Note that from $w_{\tau_0} = u_{\tau_0} - u = M_t - u > 0$ on $(\partial_p \Sigma_{\tau_0} \setminus \Pi_{\tau_0}) \cap Q_1$, we have $D_0 \subset \Sigma_{\tau_0} \cup \Pi_{\tau_0}$. Let $\nu^0 = (\nu^0_x, \nu^0_t) = (\nu^0_1, \cdots, \nu^0_n, \nu^0_t)$ be the normal vector to $\partial\{u > 0\}$ at z^0 pointing toward $\{u > 0\}$. We then have the following three possibilities:

(i) $z^0 \in D_0 \cap \Sigma_{\tau_0} \cap \{u = 0\}$.
(ii) $z^0 \in D_0 \cap \Pi_{\tau_0} \cap \{u = 0\}$ and Π_{τ_0} is orthogonal to $\partial\{u > 0\}$ at z^0 (i.e., $\nu^0_1 = 0$, $\nu^0_x \neq 0$).
(iii) $z^0 \in D_0 \cap \Pi_{\tau_0} \cap \{u = 0\}$ and Π_{τ_0} is non-orthogonal to $\partial\{u > 0\}$ at z^0 (i.e., $\nu^0_1 \neq 0$).

Before discussing the above three cases, we prove that $z^0 \in \partial\{u > 0\} \cap \partial\{u_{\tau_0} > 0\}$ and $t^0 > T_1$.

Indeed, we easily have $z^0 \in \partial\{u > 0\}$ due to $u(z^i) > u_{\tau_i}(z^i) > 0$ and $u(z^0) = 0$. Moreover, it follows from $u > 0$ on $\partial_p Q_1$ that $D_0 \cap \{u = 0\} \subset \Sigma_{\tau_0} \cup (\Pi_{\tau_0} \setminus \partial_p Q_1)$, which implies $z^i \in \Sigma_{\tau_0}$ for large i. Using $u_{\tau_0} \geq u$ in Σ_{τ_0} we also have $u_{\tau_0}(z^i) \geq u(z^i) > 0$. Those combined with $u_{\tau_0}(z^0) = w_{\tau_0}(z^0) + u(z^0) = 0$ yield $z^0 \in \partial\{u_{\tau_0} > 0\}$. It remains to prove $t^0 > T_1$. Otherwise, we have $t^0 = T_1$, since z^0 is a free boundary point. Recalling $\{u(\cdot, T_1) = 0\} = \{\mathbf{0}\}$, we see that $z^0 = (\mathbf{0}, T_1)$. From $\tau_0 > 0$, we obtain $w_{\tau_0}(z^0) = u\left((z^0)^{\tau_0}\right) - u(z^0) > 0$, which contradicts $w_{\tau_0} = 0$ on D_0, as desired.

We deal with the cases (i) and (ii) at the same time, as they both follow from the application of the result in Appendix B. As in the elliptic problem, the (parabolic) asymptotic properties (1.5) and (3.2) give

$$w_{\tau_0}(x,t) = O\left(\left(|x - x^0| + \sqrt{|t - t^0|}\right)^{2+\delta_\beta}\right). \tag{3.6}$$

For small $r > 0$,

$$P^-_{\lambda,\Lambda}(D^2 w_{\tau_0}) - \partial_t w_{\tau_0} \leq f(u_{\tau_0}) - f(u) = u^a_{\tau_0} - u^a \leq 0 \quad \text{in } A_r := \Sigma_{\tau_0} \cap \{u > 0\} \cap \tilde{Q}_r(z^0).$$

Let $(n+1)$-dimensional domain Ω^ε and the function H^ε be as in Appendix B.2, with small $\varepsilon > 0$ satisfying (B.5). Due to the $C^{1,|a|}_x \cap C^{1/2+|a|/2}_t$-regularity of $\partial\{u > 0\}$, we can take possible rotations and translations on Ω^ε and H^ε to get a domain Ω and a function H as well as a sequence $z^j \in \Omega \cap \tilde{Q}_r(z^0)$ with $z^j \to z^0$ such that for small $r > 0$

$$\begin{cases} \Omega \cap \tilde{Q}_r(z^0) \subset A_r, \\ P^-_{\lambda,\Lambda}(D^2 H) - \partial_t H = 0 \quad \text{in } \Omega \cap \tilde{Q}_r(z^0), \quad H = 0 \quad \text{on } \partial_p\Omega \cap \tilde{Q}_r(z^0), \\ H(z^j) \geq |x^j_1 - x^0_1|^{2+\sigma}, \quad w_{\tau_0}(x^j) \leq C|x^j_1 - x^0_1|^{2+\delta_\beta}. \end{cases} \tag{3.7}$$

Here, the last inequality containing w_{τ_0} follows by using (3.6). Since $H \leq \|h\|_{L^\infty(B_1)}$ on $\partial_p\Omega$ and $w_{\tau_0} > 0$ on $\overline{\partial_p\Omega \setminus \{H = 0\}}$, we have for some constant $C^* > 0$

$$H \leq C^* w_{\tau_0} \quad \text{on } \partial_p\Omega,$$

and obtain, by applying the comparison principle,

$$H \leq C^* w_{\tau_0} \quad \text{in } \Omega.$$

By taking $\sigma \in (0, \delta_\beta)$, this contradicts (3.7).

We now consider the case (iii). The assumption $\nu^0_1 \neq 0$ in (iii) and the fact that $u_{\tau_0} \geq 0$ in Σ_{τ_0} imply $\nu^0_1 > 0$.

Since the normal vector ν^0 at z^0 is not parallel to the time axis $(0, \cdots, 0, 1)$, $\partial\{u > 0\} \cap \{t = t^0\}$ and $\partial\{u > 0\} \cap \{t = t^i\}$ are $(n-1)$-dimensional surfaces on $B_1 \times \{t^0\}$ and $B_1 \times \{t^i\}$, respectively, for large i. For such i we take a point $y^i \in \mathbb{R}^n$ so that $z^i_f := (y^i, t^i) \in \partial\{u > 0\} \cap \{t = t^i\}$ is closest to $z^i = (x^i, t^i)$. In analogy to notations ν^0 and μ^0, we let $\nu^i = (\nu^i_x, \nu^i_t) = (\nu^i_1, \cdots, \nu^i_n, \nu^i_t)$ be the unit normal to $\partial\{u > 0\}$ at z^i_f pointing toward $\{u > 0\}$, and denote $\mu^i := \frac{\nu^i_x}{|\nu^i_x|}$. Then, $\mu^i = \frac{x^i - y^i}{\rho_i}$ for $\rho_i := |x^i - y^i|$. Due to the C^1-smoothness of $\partial\{u > 0\}$ near z^0, $\nu^i \to \nu^0$ and $\mu^i \to \mu^0$. In particular, $\nu^i_1 \geq \nu^0_1/2 > 0$ for large i. Next, we take a point $\bar{y}^i \in \mathbb{R}^n$ so that $\bar{z}^i_f := (\bar{y}^i, t^i) \in \partial\{u_{\tau_i} > 0\} \cap \{t = t^i\}$ is closest to $z^i = (x^i, t^i)$. As before, let $\bar{\nu}^i = (\bar{\nu}^i_x, \bar{\nu}^i_t) = (\bar{\nu}^i_1, \cdots, \bar{\nu}^i_n, \bar{\nu}^i_t)$ be the unit normal to $\partial\{u_{\tau_i} > 0\}$ at $\bar{z}^i_f$ pointing toward $\{u_{\tau_i} > 0\}$, and denote $\bar{\mu}^i := \frac{\bar{\nu}^i_x}{|\bar{\nu}^i_x|}$, so that $\bar{\mu}^i = \frac{x^i - \bar{y}^i}{\bar{\rho}_i}$ for $\bar{\rho}_i := |x^i - \bar{y}^i|$. Since $\bar{z}^i_f \to z^0 \in \partial\{u_{\tau_0} > 0\}$, we have $\bar{\nu}^i \to \bar{\nu}^0$, where $\bar{\nu}^0 := (\bar{\nu}^0_x, \nu^0_t) = (-\nu^0_1, \nu^0_2, \cdots, \nu^0_n, \nu^0_t)$ and $\bar{\mu}^i \to \bar{\mu}^0 := \frac{\bar{\nu}^0_x}{|\bar{\nu}^0_x|}$. That is, $\bar{\mu}^0$ is the reflection of

μ^0 with respect to $\{x_1 = 0\}$. Notice that $\rho_i \le \bar\rho_i$, $\bar\nu^i \le 0$, and that z^i, z^i_f, $\bar z^i_f$ $(\bar z^i_f)^{\tau_i}$ all converge to z^0.

We claim that $\frac{\bar\rho_i}{\rho_i} \le 2$ for large i. Otherwise, we have that over a subsequence $\frac{\bar\rho_i}{\rho_i} > 2$. Applying the asymptotic property of u at z^i_f and u_{τ_i} at $\bar z^i_f$, respectively, we obtain for large i

$$u(z^i) \le A_{z^i_f}((x^i - y^i)\cdot\mu^i)^\beta_+ + C_0|x^i - y^i|^{2+\delta_\beta} = A_{z^i_f}\rho_i^\beta + C_0\rho_i^{2+\delta_\beta} \le \sqrt{2}A_{z^0}\rho_i^\beta,$$
$$u_{\tau_i}(z^i) \ge A_{(\bar z^i_f)^{\tau_i}}((x^i - \bar y^i)\cdot\bar\mu^i)^\beta_+ - C_0|x^i - \bar y^i|^{2+\delta_\beta} = A_{\bar z^i_f}\bar\rho_i^\beta - C_0\bar\rho_i^{2+\delta_\beta} \ge \frac{A_{z^0}}{\sqrt{2}}\bar\rho_i^\beta.$$

Then, we obtain

$$w_{\tau_i}(z^i) = u_{\tau_i}(z^i) - u(z^i) \ge \sqrt{2}A_{z^0}\rho_i^\beta\left(\frac{1}{2}\left(\frac{\bar\rho_i}{\rho_i}\right)^\beta - 1\right) > 0,$$

which contradicts that $z^i \in D_i = \{w_{\tau_i} < 0\} \cap \Sigma_{\tau_i}$.

Next, we consider the homogeneous rescalings

$$q_i(x,t) := \frac{u(\rho_i x + y^i, \rho_i^2 t + t^i)}{\rho_i^\beta}, \qquad \bar q_i(x,t) := \frac{u_{\tau_i}(\rho_i x + y^i, \rho_i^2 t + t^i)}{\rho_i^\beta}.$$

Again, we use the asymptotic property of u at z^i_f and u_{τ_i} at $\bar z^i_f$, respectively, to get

$$|q_i(x,t) - A_{z^i_f}(x\cdot\mu^i)^\beta_+| \le C_0\rho_i^{2+\beta_\beta-\beta}\left(|x| + |t|^{1/2}\right)^{2+\delta_\beta},$$
$$\left|\bar q_i(x,t) - A_{(\bar z^i_f)^{\tau_i}}\left(\left(x + \frac{y^i - \bar y^i}{\rho_i}\right)\cdot\bar\mu^i\right)^\beta_+\right| \le C_0\rho_i^{2+\delta_\beta-\beta}\left(\left|x + \frac{y^i - \bar y^i}{\rho_i}\right| + |t|^{1/2}\right)^{2+\delta_\beta}.$$

For large i, these estimates hold in $\tilde Q_{1/2}(\mu^i, 0)$, and thus in $\tilde Q_{1/4}(\mu^0, 0)$ as well. From $\frac{\bar\rho_i}{\rho_i} \le 2$, we see that

$$\frac{|y^i - \bar y^i|}{\rho_i} \le \frac{|y^i - x^i| + |x^i - \bar y^i|}{\rho_i} = \frac{\rho_i + \bar\rho_i}{\rho_i} \le 3.$$

Thus, over a subsequence,

$$q_i \to q_0(x,t) := A_{z^0}(x\cdot\mu^0)^\beta_+ \quad \text{uniformly in } B_{1/4}(\mu^0, 0),$$
$$\bar q_i \to \bar q_0(x,t) := A_{z^0}((x + \eta^0)\cdot\bar\mu^0)^\beta_+ \quad \text{uniformly in } B_{1/4}(\mu^0, 0)$$

for some $\eta^0 \in \overline{B_3}$.

In fact, we have the above convergence in the C^1-sense as well. For this purpose, we observe that by the asymptotic property, there exist constants $C_1 > c_1 > 0$, independent of i, such that

$$c_1 \le \frac{u}{\rho_i^\beta} \le C_1, \quad c_1 \le \frac{u_{\tau_i}}{\bar\rho_i^\beta} \le C_1 \quad \text{in } \tilde Q_{\rho_i/2}(z^i).$$

This is equivalent to

$$c_1 \le q_i \le C_1, \quad c_1 \le \left(\frac{\rho_i}{\bar\rho_i}\right)^\beta \bar q_i \le C_1 \quad \text{in } \tilde Q_{1/2}(\mu^i, 0).$$

From $1 \le \frac{\bar\rho_i}{\rho_i} \le 2$, it follows that q_i and $\bar q_i$ are uniformly bounded above and below by positive constants, independent of i, in $\tilde Q_{1/3}(\mu^0, 0)$. In addition, since

$F(D^2u) - \partial_t u = u^a$ in $\{u > 0\}$ near the free boundary, we have $F(D^2 q_i) - \partial_t q_i = q_i^a$ and $F(D^2\bar{q}_i) - \partial_t \bar{q}_i = \bar{q}_i^a$ in $\tilde{Q}_{1/3}(\mu^0, 0)$. Thus, for any $\alpha \in (0,1)$,

$$\|q_i\|_{C_x^{1,\alpha}(\tilde{Q}_{1/4}(\mu^0,0))} \le C(n,\lambda,\Lambda,\alpha)\left(\|q_i\|_{L^\infty(\tilde{Q}_{1/3}(\mu^0,0))} + \|q_i^a\|_{L^\infty(\tilde{Q}_{1/3}(\mu^0,0))}\right) \le C,$$

and similarly,

$$\|\bar{q}_i\|_{C_x^{1,\alpha}(\tilde{Q}_{1/4}(\mu^0,0))} \le C.$$

Therefore,

$$\begin{aligned} q_i &\to q_0 \quad \text{in } C^1_{\text{loc}}(\tilde{Q}_{1/4}(\mu^0,0)),\\ \bar{q}_i &\to \bar{q}_0 \quad \text{in } C^1_{\text{loc}}(\tilde{Q}_{1/4}(\mu^0,0)). \end{aligned}$$

To reach a contradiction, we recall that $\nabla w_{\tau_i}(z^i) = 0$, which implies $\nabla(q_i - \bar{q}_i)(\mu^i, 0) = 0$. By the C^1-convergence we have $\nabla q_0(\mu^0,0) = \nabla \bar{q}_0(\mu^0,0)$. Comparing their first components, we get

$$\mu_1^0(\mu^0\cdot\mu^0)_+^{\beta-1} = -\mu_1^0((\mu^0+\eta^0)\cdot\bar{\mu}^0)_+^{\beta-1},$$

where μ_1^0 is the first component of μ^0. From $\nu_1^0 > 0$, we have $\mu_1^0 > 0$, thus the left-hand side of the equation is strictly positive, while the right-hand side is nonpositive. This is a contradiction.

Appendix A. Properties of A_{x^0} and A_{z^0}

In this section we discuss values and properties of $A_{x^0} > 0$ and $A_{z^0} > 0$, when $q_{x^0}(x) = A_{x^0}(x\cdot\nu^{x^0})_+^\beta$ and $q_{z^0}(x,t) = A_{z^0}(x\cdot\mu^{z^0})_+^\beta$ are solutions to

$$\begin{aligned} F(D^2 q_{x^0}) &= q_{x^0}^a \quad \text{in } B_1,\\ F(D^2 q_{z^0}) - \partial_t q_{z^0} &= q_{z^0}^a \quad \text{in } \tilde{Q}_1. \end{aligned}$$

Here, ν^{x^0} and μ^{z^0} are unit vectors in $\mathbb{R}^n$. To get A_{x^0}, we compute

$$\begin{aligned} F(D^2 q_{x^0}) &= A_{x^0}\beta(\beta-1)(x\cdot\nu^{x^0})_+^{\beta-2}F(\nu_{x^0}\otimes\nu_{x^0}),\\ q_{x^0}^a &= A_{x^0}^a(x\cdot\nu^{x^0})_+^{\beta a} = A_{x^0}^a(x\cdot\nu^{x^0})_+^{\beta-2}. \end{aligned}$$

They readily give

$$A_{x^0} = [\beta(\beta-1)F(\nu^{x^0}\otimes\nu^{x^0})]^{-\beta/2}.$$

For its bounds, we observe that $\nu^{x^0}\otimes\nu^{x^0}$ has the eigenvalues $1,0,\cdots,0$, which implies $\lambda \le F(\nu^{x^0}\otimes\nu^{x^0}) \le \Lambda$. Thus, we obtain the uniform lower and upper bounds on A_{x^0}

$$[\beta(\beta-1)\Lambda]^{-\beta/2} \le A_{x^0} \le [\beta(\beta-1)\lambda]^{-\beta/2}.$$

Since q_{z^0} is time independent, it also satisfies $F(D^2 q_{z^0}) = q_{z^0}^a$. Thus, repeating the above process with q_{z^0} will yield

$$A_{z^0} = [\beta(\beta-1)F(\mu^{z^0}\otimes\mu^{z^0})]^{-\beta/2}$$

and

$$[\beta(\beta-1)\Lambda]^{-\beta/2} \le A_{z^0} \le [\beta(\beta-1)\lambda]^{-\beta/2}.$$

Finally, if ν, $\mu \in \partial B_1$ with $F(\nu\otimes\nu) \ge F(\mu\otimes\mu)$, then for the eigenvalues e_i's of $\nu\otimes\nu - \mu\otimes\mu$,

$$\begin{aligned} F(\nu\otimes\nu) - F(\mu\otimes\mu) &\le P^+_{\lambda,\Lambda}(\nu\otimes\nu-\mu\otimes\mu) \le \Lambda\Sigma|e_i|\\ &\le n\Lambda\|\nu\otimes\nu-\mu\otimes\mu\| \le C(n,\Lambda)\|\nu-\mu\|. \end{aligned}$$

Therefore, $\nu \mapsto A_\nu$ is continuous.

Appendix B. A non-standard comparison principle

For small $\sigma > 0$, we construct[3] in Appendix B.1 an n-dimensional Lipschitz cone K, centered at the origin and contained in $\{x_1 > 0,\ x_2 > 0\}$, and a function $H : K \to \mathbb{R}$ satisfying the following: for $1 \le \Lambda/\lambda \le \eta_0$ with $\eta_0 > 1$ depending only on σ,

$$\begin{cases} \text{- } P^-_{\lambda,\Lambda}(D^2 H) = 0, \quad H \ge 0 \ \text{ in } K, \quad H = 0 \ \text{ on } \partial K, \\ \text{- } H \text{ does not decay faster than } |x|^{2+\sigma} \text{ near the center of } K, \ \text{see (B.2)}. \end{cases}$$

We prove its parabolic counterpart in Appendix B.2, see (B.5).

B.1. **Elliptic Case.** We fix a small constant $\sigma_0 \in (0, \sigma)$, and consider a function $\tilde{h}$, defined in a 2-dimensional cone $\mathcal{C}$, such that

$$\begin{cases} \text{- } \mathcal{C} \subset \{x_1 > 0,\ x_2 > 0\} \subset \mathbb{R}^2 \text{ with opening } \pi/2 - \mu, \\ \text{- } \tilde{h} \,:\, \mathcal{C} \to \mathbb{R} \text{ is a positive solution to } P^-_{\lambda,\Lambda}(D^2\tilde{h}) = 0 \text{ in } \mathcal{C} \text{ and } \tilde{h} = 0 \text{ on } \partial\mathcal{C}, \\ \text{- } \tilde{h} \text{ is homogeneous of degree } 2 + \sigma_0. \end{cases}$$

For $1 \le \Lambda/\lambda \le \eta_0 = \eta_0(\sigma_0)$ and small $\mu > 0$, this follows from [17]. Indeed, thanks to Theorem 2.4 in [17] we can find a nonnegative α-homogeneous solution $\tilde{h}$ of $P^-_{\lambda,\Lambda}(D^2\tilde{h}) = 0$ in $\mathcal{C}$ with zero boundary value. Here, the homogeneity α is determined from the equation

$$g_\eta(\alpha) = \frac{1}{2}(\pi/2 - \mu),$$

where

$$g_\eta(\alpha) = \arctan\sqrt{\eta} + \frac{2-\alpha}{\sqrt{(\alpha - 1 + 1/\eta)(\alpha - 1 + \eta)}} \arctan\sqrt{\frac{\eta(\alpha - 1 + 1/\eta)}{\alpha - 1 + \eta}}, \quad \eta = \Lambda/\lambda.$$

We can see that α is completely determined by the opening of the cone, $\pi/2 - \mu$, and the value $\eta = \Lambda/\lambda$. To see if we can make $\alpha = 2 + \sigma_0$, we first take $\eta = 1$ and compute $g_1(\alpha) = \frac{\pi}{4}\left(\frac{1}{\alpha-1}\right)$. Solving $\frac{\pi}{4}\left(\frac{1}{\alpha-1}\right) = \frac{1}{2}(\pi/2 - \mu)$ gives $\alpha = 2 + \frac{2\mu}{\pi - 2\mu}$, thus $\alpha = 2 + \sigma_0$ holds for small $\mu > 0$ (specifically, $\mu = \frac{\sigma_0\pi}{2(\sigma_0+1)}$. This computation implies that for $\eta = \Lambda/\lambda$ larger than 1, say $1 < \eta < \eta_0 = \eta_0(\sigma_0)$, we can still find small $\mu > 0$ to have $\alpha = 2 + \sigma_0$.

Next, by possibly multiplying a constant on $\tilde{h}$, we assume $|x|^{2+\sigma} < \tilde{h}/2$ on the line $\{0 < x_1 = x_2 < 1\}$. We then extend $\tilde{h}$ from $\mathcal{C}$ to the cylindrical cone $\mathcal{C} \times \mathbb{R}^{n-2}$ by defining

$$h(x_1, x_2, x'') := \tilde{h}(x_1, x_2), \quad (x_1, x_2) \in \mathcal{C},\ x'' \in \mathbb{R}^{n-2}.$$

For small $\varepsilon > 0$ we define n-dimensional cones

$$K^\varepsilon := \{h > 0\} \cap \{x_1 + x_2 > \varepsilon|x''|\}, \qquad K^0 = \{h > 0\},$$

and observe that $\lim_{\varepsilon\to 0} K^\varepsilon = K^0$.

Let now H^ε be a solution of

$$\begin{cases} \text{- } P^-_{\lambda,\Lambda}(D^2 H^\varepsilon) = 0 \text{ in } K^\varepsilon \cap B_1, \\ \text{- } H^\varepsilon = h \text{ on } K^\varepsilon \cap \partial B_1, \\ \text{- } H^\varepsilon = 0 \text{ on } \partial K^\varepsilon \cap B_1. \end{cases}$$

By applying the comparison principle, $H^\varepsilon \le h$ in $K^\varepsilon \cap B_1$, and moreover

$$H^\varepsilon \to h \quad \text{in } C^1_{\text{loc}}(K^0 \cap B_1), \quad \text{and} \quad H^\varepsilon \nearrow h \quad \text{pointwise in } K^0 \cap B_1. \tag{B.1}$$

[3] After finalizing this paper we found out that a different (but similar in nature) construction for elliptic case has been done by Silvestre-Sirakov [20], that hinges on an earlier result by Armstrong-Sirakov-Smart [5], where an higher dimensional homogeneous solution in cones are constructed.

We claim now that for $\varepsilon > 0$ small enough there is a sequence $x^j \in L_{r_j} := \{x_1 = x_2 > 0,\ x'' = 0\} \cap \overline{B_{r_j}}$, where $r_j := |x^j| \searrow 0$ and such that

$$H^\varepsilon(x^j) \geq |x^j|^{2+\sigma}. \tag{B.2}$$

To prove claim (B.2) we argue by contradiction. If (B.2) is not true, then using (B.1) we can find sequences $\varepsilon_j \searrow 0$ and $x^j \in L_{r_j}$ with $r_j = |x^j| \to 0$ such that

$$H^{\varepsilon_j}(x^j) = |x^j|^{2+\sigma}, \quad \text{and} \quad H^{\varepsilon_j}(x) < |x|^{2+\sigma} \quad \forall x \in L_{r_j}.$$

Consider

$$\tilde{H}_j(x) := \frac{H^{\varepsilon_j}(r_j x)}{(r_j)^{2+\sigma}},$$

which satisfies

$$\begin{cases} \text{-}\ P^-_{\lambda,\Lambda}(D^2\tilde{H}_j) = 0 & \text{in } K^{\varepsilon_j} \cap B_{1/r_j}, \\ \text{-}\ \tilde{H}_j = 0 & \text{on } \partial K^{\varepsilon_j} \cap B_{1/r_j}. \end{cases}$$

Set $y^0 := \frac{x^j}{r_j} \in L_1 \cap \partial B_1$ (note that $L_1 \cap \partial B_1$ is a singleton). Then by construction, we have

$$\tilde{H}_j(y^0) = |y^0|^{2+\sigma} = 1, \qquad \tilde{H}_j(x) \leq |x|^{2+\sigma} \quad \text{for } x \in L_1.$$

This, combined with Harnack inequality, implies that in every compact subset of K^0, $\tilde{H}_j$ is uniformly bounded in j. Thus, $\tilde{H}_j \to \tilde{H}_0$ in K^0, where $\tilde{H}_0$ solves the same PDE as above, and satisfies

$$\tilde{H}_0(y^0) = 1, \qquad \tilde{H}_0(x) \leq |x|^{2+\sigma} \quad \text{for } x \in L_1. \tag{B.3}$$

Applying Boundary Harnack Principle to $\tilde{H}_0$ and h in $K^0 \cap B_1$ implies

$$ch \leq \tilde{H}_0 \quad \text{in } K^0 \cap B_{1/2}.$$

This in combination with (B.3) implies

$$c|x|^{2+\sigma_0} \approx ch \leq \tilde{H}_0 \leq |x|^{2+\sigma}, \qquad x \in L_{1/2},$$

which is a contradiction, since $\sigma > \sigma_0$.

B.2. **Parabolic case.** Let the function $h : \mathbb{R}^n \to \mathbb{R}$ and n-dimensional cone $K^\varepsilon = \{h > 0\} \cap \{x_1 + x_2 > \varepsilon|x''|\}$ be as in the elliptic case. For a spatial unit vector $\nu^0_x \in \partial B_1$, we define $(n+1)$-dimensional domain

$$\Omega^\varepsilon := \{(x,t) \in \mathbb{R}^{n+1} \ :\ -1 < t < 1,\ x \in (K^\varepsilon \cap B_1) + \varepsilon|t|^{1/2}\nu^0_x\}.$$

Note that every time-slice of Ω^ε is a truncated cone of shape $K^\varepsilon \cap B_1$ with vertex $\varepsilon|t|^{1/2}\nu^0_x$. We can observe that as $\varepsilon \searrow 0$, K^ε and Ω^ε converge to n-dimensional cylinder $K^0 = \{h > 0\}$ and $(n+1)$-dimensional cylinder $\Omega^0 := (K^0 \cap B_1) \times (-1, 1)$, respectively.

Let H^ε be a solution of

$$\begin{cases} \text{-}\ P^-_{\lambda,\Lambda}(D^2H^\varepsilon) - \partial_t H^\varepsilon = 0 & \text{in } \Omega^\varepsilon, \\ \text{-}\ \text{For } -1 < t < 1,\ H^\varepsilon(\cdot,t) = h(\cdot - \varepsilon|t|^{1/2}\nu^0_x) & \text{on } \partial(K^\varepsilon \cap B_1) + \varepsilon|t|^{1/2}\nu^0_x, \\ \text{-}\ \text{For } t = -1,\ H^\varepsilon(\cdot,-1) = h(\cdot - \varepsilon\nu^0_x) & \text{on } (K^\varepsilon \cap B_1) + \varepsilon\nu^0_x. \end{cases}$$

The Maximum Principle yields $H^\varepsilon \leq \|h\|_{L^\infty(B_1)}$ in Ω^ε, thus for some function H^0 we have over a subsequence

$$H^\varepsilon \to H^0 \ \text{ in } C^1_{\mathrm{loc}}(\Omega^0), \ \text{ and } \ H^\varepsilon \to H^0 \ \text{ pointwise in } \Omega^0. \tag{B.4}$$

The fact that Ω^0 is a cylinder gives

$$\begin{cases} P^-_{\lambda,\Lambda}(D^2H^0) - \partial_t H^0 = 0 & \text{in } \Omega^0, \\ H^0 = h \quad \text{on } \partial_p\Omega^0, \end{cases}$$

thus we have by the uniqueness $H^0 = h$ in Ω^0.

We now claim that for $\varepsilon > 0$ small enough there exists a sequence $z^j = (x^j, t^j) = (x_1^j, \cdots, x_n^j, t^j) \in A_{r_j} := \{(a, a, 0, \cdots, 0, a^2) : 0 < a < r_j\}$, where $r_j \searrow 0$, and such that

$$H^\varepsilon(z^j) \geq (x_1^j)^{2+\sigma}. \tag{B.5}$$

To prove (B.5) we argue by contradiction. If (B.5) is not true, then, using (B.4) we can find sequences $\varepsilon_j \searrow 0$ and $z^j \in A_{r_j}$ with $r_j = |x_1^j| \to 0$ such that

$$H^{\varepsilon_j}(z^j) = (x_1^j)^{2+\sigma}, \quad \text{and } H^{\varepsilon_j}(z) < (x_1)^{2+\sigma} \text{ for all } z \in A_{r_j}.$$

Consider

$$\tilde{H}_j(x,t) := \frac{H^{\varepsilon_j}(r_j x, r_j^2 t)}{r_j^{2+\sigma}},$$

which satisfies

$$\begin{cases} P^-_{\lambda,\Lambda}(D^2\tilde{H}_j) - \partial_p \tilde{H}_j = 0 & \text{in } \tilde{\Omega}^j, \\ \text{For } |t| < 1/r_j^2, \ \tilde{H}_j(\cdot, t) = 0 \quad \text{on } (\partial K^{\varepsilon_j} \cap B_{1/r_j}) + \varepsilon_j |t|^{1/2} \nu_x^0, \end{cases}$$

where

$$\tilde{\Omega}^j := \left\{ (x,t) \in \mathbb{R}^{n+1} : -\frac{1}{r_j^2} < t < \frac{1}{r_j^2}, \ x \in (K^{\varepsilon_j} \cap B_{1/r_j}) + \varepsilon_j |t|^{1/2} \nu_x^0 \right\}.$$

Set $y^0 := (1, 1, 0, \cdots, 0, 1) \in \mathbb{R}^{n+1}$. By construction

$$\tilde{H}_j(y^0) = 1, \quad \tilde{H}_j(z) < (x_1)^{2+\sigma} \text{ for } z \in A_1.$$

By parabolic Harnack inequality, in every compact subset of $(K^0 \cap B_{1/2}) \times (-1/2, 1/2)$, $\tilde{H}^j$ is uniformly bounded in j. Thus, over a subsequence $\tilde{H}^j \to \tilde{H}^0$ in $(K^0 \cap B_{1/2}) \times (-1/2, 1/2)$, where $\tilde{H}^0$ satisfies

$$\begin{cases} P^-_{\lambda,\Lambda}(D^2\tilde{H}^0) - \partial_t \tilde{H}^0 = 0 & \text{in } (K^0 \cap B_{1/2}) \times (-1/2, 1/2), \\ \tilde{H}^0 = 0 \quad \text{on } (\partial K^0 \cap B_{1/2}) \times (-1/2, 1/2), \\ \tilde{H}^0(z) \leq (x_1)^{2+\sigma} \text{ for } z \in A_{1/4}. \end{cases}$$

Applying parabolic Boundary Harnack Principle (see e.g. [6]) to $\tilde{H}^0$ and h in $(K^0 \cap B_{1/2}) \times (-1/2, 1/2)$ implies that

$$ch \leq \tilde{H}^0 \quad \text{in } (K^0 \cap B_{1/4}) \times (-1/4, 1/4).$$

Therefore

$$c(x_1)^{2+\sigma_0} \approx ch(z) \leq \tilde{H}^0(z) \leq (x_1)^{2+\sigma}, \quad z \in A_{1/8},$$

which is a contradiction since $\sigma > \sigma_0$.

Declarations

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