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Weak type (1,1) estimates for inverses of discrete rough singular integral operators

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Abstract. We obtain weak type (1,1) estimates for the inverses of truncated discrete rough Hilbert transform. We include an example showing that our result is sharp. One of the ingredients of the proof are regularity estimates for convolutions of singular measure associated with the sequence $[m^\alpha]$; see [18].

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1. Introduction

Suppose $1 < \alpha \leq 1 + \frac{1}{1000}$, $0 < \theta < 1$ are fixed parameters. For a non-negative number M we consider a family of operators on $\ell^2(\mathbb{Z})$

$$\begin{aligned} \mathbb{H}_M f(x) &= \sum_{\substack{M^\theta \leq s \leq M \\ s\text{-dyadic}}} \mathcal{H}_s f(x) \\ &= \sum_{\substack{M^\theta \leq s \leq M \\ s\text{-dyadic}}} \sum_{m>0} \varphi_s\left(\frac{m^\alpha}{s}\right) \frac{f(x-[m^\alpha]) - f(x+[m^\alpha])}{m}, \quad x \in \mathbb{Z} \end{aligned} \tag{1.1}$$

for some sequence φ_s which is uniformly in $C_c^\infty(\frac{1}{2}, 2)$. It is by now a routine fact that the operators $\mathbb{H}_M$, the truncated Hilbert transforms, are bounded on ℓ^p, $1 < p < \infty$ with norm estimates uniform in M and θ. The analogous weak type $(1,1)$ estimate seems to be unknown. For a fixed θ, by a rather routine application of the methods of [4, 16, 18] the operators $\mathbb{H}_M$ can be shown to be of weak type (1,1) uniformly in M. We intend to return to this issue in the future. The subject of the current paper has been inspired by [3]. There, a theorem has been proved [3, Theorem 3], which for our purposes can be formulated as follows:

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Theorem 1.1. *Suppose K is a kernel in $\mathbb{R}^d$ satisfying $K(x) = \Omega(x)/|x|^d$, where Ω is homogeneous of degree 0, $\Omega \in L^q(S^{d-1})$ and has mean 0. Denote $Kf = K * f$. Suppose further that for some $\lambda \in \mathbb{C}$ the operator $\lambda \operatorname{Id} + K$ is invertible in $L^2(\mathbb{R}^d)$. Then $(\lambda \operatorname{Id} + K)^{-1}$ is of the form $\Lambda \operatorname{Id} + K'$, where the kernel K' satisfies the same assumptions as K.*

It immediately implies:

Corollary 1.2 ([3,4,6,15]). *In the setting of the above theorem, the operator $(\lambda \operatorname{Id} + K)^{-1}$ is of weak type $(1, 1)$.*

The principal object of the current work is to extend the above theorem to the case of discrete rough Hilbert transforms $\mathbb{H}_M$. For a fixed θ we prove the uniform in M estimates for $\|(\lambda \operatorname{Id} + \mathbb{H}_M)^{-1}\|_{\ell^1 \to \ell^{1,\infty}}$, provided such an estimate exists in the sense of ℓ^2. By the previous general remark, this goal is accomplished through the following representation theorem, which is the main result of this paper:

Theorem 1.3. *Suppose $1 < \alpha \le 1 + \frac{1}{1000}$ and let θ be such, that $\alpha - 1 < \theta < 1$. Fix $\lambda \in \mathbb{C}$ and suppose that for some constant C_I we have*

$$\left\|(\lambda \operatorname{Id} + \mathbb{H}_M)^{-1}\right\|_{\ell^2 \to \ell^2} \le C_I, \quad \text{for } M \ge M_0. \tag{1.2}$$

Then, there exists $M_1 = M_1(C_I, \lambda)$ such that for $M \ge M_1$ the kernel of the operator $(\lambda \operatorname{Id} + \mathbb{H}_M)^{-1}$ has the form

$$\lambda_I \operatorname{Id} + \beta_I \, \mathbb{H}_M + K, \tag{1.3}$$

where K is the classical discrete Calderón-Zygmund kernel, and we have an estimate uniform in $M \ge M_1$:

$$\left|\lambda_I\right| + \left|\beta_I\right| + \|K\|_{\ell^2 \to \ell^2} + \|K\|_{CZ} \le C_1(C_I, \lambda), \tag{1.4}$$

where

$$\|K\|_{CZ} = \sup_y \sum_{|x| \ge 2|y|} |K(x - y) - K(x)|.$$

Moreover, the above restriction on θ is sharp (we make this statement precise in Theorem 2.3 in the next section).

Applying standard Banach algebras arguments (see, *e.g.*, [8]), for each fixed M, the kernel of the operator $(\lambda \operatorname{Id} + \mathbb{H}_M)^{-1}$ is in $\ell^1((1 + |x|)^N)$ for any $N \ge 0$. In particular $(\lambda \operatorname{Id} + \mathbb{H}_M)^{-1}$ is bounded on ℓ^1, but the weak type $(1, 1)$ estimate obtained in this way becomes unbounded when $M \to \infty$. Also, by selfduality of the multiplier problem, the uniform in M upper bound for $\|(\lambda \operatorname{Id} + \mathbb{H}_M)^{-1}\|_{\ell^1 \to \ell^{1,\infty}}$ requires assumption (1.2).

It is worthwhile to put our result in a more general context. First we note that for the convolution Calderón-Zygmund operators in the continuous setting, the invertibility theorems are by now classical. Similarly, the resolvent of the discrete Hilbert transform, if it exists as an operator on $\ell^2(\mathbb{Z})$, is a discrete Calderón-Zygmund operator. This fact seems to be folklore and can be proved by an application of Fourier transform or by the method of [3]. The discrete analogues of the classical singular integrals have been studied intensively, see some examples in [1,2,5,10,11,13]. We believe, that our results fit well within this line of research.

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2. Main Theorem

Let us recall that we have fixed parameters $\alpha,\ \theta$ with $1 < \alpha \le 1 + \frac{1}{1000}, 0 < \theta < 1$. We introduce a family of algebras, which are subalgebras of the algebra of operators on ℓ^2.

Definition 2.1. We consider the family of operators T, which are convolution operators on $\mathbb{Z}$, with kernels of the form

$$T = \lambda\,\mathrm{Id} + \beta\,\mathbb{H}_M + \sum_{\substack{M^\theta \le s < \infty \\ s\text{-dyadic}}} K_s, \tag{2.1}$$

(we identify convolution operator with its kernel), where the operator $\mathbb{H}_M$ is the truncated Hilbert transform:

$$\mathbb{H}_M f(x) = \sum_{\substack{M^\theta \le s \le M \\ s\text{-dyadic}}} \mathcal{H}_s f(x) \tag{2.2}$$

with

$$\mathcal{H}_s f(x) = \sum_{m>0} \varphi_s\left(\frac{m^\alpha}{s}\right) \frac{f(x - [m^\alpha]) - f(x + [m^\alpha])}{m} \tag{2.3}$$

for some sequence φ_s which is uniformly in $C_c^\infty(\frac{1}{2}, 2)$. We require that the kernels K_s satisfy:

(i)$_s$ $\sum_x K_s(x) = 0$;
(ii)$_s$ $\operatorname{supp} K_s \subset [-s, s]$;
(iii)$_s$ $\sum_x |K_s(x)|^2 \le \frac{D_s^2}{s}$;
(iv)$_s$ $\sum_x |K_s(x+h) - K_s(x)|^2 \le \frac{D_s^2}{s}\left(\frac{|h|}{s}\right)^{\gamma_0}$;

for some small positive γ_0 depending only on $\delta = \theta - (\alpha - 1)$.

For a fixed M we put

$$\|\{K_s\}\|_{A_M} = \sup_{\substack{M^\theta \le s < \infty \\ s\text{-dyadic}}} D_s,$$

and

$$\|T\|_{A_M} = \inf\{|\lambda| + |\beta| + \|\{K_s\}\|_{A_M}\}, \tag{2.4}$$

where the infimum is taken over all representations of the operator T in the form (2.1).

In fact A_M is a Banach algebra with the norm $C\|T\|_{A_M}$ for a certain constant C independent of M. Moreover

$$K = \sum_{\substack{M^\theta \le s < \infty \\ s\text{-dyadic}}} K_s$$

is Calderón-Zygmund kernel with constant controlled by $\|T\|_{A_M}$.

We are now ready to formulate the two theorems leading immediately to Theorem 1.3.

Theorem 2.2. *Let $\theta > \alpha - 1$. Assume that for some fixed $\lambda \in \mathbb{C}$ and a constant C_I all operators $\lambda\,\mathrm{Id} + \mathbb{H}_M$ are invertible for $M \ge M_0$ and $\|(\lambda\,\mathrm{Id} + \beta\mathbb{H}_M)^{-1}\|_{\ell^2 \to \ell^2} \le C_I$. Then for $M \ge M_1$ we have $\|(\lambda\,\mathrm{Id} + \beta\,\mathbb{H}_M)^{-1}\|_{A_M} \le C(C_I, \lambda)$.*

Theorem 2.3. *Let $\theta < \alpha - 1$. There exists a sequence of functions φ_s and a compact set $\Gamma \subset \mathbb{C}$ such that the corresponding Hilbert transform* (2.3) *satisfies $\|(\lambda\,\mathrm{Id} + \mathbb{H}_M)^{-1}\|_{\ell^2 \to \ell^2} \le C_I$ for all M and $\lambda \in \Gamma$, and the estimate $\|(\lambda\,\mathrm{Id} + \mathbb{H}_M)^{-1}\|_{\ell^1 \to \ell^{1,\infty}} \le C$, does not, for any C, hold uniformly in $\lambda \in \Gamma$ and M.*

Remark 2.4.

(i) The range of α's considered in Theorem 2.2 is not optimal, and can be improved using the methods from [12, 18] or a variant of the argument used in this work to prove Lemma 3.1;
(ii) Theorem 2.2 is probably also true with $[m^\alpha]$ replaced by $[m^\alpha \varphi(m)]$, where φ is a function of the Hardy class considered in [12];
(iii) For values of $\theta < 1$ close to 1 Theorem 2.2 could be proved using regularising effect in ℓ^2 of the kernel $\mathbb{H}_M$. Known estimates for the Fourier transform $\hat{\mathbb{H}}_M$ seem, however, to be too weak to cover the entire range of θ considered in this paper;
(iv) In the proof of Lemma 3.7 we could have used a weaker statement of Lemma 3.1, at a cost of a more sophisticated argument. We believe that Lemma 3.1 is of some independent interest, because of its relation to certain type of Waring problem (see [7, 17]). This is one reason we have chosen the variant of proof we present;

(v) Condition (1.2) is always satisfied for sufficiently large $|\lambda|$. If we only consider real valued φ_s, more can be said. Since the kernels $\mathbb{H}_M$ are anti-symmetric, the Fourier transform $\widehat{\mathbb{H}_M}$ is purely imaginary and also anti-symmetric. Thus (1.2) is equivalent to $\lambda \notin [-i N, i N]$, where $N \geq 0$. Using the estimates from [7] it can be shown that

$$N = \limsup_{M \to \infty} \sup_{\xi \in \mathbb{R}} \left| c_\alpha \sum_{\substack{M^\theta \leq s \leq M \\ s\text{-dyadic}}} \int_0^\infty \sin(\xi t s^\alpha)\, \varphi_s(t^{1/\alpha}) \frac{dt}{t^{1-1/\alpha}} \right|;$$

(vi) We refer the reader to our subsequent paper [14] for a sharper version of Theorem 2.3.

Theorem 2.2 is an immediate consequence of the following result, which exploits the mixed-norm submultiplicity properties of algebras A_M. The idea of using such estimates to solve the problem of invertibility of singular integral operators first appeared in [3].

Theorem 2.5. *Let A_M, $M \geq M_0 \geq 1$ be a family of algebras, consisting of bounded convolution operators on ℓ^2, with norms $\|\cdot\|_{A_M}$, satisfying*

$$\begin{aligned} \|T_1 T_2\|_{A_M} \leq C_A \left(\|T_1\|_{\ell^2 \to \ell^2} \|T_2\|_{A_M} + \|T_1\|_{A_M} \|T_2\|_{\ell^2 \to \ell^2}\right) \\ + C_A \epsilon(M) \|T_1\|_{A_M} \|T_2\|_{A_M}, \end{aligned} \tag{2.5}$$

$$\|T_1 T_2\|_{A_M} \leq C_A \|T_1\|_{A_M} \|T_2\|_{A_M}, \tag{2.6}$$

where the constant C_A does not depend on M and $\epsilon(M) \to 0$ as $M \to \infty$. Suppose all operators from the sequence $T^{(M)}$ are invertible on ℓ^2 and satisfy:

$$\begin{aligned} \|(T^{(M)})^{-1}\|_{\ell^2 \to \ell^2} + \|T^{(M)}\|_{A_M} \leq K \qquad & K \text{ independent of } M \geq M_0, \\ \|T^{(M)}\|_{\ell^2 \to \ell^2} \leq \delta < 1. & \end{aligned} \tag{2.7}$$

Then for an $M_1 \geq M_0$, sufficiently large and depending only on K and δ, and all $M \geq M_1$, $T^{(M)}$ are invertible in A_M, with

$$\left\| (T^{(M)})^{-1} \right\|_{A_M} \leq C = C(K, \delta),$$

with $C(K, \delta)$ independent of $M \geq M_1$.

Proof. We will drop the superscript M and denote $T^{(M)}$ by T. We first prove that there exist constants C, N_0 and $\delta_1 < 1$, depending only on K, δ, C_A, such that

$$\left\| \left(T^{(M)}\right)^n \right\|_{A_M} \leq C\, \delta_1^n, \quad n \geq N_0. \tag{2.8}$$

A simple inductive argument shows an estimate

$$\left\|T^{2^N}\right\|_{A_M} \le 2^N\, C_A^N \left\|T^{2^{N-1}}\right\|_{\ell^2\to\ell^2} \cdots \|T\|_{\ell^2\to\ell^2}\|T\|_{A_M} + \\ +\epsilon\, G_N\left(\|T\|_{A_M}, \|T\|_{\ell^2\to\ell^2}\right),$$

where G_N is a polynomial of degree $\le 2^N$, with non-negative coefficients. Suppose an operator T satisfies (2.7). Then, clearly

$$\left\|T^{2^N}\right\|_{A_M} \le (2\,C_A)^N\, \delta^{2^N}\, K + \epsilon\, G_N(K,\delta).$$

Choose N_0 such, that

$$(2\,C_A)^{N_0}\, \delta^{2^{N_0}}\, K \le \frac{1}{4\,C_A},$$

and $M_1 \ge M_0$ so that also

$$\epsilon(M)\, G_{N_0}(K,\delta) \le \frac{1}{4\,C_A}, \qquad M \ge M_1.$$

We get

$$\left\|T^{2^{N_0}}\right\|_{A_M} \le \frac{1}{2\,C_A}, \qquad M \ge M_1.$$

By (2.6) and a standard Banach algebras consideration we get

$$\left\|T^n\right\|_{A_M} \le \left(\frac{1}{2}\right)^{\frac{n}{2^{N_0}}} \cdot C_{C_A,K,\delta}. \tag{2.9}$$

Suppose that the positive invertible on ℓ^2 operator T satisfies (2.7). Then $\delta \le I - T \le 1 - K^{-1}$ so $I - T$ satisfies (2.7). Applying (2.9) to the Neumann series representation of T^{-1} we get an estimate $\|T^{-1}\|_{A_M} \le C_{K,\delta,C_A}$.

Now, if T is an arbitrary operator, invertible on ℓ^2 and satisfying (2.7), we apply the above conclusion to $T^*\,T$ and $T\,T^*$ and the proof of the theorem is concluded. □

The fact that the algebra norms $\|\cdot\|_{A_M}$ satisfy the hypotheses (2.5) and (2.6) will follow from a series of lemmas, which are gathered in the next section.

3. Lemmas

In this section we fix $\theta = \alpha - 1 + \delta$, $\delta > 0$. Let $\varphi \in C_c^\infty(\frac{1}{2}, 2)$, and, for convenience let us introduce an operator H_s:

$$H_s f(x) = \mathcal{H}_{s^\alpha} f(x) = \sum_{m>0} \varphi\left(\frac{m}{s}\right) \frac{f(x-[m^\alpha]) - f(x+[m^\alpha])}{m}, \tag{3.1}$$

where $\mathcal{H}_s$ corresponds to the functions $\tilde{\varphi}_s(t) = \varphi(t^{1/\alpha})$. Let us denote by $H_s(x)$ the kernel of this operator.

Lemma 3.1. *Fix $1 < \alpha < 1 + \frac{1}{1000}$ and $\delta_L > 0$. Then there exist functions $G_s(x)$, $E_s(x)$ and an exponent $\gamma(\delta_L)$ independent of s, such that*

$$H_s * H_s(x) = G_s(x) + E_s(x) + \frac{C}{s}\,\delta_0(x) \tag{3.2}$$

where

$$|G_s(x)| + |E_s(x)| \le C s^{-\alpha}, \qquad \text{supp } E_s \subset \left[-s^{\alpha-1+\delta_L}, s^{\alpha-1+\delta_L}\right] \tag{3.3}$$

and

$$|G_s(x+u) - G_s(x)| \le C s^{-\alpha} |u\, s^{-\alpha}|^{-\gamma(\delta_L)} \tag{3.4}$$

where the constants C depends only on φ.

This lemma is the main technical tool we use. We postpone its proof to the next section. In this section we will apply this lemma to $\mathcal{H}_s$, that is with s replaced by $s^{\frac{1}{\alpha}}$.

Lemma 3.2. *Let $\psi \in C_c^\infty(\mathbb{R})$, $\psi \equiv 1$ for $|x| \le 1$, $\psi \equiv 0$ for $|x| \ge 2$. For a given convolution kernel K on $\mathbb{Z}$ we define truncated kernels:*

$$K_R(x) = K(x) \cdot \psi\left(\frac{x}{R}\right).$$

Then for $R \ge 1$ we have

$$\|K_R\|_{\ell^2 \to \ell^2} \le C\, \|K\|_{\ell^2 \to \ell^2},$$

where the constant C is independent of R.

Proof. This is immediate by taking the Fourier transform. □

Lemma 3.3. *For an operator T as in* (2.1)*, we have*

$$|\lambda| \le \|T\|_{\ell^2 \to \ell^2} + \epsilon(M) \|T\|_{A_M}.$$

Proof. It suffices to observe, that

$$\langle \mathbb{H}_M\, \delta_0, \delta_0 \rangle = 0,$$

and by $(\text{iii})_s$ of Definition 2.1

$$|K_s(0)|^2 \le \frac{\|T\|_{A_M}^2}{s}.$$

Then, for $\epsilon(M) \le C\, M^{-\theta/2}$ the conclusion follows from

$$\lambda = < T\,\delta_0, \delta_0 > - \sum_{\substack{M^\theta \le s < \infty \\ s\text{-dyadic}}} K_s(0).$$ □

Lemma 3.4. *Let T be the kernel of the form* (2.1). *Then T admits a representation*

$$\lambda\,\mathrm{Id} + \beta \sum_{\substack{M^\theta \le s \le M \\ s\text{-dyadic}}} \mathcal{H}_s + \sum_{\substack{M^\theta \le s < \infty \\ s\text{-dyadic}}} K'_s,$$

where:

$$\mathcal{H}_s(x) = \left(\psi\left(\frac{x}{2s}\right) - \psi\left(\frac{x}{2s}\right)\right) \mathbb{H}_M(x), \qquad s \ge M^\theta,\ \textit{dyadic},$$

the function ψ is the same smooth cutoff function as in the previous lemma, the kernels K'_s satisfy conditions $(\mathrm{i})_s$, $(\mathrm{ii})_s$ *and* $(\mathrm{iv})_s$ *from Definition* 2.1, *and we have:*

$$|\lambda| + |\beta| + \|\{K'_s\}\|_{A_M} \le C\,\|T\|_{A_M}.$$

Moreover

$$\left\| \lambda\,\mathrm{Id} + \sum_{\substack{M^\theta \le s < s_0 \\ s\text{-dyadic}}} (\beta\,\mathcal{H}_s + K'_s) \right\|_{\ell^2 \to \ell^2} \le C\|T\|_{\ell^2\to\ell^2} + \epsilon(M)\|T\|_{A_M}.$$

Proof. This lemma is standard and we include the proof for the reader's convenience. Let ψ be the smooth symmetric cutoff function as in the Lemma 3.2, and let s' be the largest dyadic integer satisfying $s' \le M^\theta/2$. We let

$$\psi^{s'}(x) = \psi\left(\frac{x}{s'}\right), \text{ and } \psi^s(x) = \psi\left(\frac{x}{s}\right) - \psi\left(\frac{2x}{s}\right) \quad \text{for } s > s',$$

and thus

$$\sum_{\substack{s_0 \ge s \ge s' \\ s\text{-dyadic}}} \psi^s(x) = \psi\left(\frac{x}{s_0}\right) = \psi_{s_0}(x),$$

with

$$\operatorname{supp} \psi^{s'} \subset \{|x| \le M^\theta\} \quad \operatorname{supp} \psi^s \subset \{s/2 \le |x| \le 2s\}, \quad s > s'.$$

Given an operator T with kernel of the form (2.1):

$$T = \lambda\,\mathrm{Id} + \beta\,\mathbb{H}_M + \sum_{\substack{M^\theta \le s < \infty \\ s\text{-dyadic}}} K_s,$$

we can write the decomposition of its kernel

$$\psi_{s_0} \cdot T = \lambda \,\mathrm{Id} + \beta \sum_{\substack{s_0 \geq s \geq 2s' \\ s\text{-dyadic}}} \psi^s \cdot \mathbb{H}_M + \sum_{\substack{s_0 \geq s \geq s' \\ s\text{-dyadic}}} \psi^s \cdot K,$$

where

$$K = \sum_{\substack{M^\theta \leq s < \infty \\ s\text{-dyadic}}} K_s.$$

Now we let

$$\begin{aligned} \mathcal{H}_s &= \psi^s \cdot \mathbb{H}_M, \qquad & s > s', \\ \tilde{K}_s &= \psi^s \cdot K, \qquad & s \geq s'. \end{aligned}$$

Observe, that the kernels $\tilde{K}_s$ satisfy the requirements in the definition of the algebra A_M, except, possibly, for the vanishing means. We let

$$K_s'(x) = \tilde{K}_s(x) - \frac{c_s}{s} \psi\left(\frac{x}{s}\right) \sum_{y \in \mathbb{Z}} \tilde{K}_s(y),$$

where the constants c_s have been chosen so that

$$\frac{c_s}{s} \sum_{x \in \mathbb{Z}} \psi\left(\frac{x}{s}\right) = 1.$$

Note, that the kernels K_s' do have vanishing means, and satisfy all the requirements of the definition of the algebra A_M, with $\|\{K_s'\}\|_{A_M}$ bounded by $\|\{K_s\}\|_{A_M}$. Now we write the decomposition of kernel $T(x)$

$$\begin{aligned} \psi_{s_0}(x) \cdot T(x) = \lambda \,\mathrm{Id}(x) &+ \beta \sum_{\substack{s_0 \geq s \geq 2s' \\ s\text{-dyadic}}} \mathcal{H}_s(x) + \sum_{\substack{s_0 \geq s \geq s' \\ s\text{-dyadic}}} K_s'(x) \\ &+ \sum_{\substack{s_0/2 \geq s \geq s' \\ s\text{-dyadic}}} J_s \left(\frac{c_s}{s} \psi\left(\frac{x}{s}\right) - \frac{c_{2s}}{2s} \psi\left(\frac{x}{2s}\right)\right) + J_{s_0} \frac{c_{s_0}}{s_0} \psi\left(\frac{x}{s_0}\right), \end{aligned}$$

where

$$J_s = \sum_{\substack{s \geq l \geq s' \\ s\text{-dyadic}}} \sum_y K_l'(y) = \sum_y K(y) \psi\left(\frac{y}{s}\right)$$

and $J_{s'/2}=0$. Let

$$K_s''(x)=K_s'(x)+J_{s/2}\left(\frac{2c_{s/2}}{s}\,\psi\left(\frac{2x}{s}\right)-\frac{c_s}{s}\,\psi\left(\frac{x}{s}\right)\right).$$

We will prove below that $|J_s|\le|\lambda|+C\|T\|_{\ell^2\to\ell^2}$. This immediately imply

$$T=\lambda\,\mathrm{Id}+\beta\sum_{\substack{s\ge 2s'\\ s\text{-dyadic}}}\mathcal{H}_s+\sum_{\substack{s\ge s'\\ s\text{-dyadic}}}K_s''$$

in a weak sense. Moreover, by Lemma 3.2 applied to $\psi_{s_0}\cdot T$ and estimate on λ provided by Lemma 3.3, the partial sums

$$\lambda\,\mathrm{Id}+\beta\sum_{\substack{s_0\ge s\ge 2s'\\ s\text{-dyadic}}}\mathcal{H}_s+\sum_{\substack{s_0\ge s\ge s'\\ s\text{-dyadic}}}K_s''$$

represent an operator with $\ell^2\to\ell^2$ bounded by $C\|T\|_{\ell^2\to\ell^2}+\epsilon(M)\|T\|_{A_M}$, and by the construction $\|K''\|_{A_M}\le C\|T\|_{A_M}$.

We will now show the required estimate for J_s, that is

$$\left|\sum_{y\in\mathbb{Z}}K(y)\,\psi\left(\frac{y}{s}\right)\right|\le c\|T\|_{\ell^2\to\ell^2}+|\lambda|.$$

We let

$$K^s=(K+\mathbb{H}_M)\cdot\psi_s,\qquad \chi_s=\frac{1}{2s+1}\chi_{[-s,s]},$$

and, since the kernel $\mathbb{H}_M$ is antysymmetric,

$$\begin{aligned}
\left|\sum_{y\in\mathbb{Z}}K(y)\,\psi_s(y)\right|^2&=\left|\sum_{y\in\mathbb{Z}}K^s(y)\sum_{y_1\in\mathbb{Z}}\chi_s(y_1)\right|^2\\
&=\left|\sum_{y\in\mathbb{Z}}K^s*\chi_s(y)\right|^2\\
&\le 8\,s\sum_{y\in\mathbb{Z}}\left|K^s*\chi_s(y)\right|^2\\
&\le 8\,s\,\|K^s\|^2_{\ell^2\to\ell^2}\|\chi_s\|^2_{\ell^2}\\
&\le\frac{8\,s}{2s+1}\,\|K^s\|^2_{\ell^2\to\ell^2}\\
&\le c\,\|K+\mathbb{H}_M\|^2_{\ell^2\to\ell^2}\\
&\le 2c\|T\|^2_{\ell^2\to\ell^2}+2|\lambda|^2,
\end{aligned}$$

where the estimate for $\|K^s\|_{\ell^2\to\ell^2}$ follows by Lemma 3.2. Now we apply Lemma 3.3. □

Lemma 3.5. *Let* $0 \le \varphi \in C_c^\infty(\mathbb{R})$ *and* $\varphi_s = \frac{c_s}{s}\varphi(\frac{\cdot}{s})$*, with constants* $c_s > 0$ *such that* $\|\varphi_s\|_1 = 1$*. For a given* $\delta > 0$ *and a positive dyadic integer* s *let* s_1 *be such that* $s^{\frac{\alpha-1+\delta}{\alpha}} \le s_1 \le s$*. Then for* $0 < \gamma \le \gamma_0(\delta)$ *we have:*

(i) $\|\varphi_{s_1} * \mathcal{H}_s\|_{\ell^2}^2 \le \frac{c}{s}$;

(ii) $\|\varphi_{s_1} * \mathcal{H}_s(\cdot + h) - \varphi_{s_1} * \mathcal{H}_s\|_{\ell^2}^2 \le \frac{c}{s}\left(\frac{h}{s}\right)^\gamma$.

We can take $\gamma_0(\delta) = \min\{\frac{\delta}{4\alpha}, \gamma(\frac{\delta}{2})\}$*, where* $\gamma(\delta)$ *is defined by* (3.4).

Proof. It suffices to prove (ii) with $|h| \le Cs$ since it implies (i). For the moment, the superscript ^{h} denotes the translation of a function by h. We have:

$$\begin{aligned}&\Big\langle\Big(\varphi_{s_1}^h - \varphi_{s_1}\Big) * \mathcal{H}_s, \Big(\varphi_{s_1}^h - \varphi_{s_1}\Big) * \mathcal{H}_s\Big\rangle = \Big\langle\Big(\varphi_{s_1}^h - \varphi_{s_1}\Big) * G_s, \varphi_{s_1}^h - \varphi_{s_1}\Big\rangle \\ &\quad + \|\varphi_{s_1}^h - \varphi_{s_1}\|_{\ell^2}^2 \frac{1}{s^{1/\alpha}} + \Big\langle\Big(\varphi_{s_1}^h - \varphi_{s_1}\Big) * E_s, \Big(\varphi_{s_1}^h - \varphi_{s_1}\Big)\Big\rangle \\ &= I + II + III.\end{aligned}$$

In the above we have applied Lemma 3.1 with $\delta_l = \delta/2$ to obtain the decomposition $\mathcal{H}_s * \mathcal{H}_s = G_s + \frac{C\delta_0}{s^{1/\alpha}} + E_s$, satisfying estimates (3.3), (3.4). We have, for $\gamma \le \gamma(\delta/2)$, where $\gamma(\delta)$ is defined by (3.4), that:

$$\begin{aligned}|I| &= \Big|\Big\langle\Big(\varphi_{s_1}^h - \varphi_{s_1}\Big) * G_s, \varphi_{s_1}^h - \varphi_{s_1}\Big\rangle\Big| \\ &= \Big\langle\varphi_{s_1} * \Big(G_s^h - G_s\Big), \varphi_{s_1}^h - \varphi_{s_1}\Big\rangle \\ &\le C\,\frac{1}{s}\left(\frac{|h|}{s}\right)^\gamma \|\varphi_{s_1}\|_{\ell^1}^2. \\ |II| &\le C\,\frac{1}{s^{1/\alpha}}\cdot\frac{1}{s_1}\cdot\left(\frac{|h|}{s_1}\right)^\gamma \\ &\le C\,\frac{1}{s^{1/\alpha}}\cdot\frac{|h|^\gamma}{s_1} \\ &\le C\,\frac{1}{s^{1/\alpha}}\cdot\frac{|h|^\gamma}{s^{1-1/\alpha+\delta/\alpha}} \\ &\le C\,\frac{1}{s}\left(\frac{|h|}{s}\right)^{\delta/2\alpha},\end{aligned}$$

for $\gamma \le \delta/2\alpha$ and $s_1 \ge s^{1-1/\alpha+\delta/\alpha}$. By the Hölder regularity of φ_{s_1} we get

$$\begin{aligned}|III| &\le C\,\left\|\varphi_{s_1}^h - \varphi_{s_1}\right\|_{\ell^\infty}\|E_s\|_{\ell^1}\\ &\le C\left(\frac{|h|}{s_1}\right)^\gamma \cdot \frac{1}{s_1}\cdot\frac{1}{s}\cdot s^{1-1/\alpha+\delta/2\alpha}\\ &\le \frac{C}{s^{1/\alpha}}\cdot\frac{1}{s_1}|h|^\gamma|s|^{\delta/2\alpha}\\ &\le \frac{C}{s^{1/\alpha}}\frac{1}{s^{1-1/\alpha+\delta/\alpha}}\cdot s^{\delta/2\alpha}|h|^\gamma\\ &\le \frac{C}{s}\left(\frac{|h|}{s}\right)^{\delta/4\alpha}\cdot s^{-\delta/4\alpha},\end{aligned}$$

for $\delta/4\alpha \ge \gamma$ and s_1 as in II. □

Let

$$\tilde{\mathbb{T}}_s = \sum_{M^\theta < s' < s}\left(\tilde{\mathcal{H}}_{s'} + \tilde{K}'_{s'}\right),$$

where the kernels $\tilde{\mathcal{H}}_{s'}$, $\tilde{K}'_{s'}$ come from the representation of $\tilde{T}$ in the sense of Lemma 3.4.

Lemma 3.6. *For $\gamma \le \gamma_0(\delta)$ and $s^{1-1/\alpha+\delta/\alpha} \le s_1 \le s$ we have*

(i) $\|\varphi_{s_1} * \mathcal{H}_s * \tilde{\mathbb{T}}_s\|_{\ell^2}^2 \le \frac{C}{s}(\|\tilde{T}\|_{\ell^2\to\ell^2}^2 + \frac{C}{M^\theta}\|\tilde{T}\|_A^2)$;

(ii) $\|\varphi_{s_1} * \mathcal{H}_s * \tilde{\mathbb{T}}_s(\cdot + h) - \varphi_{s_1} * \mathcal{H}_s * \tilde{\mathbb{T}}_s\|_{\ell^2}^2 \le \frac{C}{s}\left(\frac{|h|}{s}\right)^\gamma(\|\tilde{T}\|_{\ell^2\to\ell^2}^2 + \frac{C}{M^\theta}\|\tilde{T}\|_A^2)$.

Proof. Immediate, from Lemmas 3.4 and 3.5. □

Lemma 3.7. *Let* $0 \le l \le s^{1-1/\alpha+\delta/\alpha}$, $s^\theta = s^{\alpha-1+\delta} \le s_1 \le s$ *and* $\psi_l = \varphi_l - \varphi_{2l}$, *where* φ_l *has been defined in Lemma* 3.5. *We have for* $\gamma \le \gamma_0(\delta)$:

(i) $\|\psi_l * \mathcal{H}_s * \mathcal{H}_{s_1}\|_{\ell^2}^2 \le \frac{C}{|s|^{1+\delta/2}}$;

(ii) $\|\psi_l * \mathcal{H}_s * \mathcal{H}_{s_1}(\cdot + h) - \psi_l * \mathcal{H}_s * \mathcal{H}_{s_1}\|_{\ell^2}^2 \le \frac{C}{|s|^{1+\delta/4\alpha}}\cdot\left(\frac{|h|}{|s|}\right)^\gamma$;

(iii) $\|\psi_l * \mathcal{H}_s * K_{s_1}\|_{\ell^2}^2 \le \frac{C}{|s|^{1+\delta/2\alpha}}$;

(iv) $\|\psi_l * \mathcal{H}_s * K_{s_1}(\cdot + h) - \psi_l * \mathcal{H}_s * K_{s_1}\|_{\ell^2}^2 \le \frac{C}{|s|^{1+\delta/4\alpha}}\cdot\left(\frac{|h|}{|s|}\right)^\gamma$.

Proof. (ii) and (iv) follow from (i) and (iii), since $|h| \ge 1$. We will now prove (i). We again use Lemma 3.1 with $\delta_L = \delta/2$.

$$\begin{aligned}\|\psi_l * \mathcal{H}_s * \mathcal{H}_{s_1}\|_{\ell^2}^2 &= \langle\psi_l * G_s, \mathcal{H}_{s_1} * \mathcal{H}_{s_1} * \psi_l\rangle\\ &\quad + \langle\psi_l * E_s, \mathcal{H}_{s_1} * \mathcal{H}_{s_1} * \psi_l\rangle\\ &\quad + \langle\psi_l \cdot \frac{1}{s^{1/\alpha}}, \mathcal{H}_{s_1} * \mathcal{H}_{s_1} * \psi_l\rangle\\ &= I + II + III.\end{aligned}$$

We estimate each part.

$$\begin{aligned}
|I| &\le \|\psi_l * G_s\|_{\ell^\infty} \cdot \|\mathcal{H}_{s_1} * \mathcal{H}_{s_1} * \psi_l\|_{\ell^1} \\
&= C\left(\frac{|s|^{1-1/\alpha+\delta/\alpha}}{|s|}\right)^{\gamma} \cdot \frac{1}{|s|} \\
&\le \frac{C}{|s|} \cdot \frac{1}{|s|^{\delta_1}}. \\
|III| &= |\langle \psi_l * \mathcal{H}_{s_1}, \mathcal{H}_{s_1} * \psi_l\rangle| \cdot \frac{1}{s^{1/\alpha}} \\
&\le \|\mathcal{H}_{s_1}\|_{\ell^2}^2 \, \|\psi_l\|_{\ell^1}^2 \cdot \frac{1}{s^{1/\alpha}} \\
&\le \frac{C}{s_1^{1/\alpha}} \cdot \frac{1}{s^{1/\alpha}} \\
&\le \frac{1}{s^{1/\alpha}} \cdot \frac{1}{s^{1-1/\alpha+\delta/\alpha}} \\
&\le \frac{1}{s^{1+\delta/\alpha}}. \\
|II| &= |\langle E_s, \mathcal{H}_{s_1} * \mathcal{H}_{s_1} * \psi_l * \psi_l\rangle| \\
&\le |\langle E_s * \mathcal{H}_{s_1}, \mathcal{H}_{s_1} * \psi_l * \psi_l\rangle| \\
&\le \|E_s\|_{\ell^1} \cdot \|\mathcal{H}_{s_1}\|_{\ell^2}^2 \\
&\le \frac{s^{1-1/\alpha+\delta/2\alpha}}{s} \cdot \frac{1}{s_1^{1/\alpha}} \\
&\le \frac{s^{1-1/\alpha+\delta/2\alpha}}{s \cdot s^{1-1/\alpha+\delta/\alpha}} \\
&\le \frac{1}{s^{1+\delta/2\alpha}}.
\end{aligned}$$

The estimates of $|II|$ is very crude but it suffices for our purposes. The proof of (iii) is identical. □

Lemma 3.8. *We have that*

$$\|\mathcal{H}_s * \tilde{\mathbb{T}}_s\|_{\ell^2}^2 \le \frac{C}{s}\left(\|\tilde{T}\|_{\ell^2}^2 + \|\tilde{T}\|_A \cdot \left(\frac{1}{s^{\delta/4\alpha}} + \epsilon(s)\right)\right), \tag{3.5}$$

$$\|\mathcal{H}_s * \tilde{\mathbb{T}}_s(\cdot + h) - \mathcal{H}_s * \tilde{\mathbb{T}}_s\|_{\ell^2}^2 \le \frac{C}{|s|}\left(\frac{|h|}{|s|}\right)^{\gamma}\left(\|\tilde{T}\|_{\ell^2}^2 + \|\tilde{T}\|_A \cdot \left(\frac{1}{s^{\delta/4\alpha}} + \epsilon(s)\right)\right), \tag{3.6}$$

where $\mathbb{T}_s, \tilde{\mathbb{T}}_s$ *have been defined before Lemma* 3.6.

Proof. It is a corollary of Lemmas 3.6 and 3.7. Let $s_1 = s^{1-1/\alpha+\delta/\alpha}$, $\varphi \in C_c^\infty(-\frac{1}{2}, \frac{1}{2})$ and φ_{s_1} ψ_l be as in Lemma 3.7. Then $\delta_0 = \varphi_{s_1} + \sum_{\substack{l=1\\ l\text{-dyadic}}}^{\frac{1}{2}s_1} \psi_l$. The conclusion of the Lemma follows directly from the formula:

$$\mathcal{H}_s * \tilde{\mathbb{T}}_s = \varphi_{s_1} * \mathcal{H}_s * \tilde{\mathbb{T}}_s + \sum_{\substack{M^\theta \le s' \le s\\ s'\text{-dyadic}}} \sum_{\substack{l=1\\ l\text{-dyadic}}}^{\frac{1}{2}s_1} \psi_l * \mathcal{H}_s * \big(\tilde{\mathcal{H}}_{s'} + \tilde{K}_{s'}\big). \tag{3.7}$$

Since the kernels $\mathbb{T}_s, \tilde{\mathbb{T}}_s$ are supported in $[-Cs, Cs]$ for some constant C, from Lemma 3.6 we conclude that

$$\varphi_{s_1} * \mathcal{H}_s * \tilde{\mathbb{T}}_s$$

satisfies (3.5) and (3.6), that is $(\text{i})_{s_2}$, $(\text{ii})_{s_2}$, $(\text{iii})_{s_2}$, $(\text{iv})_{s_2}$ of the Definition 2.1 for some $s_2 = Cs$ and with the constant $D_{s_2} \le C\|\tilde{\mathbb{T}}_s\|_{\ell^2\to\ell^2} \le C\|\tilde{T}\|_{\ell^2\to\ell^2} + C\epsilon(s)\|\tilde{T}\|_{A_M}$. Since $s^\theta \le s' \le s$, each of the kernels

$$\psi_l * \mathcal{H}_s * \big(\tilde{\mathcal{H}}_{s'} + \tilde{K}_{s'}\big), \tag{3.8}$$

by Lemma 3.7, satisfies (3.5) and (3.6), that is $(\text{i})_{s_2}$, $(\text{ii})_{s_2}$, $(\text{iii})_{s_2}$, $(\text{iv})_{s_2}$ of the Definition 2.1 with $s_2 = Cs$ and $D_{s_2} \le Cs^{-\delta/8\alpha}\|T\|_{A_M} \le CM^{-\frac{\delta(\alpha-1+\delta)}{8\alpha}}\|T\|_{A_M}$. Since the number of summands in (3.7) is at most $C(\log M)^2$, the lemma follows. □

Lemma 3.9. *We have:*

$$\|T\tilde{T}\|_{A_M} \le C\left(\|T\|_{\ell^2\to\ell^2}\|\tilde{T}\|_{A_M} + \|T\|_{A_M}\|\tilde{T}\|_{\ell^2\to\ell^2}\right) + \epsilon_1(M)\|T\|_{A_M}\|\tilde{T}\|_{A_M},$$

where $\epsilon_1(M) \le CM^{-\frac{\delta(\alpha-1+\delta)}{16\alpha}}$, *and the constant* C *does not depend on* M.

Proof. We use the identity

$$T\tilde{T} = \qquad \lambda\tilde{T} + \tilde{\lambda}T + \sum_s (K_s + \mathcal{H}_s) * \tilde{\mathbb{T}}_s + \sum_s (\tilde{K}_s + \tilde{\mathcal{H}}_s) * \mathbb{T}_{2s},$$

(where $\mathbb{T}_s$, $\tilde{\mathbb{T}}_s$ are defined as in the previous Lemma). We apply Lemma 3.8, and obtain the estimates in the case $s \le M$. The case $s > M$ is immediate, since then $\mathcal{H}_s$ vanish and by the ℓ^2 boundedness of $\mathbb{T}_s, \tilde{\mathbb{T}}_s$, the kernels $K_s * \tilde{\mathbb{T}}_s$, $\tilde{K}_s * \mathbb{T}_s$ satisfy conditions $(\text{i})_{Cs}$, $(\text{ii})_{Cs}$, $(\text{iii})_{Cs}$, $(\text{iv})_{Cs}$ of Definition 2.1 with appropriate norm control. □

4. Proof of Lemma 3.1.

In this section we slightly abuse the notation and denote a generic s by M. We note that H_M, introduced in (3.1), is supported in $[-CM^\alpha, CM^\alpha]$. Denote $G_M = H_M * H_M$. The estimates (3.4) and (3.3) on G_M have been proved in [18], the estimate (3.4) under additional restriction $M^{\frac{99}{100}} \le |x|, |x+u|$ and the estimate (3.3) for any $x \ne 0$. In what follows we will prove (3.4) for the remaining case $M^{\alpha-1+\delta_L} \le x, x+u \le M^{\frac{99}{100}}$. Then the new function $\tilde{G}_M$ defined on the whole $\mathbb{Z}$ by $\tilde{G}_M(x) = G_M(x)$ for $|x| \ge M^{\alpha-1+\delta_L}$ and $\tilde{G}_M(x) = G_M([M^{\alpha-1+\delta_L}])$ for $|x| \le M^{\alpha-1+\delta_L}$ satisfies (3.4). Since $G_M(x) = G_M(-x)$, for $|x| \ge M^{\alpha-1+\delta_L}$ we obviously have, for those x, $\tilde{G}_M(x) = G_M(x)$. We will denote $\tilde{G}_M$ again by G_M and define $E_M(x)$ by equation (3.2) with additional condition $G_M(0)+E_M(0) = 0$. Then $E_M(x)$ obviously satisfies (3.3).

We will apply the method of trigonometric polynomials and we refer the reader to [9] for all background facts. We begin with some definitions used in the sequel.

Definition 4.1. Let $\delta > 0$ be small, and $\delta_0 = \frac{\delta}{100}$. We consider the partition of the interval [0, 1) into intervals of the form

$$I_r = \left[\frac{r}{M^{\delta_0}}, \frac{r+1}{M^{\delta_0}}\right) \subset [0,1), \qquad 0 \le r < M^{\delta_0}.$$

For a number $\Delta \in [0, 1)$ we will denote by $I(\Delta)$ the unique interval of the above form such that $\Delta \in I(\Delta)$. We will write $I_r = [a(I_r), b(I_r))$ and denote by $l(\Delta) = l(I(\Delta)) = b(I(\Delta)) - a(I(\Delta))$ the length of $I(\Delta)$.

Furthermore, we let $m(h, x, \Delta)$ be the unique, if it exists, non-negative solution of

$$(m+h)^\alpha - m^\alpha = x + \Delta, \tag{4.1}$$

where $x, h \in \mathbb{N}$ and $0 \le \Delta < 1$. Let

$$H = \frac{x}{M^{\alpha-1}}, \quad x \in \mathbb{N}, \quad M^{\alpha-1+\delta_L} \le x \le M^{\frac{99}{100}}, \tag{4.2}$$

$$\|w\| = \inf_{k \in \mathbb{Z}} |k - w|, \quad w \in \mathbb{R}. \tag{4.3}$$

We will consider the following condition for (h, x, Δ, k):

$$\begin{aligned} \forall\, m, \quad & m(h, x, a(I(\Delta))) \le m \le m(h, x, b(I(\Delta))) \\ & \Longrightarrow \left\|\alpha \cdot k \cdot m^{\alpha-1}\right\| \ge M^{-\delta_0/2}. \end{aligned} \tag{4.4}$$

Lemma 4.2. *If* $\frac{M}{2} \le m \le 2M$, *satisfies* (4.1), *and* $H,\ x,\ h,\ \Delta$ *are as above, then*

$$C^{-1}H \le h \le CH \tag{4.5}$$

for some constant C independent of M, x, h, Δ. Moreover we have the following estimates:

$$\begin{aligned} &m(h,x,b(I(\Delta))) - m(h,x,a(I(\Delta))) \\ &= c_\alpha \frac{l(I(\Delta))}{h} m(h,x,0)^{2-\alpha}\left(1+O(M^{-\delta_0})\right), \end{aligned} \tag{4.6}$$

$$m(h,x,0) = \left(\frac{x}{h\,\alpha}\right)^{\rho}\left(1+O(M^{-\delta_0})\right),\ \textit{where } \rho = \tfrac{1}{\alpha-1} \tag{4.7}$$

$$\begin{aligned} S &= \sum_{\substack{H/C\le h\le CH \\ I_r\subset[0,1)}} \varphi\left(\frac{m(h,x,b_r)}{M}\right)^2 \frac{l(I_r)}{h} m(h,x,b_r)^{2-\alpha} \\ &= c_\alpha M^{2-\alpha}\left(1+O\big(M^{-\delta_0}\big)\right), \end{aligned} \tag{4.8}$$

where the choice of $b_r \in I_r$ is arbitrary, and $\varphi \in C_c^\infty(\frac{1}{2}, 2)$.

Proof. The estimate (4.5) follows immediately from Taylor's formula. In order to prove (4.6) we use the mean value theorem and the definition of $m(h,x,t)$:

$$\frac{\partial m(h,x,t)}{\partial t} = \frac{\partial m(h,x,t)}{\partial x} = \frac{m(h,x,t)^{2-\alpha}}{\alpha(\alpha-1)h}\left(1+O\left(\frac{h}{M}\right)\right) = O\left(\frac{M}{x}\right), \tag{4.9}$$

$$\frac{m(h,x,t)^{2-\alpha} - m(h,x,0)^{2-\alpha}}{m(h,x,0)^{2-\alpha}} = \frac{(2-\alpha)m(h,x,t_1)^{1-\alpha}\frac{\partial m(h,x,t_1)}{\partial x}}{m(h,x,0)^{2-\alpha}} = O\left(\frac{1}{x}\right). \tag{4.10}$$

Hence:

$$\begin{aligned} m(h,x,b(I(\Delta))) - m(h,x,a(I(\Delta))) &= l(\Delta)\cdot\frac{\partial m(h,x,t)}{\partial x} \\ = l(\Delta)\frac{m(h,x,t)^{2-\alpha}}{\alpha(\alpha-1)h}\left(1+O\left(\frac{h}{M}\right)\right) \\ = l(\Delta)\frac{m(h,x,0)^{2-\alpha}}{\alpha(\alpha-1)h}\left(1+O\left(\frac{h}{M}\right)\right)\left(1+O\left(\frac{1}{x}\right)\right). \end{aligned}$$

Now we prove (4.7). Let x_1 be such that

$$m(h,x_1,0) = \left(\frac{x}{h\alpha}\right)^{\rho},$$

that is

$$x_1 = \left(\left(\frac{x}{h\alpha}\right)^{\rho} + h\right)^{\alpha} - \left(\frac{x}{h\alpha}\right)^{\rho\alpha}.$$

Using the Taylor's formula applied to (4.1) we obtain $|x_1 - x| \le xM^{-1/100}$. We have:

$$\left|\frac{m(h,x_1,0)-m(h,x,0)}{m(h,x_1,0)}\right| \le \frac{C}{M}\,\frac{\partial m(h,x_1,b)|x_1-x|}{\partial x_1}$$
$$\le \frac{C_1}{M}\,\frac{M}{x}\,|x-x_1| \le M^{-\frac{1}{100}}.$$

We now prove the last part, (4.8). Using the estimate (4.9) it is straightforward to check that

$$S = \left(\sum_{\substack{H/C\le h\le CH\\ I_r\subset[0,1)}} \varphi\left(\frac{m(h,x,0)}{M}\right)^2 \frac{m(h,x,0)^{2-\alpha}}{h} l(I_r)\right)\left(1+O\left(M^{-(\alpha-1+\delta)}\right)\right)$$
$$= \left(\sum_{H/C\le h\le CH} \varphi\left(\frac{m(h,x,0)}{M}\right)^2 \frac{m(h,x,0)^{2-\alpha}}{h}\right)\left(1+O\left(M^{-(\alpha-1+\delta)}\right)\right).$$

We apply (4.7) and replace $m(h,x,0)$ by $m(h,x_1,0)$. We get

$$= \left(\sum_{H/C\le h\le CH} \varphi\left(\frac{1}{M}\left(\frac{x}{\alpha h}\right)^{\rho}\right)^2 \left(\frac{x}{\alpha h}\right)^{\rho(2-\alpha)} \frac{1}{h}\right)\left(1+O\left(M^{-(\alpha-1+\delta)}\right)\right)$$
$$= \left(\int_0^{\infty} \varphi\left(\frac{1}{M}\left(\frac{x}{\alpha h}\right)^{\rho}\right)^2 \left(\frac{x}{\alpha h}\right)^{\rho(2-\alpha)} \frac{dh}{h}\right)\left(1+O\left(M^{-\delta}\right)\right).$$

The last equality follows from (4.5), and the fact, that by (4.5)

$$\varphi\left(\frac{1}{M}\left(\frac{x}{\alpha h}\right)^{\rho}\right) = 0 \qquad \text{for } h \le C^{-1}H \text{ or } h \ge CH,$$

and by Taylor's formula. Now, by the change of variables, the last integral equals to $c_\alpha M^{2-\alpha}$ and (4.8) follows. □

Lemma 4.3. *Let $M^{\alpha-1+\delta_L} \le x \le M^{\frac{99}{100}}$. We then have:*

$$M^2 H_M * H_M(x) = \sum_{\substack{H/C\le h\le CH\\ I_r\subset[0,1)}} \varphi\left(\frac{m(h,x,a(I_r))}{M}\right)^2 \left(\left|\mathcal{J}^-_{h,x,I_r}\right| + \left|\mathcal{J}^+_{h,x-1,I_r}\right|\right) + Er(x),$$

where $\mathcal{J}^-_{h,x,I_r}$, and $\mathcal{J}^+_{h,x,I_r}$ are sets satisfying the inclusions:

$$\mathcal{J}^-_{h,x,I_r} \supset \{m \in [m(h,x,a(I_r)), m(h,x,b(I_r)) : \{m^\alpha\} \ge 1 - a(I_r)\},$$
$$\mathcal{J}^-_{h,x,I_r} \subset \{m \in [m(h,x,a(I_r)), m(h,x,b(I_r)) : \{m^\alpha\} \ge 1 - b(I_r)\},$$
$$\mathcal{J}^+_{h,x,I_r} \supset \{m \in [m(h,x,a(I_r)), m(h,x,b(I_r)) : \{m^\alpha\} \le 1 - b(I_r)\},$$
$$\mathcal{J}^+_{h,x,I_r} \subset \{m \in [m(h,x,a(I_r)), m(h,x,b(I_r)) : \{m^\alpha\} \le 1 - a(I_r)\}.$$

Moreover, for the error function $Er(x)$ we have $|Er(x)| \leq CM^{1-\alpha}M^{2-\alpha}$ so it satisfies conditions (3.3) *and* (3.4) *required for G.*

Proof. By the definition of H_M, we have:

$$M^2\, H_M * H_M(x) = \sum_{m_1,m_2\in\mathbb{Z}} \varphi\left(\frac{m_1}{M}\right)\frac{M}{m_1}\varphi\left(\frac{m_2}{M}\right)\frac{M}{m_2}\delta_{\pm[m_1^\alpha]} * \delta_{\pm[m_2^\alpha]}(x)$$
$$= 2\sum_{m_1,m_2\in\mathbb{Z}} \tilde{\varphi}\left(\frac{m_1}{M}\right)\tilde{\varphi}\left(\frac{m_2}{M}\right)\delta_{[m_1^\alpha]-[m_2^\alpha]}(x) = (\dagger)$$

where we have denoted $\tilde{\varphi}(t) = \operatorname{sgn}(t)|t|^{-1}\varphi(t)$, and used the fact that for $m_1 > m_2$ and $0 < x \leq M^{\frac{99}{100}}$ the equation $\pm[m_1^\alpha] \pm [m_2^\alpha] = x$ can be solved only when $[m_1^\alpha] - [m_2^\alpha] = x$.

Now we fix $h > 0$ and consider solutions to the equation:

$$x = [m_1^\alpha] - [m_2^\alpha], \quad m_1 - m_2 = h, \quad \frac{M}{2} \leq m_1 \leq 2M.$$

Each solution is a pair m_1, m_2, but it is determined uniquely by its larger component m_1. In the following we refer to m_1 as "the solution". The set $\mathcal{J}^+_{h,x,I_r}$ consists of solutions with additional condition

$$m_1^\alpha - m_2^\alpha = x + \Delta, \qquad \Delta \in I_r \subset [0,1).$$

The complementary set, $\mathcal{J}^-_{h,x,I_r}$ consists of solutions with additional condition

$$m_1^\alpha - m_2^\alpha = x - 1 + \Delta, \qquad \Delta \in I_r \subset [0,1).$$

It is immediate, that if $\left[(m+h)^\alpha\right] - \left[m^\alpha\right] = x$ then

$$(m+h)^\alpha - m^\alpha = x + \Delta,$$

or

$$(m+h)^\alpha - m^\alpha = x - 1 + \Delta,$$

for some $\Delta \in [0,1)$. Hence

$$\left\{\frac{1}{2}M \leq m \leq 2M : (\exists\, k)\, x = [m^\alpha] - [k^\alpha]\right\} = \bigcup_{\substack{H/C\leq h\leq CH\\ I_r\subset[0,1)}} \mathcal{J}^+_{h,x,I_r} \cup \mathcal{J}^-_{h,x,I_r}.$$

Hence, we have

$$(\dagger) = 2\sum_{I_r\subset[0,1)}\ \sum_{H/C\leq h\leq CH}\ \sum_{m_1\in\mathcal{J}^+_{h,x,I_r}\cup\mathcal{J}^-_{h,x,I_r}} \tilde{\varphi}\left(\frac{m_1}{M}\right)\tilde{\varphi}\left(\frac{m_2}{M}\right)\delta_{[m_1^\alpha]-[m_2^\alpha]}(x) = (\ddagger).$$

Since for $m_1 \in \mathcal{J}^+_{h,x,\Delta} \uplus \mathcal{J}^-_{h,x,\Delta}$ we have, by (4.6),

$$\begin{aligned}|m_1 - m(h,x,a(\Delta))| \le CM^{2-\alpha}, |m_2 - m(h,x,a(\Delta))| \\ \le CM^{2-\alpha} + C|m_2 - m_1| \le CM^{2-\alpha} + CH \le CM^{2-\alpha},\end{aligned}$$

applying Taylor's formula for φ we get

$$(\ddagger) = 2 \sum_{I_r \subset [0,1)} \sum_{H/C \le h \le CH} \tilde{\varphi}\left(\frac{m(h,x,a(I_r))}{M}\right)^2 \sum_{m_1 \in \mathcal{J}^+_{h,x,I_r} \uplus \mathcal{J}^-_{h,x,I_r}} 1 + Er(x),$$

where the error term $Er(x)$ satisfies

$$|Er| \le CM^{1-\alpha} \# \left\{ \frac{1}{2}M \le m \le 2M : (\exists k) x = \left[m^\alpha\right] - \left[k^\alpha\right] \right\} \le CM^{1-\alpha} M^{2-\alpha}. \quad (4.11)$$

The last inequality, by [18] is true for every $x \in \mathbb{Z}$. The first statement of Lemma follows.

If for some $\Delta \in I(\Delta) \subset [0, 1)$ we have

$$(m+h)^\alpha - m^\alpha = x + \Delta, \qquad x \in \mathbb{N},$$

and

$$\{m^\alpha\} \le 1 - b(I(\Delta)),$$

then

$$\left[(m+h)^\alpha\right] - \left[m^\alpha\right] = x.$$

So

$$\{m^\alpha\} + \{(m+h)^\alpha - m^\alpha\} \le 1 - b(I(\Delta)) + \Delta,$$

and thus

$$\{(m+h)^\alpha\} = \{m^\alpha\} + \{(m+h)^\alpha - m^\alpha\} = \{m^\alpha\} + \Delta.$$

So,

$$\left[(m+h)^\alpha\right] - \left[m^\alpha\right] = x + \Delta - \left(\{(m+h)^\alpha\} - \{m^\alpha\}\right) = x.$$

Analogously:

$$\begin{aligned}\{m^\alpha\} \ge 1 - a(I(\Delta)) \;&\Rightarrow\; \{m^\alpha\} + \{(m+h)^\alpha - m^\alpha\} > 1 \\ &\Rightarrow\; \{(m+h)^\alpha\} = \{m^\alpha\} + \Delta - 1,\end{aligned}$$

and then

$$\left[(m+h)^\alpha\right] - \left[m^\alpha\right] = x - 1.$$

It follows that

$$\left[(m+h)^\alpha\right] - \left[m^\alpha\right] = x \;\Rightarrow\; \{m^\alpha\} \le 1 - a(I(\Delta)).$$

The required inclusions now follow. □

Let us introduce the following four functions. Given an interval $I_r \subset [0,1)$, let

$$\chi_1 = \chi_{[1-a(I_r),1-M^{-\delta_0}]}, \quad \chi_2 = \chi_{[1-b(I_r),1]}.$$

Also, choose a function φ, smooth, even, positive, monotone on $\mathbb{R}^+$, with support contained in $[-M^{-\delta_0}, M^{-\delta_0}]$, and with integral 1. Extend these three functions as 1-periodic on $\mathbb{R}$ ($M^{-\delta_0} << 1$), and let

$$\psi^{-,-}_{M,I_r} = \chi_1 * \varphi, \quad \psi^{-,+}_{M,I_r} = \chi_2 * \varphi,$$

where the convolutions are on the torus. Using Lemma 4.5 we have the following obvious estimates:

$$\sum_{m(h,x,a(I_r))\le m\le m(h,x,b(I_r))} \psi^{-,-}_{M,I_r}(m^\alpha) \le \left|\mathcal{J}^-_{h,x,I_r}\right|,$$

$$\left|\mathcal{J}^-_{h,x,I_r}\right| \le \sum_{m(h,x,a(I_r))\le m\le m(h,x,b(I_r))} \psi^{-,+}_{M,I_r}(m^\alpha).$$

Now we choose new

$$\chi_1 = \chi_{[M^{-\delta_0},1-b(I_r)]}, \quad \chi_2 = \chi_{[0,1-a(I_r)]},$$

and let

$$\psi^{+,-}_{M,I_r} = \chi_1 * \varphi, \quad \psi^{+,+}_{M,I_r} = \chi_2 * \varphi.$$

In this case, we have

$$\sum_{m(h,x,a(I_r))\le m\le m(h,x,b(I_r))} \psi^{+,-}_{M,I_r}(m^\alpha) \le \left|\mathcal{J}^+_{h,x,I_r}\right|,$$

$$\left|\mathcal{J}^+_{h,x,I_r}\right| \le \sum_{m(h,x,a(I_r))\le m\le m(h,x,b(I_r))} \psi^{+,+}_{M,I_r}(m^\alpha).$$

It is straightforward to see that if ψ is any one of the above introduced functions, we have the estimates:

$$\sum_{k\in\mathbb{Z}} \left|\hat{\psi}(k)\right| \le C \log M, \tag{4.12}$$

$$\sum_{|k|>M^{2\delta_0}} \left|\hat{\psi}(k)\right| \le C\, M^{-\delta_0}. \tag{4.13}$$

Lemma 4.4. *We have the estimate*

$$\left|\sum_{m(h,x,a(I_r)\le m\le m(h,x,b(I_r))} \psi(m^\alpha) - (m(h,x,b(I_r)) - m(h,x,a(I_r)))\int_0^1 \psi(t)\,dt\right|$$

$$\le \sum_{0<|k|\le M^{2\delta_0}} \left|\hat{\psi}(k)\right| \left|S_k(h,x,I_r)\right| + \frac{C}{M^{\delta_0/4}} |m(h,x,b(I_r)) - m(h,x,a(I_r))|,$$

where ψ is any of the functions $\psi^{\pm}_{M,I_r}$.

We further have

$$\big|S_k(h,x,I_r)\big| \le \frac{1}{M^{\delta_0/4}}|m(h,x,b(I_r)) - m(h,x,a(I_r))| \tag{4.14}$$

and if (h,x,Δ,k) *satisfies* (4.4)*, moreover we have*

$$\big|S_k(h,x,I_r)\big| \le C|m(h,x,b(I_r)) - m(h,x,a(I_r))|. \tag{4.15}$$

Proof. Let us denote

$$\mathcal{J} = \{m(h,x,a(I_r)) \le m \le m(h,x,b(I_r))\}. \tag{4.16}$$

We have

$$\begin{aligned}
&\left|\sum_{m\in\mathcal{J}} \psi(m^\alpha) - \sum_{m\in\mathcal{J}} \hat{\psi}(0)\right| \\
&\le \sum_{0<|k|\le M^{2\delta_0}} \big|\hat{\psi}(k)\big| \left|\sum_{m\in\mathcal{J}} e^{2\pi i\, m^\alpha\cdot k}\right| + |\mathcal{J}|\cdot \sum_{|k|>M^{2\delta_0}} \big|\hat{\psi}_{M,I_r}(k)\big| \\
&= I + II.
\end{aligned}$$

It follows from (4.13) that $II \le |\mathcal{J}|\, M^{-\delta_0}$. We will estimate I. We have, as in the proof of Van der Corput's difference lemma (see [9]):

$$\begin{aligned}
\left|\sum_{m\in\mathcal{J}} e^{2\pi i\, m^\alpha k}\right| &\le \frac{1}{D}\sum_{m\in\mathcal{J}} \left|\sum_{s=0}^{D-1} e^{2\pi i\, ((m+s)^\alpha - m^\alpha)\cdot k}\right| + C\cdot D \\
&\le \frac{1}{D}\sum_{m\in\mathcal{J}} \left|\sum_{s=0}^{D-1} e^{2\pi i\, ks\alpha m^{\alpha-1}}\right| + C|\mathcal{J}|\left(\cdot\frac{D^2M^{2\delta_0}}{M^{2-\alpha}} + \frac{D}{|\mathcal{J}|}\right),
\end{aligned}$$

with the second term of the last expression estimated by $|\mathcal{J}|\big(\frac{M^{4\delta_0}}{M^{2-\alpha}} + M^{\delta_0-\frac{1}{100}}\big) \le |\mathcal{J}|M^{-\delta_0}$ if we have $D = M^{\delta_0}$. We have used in the above the the following obvious consequence of the Taylor's formula

$$e^{2\pi i((m+s)^\alpha - m^\alpha)} = e^{2\pi i\alpha s\, m^{\alpha-1}} + O\left(\frac{s^2\,k}{m^{2-\alpha}}\right).$$

We continue the original estimate:

$$\le \frac{1}{D}\sum_{m\in\mathcal{J}} \min\left\{D, \frac{2}{\|\alpha k m^{\alpha-1}\|}\right\} + \frac{C|\mathcal{J}|}{M^{\delta_0}}.$$

Now, if (h,x,Δ,k) satisfies the (4.4) condition, then

$$\frac{1}{D}\sum_{m\in\mathcal{J}} \min\left\{D, \frac{2}{\|\alpha k m^{\alpha-1}\|}\right\} \le M^{-\delta_0/2}|\mathcal{J}|. \qquad \square$$

Lemma 4.5. *Assume $|k| \le M^{2\delta_0}$. We have the estimates*

$$\sum_{1/C\,H \le h \le C\,H} \left| S_k(h, x, I_r) \right| \le \frac{C\,H}{M^{\delta_0/4}} \left| m(h, x, b(I_r)) - m(h, x, a(I_r)) \right|$$
$$\le C\, l(I_r)\, M^{2-\alpha-\delta_0/4}.$$

Proof. The last inequality is an obvious consequence of (4.6). Based on (4.14) and (4.15), it is enough to prove the estimate

$$\#\{h :\ (h, x, \Delta, k) \text{ does not satisfy (4.4)}\} \le C H\, M^{-\delta_0/4}.$$

To do so, let us momentarily fix h, x, Δ, k which do not satisfy (4.4). Thus there exists $m \in \mathcal{J}$ such that

$$\left\| \alpha\, k\, m^{\alpha-1} \right\| < M^{-\frac{\delta_0}{2}}.$$

Let $|k| \le M^{2\delta_0}$. We will show the estimate

$$\alpha\, k\, m^{\alpha-1} = \frac{kx}{h} + O\left(M^{-\frac{\delta}{2}}\right).$$

Since $m \in J$, it satisfies the equation

$$(m+h)^\alpha - m^\alpha = x + \Delta, \qquad a(I(\Delta)) \le \Delta < b(I(\Delta)),$$

and by the mean value theorem

$$\alpha\, h\, m^{\alpha-1} = x + \Delta + O\left(\frac{h^2\, M^\alpha}{M^2}\right).$$

By (4.2) we have $M^\delta \le H \le M^{99/100}$, and consequently, since $|k| \le M^{2\delta_0}$ and $2\delta_0 < \delta/2$,

$$\alpha\, k\, m^{\alpha-1} = \frac{k\,x}{h} + O\left(M^{-\delta/2}\right).$$

We have

$$\left\| \frac{k\,x}{h} \right\| \le \left\| \alpha\, k\, m^{\alpha-1} \right\| + M^{-\delta/2} \le 2M^{-\delta_0/2}.$$

Now, let $w \in \mathbb{N}$ be the integer approximation of $\frac{kx}{h}$, thus

$$\frac{kx}{h} = w + e, \qquad |e| \le 2\, M^{-\delta_0/2}.$$

Now we assume that we have at least $H\, M^{-\delta_0/4}$ different h_i's, and that (4.4) is false. Thus, each of these h_i's satisfies

$$k\,x = h_i\, w_i + e_i\, h_i, \tag{4.17}$$

and since kx and $h_i w_i$ are integers, so are $e_i h_i$, and

$$|e_i\, h_i| \le 2H\, M^{-\delta_0/2}.$$

Now, for given number z with $|z| \le 2HM^{-\delta_0/2}$ we consider the set

$$\mathcal{A}_z = \{h_i : kx = h_i w_i + z\}.$$

If for each z the number of elements of $\mathcal{A}_z$ is less than $\frac{1}{2}M^{\delta_0/4}$, that the total number of h_i's satisfying (4.17) would be $< \frac{1}{2}M^{\delta_0/4} \cdot 2HM^{-\delta_0/2} = HM^{-\delta_0/4}$, which is a contradiction. Thus, there must be a z, for which

$$\#\{h_i : kx = h_i w_i + z\} \ge \frac{1}{2}\, M^{\delta_0/4}. \tag{4.18}$$

Now, since $|z| \le \frac{Cx}{M^{\alpha-1}}$, $k \ne 0$ we have $0 \ne |kx - z| \le M^{\delta_0/2+1}$ and by (4.18) $kx - z$ has at least $M^{\delta_0/4}$ divisors, which is impossible by a well known estimate on the number of divisors. □

Corollary 4.6. *We have*

$$\sum_{I_r}\sum_{h\sim H}\left|\left|\mathcal{J}^{+}_{h,x,I_r}\right| + \left|\mathcal{J}^{-}_{h,x,I_r}\right| - (m(h,x,b(I_r)) - m(h,x,a(I_r)))\right| \le C\, M^{2-\alpha-\delta_0/4}.$$

$$\sum_{I_r}\sum_{h\sim H}\varphi\left(\frac{m(h,x,a(I_r))}{M}\right)^2\left(\left|\mathcal{J}^{+}_{h,x,I_r}\right| + \left|\mathcal{J}^{-}_{h,x,I_r}\right|\right) = S + O\left(M^{2-\alpha-\delta_0/4}\right)$$

where S is defined by (4.8).

Proof. The first formula is an immediate consequence of Lemmas (4.5) and (4.4). For the second formula we apply (4.6) and the first part. □

5. A counterexample

In this section we prove Theorem 2.3. Fix $1 < \alpha < 1 + \frac{1}{1000}$, $0 < \delta \le \frac{(\alpha-1)^2}{\alpha}$ and $\kappa = c\delta$, where c will be specified later. Let $\{M_l\}_l$ be a sequence of integers satisfying $10M_l \le M_{l+1}^{\alpha-1-1.1\delta}$, with $\varphi \in C_c^\infty(1,2)$ real valued. We put $\varphi_s = \varphi$ if for some l we have (recall s is dyadic) $s \in U_- = [M_l^{\alpha-1-1.1\delta}, M_l^{\alpha-1-\delta}]$ or $s \in U_+ = [M_l^{1-0.1\kappa}, M_l]$, and $\varphi_s = 0$ otherwise. We will consider Hilbert transform

$$\mathbb{H}_{M^\alpha} = \sum_{\substack{M^{\alpha-1-1.1\delta} \le s \le M \\ s\text{-dyadic}}} H_s$$

(we use more convienient H_s instead of $\mathcal{H}_s$) corresponding to this sequence $\{\varphi_s\}$, and $\theta = \alpha - 1 - 1.1\delta$.

Fix l and denote $M = M_l$. By (2.2), $\mathbb{H}_{M^\alpha}$ contains two large blocks $\mathbb{H}_+, \mathbb{H}_-$ corresponding to summation indices in U_+, U_- respectively. For $P = M^{\alpha(\alpha-1-\delta)}$ and an integer j satisfying, for C sufficiently large, $\frac{1}{C}M^{\alpha(2+0.9\delta-\alpha)} \leq j \leq CM^{\alpha(2+\delta-\alpha)}$, let $I_j = [(j-1)P, (j+1)P]$. Consider A_j, the set of $n \in U_-$ such that for some $x \in I_j$ the equation

$$[m^\alpha] \pm [n^\alpha] = x \tag{5.1}$$

has more than 1 solution (a pair m, n, with $m \in U_+$ and $n \in U_-$); we allow the different choice of $\pm$ signs for different solutions. Let m_1 and m_2 satisfy (5.1) possibly with different $x_1, x_2 \in I_j$ and $n_1, n_2 \in U_-$. We define $h = m_1 - m_2$ and estimate using $m_1, m_2 \in U_+$ and the Taylor's formula:

$$|m_1^\alpha - m_2^\alpha| \leq P \;\Rightarrow\; hM^{(1-0.1\delta)(\alpha-1)} \leq CM^{\alpha(\alpha-1-\delta)}.$$

Let $H = \frac{CM^{\alpha(\alpha-1-\delta)}}{M^{(1-0.1\delta)(\alpha-1)}}$; hence $|h| \leq H$, that is m_1, m_2 are contained in the interval of length H containing some m_0 satisfying (5.1). If $n_1 \in A_j$, then for some $n_2 \neq n_1$ we have two pairs m_1, n_1 and m_2, n_2 satisfying (5.1). In what follows we assume that the $\pm$ signs corresponding to both pairs are minus. By (5.1) we obtain

$$\left[n_1^\alpha\right] - \left[n_2^\alpha\right] = \left[m_1^\alpha\right] - \left[m_2^\alpha\right] = \left[m_1^\alpha - m_2^\alpha\right] + \Delta, \qquad \Delta \in \{-1, 0, 1\}. \tag{5.2}$$

We have:

$$\begin{aligned} m_1^\alpha - m_2^\alpha &= m_1^\alpha - m_0^\alpha + m_0^\alpha - m_2^\alpha \\ &= \alpha h_1 m_0^{\alpha-1} - \alpha h_2 m_0^{\alpha-1} + O(H^2 M^{\alpha-2}), \quad H^2 M^{\alpha-2} \leq 1. \end{aligned}$$

From this:

$$[m_1^\alpha - m_2^\alpha] + \Delta = [\alpha\,(h_1 - h_2)\, m_0^{\alpha-1}] + \Delta_1 \tag{5.3}$$

$$\Delta_1 \in \{-2, -1, 0, 1, 2\}, \quad -H \leq h_1, h_2 \leq H. \tag{5.4}$$

There are at most $5(4H+1)$ different numbers represented by right hand side of (5.3). By Lemma 3.1, the number of solutions to

$$\left[n_1^\alpha\right] \pm \left[n_2^\alpha\right] = k, \qquad 0 < n_1, n_2 \leq M^{\alpha-1-\delta}$$

is at most $CM^{(\alpha-1-\delta)(2-\alpha)}$. Thus the number of pairs (n_1, n_2) with n_1, m_1 and n_2, m_2 satisfying (5.2), that is (5.1) for the same x, does not exceed

$$M^{(\alpha-1-\delta)(2-\alpha)} \cdot 21\, H \leq C \cdot M^{\alpha-1-1.9\delta}.$$

The case of other choices of $\pm$ signs follows exactly the same way. So we obtained $|A_j| \leq M^{\alpha-1-1.9\delta}$.

Let x be of the form

$$x = [m^\alpha] \pm [n^\alpha], \qquad n \notin A_j \cup A_{j-1} \cup A_{j+1}, [m^\alpha] \in I_j. \tag{5.5}$$

Then one can easily verify, that $x \in I_j \cup I_{j-1} \cup I_{j+1}$. We infer that the representation (5.5) is unique, and it remains unique if we drop the assumption $[m^\alpha] \in I_j$ (we remark that if $n \le M^{\frac{\alpha-1-1.1\delta}{\alpha}}$ than this statement is immediate and do not require an argument above). In particular for x, m, n related by (5.5)

$$\begin{aligned} \left|\mathbb{H}_+ * \mathbb{H}_-(x)\right| &\ge \frac{1}{m \cdot n}, \\ \mathbb{H}_- * \mathbb{H}_-(x) &= 0. \end{aligned} \tag{5.6}$$

Thus (we leave the proof for the reader)

$$\left\|\mathbb{H}_+ * \mathbb{H}_-\right\|_{\ell^p} \ge C \left(\frac{\delta\kappa}{100}\right)^{\frac{1}{p}} (\log M)^2, \qquad p = 1 + \frac{1}{\log M}. \tag{5.7}$$

We will show the estimate

$$\left\|\mathbb{H}_+ * \mathbb{H}_+\right\|_{\ell^p} \le C\kappa^{\frac{2}{p}} (\log M)^2 \tag{5.8}$$

where p is as in (5.7). We have

$$\mathbb{H}_+ * \mathbb{H}_+ = \sum_{\substack{M^{1-0.1\kappa} \le s_1, s_2 \le M \\ s_1, s_2\text{-dyadic}}} H_{s_1} * H_{s_1}.$$

Since this expression contains at most $C\kappa^2(\log M)^2$ summands, it suffices to prove that $\|H_{s_1} * H_{s_2}\|_{\ell^p} \le C$. Assume $s_1 \ge s_2$. Since $H_{s_1} * H_{s_2}$ is supported in $[-Cs_1^\alpha, Cs_1^\alpha]$, by Cauchy-Schwartz, it suffices to have $\|H_{s_1} * H_{s_2}\|_{\ell^2}^2 \le Cs_1^{-\alpha}$. We have

$$\left\|H_{s_1} * H_{s_2}\right\|_{\ell^2}^2 = \left\langle H_{s_1} * H_{s_1}, H_{s_2} * H_{s_2}\right\rangle \le C\left(\frac{1}{s_1 s_2} + \frac{s_2^\alpha}{s_1^\alpha s_2^\alpha}\right)$$

where, since $H_{s_2} * H_{s_2}$ is supported in $[-Cs_2^\alpha, Cs_2^\alpha]$, the last estimate follows from the Lemma 3.1. Fix sufficiently small $c > 0$ and $\kappa = c\delta$. From the (5.7), (5.8) and (5.6) we infer that the estimate

$$\|(\mathbb{H}_+ + \mathbb{H}_-) * (\mathbb{H}_+ + \mathbb{H}_-)\|_{\ell^p} \le \frac{C}{p-1}$$

cannot hold uniformly with M and $p > 1$. By the definition, $\mathbb{H}_{M^\alpha}$ is antisymmetric with $\ell^2 \to \ell^2$ operator norm controlled independently of M, so it has purely imaginary spectrum contained in some fixed interval $D \subset i\mathbb{R}$. Let Γ be a contour in $\mathbb{C}$ enclosing D. Then we have $\|(\lambda I + \mathbb{H}_{M^\alpha})^{-1}\|_{\ell^2\to\ell^2} \le C$. Now, if we have $\|(\lambda I + \mathbb{H}_{M^\alpha})^{-1}\|_{\ell^1\to\ell^{1,\infty}} \le C$, uniformly for M and $\lambda \in \Gamma$, we should have $\|(\lambda I + \mathbb{H}_{M^\alpha})^{-1}\|_{\ell^p\to\ell^p} \le \frac{C}{p-1}$. The formula $\mathbb{H}_{M^\alpha}^2 = \frac{-1}{2\pi i}\oint_\Gamma \lambda^2(\lambda I + \mathbb{H}_{M^\alpha})^{-1} d\lambda$ implies that the estimate $\|\mathbb{H}_{M^\alpha}^2\|_{\ell^p\to\ell^p} \le \frac{C}{p-1}$ holds uniformly in M. This is a contradiction.

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