THIN ACTIONS ON CAT(0)-SPACES

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Abstract. We study packed, geodesically complete, CAT(0)-spaces X admitting a *thin action*: that is, a discrete isometry group Γ of X whose systole at every point is sufficiently small (smaller than a universal constant only depending on the packing constants of X –the *Margulis' constant*). We show that if X admits an almost-cocompact thin action then it can be decomposed as a union $\bigcup_i \text{Min}(g_i)$ of minimal sets of isometries with small displacement, with the property that if $\text{Min}(g_i) \cap \text{Min}(g_j) \neq \emptyset$ then g_i, g_j generate a virtually nilpotent group. This generalizes well-known structure theorems for manifolds, due to Buyalo and Cao-Cheeger-Rong, to CAT(0)-spaces. We then deduce some splitting and rigidity results for thin actions by slim groups (i.e. isometry groups whose elliptic elements have fixed point set with empty interior): for instance, every packed, geodesically complete, CAT(0)-space admitting a thin, almost cocompact action by a slim group Γ always splits a non-trivial Euclidean factor, provided that Γ does not contain a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$; also, if $\dim(X) \leq 2$ then X is isometric to the product of $\mathbb{R}$ with a (possibly trivial) tree, while in the cocompact case we always deduce that $X = \mathbb{R}$ necessarily. Moreover, we show that if a space X as above has some strictly negative curvature behaviour (e.g. if X is a visibility space, or it has an open subset which is CAT$(-\varepsilon)$) then it cannot support a thin action; this generalizes the classical Margulis' Lemma to a broader class of singular spaces.

1. Introduction

The classical Margulis-Heintze Lemma is a cornerstone in the study of negatively curved manifolds: it asserts that there exists some positive constant $\varepsilon(n, \kappa)$ such that for every torsionless lattice Γ of a complete, Riemannian n-dimensional manifold X with sectional curvature $-\kappa^2 \leq K(X) < 0$ there exists a point x such that

$$\text{sys}(\Gamma, x) := \inf_{g \in \Gamma^*} d(x, gx) > \varepsilon(n, \kappa),$$

where Γ^* is the subset of nontrivial elements of Γ ([Hei76], [BGS13], [BZ88]). Accordingly, for every discrete group Γ acting by isometries on a metric space X we will call *diastole* of the action (or of the quotient $M = \Gamma \backslash X$), the value

$$\text{dias}(\Gamma, X) = \sup_{x \in X} \inf_{g \in \Gamma^*} d(x, gx).$$

We say that the action of Γ is ε*-thin* if $\text{dias}(\Gamma, X) < \varepsilon$ (since the subset of points x where $\text{sys}(\Gamma, x) < \varepsilon$ is classically known as the ε-thin subset of X). Buyalo [Buy90a]-[Buy90b] first, in dimension smaller than 5, and Cao-Cheeger-Rong [CCR01] later, in every dimension, studied the possibility of extending the Margulis-Heintze Lemma to *uniform lattices of non-positively curved* manifolds. They proved the following alternative: either the diastole is bounded below by a universal positive constant $\eta(n, \kappa) > 0$, or the action of Γ is η-thin and the manifold $\Gamma \backslash X$ admits a so-called *abelian local splitting*

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structure. This is, roughly speaking, a decomposition of X into a union of minimal sets of hyperbolic isometries with the additional property that if two minimal sets intersect then the corresponding isometries commute.

Example 1.1 ([Gro78, §5] and [Buy81, §4])**.** Consider two copies Σ_1, Σ_2 of a hyperbolic surface with connected, geodesic boundary of length ℓ, then take the products $M_i = \Sigma_i \times S^1$ with a circle of length ℓ, and glue M_1 to M_2 by means of an isometry φ of the boundary tori $\partial M_i = \partial \Sigma_i \times S^1$ which interchanges the circles $\partial \Sigma_i$ with S^1. This yields a nonpositively curved 3-manifold M (a *graph manifold*) with sectional curvature $-1 \leq k(M) \leq 0$ whose diastole can be chosen arbitrarily small, by taking $\ell \to 0$. This manifold is the prototypical example of a nonpositively curved manifold with an abelian local splitting structure: calling g_1, g_2 the hyperbolic isometries of $X = \widetilde{M}$ corresponding to the boundary loops $\partial \Sigma_i$, the universal cover $\widetilde{X}$ is the union of the two closed, minimum sets $\mathrm{Min}(g_i) \cong \widetilde{\Sigma}_i \times \mathbb{R}$ (where g_i acts as *id* times a translation) and $\mathrm{Min}(g_1) \cap \mathrm{Min}(g_2)$ is the gluing torus, on which $\langle g_1, g_2 \rangle$ acts on as an abelian group.

In this paper, we will generalize this result to groups acting (not necessarily cocompactly) on CAT(0)-spaces and we draw from it a description of spaces with thin actions. In the framework of CAT(0)-spaces, the lower sectional curvature bound appearing in the assumptions of the classical Margulis-Heintze Lemma can be naturally replaced by a uniform, local packing condition: we say that a metric space (X, d) satisfies the *P_0-packing condition at scale* $r_0 > 0$ (or that X is *(P_0, r_0)-packed*, for short) if all balls of radius $3r_0$ in X contain at most P_0 points that are $2r_0$-separated from each other. In [CS21] it is proved that, for complete and geodesically complete CAT(0)-spaces, a packing condition at some scale r_0 yields an explicit, uniform control of the packing function at any other scale r: therefore, for these spaces, this condition is equivalent to similar conditions which have been considered by other authors with different names ("uniform compactness of the family of r-balls" in [Gro81]; "geometrical boundedness" in [DY05], etc.). This packing condition should be thought of as a macroscopic replacement of a lower bound on the curvature: for Riemannian manifolds it is strictly weaker than a lower bound on the Ricci curvature, and for general metric spaces it is weaker than asking that X is a space with curvature bounded below in the sense of Alexandrov (see [BCGS17, §3.3], for different examples and comparison between packing and curvature conditions). Although much weaker than a curvature bound, the packing condition implies, by Breuillard-Green-Tao's work, an analogue of the celebrated Margulis' Lemma for a general metric space X: *there exists a constant $\varepsilon_0 > 0$, only depending on the packing constants P_0, r_0, such that for every discrete group of isometries Γ of X the ε_0-almost stabilizer $\Gamma_{\varepsilon_0}(x)$ of every point x is virtually nilpotent* (cp. [BGT11] and Section 2.3 below for details). We call this $\varepsilon_0 = \varepsilon_0(P_0, r_0)$ the *Margulis constant*, since it plays the role of the classical Margulis constant in this metric setting.
There are a lot of non-manifold examples in the class of packed, CAT(0)-spaces: limits of Hadamard manifolds with curvature bounded below; simplicial complexes with locally constant curvature and bounded geometry (also

called M_κ-complexes, cp. [BH99] and [CS21]), Euclidean and hyperbolic buildings with bounded geometry in particular; nonpositively curved cones over compact CAT(1)-spaces, etc.

The first theorem of this paper generalizes Cao-Cheeger-Rong's result to this broader setting, taking into account also non-cocompact actions of groups, possibly with torsion:

Theorem A (Extract from Theorem 3.1)**.**
There exist two positive functions $\lambda_0 = \lambda_0(P_0, r_0)$, $\eta_0 = \eta_0(P_0, r_0) < \varepsilon_0$ with the following property: for every complete, geodesically complete, (P_0, r_0)-packed CAT(0)*-space X and every almost-cocompact discrete group of isometries Γ of X, if Γ is η_0-thin then X can be decomposed as*

$$X = \bigcup_{g \in \Sigma_{\lambda_0}} \mathrm{Min}(g) \tag{1}$$

where Σ_{λ_0} is the subset of $g \in \Gamma^*$ with minimal displacement $\ell(g) \leq \lambda_0$. Moreover, if two minimum sets $\mathrm{Min}(g_i)$, $\mathrm{Min}(g_j)$ *of this decomposition intersect, then g_i, g_j generate a virtually nilpotent subgroup of* Γ.

By *almost-cocompact* group, we mean a group which acts cocompactly on the δ-thick subsets of X. The same notion was introduce in [BGS13, §8.4] with the notation $\mathrm{InjRad}(\Gamma \backslash X) \to 0$. In particular cocompact groups or groups acting with quotient of finite measure are almost cocompact (see Section 2 for the definition and the natural measure of CAT(0)-spaces under consideration).

The proof of Theorem A is different from that for manifolds given in [CCR01]: actually, the proof of Cao-Cheeger-Rong is based on the fact that hyperbolic elements *stabilize*, a property which does not hold for CAT(0)-spaces. Recall that an isometry g is called *stable* (resp. *s-stable*) if $\mathrm{Min}(g) = \mathrm{Min}(g^k)$ for all $k > 0$ (resp. for all $1 \leq k \leq s$); accordingly, one says that an element g stabilizes (resp. s-stabilizes) if there exists a power g^n which is stable (resp. s-stable). The following easy example shows that isometries of CAT(0)-spaces in general do not even 2-stabilize:

Example 1.2. Let T be a 3-regular metric tree T, with edge lengths 1. We will define an elliptic isometry g_T of T with infinite order as follows. Fix a basepoint $v_0 \in T$, choose a geodesic ray ξ starting from v_0, and order the vertices of ξ with respect to the distance from v_0 as $\{v_0, v_1, v_2, \ldots\}$. For $i \geq 1$, consider the three subtrees given by the closures of the connected components of $T - \{v_i\}$, and call T_i the one containing $[v_0, v_i]$ and T_i', T_i'' the other two. Then let σ_i be the involution of T that fixes T_i, and interchanges T_i' with T_i''. The sequence of isometries $(\sigma_n \circ \sigma_{n-1} \circ \ldots \circ \sigma_1)_{n \in \mathbb{N}}$ converges uniformly on compact subsets to a limit isometry g_T. We claim that, for every $n \in \mathbb{N}$, the isometry g_T^n is not 2-stable, i.e. $\mathrm{Fix}(g_T^{2n}) \neq \mathrm{Fix}(g_T^n)$. In order to prove this, write $n = 2^k \cdot m$ with m odd and set $v_{k+2}^- := \sigma_{k+1}(v_{k+2})$. Observe that:

(i) $\mathrm{Fix}(g_T^n) \supseteq \mathrm{Fix}(g_T^{2^k}) \supseteq B(v_0, k+1)$;
(ii) $\mathrm{Fix}(g_T^n) \not\supseteq B(v_0, k+2)$ because $g_T^n(v_{k+2}) = v_{k+2}^-$.

(The second assertion follows since $g_T^{2^k}$ fixes v_{k+1} and interchanges v_{k+2} with v_{k+2}^-; therefore, $g_T^{2^k}$ acts on $S := \{v_{k+2}, v_{k+2}^-\}$ as a non-trivial transposition and $g_T^n = (g_T^{2^k})^m$ also acts on S non-trivially since m is odd.)
Then $\mathrm{Fix}(g_T^{2n}) \supseteq B(v_0, k+2)$ by (i), while $\mathrm{Fix}(g_T^n) \not\supseteq B(v_0, k+2)$ by (ii), proving the claim.
To produce an example of a *discrete* group with a hyperbolic element which does not stabilizes, just consider the product $X = T \times \mathbb{R}$ and the group G generated by the isometry g acting on X as (g_T, τ), where τ is a non-trivial translation of $\mathbb{R}$.

A major difficulty for this is that, in this extended metric setting, there exist different isometries which coincide on open sets. This problem is related to the existence of non-trivial isometries whose sets of fixed points have non-empty interior. For many applications of Theorem A we will exclude this case, restricting our attention to what we called *slim groups* (we refer to Section 2.2 for more details). By definition two isometries of a slim group coinciding on an open set must coincide globally: this gives a better control on the geometry of the group action. Every torsion-free group is trivially slim, as well as any discrete group acting on a CAT(0)-homology manifold (cp. Lemma 2.2). Moreover, opposite to [CCR01], we also have to deal with parabolic and elliptic isometries, as Γ is allowed to have torsion.
On the other hand, our proof has some similarities with a minimality argument used by Cao-Cheeger-Rong (see their proof of [CCR01, Theorem 0.3]), and it is inspired by statements of Ballmann-Gromov-Schroeder ([BGS13, Lemma 7.11 and §13]) holding for *analytic* manifolds (we stress that analiticity is crucial in [BGS13] for their argument to work); similar arguments, which use analiticity or strictly negative curvature, can be found in [Gel11, §2] and [BGLS10, §2]. The main innovation in our proof relies in the choice of the good minimal point for the function Ψ (cp. Step 3 in the proof of Theorem 3.1), which allows us to overcome any analyticity or negative curvature assumption.

In Theorem 3.1 we will see a stronger statement than Theorem A: the same alternative holds for any λ smaller than the Margulis' constant ε_0 and an explicit $\eta = \eta(P_0, r_0, \lambda)$. Here we want just to stress that the equality (1), in particular the fact the the union runs over minimum sets of isometries with minimal displacement less than the Margulis constant, imposes a very strong condition on the topology of X. For instance, if X is a non-elementary Gromov hyperbolic space and Γ is torsionless, then X cannot satisfy (1) with $\lambda_0 \leq \varepsilon_0$, and the diastole is always universally bounded away from zero. This was already proved in [BCGS17, Proof of Proposition 5.24] in the cocompact case, and in [CS24, Corollary 1.4] for non-cocompact actions. The following corollary yields a simple algebraic obstruction.

Corollary B. *Let X be a complete, geodesically complete, (P_0, r_0)-packed,* CAT(0)*-space. Assume that Γ is a discrete, slim, almost-cocompact group of isometries of X. If $\mathrm{dias}(\Gamma, X) < \eta_0$, then*

(i) *either Γ contains a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$*
(ii) *or X splits a non-trivial Euclidean factor.*

If moreover Γ *is cocompact then (ii) can be replaced by*

(ii') $X = \mathbb{R}$.

(The constant η_0 *here and in the following is the same as in Theorem A.)*

The following theorem generalizes the lower bounds for $\mathrm{dias}(\Gamma, X)$ of [BCGS17] and [CS24], and completely determines the structure of packed (complete, geodesically complete) CAT(0)-spaces with diastole small enough in some remarkable cases:

Theorem C (Extract from Proposition 4.1 and Corollaries 4.2 & 4.5)**.** *Let* X *be a complete, geodesically complete,* (P_0, r_0)*-packed,* CAT(0)*-space. Assume moreover that one of the following holds:*

- (i) X *has a point of dimension* 1*,*
- (ii) *or* X *has a point with an open neighbourhood which is* CAT$(-\varepsilon)$ *for some* $\varepsilon > 0$*,*
- (iii) *or* X *is a visibility space.*

If X *has a discrete, slim, almost-cocompact group of isometries* Γ *such that* $\mathrm{dias}(\Gamma, X) < \eta_0$ *then* X *is isometric to* $\mathbb{R}$*.*
(See Section 2.1 for the notion of *points of dimension* k and generalites on the dimension for CAT(0)-spaces).

The following is another structural consequence of a small diastole for CAT(0)-spaces in low dimension.

Theorem D. *Let* X *be a complete, geodesically complete,* (P_0, r_0)*-packed,* CAT(0)*-space of dimension at most* 2 *and let* Γ *be a discrete, slim, almost-cocompact group of isometries of* X*. If* $\mathrm{dias}(\Gamma, X) < \eta_0$ *then* X *is isometric to* $T \times \mathbb{R}$*, where* T *is a tree (possibily a point). If moreover* Γ *is cocompact then it virtually splits as* $\Gamma_T \times \mathbb{Z}$*, where* Γ_T *(resp.* $\mathbb{Z}$*) acts discretely and cocompactly on* T *(resp.* $\mathbb{R}$*).*

This splitting result is false in higher dimensions as showed by Example 1.1.

Remark 1.3. This last theorem has two interesting consequences. In dimension ≤ 2, if the diastole is smaller than the universal constant η_0 then:

- (a) the space is *pure dimensional*, that is the dimension of every point is the same;
- (b) the space is *collapsible*, in the following sense: one can define new CAT(0) metrics d_n on X in such a way that (X, d_n) is still uniformily packed at scale r_0 and Γ still acts by isometries with respect to d_n, but the diastole of Γ with respect to the metric d_n is smaller than $\frac{1}{n}$. Therefore, if the $\mathrm{dias}(\Gamma, X)$ is sufficiently small, then it can be taken arbitrarily small by changing the metric of X remaining in the class of uniformly packed CAT(0) metrics.

The result of (a) is not true in higher dimensions: it is enough to consider the product $X = Y \times \mathbb{R}$, where Y is any non-pure dimensional, complete, geodesically complete, packed CAT(0)-space admitting a cocompact group of isometries Γ_Y, and taking $\Gamma = \Gamma_Y \times \langle \tau_\varepsilon \rangle$, where τ_ε is a translation by ε. On the other hand, we have no clue if (b) is true in higher dimensions.
In [CS23] and [Cav23] we give a positive answer in case of cocompact actions

with bounded codiameter, provided the diastole is sufficiently small in terms of the diameter and the packing constants.

We remark that Corollary B, Theorem C and Theorem D are false if the group is not slim, as the following example due to A.Lytchak shows.

Example 1.4. Let X be the Cayley graph of the free group on two generators $F(a,b)$. Let e be the identity point on this graph and $\tilde{a},\tilde{b}$ the axes of the isometries a and b. We consider the standard topological immersion of X in $\mathbb{R}^2$ with edges labelled by a parallel to the x-axis, and edges labelled by b parallel to the y-axis. Let σ_a, σ_b be the Euclidean reflections with respect to $\tilde{a}$ and $\tilde{b}$: they naturally define isometric automorphisms of X. Let Γ be the group of isometries of X generated by $\{a, b, \sigma_a, \sigma_b\}$. It is clear that every point of X is stabilized by some element of Γ, so $\text{dias}(\Gamma, X) = 0$. The only non-trivial fact is to show that Γ is discrete, which is equivalent to say that the stabilizer of e is finite. Let $\mathbb{S}^1 * \mathbb{S}^1$ be the quotient of X with respect to the action of $F(a,b)$. Every element of Γ defines an isometry of $\mathbb{S}^1 * \mathbb{S}^1$ fixing the gluing point. By the standard cover theory each isometry of $\mathbb{S}^1 * \mathbb{S}^1$ defines a unique isometry $\tilde{f}$ of X such that $\tilde{f}(e) = e$. So there exists an injective map from $\text{Stab}_\Gamma(e)$ to $\text{Isom}(\mathbb{S}^1 * \mathbb{S}^1)$. The latter group is finite, so also $\text{Stab}_\Gamma(e)$ is finite.

In this case the decomposition given by Theorem A contains only elliptic elements and this is why the slim assumption plays a role. The next example shows instead that the decomposition given by Theorem A is much stronger than a decomposition as union of minimal sets of *some* isometries.

Example 1.5. Let Y be a flat torus with a loop of length ℓ glued at some point. Let X be its universal cover, which is a geodesically complete, CAT(0) space which is (P_0, r_0)-packed for some P_0, r_0. It appears as a tree of $\mathbb{R}^2$. Let Γ be the fundamental group of Y. It is clear that X can be written as the union of the minimal sets of the non-trivial isometries of Γ, which is slim because it is torsion-free. However the conclusions of Theorem C do not hold for X, showing that it cannot be decomposed as in Theorem A.

We would also bring to the attention of the reader a general observation. When Γ is a discrete, torsionless group of isometries of our CAT(0)-space X, then the quotient space $M = \Gamma \backslash X$ is a locally CAT(0)-space which is endowed with a natural measure μ_M (defined, locally, as explained in Section 2.1). If $\text{dias}(\Gamma, X) \geq \eta_0$ then there is a ball of M of radius $\frac{\eta_0}{2}$ which is isometric to a ball of X and in particular (using the volume estimates recalled in Section 2.3) we get

$$\mu_M(M) \geq v_0 = v_0(P_0, r_0) > 0.$$

Therefore, we can read all the above results at level of quotient spaces as a "small volume versus rigidity" alternative. By the way of example, we just express Theorem C(ii) in this terms.

Corollary E (Thm. C(ii), revisited)**.** *Let M be a compact, locally geodesically complete, locally* CAT(0)*-space whose universal cover is (P_0, r_0)-packed. Suppose there exists an (arbitrarily small) open set $U \subset M$ which is* CAT$(-\varepsilon)$ *for some $\varepsilon > 0$. If $\mu_M(M) < v_0$ then M, up to renormalization, is isometric to $\mathbb{S}^1$.*

2. Preliminaries

The open (resp. closed) ball of center x and radius r in a metric space X will be denoted by $B(x,r)$ (resp. $\overline{B}(x,r)$). A geodesic in a metric space X is an isometric embedding $c\colon [a,b] \to X$, where $[a,b]$ is an interval of $\mathbb{R}$. The endpoints of the geodesic c are the points $c(a)$ and $c(b)$; a geodesic with endpoints $x, y \in X$ is also denoted by $[x,y]$. A geodesic ray is an isometric embedding $c\colon [0,+\infty) \to X$ and a geodesic line is an isometric embedding $c\colon \mathbb{R} \to X$. A metric space X is called *geodesic* if for every two points $x, y \in X$ there is a geodesic with endpoints x and y.

2.1. **Geodesically complete CAT(0)-spaces.**

A metric space X is CAT(0) if it is geodesic and every geodesic triangle $\Delta(x,y,z)$ is thinner than its Euclidean comparison triangle $\overline{\Delta}(\bar{x},\bar{y},\bar{z})$, that is for every pair of points $p \in [x,y]$ and $q \in [x,z]$ we have $d(p,q) \leq d(\bar{p},\bar{q})$ where $\bar{p},\bar{q}$ are the comparison points of p,q in $\overline{\Delta}(\bar{x},\bar{y},\bar{z})$ (see for instance [BH99] for the basics of CAT(0)-geometry).
As a consequence of the definition every CAT(0)-space is uniquely geodesic: for every two points x, y there exists a unique geodesic with endpoints x and y. A CAT(0)-metric space is *geodesically complete* if every geodesic segment $c\colon [a,b] \to X$ can be extended to a geodesic line. For instance a complete CAT(0)-homology manifold is always geodesically complete ([BH99, Proposition II.5.12]).
A subset D of a complete, geodesically complete CAT(0)-space X is *convex* if for every two points $x, y \in D$ the geodesic $[x,y]$ is contained in D.
The space of directions $\Sigma_x X$ at a point $x \in X$ is the set of geodesic segments starting from x modulo the equivalence relation $c \sim c'$ if and only if $\angle_x(c,c') = 0$. It is a CAT(1) metric space when endowed with the distance $([c],[c']) \mapsto \angle_x(c,c')$. Given $x, y \in X$ we denote by $[y]_x$ the element of $\Sigma_x X$ defined by the geodesic segment $[x,y]$.
For a CAT(0) space we denote by GD(X) the *geometric dimension* of X as defined in [Kle99]: by definition, it is the smallest function assigning to each CAT(1)-space X a number GD(X) such that GD$(X) = 0$ if and only if X is discrete and

$$\mathrm{GD}(X) \geq 1 + \sup_{x \in X} \mathrm{GD}(\Sigma_x X).$$

By [Kle99, Theorem A] we also have

$$\mathrm{GD}(X) = \sup\{k : \exists \text{ an open set } U \subseteq \mathbb{R}^k \text{ and a } \tfrac{1}{2}\text{-biHölder embedding } \iota : U \hookrightarrow X\} \tag{2}$$

$$\mathrm{GD}(X) = \sup\{\mathrm{TD}(K) : K \subseteq X \text{ is compact}\} \tag{3}$$

where TD stands for the topological dimension. In particular if a CAT(0)-space has geometric dimension 1 then all its spaces of directions are discrete. Moreover, for CAT(0)-spaces it holds:

Lemma 2.1. $\mathrm{GD}(X \times Y) = \mathrm{GD}(X) + \mathrm{GD}(Y)$.

Proof. Let us take a compact set $K \subseteq X \times Y$. Then $K \subseteq K_X \times K_Y$, where K_X (resp. K_Y) is the projection of K on X (resp. on Y). The sets K_X and

K_Y are compact, so $\mathrm{TD}(K_X) \leq \mathrm{GD}(X)$ and $\mathrm{TD}(K_Y) \leq \mathrm{GD}(Y)$. By [Pea75, Proposition 3.1.5, Theorem 4.5.4 and Proposition 4.5.5] we get

$$\mathrm{TD}(K) \leq \mathrm{TD}(K_X \times K_Y) \leq \mathrm{TD}(K_X) + \mathrm{TD}(K_Y) \leq \mathrm{GD}(X) + \mathrm{GD}(Y).$$

By the arbitrariness of K we get $\mathrm{GD}(X \times Y) \leq \mathrm{GD}(X) + \mathrm{GD}(Y)$.
The monotonicity of the topological dimension in the form of [Pea75, Proposition 3.1.5] implies that if $\mathrm{GD}(X) = +\infty$ or $\mathrm{GD}(Y) = +\infty$ then also $\mathrm{GD}(X \times Y) = +\infty$. So, in order to show the other inequality, we can suppose that both $n := \mathrm{GD}(X)$ and $m := \mathrm{GD}(Y)$ are finite. Let $U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^m$ be open set admitting $\frac{1}{2}$-biHölder embeddings $\varphi\colon U \to X$, $\psi\colon V \to Y$. We claim that the map $(\varphi, \psi)\colon U \times V \subseteq \mathbb{R}^{n+m} \to X \times Y$ is a $\frac{1}{2}$-biHölder embedding. We have, by the definition of product metric,

$$\begin{aligned} d_{X\times Y}((\varphi(u), \psi(v)), (\varphi(u'), \psi(v'))) &= \sqrt{d_X(\varphi(u), \varphi(u'))^2 + d_Y(\psi(v), \psi(v'))^2} \\ &\leq \sqrt{C(\|u - u'\| + \|v - v'\|)} \\ &\leq C' \sqrt{d_{\mathbb{R}^{n+m}}((u, v), (u', v'))}, \end{aligned}$$

where the last step follows by the equivalence of the L^1-product metric and the L^2 one on $\mathbb{R}^n \times \mathbb{R}^m$. This shows that the map (φ, ψ) is $\frac{1}{2}$-Hölder. In the same way we can prove that $(\varphi^{-1}, \psi^{-1})$ is $\frac{1}{2}$-Hölder as well. Therefore we deduce by (2) that $\mathrm{GD}(X \times Y) \geq \mathrm{GD}(X) + \mathrm{GD}(Y)$, concluding the proof. $\square$

In the following, we will always assume that X is a proper, geodesically complete CAT(0)*-space.* By [LN19] we know that every point $x \in X$ has a well defined integer dimension in the following sense: there exists $n_x \in \mathbb{N}$ such that every small enough ball around x has Hausdorff dimension equal to n_x. This defines a stratification of X into pieces of different integer dimensions. Namely, if X^k denotes the subset of points of X with dimension k, then

$$X = \bigcup_{k \in \mathbb{N}} X^k.$$

The set X^k contains a dense subset M_k, which is open in X and locally bi-Lipschitz homeomorphic to $\mathbb{R}^k$ ([LN19, Theorem 1.2]). In particular if $X^k \neq \emptyset$ then there exists some ball in X which is a k-dimensional Lipschitz manifold. The *dimension* of X is the supremum of the dimensions of its points: it coincides with the *topological dimension* of X, cf. [LN19, Theorem 1.1], and with the geometric dimension of X. The formula

$$\mu_X := \sum_{k \in \mathbb{N}} \mathcal{H}^k \llcorner X^k$$

where $\mathcal{H}^k \llcorner X^k$ is the k-dimensional Hausdorff measure $\mathcal{H}^k$ restricted to the set X^k, defines a natural measure on X which is locally positive and locally finite.

2.2. **Isometries of CAT(0)-spaces.**

The group of isometries of X will be denoted by $\mathrm{Isom}(X)$. The translation length of $g \in \mathrm{Isom}(X)$ is by definition

$$\ell(g) := \inf_{x \in X} d(x, gx).$$

If the infimum is realized then g is called *elliptic* if $\ell(g) = 0$ and *hyperbolic* otherwise; g is called *parabolic* when the infimum is not realized.
It is well known that the function $d_g \colon X \to [0, +\infty)$, $x \mapsto d(x, gx)$ is convex. Its τ-level sets, for $\tau \geq 0$, are by definition the subsets

$$M_\tau(g) := \{x \in X \text{ s.t. } d(x, gx) \leq \tau\}$$

which are closed, convex subsets of X. We recall that the *minimum set of* g is defined as

$$\mathrm{Min}(g) := M_{\ell(g)}(g)$$

that is, $\mathrm{Min}(g)$ is the subset of points of X realizing the minimum in the definition of the translation length of g. Notice that if g is parabolic then $\mathrm{Min}(g) = \emptyset$, while if g is elliptic then $\mathrm{Min}(g)$ is the set of points fixed by g, in which case it will be denoted also by $\mathrm{Fix}(g)$.
It is known that if g in an hyperbolic isometry then $\mathrm{Min}(g)$ splits isometrically as a product $D(g) \times \mathbb{R}$, where $D(g)$ is a convex subset of X, and g acts on $D(g) \times \mathbb{R}$ respecting the product decomposition and acting as the identity on $D(g)$ and as a translation of length $\ell(g)$ on $\mathbb{R}$, see [BH99, Theorem II.6.8].
When dealing with torsion there is a class of elliptic isometries which deserves a name. We say that an elliptic isometry g of X is *slim* if $\mathrm{Fix}(g)$ has empty interior. A group of isometries Γ is said to be *slim* if every non-trivial elliptic isometry of Γ is slim. For instance every torsion-free group is trivially slim, as well as any discrete group of isometries of a CAT(0)-homology manifold.

Lemma 2.2. *Let X be a proper* CAT(0)*-space which is a homology manifold, and let Γ be a discrete group of isometries of X. Then every elliptic isometry of Γ is slim.*

Proof. Let n be the dimension of X as homology manifold. Let us denote by E the set of non-manifold points of X, i.e. those points which do not have a neighbourhood homeomorphic to an open subset of $\mathbb{R}^n$. By [LN22, Theorem 1.2] the set E is locally finite. Moreover E is preserved by the isometry group of X. We can assume also that $n \geq 3$, otherwise X is already a topological manifold. In this case the topological manifold $X \setminus E$ is connected. Let g be an elliptic isometry of Γ. Since Γ is discrete, the order of g is finite. After taking a power of g, whose fixed point set contains the fixed point set of g, we can suppose g has prime order. By Newman's second theorem [New31] if g fixes an open subset of $X \setminus E$ then it is the identity. This shows that $\mathrm{Fix}(g) \cap (X \setminus E)$ has empty interior. Also E has empty interior, so $\mathrm{Fix}(g)$ has empty interior. □

Let now Γ be a subgroup of $\mathrm{Isom}(X)$. For $x \in X$ and $\eta \geq 0$ we set

$$\Sigma_\eta(x) := \{g \in \Gamma \text{ s.t. } d(x, gx) < \eta\}$$

$$\overline{\Sigma}_\eta(x) := \{g \in \Gamma \text{ s.t. } d(x, gx) \leq \eta\}$$

and we define $\Gamma_\eta(x) := \langle \Sigma_\eta(x) \rangle$, $\overline{\Gamma}_\eta(x) = \langle \overline{\Sigma}_\eta(x) \rangle$. For instance when $\eta = 0$ we have $\overline{\Sigma}_0(x) = \overline{\Gamma}_0(x) = \mathrm{Stab}_\Gamma(x)$, the stabilizer of x. The subgroup Γ is discrete if for all $x \in X$ and all $\eta \geq 0$ the set $\Sigma_\eta(x)$ is finite.
The *systole* of Γ *at a point* $x \in X$ is

$$\mathrm{sys}(\Gamma, x) := \inf_{g \in \Gamma^*} d(x, gx)$$

where $\Gamma^* = \Gamma \setminus \{\mathrm{id}\}$. Finally, the central notion of this paper is the *diastole* of Γ, that is the quantity:

$$\mathrm{dias}(\Gamma, X) = \sup_{x\in X} \mathrm{sys}(\Gamma, x).$$

We end this section with some facts about finite sets of isometries we will need later. The first one is a well-known fact about commuting isometries.

Proposition 2.3 ([Gel11, Lemma 2.7]). *Let X be a complete* CAT(0)*-space and let $g_1, \ldots, g_n$ be commuting isometries of X. Let $\tau_1, \ldots, \tau_n \geq 0$ such that $M_{\tau_i}(g_i) \neq \emptyset$. Then $\bigcap_{i=1}^n M_{\tau_i}(g_i) \neq \emptyset$.*

The second one is a similar statement for elliptic isometries:

Proposition 2.4 ([BH99, Corollary II.2.8]). *Let X be a complete* CAT(0)*-space and let F be a finite group of isometries of X. Then $\bigcap_{g\in F} \mathrm{Fix}(g) \neq \emptyset$.*

2.3. **Packing conditions on CAT(0)-spaces.**
Let $P_0, r_0 > 0$: we say X is P_0*-packed at scale* r_0 (or (P_0, r_0)-packed) if for every $x \in X$ the cardinality of every r_0-separated subset of $\overline{B}(x, 3r_0)$ is at most P_0 (recall that a subset $Y \subseteq X$ is r_0*-separated* subset if $d(y, y') > r_0$ for all $y, y' \in Y$). We will simply say that X *is packed* if it is (P_0, r_0)-packed for some $P_0, r_0 > 0$. If X is a complete, geodesically complete, (P_0, r_0)-packed, CAT(0)-space then the dimension of X is at most $\frac{P_0}{2}$, and for every $R \geq 0$ and for every $x \in X$ it holds:

$$v(R) \leq \mu_X(B(x, R)) \leq V(R), \tag{4}$$

where $v(R)$ and $V(R)$ are functions only depending on the packing constants P_0, r_0, as showed in [CS21]. Another remarkable consequence of a packing condition at some fixed scale is the following version of the Margulis' Lemma due to Breuillard-Green-Tao, that for geodesically complete CAT(0)-spaces can be written as follows.

Theorem 2.5 ([BGT11, Corollary 11.17 and Corollary 11.2]). *Let X be a complete, geodesically complete, (P_0, r_0)-packed,* CAT(0)*-space and Γ be a discrete group of isometries of X. There exist $\varepsilon_0 = \varepsilon_0(P_0, r_0) > 0$, $I_0 = I_0(P_0, r_0) \geq 0$ and $S_0 = S_0(P_0, r_0) \geq 0$ such that the following holds true: for every $x \in X$ and every positive $\eta \leq \varepsilon_0$ there exist normal subgroups Δ_x, Λ_x of $\overline{\Gamma}_\eta(x)$ such that*

(i) *$[\overline{\Gamma}_\eta(x) : \Delta_x] \leq I_0$;*
(ii) *Λ_x is a finite subgroup of Δ_x;*
(iii) *Δ_x/Λ_x is nilpotent of step $\leq S_0$.*

In particular $\overline{\Gamma}_\eta(x)$ is virtually nilpotent. Moreover if Γ is cocompact then $\overline{\Gamma}_\eta(x)$ is virtually abelian (cp. [BH99, Theorem II.7.8]*).*

We call this $\varepsilon_0(P_0, r_0)$ the *Margulis constant*, since it plays the role of the classical Margulis constant in this metric setting. This constant and Theorem 2.5 will play a crucial role in our main theorem.
We end this section describing more carefully the structure of the groups $\overline{\Gamma}_\eta(x)$ appearing in Theorem 2.5.

Lemma 2.6 ([BGS13, Lemma 7.4 and Remark 7.2])**.** *Let X be a complete,* CAT(0)*-space and let N be a nilpotent group of isometries of X. If N has a semisimple isometry then there exists a semisimple isometry in the center of N. The set of semisimple isometries in the center of N forms an abelian normal subgroup of N.*

Proposition 2.7. *Let X be a complete,* CAT(0)*-space with* $\mathrm{GD}(X) < +\infty$. *Let Γ be a finitely generated, discrete, virtually nilpotent group of isometries of X. Then:*

- (i) *there exists a Γ-invariant convex subset $W \times \mathbb{R}^k$, $k \geq 0$, of X such that each $\gamma \in \Gamma$ acts as (γ', γ''), where γ' is not hyperbolic;*
- (ii) *a finite index, normal subgroup of Γ splits as $\Gamma_W \times \mathbb{Z}^k$, where Γ_W is a finitely generated, discrete, nilpotent group acting on W without hyperbolic isometries and $\mathbb{Z}^k$ acts as a lattice on $\mathbb{R}^k$;*
- (iii) *the set of semisimple isometries of $\Gamma_W \times \mathbb{Z}^k$ is a virtually abelian, normal subgroup of Γ. It coincides with the set of elements whose first component is elliptic;*
- (iv) *if Γ does not contain parabolic elements then it is virtually abelian.*

Proof. We show (i) and (ii) by induction on $n := \mathrm{GD}(X)$, where the case $n = 0$ is trivial. If Γ does not contain hyperbolic isometries it is enough to take $W = X$ and $k = 0$. Suppose Γ has a hyperbolic isometry and denote by N a nilpotent subgroup of finite index. Since N is finitely generated and nilpotent it is not restrictive to suppose N torsion-free and normal in Γ. Clearly also N has hyperbolic isometries, so there exists a hyperbolic isometry in the center of N by Lemma 2.6. We consider the set H of semisimple isometries in the center of N: it is an abelian subgroup of N by Lemma 2.6. Let $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that $\Gamma = \bigcup_{i=1}^n \gamma_i N$. Observe that the isometries of $\gamma_i H \gamma_i^{-1}$ are still semisimple and belong to the center of N. Indeed the center of N is characteristic in N, so it is normal in Γ. The isometries $\{\gamma_i H \gamma_i^{-1}\}_{i=1,\ldots,n}$ generate an abelian group which is clearly normal in Γ and has positive rank. Up to passing to a characteristic subgroup of this abelian group we can suppose to have an abelian, torsion-free, normal subgroup A of Γ. By the flat torus Theorem (cp. [BH99, Theorem II.7.1]) we know that there exists a convex subset $W' \times \mathbb{R}^j$ of X which is preserved by Γ, and moreover each $\gamma \in \Gamma$ acts as (γ', γ''). Furthermore Γ has a subgroup of finite index which splits as $\Gamma_{W'} \times A \cong \Gamma_{W'} \times \mathbb{Z}^j$. Let $p(\Gamma)$ be the projection of Γ on $\mathrm{Isom}(W')$. Clearly it is virtually nilpotent and finitely generated. We need to show it is discrete. It contains $\Gamma_{W'}$ as finite index subgroup and it is easy to see that $\Gamma_{W'}$ is discrete, so also $p(\Gamma)$ is discrete. Finally observe that since the rank of A is strictly positive we have $\mathrm{GD}(W') < \mathrm{GD}(X)$ by Lemma 2.1, so we can apply induction on $p(\Gamma)$. There exists a convex subset $W \times \mathbb{R}^k$ which is $p(\Gamma)$-invariant and each isometry of $p(\Gamma)$ preserves the product decomposition, i.e. each $\gamma \in p(\Gamma)$ decomposes as (γ', γ'') with γ' not hyperbolic. This means that the set $W \times \mathbb{R}^k \times \mathbb{R}^j$ does the job for Γ. Moreover by induction we can also assume that a finite index subgroup of $\Gamma_{W'}$ splits as $\Gamma_W \times \mathbb{Z}^k$, so a finite index subgroup of Γ splits as $\Gamma_W \times \mathbb{Z}^{k+j}$. This concludes the proof of (i) and (ii).

We now move to the proof of (iii). An isometry (γ', γ'') of $\Gamma_W \times \mathbb{Z}^k$ is semisimple if and only if both γ' and γ'' are semisimple ([BH99, Proposition II.6.9]). Since γ'' is always semisimple because it is an isometry of $\mathbb{R}^k$, we conclude that $(\gamma', \gamma'') \in \Gamma_W \times \mathbb{Z}^k$ is semisimple if and only if γ' has finite order. Since the finite order elements of a nilpotent group form a subgroup, we see that the set of semisimple isometries of $\Gamma_W \times \mathbb{Z}^k$ are a subgroup. Let (γ', γ'') be a semisimple isometry of $\Gamma_W \times \mathbb{Z}^k$ and (g', g'') be an element of Γ. Then

$$(g', g'')(\gamma', \gamma'')(g'^{-1}, g''^{-1}) = (g'\gamma' g'^{-1}, g''\gamma'' g''^{-1}).$$

Since $\Gamma_W \times \mathbb{Z}^k$ is normal in Γ we know that $(g'\gamma' g'^{-1}, g''\gamma'' g''^{-1})$ still belongs to $\Gamma_W \times \mathbb{Z}^k$. Moreover $g'\gamma' g'^{-1}$ is still of finite order: this shows that the subgroup of semisimple isometries of $\Gamma_W \times \mathbb{Z}^k$ is normal in Γ. Clearly the quotient of this group by the torsion group of Γ_W, which is finite, is abelian, so this group is virtually abelian.

Finally (iv) follows immediately: indeed if Γ does not have parabolic elements then also $\Gamma_W \times \mathbb{Z}^k$ does not have parabolic elements. This means that Γ_W is finite, so $\Gamma_W \times \mathbb{Z}^k$ is virtually abelian. Therefore also Γ is virtually abelian. □

2.4. **Almost cocompact groups of CAT(0)-spaces.**

A group of isometries Γ of a complete, geodesically complete CAT(0)-space X is called *cocompact* if there exists a compact subset $K \subset X$ such that $X = \bigcup_{g\in\Gamma} gK$. The main theorem of the paper will be proved under a weaker assumption, which we call *almost-cocompactness*, which we now describe.

For $\delta > 0$ we define the *δ-thick subset* of X as

$$\text{Thick}_\delta(\Gamma, X) := \{x \in X \text{ s.t. } \text{sys}(\Gamma, x) \geq \delta\}.$$

This is a Γ-invariant subset of X; accordingly, the δ-thick subset of the quotient space $M = \Gamma\backslash X$ is defined as $\text{Thick}_\delta(M) := \Gamma\backslash\text{Thick}_\delta(\Gamma, X)$. We say that Γ is *almost-cocompact* if the action of Γ on $\text{Thick}_\delta(\Gamma, X)$ is cocompact (i.e. $\text{Thick}_\delta(M)$ is compact) for all $\delta > 0$. In [BGS13, §8], the same notion for manifolds was introduced with the notation $\text{InjRad}(\Gamma\backslash X) \to 0$.

It is clear that if Γ is cocompact then it is almost-cocompact. Moreover if the global systole of the action is strictly positive then almost-cocompactness coincides with cocompactness. Among almost-cocompact actions the most important ones are those with finite covolume:

Definition 2.8. Let Γ be a discrete subgroup of isometries of a complete, geodesically complete CAT(0)-space X. An open subset $\mathcal{D} \subset X$ is a *fundamental domain* for Γ if :

(i) $g\mathcal{D} \cap h\mathcal{D} = \emptyset$ for all $g, h \in \Gamma$, $g \neq h$;
(ii) $\bigcup_{g\in\Gamma} g\overline{\mathcal{D}} = X$.

The group Γ is said to have *finite covolume* if there exists a fundamental domain $\mathcal{D}$ for Γ with $\mu_X(\mathcal{D}) < +\infty$.

Lemma 2.9 (Compare with [BGS13, §8.4])**.** *Let X be a complete, geodesically complete, packed,* CAT(0) *space X and let Γ be a discrete group of isometries of X with finite covolume. Then Γ is almost-cocompact.*

Proof. Suppose $\Gamma\backslash\text{Thick}_\delta(\Gamma, X)$ is not compact for some $\delta > 0$. It is complete, so it is not totally bounded. This means we can find infinitely many disjoint balls of radius $0 < \varepsilon \leq \frac{\delta}{2}$ centered in points of $\Gamma\backslash\text{Thick}_\delta(\Gamma, X)$. By definition these balls are isometric to the corresponding balls in X, in particular their volume is bigger than a universal number $v > 0$ depending on ε and the packing constants of X, by (4). This contradicts the finite covolume assumption. □

Remark 2.10. It is easy to construct almost-cocompact actions with infinite covolume, as in [BGS13, §11.1]. Indeed, let $\varepsilon_i = 1/i$, let Σ_1 be a hyperbolic, genus 1 surface with one geodesic boundary component of lengh $\frac{1}{2}$, and for each $i \geq 2$ consider a hyperbolic surface Σ_i of genus 1 with two geodesic boundary components of lengths $\varepsilon_{i-1}, \varepsilon_{i+1}$. Then, take the product manifolds $M_i = \Sigma_i \times S^1_{\varepsilon_i}$, where $S^1_{\varepsilon_i}$ is a circle of length ε_i, and glue each M_i to M_{i+1} along the boundary tori $S^1_{\varepsilon_{i+1}} \times S^1_{\varepsilon_i} \cong S^1_{\varepsilon_i} \times S^1_{\varepsilon_{i+1}}$ by interchanging the circles. The universal cover of the resulting manifold M is packed since it has sectional curvature $-1 \leq k_M \leq 0$. Moreover M has infinite volume (as $\text{Vol}(M) = 2\pi + 4\pi \sum_{i\geq 2} \varepsilon_i = +\infty$), but for each δ the thick subset $\text{Thick}_\delta(M)$ is compact, as it is included in the union of finitely many M_i.

The existence of a fundamental domain is discussed in the following result.

Proposition 2.11. *Let X be a complete* CAT(0)*-space and let Γ be a discrete group of isometries. Then there exists a fundamental domain for Γ if and only if* $\text{dias}(\Gamma, X) > 0$*, i.e. if there exists $x \in X$ such that* $\text{Stab}_\Gamma(x) = \{\text{id}\}$*. In particular fundamental domains exist when Γ is slim, e.g. when it is torsion-free or X is a homology manifold by Lemma 2.2.*

Proof. Clearly $\text{dias}(\Gamma, X) > 0$ if and only if there exists $x \in X$ such that $\text{Stab}_\Gamma(x) = \{\text{id}\}$. If this is the case then the set

$$\mathcal{D} = \{y \in X \text{ s.t. } d(y, x) < d(gy, x) \text{ for all } g \in \Gamma\}$$

is a fundamental domain. Property (i) of Definition 2.8 is satisfied by definition, while property (ii) follows since X is a geodesic space. Indeed for every $y \in X$ we find $g \in \Gamma$ such that $d(y, gx) \leq d(y, hx)$ for every $h \in \Gamma$. If the inequality is strict for every $h \neq g$, then $y \in g\mathcal{D}$. Otherwise we take the geodesic $c = [gx, y]$ and we consider points $c(t)$ approaching y. We claim that $c(t)$ belongs to $g\mathcal{D}$ for all $t < d(gx, y)$, implying that $y \in g\overline{\mathcal{D}}$. Actually, let $h \neq g$: if $c(t), y$ and hx are along a geodesic then y is in between $c(t)$ and hx because $hx \neq gx$, hence $d(c(t), gx) < d(y, gx) < d(c(t), hx)$ clearly. On the other hand if $c(t), y$ and hx are not along a geodesic then

$$d(c(t), hx) > d(y, hx) - d(y, c(t)) \geq d(y, gx) - d(y, c(t)) = d(c(t), gx).$$

This shows that $c(t) \in g\mathcal{D}$. The assumption $\text{dias}(\Gamma, X) > 0$ is satisfied if Γ is slim: indeed by Baire theorem $\bigcup_{g \in \Gamma^*} \text{Fix}(g)$ has empty interior, because Γ is countable. So there exists a point whose stabilizer is trivial.
Viceversa let us suppose that $\text{Stab}_\Gamma(x) \neq \{\text{id}\}$ for every $x \in X$. If $\mathcal{D}$ is a subset of X and $x \in \mathcal{D}$ then there exists a non-trivial $g \in \Gamma$ such that $gx = x$, so $g\mathcal{D} \cap \mathcal{D} \neq \emptyset$. Therefore fundamental domains cannot exist. □

Example 1.4 shows that there exist cocompact, discrete groups of isometries Γ of complete, geodesically complete, CAT(0)-spaces X with $\text{dias}(\Gamma, X) = 0$.

We conclude with an important property of almost-cocompact, slim groups. A group of isometries Γ of a CAT(0)-space X is said to be *minimal* if there are no convex, closed, Γ-invariant proper subsets $C \subsetneq X$.

Proposition 2.12 (Compare with [BGS13, Proof of Lemma 2 at page 195])**.** *Let X be a proper, geodesically complete,* CAT(0)*-space. Let Γ be a discrete, slim, almost-cocompact group of isometries of X. Then Γ is minimal.*

Proof. Suppose we have a closed, convex, Γ-invariant subset $C \subsetneq X$. Let $x \in X \setminus C$ and $y \in C$ be its projection on C. Consider the geodesic ray $[y, x]$ and extend it to a geodesic ray c_x. By Γ-invariance of C we know that d_g is non-decreasing along c_x for all $g \in \Gamma$. Indeed it is a convex function on $[0, +\infty)$ and it attains its minimum at $t = 0$ since the projection map $\pi_C : X \to C$ is 1-Lipschitz ([BH99, Corollary 2.5]) and $\pi_C(c_x(t)) = y$, $\pi_C(gc_x(t)) = gy$ for every $t \geq 0$ by Γ-invariance of C. Suppose first there exist $\delta > 0$ and a time $t_0 \geq 0$ such that $d(c_x(t_0), gc_x(t_0)) \geq \delta > 0$ for all $g \in \Gamma^*$. The same holds for all $t \geq t_0$, so there exists a whole geodesic ray which is contained in $\text{Thick}_\delta(\Gamma, X)$. Clearly this ray does not project to a compact subset of $\Gamma \backslash X$, a contradiction to the almost-cocompactness assumption. In the remaining case we have that the ray c_x is entirely contained in $\bigcup_{g \in \Gamma^*} \text{Fix}(g)$, and this must happen for all possible choices of $x \in X \setminus C$. In other words $X \setminus C \subseteq \bigcup_{g \in \Gamma^*} \text{Fix}(g)$. Now, $X \setminus C$ is an open, non-empty subset of X while each $\text{Fix}(g)$ is a closed subset with empty interior. The union is countable because Γ is discrete, so by Baire Theorem $\bigcup_{g \in \Gamma^*} \text{Fix}(g)$ has empty interior, impossible. □

3. Thin actions by almost-cocompact groups

From now on we fix $P_0, r_0 > 0$ and we call ε_0, I_0, S_0 the constants given by Theorem 2.5. For $\lambda \geq 0$ we introduce the set

$$\text{Min}_\lambda(\Gamma, X) := \bigcup_{g \in \Gamma^* \,:\, \ell(g) \leq \lambda} \text{Min}(g).$$

In particular when $\lambda = 0$ we denote the set $\text{Min}_\lambda(\Gamma, X)$ by $\text{Fix}(\Gamma, X)$, which is the set of points that are fixed by some non-trivial (and necessarely elliptic) isometry. Observe that $\text{Min}_\lambda(\Gamma, X)$ contains $\text{Fix}(\Gamma, X)$ for every $\lambda \geq 0$.
We state here the version of the main theorem we will prove.

Theorem 3.1. *Let X be a complete, geodesically complete, (P_0, r_0)-packed,* CAT(0)*-space. Let Γ be a discrete, almost-cocompact group of isometries of X and let $0 < \lambda \leq \varepsilon_0$. Assume that*

$$\text{dias}(\Gamma, X) < \eta(P_0, r_0, \lambda) := \frac{\lambda}{(2I_0 - 1)(3 \cdot 2^{S_0 - 1} - 2)}. \tag{5}$$

Then $X = \text{Min}_\lambda(\Gamma, X)$.

Example 1.4 in which $\text{dias}(\Gamma, X) = 0$ shows that the elements of the decomposition can be all elliptic. Notice that if in the above decomposition $\text{Min}(g)$ intersects $\text{Min}(g')$ then $\{g, g'\}$ generate a virtually nilpotent group (virtually abelian, if Γ is cocompact), by Theorem 2.5.
To prove Theorem 3.1 we need a couple of preparatory results. The first one is a group-theoretic result for finite index subgroups:

Lemma 3.2 ([SW92, Lemma 3.4]). *Let G be a group generated by a finite set S. Let H be a subgroup of G with finite index I. Then H can be generated by a set S' whose elements have word length with respect to S at most $(2I-1)$.*

(We recall that, in the above setting, the word length with respect to S of an element $g \in G$ is the infimum of the natural numbers n such that g can be written as a product of n elements of S.)

Lemma 3.3. *For every $x \in X$ and $r > 0$ there exists an open set $U \ni x$ such that $\Sigma_\eta(y) \supseteq \Sigma_\eta(x)$ and $\overline{\Sigma}_\eta(y) \subseteq \overline{\Sigma}_\eta(x)$ for all $y \in U$.*

Proof. By discreteness $\Sigma_\eta(x) = \{g_1, \ldots, g_k\}$ and $d(x, g_i x) < \eta$ for all i. Hence there exists $\varepsilon > 0$ such that $d(x, g_i x) < \eta - 2\varepsilon$ for every $i = 1, \ldots, k$. So we have $\Sigma_\eta(y) \supseteq \Sigma_\eta(x)$ for every point $y \in B(x, \varepsilon)$.
In order to prove the second part we suppose the opposite: there is a sequence x_n converging to x such that $\overline{\Sigma}_\eta(x_n) \supset \overline{\Sigma}_\eta(x)$ for every n: in particular there is $g_n \in \Gamma$ such that $d(x_n, g_n x_n) \leq \eta$ but $d(x, g_n x) > \eta$. By discreteness the sets of possible g_n's is finite, so we can take a subsequence where g_n is constantly equal to a fixed $g \in \Gamma$. Therefore $d(x, gx) = \lim_{n \to +\infty} d(x_n, g x_n) \leq \eta$ by continuity, which is a contradiction. □

We are ready to prove Theorem 3.1. We present it in the cocompact case and we will see after the proof how to do the general case.

Proof of Theorem 3.1.
We suppose $X \neq \mathrm{Min}_\lambda(\Gamma, X)$. Our aim is to find a point $x \in X$ such that $\mathrm{sys}(\Gamma, x) \geq \eta = \eta(P_0, r_0, \lambda)$. Let $f \colon [0, +\infty) \to [0, +\infty)$ be a continuous, decreasing function such that $\lim_{t \to 0} f(t) = +\infty$ and $f(t) = 0$ for all $t \geq \eta$. We define the map $\Psi \colon X \setminus \mathrm{Min}_\lambda(\Gamma, X) \to [0, +\infty)$ by

$$\Psi(x) = \sum_{g \in \Sigma_\eta(x) \setminus \{\mathrm{id}\}} f(d(x, gx)) = \sum_{g \in \overline{\Sigma}_\eta(x) \setminus \{\mathrm{id}\}} f(d(x, gx)), \tag{6}$$

where the sum is equal to 0 if $\Sigma_\eta(x) = \{\mathrm{id}\}$. This map is well defined on every point x which does not belong to $\mathrm{Fix}(\Gamma, X)$, in particular it is well defined outside $\mathrm{Min}_\lambda(\Gamma, X)$. In order to prove the thesis it is enough to show that $\min \Psi = 0$. Indeed if a point $x \in X \setminus \mathrm{Min}_\lambda(\Gamma, X)$ satisfies $\Psi(x) = 0$ then $\mathrm{sys}(\Gamma, x) \geq \eta$.

Step 1: Ψ is continuous and Γ-invariant.
The Γ-invariance is obvious. Let us take a sequence x_n converging to x_∞. By Lemma 3.3 we have $\Sigma_\eta(x_n) \supseteq \Sigma_\eta(x_\infty)$ and $\overline{\Sigma}_\eta(x_n) \subseteq \overline{\Sigma}_\eta(x_\infty)$, for n big enough. Therefore, using the two expressions of Ψ in (6):

$$\begin{aligned}
\liminf_{n \to +\infty} \Psi(x_n) &= \liminf_{n \to +\infty} \sum_{g \in \Sigma_\eta(x_n) \setminus \{\mathrm{id}\}} f(d(x_n, g x_n)) \\
&\geq \liminf_{n \to +\infty} \sum_{g \in \Sigma_\eta(x_\infty) \setminus \{\mathrm{id}\}} f(d(x_n, g x_n)) \\
&= \sum_{g \in \Sigma_\eta(x_\infty) \setminus \{\mathrm{id}\}} f(d(x_\infty, g x_\infty)) = \Psi(x_\infty)
\end{aligned}$$

and

$$\begin{aligned}\limsup_{n\to+\infty}\Psi(x_n) &= \limsup_{n\to+\infty}\sum_{g\in\overline{\Sigma}_\eta(x_n)\setminus\{\mathrm{id}\}} f(d(x_n,gx_n))\\ &\leq \limsup_{n\to+\infty}\sum_{g\in\overline{\Sigma}_\eta(x_\infty)\setminus\{\mathrm{id}\}} f(d(x_n,gx_n))\\ &= \sum_{g\in\overline{\Sigma}_\eta(x_\infty)\setminus\{\mathrm{id}\}} f(d(x_\infty,gx_\infty)) = \Psi(x_\infty).\end{aligned}$$

Step 2: Ψ has a minimum.
Let us take a sequence $x_n \in X \setminus \mathrm{Min}_\lambda(\Gamma, X)$ such that $\Psi(x_n)\to\inf\Psi$. By cocompactness of the action we can suppose that x_n converges to x_∞ and, since Ψ is continuous, $\Psi(x_\infty)=\inf\Psi$.

In the next step we will find a good point realizing the minimum of Ψ. In order to do that we introduce the partial order $\preceq$ on $\Psi^{-1}(m)$, $m\in[0,+\infty)$, defined by $x\preceq y$ if and only if $\overline{\Sigma}_\eta(x)\subseteq\overline{\Sigma}_\eta(y)$.

Step 3: the mininum of Ψ is realized at a point x maximal for $\preceq$.
Set $m := \min\Psi$. We consider the set $\Psi^{-1}(m)$ with the partial order $\preceq$. The function $x\mapsto \#\overline{\Sigma}_\eta(x)$ is upper semicontinuous by Lemma 3.3 and Γ-invariant, so it has a maximum on X. Therefore it is enough to take a point x of $\Psi^{-1}(m)$ for which $\#\overline{\Sigma}_\eta(x)$ is maximal.

Our aim is to show that $m=0$, so we suppose $m>0$ and we look for a contradiction. Indeed we will find a point $y\in X\setminus\mathrm{Min}_\lambda(\Gamma,X)$ such that $\Psi(y)<m$. Since $m>0$ the set $\overline{\Sigma}_\eta(x)$ contains some non-trivial element. We write $\overline{\Sigma}_\eta(x)=\{\sigma_1,\ldots,\sigma_n\}$ and we consider the group $\overline{\Gamma}_\eta(x)=\langle\overline{\Sigma}_\eta(x)\rangle$.

Step 4: there exists a non-empty, closed, convex, $\overline{\Gamma}_\eta(x)$-invariant subset Y of X not containing x.
By Theorem 2.5 there are groups $\Lambda_x,\Delta_x \lhd \overline{\Gamma}_\eta(x)$ such that:

(i) $[\overline{\Gamma}_\eta(x):\Delta_x]\leq I_0$;
(ii) Λ_x is a finite subgroup of Δ_x;
(iii) Δ_x/Λ_x is nilpotent of step $\leq S_0$.

Here we divide the proof in two cases: (a) $\Lambda_x\neq\{\mathrm{id}\}$ and (b) $\Lambda_x=\{\mathrm{id}\}$ and hence Δ_x is nilpotent of step $\leq S_0$.
In case (a) we define the set $Y=\bigcap_{g\in\Lambda_x}\mathrm{Min}(g)$. It is closed, convex and non-empty by Proposition 2.4 because Λ_x is finite. Moreover it is $\overline{\Gamma}_\eta(x)$-invariant since Λ_x is normal in $\overline{\Gamma}_\eta(x)$. Finally $x\notin Y$ because $x\notin\mathrm{Fix}(\Gamma,X)$.
In case (b) we know that the group Δ_x is nilpotent of step $\leq S_0$ and it has index at most I_0 in $\overline{\Gamma}_\eta(x)$. We apply Lemma 3.2 to find a generating set Σ' of Δ_x such that each $\sigma'\in\Sigma'$ has length at most $(2I_0-1)$ in the alphabet $\{\sigma_1,\ldots,\sigma_n\}$. Since the lower central series of Δ_x has length $S\leq S_0$ we can find elements $\sigma'_1,\ldots,\sigma'_S\in\Sigma'$ such that $[\sigma'_1,[\sigma'_2,\cdots[\sigma'_{S-1},\sigma'_S]]]$ is not trivial and belongs to the center of Δ_x. We call g_0 such an element and we notice that its length in the alphabet $\{\sigma_1,\ldots,\sigma_n\}$ is at most $(2I_0-1)(3\cdot 2^{S_0-1}-2)$. We consider the finite set $\mathrm{Conj}_{\overline{\Gamma}_\eta(x)}(g_0)=\{hg_0h^{-1}\text{ s.t. } h\in\overline{\Gamma}_\eta(x)\}$. Since the center of Δ_x is characteristic in Δ_x and Δ_x is normal in $\overline{\Gamma}_\eta(x)$, then

the center of Δ_x is normal in $\overline{\Gamma}_\eta(x)$. Hence all the elements of $\mathrm{Conj}_{\overline{\Gamma}_\eta(x)}(g_0)$ are in the center of Δ_x and commute. The isometries of $\mathrm{Conj}_{\overline{\Gamma}_\eta(x)}(g_0)$ are of the same type of g_0. If g_0 is either hyperbolic or elliptic we take $Y = \bigcap_{h\in\overline{\Gamma}_\eta(x)} \mathrm{Min}(hg_0h^{-1})$. It is non-empty by Proposition 2.3, convex, closed and clearly $\overline{\Gamma}_\eta(x)$-invariant. Moreover by triangle inequality we have

$$d(x, g_0x) \leq (2I_0 - 1)(3 \cdot 2^{S_0-1} - 2) \cdot \eta = \lambda. \tag{7}$$

This shows that $\ell(g_0) \leq \lambda$, so by definition $x \notin \mathrm{Min}(g_0)$ and in particular $x \notin Y$. If g_0 is parabolic we take τ such that $\ell(g_0) < \tau < d(x, g_0x)$ and we consider the set $Y = \bigcap_{h\in\overline{\Gamma}_\eta(x)} M_\tau(hg_0h^{-1})$. It is again convex, closed, non-empty by Proposition 2.3 and $\overline{\Gamma}_\eta(x)$-invariant. Clearly $x \notin Y$ by definition of τ.

The denominator in the definition of η in (5) comes from (7). It remains the last step.

Step 5: the minimum m of Ψ is 0.
Let Y be the subset of the previous step. We consider the projection y of x on Y and we extend the geodesic segment $[y, x]$ to a geodesic ray c beyond x. Since Y is $\overline{\Gamma}_\eta(x)$-invariant then d_{σ_i} is a non-decreasing function along c for every $i = 1, \ldots, n$, by the same argument of Proposition 2.12.
We now claim that the functions d_{σ_i} are not all constant along c. Otherwise $\sigma_i c$ is parallel to c for every $i = 1, \ldots, n$. Therefore gc is parallel to c for all $g \in \overline{\Gamma}_\eta(x)$. In case (a), i.e. $\Lambda_x \neq \{\mathrm{id}\}$, we have the stronger conclusion $gc = c$ for all $g \in \Lambda_x$. In particular x is fixed by all the elements of Λ_x, which is in contradiction to the fact that $x \notin \mathrm{Min}_\lambda(\Gamma, X) \supseteq \mathrm{Fix}(\Gamma, X)$. In case (b), i.e. $\Lambda_x = \{\mathrm{id}\}$, the element g_0 defined in Step 4 cannot be parabolic: indeed, by our choice of τ in the definition of Y, we see that g_0c cannot be parallel to c. Then g_0 is hyperbolic and we have again a contradiction, since Y is contained in $\mathrm{Min}(g_0)$, and then also x should belong to $\mathrm{Min}(g_0)$ which is impossible because $\ell(g_0) \leq \lambda$ and $x \notin \mathrm{Min}_\lambda(\Gamma, X)$.
Therefore the functions d_{σ_i} are convex, non-decreasing along c and at least one of them, say d_{σ_1}, is non-constant, hence strictly increasing when restricted to some interval $[t_1, +\infty)$. Thus the quantity

$$\bar{t} = \max\{t \in [0, +\infty) \text{ s.t. } d_{\sigma_i}(c(t)) \leq d_{\sigma_i}(x)\ \forall i = 1, \ldots, n\} \leq t_1$$

is finite. We denote by $\bar{x}$ the point $c(\bar{t})$. We claim that $\overline{\Sigma}_\eta(\bar{x}) = \overline{\Sigma}_\eta(x)$ and consequently $\Psi(\bar{x}) = \Psi(x) = m$. Let us define the set $A = \{y \in [x, \bar{x}] \text{ s.t. } \overline{\Sigma}_\eta(y) = \overline{\Sigma}_\eta(x)\}$. By Lemma 3.3 and by definition of $\bar{t}$ we conclude that A is open. Indeed for y' in a open neighbourhood U of y we have $\overline{\Sigma}_\eta(y') \subseteq \overline{\Sigma}_\eta(y) = \overline{\Sigma}_\eta(x)$. By definition of $\bar{t}$ we know that in $c \cap U$, which is an open set of $[x, \bar{x}]$ the other inclusion holds. Furthermore A is closed: indeed suppose to have a sequence of points $y_n \in A$ converging to y. Therefore $\overline{\Sigma}_\eta(y) \supseteq \overline{\Sigma}_\eta(y_n) = \overline{\Sigma}_\eta(x)$ for n big enough, again by Lemma 3.3. But this containement is actually an equality, by the maximality of x with respect to $\preceq$ implying $y \in A$. Indeed observe that by definition $\Psi(y_n) = \Psi(x)$ for every n and by continuity of Ψ the same holds for y. Now we can use the maximality of x with respect to $\preceq$ to conclude the equality. So $A = [x, \bar{x}]$, i.e. $\overline{\Sigma}_\eta(\bar{x}) = \{\sigma_1, \ldots, \sigma_n\}$.

By Lemma 3.3 it holds $\overline{\Sigma}_\eta(\bar{x}) \supseteq \overline{\Sigma}_\eta(c(\bar{t}+t))$ for every $t > 0$ small enough. By definition of $\bar{x}$ for such t's there is at least one index $i \in \{1, \ldots, n\}$ such that $d(\sigma_i c(\bar{t}+t), c(\bar{t}+t)) > d(\sigma_i \bar{x}, \bar{x})$. So

$$\Psi(c(\bar{t}+t)) \leq \sum_{i=1}^{n} f(d(\sigma_i c(\bar{t}+t), c(\bar{t}+t))) < \sum_{i=1}^{n} f(d(\sigma_i \bar{x}, \bar{x})) = \Psi(\bar{x}) = m.$$

Observe that the sum is made among the same set of isometries because of the inclusion $\overline{\Sigma}_\eta(\bar{x}) \supseteq \overline{\Sigma}_\eta(c(\bar{t}+t))$. This contradicts the fact that m is the minimum, concluding the proof. □

Remark 3.4. *The proof of Theorem 3.1 in the almost-cocompact case.*
We gave the proof in the cocompact case to avoid obscuring the main arguments with other technical details. However, the *same* proof works for almost cocompact actions (and in particular for finite volume ones): the only steps where we used cocompactness are Step 2 and Step 3. Step 2 can be replaced in the almost-cocompact case by the following argument:
– take again a sequence x_n such that $\Psi(x_n) \to \inf \Psi$. The sequence $\Psi(x_n)$ is bounded, therefore there exists $0 < \delta < \eta$ such that $d(x_n, gx_n) \geq \delta$ for every $g \in \Sigma_\eta(x_n)$ and for every n;
– this means that $\operatorname{sys}(\Gamma, x_n) \geq \delta$ for every n, so all the x_n's are contained in the δ-thick part $\operatorname{Thick}_\delta(\Gamma, X)$ of X, which is cocompact by almost cocompactness;
– then, we can find a subsequence, called again x_n, that converges to x_∞ and, since Ψ is continuous, $\Psi(x_\infty) = \inf \Psi$.
Step 3 can be replaced by a similar argument: the map $x \mapsto \overline{\Sigma}_\eta(x)$ is still upper semi-continuous and Γ-invariant. The set $\Psi^{-1}(m)$ is contained in a thick part of X, as said above. By almost cocompactness there exists L such that $\overline{\Sigma}_\eta(x) \leq L$ for all $x \in \Psi^{-1}(m)$. The rest of the proof does not change.

Remark 3.5. An interesting, though simple, topological fact is that when $X = \operatorname{Min}_\lambda(\Gamma, X)$, since this family of closed subsets is locally finite then we can take the union only among those which have non-empty interior, as already observed in [CCR01, Proposition 0.2]. This is of particular interest for *slim* actions of groups with torsion, for instance for actions on homology manifolds.

4. Splitting and rigidity of CAT(0) spaces under thin, slim actions

Our aim is to find several rigidity results for spaces with small diastole. Let $I_0(P_0, r_0)$ be the constant given by Theorem 2.5 and let J_0 be the constant given by Bieberbach's Theorem in dimension 2: every discrete, cocompact group of isometries of $\mathbb{R}^2$ contains a lattice of index at most J_0. We set

$$\lambda_0 = \min\left\{\varepsilon_0, \frac{4r_0}{\max\{I_0, J_0\} \cdot (P_0+1)}\right\} \tag{8}$$

and we call $\eta_0 = \eta_0(P_0, r_0, \lambda_0) = \eta_0(P_0, r_0) > 0$ the associated constant given by Theorem 3.1. Let X be a complete, geodesically complete, (P_0, r_0)-packed CAT(0)-space and let Γ be a discrete, almost-cocompact group of isometries of X. We denote by Λ the set of hyperbolic isometries g of Γ with $\ell(g) \leq \lambda_0$

and such that $\mathrm{Min}(g)$ has non-empty interior. Clearly $h\Lambda h^{-1} = \Lambda$ for all $h \in \Gamma$. Theorem 3.1 and Remark 3.5 implies that if the diastole is smaller than η_0 and the group is slim then $X = \bigcup_{g\in\Lambda} \mathrm{Min}(g)$.

We start studying spaces with points of dimension 1:

Proposition 4.1. *Let X be a complete, geodesically complete, (P_0, r_0)-packed, $\mathrm{CAT}(0)$-space and suppose that X has a point of dimension 1. Let Γ be a discrete, slim, almost-cocompact group of isometries of X. If $\mathrm{dias}(\Gamma, X) < \eta_0$ then X is isometric to $\mathbb{R}$.*

Proof. We have $X = \bigcup_{g\in\Lambda} \mathrm{Min}(g)$. Since X has points of dimension 1 then there is an open subset $U \subseteq X$ such that every point of U has dimension 1. In particular we can find a point $x \in X$ such that $\overline{B}(x, \varepsilon)$ is isometric to an interval for some $\varepsilon > 0$, by the discussion of Section 2.1. We take $g \in \Lambda$ such that $x \in \mathrm{Min}(g)$. We can identify x with a point $(y, 0) \in D(g) \times \mathbb{R}$. Since x is of dimension 1 we conclude that $(y, 0)$ is a point of dimension 1, and therefore that y is a point of dimension 0: an isolated point. Since $D(g)$ is convex, and in particular connected, we conclude that $D(g)$ is a point. Hence $\mathrm{Min}(g)$ is isometric to $\mathbb{R}$. Suppose $X \neq \mathrm{Min}(g)$. Then we can find a point x' at distance r_0 from $\mathrm{Min}(g)$, because geodesics are extendible. We consider the projection y of x' on $\mathrm{Min}(g)$. We focus on the segment $[y, gy]$. Clearly some element in the orbit of $\langle g\rangle x$ must lie inside this segment. Therefore we can find a point, called again x, such that $x \in [y, gy]$ and $\overline{B}(x, \varepsilon)$ is isometric to an interval. We claim that the geodesic $[x', gx']$ has length at least $2r_0$. Let $c\colon [0, d(x', gx')] \to X$ be the geodesic $[x', gx']$. For every $t \in [0, d(x', gx')]$ we consider the geodesic $[c(t), x]$. There are only three cases: $[c(t)]_x = v_-$, $[c(t)]_x = v_+$ or $c(t) = x$, where $v_\pm$ are the only two elements of $\Sigma_x X$. The function $t \mapsto [c(t)]_x$ is continuous on the set $\{t \in [0, d(x', gx')] \text{ s.t. } c(t) \neq x\}$ (cp. [CS21, Lemma 2.3]) and takes values in the discrete space $\{v_-, v_+\}$. If $c(t) \neq x$ for every $t \in [0, d(x', gx')]$ one concludes that either $[c(t)]_x = v_+$ for all $t \in [0, d(x', gx')]$ or $[c(t)]_x = v_-$ for all $t \in [0, d(x', gx')]$. We claim that both these two cases are impossible since $[x']_x \neq [gx']_x$. Indeed we consider the convex map $d(x', \cdot)$ on the geodesic $\mathrm{Min}(g)$ which has its minimum at y. This forces to have $[x']_x = [y]_x$. Indeed let $x_{\pm\varepsilon}$ be the points along $\mathrm{Min}(g)$ at distance ε from x in direction $v_\pm$ and suppose $[y]_x = v_-$. Then $d(x_{-\varepsilon}, x') < d(x_{+\varepsilon}, x')$. The geodesic $[x', x]$ is either $[x', x_{-\varepsilon}] \cup [x_{-\varepsilon}, x]$ or $[x', x_{+\varepsilon}] \cup [x_{+\varepsilon}, x]$, so the inequality above says it is the first one. In other words $[x']_x = [y]_x$. In the same way $[gx']_x = [gy]_x$ and clearly $[y]_x \neq [gy]_x$. Therefore there must be some $t \in [0, d(x', gx')]$ such that $c(t) = x$. This forces the length of c to be bigger than $2r_0$. Analogously we have that $d(g^n x', g^m x') > 2r_0$ for all distincts $n, m \in \mathbb{Z}$. Moreover $d(g^n x', y) \leq r_0 + |n|\ell(g) \leq r_0 + |n| \cdot \frac{4r_0}{P_0+1}$ for all $n \in \mathbb{Z}$. Therefore we can find $P_0 + 1$ distinct points of the form $g^n x'$ inside $\overline{B}(y, 3r_0)$. They form a $2r_0$-separated subset of $\overline{B}(y, 3r_0)$ and this is a contradiction to the packing assumption on X. □

Another application is a characterization of spaces with small diastole admitting a strictly negatively curved open subset. It resembles [CCR01, Corollary 0.7].

Corollary 4.2. *Let X be a complete, geodesically complete, (P_0, r_0)-packed, $\mathrm{CAT}(0)$-space and suppose that there exists some (arbitrarily small) open set $U \subset X$ which is $\mathrm{CAT}(-\varepsilon)$, for some $\varepsilon > 0$. Let Γ be a discrete, slim, almost-cocompact group of isometries of X: if $\mathrm{dias}(\Gamma, X) < \eta_0$ then X is isometric to $\mathbb{R}$.*

Proof. Again we know that $X = \bigcup_{g\in\Lambda} \mathrm{Min}(g)$ and each of these minimal sets has non-empty interior. If there exists a point of dimension 1 then X is isometric to $\mathbb{R}$ by Proposition 4.1. Therefore we can suppose that every point of X has dimension ≥ 2. We then take a point $x \in U$ and an isometry $g \in \Lambda$, such that $x \in \mathrm{Min}(g) = D(g) \times \mathbb{R}$. We identify x with a point $(y, 0) \in D(g) \times \mathbb{R}$. Since every point of X has dimension ≥ 2 then $D(g)$ is not a single point, i.e. there is $y \neq y' \in D(g)$. By the quadrangle flat theorem ([BH99, Theorem II.2.11]) we have that the non-degenerate quadrangle $(y, 0), g(y, 0), g(y', 0), (y', 0)$ is isometric to a rectangle in $\mathbb{R}^2$. It is then possible to find inside U a non-degenerate quadrangle isometric to a euclidean rectangle. This contradicts the assumption on U. □

For the next applications we need the following splitting criterion.

Lemma 4.3. *Let X be a complete, geodesically complete, (P_0, r_0)-packed, $\mathrm{CAT}(0)$-space. Let Γ be a discrete, slim, almost-cocompact group of isometries of X. Suppose that $X = \bigcup_{g\in\Lambda} \mathrm{Min}(g)$ and that if $g, h \in \Lambda$ are such that $\mathrm{Min}(g) \cap \mathrm{Min}(h) \neq \emptyset$ then $\langle g, h\rangle$ is virtually cyclic.*
Then X splits isometrically as $Y \times \mathbb{R}$. Moreover if Γ is finitely generated then Γ virtually splits as $\Gamma_Y \times \mathbb{Z}$, where Γ_Y (resp. $\mathbb{Z}$) acts discretely on Y (resp. $\mathbb{R}$).

The first part of the proof should be compared to [BGS13, Appendix 2, Lemma 1 and Lemma 2].

Proof. First, if K is a compact subset of X then the set

$$\Lambda_K := \{g \in \Lambda \text{ s.t. } \mathrm{Min}(g) \cap K \neq \emptyset\}$$

is finite. Indeed if x is a point of K then $d(x, gx) \leq 2\mathrm{Diam}(K) + \lambda_0$ for all $g \in \Lambda_K$ and the claim follows by discreteness of Γ.
Secondly, if $K \subseteq X$ is compact, connected and $\Lambda_K = \{g_1, \ldots, g_n\}$, we claim that for every $i \in \{1, \ldots, n\}$ there exists a sequence $f_1 = g_1, f_2, \ldots, f_\ell = g_i$ such that $f_j \in \Lambda_K$ and $\mathrm{Min}(f_j) \cap \mathrm{Min}(f_{j+1}) \neq \emptyset$ for every j. In order to prove this we define the intersection graph $\mathcal{G}$ of these subsets, i.e. the graph with vertices $\{1, \ldots, n\}$ and with an edge between the vertices i, j if and only if $\mathrm{Min}(g_i) \cap \mathrm{Min}(g_j) \neq \emptyset$. For every connected component $A \subseteq \mathcal{G}$ the subset $M_A := \bigcup_{i\in A}(\mathrm{Min}(g_i) \cap K)$ is a closed subset of K. If we have two connected components A, B of $\mathcal{G}$ then $M_A \cap M_B = \emptyset$, so if the graph is not connected then we can write the connected set K as union of disjoint, non-empty, closed subsets. This implies that $\mathcal{G}$ is connected, hence the claim.
We now show that the isometries of Λ have all parallel axes. Let $g, h \in \Lambda$ and take $x \in \mathrm{Min}(g), y \in \mathrm{Min}(h)$. Based on the two observations above, setting $K = [x, y]$ we find a finite sequence $f_1 = g, f_2, \ldots, f_\ell = h$ of elements of Λ such that $\mathrm{Min}(f_j) \cap \mathrm{Min}(f_{j+1}) \neq \emptyset$ for all $j = 1, \ldots, n-1$. We prove that the axes of g and h are parallel by induction on ℓ. If $\ell = 0$ it is trivial. If $\ell = 1$,

since $\langle g, h\rangle$ is virtually cyclic, there exist $p, q \in \mathbb{Z}^*$ such that $g^p = h^q$ and the claim follows. The induction step is similar: the axes of g are parallel to the axes of $f_{\ell-1}$ and there exist $p, q \in \mathbb{Z}^*$ such that $f_{\ell-1}^p = h^q$.
Let now c be one axis of some $g \in \Lambda$ and consider the subset P_c of X of all geodesic lines parallel to c. It is closed, convex and Γ-invariant. Indeed if $\gamma \in \Gamma$ then γc is an axis of $\gamma g \gamma^{-1} \in \Lambda$, so γc is parallel to c. By minimality of the action, Proposition 2.12, we conclude that $X = P_c$ and P_c canonically splits isometrically as $Y \times \mathbb{R}$, cp. [BH99, Theorem II.2.14].
For the finitely generated case we need to show a preliminary fact: for every compact, connected $K \subseteq X$ there exists a hyperbolic isometry $\gamma \in \Gamma^*$ such that $K \subseteq \mathrm{Min}(\gamma)$ and the axes of γ are parallel to c. Indeed, we know that Λ_K is finite, say $\Lambda_K = \{g_1, \ldots, g_n\}$, so $K \subseteq \bigcup_{i=1}^n \mathrm{Min}(g_i)$. Again, for every $i \in \{1, \ldots, n\}$ there exists a sequence $f_1 = g_1, \ldots, f_\ell = g_i$ such that $f_j \in \Lambda_K$ and $\mathrm{Min}(f_j) \cap \mathrm{Min}(f_{j+1}) \neq \emptyset$ for every j. By induction, as before, we can find q_i, $p_i \in \mathbb{Z}^*$ such that $g_1^{q_i} = g_i^{p_i}$ for all $i \in \{1, \ldots, n\}$. We define $\gamma = \prod_{i=1}^n g_1^{q_i}$. Then

$$K \subseteq \bigcup_{i=1}^n \mathrm{Min}(g_i) \subseteq \bigcup_{i=1}^n \mathrm{Min}(g_i^{p_i}) \subseteq \mathrm{Min}(\gamma),$$

and clearly the axes of γ are parallel to c.
Suppose Γ is generated by $\{h_1, \ldots, h_n\}$ and set $R := \max_{i=1,\ldots,n} d(x_0, h_i x_0)$ for some point $x_0 \in X$. We apply the claim above to $K = \overline{B}(x_0, 2R)$, finding a hyperbolic isometry $\gamma \in \Gamma^*$ with axes parallel to c and $\overline{B}(x_0, 2R) \subseteq \mathrm{Min}(\gamma)$. We recall that Γ sends geodesic lines parallel to c to geodesic lines parallel to c. In particular the axes of $h_i \gamma h_i^{-1}$ are parallel to the axes of γ. By definition of R and γ we observe that $B(h_i x_0, R) \subseteq \mathrm{Min}(\gamma) \cap \mathrm{Min}(h_i \gamma h_i^{-1})$. Therefore these two isometries have the same axis passing through every point of $B(h_i x_0, R)$ and they have the same translation length. This implies that $h_i \gamma h_i^{-1} = \gamma$ on $B(h_i x_0, R)$. Hence $h_i \gamma h_i^{-1} = \gamma$ because they coincide on an open subset. This shows that $\langle \gamma \rangle$ is normal, and even central, in Γ. So Γ preserves $\mathrm{Min}(\gamma) = X = Y \times \mathbb{R}$ because Γ is minimal by Proposition 2.12 and Γ virtually splits as $\Gamma_Y \times \mathbb{Z}$ by [BH99, Theorem II.7.1], with each factor acting discretely on the corresponding space. □

Proof of Corollary B. As usual we know that $X = \bigcup_{g \in \Lambda} \mathrm{Min}(g)$. For every $x \in X$ let $\overline{\Gamma}_{\lambda_0}(x)$ be the non-trivial group generated by $\overline{\Sigma}_{\lambda_0}(x)$. By Proposition 2.7 we can find a convex subset $W_x \times \mathbb{R}^{k_x}$ and a finite index subgroup $\Gamma_{W_x} \times \mathbb{Z}^{k_x}$ of $\overline{\Gamma}_{\lambda_0}(x)$, where Γ_{W_x} acts on W_x without hyperbolic elements and $\mathbb{Z}^{k_x}$ acts on $\mathbb{R}^{k_x}$ as a lattice. The existence of a hyperbolic isometry displacing x by at most λ_0 forces to have $k_x \geq 1$. If $k_x \geq 2$ we have the desired $\mathbb{Z} \times \mathbb{Z}$, so we can suppose $k_x = 1$. If Γ_{W_x} contains some parabolic element we have again $\mathbb{Z} \times \mathbb{Z}$ inside Γ. In the remaining case we have $k_x = 1$ and Γ_{W_x} finite, so $\overline{\Gamma}_{\lambda_0}(x)$ is virtually cyclic for every $x \in X$. The assumptions of Lemma 4.3 are clearly satisfied, so X splits as $Y \times \mathbb{R}$. If moreover Γ is cocompact (hence finitely generated) then it has a finite index subgroup which splits as $\Gamma_Y \times \mathbb{Z}$, and clearly Γ_Y acts cocompactly on Y. By [Swe99, Theorem 11] either Y is a point or Γ_Y contains an infinite order element, because Y is also geodesically complete. In the second case we have again $\mathbb{Z} \times \mathbb{Z}$ inside Γ, in the first one $X = \mathbb{R}$.

□

Remark 4.4. If in Corollary B we consider finitely generated almost-cocompact groups in place of cocompact ones we can still apply Lemma 4.3 to get a finite index subgroup of the form $\Gamma_Y \times \mathbb{Z}$. Moreover we could get the same conclusion of Corollary B under these weaker assumptions if we knew that Γ_Y contains an isometry of infinite order. This last statement is the content of the following conjecture.

Conjecture. *Let X be a proper,* CAT(0)*-space and let Γ be a discrete group of isometries of X, not necessarely cocompact. If Γ is torsion, i.e. it contains only finite order elements, then Γ is finite.*
An easier version for us would be: if Γ is almost-cocompact then it contains an infinite order element.

By the flat torus theorem ([BH99, Theorem I.7.1]), if Γ contains $\mathbb{Z} \times \mathbb{Z}$ then X contains an isometrically embedded flat $\mathbb{R}^2$. The same holds if X splits as $Y \times \mathbb{R}$, with Y different from a point. This is impossible for instance if X is a visibility space (see [BH99, Definition I.9.28]), therefore we obtain:

Corollary 4.5. *Let X be a complete, geodesically complete, (P_0, r_0)-packed,* CAT(0)*-space and assume that X is a visibility space. Let Γ be a discrete, slim, almost-cocompact group of isometries of X: if $\text{dias}(\Gamma, X) < \eta_0$ then X is isometric to $\mathbb{R}$.*

The last example of characterization of spaces with small diastole is the following, holding for spaces of dimension at most 2.

Proposition 4.6. *Let X be a complete, geodesically complete,* CAT(0)*-space which is P_0-packed at scale r_0, and assume that the dimension of X is at most 2. Let Γ be a discrete, slim, almost-cocompact group of isometries of X: if $\text{dias}(\Gamma, X) < \eta_0$ then X is isometric to a product $T \times \mathbb{R}$, where T is a geodesically complete, simplicial tree (possibily reduced to a point). Moreover if Γ is cocompact then it has a finite index subgroup which splits as $\Gamma_T \times \mathbb{Z}$, where Γ_T (resp. $\mathbb{Z}$) acts discretely and cocompactly on T (resp. $\mathbb{R}$).*

First of all we need to characterize trees as those CAT(0)-spaces with geometric dimension 1. We recall that a metric space Y is a real tree if for every two points $x, z \in Y$ the image of every continuous injective map $\alpha \colon [a, b] \to X$ such that $\alpha(a) = x$ and $\alpha(b) = z$ coincides with the image of $[x, z]$.

Lemma 4.7 (See also the proof of Lemma 6.3 of [OP22])**.**
Let Y be a CAT(0)*-space. The following are equivalent:*
(i) $\text{GD}(Y) = 1$*;*
(ii) *$\Sigma_y Y$ is discrete and not empty for every $y \in Y$;*
(iii) *Y is a non-trivial real tree.*

If Y satisfies (i)-(iii) and it is geodesically complete and proper then it is a simplicial tree.

Proof. If $\text{GD}(Y) = 1$ then $\Sigma_y Y$ is discrete for every $y \in Y$, as follows directly from the definition recalled in Section 2.1. If $\Sigma_y Y = \emptyset$ for some $y \in Y$ then there are no geodesics starting from y, that is y is an isolated point and

$Y = \{y\}$ (since Y is supposed geodesic), so $\mathrm{GD}(Y) = 0$. This shows that (i) implies (ii).
Now let us suppose (ii). Let $x, z \in Y$, $c = [x, z]$ and α be a continuous injective map $\alpha\colon [a, b] \to Y$ such that $\alpha(a) = x$ and $\alpha(b) = z$. We want to show that $\mathrm{Im}(\alpha) = \mathrm{Im}(c)$, where $c\colon [0, d(x, z)] \to Y$ is the geodesic $[x, z]$.
We first prove that $\mathrm{Im}(c) \subseteq \mathrm{Im}(\alpha)$. Let $s \in (0, d(x, z))$. The set $\Sigma_{c(s)}Y$ is discrete, say equal to $\{v_i\}_{i\in I}$. We consider the sets

$$A_i = \{t \in [a, b] \text{ s.t. } [\alpha(t)]_{c(s)} = v_i\}, \text{ for } i \in I.$$

Observe that the quantity $[\alpha(t)]_{c(s)}$ is defined as soon as $\alpha(t) \neq c(s)$. By discreteness of $\Sigma_{c(s)}Y$ it is easy to check that each A_i is open and closed, provided $c(s) \notin \mathrm{Im}(\alpha)$. Therefore by connectedness of $[a, b]$ there are two possibilities: either there exists $i \in I$ such that $A_i = [a, b]$ or there is some $t \in (a, b)$ such that $\alpha(t) = c(s)$. The first possibility has to be excluded since clearly $[\alpha(a)]_{c(s)} \neq [\alpha(b)]_{c(s)}$, so $c(s) \in \mathrm{Im}(\alpha)$.
We now prove the other inclusion $\mathrm{Im}(\alpha) \subseteq \mathrm{Im}(c)$. Set

$$A = \{t \in [a, b] \text{ s.t. } \alpha(t) \in \mathrm{Im}(c)\}.$$

It is a closed subset of $[a, b]$, so the complement is the disjoint union of countably many intervals, $A^c = \bigcup_{i=1}^{N}(a_i, b_i)$, $N \in \mathbb{N} \cup \{\infty\}$. In the interval (a_i, b_i) we have $\mathrm{Im}(\alpha|_{[a_i,b_i]}) \cap c = \{\alpha(a_i), \alpha(b_i)\}$. By injectivity of α we know that $\alpha(a_i) \neq \alpha(b_i)$, so the subsegment of c joining these two points is not trivial. However applying the first part of the proof to the continuous map $\alpha|_{[a_i,b_i]}$ we obtain $\mathrm{Im}(\alpha|_{[a_i,b_i]}) \cap c \supseteq [\alpha(a_i), \alpha(b_i)]$, a contradiction. This shows that $A = [a, b]$ and therefore that (ii) implies (iii).
Suppose now that Y is a non-trivial real tree. Every compact subset K of Y is a $\mathbb{R}$-tree as well as follows by [AB07, Theorem 1]. By [AB07, Theorem 2] we deduce that $\mathrm{TD}(K) = 1$. By using the characterization (3) of the geometric dimension recalled in Section 2.1, we deduce that $\mathrm{GD}(Y) = 1$.
Finally, if a real tree Y is proper and geodesically complete then it is simplicial. Indeed a real tree is simplicial if and only if the set of branching points S is closed and discrete (cp. [Chi01, Theorem 2.2.10]). Suppose y_n is a sequence of branching points of Y converging to y. Then each y_n is in the interior of at least three different geodesics that we can suppose of length 1 by geodesic completeness. This implies that also y is in the interior of at least three different geodesics, i.e. $y \in S$. In other words S is closed. Similarly one uses the geodesic completeness to show that S is discrete, otherwise there are infinitely many points at pairwise distance 1 inside a ball of radius 3 contradicting the properness of Y. □

The second tool we need is a better description of the almost stabilizers in dimension at most 2.

Lemma 4.8. *Let X be a complete, geodesically complete, (P_0, r_0)-packed, $\mathrm{CAT}(0)$-space and assume that the dimension of X is at most 2. Let Γ be a discrete group of isometries of X and let ε_0 be the Margulis constant given by Theorem 2.5. Then for every $x \in X$ one of the following three mutually exclusive possibilities occur:*

(i) *$\overline{\Gamma}_{\varepsilon_0}(x)$ has no hyperbolic isometries;*
(ii) *$\overline{\Gamma}_{\varepsilon_0}(x)$ has no parabolic isometries and it is virtually $\mathbb{Z}$;*

(iii) $\overline{\Gamma}_{\varepsilon_0}(x)$ *has no parabolic isometries and it has a subgroup of index at most* J_0 *which is isomorphic to* $\mathbb{Z}^2$.

Proof. As $\overline{\Gamma}_{\varepsilon_0}(x)$ is virtually nilpotent, by Proposition 2.7 we can find a $\overline{\Gamma}_{\varepsilon_0}(x)$-invariant convex subset $W \times \mathbb{R}^k$ of X and a finite index subgroup of $\overline{\Gamma}_{\varepsilon_0}(x)$ of the form $\Gamma_W \times \mathbb{Z}^k$, where Γ_W acts on W by parabolic or elliptic isometries. If there is at least one hyperbolic isometry in $\overline{\Gamma}_{\varepsilon_0}(x)$ then $k \geq 1$ and of course $k \leq 2$. We start with the case $k = 2$. By Lemma 2.1 W must have dimension 0, so it must be a point. This means that $\overline{\Gamma}_{\varepsilon_0}(x)$ acts on $\mathbb{R}^2$ cocompactly and discretely because it contains a lattice of $\mathbb{R}^2$ as finite index subgroup. By Bieberbach's Theorem we conclude that $\overline{\Gamma}_{\varepsilon_0}(x)$ has a subgroup of index at most J_0 which is isomorphic to $\mathbb{Z}^2$.
Now suppose $k = 1$. Then W is a CAT(0)-space of geometric dimension at most 1 by Lemma 2.1, so it is a real tree by Lemma 4.7 or a point. In any case W has no parabolic isometries (cp. [CDP90, Corollary 9.3.2]), so Γ_W contains only elliptic elements. As a consequence Γ_W is finite, so $\overline{\Gamma}_{\varepsilon_0}(x)$ has no parabolic isometries and it is virtually cyclic. □

Proof of Proposition 4.6. We know that $X = \bigcup_{g\in\Lambda} \mathrm{Min}(g)$. If X has a point of dimension 1 then $X = \mathbb{R}$ by Proposition 4.1. We can therefore suppose that every point of X is of dimension 2. The minimal set of $g \in \Lambda$ is of the form $\mathrm{Min}(g) = D(g) \times \mathbb{R}$. Since every point of X has dimension 2 and $\mathrm{Min}(g)$ has non-empty interior we deduce that $D(g)$ is not a point, and by Lemma 4.7 it is a (maybe compact) real tree since it has geometric dimension 1 by Lemma 2.1. If $g, h \in \Lambda$ are such that $\mathrm{Min}(g) \cap \mathrm{Min}(h) \neq \emptyset$ then either $\langle g, h\rangle$ is virtually cyclic or $\langle g^{J_0}, h^{J_0}\rangle \cong \mathbb{Z}^2$ by Lemma 4.8. Let us study the second case. Since g^{J_0} and h^{J_0} commute, the isometry h^{J_0} acts on $\mathrm{Min}(g^{J_0}) = D(g^{J_0}) \times \mathbb{R}$ preserving the product decomposition. Let h' be the isometry of $D(g^{J_0})$ induced by h^{J_0}. The isometry h' cannot have fixed points otherwise h^{J_0} and g^{J_0} have a common axis, and they would generate a virtually cyclic group. Therefore h' is a hyperbolic isometry of $D(g^{J_0})$ and we call c its axis. We want to show that $D(g^{J_0}) = c$. Suppose there exists a point $x \in D(g^{J_0})$ which is not in c. Let y be its projection on c. As $D(g^{J_0})$ is a tree, the geodesic in $D(g^{J_0})$, and so in X, between x and $h'x$ is the concatenation $[x, y] \cup [y, h'y] \cup [h'y, h'x]$. We extend the geodesic segment $[y, x]$ in X beyond x, defining a geodesic ray c_y. We do the same for the geodesic segment $[h'y, h'x]$ defining a geodesic ray $c_{h'y}$. We observe that the concatenation $c_y \cup [y, h'y] \cup c_{h'y}$ is a geodesic line in X. In particular the distance between the two points along c_y and $c_{h'y}$ at distance r_0 from y and $h'y$ is at least $2r_0$. We can repeat the construction for all points $(h')^n x$, extending the geodesic segments $[(h')^n y, (h')^n x]$ beyond $(h')^n x$ and finding points that are at distance at least $2r_0$ one from the other. Clearly $\ell(h') \leq \ell(h^{J_0}) \leq J_0 \cdot \lambda_0$, so arguing as in the proof of Proposition 4.1 we conclude we can find more than P_0 points inside $\overline{B}(y, 3r_0)$ that are r_0-separated as soon as $\lambda_0 \leq \frac{4r_0}{J_0(P_0+1)}$ which is the case by (8). This violates the packing assumption on X, so $D(g^{J_0}) = \mathbb{R}$ and $\mathrm{Min}(g^{J_0})$ must be isometric to $\mathbb{R}^2$. The same conclusion holds for $\mathrm{Min}(h^{J_0})$. By the flat torus theorem we know that

$$Y \times \mathbb{R}^2 = \mathrm{Min}(g^{J_0}) \cap \mathrm{Min}(h^{J_0}) \subseteq \mathrm{Min}(g^{J_0}) = \mathbb{R}^2$$

for some convex subset Y of X. Since the dimension of X is at most 2 the set Y must be a point by Lemma 2.1, and the factor $\mathbb{R}^2$ on the left is isometrically embedded into the $\mathbb{R}^2$ on the right, implying that the inclusion above is an equality. This shows that $\mathrm{Min}(g^{J_0}) = \mathrm{Min}(h^{J_0}) = \mathbb{R}^2$. The same argument shows that also $\mathrm{Min}(g^{pJ_0}) = \mathrm{Min}(g^{J_0}) = \mathbb{R}^2$ for all $p \neq 0$, because h^{J_0} acts on $\mathrm{Min}(g^{pJ_0})$. In the same way we have $\mathrm{Min}(h^{pJ_0}) = \mathrm{Min}(h^{J_0}) = \mathbb{R}^2$ for all $p \neq 0$.
We claim that $\mathrm{Min}(f) \subseteq \mathbb{R}^2$ for all $f \in \Lambda$. It is not difficult to see it by taking a path $f_1 = g, f_2, \ldots, f_\ell = f$ such that $f_j \in \Lambda$ and $\mathrm{Min}(f_j) \cap \mathrm{Min}(f_{j+1}) \neq \emptyset$ for all $i = 1, \ldots, \ell - 1$, as in the proof of Lemma 4.3. Therefore $X = \bigcup_{g\in\Lambda} \mathrm{Min}(g) = \mathbb{R}^2$ which is $\mathbb{R} \times \mathbb{R}$. If moreover Γ is cocompact then the conclusion is implied by Bieberbach's Theorem in dimension 2.
The remaining case is the one where $\langle g, h\rangle$ is virtually cyclic for all $g, h \in \Lambda$ such that $\mathrm{Min}(g) \cap \mathrm{Min}(h) \neq \emptyset$. An application of Lemma 4.3 gives the splitting of X as $T \times \mathbb{R}$. Here T is necessarely a simplicial tree because of Lemma 4.7 and Lemma 2.1. If Γ is cocompact then the conclusion follows from the second part of Lemma 4.3.

□

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