UNIQUENESS OF POSITIVE SOLUTIONS TO ELLIPTIC EQUATIONS WITH CRITICAL EXPONENTIAL GROWTH ON THE UNIT DISC AND ITS APPLICATIONS

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Abstract. In the past few decades, uniqueness of positive solutions to elliptic equations with polynomial growth has been extensively studied. However, the corresponding problems associated with elliptic equation with critical exponential growth given by the Trudinger-Moser inequalities still remains open. For this kind of equations, the classic non-degeneracy method based on the Pohozaev identity and the study of the linearized equation do not seem to work. In this paper, we will solve this uniqueness problem. More precisely, we obtain uniqueness of positive solutions to equations of the form

$$\begin{cases} -\Delta u = \lambda u e^{u^2}, & x \in B_1 \subset \mathbb{R}^2, \\ u > 0, & x \in B_1, \\ u = 0, & x \in \partial B_1, \end{cases}$$

where $0 < \lambda < \lambda_1(B_1)$ and $\lambda_1(B_1)$ denotes the first eigenvalue of the operator $-\Delta$ with Dirichlet boundary condition. This uniqueness result is given in Theorem 1.1. Our method relies on a delicate and difficult analysis of radial solutions to the above equation and a careful asymptotic expansion of solutions near the boundary. Furthermore, building on this uniqueness result, we develop a new strategy to establish a quantization property for elliptic equations with critical exponential growth in the balls of hyperbolic spaces, and obtain the multiplicity and non-existence of positive critical points for super-critical Trudinger-Moser functional. Our method for the quantization property and non-existence of the critical points avoids using the complicated blow-up analysis used in the literature. This method can also be applied to study the similar problems in balls of high dimensional Euclidean space $\mathbb{R}^n$ or hyperbolic spaces provided the uniqueness for the corresponding quasilinear elliptic equations with the critical exponential growth is established.

Keywords: Uniqueness; Critical points; Multiplicity; Trudinger-Moser growth; Quantization analysis.

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1. Introduction

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The main content of this paper is an uniqueness theorem for positive solutions to elliptic equations with Trudinger-Moser growth and application of this result to quantization analysis, multiplicity and non-existence of critical points of Trudinger-Moser functional in balls of Euclidean or hyperbolic spaces. Uniqueness problems and quantization analysis for elliptic equations have attracted much attention due to their importance in applications to PDEs and geometric analysis. Let us first present a brief history of the main results in this direction.

In the past few decades, much attention has been paid to uniqueness of solutions to elliptic equations with the nonlinearity f of polynomial growth, namely,

$$\begin{cases} -\Delta u = f(u), & x \in B_1, \\ u > 0, & x \in B_1, \\ u = 0, & x \in \partial B_1, \end{cases} \tag{1.1}$$

where B_1 is the unit ball in $\mathbb{R}^n$ $(n \geq 2)$. By the classical moving-plane method, one knows that every solution of problem (1.1) must be radially decreasing. Hence Problem (1.1) can be reduced to the following radial equation:

$$\begin{cases} -(r^{n-1}u')' = r^{n-1}f(u), & r \in (0,1), \\ u > 0, & r \in (0,1), \\ u'(0) = u(1) = 0. \end{cases} \tag{1.2}$$

Now, we recall some important results for the nonlinearity $f(u)$ of the form $f(u) = \lambda u + u^p$, with $1 < p < +\infty$ for $\lambda \geq 0$ and $n \geq 3$. When $\lambda = 0$ and $1 < p < \frac{n+2}{n-2}$, Gidas, Ni and Nirenberg [17] proved that problem (1.1) admits only one radial solution through homogeneity. (By the Pohozaev identity, problem (1.1) does not admit any solution for $\lambda = 0$ and $p \geq \frac{n+2}{n-2}$). When $\lambda > 0$ and $1 < p \leq \frac{n}{n-2}$, uniqueness of a positive solution was obtained by Ni and Nussbaum [30]. Kwong and Li [31] extended this uniqueness result to the case $\lambda > 0$ and $1 < p < \frac{n+2}{n-2}$, while uniqueness for the critical case ($p = \frac{n+2}{n-2}$, the Brezis-Nirenberg problem [6]) was proved by Srikanth [32]. In the aforementioned papers, the main idea to prove uniqueness result is to show that the corresponding linearized equation has only one zero point in $(0,1)$. This is the so-called non-degeneracy method. Subsequently, Adimurthi [3] provided an elementary proof for uniqueness when $\lambda \geq 0$ and $1 < p \leq \frac{n+2}{n-2}$ by exploiting a generalized Pohozaev variational identity. For $\lambda > 0$ and $p > \frac{n+2}{n-2}$, uniqueness cannot be expected to hold. Indeed, it has been shown by Budd and Norbury [7] that there exists $\lambda > 0$ such that probelm (1.2) has infinitely many solutions when $3 \leq n \leq 9$.

It should be noted that by the Sobolev imbedding theorem, critical growth means that the nonlinearity cannot exceed the polynomial of degree $\frac{n+2}{n-2}$ when $n \geq 3$. While in the case $n = 2$, we say that $f(s)$ has *critical exponential growth* at infinity if there exists

$\alpha_0 > 0$ such that

$$\lim_{t\to\infty} \frac{f(t)}{\exp(\alpha t^2)} = \begin{cases} 0, & \text{for } \alpha > \alpha_0 \\ +\infty, & \text{for } \alpha < \alpha_0 \end{cases}, \tag{1.3}$$

which is given by the famous Trudinger-Moser inequality ([28, 37])

$$\sup_{\|\nabla u\|_2^2 \le 1, u \in W_0^{1,2}(\Omega)} \int_\Omega \exp(\alpha |u|^2) dx < \infty, \text{iff } \alpha \le 4\pi, \tag{1.4}$$

where $\Omega \subseteq \mathbb{R}^2$ is a bounded domain and $W_0^{1,2}(\Omega)$ denotes the usual Sobolev space.

Thus, the maximal growth in the case $n = 2$ is of exponential type. A natural but nontrivial problem arises: Can the uniqueness result still hold if we replace the nonlinearity of equation (1.1) with exponential growth, and in particular for $f(t) = \lambda t e^{t^\mu}$, with $0 < \mu \le 2$ and $\lambda > 0$?

When $\mu = 1$, by using a new identity from the beautiful analysis developed by Atkinson and Peletier [1], Adimurthi [4] obtained uniqueness for the subcritical case $f(t) = te^t$. Tang [35] further showed that the uniqueness is still true for more general nonlinearity of the type $f(t) = \lambda g(t) e^t$, where $g(t)$ is a polynomial and satisfies certain conditions. However, this method cannot be extended for the case $\mu > 1$. Recently, under the assumption that $\|u\|_\infty$ is large enough, Adimurthi, Karthik A and Giacomoni [5] proved uniqueness of positive radial solution under suitable growth conditions on the nonlinearity. We remark that the nonlinearity in [5] includes the subcritical case $1 < \mu < 2$ and partially critical case such as $f(t) = t^p e^{t^2 + \beta t}$ with $\beta > 0$. However, their results do not include the standard critical case $\lambda t e^{t^2}$. We also mention that uniqueness of solutions to a nonlocal equation of the form $-\Delta u = \rho \frac{e^u}{\int_\Omega e^u}$ with Dirichlet boundary condition on bounded domains Ω in $\mathbb{R}^2$ was established in [9] and [33].

The first purpose of this paper is to solve the uniqueness problem for the elliptic equation with the standard critical exponential growth:

$$\begin{cases} -\Delta u = \lambda u e^{u^2}, & x \in B_1, \\ u > 0, & x \in B_1, \\ u = 0, & x \in \partial B_1, \end{cases} \tag{1.5}$$

where $\lambda > 0$ and B_1 is the unit disk in $\mathbb{R}^2$.

Adimurthi in [2] proved that the above equation (1.5) has a positive solution if and only if $\lambda \in (0, \lambda_1(B_1))$, where $\lambda_1(B_1)$ denotes the first eigenvalue of Laplace operator on the unit ball with the Dirichlet boundary. By the method of moving-plane, we know that every positive solution of equation (1.5) must be radially decreasing and hence satisfies

$$\begin{cases} -(ru')' = r\lambda u e^{u^2}, & r \in (0,1), \\ u > 0, & r \in (0,1), \\ u'(0) = 0, u(1) = 0. \end{cases} \tag{1.6}$$

Our first main result can be stated as follows:

Theorem 1.1. *For any $\lambda > 0$, problem* (1.6) *admits at most one solution.*

Remark 1.2. *Classical approaches based on the non-degeneracy method has been successfully applied to solve the uniqueness problem if $f(t)$ has the subcritical or critical polynomial growth. However, this method fails to deal with the exponential nonlinearity $f(t)$ like $te^{t^{\mu}}$. It is mainly because in this case we have $\lim\limits_{t\to+\infty} \frac{f'(t)}{f(t)} = +\infty$, which is significantly different from that in the case of polynomial nonlinearity. To handle the critical exponential case, we will establish an elementary, but deep and powerful result (see Lemma 2.3). More precisely, we can show that there exists some $0 < t < 1$ such that $w = u - tv$ satisfies $w > 0$, $w' < 0$ in $(0,1)$ and $w(1) = w'(1) = 0$, provided there exist two solutions u and v. Then we can deduce a contradiction through the local Pohozaev-type variational identity and a careful asymptotic expansion of u and v at the boundary.*

It was shown by Carleson-Chang [8] that if $\alpha = 4\pi$ and Ω is a ball, then the supremum in (1.4) can be achieved by a radial function u_0 satisfying the equation

$$\begin{cases} -\Delta u = \lambda_0 u e^{u^2}, & x \in B_1, \\ u > 0, & x \in B_1, \\ u = 0, & x \in \partial B_1, \end{cases}$$

for some $0 < \lambda_0 < \lambda_1(B_1)$. Hence, uniqueness result for ODE equation (1.6) will be an important step towards solving the following uniqueness conjecture about the maximizers of the Trudinger-Moser inequality.

Conjecture 1.3. *If $\alpha = 4\pi$ and Ω is a ball, then the maximizers of the Trudinger-Moser inequality* (1.4) *are unique.*

Remark 1.4. *We remark that the maximizers of (1.4) on the unit disk must satisfy the following nonlocal equation:*

$$\begin{cases} -\Delta u = \lambda_u u e^{u^2}, & x \in B_1, \\ u > 0, & x \in B_1, \\ u = 0, & x \in \partial B_1, \end{cases}$$

where Lagrange multiplier $\lambda_u = \frac{4\pi}{\int_{B_1} u^2 e^{u^2}}$, which is a parameter associated with u. If one can furthermore prove that all maximizers of Trudinger-Moser inequality correspond to the same Lagrange multiplier, which together with the uniqueness of positive solution of Theorem 1.1 will give uniqueness of maximizer of Trudinger-Moser inequality on the disk.

From [2], we know that equation (1.5) admits a positive ground-state solution u_λ with functional energy

$$I_\lambda(u) = \frac{1}{2}\int_{B_1} |\nabla u|^2 dx - \frac{\lambda}{2}\int_{B_1} e^{u^2} dx < 2\pi$$

for any $0 < \lambda < \lambda_1(B_1)$. In [14], del Pino, Musso and Ruf proved that if Ω in problem (1.5) is not a simply connected domain, then one can construct a family of positive solutions u_λ satisfying the equation

$$\begin{cases} -\Delta u = \lambda u e^{u^2}, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ 0 < \lambda < \lambda_1(\Omega), \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.7}$$

such that

$$I_\lambda(u_\lambda) = \frac{1}{2}\int_\Omega |\nabla u_\lambda|^2 dx - \frac{\lambda}{2}\int_\Omega e^{u_\lambda^2} dx \to 4\pi$$

as $\lambda \to 0$. From their proof, one may conjecture that if the domain of problem (1.5) is a simply connected or even a convex one, there is no positive solution u_λ to (1.7) such that the above property holds. When Ω is a ball, we can give a positive answer to this problem. Indeed, through our uniqueness result, we see that each positive solution of (1.5) must be a ground-state solution. Consequently, this implies that one cannot construct a family of solutions u_λ such that its functional energy exceeds 2π. More precisely, it can be stated as follows:

Corollary 1.5. *Given any family of positive solutions u_λ satisfying the equation*

$$\begin{cases} -\Delta u = \lambda u e^{u^2}, & x \in B_1, \\ u > 0, & x \in B_1, \\ 0 < \lambda < \lambda_1(B_1), \\ u = 0, & x \in \partial B_1, \end{cases} \tag{1.8}$$

there must hold

$$I_\lambda(u_\lambda) = \frac{1}{2}\int_{B_1} |\nabla u_\lambda|^2 dx - \frac{\lambda}{2}\int_{B_1} e^{u_\lambda^2} dx < 2\pi.$$

Slightly modifying the proof of Theorem 1.1, we can also obtain uniqueness of positive solutions for elliptic equations with the critical exponential growth in the ball of hyperbolic space. Using this uniqueness result, we can establish the quantization property of positive solutions for corresponding elliptic equations. Quantization property for elliptic equations with the critical exponential growth dates back to the work of Druet in [15], which can be stated as follows:

Let $\{u_k\}_k$ be a sequence of positive solutions of problem (1.7) with λ replaced by λ_k. Suppose that u_k is bounded in $W_0^{1,2}(\Omega)$ and u_k blows up. Then after passing to a subsequence, one has $\lambda_k \to \lambda_0$, $u_k \rightharpoonup u_0$ and there exists some integer N such that

$$\lim_{k \to +\infty} \|\nabla u_k\|_2^2 = \|\nabla u_0\|_2^2 + 4\pi N.$$

When Ω is a disk, the positive solution u_k must be radially symmetric through standard moving-plane method. (see e.g. [13], [17].) By the ODE technique, Malchiodi and

Martinazzi [25] proved that $u_0 = 0$ and $N = 1$. Furthermore, they showed that the functional I_λ under the constraint $\int_{B_1} |\nabla u|^2 dx = \gamma$ does not admit any positive critical point for γ sufficiently large. More recently, Druet and Thizy [16] showed for more general domain Ω that $u_0 = 0$ and N is equal to the number of concentration points by the complicated blow-up analysis technique combining with a comparison theorem.

In this paper, we will also utilize the uniqueness result to develop a new strategy to establish the quantization property for elliptic equations with the critical exponential growth in the hyperbolic space $\mathbb{H}$. Before we state our main results, we introduce some notations about two-dimensional hyperbolic space. Let $\mathbb{H}$ denote the standard hyperbolic space $\mathbb{H} = (B_1, \left(\frac{2}{1-|x|^2}\right)^2 dx^2)$ which is a unit disk equipped with the Poincaré metric $(\frac{2}{1-|x|^2}))^2 dx^2$, $B_{\mathbb{H}}(0,R)$ denote the hyperbolic ball centered at the origin with the geodesic radius equal to R, $-\Delta_{\mathbb{H}}$ denote Laplace-Beltrami operator in $\mathbb{H}$, $\lambda_1(B_{\mathbb{H}}(0,R))$ denote the first eigenvalue of the operator $-\Delta_{\mathbb{H}}$ with the Dirichlet boundary in $B_{\mathbb{H}}(0,R)$.

Our second main result in this paper reads as follows:

Theorem 1.6. *Assume that u_λ is a family of solutions satisfying equation*

$$\begin{cases} -\Delta_{\mathbb{H}} u = \lambda u e^{u^2}, & x \in B_{\mathbb{H}}(0,R), \\ u > 0, & x \in B_{\mathbb{H}}(0,R), \\ 0 < \lambda < \lambda_1(B_{\mathbb{H}}(0,R)), & \\ u = 0, & x \in \partial B_{\mathbb{H}}(0,R). \end{cases} \tag{1.9}$$

Then u_λ is radially symmetric and unique. Furthermore, when $\lambda \to \lambda_0$, we have:

(i) If $\lambda_0 = 0$, then u_λ blows up at the origin, and $|\nabla_{\mathbb{H}} u_\lambda|^2 dV_{\mathbb{H}} \rightharpoonup 4\pi\delta_0$, $\lambda u_\lambda e^{u_\lambda^2} \rightharpoonup 4\pi\delta_0$;

(ii) If $\lambda_0 \in (0, \lambda_1(B_{\mathbb{H}}(0,R)))$, then $u_\lambda \to u_0$ in $C^2(B_{\mathbb{H}}(0,R)))$ and u_0 is a positive radial solution of the equation

$$\begin{cases} -\Delta_{\mathbb{H}} u = \lambda_0 u e^{u^2}, & x \in B_{\mathbb{H}}(0,R), \\ u > 0, & x \in B_{\mathbb{H}}(0,R), \\ u = 0, & x \in \partial B_{\mathbb{H}}(0,R); \end{cases} \tag{1.10}$$

(iii) If $\lambda_0 = \lambda_1(B_{\mathbb{H}}(0,R))$, then $u_\lambda \to 0$ in $C^2(B_{\mathbb{H}}(0,R))$.

Remark 1.7. *Existence of a ground state solution (and consequently positive solution) for elliptic equation* (1.6) *can be verified following the same line as in* [2, 10, 11].

Remark 1.8. *Our proofs of quantization result on the hyperbolic spaces are based on the uniqueness result of positive solutions of equation* (1.9)*. Once the uniqueness property is obtained, we can study the quantization properties of the least energy solutions instead of the positive solutions. Our method is very simple and we can avoid using the complicated blow-up analysis of ODEs. We stress that this method can be also applied to study the quantization result for high dimensional ball of $\mathbb{R}^n$ or general hyperbolic space $\mathbb{H}^n$, provided* *the uniqueness for solutions of the corresponding equations is established. This will be carried out in our forthcoming work.*

Remark 1.9. *Since the uniqueness result for problem* (1.9) *holds for any fixed* λ*, hence we need not to choose a subsequence of* $\{u_\lambda\}_\lambda$ *to obtain quantization result. Furthermore, in our proofs for the quantization result, we get rid of the assumption that* u_λ *is uniformly bounded in* $W^{1,2}(B_{\mathbb{H}}(0,R))$*, which was required in* [15, 16]*, hence we can directly obtain the non-existence and multiplicity of positive critical point for supercritical Trudinger-Moser functional in* $B_{\mathbb{H}}(0,R)$*.*

Theorem 1.10. *There exists* $\gamma^* > 4\pi$ *such that the Trudinger-Moser functional* $F(u) = \int_{B_{\mathbb{H}}(0,R)}(e^{u^2}-1)dV_{\mathbb{H}}$ *under the constraint* $\int_{B_{\mathbb{H}}(0,R)} |\nabla_{\mathbb{H}} u_\lambda|^2 dV_{\mathbb{H}} = \gamma$ *has at least two positive critical points for* $\gamma \in (4\pi, \gamma^*)$*, at least one critical point for* $\gamma = 4\pi$ *or* $\gamma = \gamma^*$*, no positive critical point for* $\gamma \in (\gamma^*, +\infty)$*.*

Remark 1.11. *Exploiting the existence of the critical points of the Trudinger-Moser functional* $M(u) = \int_\Omega (e^{u^2}-1)dx$ *under the constraint* $\int_\Omega |\nabla u|^2 dx = \sigma$ *for* $\sigma > 4\pi$ *has been a challenging problem. Using a variational method and the monotonicity of the functional* $M(u)$*, Struwe* [34] *proved that there exits* $\sigma^* > 4\pi$ *such that* $M(u)$ *has at least two positive critical points for almost* $\gamma \in (4\pi, \sigma^*)$*. Later, Lamm, Robert and Struwe* [18] *introduced the Trudinger-Moser flow and strengthened it to every* $\gamma \in (4\pi, \sigma^*)$*. When* Ω *is a disk, Malchiodi and Martinazzi* [25] *applied refined blow-up analysis for radial critical point of* M *to derive that* $M(u)$ *does not admit any positive critical point for* σ *sufficiently large. It is conjectured that if* Ω *is a simply connected domain, the above non-existence result still holds. We also mention that recently Marchis, Malchiodi, Martinazzi and Thizy* [24] *proved when* Ω *is a closed manifold that the Trudinger-Moser functional* $M(u) = \int_\Omega (e^{u^2}-1)dx$ *under the constraint* $\int_\Omega \left(|\nabla u|^2 + |u|^2\right)dV_g = \sigma$ *always has a nontrivial positive critical point for any* $\sigma > 0$*.*

Finally, we note that the following improved Trudinger-Moser inequality (see [36]) still holds:

$$\sup_{u\in W_0^{1,2}(B_1),\int_{B_1}(|\nabla u|^2-\lambda|u|^2)dx\leq 1} \int_{B_1} e^{4\pi u^2}dx < +\infty \tag{1.11}$$

if $\lambda < \lambda_1(B_1)$. Furthermore, it is proved in [38] that the improved Trudinger-Moser inequality (1.11) admits an extremal function through the blow-up technique. Hence it is also interesting to consider the uniqueness for the extremal of (1.11) and the problems of positive critical point for super-critical Trudinger-Moser functional $H(u) = \int_{B_1}(e^{u^2}-1)dx$ under the constraint $\int_{B_1}(|\nabla u|^2 - \lambda|u|^2)dx = \beta$ for $\beta > 4\pi$. In fact, slightly modifying the proofs of our Theorems 1.1, 1.6 and 1.10, we can also obtain uniqueness of the problem (1.9), the quantization results, multiplicity and non-existence for positive critical point of supercritical Trudinger-Moser functional. We only list these results without giving the detailed proof.

Theorem 1.12. *For any $\lambda < \lambda_1(B_1) = 2\pi$. Let u_θ be a family of solutions satisfying*

$$\begin{cases} -\Delta u - \lambda u = \theta u e^{u^2}, & x \in B_1, \\ u > 0, & x \in B_1, \\ u = 0, & x \in \partial B_1, \end{cases} \tag{1.12}$$

where θ satisfies $0 < \theta < \lambda_1(B_1) - \lambda$. Then u_θ is radially decreasing and unique. Furthermore, when $\theta \to \theta_0$, there holds

(i) If $\theta_0 = 0$, then u_θ blows up at the origin, $\lambda u_\theta^2 \exp(u_\theta^2) dx \rightharpoonup 4\pi\delta_0$ and $\|\nabla u_\theta\|_2^2 \to 4\pi$ as $\theta \to 0$.

(ii) if $\theta_0 \in (0, \lambda_1(B_1) - \lambda)$, then $u_\theta \to u_0$ in $C^2(\bar{B}_1)$ and u_0 is a positive radial solution of the equation

$$\begin{cases} -\Delta u_0 - \lambda u_0 = \theta_0 u_0 e^{u_0^2} & x \in B_1, \\ u_0 > 0, & x \in B_1, \\ u_0 = 0, & x \in \partial B_1. \end{cases} \tag{1.13}$$

(iii) if $\theta_0 = \lambda_1(B_1) - \lambda$, then $u_\theta \to 0$ in $C^2(\bar{B}_1)$.

Theorem 1.13. *There exists $\beta^* > 4\pi$ such that the Trudinger-Moser functional $H(u) = \int_{B_1} \left(e^{u^2} - 1\right) dx$ under the constraint $\|\nabla u\|_2^2 - \lambda \|u\|_2^2 = \beta$ does not admit any positive critical point for $\beta > \beta^*$ and admits at least two positive critical points for $\beta \in (4\pi, \beta^*)$.*

Remark 1.14. *When $\lambda = 0$, the multiplicity and non-existence of positive critical point of super-critical Trudinger-Moser functional have been obtained by Malchiodi and Martinazzi in* [25] *through the accurate Dirichlet energy expansion formula obtained by the ODE technique. It should be noted that even if $\lambda = 0$, there is no upper bound estimate for β^*. A famous conjecture proposed by Malchiodi is that β^* should be equal to 8π. Using the method we developed in this paper, we can deduce that $\beta^* < 8\pi$ if we consider the perturbed Trudinger-Moser functional $\tilde{H}(u) = \int_{B_1} (e^{u^2} - 1 - u^2) dx$ under the constraint $\int_{B_1} |\nabla u|^2 dx = \beta$.*

Theorem 1.15. *The perturbed supercritical Trudinger-Moser functional $\tilde{H}(u) = \int_{B_1} (e^{u^2} - 1 - u^2) dx$ under the constraint $\int_{B_1} |\nabla u|^2 dx = \beta$ does not admit any positive critical point if $\beta \geq 8\pi$.*

2. Proof of Theorem 1.1

In order to prove uniqueness of positive solutions of equation (1.6), we need some lemmas.

Lemma 2.1. *Let u and v be non-negative C^1 functions on $[R_1, R_2]$, with $v \neq 0$ and $u > 0$ in (R_1, R_2). Assume that $v(R_i) = 0$ and $u'(R_i) \neq 0$ if $u(R_i) = 0$, where $i = 1, 2$. Then there exists a unique $t \in (0, \infty)$ such that $w = u - tv \geq 0$ in $[R_1, R_2]$ and $w(\xi) = 0$ for some $\xi \in [R_1, R_2]$. Furthermore* *(i) if $\xi \in (R_1, R_2)$ and w is in C^2 in a neighbourhood of ξ, then $w(\xi) = w'(\xi) = 0$, $w''(\xi) \geq 0$;*

(ii) if $\xi = R_i$ and $u(R_i) = 0$ for some $i \in \{1, 2\}$, then $w'(\xi) = 0$;

(iii) if $u(R_i) \neq 0$ and $v(R_i) = 0$, then $\xi \neq R_i$, where $i = 1, 2$.

Proof. This result was first essentially established by Rabinowitz [26] (see also [3]). Since the proof is short, for the convenience of the reader, we prefer to give the proof. By the hypotheses, v/u is a continuous function on $[R_1, R_2]$. Let $w = u - tv$, where

$$t = \max\left\{s : \frac{sv}{u} \leq 1\right\} = \left[\max \frac{v}{u}\right]^{-1}.$$

Then $w \geq 0$ and there exists a $\xi \in [R_1, R_2]$ such that $w(\xi) = 0$. If $\xi \in (R_i, R_2)$, then ξ is a local minimizer and hence (i) holds. If $\xi = R_i$ for some $i \in \{1, 2\}$, then by 1'Hôpital's rule

$$\frac{1}{t} = \lim_{r \to R_i} \frac{v(r)}{u(r)} = \frac{v'(R_i)}{u'(R_i)}$$

and hence $w'(\xi) = 0$. This proves (ii). We prove (iii) by contradiction. If $\xi = R_i$, then we have $\frac{1}{t} = \frac{v(R_i)}{u(R_i)} = 0$, this is impossible since $t \in (0, \infty)$. Hence (iii) holds.

□

Lemma 2.2. *Let u and v be two solutions of problem (1.6) and $r_1, r_2 \in (0, 1)$. For any $t > 0$, set $w = u - tv$. Then we have*

(i) if $w(r_1) = w'(r_1) = 0$ and $w''(r_1) \geq 0$, then $t \leq 1$.

(ii) if $t = u(0)/v(0)$ and $w > 0$ in $(0, r_2)$, then $t \leq 1$.

In either case, $t = 1$ implies that $u \equiv v$.

Proof. If $w(r_1) = w'(r_1) = 0$, then $u(r_1) = tv(r_1)$. Combining this, (1.6) and the assumption of (i), we have

$$0 \geq -\left(w'' + \frac{1}{r} w'\right)(r_1) = \lambda t v(r_1)(e^{u^2(r_1)} - e^{v^2(r_1)}). \tag{2.1}$$

Hence $e^{t^2 v^2(r_1)} = e^{u^2(r_1)} \leq e^{v^2(r_1)}$, which implies that $t \leq 1$. This proves (i).

Now we prove (ii) by contradiction. Suppose that (ii) is not true. From the assumption that $w > 0$ in $(0, r_2)$, we observe that for any $0 < \eta < r_2$, there exists a $\xi \in (0, \eta)$ such that $w'(\xi) > 0$. Now, we claim that there exists $\varepsilon_0 > 0$ such that $w' \geq 0$ in $[0, \varepsilon_0]$. Indeed, if this is not true, then there exists a sequence $\xi_k \to 0$ such that $w'(\xi_k) < 0$. This implies that w' changes sign infinitely often near zero and then we can find a sequence $\eta_k \to 0$

such that $w'(\eta_k) \geq 0$, $w''(\eta_k) \geq 0$. Similar as (2.1), from the assumption that $w > 0$ in $(0, r_2)$ and (1.6), we have

$$0 \geq -\left(w'' + \frac{1}{r}w'\right)(\eta_k) > \lambda t v(\eta_k)(e^{u^2(\eta_k)} - e^{v^2(\eta_k)}). \tag{2.2}$$

Letting $\eta_k \to 0$, we obtain $e^{u^2(0)} \leq e^{v^2(0)}$, and hence $t = u(0)/v(0) \leq 1$. This contradict with $t > 1$, and the claim is proved. Since $t > 1$, it follows from the claim that $w' \geq 0$ in $[0, \varepsilon_0]$ and $w'(0) = 0$. Hence there exists a sequence $\eta_k \to 0$ such that $w''(\eta_k) \geq 0$. Now by repeating the same argument at η_k as (2.2), we conclude that $t \leq 1$, which is a contradiction. This proves (ii).

Finally, if $t = 1$, then $u(p) = v(p)$ and $u'(p) = v'(p)$, where $p \in \{r_1, 0\}$, which together with uniqueness of the second order ODE equation with the Cauchy initial value implies that $u \equiv v$. □

Lemma 2.3. *Let $u \neq v$ be two solutions of problem* (1.6)*. Then there exist $t \in (0,1)$, $\xi \in (0,1)$ such that for $w = u - tv$, one of the following condition holds:*

(a) $u(0) = tv(0)$,

(b) $w \geq 0$ in $[0,\xi], w' \leq 0$ in $[0,\xi], w(\xi) = w'(\xi) = 0$,

(c) $w \geq 0, w' \leq 0$ in $[0,1], w'(1) = 0$.

Proof. We first claim that there exist $t \in (0,1)$ and $\xi \in (0,1)$ such that for $w = u - tv$, one of the following conditions holds:

(I) $u(0) = tv(0)$,

(II) $w \geq 0$ in $[0,\xi], w(\xi) = w'(\xi) = 0, w''(\xi) \geq 0$,

(III) $w \geq 0, w' \leq 0$ in $[0,1], w'(1) = 0$.

From Lemma 2.1, we can choose $t_1 > 0$, $\xi_1 \in [0,1]$ such that $w_1 = u - t_1 v$ satisfies $w_1 \geq 0$ in $[0,1]$, $w_1(\xi_1) = w_1'(\xi_1) = 0$. Indeed, we have $t_1 < 1$. This is because if $t_1 \geq 1$, it holds $v \leq t_1 v \leq u$. Then we can deduce that $u \equiv v$, which is a contradiction with u and v being the different solutions of equation (1.6). In fact, if $u > v$ in some interval $(a,b) \subseteq (0,1)$, since u and v satisfy equation (1.6), we have

$$\int_{B_1} (-\Delta u v + \Delta v u) dx = \lambda \int_{B_1} uv(e^{u^2} - e^{v^2}) dx > 0.$$

On the other hand, integration by parts directly gives that

$$\int_{B_1} (-\Delta u v + \Delta v u) dx = 2\pi(v'(1)u(1) - u'(1)v(1)) = 0,$$

which is a contradiction.

If $\xi_1 = 0$, then $u(0) = t_1 v(0)$ and (I) holds. If $\xi_1 \in (0,1)$, then ξ_1 is a local minimum and (II) holds.

Again from Lemma 2.1, for every $\eta \in (0,\infty)$, one can choose $t_\eta > 0$, $\xi_\eta \in [0,\eta]$, such that $w_\eta = u - t_\eta v$ satisfies $w_\eta \geq 0$ in $[0,\eta]$, $w_\eta(\xi_\eta) = 0$. We claim that if (I) and (II) do not hold, then $\xi_\eta = \eta$. We prove this by contradiction. Indeed, if $\xi_\eta \in (0,\eta)$, then ξ_η is a local minimizer for w_η and therefore $w_\eta'(\xi_\eta) = 0$, $w_\eta''(\xi_\eta) \geq 0$. Hence from (i) of Lemma 2.2, we get $t_\eta < 1$, which contradicts the assumption that (II) does not hold. If $\xi_\eta = 0$, from (ii) of Lemma 2.2, we conclude that $t_\eta < 1$. This contradicts the assumption that (I) does not hold.

Now, we show that if (I) and (II) do not hold, then $w'(r) \leq 0$ for $r \in [0,1]$. We prove this by contradiction. Suppose that there exists $\eta_0 \in (0,1)$ such that $w_1'(\eta_0) > 0$. We consider the interval $[0,\eta_0]$. From the above argument, we see that $\xi_{\eta_0} = \eta_0$, thus we have $t_{\eta_0} = u(\eta_0)/v(\eta_0)$. Hence for $0 \leq r < \eta_0 \leq \infty$, we deduce from $w_{\eta_0}(r) \geq 0$ that

$$\frac{u(r)}{v(r)} \geq \frac{u(\eta_0)}{v(\eta_0)}.$$

Thus u/v is a decreasing function, this implies that $u/v \leq u'/v'$. Let $1 > r > \eta_0$, so that

$$\frac{u(r)}{v(r)} \leq \frac{u(\eta_0)}{v(\eta_0)} \leq \frac{u'(\eta_0)}{v'(\eta_0)} < t_1$$

and hence $w_1(r) < 0$, which is a contradiction. Therefore, $w_1' \leq 0$ in $[0,1]$ and this proves (III), and the claim is proved.

Next, we show that if (I) and (III) do not hold, then (b) holds. Let

$$\eta_0 = \inf\{\eta > 0;\ \text{(II) holds in } [0,\eta]\}.$$

We claim that $\eta_0 > 0$. We prove this by contradiction. Suppose that $\eta_0 = 0$, then there exists a sequence $\eta_k \to 0$ with $t_{\eta_k} \to t_0$ for some $t_0 \leq 1$ such that $w_{\eta_k}(\eta_k) = 0$. This implies that $u(0) = t_0 v(0)$. If $t_0 = 1$, then $u \equiv v$, which is a contradiction. If $t_0 < 1$, then (I) holds, which contradicts our assumption, hence $\eta_0 > 0$. Now by the definition of η_0, we have $w_{\eta_0}(\eta_0) = w_{\eta_0}'(\eta_0) = 0$, $w_{\eta_0}''(\eta_0) \geq 0$. Then it follows from (i) of Lemma 2.2 that $t_{\eta_0} < 1$.

Now, we show that $w_{\eta_0}' \leq 0$ in $[0,\eta_0]$. We assert that for any $0 < \eta < \eta_0$, one has $\xi_\eta = \eta$, $t_\eta = u(\eta)/v(\eta)$. Indeed, if $\xi_\eta \in (0,\eta)$, then ξ_η is a local minimizer of w_η, and hence $w_\eta'(\xi_\eta) = 0$, $w_\eta''(\xi_\eta) \geq 0$. Therefore from (i) of Lemma 2.2, $t_\eta < 1$, but this is impossible from the definition of η_0. If $\xi_\eta = 0$, then $t_\eta = u(0)/v(0)$ and hence from (ii) of Lemma 2.2, $t_\eta < 1$. Thus (I) holds, which contradicts our assumption. Therefore we have $\xi_\eta = \eta$ and $t_\eta = u(\eta)/v(\eta)$. This implies that $w_\eta(r) > 0$ for $0 < r < \eta$, and hence $u(r)/v(r) > u(\eta)/v(\eta)$. Thus u/v is a decreasing function in $[0,\eta_0]$ and $u/v \leq u'/v'$ in $[0,\eta_0]$. Suppose that there exists $\eta_1 < \eta_0$ such that $w_{\eta_0}'(\eta_1) > 0$, then for $\eta_1 < r < \eta_0$ we have

$$\frac{u(r)}{v(r)} \leq \frac{u(\eta_1)}{v(\eta_1)} \leq \frac{u'(\eta_1)}{v'(\eta_1)} < t_{\eta_0}.$$

Hence $w_{\eta_0}(r) < 0$, which is a contradiction. Therefore $w'_{\eta_0} \leq 0$ in $[0, \eta_0]$, and (b) holds in $[0, \eta_0]$. The proof of this lemma is finished.

□

Lemma 2.4. *Let u and u_2 be two solutions of* (1.6) *with* $u(0) < u_2(0)$*. Then there exists a solution u_1 of* (1.6) *such that* $u_1(0) < u_2(0)$*. Furthermore, u_1 and u_2 intersects at most once in* $(0, 1)$*.*

Proof. Let $\alpha > 0$ and $w(\cdot, \alpha)$ be the unique solution of the following initial-value problem,

$$\begin{cases} -(rw')' = rf(w), \\ w(0) = \alpha, w'(0) = 0, \end{cases} \tag{2.3}$$

where $f(w) = \lambda w e^{w^2}$. We also denote $R(\alpha)$ the first zero of $w(\cdot, \alpha)$ defined by

$$R(\alpha) = \sup\{r : w(s, \alpha) > 0 \text{ for all } s \in [0, r]\}.$$

Assume that u and u_2 intersect at least twice (otherwise there is nothing to prove). Let $0 < R_1 < R_2 < 1$ be the first two consecutive points of intersection with $u(r) < u_2(r)$ for all $r \in (0, R_1)$. Clearly, $w(r, \alpha_0) = u(r)$ and $w(r, \alpha_2) = u_2(r)$, where $\alpha_0 = u(0)$, $\alpha_2 = u_2(0)$ with $R(\alpha_0) = R(\alpha_2) = 1$. Let $\alpha < \alpha_0$ and be close to α_0, and $0 < R_1(\alpha) < R_2(\alpha) < \infty$ be the first two consecutive points of intersections of $w(\cdot, \alpha)$ with u_2 (which exist by continuity) such that $w(r, \alpha) \leq u_2(r)$ for $r \in (0, R_1(\alpha))$.

Now, as α moves towards zero, one of the following three possibilities holds.

(i) There exists a $\alpha_1 \in (0, \alpha_0)$ such that $R_2(\alpha_1) = 1$. Then the conclusion of the lemma holds.

(ii) There exists a $\alpha_1 \in (0, \alpha_0)$ and a point $R \in (0, 1)$ such that

$$w(R, \alpha_1) = u_2(R), \qquad w'(R, \alpha_1) = u_2'(R).$$

Then, by uniqueness of the initial-value problem, $w(r, \alpha_1) = u_2(r)$ for all $r \in (0, 1)$, which is a contradiction.

(iii) $0 < R_1(\alpha) < R_2(\alpha) < 1$ for all α and $\lim_{\alpha \to 0}(R_2(\alpha) - R_1(\alpha)) = 0$. In this cases, let $I(\alpha) = [R_1(\alpha), R_2(\alpha)]$ and $v(r) = w(r, \alpha) - u_2(r)$. Then $v(r)$ satisfies

$$\begin{cases} -(rv')' = Qv, & r \in I(\alpha), \\ v > 0, & r \in I(\alpha), \\ v(R_1(\alpha)) = v(R_2(\alpha)) = 0, \end{cases} \tag{2.4}$$

where

$$Q(r) = r\frac{f(w(r, \alpha)) - f(u_2(r))}{w(r, \alpha) - u_2(r)}.$$

Hence v is the first eigenfunction with eigenvalue $\mu_1 = 1$ of the following eigenvalue problem:

$$\begin{cases} -(r\varphi')' = \mu Q\varphi, & r \in I(\alpha), \\ \varphi = 0, & r \in \partial I(\alpha). \end{cases}$$

For $\alpha \in (0, \alpha_0)$ and $0 < R_1 \leq R_1(\alpha)$, we have

$$M = \sup\{Q(r) : r \in I(\alpha), \alpha \in (0, \alpha_0)\} < \infty. \tag{2.5}$$

Let $\lambda_1(\alpha)$ be the first eigenvalue of

$$\begin{cases} -\frac{d^2\varphi}{dr^2} = \lambda\varphi, & x \in I(\alpha), \\ \varphi = 0, & x \in \partial I(\alpha). \end{cases}$$

Then, from (2.4)-(2.6), we get

$$\begin{aligned} 1 &= \inf\left\{\frac{\int_{I(\alpha)} r(\varphi')^2 dr}{\int_{I(\alpha)} Q\varphi^2 dr}; \varphi \in H_0^1(I(\alpha))\right\} \\ &\geq \frac{R_1}{M}\left\{\frac{\int_{I(\alpha)} (\varphi')^2 dr}{\int_{I(\alpha)} \varphi^2 dr}; \varphi \in H_0^1(I(\alpha))\right\} \\ &\geq \frac{R_1}{M}\lambda_1(\alpha). \end{aligned} \tag{2.6}$$

Since $R_2(\alpha) - R_1(\alpha) \to 0$ as $\alpha \to 0$, we derive that $\lambda_1(\alpha) \to \infty$, which contradicts (2.6). This proves the lemma.

□

Lemma 2.5. *Let u and u_2 be two solutions of* (1.6) *with $u(0) < u_2(0)$, then there exists u_1 of* (1.6) *such that $u_1(0) < u_2(0)$, u_1 and u_2 intersects only once in* $(0, 1)$*. Moreover, $\frac{u_1(r)}{u_2(r)}$ is strictly increasing.*

Proof. According to Lemma 2.4, in order to prove that u_1 and u_2 intersects only once in $(0, 1)$, we only need to prove that u_1 and u_2 intersects at least once. We argue this by contradiction. Assume that u_1 and u_2 does not intersect in $(0, 1)$, then $u_2 > u_1$ in $(0, 1)$. Since u_1 and u_2 satisfies equation (1.6), we have

$$\int_{B_1} (-\Delta u_2 u_1 + \Delta u_1 u_2) dx = \lambda \int_{B_1} u_2 u_1 (e^{u_2^2} - e^{u_1^2}) dx > 0.$$

On the other hand, integration by parts directly gives that

$$\int_{B_1} (-\Delta u_2 u_1 + \Delta u_1 u_2) dx = 2\pi(u_1'(1)u_2(1) - u_2'(1)u_1(1)) = 0,$$

which is obviously a contradiction.

Now, we start to prove that $\frac{u_1(r)}{u_2(r)}$ is strictly increasing. Assume that the only intersection point of u_1 and u_2 is r_0, then $u_2(r) \geq u_1(r)$ in $(0, r_0]$. By equation (1.6), one can directly obtain that for any $r \in (0, r_0]$,

$$\int_{B_r} (-\Delta u_2 u_1 + \Delta u_1 u_2) dx = \int_{B_r} u_2 u_1 (e^{u_2^2} - e^{u_1^2}) dx > 0.$$

On the other hand, integration by parts directly gives

$$\int_{B_r} (-\Delta u_2 u_1 + \Delta u_1 u_2) dx = 2\pi (u_1'(r) u_2(r) - u_2'(r) u_1(r)) r.$$

Combining the above estimate, we derive that $u_1'(r)u_2(r) - u_2'(r)u_1(r) > 0$ for $r \in (0, r_0]$, that is $\frac{u_1(r)}{u_2(r)}$ is strictly increasing in $(0, r_0]$. Now, we will prove that $u_1'(r)u_2(r) - u_2'(r)u_1(r) > 0$ holds for $r \in (r_0, 1)$. By equation (1.6), one can similarly obtain

$$\int_{B_1 \setminus B_r} (-\Delta u_2 u_1 + \Delta u_1 u_2) dx = \int_{B_1 \setminus B_r} u_2 u_1 (e^{u_2^2} - e^{u_1^2}) dx < 0.$$

Again applying integration by parts, one can derive that

$$2\pi (u_1'(r) u_2(r) - u_2'(r) u_1(r)) r = -\int_{B_1 \setminus B_r} (-\Delta u_2 u_1 + \Delta u_1 u_2) dx > 0.$$

Then we accomplish the proof of Lemma 2.5. □

Now, we are in position to give the proof of uniqueness of positive radial solution to equation (1.6).

Proof of Theorem 1.1. Assume that u and v are two positive solutions of equation (1.6) with $v(0) < u(0)$, u and v intersecting only once in $(0, 1)$. According to Lemma 2.3, there exist $t \in (0, 1)$, $\xi \in (0, 1)$ such that for $w = u - tv$ one of the following holds:

(a) $u(0) = tv(0)$,

(b) $w \geq 0, w' \leq 0$ in $[0, \xi], w(\xi) = w'(\xi) = 0$,

(c) $w \geq 0, w' \leq 0$ in $[0, 1], w'(1) = 0$.

Obviously, (a) does not occur. According to Lemma 2.5, we know that $\frac{v(r)}{u(r)}$ is strictly increasing in $(0, 1)$. This gives that for any $r \in (0, 1)$, $\frac{v'(r)}{u'(r)} < \frac{v(r)}{u(r)}$. Hence it follows that there does not exists $\xi \in (0, 1)$ such that $w(\xi) = w'(\xi) = 0$, that is to say that (b) is also impossible. In order to prove the uniqueness theorem, we only need to prove that (c) does not happen. Assume that (c) happens, then $w(1) = w'(1) = 0$. A simple calculation combining with $u'(1) = tv'(1)$ and $u(1) = v(1) = 0$ yields that

$$u^{(2)}(1) = tv^{(2)}(1), \ u^{(3)}(1) = tv^{(3)}(1), \ u^{(4)}(1) = tv^{(4)}(1) \tag{2.7}$$

and

$$u^{(5)}(1) - tv^{(5)}(1) = -6\big((u')^3(1) - t(v')^3(1)\big) = -6u'(1)\big((u')^2(1) - (v')^2(1)\big). \tag{2.8}$$

Furthermore, we also have

$$u^{(6)}(1) - tv^{(6)}(1) + u^{(5)}(1) - tv^{(5)}(1) + 36(u')^2(1)u''(1) - 36t(v')^2(1)v''(1) = 0.$$

This together with $u''(1) + u'(1) = 0$ and (2.8) gives that

$$\begin{aligned} u^{(6)}(1) - tv^{(6)}(1) &= 36(u')^3(1) - 36t(v')^3(1) - \big(u^{(5)}(1) - tv^{(5)}(1)\big) \\ &= 36u'(1)\big((u')^2(1) - (v')^2(1))\big) + 6u'(1)\big((u')^2(1) - (v')^2(1)\big) \\ &= 42u'(1)\big((u')^2(1) - (v')^2(1)\big) \end{aligned} \tag{2.9}$$

Since u and v satisfy equation (1.7), using the Pohozaev identity and combining the radial symmetry of u and v, we have

$$\int_{B_r} -\Delta u(x\cdot\nabla u)dx = -\frac{1}{2}\int_{\partial B_r} |\nabla u|^2(x\cdot\nu)d\sigma = -\pi r^2(u')^2(r)$$

and

$$\begin{aligned} \int_{B_r} ue^{u^2}(x\cdot\nabla u) =& \frac{1}{4}\int_{B_r} \nabla(e^{u^2}-1)\nabla(|x|^2) \\ &= -\frac{1}{4}\int_{B_r}(e^{u^2}-1)\Delta(|x|^2)dx + \frac{1}{4}\int_{\partial B_r}(e^{u^2}-1)\frac{\partial|x|^2}{\partial\nu}ds \\ &= -\int_{B_r}(e^{u^2}-1)dx + \pi r^2(e^{u^2(r)}-1). \end{aligned}$$

That is

$$\pi r^2(u')^2(r) = \int_{B_r}(e^{u^2}-1)dx - \pi r^2(e^{u^2(r)}-1).$$

This deduces that

$$\pi(u')^2(1) - \pi r^2(u')^2(r) = \int_{B_1\setminus B_r}(e^{u^2}-1)dx + \pi r^2(e^{u^2(r)}-1). \tag{2.10}$$

Similarly, we also have

$$\pi(v')^2(1) - \pi r^2(v')^2(r) = \int_{B_1\setminus B_r}(e^{v^2}-1)dx + \pi r^2(e^{v^2(r)}-1). \tag{2.11}$$

Multiplying (2.11) by t^2, and then subtracting (2.10), we can obtain from $u'(1) = tv'(1)$ that

(2.12)

$$\begin{aligned} \pi r^2((u')^2(r) - t^2(v')^2(r)) &= \int_{B_1\setminus B_r}\big(t^2(e^{v^2}-1) - (e^{u^2}-1)\big)dx + \pi r^2\big(t^2(e^{v^2}-1) - (e^{u^2}-1)\big) \\ &= \pi r^2(I + II). \end{aligned}$$

In view of (2.7), (2.8) and (2.9), we have

(2.13)
$$
\begin{aligned}
&(u')^2(r)-t^2(v')^2(r)=(u'(r)+tv'(r))(u'(r)-tv'(r))\\
=&\big(2u'(1)+2u''(1)(r-1)+O((r-1)^2)\big)\times\big(\frac{1}{4!}(u^{(5)}(1)-tv^{(5)}(1))(r-1)^4+\\
&+\frac{1}{5!}(u^{(6)}(1)-tv^{(6)}(1)(r-1)^5+O((r-1)^6)\big)\\
=&-\frac{1}{2}(u')^2(1)\big((u')^2(1)-(v')^2(1)\big)(r-1)^4+\frac{1}{2}(u')^2(1)\big((u')^2(1)-(v')^2(1)\big)(r-1)^5+\\
&+\frac{84}{120}(u')^2(1)\big((u')^2(1)-(v')^2(1)\big)(r-1)^5+O((r-1)^6).
\end{aligned}
$$

For II, we have

$$
\begin{aligned}
II&=\big(t^2v^2(r)-u^2(r)+\frac{1}{2}t^2v^4(r)-\frac{1}{2}u^4(r)\big)+O((r-1)^6)\\
&=\big((tv(r)+u(r))(tv(r)-u(r))+\frac{1}{2}\big((tv)^2v^2(r)-u^4(r)\big)\big)+O((r-1)^6)\\
&=\big(2u'(1)(r-1)+O((r-1)^2)\big)\big((\frac{tv^{(5)}(1)-u^{(5)}(1)}{5!})(r-1)^5+O((r-1)^6)\big)+\\
&\quad+\frac{1}{2}\big(u'(1)(r-1)+\frac{1}{2}u''(1)(r-1)^2+O((r-1)^3)\big)^2\big(v'(1)(r-1)+\frac{v''(1)}{2}(r-1)^2+\\
&\quad+O((r-1)^3)\big)^2-\frac{1}{2}\big(u'(1)(r-1)+\frac{1}{2}u''(1)(r-1)^2+O((r-1)^3)\big)^4+O((r-1)^6)\\
&=\frac{1}{2}\big(u'(1)(r-1)+\frac{1}{2}u''(1)(r-1)^2+O((r-1)^3)\big)^2\times\Big(\big(v'(1)(r-1)+\frac{1}{2}v''(1)(r-1)^2+\\
&\quad+O((r-1)^3)\big)^2-\big(u'(1)(r-1)+\frac{u''(1)}{2}(r-1)^2+O((r-1)^3)\big)^2\Big)+O((r-1)^6)\\
&=\frac{1}{2}(u')^2(1)((v')^2(1)-(u')^2(1))(r-1)^4+\frac{\pi r^2}{2}u(1)u''(1)\big((v')^2(1)-(u')^2(1)\big)(r-1)^5\\
&\quad+\frac{1}{2}(u')^2(1)(v'(1)v''(1)-u'(1)u''(1))(r-1)^5+O((r-1)^6)\\
&=-\frac{1}{2}(u'(1))^2((u')^2(1)-(v')^2(1))(r-1)^4+(u')^2(1)((u')^2(1)-(v')^2(1))(r-1)^5+\\
&\quad+O((r-1)^6),
\end{aligned}
$$

where in the last step, we have used the fact that $u''(1)+u'(1)=0$.

For I, we have

$$
\begin{aligned}
I&=-\frac{1}{2\pi r^2}(u')^2(1)((u')^2(1)-(v')^2(1))(r-1)^4\pi(1-r^2)+O((r-1)^6)\\
&=(u')^2(1)((u')^2(1)-(v')^2(1))(r-1)^5+O((r-1)^6).
\end{aligned}
$$

Combining (2.12), (2.13) and the estimates of I and II, we derive that

$$(\frac{1}{2}+\frac{84}{120})(u')^2(1)\big((u')^2(1)-(v')^2(1)\big)(r-1)^5 = 2(u')^2(1)\big((u')^2(1)-(v')^2(1)\big)(r-1)^5,$$

which is a contradiction. Hence (c) is impossible. This accomplishes the proof of Theorem 1.1. □

3. proof of Theorem 1.6

In this section, we will prove uniqueness of positive solution for elliptic equation in the ball of hyperbolic space and give quantization result for positive critical point of Trudinger-Moser functional in the ball of hyperbolic space.

Assume that u_λ is a positive solution of equation (1.9). By the conformal invariance between $(B, dV_{\mathbb{H}})$ and (B, dx), the elliptic equation in hyperbolic ball $B_{\mathbb{H}}(0,R)$ can be rewritten as the elliptic equation in the ball $B_{\tilde{R}}$ of $\mathbb{R}^2$, with radius $\tilde{R}$ equal to $\frac{e^R-1}{e^R+1}$:

$$\begin{cases} -\Delta u_\lambda = \lambda u_\lambda e^{u_\lambda^2}(\frac{2}{1-|x|^2})^2, & x \in B_{\tilde{R}}, \\ u_\lambda > 0, & x \in B_{\tilde{R}}, \\ 0 < \lambda < \lambda_1(B_{\mathbb{H}}(0,R)), & \\ u_\lambda = 0, & x \in \partial B_{\tilde{R}}. \end{cases} \tag{3.1}$$

Using the moving plane method on hyperbolic spaces, we can show that u_λ must be radially symmetric about the origin and decreasing. (see e.g., [19]). Hence u_λ satisfies the following ODE equation

$$\begin{cases} -(ru_\lambda')' = r\lambda u_\lambda e^{u_\lambda^2}(\frac{2}{1-r^2})^2, & r \in (0, \tilde{R}), \\ u_\lambda > 0, & r \in (0, \tilde{R}), \\ u_\lambda'(0) = 0, u_\lambda(\tilde{R}) = 0. & \end{cases} \tag{3.2}$$

We remark that in the work of Naito and Suzuki [29], they also considered the radial symmetry of positive solutions for a class of semilinear elliptic equations with some weight on the unit ball, however, our equation (3.1) does not satisfy the hypothesis made in the paper of Naito and Suzuki.

Since the weight function $(\frac{2}{1-r^2})^2$ has no singular points on the corresponding defined interval, the weight function and its derivatives don't contribute to any vanishing factor around $r = \tilde{R}$, hence a slight modification of the proof of Theorem 1.1 can show that the positive solution u_λ is unique. Therefore, u_λ is also a least-energy solution of elliptic equation (3.1). Now we are in position to give quantization result for least energy solution of elliptic equation (3.1).

Proof of (i) in Theorem 1.6: We recall that u_λ is a least energy solution of elliptic equation (3.1) if its functional energy $I_\lambda(u)$ defined by

$$I_\lambda(u) = \frac{1}{2}\int_{B_{\mathbb{H}}(0,R)} |\nabla_{\mathbb{H}} u|^2 dV_{\mathbb{H}} - \frac{\lambda}{2}\int_{B_{\mathbb{H}}(0,R)} (e^{u^2}-1) dV_{\mathbb{H}}$$

is equal to $m_\lambda := \min\{I_\lambda(u): \ I'_\lambda(u)=0\}$, which is equivalent to say $I_\lambda(u_\lambda)=m_\lambda$. We claim that $\lim\limits_{\lambda\to 0} m_\lambda$ has the positive lower bound. We argue this by contradiction. Suppose not, then $\lim\limits_{\lambda\to 0} m_\lambda = 0$, which together with $I'_\lambda(u_\lambda)u_\lambda = 0$ implies that $\lim\limits_{\lambda\to 0}\int_{B_{\mathbb{H}}(0,R)}|\nabla_{\mathbb{H}}u_\lambda|^2dV_{\mathbb{H}} = 0$. Then it follows that $e^{u_\lambda^2}$ is bounded in $L^q(B_{\mathbb{H}}(0,R))$ for any $q>1$. Let $v_\lambda = \frac{u_\lambda}{\|\nabla_{\mathbb{H}}u_\lambda\|_2}$, then v_λ is bounded in $W^{1,2}(B_{\mathbb{H}}(0,R))$. Hence, there exists $v_0\in W^{1,2}(B_{\mathbb{H}}(0,R))$ such that v_λ strongly converges to v_0 in $L^q(B_{\mathbb{H}}(0,R))$ for any $q>1$. Noticing that $I'_\lambda(u_\lambda)u_\lambda=0$, one can write $1=\lambda\int_{B_{\mathbb{H}}(0,R)} v_\lambda^2 e^{u_\lambda^2}dV_{\mathbb{H}}$. This together with the boundedness of v_λ^2 and $e^{u_\lambda^2}$ in $L^q(B_{\mathbb{H}}(0,R))$ yields that

$$\lim_{\lambda\to 0}\lambda\int_{B_{\mathbb{H}}(0,R)} v_\lambda^2 e^{u_\lambda^2}dV_{\mathbb{H}} = 0,$$

which is a contradiction. This proves that there exists $c_0>0$ such that $\lim\limits_{\lambda\to 0}\int_{B_{\mathbb{H}}(0,R)}|\nabla_{\mathbb{H}}u_\lambda|^2dV_{\mathbb{H}} \geq c_0$. Through $I'_\lambda(u_\lambda)u_\lambda=0$, we have that

$$\int_{B_{\mathbb{H}}(0,R)}|\nabla_{\mathbb{H}}u_\lambda|^2 dV_{\mathbb{H}} = \lambda\int_{B_{\mathbb{H}}(0,R)} u_\lambda^2\exp(u_\lambda^2)dV_{\mathbb{H}}, \tag{3.3}$$

which implies that

$$\lim_{\lambda\to 0}\int_{B_{\mathbb{H}}(0,R)} u_\lambda^2\exp(u_\lambda^2)dV_{\mathbb{H}} = +\infty. \tag{3.4}$$

This deduces $c_\lambda := \lim\limits_{\lambda\to 0} u_\lambda(0) = +\infty$, that is to say that u_λ blows up at the origin. Now, we will prove that

$$\lim_{\lambda\to 0}\int_{B_{\mathbb{H}}(0,R)}|\nabla_{\mathbb{H}}u_\lambda|^2dV_{\mathbb{H}} = 4\pi. \tag{3.5}$$

Since u_λ is the least energy critical point of functional $I_\lambda(u)$, one can deduce that

$$0 < I_\lambda(u_\lambda) < 2\pi \tag{3.6}$$

by the Trudinger-Moser inequality on compact manifold (see the Appendix). This together with $I'(u_\lambda)u_\lambda=0$ yields that u_λ is bounded in $W_0^{1,2}(B_{\mathbb{H}}(0,R))$ if $\lambda\to 0$. Then there exists some u_0 such that $u_\lambda \rightharpoonup u_0$ weakly in $W_0^{1,2}(B_{\mathbb{H}}(0,R))$ and $u_\lambda\to u_0$ strongly in $L^q(B_{\mathbb{H}}(0,R))$ for any $q>1$. Now we claim that

$$\lim_{\lambda\to 0}\int_{B_{\mathbb{H}}(0,R)}|\nabla_{\mathbb{H}}u_\lambda|^2dV_{\mathbb{H}} = \lim_{\lambda\to 0} 2I_\lambda(u_\lambda) \leq 4\pi. \tag{3.7}$$

This is mainly because

$$\lim_{\lambda\to 0}\lambda\int_{B_{\mathbb{H}}(0,R)} e^{u_\lambda^2}dV_{\mathbb{H}} = 0. \tag{3.8}$$

Indeed, we can write

$$\lambda \int_{B_{\mathbb{H}}(0,R)} e^{u_\lambda^2} dV_{\mathbb{H}} = \lambda \int_{\{|u_\lambda|>M\}} e^{u_\lambda^2} dV_{\mathbb{H}} + \lambda \int_{\{|u_\lambda|\le M\}} e^{u_\lambda^2} dV_{\mathbb{H}} \\ = I_1 + I_2. \tag{3.9}$$

For I_1, through $\lambda \int_{B_{\mathbb{H}}(0,R)} u_\lambda^2 e^{u_\lambda^2} dV_{\mathbb{H}} \lesssim 1$, we obtain

$$I_1 \le \frac{\lambda}{M^2} \int_{B_{\mathbb{H}}(0,R)} u_\lambda^2 e^{u_\lambda^2} dV_{\mathbb{H}} \lesssim \frac{1}{M^2}, \tag{3.10}$$

which implies $\lim\limits_{M\to\infty} \lim\limits_{\lambda\to+\infty} I_1 = 0$. For I_2, Obviously $I_2 \le \lambda e^{M^2} \mathrm{Vol}_{\mathbb{H}}(B_{\mathbb{H}}(0,R))$, hence $\lim\limits_{M\to\infty} \lim\limits_{k\to+\infty} I_2 = 0$. Combining the estimates of I_1 and I_2, we conclude that

$$\lim_{k\to+\infty} \lambda \int_{B_{\mathbb{H}}(0,R)} e^{u_\lambda^2} dV_{\mathbb{H}} = 0,$$

hence the claim is proved.

Now, we are in position to prove that $\lim\limits_{\lambda\to 0} \int_{B_{\mathbb{H}}(0,R)} |\nabla_{\mathbb{H}} u_\lambda|^2 dV_{\mathbb{H}} = 4\pi$. We argue this by contradiction. Suppose that $\lim\limits_{\lambda\to 0} \int_{B_{\mathbb{H}}(0,R)} |\nabla_{\mathbb{H}} u_\lambda|^2 dV_{\mathbb{H}} < 4\pi$, it follows from Trudinger-Moser inequalities (see e.g., [23], [27]) that $\int_{B_{\mathbb{H}}(0,R)} u_\lambda^2 e^{u_\lambda^2} dV_{\mathbb{H}}$ is bounded, which is a contradiction with (3.4). Hence (3.8) holds.

Finally, we claim

$$|\nabla_{\mathbb{H}} u_\lambda|^2 dV_{\mathbb{H}} \rightharpoonup 4\pi\delta_0.$$

We argue this by contradiction. Suppose not, then there exists some $\delta > 0$ such that

$$\int_{B_{\mathbb{H}}(0,\delta)} |\nabla_{\mathbb{H}} u_\lambda|^2 dV_{\mathbb{H}} < 4\pi.$$

Hence, using the Trudinger-Moser inequality again, we derive that $u_\lambda e^{u_\lambda^2} \in L^p(B_{\mathbb{H}}(0,\delta))$, for some $p > 1$. Then it follows from the standard elliptic estimates that

$$\sup_\lambda \|u_\lambda\|_{C^1(B_{\mathbb{H}}(0,\delta))} < \infty,$$

which is a contradiction with $\lim\limits_{\lambda\to+\infty} u_\lambda(0) = +\infty$.

Proof of (ii) in Theorem 1.6: when $\lambda \to \lambda_0 \in (0, \lambda_1(B_{\mathbb{H}}(0,R))$, one can similarly obtain that there exist $0 < \rho_1 < \rho_2 < 2\pi$ such that

$$\rho_1 \le \lim_{\lambda\to\lambda_0} I_\lambda(u_\lambda) \le \rho_2 < 2\pi,$$

as $\lambda \to \lambda_0$. Gathering this and $I'(u_\lambda)u_\lambda = 0$, we deduce that u_λ is bounded in $W_0^{1,2}(B_{\mathbb{H}}(0,R)))$. Then there exists some u_0 such that $u_\lambda \rightharpoonup u_0$ weakly in $W_0^{1,2}(B_{\mathbb{H}}(0,R))$. Next, we will show that u_0 satisfies equation

$$\begin{cases} -\Delta_{\mathbb{H}} u_0 = \lambda_0 u_0 e^{u_0^2}, & x \in B_{\mathbb{H}}(0,R), \\ u_0 > 0, & x \in B_{\mathbb{H}}(0,R), \\ u_0 = 0, & x \in \partial B_{\mathbb{H}}(0,R). \end{cases} \tag{3.11}$$

For this purpose, according to the definition of weak solution, we only need to prove that

$$\lim_{\lambda\to\lambda_0} \lambda \int_{B_{\mathbb{H}}(0,R)} u_\lambda e^{u_\lambda^2} dV_{\mathbb{H}} = \lambda_0 \int_{B_{\mathbb{H}}(0,R)} u_0 e^{u_0^2} dV_{\mathbb{H}}.$$

Indeed, we can write

$$\begin{aligned} \lambda \int_{B_{\mathbb{H}}(0,R)} u_\lambda e^{u_\lambda^2} dV_{\mathbb{H}} &= \lambda \int_{\{|u_\lambda|>M\}} u_\lambda e^{u_\lambda^2} dV_{\mathbb{H}} + \lambda \int_{\{|u_\lambda|\le M\}} u_\lambda e^{u_\lambda^2} dV_{\mathbb{H}} \\ &= I_1 + I_2. \end{aligned} \tag{3.12}$$

For I_1, through $\lambda \int_{B_{\mathbb{H}}(0,R)} u_\lambda^2 e^{u_\lambda^2} dV_{\mathbb{H}} \lesssim 1$, we obtain

$$I_1 \le \frac{\lambda}{M} \int_{B_{\mathbb{H}}(0,R)} u_\lambda^2 e^{u_\lambda^2} dV_{\mathbb{H}} \lesssim \frac{1}{M^2}, \tag{3.13}$$

which implies $\lim\limits_{M\to\infty} \lim\limits_{\lambda\to\lambda_0} I_1 = 0$. For I_2, through Lebesgue dominated convergence theorem, we can derive that $\lim\limits_{M\to\infty} \lim\limits_{\lambda\to\lambda_0} I_2 = \lambda_0 \int_{B_{\mathbb{H}}(0,R)} u_0 e^{u_0^2} dV_{\mathbb{H}}$, which together with the estimate of I_1 gives

$$\lim_{\lambda\to\lambda_0} \lambda \int_{B_{\mathbb{H}}(0,R)} u_\lambda e^{u_\lambda^2} dV_{\mathbb{H}} = \lambda_0 \int_{B_{\mathbb{H}}(0,R)} u_0 \exp(u_0^2) dV_{\mathbb{H}}. \tag{3.14}$$

Similarly, we can also prove that

$$\lim_{\lambda\to\lambda_0} \lambda \int_{B_{\mathbb{H}}(0,R)} e^{u_\lambda^2} dV_{\mathbb{H}} = \lambda_0 \int_{B_{\mathbb{H}}(0,R)} e^{u_0^2} dV_{\mathbb{H}}. \tag{3.15}$$

Now, we claim that $u_0 \neq 0$. We argue this by contradiction. If $u_0 = 0$, then from equality (3.14), we see that $\lim\limits_{\lambda\to\lambda_0} \int_{B_{\mathbb{H}}(0,R)} |\nabla_{\mathbb{H}} u_\lambda|^2 dV_{\mathbb{H}} = 2 \lim\limits_{\lambda\to\lambda_0} I_\lambda(u_\lambda) \le 2\rho_1 < 4\pi$. Using the Trudinger-Moser inequality and Vitali convergence theorem, we derive that

$$\lim_{\lambda\to\lambda_0} \lambda \int_{B_{\mathbb{H}}(0,R)} u_\lambda^2 e^{u_\lambda^2} dV_{\mathbb{H}} = \lambda_0 \int_{B_{\mathbb{H}}(0,R)} u_0^2 e^{u_0^2} dV_{\mathbb{H}} = 0,$$

which implies that $\lim\limits_{\lambda\to\lambda_0} I_\lambda(u_\lambda) = 0$. This poses a contradiction to the following fact:

$$0 < \rho_1 \le \lim_{\lambda\to\lambda_0} I_\lambda(u_\lambda).$$

So $u_0 \neq 0$.

Next, we will prove that $u_\lambda \to u_0$ in $W_0^{1,2}(B_{\mathbb{H}}(0,R))$. According to the convexity of norm in $W_0^{1,2}(B_{\mathbb{H}}(0,R))$, we only need to prove that

$$\lim_{\lambda\to\lambda_0}\int_{B_{\mathbb{H}}(0,R)}|\nabla_{\mathbb{H}}u_\lambda|^2 dV_{\mathbb{H}} > \int_{B_{\mathbb{H}}(0,R)}|\nabla_{\mathbb{H}}u_0|^2 dV_{\mathbb{H}}$$

is impossible. We argue this by contradiction. Set

$$v_\lambda := \frac{u_\lambda}{\lim\limits_{\lambda\to\lambda_0}\|\nabla_{\mathbb{H}}u_\lambda\|^2_{L^2(B_{\mathbb{H}}(0,R))}} \text{ and } v_0 := \frac{u_0}{\lim\limits_{\lambda\to\lambda_0}\|\nabla_{\mathbb{H}}u_\lambda\|^2_{L^2(B_{\mathbb{H}}(0,R))}}.$$

We claim that there exists $q_0 > 1$ sufficiently close to 1 such that

$$q_0\lim_{\lambda\to\lambda_0}\|\nabla_{\mathbb{H}}u_\lambda\|^2_{L^2(B_{\mathbb{H}}(0,R))} < \frac{4\pi}{1-\|\nabla_{\mathbb{H}}v_0\|^2_{L^2(B_{\mathbb{H}}(0,R))}}. \tag{3.16}$$

Indeed, by (3.15) and (3.6), we have

$$\begin{aligned}
&\lim_{\lambda\to\lambda_0}\|\nabla_{\mathbb{H}}u_\lambda\|^2_{L^2(B_{\mathbb{H}}(0,R))}\big(1-\|\nabla_{\mathbb{H}}v_0\|^2_{L^2}\big)\\
&= \lim_{\lambda\to\lambda_0}\|\nabla_{\mathbb{H}}u_\lambda\|^2_{L^2(B_{\mathbb{H}}(0,R))}\left(1-\frac{\|\nabla_{\mathbb{H}}u_0\|^2_{L^2(B_{\mathbb{H}}(0,R))}}{\|\nabla_{\mathbb{H}}u_\lambda\|^2_{L^2(B_{\mathbb{H}}(0,R))}}\right)\\
&= 2\lim_{\lambda\to\lambda_0}I_\lambda(u_\lambda)+\lambda\int_{B_{\mathbb{H}}(0,R))}(e^{u_\lambda^2}-1)dV_{\mathbb{H}}-2I_{\lambda_0}(u_0)-\lambda_0\int_{B_{\mathbb{H}}(0,R)}(e^{u_0^2}-1)dV_{\mathbb{H}}\\
&< 4\pi,
\end{aligned} \tag{3.17}$$

and then the claim is proved. Through the concentration-compactness principle for the Trudinger-Moser inequality [20], one can derive that there exists $p_0 > 1$ such that

$$\sup_{\lambda}\int_{B_{\mathbb{H}}(0,R)}\big(u_\lambda^2 e^{u_\lambda^2}\big)^{p_0}dV_{\mathbb{H}} < \infty. \tag{3.18}$$

Then it follows from the Vitali convergence theorem that

$$\lim_{\lambda\to\lambda_0}\lambda\int_{B_{\mathbb{H}}(0,R)}u_\lambda^2\exp(u_\lambda^2)dV_{\mathbb{H}} = \lambda_0\int_{B_{\mathbb{H}}(0,R)}u_0^2\exp(u_0^2)dV_{\mathbb{H}}.$$

Hence, we conclude that $u_\lambda \to u_0$ in $W_0^{1,2}(B_{\mathbb{H}}(0,R))$ from (3.3). Using the Trudinger-Moser inequality in $W_0^{1,2}(B_{\mathbb{H}}(0,R))$, we derive that for any $p > 1$, there holds

$$\int_{B_{\mathbb{H}}(0,R)}\big(u_\lambda^2\exp(u_\lambda^2)\big)^p dV_{\mathbb{H}} \lesssim 1.$$

Since u_λ satisfies equation (1.9), standard elliptic estimate gives $u_\lambda \to u_0$ in $C^2(B_{\mathbb{H}}(0,R))$. Then the proof of (ii) in Theorem 1.6 is accomplished.

Proof of (iii) in Theorem 1.6: We will prove that if $\lambda_0 = \lambda_1(B_{\mathbb{H}}(0,R))$, then $u_\lambda \to 0$ in $C^2(B_{\mathbb{H}}(0,R))$. We first show that $\int_{B_{\mathbb{H}}(0,R)}|\nabla_{\mathbb{H}}u_\lambda|^2 dV_{\mathbb{H}}$ is bounded. We argue this by contradiction. Assume that $\lim\limits_{\lambda\to\lambda_0}|\nabla u_\lambda|^2dV_{\mathbb{H}}=+\infty$, then it follows from $I'_\lambda(u_\lambda)u_\lambda=0$ and $I_\lambda(u_\lambda)<2\pi$ that

$$\lim_{\lambda\to\lambda_0}\lambda\int_{B_{\mathbb{H}}(0,R)}u_\lambda^2e^{u_\lambda^2}dV_{\mathbb{H}}=+\infty,\quad \lim_{\lambda\to\lambda_0}\lambda\int_{B_{\mathbb{H}}(0,R)}e^{u_\lambda^2}dV_{\mathbb{H}}=+\infty.$$

On the other hand, we can also derive that

$$\lim_{\lambda\to\lambda_0}\lambda\int_{B_{\mathbb{H}}(0,R)}\big(u_\lambda^2e^{u_\lambda^2}-(e^{u_\lambda^2}-1)\big)dV_{\mathbb{H}}=\lim_{\lambda\to\lambda_0}2I_\lambda(u_\lambda)<4\pi,$$

which implies that $\int_{B_{\mathbb{H}}(0,R)}\big(e^{u_\lambda^2}-1-u_\lambda^2\big)dV_{\mathbb{H}}$ is bounded, hence $\int_{B_{\mathbb{H}}(0,R)}\big(e^{u_\lambda^2}-1\big)dV_{\mathbb{H}}$ is bounded since $\|u_\lambda\|_{L^2(B_{\mathbb{H}}(0,R))}\lesssim\|u_\lambda\|_{L^4(B_{\mathbb{H}}(0,R))}$. This arrives at a contradiction with the fact

$$\lim_{\lambda\to\lambda_0}\lambda\int_{B_{\mathbb{H}}(0,R)}\big(e^{u_\lambda^2}-1\big)dV_{\mathbb{H}}=+\infty.$$

Therefore, $\int_{B_{\mathbb{H}}(0,R)}|\nabla_{\mathbb{H}}u_\lambda|^2dV_{\mathbb{H}}$ is bounded. Hence, there exists some non-negative function $u_0\in W_0^{1,2}(B_{\mathbb{H}}(0,R))$ such that $u_\lambda\rightharpoonup u_0$ weakly in $W_0^{1,2}(B_{\mathbb{H}}(0,R))$. As what we did in the previous proof for (ii), we can similarly derive that

$$\lim_{\lambda\to\lambda_0}\lambda\int_{B_{\mathbb{H}}(0,R)}u_\lambda\exp(u_\lambda^2)dV_{\mathbb{H}}=\lambda_0\int_{B_{\mathbb{H}}(0,R)}u_0\exp(u_0^2)dV_{\mathbb{H}}$$

and

$$\lim_{\lambda\to\lambda_0}\lambda\int_{B_{\mathbb{H}}(0,R)}e^{u_\lambda^2}dV_{\mathbb{H}}=\lambda_0\int_{B_{\mathbb{H}}(0,R)}e^{u_0^2}dV_{\mathbb{H}}.$$

Through equation (1.9) and the definition of weak solution, we see that u_0 satisfies equation

$$\begin{cases}-\Delta_{\mathbb{H}}u=\lambda_0ue^{u^2}, & x\in B_1,\\ u>0, & x\in B_1,\\ u=0, & x\in\partial B_1.\end{cases}\tag{3.19}$$

Noticing $\lambda_0=\lambda_1(B_{\mathbb{H}}(0,R))$ is the first eigenvalue of $-\Delta_{\mathbb{H}}$ in $B_{\mathbb{H}}(0,R)$ with the Dirichlet boundary, hence one can easily obtain $u_0=0$ through Pohozaev identity. This deduces that

$$\lim_{\lambda\to\lambda_0}\lambda\int_{B_{\mathbb{H}}(0,R)}\big(\exp(u_\lambda^2)-1)dV_{\mathbb{H}}=\lambda_0\int_{B_{\mathbb{H}}(0,R)}\big(\exp(u_\lambda^2)-1)dV_{\mathbb{H}}=0.$$

Hence it follows that $\lim\limits_{\lambda\to\lambda_0}\int_{B_{\mathbb{H}}(0,R)}|\nabla_{\mathbb{H}}u_\lambda|^2dV_{\mathbb{H}}=2\lim\limits_{\lambda\to\lambda_0}I_\lambda(u_\lambda)<4\pi$. Combining this and the Trudinger-Moser inequality, we find that there exists some $p_0>1$ such that $\int_{B_{\mathbb{H}}(0,R)}\big(u_\lambda^2e^{u_\lambda^2}\big)^{p_0}dV_{\mathbb{H}}$ is bounded. Using the Vitali convergence theorem, we derive that

$$\lim_{\lambda\to\lambda_0}\lambda\int_{B_{\mathbb{H}}(0,R)}u_\lambda^2\exp(u_\lambda^2)dV_{\mathbb{H}}=\lambda_0\int_{B_{\mathbb{H}}(0,R)}u_0^2\exp(u_0^2)dV_{\mathbb{H}}=0,$$

which implies that

$$\lim_{\lambda\to\lambda_0}\int_{B_{\mathbb{H}}(0,R)}|\nabla_{\mathbb{H}}u_\lambda|^2dV_{\mathbb{H}}=0 \text{ and } \lim_{\lambda\to\lambda_0}I_\lambda(u_\lambda)=0. \tag{3.20}$$

That is to say $u_\lambda \to 0$ in $W_0^{1,2}(B_{\mathbb{H}}(0,R))$. Using the Trudinger-Moser inequality again, we derive that for any $p>1$, there holds $\int_{B_{\mathbb{H}}(0,R)}\left(u_\lambda^2\exp(u_\lambda^2)\right)^p dV_{\mathbb{H}}\lesssim 1$. Since u_λ satisfies elliptic equation (3.1), standard elliptic estimate gives that $u_\lambda\to u_0=0$ in $C^2(B_{\mathbb{H}}(0,R))$. Then the proof of (iii) in Theorem 1.6 is accomplished.

4. proof of Theorem 1.10

In this section, we will prove the multiplicity and non-existence result for the Trudinger-Moser functional

$$F(u)=\int_{B_{\mathbb{H}}(0,R)}(e^{u^2}-1)dV_{\mathbb{H}}$$

under the constraint $\int_{B_{\mathbb{H}}(0,R)}|\nabla_{\mathbb{H}}u_\lambda|^2dV_{\mathbb{H}}=\gamma$ for $\gamma>4\pi$, namely we shall give the proof of Theorem 1.10.

Obviously, the positive critical points u_0 of the Trudinger-Moser functional $F(u)$ under the constraint $\int_{B_{\mathbb{H}}(0,R)}|\nabla_{\mathbb{H}}u_\lambda|^2dV_{\mathbb{H}}=\gamma$ must satisfy

$$\begin{cases}-\Delta_{\mathbb{H}}u=\lambda_0 ue^{u^2}, & x\in B_{\mathbb{H}}(0,R),\\ u\geq 0, & x\in B_{\mathbb{H}}(0,R),\\ u=0, & x\in\partial B_{\mathbb{H}}(0,R),\\ \int_{B_{\mathbb{H}}(0,R)}|\nabla_{\mathbb{H}}u|^2dV_{\mathbb{H}}=\gamma, & \end{cases} \tag{4.1}$$

where $\lambda_0\in(0,\lambda_1(B_{\mathbb{H}}(0,R)))$. Set $\Lambda_\lambda=\int_{B_{\mathbb{H}}(0,R)}|\nabla_{\mathbb{H}}u_\lambda|^2dV_{\mathbb{H}}$, where u_λ is the positive solution of equation

$$\begin{cases}-\Delta_{\mathbb{H}}u_\lambda=\lambda u_\lambda e^{u_\lambda^2}, & x\in B_{\mathbb{H}}(0,R),\\ u_\lambda\geq 0, & x\in B_{\mathbb{H}}(0,R),\\ u_\lambda=0, & x\in\partial B_{\mathbb{H}}(0,R).\end{cases} \tag{4.2}$$

The definition of Λ_λ is well-defined because the positive solution of (4.2) is unique. Through Theorem 1.6, we see that Λ_λ is continuous with the respect to the parameter $\lambda\in(0,\lambda_1(B_{\mathbb{H}}(0,R)))$ and

$$\lim_{\lambda\to 0}\int_{B_{\mathbb{H}}(0,R)}|\nabla_{\mathbb{H}}u_\lambda|^2dV_{\mathbb{H}}=4\pi,\quad \lim_{\lambda\to\lambda_1(B_{\mathbb{H}}(0,R))}\int_{B_{\mathbb{H}}(0,R)}|\nabla_{\mathbb{H}}u_\lambda|^2dV_{\mathbb{H}}=0.$$

Hence Λ_λ is bounded in $(0,\lambda_1(B_{\mathbb{H}}(0,R)))$. Define $\gamma^*=\sup\{\Lambda_\lambda:\lambda\in(0,\lambda_1(B_{\mathbb{H}}(0,R))\}$, we see that for any $\gamma>\gamma^*$, Trudinger-Moser functional $F(u)$ under the constraint $\int_{B_{\mathbb{H}}(0,R)}|\nabla_{\mathbb{H}}u_\lambda|^2dV_{\mathbb{H}}=\gamma$ does not admit any positive critical point if $\gamma>\gamma^*$.

Now, in order to finish the proof of Theorem 1.10, we only need to prove that $F(u)$ under the constraint $\int_{B_{\mathbb{H}}(0,R)}|\nabla_{\mathbb{H}}u_\lambda|^2dV_{\mathbb{H}}=\gamma$ has at least two positive critical points if

$\gamma \in (4\pi, \gamma^*)$. Since Λ_λ is continuous with respect to the parameter $\lambda \in (0, \lambda_1(B_{\mathbb{H}}(0,R)))$ and $\Lambda(0) = 4\pi$, $\Lambda(\lambda_1(B_{\mathbb{H}}(0,R))) = 0$, hence it suffices to prove that there exists a $\lambda_* \in (0, \lambda_1(B_{\mathbb{H}}(0,R)))$ such that $\Lambda(\lambda_*) > 4\pi$. This will be easily verified by showing that the Trudinger-Moser functional $F(u)$ under the constraint $\int_{B_{\mathbb{H}}(0,R)} |\nabla_{\mathbb{H}} u_\lambda|^2 dV_{\mathbb{H}} = \gamma$ for γ sufficiently close to 4π has a local maximum point. The argument is essentially similar to the one for the local maximum point of the super-critical Trudinger-Moser functional on bounded domain of $\mathbb{R}^2$, which was proved by Struwe in [34]. For simplicity, we only give the outline of the proof.

Step 1: Set

$$\beta^*_{4\pi} = \sup_{\int_{B_{\mathbb{H}}(0,R)} |\nabla_{\mathbb{H}} u_\lambda|^2 dV_{\mathbb{H}} = 1} \int_{B_{\mathbb{H}}(0,R)} (e^{4\pi u^2} - 1) dV_{\mathbb{H}},$$

then the set

$$K_{4\pi} = \{u \in W_0^{1,2}(B_{\mathbb{H}}(0,R)) : \int_{B_{\mathbb{H}}(0,R)} |\nabla_{\mathbb{H}} u|^2 dV_{\mathbb{H}} = 1, \int_{B_{\mathbb{H}}(0,R)} e^{4\pi u^2} dV_{\mathbb{H}} = \beta^*_{4\pi}\}$$

is compact. The proof of compactness is essential to the proof of existence of extremal functional for critical Trudinger-Moser functional on two dimensional compact manifold which is established by Y. X. Li [21, 22].

Step 2: Let Σ be the set consisting of all functions $u \in W_0^{1,2}(B_{\mathbb{H}}(0,R))$ satisfying $\int_{B_{\mathbb{H}}(0,R)} |\nabla_{\mathbb{H}} u_\lambda|^2 dV_{\mathbb{H}} = 1$ and define the Dirichlet norm neighborhoods of $K_{4\pi}$ in Σ by

$$N_\epsilon = \{u \in \Sigma |\ \exists v \in K_{4\pi} \ \ s.t. \int_{B_{\mathbb{H}}(0,R)} |\nabla_{\mathbb{H}} (u - v)|^2 dV_{\mathbb{H}} < \epsilon\}.$$

Similarly, we can show that for sufficiently small $\epsilon > 0$, there holds

$$\sup_{u \in N_{2\epsilon} \setminus N_\epsilon} \int_{B_{\mathbb{H}}(0,R)} (e^{4\pi u^2} - 1) dV_{\mathbb{H}} < \beta^*_{4\pi} = \sup_{u \in N_\epsilon} \int_{B_{\mathbb{H}}(0,R)} (e^{4\pi u^2} - 1) dV_{\mathbb{H}}. \tag{4.3}$$

Step 3: Through compactness of $K_{4\pi}$ and uniformly local continuity of F, we can show that there exists $\alpha^* > 4\pi$ and $\varepsilon > 0$ such that for any $\alpha \in [4\pi, \alpha^*)$, there holds

$$\sup_{u \in N_{2\epsilon} \setminus N_\epsilon} \int_{B_{\mathbb{H}}(0,R)} (e^{\alpha u^2} - 1) dV_{\mathbb{H}} < \sup_{u \in N_\epsilon} \int_{B_{\mathbb{H}}(0,R)} (e^{\alpha u^2} - 1) dV_{\mathbb{H}} =: \beta^*_\alpha$$

Combining Steps 1-3, one can easily obtain that Trudinger-Moser functional $F(u) = \int_{B_{\mathbb{H}}(0,R)} (e^{u^2} - 1) dV_{\mathbb{H}}$ under the constraint $\int_{B_{\mathbb{H}}(0,R)} |\nabla_{\mathbb{H}} u_\lambda|^2 dV_{\mathbb{H}} = \gamma$ for γ sufficiently close to 4π has a local maximum point. This accomplishes the proof of Theorem 1.10.

5. Proof of Theorem 1.15

Obviously, the positive critical points u_0 of the perturbed Trudinger-Moser functional $\tilde{H}(u) = \int_{B_1}(e^{u^2} - 1 - u^2)dx$ under the constraint $\int_{B_1} |\nabla u|^2 dx = \beta$ must satisfy

$$\begin{cases} -\Delta u = \lambda_0 u(e^{u^2} - 1), & x \in B_1 \\ u > 0, & x \in B_1 \\ u = 0, & x \in \partial B_1 \\ \int_{B_1} |\nabla u|^2 dx = \beta. \end{cases} \tag{5.1}$$

Through the moving-plane method, it is easy to check that u_0 is radially decreasing. Then u_0 satisfies the following ODE equation

$$\begin{cases} -(ru')' = r\lambda_0 u(e^{u_\lambda^2} - 1), & r \in (0,1), \\ u > 0, & r \in (0,1), \\ u'(0) = 0, u(1) = 0. \end{cases} \tag{5.2}$$

Using the argument of Theorem 1.1 again, we can deduce that u_0 is the least energy critical point of the functional

$$I_{\lambda_0}(u) = \frac{1}{2}\int_{B_1} |\nabla u|^2 dx - \frac{\lambda_0}{2}\int_{B_1}(e^{u^2} - 1 - u^2)dx.$$

By using the Trudinger-Moser inequality and Nehari manifold method (see the Appendix), one can deduce that $I_{\lambda_0}(u_0) < 2\pi$. Using $I'_{\lambda_0}(u_0)u_0 = 0$, we obtain

$$\begin{aligned} I_{\lambda_0}(u_0) &= \frac{\lambda_0}{2}\int_{B_1} u_0^2(e^{u_0^2} - 1)dx - \frac{\lambda_0}{2}\int_{B_1}(e^{u_0^2} - 1 - u_0^2)dx \\ &\geq \frac{\lambda_0}{4}\int_{B_1} u_0^2(e^{u_0^2} - 1)dx = \frac{1}{4}\int_{B_1} |\nabla u_0|^2 dx. \end{aligned} \tag{5.3}$$

This together with $I(u_0) < 2\pi$ yields $\int_{B_1} |\nabla u_0|^2 dx < 8\pi$. Hence the perturbed supercritical Trudinger-Moser functional $\tilde{H}(u) = \int_{B_1}(e^{u^2} - 1 - u^2)dx$ under the constraint $\int_{B_1} |\nabla u|^2 dx = \beta$ does not admit any positive critical point if $\beta \geq 8\pi$.

6. Appendix

In this section, we will show that the functional energy $I_\lambda(u_\lambda) < 2\pi$, if u_λ is the least energy point of functional

$$I_\lambda(u) = \frac{1}{2}\int_{B_{\mathbb{H}}(0,R)} |\nabla_{\mathbb{H}} u|^2 dV_{\mathbb{H}} - \frac{\lambda}{2}\int_{B_{\mathbb{H}}(0,R)}(e^{u^2} - 1)dV_{\mathbb{H}}.$$

Recalling the definition of the least energy critical point of functional $I_\lambda(u)$, we know that $I_\lambda(u_\lambda) = m_\lambda := \min\{I_\lambda(u) : \ I'_\lambda(u)u = 0\}$.

We first show that $m_\lambda > 0$. We argue this by contradiction. Assume that $m_\lambda = 0$, then exists a sequence $\{u_k\}_k \in W_0^{1,2}(B_{\mathbb{H}}(0,R))$ such that

$$\int_{B_{\mathbb{H}}(0,R)} |\nabla_{\mathbb{H}} u_k|^2 dV_{\mathbb{H}} - \lambda \int_{B_{\mathbb{H}}(0,R)} u_k^2 e^{u_k^2} dV_{\mathbb{H}} = 0,\ \forall k \in \mathbb{N}$$

and

$$\lim_{k\to\infty} \left(\frac{1}{2} \int_{B_{\mathbb{H}}(0,R)} |\nabla_{\mathbb{H}} u_k|^2 dV_{\mathbb{H}} - \frac{\lambda}{2} \int_{B_{\mathbb{H}}(0,R)} \big(e^{u_k^2} - 1\big) dV_{\mathbb{H}} \right) = 0.$$

Direct computations show that

$$\begin{aligned} m_\lambda &= \lim_{k\to\infty} \left(\frac{1}{2} \int_{B_{\mathbb{H}}(0,R)} |\nabla_{\mathbb{H}} u_k|^2 dV_{\mathbb{H}} - \frac{\lambda}{2} \int_{B_{\mathbb{H}}(0,R)} (e^{u_k^2} - 1) dV_{\mathbb{H}} \right) \\ &= \lim_{k\to\infty} \left(\frac{\lambda}{2} \int_{B_{\mathbb{H}}(0,R)} u_k^2 e^{u_k^2} dV_{\mathbb{H}} - \frac{\lambda}{2} \int_{B_{\mathbb{H}}(0,R)} (e^{u_k^2} - 1) dV_{\mathbb{H}} \right) \\ &\geq \frac{\lambda}{4} \lim_{k\to\infty} \int_{B_{\mathbb{H}}(0,R)} u_k^2 (e^{u_k^2} - 1) dV_{\mathbb{H}} \\ &= \frac{1}{4} \lim_{k\to\infty} \int_{B_{\mathbb{H}}(0,R)} \big(|\nabla_{\mathbb{H}} u_k|^2 - \lambda |u_k|^2\big) dV_{\mathbb{H}}. \end{aligned} \tag{6.1}$$

Since $\lambda < \lambda_1(B_{\mathbb{H}}(0,R))$, it follows from the Sobolev imbedding theorem that

$$u_k \to 0 \text{ in } W_0^{1,2}(B_{\mathbb{H}}(0,R)) \ \text{ and } \ u_k \to 0 \ \text{ in } L^p(B_{\mathbb{H}}(0,R)) \ \text{ for any } p \geq 1.$$

Let $v_k = \frac{u_k}{\|\nabla_{\mathbb{H}} u_k\|_2}$, then $v_k \rightharpoonup v$ in $W_0^{1,2}(B_{\mathbb{H}}(0,R))$ with $\|v\|_2^2 \leq \lim \|v_k\|_2^2 < \frac{1}{\lambda_1(B_{\mathbb{H}}(0,R))}$.

Since $u_k \to 0$ in $W_0^{1,2}(B_{\mathbb{H}}(0,R))$, by Trudinger-Moser inequality in $W_0^{1,2}(B_{\mathbb{H}}(0,R))$, we have $e^{u_k^2} \in L^p(B_{\mathbb{H}}(0,R))$ for any $p > 1$. Then it follows from the Vitali convergence theorem that

$$\begin{aligned} 1 &= \lim_{k\to+\infty} \int_{B_{\mathbb{H}}(0,R)} \frac{\lambda u_k^2}{\|\nabla_{\mathbb{H}} u_k\|_2^2} e^{u_k^2} dV_{\mathbb{H}} \\ &= \lim_{k\to+\infty} \lambda \int_{B_{\mathbb{H}}(0,R)} e^{u_k^2} |v_k|^2 dx \\ &= \lambda \|v\|_2^2 \leq \frac{\lambda}{\lambda_1(B_{\mathbb{H}}(0,R))} < 1, \end{aligned} \tag{6.2}$$

which is a contradiction.

Next, we start to prove that $m_\lambda < 2\pi$. Let $w \in W_0^{1,2}(B_{\mathbb{H}}(0,R))$ such that $\|\nabla_{\mathbb{H}} w\|_2^2 - \lambda \|w\|_2^2 = 1$. Then there exists some $\gamma_w > 0$ such that

$$\int_{B_{\mathbb{H}}(0,R)} (|\nabla_{\mathbb{H}} \gamma_w w|^2 - \lambda |\gamma_w w|^2) dV_{\mathbb{H}} - \lambda \int_{B_{\mathbb{H}}(0,R)} (\gamma_w w)^2 \big(e^{(\gamma_w w)^2} - 1\big) dV_{\mathbb{H}} = 0,$$

which implies that

$$(6.3)\qquad \begin{aligned} m_\lambda &\le \frac{1}{2}\int_{B_{\mathbb{H}}(0,R)} \big(|\nabla_{\mathbb{H}}\gamma_w w|^2 - \lambda|\gamma_w w|^2\big)dV_{\mathbb{H}} - \frac{\lambda}{2}\int_{B_{\mathbb{H}}(0,R)} \big(e^{(\gamma_w w)^2} - 1 - (\gamma_w w)^2\big)dV_{\mathbb{H}} \\ &< \frac{\gamma_w^2}{2}\int_{B_{\mathbb{H}}(0,R)} \big(|\nabla_{\mathbb{H}} w|^2 - \lambda|w|^2\big)dV_{\mathbb{H}} = \frac{\gamma_w^2}{2}. \end{aligned}$$

Set $m_\lambda = \frac{\gamma_\infty^2}{2}$. Since $\big(e^{(\gamma w)^2} - 1\big)w^2$ is monotone increasing about the variable γ, we derive that

$$(6.4)\qquad \begin{aligned} \int_{B_{\mathbb{H}}(0,R)} \big(e^{(\gamma_\infty w)^2} - 1\big)w^2 dV_{\mathbb{H}} &\le \int_{B_{\mathbb{H}}(0,R)} \big(e^{(\gamma_w w)^2} - 1\big)w^2 dV_{\mathbb{H}} \\ &= \int_{B_{\mathbb{H}}(0,R)} (|\nabla_{\mathbb{H}} w|^2 - \lambda|w|^2)dV_{\mathbb{H}} = 1, \end{aligned}$$

which implies that

$$\sup_{\int_{B_{\mathbb{H}}(0,R)}(|\nabla_{\mathbb{H}} w|^2 - \lambda|w|^2)dV_{\mathbb{H}} = 1} \int_{B_{\mathbb{H}}(0,R)} \big(e^{(\gamma_\infty w)^2} - 1\big)w^2 dV_{\mathbb{H}} < \infty.$$

Noticing

$$\int_{B_{\tilde{R}}(0)} |\nabla w|^2 dx = \int_{B_{\mathbb{H}}(0,R)} |\nabla_{\mathbb{H}} w|^2 dV_{\mathbb{H}},$$

and $dV_{\mathbb{H}} = \big(\frac{2}{1-|x|^2}\big)^2 dx$, we obtain that

$$\sup_{\int_{B_{\tilde{R}}(0)} |\nabla w|^2 dx = 1} \int_{B_{\tilde{R}}(0)} \big(e^{(\gamma_\infty w)^2} - 1\big)w^2 dx < \infty,$$

where $B_{\tilde{R}}(0)$ denotes the ball with radius $\tilde{R}$ equal to $\frac{e^R-1}{e^R+1}$ in $\mathbb{R}^2$. Then one can construct well-known Moser sequence (that is a concentration sequence which blows up at some point) to deduce that $m_\lambda = \frac{\gamma_\infty^2}{2} < 2\pi$.

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