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Elliptic and parabolic problems for a class of operators with discontinuous coefficients

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Abstract. We study elliptic and parabolic problems associated to the second order elliptic operator

$$L = \Delta + (a-1)\sum_{i,j=1}^{N} \frac{x_i x_j}{|x|^2} D_{ij} + c\frac{x}{|x|^2} \cdot \nabla - b|x|^{-2}$$

with $a > 0$ and $b,\ c$ real coefficients. We prove generation of analytic semigroup and domain characterization.

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1. Introduction

In this paper we study elliptic and parabolic problems in $L^p(\mathbb{R}^N)$ associated to the second order elliptic operator

$$L = \Delta + (a-1)\sum_{i,j=1}^{N} \frac{x_i x_j}{|x|^2} D_{ij} + c\frac{x}{|x|^2} \cdot \nabla - b|x|^{-2}$$

with $a > 0$ and $b,\ c$ real coefficients. The leading coefficients are uniformly elliptic but discontinuous at 0, if $a \neq 1$, and singularities in the lower order terms appear when b or c is different from 0. The operator commutes with dilations, in the sense that $I_s^{-1} L I_s = s^2 L$, if $I_s u(x) = u(sx)$.

In the special case $b = c = 0$, these operators have already been introduced to provide counterexamples to the elliptic regularity. Positive results have also been obtained by Manselli and Ragnedda, see [5–7], who proved existence and uniqueness results in Sobolev spaces in a bounded domain containing the origin and spectral properties in the two-dimensional case.

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When $a = 1$, that is when the discontinuity in the second order coefficients disappears, generation results can be deduced by [3], or by [8, 15, 18, 19], where more general operators have been studied and the domain is explicitly described. Similar operators, but with singular coefficients only at infinity, have been studied also in [16,17,20].

If $1 < p < \infty$, we define the maximal operator $L_{p,\max}$ through the domain

$$D(L_{p,\max}) = \left\{u \in L^p\left(\mathbb{R}^N\right) \cap W^{2,p}_{\mathrm{loc}}\left(\mathbb{R}^N \setminus \{0\}\right)\, ;\ Lu \in L^p\left(\mathbb{R}^N\right)\right\}$$

and note that, by local elliptic regularity, $L_{p,\max}$ is closed and

$$D(L_{p,\max}) = \left\{u \in L^p\left(\mathbb{R}^N\right)\, ;\ Lu \in L^p\left(\mathbb{R}^N\right) \text{ as a distribution in } \mathbb{R}^N \setminus \{0\}\right\}.$$

The operator $L_{p,\min}$ is defined as the closure, in $L^p(\mathbb{R}^N)$ of $(L, C_c^\infty(\mathbb{R}^N \setminus \{0\})$ (the closure exists since this operator is contained in the closed operator $L_{p,\max}$) and it is clear that $L_{p,\min} \subset L_{p,\max}$.

This paper contains results on generation theorems and domain characterization. To state them and explain how they are proved we introduce some notation.

The equation $Lu = 0$ has radial solutions $|x|^{-s_1}$, $|x|^{-s_2}$ where s_1, s_2 are the roots of the indicial equation $f(s) = -as^2 + (N - 1 + c - a)s + b = 0$ given by

$$s_1 := \frac{N-1+c-a}{2a} - \sqrt{D}, \quad s_2 := \frac{N-1+c-a}{2a} + \sqrt{D}, \tag{1.1}$$

where

$$D := \frac{b}{a} + \left(\frac{N-1+c-a}{2a}\right)^2. \tag{1.2}$$

The above numbers are real if and only if $D \geq 0$. When $D < 0$ the equation $u - Lu = f$ cannot have positive distributional solutions for certain positive f, see Proposition 3.11. In the case of Schrödinger operators with inverse square potential, a famous result due to Baras and Goldstein [2] states the instantaneous blow up of positive distributional solutions of the corresponding heat equation, when $D = b + (N-2)^2/4 < 0$. Our result can be seen as an elliptic counterpart in the general situation.

We point out, however, that even when $b + (N - 2)^2/4$ is negative there are realizations of the operator L in $L^2(\mathbb{R}^N)$ which generate analytic semigroups. Such semigroups are not positive and these realizations are necessarily non self-adjoint, see [9].

Assuming $D \geq 0$ we show that there exists an intermediate operator $L_{p,\min} \subset L_{p,\text{int}} \subset L_{p,\max}$ which generates an (analytic) semigroup in $L^p(\mathbb{R}^N)$ if and only if $\frac{N}{p} \in (s_1, s_2 + 2)$. An intuitive explanation (for $D > 0$) of this result is the following. If $u(x) = \sum u_j(r)P_j(\omega)$ and $f = \sum f_j(r)P_j(\omega)$ where (P_j) are spherical harmonics, then the equation $\omega^2 u - Lu = f$, $\operatorname{Re}\omega > 0$ can be reduced to the infinite system ODE of Bessel type

$$\omega^2 u_j(r) - \left(a u_j''(r) + \frac{N-1+c}{r} u_j'(r) - (b+\lambda_n)\frac{u_j(r)}{r^2} \right) = f_j(r), \tag{1.3}$$

where n is the degree of P_j and $\lambda_n = n^2 + (N-2)n$ are the eigenvalues of the Laplace-Beltrami operator on the sphere S^{N-1}. Each of the above equation has characteristic numbers $s_1^{(n)}, s_2^{(n)}$, defined as in (1.1), (1.2) with $b + \lambda_n$ instead of b. The numbers $s_1^{(n)}$ decrease to $-\infty$, whereas $s_2^{(n)}$ increase to $+\infty$. The equations (1.3) have more regularizing effect as n increases, since the potentials $(-b+\lambda_n)r^{-2}$ become more and more negative and therefore the most critical equation appears for $n = 0$ and corresponds to radial functions. For positive ω, (1.3) with $n = 0$ and $f_0 = 0$ has two linearly independent solutions $v_{\omega,1}, v_{\omega,2}$ with the following properties: $v_{\omega,1}$ is exponentially increasing at ∞ and behaves like r^{-s_1} as $r \to 0$, $v_{\omega,2}$ is exponentially decreasing at ∞ and behaves like r^{-s_2} as $r \to 0$. Using these function one can construct a Green function as for Sturm-Liouville problems. However, if $N/p \leq s_1$, then neither $v_{\omega,1}$ or $v_{\omega,2}$ belong to $L^p((0,1), r^{N-1}\,dr)$ and equation (1.3) with $n = 0$ cannot be solved for suitable f_0. If $N/p \geq s_2 + 2$, the function $v_{\omega,2}$ belongs to the domain of the minimal operator $L_{p,\min}$ and is therefore an eigenfunction of any of its extensions. These facts explain the negative part of our result.

If $N/p \in (s_1, s_2)$, then $v_{\omega,1}$ is the only solution of the homogeneous equation which is in L^p near 0 and $v_{\omega,2}$ is the only solution of the homogeneous equation which is in L^p near ∞ (in both cases with respect to the measure $r^{N-1}\,dr$). This means that there is only one way to construct a resolvent and hence $L_{p,\max}$ is a generator. By duality, $L_{p,\min}$ is a generator when $N/p \in (s_1 + 2, s_2 + 2)$. Therefore $L_{p,\text{int}} = L_{p,\max}$ if $N/p \in (s_1, s_2]$ and $L_{p,\text{int}} = L_{p,\min}$ if $N/p \in [s_1 + 2, s_2 + 2)$ and $L_{p,\text{int}}$ is the unique realization of L between $L_{p,\min}$ and $L_{p,\max}$ which generates a semigroup, when these two intervals overlap, that is when $s_1 + 2 \leq s_2$, since it coincides either with $L_{p,\min}$ or with $L_{p,\max}$ (and with both when $N/p \in [s_1 + 2, s_2]$). However, if $s_2 < s_1 + 2$ and N/p is in between, that is when

$$D = \frac{b}{a} + \left(\frac{N-1+c-a}{2a} \right)^2 \in [0,1) \ \text{ and } \ \frac{N}{p} \in (s_2, s_1 + 2),$$

both functions $v_{\omega,1}, v_{\omega,2}$ are in $L^p((0,1), r^{N-1}\,dr)$ and there is no uniqueness even among the generators of positive and analytic semigroups, see [12]. The choice of the domain of $L_{p,\text{int}}$ is made to preserve the consistency of the semigroup in the

L^q-scale, by extrapolating the semigroup from those $L^q(\mathbb{R}^N)$ for which there is uniqueness; namely we select $v_{\omega,1}$ to construct the Green function near 0 but other choices are possible.

The above arguments can be made rigorous in L^2 by expansion in spherical harmonics, but not directly in L^p. Instead we use a global argument based on improved Hardy and Poincaré inequalities which yield complex dissipativity on subspaces of $L^p(\mathbb{R}^N)$ generated by high order spherical harmonics and then we perform a one dimensional analysis on a finite number of cases.

Before describing in more details the content of the sections, let us discuss another way to look at the operator. In spherical coordinates we write

$$a^{-1}L = D_{rr} + \frac{N-1+\tilde{c}}{r}D_r - \frac{\tilde{b}}{r^2} + \frac{\Delta_0}{ar^2} =: C + \frac{\Delta_0}{ar^2}, \tag{1.4}$$

where $\tilde{c} = a^{-1}(c-(N-1)(a-1))$ and $\tilde{b} = a^{-1}b$. Clearly L and $a^{-1}L$ have the same operator theoretical properties and, accordingly, the corresponding numbers s_1, s_2 and $\tilde{s}_1.\tilde{s}_2$ coincide. However, when written in this form, it is easily seen from (1.1) that these numbers depend only on $\tilde{b}$ and $\tilde{c}$ and not on a. This means that all the results proved in this paper, both on the generation side and on domain characterization, are independent of $a > 0$ for an operator as in (1.4), hence they rely only on the radial part C. This remarkable fact, however, is a consequence of our results and we have no a-priori argument for showing it, since the operators C and Δ_0/r^2 do not commute. In order to explain better this point we discuss a situation where a plays a role. From [11, Theorem 3.1] it is seen that the validity of Rellich estimates

$$\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{2p}}\,dx \le M \int_{\mathbb{R}^N} \left| Cu + \frac{\Delta_0 u}{ar^2}\right|^p\,dx$$

for $u \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$ does depend on the parameter a and the same holds also for Calderón-Zygmund estimates. These facts seem to be, at first sight, in contrast with the domain characterization for (1.4) which, as said before, depends only on $\tilde{b}, \tilde{c}$. However, no contradiction arises since the generation properties and the domain characterization depend on the solvability of the set of equations (1.3), whose most degenerate one is that with $\lambda_0 = 0$ which is not affected by division by a. On the other hand, Rellich and Calderón-Zygmund inequalities can hold in (the closure of) $C_c^\infty(\mathbb{R}^N \setminus \{0\})$ which is too small, in some cases, to provide solvability. It turns out that these estimates are equivalent to a spectral problem and hold in $C_c^\infty(\mathbb{R}^N \setminus \{0\})$ except for countably many values of $a, \tilde{b}, \tilde{c}$ (explicitly computed) and the parameter a plays a role, since the eigenvalues λ_j of the Laplace-Beltrami operator Δ_0 are changed to $\lambda_j/a \neq \lambda_j$ for $j \ge 1$.

A summary of the contents of the chapters is the following. In Chapter 2 we define the spaces $L^p_{<n}$, $L^p_{\ge n}$ which are the closed linear span of functions of the form $\sum_j f_j(r)P_j(\omega)$, where the sums are finite and the spherical harmonics P_j

have degree less than n or greater or equal than n, respectively. We recall their main properties and study the action of L when restricted to them. We study carefully Sobolev regularity of functions of the form $v(r)P(\omega)$, where P is a spherical harmonics and v is defined as the mean over S^{N-1} of a Sobolev function u. We improve Poincaré and Hardy inequalities on the sphere S^{N-1} for functions which are linear combination of spherical harmonics of high order and then we improve weighted Poincaré and Hardy inequalities on $L^p_{\geq n}$, showing that the best constants tend to 0 as $n \to \infty$ (we do not make any effort to estimate these constants which are related to the eigenvalues of the p-Laplacian $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ on the sphere). Using these results, we show in Section 3 that $L_{p,\min} = L_{p,\max}$ generates an analytic semigroup of contraction in $L^p_{\geq n}$ if n is large enough and characterize the domain, using Rellich inequalities proved in [11] and [10]. The operator in $L^p_{<n}$ is reduced to a finite number of ordinary differential operators of Bessel type, whose resolvents can be written explicitly and analyzed in detail. The condition $D \geq 0$ comes from the 1-d analysis and guarantees the existence of positive solutions near the origin; in contrast, when $D < 0$ all radial solutions of the homogeneous equation $u - Lu = 0$ oscillate. Finally, we put together the results in $L^p_{\geq n}$ and in $L^p_{<n}$ to obtain necessary and sufficient conditions for generation in $L^p(\mathbb{R}^N)$. When these conditions are satisfied, the semigroup turns out to be analytic and positive. However, it need not be contractive and this explains why the variational approach based on Hardy inequality does not give the full range of p for which the semigroup exists. We refer the reader to [3] where the question of non-contractivity is treated in the case $a = 1$ and also to [8] where similar phenomena are shown for certain degenerate operators. Section 4 is devoted to analyze in detail the operator

$$L = \Delta + (a-1)\sum_{i,j=1}^{N} \frac{x_i x_j}{|x|^2} D_{ij},$$

which corresponds to $b = c = 0$ and to compare our results with those obtained in [5] where, however, the more restrictive condition $a \geq 1$ is assumed.

Notation. We denote by $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ the natural numbers including 0. Often we use Ω for $\mathbb{R}^N \setminus \{0\}$. When V is an open subset of $\mathbb{R}^N$, $C_b(V)$ is the Banach space of all continuous and bounded functions in V, endowed with the sup-norm, $C_0(V)$ its subspace consisting of functions vanishing at the boundary of V and at infinity, when V is unbounded. $C_c^\infty(V)$ denotes the space of infinitely continuously differentiable functions with compact support in V. $C_0^0(\mathbb{R}^N)$ stands for the Banach space of all continuous functions in $\mathbb{R}^N$ vanishing at $0, \infty$. The unit sphere $\{\|x\| = 1\}$ in $\mathbb{R}^N$ is denoted by S^{N-1} and Δ_0 is the Laplace-Beltrami operator on S^{N-1}. We adopt standard notation for L^p and Sobolev spaces and write L^p_{rad} for $L^p((0,\infty), r^{N-1}\,dr)$. We denote by $\mathbb{C}_+ := \{z\ ;\ \operatorname{Re} z > 0\}$ and write for nonnegative functions f, g, $f(z) \approx g(z)$ as $|z| \to 0$ ($|z| \to \infty$) if $\frac{f(z)}{g(z)}$ converges to a positive constant as $|z| \to 0$ ($|z| \to \infty$).

2. The operator L in spherical coordinates

2.1. Spherical coordinates

Introducing spherical coordinates $x = r\omega$, $r = |x|$ and $\omega = x/|x| \in S^{N-1}$, we write the Laplace operator as

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{N-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_0,$$

where Δ_0 the Laplace-Beltrami operator on the unit sphere S^{N-1}, see [22, Chapter IX]).

We recall that a spherical harmonic P_n of order n is the restriction to S^{N-1} of a homogeneous harmonic polynomial of degree n and that the linear span of spherical harmonics (which coincides with all polynomials) is dense in $C(S^{N-1})$, hence in $L^p(S^{N-1})$. We refer to [21, Chapter IV.2] for a proof of the following well-known lemma.

Lemma 2.1. *If P is a spherical harmonic of degree n, then*

$$\Delta_0 P = -\left(n^2 + (N-2)n\right) P.$$

The values $\lambda_n := n^2 + (N-2)n$ are the eigenvalues of the Laplace-Beltrami operator $-\Delta_0$ on S^{N-1} and the corresponding eigenspaces consist of all spherical harmonics of degree n and have dimension d_n where $d_0 = 1$, $d_1 = N$ and for $n \geq 2$

$$d_n = \binom{N+n-1}{n} - \binom{N+n-3}{n-2}.$$

If $u \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$, $u(x) = \sum u_j(r)P_j(\omega)$ (here we consider finite sums), then

$$Lu(r,\omega) = \sum_j \left(a u_j''(r) + \frac{N-1+c}{r} u_j'(r) - (b + \lambda_{n_j})\frac{u_j(r)}{r^2} \right) P_j(\omega), \quad (2.1)$$

where n_j is the degree of the spherical harmonic P_j.

The formal adjoint of L is given by

$$L^* = \Delta + (a-1)\sum_{i,j=1}^N \frac{x_i x_j}{|x|^2} D_{ij} + c^* \frac{x}{|x|^2}\cdot\nabla - b^*|x|^{-2}, \quad (2.2)$$

where $c^* = 2(N-1)(a-1) - c$ and $b^* = b + (N-2)(c - (N-1)(a-1))$. Observe that, in spherical coordinates $x = r\omega$,

$$\begin{aligned} L &= aD_{rr} + \frac{N-1+c}{r} D_r - \frac{b-\Delta_0}{r^2} \\ L^* &= aD_{rr} + \frac{N-1+c^*}{r} D_r - \frac{b^*-\Delta_0}{r^2}, \end{aligned} \quad (2.3)$$

where Δ_0 is the Laplace-Beltrami operator on S^{N-1}.

The following lemma follows from elliptic regularity, see [8, Proposition 2.2].

Lemma 2.2. *Let $1 < p < \infty$. Then the adjoint of $L_{p,\min}$, $L_{p,\max}$ are $L^*_{p',\max}$, $L^*_{p',\min}$, respectively.*

Let us compute the numbers s_1^*, s_2^*, D^* defined as in (1.1), (1.2) and relative to L^*. We have

$$\begin{aligned} D^* &:= \frac{b^*}{a} + \left(\frac{N-1+c^*-a}{2a}\right)^2 = D, \\ s^*_{1,2} &:= \frac{N-1+c^*-a}{2a} \mp \sqrt{D^*} = s_{1,2} + \frac{(a-1)(N-1)-c}{a} = N-2-s_{2,1}. \end{aligned} \tag{2.4}$$

Observe that $\frac{N}{p} > s_1$ is equivalent to $\frac{N}{p'} < s_2^* + 2$ and $\frac{N}{p} < s_2$ is equivalent to $\frac{N}{p'} > s_1^* + 2$. Similarly, $\frac{N}{p} > s_1 + 2$ is equivalent to $\frac{N}{p'} < s_2^*$ and $\frac{N}{p} < s_2 + 2$ is equivalent to $\frac{N}{p'} > s_1^*$.

L is formally self-adjoint, that is $L = L^*$, if and only if $c = (a-1)(N-1)$. In this case

$$s_{1,2} = \frac{N-2}{2} \mp \sqrt{\frac{b}{a} + \left(\frac{N-2}{2}\right)^2}.$$

2.2. The spaces L^p_J

We refer the reader to [11, Section 2] for further information and proofs of the following results. If X, Y are function spaces over G_1, G_2 we denote by $X \otimes Y$ the algebraic tensor product of X and Y, that is the set of all functions $u(x, y) = \sum_{i=1}^n f_i(x) g_i(y)$ where $f_i \in X$, $g_i \in Y$ and $x \in G_1$, $y \in G_2$. In what follows we denote by P a spherical harmonic and by deg P its degree. We fix a complete orthonormal system of spherical harmonics $\{P_j,\ j \in \mathbb{N}_0\}$ (which is dense in $L^p(S^{N-1})$ for every $1 \le p < \infty$) and a subset J of $\mathbb{N}_0$.

Definition 2.3. Let J be a subset of N_0 and $1 \le p < \infty$. We denote by L^p_J, the closure of

$$L^p((0,\infty), r^{N-1}dr) \otimes \operatorname{span}\{P_j : j \in J\}$$

in $L^p(\mathbb{R}^N)$. We use $L^p_{\ge n}$, $L^p_{<n}$ and L^p_n when J identifies all spherical harmonics of degree $\ge n, < n, = n$, respectively.

Let us observe that $L^p = L^p_{\mathbb{N}_0}$ and that L^p_0 consists of all radial functions in L^p. Moreover $C_c^\infty((0,\infty)) \otimes \operatorname{span}\{P_j : j \in J\}$ is dense in L^p_J. Observe that, by (2.1), the spaces L^p_J are invariant under the operator L. The next lemma clarifies the structure of the spaces L^p_J.

Lemma 2.4. *Let* $1 \le p < \infty$ *and assume that the* L^2 *orthogonal projection* $S : L^2(S^{N-1}) \to \mathrm{span}\{P_j : j \in J\}$ *extends to a bounded projection in* $L^p(S^{N-1})$. *Then*

$$L^p = L^p_J \oplus L^p_{\mathbb{N}_0 \setminus J}$$

and

$$L^p_J = \left\{ u \in L^p : \int_{S^{N-1}} u(r\,\omega) P_j(\omega)\, d\sigma(\omega) = 0 \text{ for a.e. } r > 0 \text{ and } j \notin J \right\}. \quad (2.5)$$

If J *is finite we have in addition*

$$L^p_J = \left\{ u = \sum_{j \in J} f_j(r) P_j(\omega) : f_j \in L^p((0,\infty), r^{N-1}dr) \right\}$$

and the projection $I \otimes S : L^p \to L^p_J$ *is given by*

$$(I \otimes S)u = \sum_{j \in J} T_j u(r)\, P_j(\omega),$$

where

$$T_j u(r) := \int_{S^{N-1}} u(r\,\omega) P_j(\omega)\, d\sigma(\omega).$$

Remark 2.5. Observe that the hypotheses on the above lemma are always satisfied if J (or $\mathbb{N}_0 \setminus J$) is finite. In this last case note also that if $u \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$ then $(I \otimes S)u \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$. We also remark that equality (2.5) holds without assuming the existence of a bounded projection S.

Lemma 2.6. *Under the hypotheses of the Lemma* 2.4, *the dual space of* L^p_J *is* $L^{p'}_J$.

Proof. By the lemma above L^p_J is the quotient of L^p with respect to $L^p_{\mathbb{N}_0 \setminus J}$ and hence its dual coincides with the annihilator, say E, of $L^p_{\mathbb{N}_0 \setminus J}$ in $L^{p'}$, with respect to the usual duality between L^p and $L^{p'}$, denoted by $\langle , \rangle$. Next we note that a function $v \in L^{p'}$ belongs to E if and only if $\langle w, v \rangle = 0$ for every $w = u(r)P_j(\omega) \in L^p$, $j \notin J$ or

$$\int_{S^{N-1}} v(r\,\omega) P_j(\omega)\, d\sigma(\omega) = 0 \text{ for a.e. } r > 0 \text{ and } j \notin J.$$

By Lemma 2.4, this means that $v \in L^{p'}_J$. □

2.3. Poincaré and Hardy inequalities in L^p_J

In this section we improve Poincaré-type inequalities for smooth functions on the sphere S^{N-1} which are orthogonal to some sets of spherical harmonics. Then we use these inequalities to improve Hardy-type inequalities in $L^p_{\geq n}$.

We define

$$C^\infty_{\geq n}(S^{N-1}) = \Big\{ v \in C^\infty(S^{N-1}) : \int_{S^{N-1}} v(\omega) P(\omega)\, d\omega = 0 \ \forall \text{ spherical harmonic } P \text{ of degree less than } n \Big\}.$$

For $1 \leq p < \infty, n \geq 1$, we denote by $C_{p,n}$ the best constant in the inequality

$$\int_{S^{N-1}} |v|^p\, d\omega \leq C \int_{S^{N-1}} |\nabla_\tau v|^p\, d\omega, \quad v \in C^\infty_{\geq n}(S^{N-1}),$$

and for $1 < p < \infty$ by $\widetilde{C}_{p,n}$ the best constant for which

$$\int_{S^{N-1}} |v|^p\, d\omega \leq C \int_{S^{N-1}} |\nabla_\tau v|^2 |v|^{p-2}\, d\omega, \quad v \in C^\infty_{\geq n}(S^{N-1}).$$

Note that $C_{p,n}$ is defined also for $p = 1$, whereas $\widetilde{C}_{p,n}$ only for $p > 1$. Moreover, when $1 < p < 2$ the expression $|\nabla_\tau v|^2 |v|^{p-2}$, entering the definition of $\widetilde{C}_{p,n}$, is understood as $|\nabla_\tau v|^2 |v|^{p-2} \chi_{\{v \neq 0\}}$, to give a meaning where v is equal to 0. However, if v is smooth, then $|v|^{\frac{p}{2}-1} v$ could be not, and this is source of some technical complications.

The constants $C_{p,n}$ and $\widetilde{C}_{p,n}$ satisfy the following properties.

Lemma 2.7.

(i) *For $n \geq 1$, $C_{p,n}$ and $\widetilde{C}_{p,n}$ are finite and decreasing with respect to n;*
(ii) *$C_{p,n} \to 0$, $\widetilde{C}_{p,n} \to 0$ as $n \to \infty$.*

Proof. (i) Since it is clear that $C_{p,n}$ and $\widetilde{C}_{p,n}$ are decreasing, it suffices to show, by contradiction, that $C_{p,1}$ and $\widetilde{C}_{p,1}$ are finite. Let $1 \leq p < \infty$. If $C_{p,1} = +\infty$, then there exists $(v_m)_{m \in \mathbb{N}} \subset C^\infty_{\geq 1}(S^{N-1})$ such that $\|v_m\|_p = 1$ and $\|\nabla_\tau v_m\|_p \leq 1/m$. In view of the Rellich-Kondrachov theorem, by taking a subsequence we may assume that $v_m \to v$ strongly in $L^p(S^{N-1})$ and hence, since $\|\nabla_\tau v_m\|_p \leq 1/m$, (v_m) is a Cauchy sequence in $W^{1,p}(S^{N-1})$ and therefore converges to v in $W^{1,p}(S^{N-1})$. Since $v_m \in C^\infty_{\geq 1}(S^{N-1})$ and $\|v_m\|_p = 1$, we also have $\|v\|_p = 1$ and v has zero mean on S^{N-1}. On the other hand, v is constant since

$$\|\nabla_\tau v\|_p = \lim_{m \to \infty} \|\nabla_\tau v_m\|_p = 0.$$

This is a contradiction. Concerning $\widetilde{C}_{p,1}$, we argue as above when $2 \le p < \infty$, by replacing v_m with $|v_m|^{\frac{p}{2}-1}v_m \in H^1(S^{N-1})$, see (ii), Step 2, below.

Let us now consider the case $1 < p < 2$ and prove that $\widetilde{C}_{p,n} \le (C_{p,n})^{\frac{2}{p}}$. In fact, by Hölder inequality

$$\begin{aligned}\int_{S^{N-1}} |v|^p \, d\omega &\le C_{p,n} \int_{S^{N-1}} |\nabla_\tau v|^p \, d\omega \\ &\le C_{p,n} \left(\int_{S^{N-1}} |\nabla_\tau v|^2 |v|^{p-2} \, d\omega \right)^{\frac{p}{2}} \left(\int_{S^{N-1}} |v|^p \, d\omega \right)^{1-\frac{p}{2}}\end{aligned}$$

and therefore $\widetilde{C}_{p,n} \le (C_{p,n})^{\frac{2}{p}} < \infty$.

(ii) (Step 1, $1 \le p < \infty$) We first prove the assertion for $C_{p,n}$ assuming, by contradiction, that $C_{p,n} > \delta > 0$. Then there exists $v_n \in C^\infty_{\ge n}(S^{N-1})$ such that

$$\int_{S^{N-1}} |v_n|^p \, d\omega = 1, \quad \int_{S^{N-1}} |\nabla_\tau v_n|^p \, d\omega < \delta^{-1}.$$

By taking a subsequence we may assume that $v_n \to v$ strongly in $L^p(S^{N-1})$. Since

$$\int_{S^{N-1}} v_n P \, d\omega = 0$$

if $\deg P < n$, one obtains

$$\int_{S^{N-1}} |v|^p \, d\omega = 1, \quad \int_{S^{N-1}} v P \, d\omega = 0$$

for every spherical harmonic P and hence $v = 0$, which is a contradiction.

(Step 2) We next prove that $\widetilde{C}_{p,n} \to 0$. For $1 < p < 2$ this follows from the inequality $\widetilde{C}_{p,n} \le (C_{p,n})^{\frac{2}{p}}$ proved in (i) and Step 1. If $2 \le p < \infty$ we argue by contradiction. Assuming that $\widetilde{C}_{p,n} > \delta > 0$ we take as above $v_n \in C^\infty_{\ge n}(S^{N-1})$ such that

$$\int_{S^{N-1}} |v_n|^p \, d\omega = 1, \quad \int_{S^{N-1}} |\nabla_\tau v_n|^2 |v|^{p-2} \, d\omega < \delta^{-1}.$$

By taking a subsequence, we suppose that $v_n \to v$ weakly in $L^p(S^{N-1})$ and then $v = 0$ since v is orthogonal to all spherical harmonics, as in Step 1. We define

$$w_n := |v_n|^{\frac{p}{2}-1} v_n \in C^1\big(S^{N-1}\big).$$

Then $\|w_n\|_2 = 1$ and $\nabla_\tau w_n = |v_n|^{\frac{p}{2}-3} v_n ((p/2)\mathrm{Re}(\overline{v_n} \nabla_\tau v_n) + i\,\mathrm{Im}(\overline{v_n} \nabla_\tau v_n))$ (note that the derivative of $t \mapsto |z(t)|^{\frac{p}{2}-1} z(t)$ with $z \in C^1$ is $|z|^{\frac{p}{2}-3} z[(p/2)\mathrm{Re}(\overline{z} z') +$

$i\mathrm{Im}(\overline{z}z')]$) so that $\|\nabla_\tau w_n\|_2$ is bounded because of $p \ge 2$. By taking a subsequence we may assume that $w_n \to w$ weakly in $H^1(S^{N-1})$ and strongly in $L^2(S^{N-1})$. Set

$$f_n := |w_n|^{1-\frac{2}{p}} w_n \in L^{p'}\big(S^{N-1}\big), \quad f := |w|^{1-\frac{2}{p}} w \in L^{p'}\big(S^{N-1}\big)$$

and observe that $f_n = |v_n|^{p-2}v_n$. Using the estimate

$$\Big| |z_1|^{1-\frac{2}{p}} z_1 - |z_2|^{1-\frac{2}{p}} z_2 \Big| \le \frac{2}{p'}(|z_1| + |z_2|)^{1-\frac{2}{p}} |z_1 - z_2|, \quad z_1, z_2 \in \mathbb{C},$$

we obtain by Hölder inequality with $q = 2 - (2/p) \ge 1$

$$\begin{aligned}
&\int_{S^{N-1}} |f_n - f|^{p'} \, d\omega \\
&\le \left(\frac{2}{p'}\right)^{p'} \int_{S^{N-1}} (|w_n| + |w|)^{\frac{p-2}{p-1}} |w_n - w|^{\frac{p}{p-1}} \, d\omega \\
&\le \left(\frac{2}{p'}\right)^{p'} \left(\int_{S^{N-1}} (|w_n| + |w|)^2 \, d\omega\right)^{\frac{p-2}{2(p-1)}} \left(\int_{S^{N-1}} |w_n - w|^2 \, d\omega\right)^{\frac{p}{2(p-1)}} \\
&\le \left(\frac{2}{p'}\right)^{p'} 2^{\frac{p-2}{p-1}} \left(\int_{S^{N-1}} |w_n - w|^2 \, d\omega\right)^{\frac{p}{2(p-1)}}.
\end{aligned}$$

Therefore $f_n \to f$ strongly in $L^{p'}(S^{N-1})$. Since $f_n = |v_n|^{p-2}v_n$ we have

$$\int_{S^{N-1}} v_n \overline{f}_n \, d\omega = \int_{S^{N-1}} |v_n|^p \, d\omega = 1.$$

On the other hand, since $v_n \to 0$ weakly in $L^p(S^{N-1})$ and $f_n \to f$ strongly in $L^{p'}(S^{N-1})$, we have also

$$\int_{S^{N-1}} v_n \overline{f}_n \, d\omega \to 0,$$

which is a contradiction. □

Next we prove Hardy-type inequalities for smooth functions in $L^p_{\ge n}$. Note that $p = 1$ is allowed in the second inequality.

Proposition 2.8. *For every $n \in \mathbb{N}$, $\beta \in \mathbb{R}$ and $u \in C^\infty(\mathbb{R}^N \setminus \{0\}) \cap L^p_{\ge n}(\mathbb{R}^N)$ the following inequalities hold*

$$\int_{\mathbb{R}^N} |x|^\beta |u(x)|^p \, dx \le \widetilde{C}_{p,n} \int_{\mathbb{R}^N} |x|^{\beta+2} |\nabla u(x)|^2 |u(x)|^{p-2} \, dx, \qquad 1 < p < \infty$$

$$\int_{\mathbb{R}^N} |x|^{\beta-p} |u(x)|^p \, dx \le C_{p,n} \int_{\mathbb{R}^N} |x|^\beta |\nabla u(x)|^p \, dx, \qquad 1 \le p < \infty,$$

where $C_{p,n}$, $\widetilde{C}_{p,n}$ are as in Lemma 2.7 *and tend to* 0 *as $n \to \infty$.*

Proof. For any fixed $r > 0$, $u(r\,\cdot) \in C^\infty_{\geq n}(S^{N-1})$. Hence using Lemma 2.7 we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} |x|^\beta |u(x)|^p\,dx &= \int_0^\infty r^{\beta+N-1}\left(\int_{S^{N-1}} |u(r\omega)|^p\,d\omega\right) dr\\ &\leq \widetilde{C}_{p,n}\int_0^\infty r^{\beta+N-1}\left(\int_{S^{N-1}} \left|\nabla_\tau\big(u(r\omega)\big)\right|^2 |u(r\omega)|^{p-2}\,d\omega\right) dr\\ &\leq \widetilde{C}_{p,n}\int_0^\infty r^{\beta+N+1}\left(\int_{S^{N-1}} |\nabla u(r\omega)|^2 |u(r\omega)|^{p-2}\,d\omega\right) dr\\ &= \widetilde{C}_{p,n}\int_{\mathbb{R}^N} |x|^{\beta+2}|\nabla u(x)|^2|u(x)|^{p-2}\,dx.\end{aligned}$$

The proof of the second inequality is identical. □

Remark 2.9. Lemma 2.7, combined with a Hardy-type inequality (see [10, Proposition 8.3]) which involves only the radial part of the gradient, allows to improve Proposition 2.8 in the following way:

$$\left(\frac{1}{\widetilde{C}_{p,n}} + \left(\frac{N+\beta}{p}\right)^2\right)\int_{\mathbb{R}^N} |x|^\beta |u(x)|^p\,dx \leq \int_{\mathbb{R}^N} |x|^{\beta+2}|\nabla u(x)|^2|u(x)|^{p-2}\,dx.$$

The classical Hardy inequality holds for $p \neq N$. However it holds for every u in $L^p_{\geq 1}$.

Corollary 2.10. *Let* $1 \leq p < \infty$, $u \in W^{1,p}(\mathbb{R}^N) \cap L^p_{\geq 1}$. *Then*

$$\int_{\mathbb{R}^N} |x|^{-p}|u(x)|^p\,dx \leq C_{p,1}\int_{\mathbb{R}^N} |\nabla u(x)|^p\,dx. \tag{2.6}$$

Proof. For $u \in C^\infty(\mathbb{R}^N \setminus \{0\}) \cap L^p_{\geq 1}$, (2.6) is a particular case of Proposition 2.8 with $n = 1, \beta = 0$. In the general case we observe that $0 = \int_{S^{N-1}} u(r\omega)\,d\omega = \int_{S^{N-1}} u_r(r\omega)\,d\omega$ and consider a sequence $(v_n) \subset C_c^\infty(\mathbb{R}^N)$ such that $v_n \to u$ in $W^{1,p}(\mathbb{R}^N)$. Let $u_n = v_n - w_n$ with $w_n(r) = c\int_{S^{N-1}} v_n(r\omega)d\omega = \int_{S^{N-1}}(v_n(r\omega) - u(r\omega))\,d\omega$ and c^{-1} is the measure of S^{N-1}. Then $u_n \in C^\infty(\mathbb{R}^N \setminus \{0\}) \cap L^p_{\geq 1}$ and $u_n \to u$ in $L^p(\mathbb{R}^N)$. Moreover, since $w_n'(r) = c\int_{S^{N-1}} v_{n,r}(r\omega)d\omega = c\int_{S^{N-1}}(v_{n,r}(r\omega) - u_r(r\omega))\,d\omega$, then $w_n' \to 0$ in L^p_{rad}, that is $\nabla w_n \to 0$ in $L^p(\mathbb{R}^N)$ and hence $u_n \to u$ in $W^{1,p}(\mathbb{R}^N)$. We conclude the proof by writing (2.6) for each u_n, letting $n \to \infty$ and using Fatou's lemma. □

Note that the above proof shows that $C^\infty(\mathbb{R}^N \setminus \{0\}) \cap W^{1,p}(\mathbb{R}^N) \cap L^p_{\geq 1}$ is dense in $W^{1,p}(\mathbb{R}^N) \cap L^p_{\geq 1}$.

2.4. Sobolev regularity in L^p_J

We denote by $W^{m,p}_J$ the spaces $W^{m,p}(\mathbb{R}^N) \cap L^p_J$. Most of the results in this section hold also for $p = \infty$ but sometimes one has to substitute bounded functions with bounded and continuous functions. To avoid misunderstanding and to simplify the exposition we always assume that p is finite.

The following lemma is false when $N = 1$ (consider a function which is equal to 0 in a left neighbourhood of 0 and 1 in a right neighbourhood of 0). It is used to control the singularity at the origin introduced by the spherical coordinates. Even though it is known, see, *e.g.*, [4] or [23], we provide a proof for completeness.

Lemma 2.11. *If* $1 \le p < \infty$ *and* $N \ge 2$*, then* $W^{k,p}(\Omega) = W^{k,p}(\mathbb{R}^N)$.

Proof. By induction, it is sufficient to give a proof only for $k = 1$. We fix $\eta \in C^\infty(\mathbb{R}^N)$ such that $\eta(x) = 0$ for $x \in B(0,1)$ and $\eta(x) = 1$ for $x \in \mathbb{R}^N \setminus B(0,2)$, therefore $\operatorname{supp} \frac{\partial \eta}{\partial x_j} \subset \overline{B(0,2)} \setminus B(0,1)$. Moreover, we define for $\varepsilon > 0$, $\eta_\varepsilon(x) := \eta(\frac{x}{\varepsilon})$.

Let $u \in W^{1,p}(\Omega)$. Since $\eta u \in W^{1,p}(\mathbb{R}^N)$, it suffices to prove that $v = (1 - \eta)u \in W^{1,p}(\mathbb{R}^N)$.

For a fixed $\varphi \in C_c(\mathbb{R}^N)$, from the definition of weak derivative $D_j^{(\Omega)} v$ in Ω we have

$$\mathcal{I}_\varepsilon := \int_{\mathbb{R}^N} \eta_\varepsilon v \frac{\partial \varphi}{\partial x_j}\, dx - \int_{\mathbb{R}^N} \eta_\varepsilon D_j^{(\Omega)} v \varphi \, dx = - \int_{\mathbb{R}^N} v \frac{\partial \eta_\varepsilon}{\partial x_j} \varphi \, dx.$$

If $\mathcal{I}_\varepsilon$ vanishes as $\varepsilon \to 0$, we have $D_j^{(\mathbb{R}^N)} v = D_j^{(\Omega)} v \in L^p(\mathbb{R}^N)$ and therefore the proof is complete. By the definition of η_ε, we have

$$|\mathcal{I}_\varepsilon| \le \|\nabla \eta\|_{L^\infty} \|\varphi\|_{L^\infty} \left(\frac{1}{\varepsilon} \int_{B(0,2\varepsilon)\setminus B(0,\varepsilon)} |v(x)|\, dx \right).$$

For a fixed $\varepsilon > 0$, using standard mollifier $\{\rho_n\}_n$, we see that $v_n := \rho_n * v \in C^\infty(\mathbb{R}^N \setminus B(0,\varepsilon))$ for $n > \varepsilon^{-1}$, $v_n \to v$ in $W^{1,p}(\mathbb{R}^N \setminus B(0,\varepsilon))$ as $n \to \infty$ and $\operatorname{supp} v_n \subset \operatorname{supp} v + B(0,1)$ is also compact.

Then taking $R := \operatorname{dist}(0, \operatorname{supp} v) + 1$ and $\omega := x/|x|$, we have for $x \in B(0,2\varepsilon) \setminus B(0,\varepsilon)$,

$$|v_n(x)| = \left| \int_{|x|}^R \nabla v_n(t\omega) \cdot \omega \, dt \right| \le \int_{|x|}^R |\nabla v_n(t\omega)|\, dt = \int_\varepsilon^R |\nabla v_n(t\omega)|\, dt.$$

Integrating $\varepsilon^{-1}|v_n(x)|$ over $B(0,2\varepsilon) \setminus B(0,\varepsilon)$ and using the spherical coordinates, we see

$$\begin{aligned} \frac{1}{\varepsilon} \int_{B(0,2\varepsilon)\setminus B(0,\varepsilon)} |v_n(x)|\, dx &\le \frac{1}{\varepsilon} \int_\varepsilon^{2\varepsilon} \left(\int_{S^{N-1}} \left(\int_\varepsilon^R |\nabla v_n(t\omega)|\, dt \right) d\omega \right) r^{N-1}\, dr \\ &= \frac{2^N - 1}{N} \int_{B(0,R)\setminus B(0,\varepsilon)} \left(\frac{\varepsilon}{|y|} \right)^{N-1} |\nabla v_n(y)|\, dy. \end{aligned}$$

Letting $n \to \infty$, we obtain

$$|\mathcal{I}_\varepsilon| \le \frac{2^N-1}{N}\|\nabla\eta\|_{L^\infty}\|\varphi\|_{L^\infty}\int_{B(0,R)\setminus B(0,\varepsilon)}\left(\frac{\varepsilon}{|y|}\right)^{N-1}|\nabla v(y)|\,dy$$
$$\le \frac{2^N-1}{N}\|\nabla\eta\|_{L^\infty}\|\varphi\|_{L^\infty}\int_{B(0,R)}\left(1\wedge\frac{\varepsilon}{|y|}\right)^{N-1}|\nabla v(y)|\,dy.$$

The dominated convergence theorem implies $\mathcal{I}_\varepsilon \to 0$ as $\varepsilon \to 0$. □

In the following two results we characterize when functions of the form $v(r)P(\omega)$ belong to Sobolev spaces. Note that the result depends on the order of the spherical harmonic P. In the proof of Propositions 2.13, 2.14 we shall use the following elementary result.

Lemma 2.12. *Let $u(r\omega) = v(r)P(\omega)$, where P is a spherical harmonic. Then $u \in W^{k,p}_{\mathrm{loc}}(\Omega)$ if and only if $v \in W^{k,p}_{\mathrm{loc}}((0,\infty))$.*

Proof. If n is order of P, then $Q(x) = r^n P(\omega)$ is a harmonic polynomial and we write $u(x) = |x|^{-n}v(|x|)Q(x)$. If $v(r) \in W^{k,p}_{\mathrm{loc}}((0,\infty))$, then $v(|x|) \in W^{k,p}_{\mathrm{loc}}(\Omega)$, by composition, and the same is true for u. Conversely, if $u \in W^{k,p}_{\mathrm{loc}}(\Omega)$, then all radial derivatives of u of order not greater than k are in $L^p_{\mathrm{loc}}(\Omega)$ and then $v(r) = \int_{S^{N-1}} u(r\omega)P(\omega)\,d\omega$ is in $W^{k,p}_{\mathrm{loc}}((0,\infty))$. □

Proposition 2.13. *Let $1 \le p < \infty$, $0 \ne P$ be a fixed spherical harmonic of order $n \in \mathbb{N}_0$ and $u(r\omega) = v(r)P(\omega)$, with $v \in L^p_{\mathrm{rad}}$.*

- (i) *If $n = 0$, then $u \in W^{1,p}(\mathbb{R}^N)$ if and only if v' belongs to L^p_{rad}. Moreover the norms $\|u\|_{1,p}$ and $\|v\|_{p,\mathrm{rad}} + \|v'\|_{p,\mathrm{rad}}$ are equivalent on the (closed) subspace of $W^{1,p}(\mathbb{R}^N)$ spanned by these functions;*
- (ii) *If $n \ge 1$, then $u \in W^{1,p}(\mathbb{R}^N)$ if and only if v' and $\frac{v}{r}$ belong to L^p_{rad}. Moreover, the norms $\|u\|_{1,p}$ and $\|v\|_{p,\mathrm{rad}} + \|v'\|_{p,\mathrm{rad}} + \|v/r\|_{p,\mathrm{rad}}$ are equivalent on the (closed) subspace of $W^{1,p}(\mathbb{R}^N)$ spanned by these functions.*

Proof. $Q(x) = r^n P(\omega)$ is a harmonic homogeneous polynomial of degree n and $D_h Q$ is a homogeneous polynomial of degree $n-1$. Therefore, writing $u(x) = r^{-n}v(r)Q(x)$ we see from Lemma 2.12 that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ if and only if $v \in W^{1,p}_{\mathrm{loc}}((0,\infty))$. Moreover, for $r > 0$, $h = 1,\dots,N$ we have

$$D_h u(x) = \left(v'(r) - n\frac{v(r)}{r}\right)\omega_h P(\omega) + \frac{v(r)}{r}\frac{D_h Q(x)}{r^{n-1}}. \tag{2.7}$$

If $n = 0$, since $Q = 1$, then from (2.7) and Lemma 2.11 we see that $u \in W^{1,p}(\mathbb{R}^N)$ if and only if v' belongs to L^p_{rad} and the norms $\|\nabla u\|_p$ and $\|v'\|_{p,\mathrm{rad}}$ are clearly equivalent.

If $n \geq 1$ we first note that, by (2.7) and Lemma 2.11, $u \in W^{1,p}(\mathbb{R}^N)$ if $v', v/r \in L^p_{\text{rad}}$ and $\|\nabla u\|_p \leq C(\|v'\|_{p,\text{rad}} + \|v/r\|_{p,\text{rad}})$. Conversely, let $u \in W^{1,p}(\mathbb{R}^N)$. Then noting that $u_r = \nabla u \cdot \frac{x}{r} = v'(r)P(\omega) \in L^p(\mathbb{R}^N)$, we have $\|v'\|_{p,\text{rad}} \leq C\|\nabla u\|_p$. On the other hand, Corollary 2.10 yields $\|v/r\|_{p,\text{rad}} \leq C\|\nabla u\|_p$. □

Proposition 2.14. *Let $1 \leq p < \infty$, $0 \neq P$ be a spherical harmonic of order $n \in \mathbb{N}_0$. Let moreover $u(r\omega) = v(r)P(\omega)$, with $v \in L^p_{\text{rad}}$ and assume that $u \in W^{1,p}(\mathbb{R}^N)$. Then*

(i) *If $n = 0$, then $u \in W^{2,p}(\mathbb{R}^N)$ if and only if v'' and $\frac{v'}{r}$ belong to L^p_{rad}. Moreover, the seminorms $\|D^2u\|_p$, where D^2u is the Hessian matrix of u and $\|v''\|_{p,\text{rad}} + \|v'/r\|_{p,\text{rad}}$ are equivalent on the (closed) subspace of $W^{2,p}(\mathbb{R}^N)$ spanned by these functions;*
(ii) *If $n = 1$, then $u \in W^{2,p}(\mathbb{R}^N)$ if and only if v'' and $\frac{v'}{r} - \frac{v}{r^2}$ belong to L^p_{rad}. Moreover, the seminorms $\|D^2u\|_p$, and $\|v''\|_{p,\text{rad}} + \|v'/r - v/r^2\|_{p,\text{rad}}$ are equivalent on the (closed) subspace of $W^{2,p}(\mathbb{R}^N)$ spanned by these functions;*
(iii) *If $n \geq 2$, then $u \in W^{2,p}(\mathbb{R}^N)$ if and only if v'', $\frac{v'}{r}$ and $\frac{v}{r^2}$ belong to L^p_{rad}. Moreover, the seminorms $\|D^2u\|_p$, and $\|v''\|_{p,\text{rad}} + \|v'/r\|_{p,\text{rad}} + \|v/r^2\|_{p,\text{rad}}$ are equivalent on the (closed) subspace of $W^{2,p}(\mathbb{R}^N)$ spanned by these functions.*

Proof. $Q(x) = r^n P(\omega)$ is a harmonic homogeneous polynomial of degree n. Therefore, writing $u(x) = r^{-n}v(r)Q(x)$ and using Lemma 2.12, we see from (2.7) that for $r > 0$

$$\begin{aligned} D_{hk}u(x) = &\left(v''(r) - (2n+1)\frac{v'(r)}{r} + n(n+2)\frac{v(r)}{r^2}\right)\omega_h\omega_k P(\omega) \\ &+ \left(\frac{v'(r)}{r} - n\frac{v(r)}{r}\right)\left(\delta_{hk}P(\omega) + \omega_h\frac{D_kQ(x)}{r^{n-1}} + \omega_k\frac{D_hQ(x)}{r^{n-1}}\right) \\ &+ \frac{v(r)}{r^2}\frac{D_{hk}Q(x)}{r^{n-2}}. \end{aligned} \tag{2.8}$$

Observe that D_hQ, $D_{hk}Q$ are homogeneous polynomials of degree $n-1, n-2$, respectively.

The case $n = 0$. Since $Q = 1$, then from (2.8) and Lemma 2.11 we see that $u \in W^{2,p}(\mathbb{R}^N)$ if and only if v'' and $\frac{v'}{r}$ belong to L^p_{rad}, with equivalence of the corresponding norms. Indeed, from (2.8), we immediately see that if v'' and $\frac{v'}{r}$ belong to L^p_{rad} then $D_{hk}u(x) \in L^p(\mathbb{R}^N)$. Concerning the other implication, from (2.8) we have that, if $D_{hk}u(x) \in L^p(\mathbb{R}^N)$, $h \neq k$ implies $v'' - \frac{v'}{r} \in L^p_{\text{rad}}$ and $h = k$ implies that $\frac{v'}{r} + \left(v'' - \frac{v'}{r}\right)\omega_h^2$ belongs to $L^p(\mathbb{R}^N)$. By difference, also $\frac{v'}{r}, v'' \in L^p_{\text{rad}}$.

The case $n = 1$. Q is linear, say $Q(x) = x_j$ and $D_{hk}Q = 0$. The equation (2.8) becomes

$$D_{hk}u(x) = \left(v''(r) - 3\left(\frac{v'(r)}{r} - \frac{v(r)}{r^2}\right)\right)\omega_j\omega_h\omega_k + \left(\frac{v'(r)}{r} - \frac{v(r)}{r^2}\right)\left(\delta_{hk}\omega_j + \omega_h\delta_{jk} + \omega_k\delta_{jh}\right).$$

Then $D_{hk}u \in L^p(\mathbb{R}^N)$ if v'' and $\left(\frac{v'}{r} - \frac{v}{r^2}\right) \in L^p_{\text{rad}}$ and we conclude that $u \in W^{2,p}(\mathbb{R}^N)$ by using Lemma 2.11.

Conversely, if $u \in W^{2,p}(\mathbb{R}^N)$, then

$$v''(r)P(\omega) = u_{rr}(x) = \sum_{i,j}\left[\frac{x_i x_j}{r^2}D_{ij}u + \frac{x_j}{r^2}\left(\delta_{ij} - \frac{x_i x_j}{r^2}\right)D_i u\right] = \sum_{i,j} D_{ij}u\frac{x_i x_j}{r^2} \in L^p\left(\mathbb{R}^N\right)$$

and hence $v'' \in L^p_{\text{rad}}$. Finally, since $\Delta_0 P = -(N-1)P$ we have

$$\Delta u = \left(v'' + (N-1)\left(\frac{v'}{r} - \frac{v}{r^2}\right)\right)P \in L^p\left(\mathbb{R}^N\right),$$

hence, by difference $\frac{v'}{r} - \frac{v}{r^2} \in L^p_{\text{rad}}$. The equivalence of the norms follows also by the arguments above.

The case $n \geq 2$. If v'', $\frac{v'}{r}$ and $\frac{v}{r^2}$ belong to L^p_{rad}, then $D_{hk}u \in L^p(\mathbb{R}^N)$, by (2.8) and $u \in W^{2,p}(\mathbb{R}^N)$ by Lemma 2.11.

Conversely, let $u \in W^{2,p}(\mathbb{R}^N)$. Then as in the case $n = 1$ we obtain $v'' \in L^p_{\text{rad}}$. Since $\Delta u = \left(v'' + \frac{N-1}{r}v' - \lambda_n\frac{v}{r^2}\right)P$ and $u_{rr} = v''P$ belong to $L^p(\mathbb{R}^N)$, it follows that $g = \frac{N-1}{r}v' - \lambda_n\frac{v}{r^2} \in L^p_{\text{rad}}$. Note that $v' - c\frac{v}{r} = fr$ where $f = g/(N-1)$ and $c = \lambda_n/(N-1) > 2$, since $n \geq 2$ and $\lambda_n \geq 2N$. Now we show $v/r^2 \in L^p_{\text{rad}}$. Integrating the identity $(v' - c\frac{v}{r})r^{-c} = fr^{1-c}$ between $r < s$, we obtain

$$s^{-c}v(s) - r^{-c}v(r) = \int_r^s f(t)t^{1-c}\,dt.$$

Since $f \in L^p_{\text{rad}}$ and $c > 2$, by Hölder inequality we see for every $\delta \in [0, c-2]$

$$\int_r^\infty |f(t)|t^{1-c}\,dt \leq \left(\int_r^\infty |f(t)|^p t^{N-1-p\delta}\,dt\right)^{\frac{1}{p}}\left(\int_r^\infty t^{(1-c+\delta)p' - \frac{N-1}{p-1}}\,dt\right)^{\frac{1}{p'}} = C_\delta\left(\int_r^\infty |f(t)|^p t^{N-1-p\delta}\,dt\right)^{\frac{1}{p}} r^{2-c+\delta-\frac{N}{p}}$$

and therefore, choosing $\delta = 0$, the integral on the right hand side converges at infinity. Hence $s^{-c}v(s)$ has a limit at infinity which is 0, since $v \in L^p_{\mathrm{rad}}$. Then letting $s \to \infty$, we have

$$v(r) = -r^c \int_r^\infty f(t) t^{1-c}\, dt \quad \text{and} \quad \frac{|v(r)|}{r^2} \le C_\delta \left(\int_r^\infty |f(t)|^p t^{N-1-p\delta}\, dt \right)^{\frac{1}{p}} r^{\delta - \frac{N}{p}}.$$

Then we have

$$\begin{aligned} \int_0^\infty \left| \frac{v(r)}{r^2} \right|^p r^{N-1}\, dr &\le C_\delta^p \int_0^\infty \left(\int_0^s r^{p\delta - 1}\, dr \right) |f(t)|^p t^{N-1-p\delta}\, dt \\ &= \frac{C_\delta^p}{p\delta} \int_0^\infty |f(t)|^p t^{N-1}\, dt = \|f\|^p_{L^p_{\mathrm{rad}}}. \end{aligned}$$

By difference, $v'/r \in L^p_{\mathrm{rad}}$ and the proof is complete. □

Observe that for $n = 0, 1$, a function u can be in $W^{2,p}(\mathbb{R}^N)$ even though v/r^2 does not belong to L^p_{rad}.

Conversely, now we study the regularity of radial functions defined as means of $W^{2,p}$- functions or in other words, the continuity of the operator $I \otimes S$ introduced in Lemma 2.4, with respect to the Sobolev norm.

Proposition 2.15. *Let $1 \le p < \infty$, $u \in L^p(\mathbb{R}^N)$ and P be a spherical harmonic of order n. Let us define*

$$v(r) = \int_{S^{N-1}} u(r\omega) P(\omega)\, d\omega, \qquad w(x) = v(r) P(\omega).$$

Then

(i) *If $u \in W^{1,p}(\mathbb{R}^N)$ then $w \in W^{1,p}(\mathbb{R}^N)$; moreover $\|w\|_{1,p} \le C\|u\|_{1,p}$ for a suitable $C > 0$ independent of u;*
(ii) *If $u \in D(L_{p,\max})$ then $w \in D(L_{p,\max})$ and*

$$Lw(x) = P(\omega) \left(a v_{rr} - \frac{N-1+c}{r} v_r - \frac{b + \lambda_n}{r^2} v \right) = P(\omega) \int_{S^{N-1}} Lu(r\omega)\, d\omega.$$

In particular if $u \in W^{2,p}(\mathbb{R}^N)$ then $w \in W^{2,p}(\mathbb{R}^N)$ and $\|w\|_{2,p} \le C\|u\|_{2,p}$ for a suitable $C > 0$ independent of u.

Proof. (i) Clearly $v \in L^p_{\mathrm{rad}}$ and, since $v_r = \int_{S^{N-1}} u_r(r\omega) P(\omega)\, d\omega$ and $u_r = \nabla u \cdot \frac{x}{r} \in L^p(\mathbb{R}^N)$, then also $v' \in L^p_{\mathrm{rad}}$. This completes the proof if $n = 0$, by Proposition 2.13 (i). In the case $n \ge 1$, by Proposition 2.13 (ii) we have also to prove that $v/r \in L^p_{\mathrm{rad}}$. Let $w(r) = \int_{S^{N-1}} u(r\omega)\, d\omega$ and $v_1(x) = w(|x|)$. The case $n = 0$ yields $v_1 \in W^{1,p}(\mathbb{R}^N)$ and moreover, $v(r) = \int_{S^{N-1}} (u(r\omega) - v_1(r)) P(\omega)\, d\omega$. We apply Corollary 2.10 to $u - v_1 \in W^{1,p}(\mathbb{R}^N) \cap L^p_{\ge 1}$ and obtain that $v/r \in L^p_{\mathrm{rad}}$.

(ii) From (2.3) and since Δ_0 is self-adjoint in $L^2(S^{N-1})$ and $\Delta_0 P = -\lambda_n P$ we get for $u \in D(L_{p,\max})$

$$\begin{aligned} &a v_{rr} - \frac{N-1+c}{r} v_r - \frac{b+\lambda_n}{r^2} v \\ &= \int_{S^{N-1}} \left(a u_{rr} - \frac{N-1+c}{r} u_r - \frac{b+\lambda_n}{r^2} u \right) P(\omega)\, d\omega \\ &= \int_{S^{N-1}} \left(Lu - \frac{\lambda_n}{r^2} u - \frac{\Delta_0 u}{r^2} \right) P(\omega)\, d\omega \\ &= \int_{S^{N-1}} \left(Lu - \frac{\lambda_n}{r^2} u \right) d\omega - \frac{1}{r^2} \int_{S^{N-1}} u \Delta_0 P\, d\omega \\ &= \int_{S^{N-1}} Lu\, d\omega, \end{aligned}$$

which is the formula in the statement since $Lw = P(\omega)\big(a v_{rr} - \frac{N-1+c}{r} v_r - \frac{b+\lambda_n}{r^2} v\big)$. This yields $w \in D(L_{p,\max})$. To prove that $w \in W^{2,p}(\mathbb{R}^N)$ when $u \in W^{2,p}(\mathbb{R}^N)$ we apply Proposition 2.14 and note, first of all, that $v_{rr} = \int_{S^{N-1}} u_{rr}(r\omega) P(\omega)\, d\omega \in L^p_{\text{rad}}$. Taking $L = \Delta$ in the formula above we obtain that $v_{rr} - \frac{N-1}{r} v_r + \frac{\lambda_n}{r^2} v \in L^p_{\text{rad}}$, hence $\frac{N-1}{r} v_r - \frac{\lambda_n}{r^2} v \in L^p_{\text{rad}}$. This yields, $v'/r \in L^p_{\text{rad}}$ if $n = 0$, $v'/r - v/r^2 \in L^p_{\text{rad}}$ if $n = 1$ (since $\lambda_1 = N-1$) and $v'/r, v/r^2 \in L^p_{\text{rad}}$ as in the proof of Proposition 2.14 (iii). □

Let us show that the projection $I \otimes S$ is bounded in $W^{k,p}(\mathbb{R}^N)$, $k = 1, 2$ and with respect to the graph-norm of L. We point out that this result has been already proved in [11, Section 2] for $1 < p < \infty$ and $k = 2$.

Lemma 2.16. *Let $1 \le p < \infty$ and assume that J is finite. Then the projection $I \otimes S$ of Lemma 2.4 extends to a bounded projection of $W^{k,p}(\mathbb{R}^N)$, $k = 1, 2$. Moreover, $L(I \otimes S)u = (I \otimes S)Lu$ for every $u \in D(L_{p,\max})$.*

Proof. Since

$$(I \otimes S)u = \sum_{i \in J} P_j(\omega) \int_{S^{N-1}} u(r\omega) P_j(\omega)\, d\omega,$$

all statements follow from Proposition 2.15. □

As in [11, Section 2] one deduces the following two results.

Lemma 2.17. *Let $1 \le p < \infty$, $J \subset \mathbb{N}_0$. Then $e^{t\Delta} L^p_J \subset L^p_J$. If J is finite, the projection $I \otimes S$ of Lemma 2.4 satisfies for every $u \in L^p$*

$$e^{t\Delta}(I \otimes S)u = (I \otimes S)e^{t\Delta}u.$$

Lemma 2.18. *Let* $1 \le p < \infty$. *Then* $C_c^\infty(\mathbb{R}^N)$ *functions of the form*

$$v = \sum f_j(r) P_j(\omega), \tag{2.9}$$

where the sums are finite and $j \in J$, *are dense in* $W_J^{m,p}$ *with respect to the Sobolev norm. If* $u \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$, *the approximating functions of the form* (2.9) *can be chosen to have support in* $\mathbb{R}^N \setminus \{0\}$, *too.*

The above lemma, in combination with Lemma 2.11, allows to approximate functions in $W^{m,p}$ with smooth function as in (2.9), with the f_j having compact support away from 0.

3. Generation of analytic semigroup in L^p and domain characterization

3.1. Generation in $L^p_{\ge n}$

An operator $(A, D(A))$ densely defined in a Banach space X is called regularly dissipative if the set

$$\{\langle Au, v\rangle : \ u \in D(A), \ v \in X', \|v\| = 1 \text{ and } \langle u, v\rangle = \|u\|\}$$

is contained in a sector of the complex plane symmetric with respect to the half-line $(-\infty, 0]$ and of angle less than π. It is well-known that a regularly dissipative operator, whose resolvent contains a point in $\mathbb{C}_+$, generates an analytic semigroup which is contractive in a sector around the positive axis.

Let us first prove that, if n is sufficiently large, a minimal realization of L is regularly dissipative on $L^p_{\ge n}$.

Proposition 3.1. *Let* $1 < p < \infty$. *Then there exists* $n_0 \in \mathbb{N}$ *such that for* $n \ge n_0$, L *endowed with domain* $C_c^\infty(\mathbb{R}^N \setminus \{0\}) \cap L^p_{\ge n}$ *is regularly dissipative in* $L^p_{\ge n}$.

Proof. Let $u \in C_c^\infty(\mathbb{R}^N \setminus \{0\}) \cap L^p_{\ge n}$. Note that

$$Lu(x) = \sum_{ij} D_i \left(\overline{a}_{ij} D_j u(x)\right) + (c - (N-1)(a-1)) \frac{x}{|x|^2} \cdot \nabla u(x) - \frac{b}{|x|^2} u(x),$$

with $\overline{a}_{ij}(x) = \delta_{ij} + (a-1)|x|^{-2} x_i x_j$. Multiplying $-Lu$ and $u^\star = |u|^{p-2}\overline{u}$ and integrating over $\mathbb{R}^N$, we see from integration by parts (for details, see [13]) that

$$\begin{aligned}\int_{\mathbb{R}^N} (-Lu) u^\star dx = &\int_{\mathbb{R}^N} \sum_{ij} \overline{a}_{ij} D_i u D_j(u^\star)\, dx \\ &- (c-(N-1)(a-1)) \int_{\mathbb{R}^N} \left(\frac{x}{|x|^2} \cdot \nabla u\right) u^\star\, dx + b \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^2}\, dx.\end{aligned} \tag{3.1}$$

Putting $\mathfrak{a}(\xi) = \sum_{ij} \overline{a}_{ij}\xi_i\overline{\xi_j}$ for $\xi \in \mathbb{C}$ and taking real parts in (3.1) we have

$$\begin{aligned}
&\mathrm{Re} \int_{\mathbb{R}^N} (-Lu)u^\star\, dx \\
&= (p-1)\int_{\mathbb{R}^N} \mathfrak{a}(\mathrm{Re}\,(\overline{u}\nabla u))|u|^{p-4}\, dx + \int_{\mathbb{R}^N} \mathfrak{a}(\mathrm{Im}\,(\overline{u}\nabla u))|u|^{p-4}\, dx \\
&\quad + \left(b + \frac{(c-(N-1)(a-1))(N-2)}{p}\right)\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^2}\, dx \\
&= (p-1-\delta)\int_{\mathbb{R}^N} \mathfrak{a}(\mathrm{Re}\,(\overline{u}\nabla u))|u|^{p-4}\, dx \\
&\quad + (1-\delta)\int_{\mathbb{R}^N} \mathfrak{a}(\mathrm{Im}\,(\overline{u}\nabla u))|u|^{p-4}\, dx + \delta\int_{\mathbb{R}^N} \mathfrak{a}(\nabla u)|u|^{p-2}\, dx \qquad (3.2)\\
&\quad + \left(b + \frac{(c-(N-1)(a-1))(N-2)}{p}\right)\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^2}\, dx \\
&\ge (p-1-\delta)\int_{\mathbb{R}^N} \mathfrak{a}(\mathrm{Re}\,(\overline{u}\nabla u))|u|^{p-4}\, dx \\
&\quad + (1-\delta)\int_{\mathbb{R}^N} \mathfrak{a}(\mathrm{Im}\,(\overline{u}\nabla u))|u|^{p-4}\, dx + \delta(1\wedge a)\int_{\mathbb{R}^N} |\nabla u|^2|u|^{p-2}\, dx \\
&\quad + \left(b + \frac{(c-(N-1)(a-1))(N-2)}{p}\right)\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^2}\, dx,
\end{aligned}$$

where $0 < \delta < 1 \wedge (p-1)$ and we have used $\mathfrak{a}(\xi) \ge (1\wedge a)|\xi|^2$. Applying Proposition 2.8 with $\beta = -2$ we see that

$$\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^2}\, dx \le \widetilde{C}_{p,n}\int_{\mathbb{R}^N} |\nabla u|^2|u|^{p-2}\, dx.$$

Then we have

$$\begin{aligned}
\mathrm{Re}\int_{\mathbb{R}^N}(-Lu)u^\star\, dx &\ge (p-1-\delta)\int_{\mathbb{R}^N} \mathfrak{a}(\mathrm{Re}\,(\overline{u}\nabla u))|u|^{p-4}\, dx \\
&\quad +(1-\delta)\int_{\mathbb{R}^N} \mathfrak{a}(\mathrm{Im}\,(\overline{u}\nabla u))|u|^{p-4}\, dx + K_n(\delta)\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^2}\, dx,
\end{aligned}$$

where

$$K_n(\delta) = b + \frac{(c-(N-1)(a-1))(N-2)}{p} + \frac{\delta(1\wedge a)}{\widetilde{C}_{p,n}}.$$

Note that for every $0 < \delta < 1 \wedge (p-1)$, $K_n(\delta) \to \infty$ as $n \to \infty$. On the other hand, taking imaginary parts in (3.1), we have

$$\left|\mathrm{Im} \int_{\mathbb{R}^N} (-Lu)u^\star \, dx\right| \le |p-2| \int_{\mathbb{R}^N} \sum_{ij} \left|\overline{a}_{ij} \mathrm{Re}\,(\overline{u} D_i u) \mathrm{Im}\,(\overline{u} D_j u)\right| |u|^{p-4}\, dx$$

$$+ \left|c-(N-1)(a-1)\right| \int_{\mathbb{R}^N} \frac{|\mathrm{Im}\,(\overline{u}\nabla u)||u|^{p-4}}{|x|}\, dx$$

$$\le |p-2| \left(\int_{\mathbb{R}^N} \mathfrak{a}(\mathrm{Re}(\overline{u}\nabla u))|u|^{p-4}\, dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} \mathfrak{a}(\mathrm{Im}(\overline{u}\nabla u))|u|^{p-4}\, dx\right)^{\frac{1}{2}}$$

$$+ (1\wedge a)\left|c-(N-1)(a-1)\right| \left(\int_{\mathbb{R}^N} \mathfrak{a}(\mathrm{Im}(\overline{u}\nabla u))|u|^{p-4}\, dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^2}\, dx\right)^{\frac{1}{2}}.$$

Taking $n_0 \in \mathbb{N}$ such that $K_n(\delta) > 0$ for $n \ge n_0$ we obtain

$$\left|\mathrm{Im} \int_{\mathbb{R}^N} (-Lu)u^\star \, dx\right| \le \ell \left(\mathrm{Re} \int_{\mathbb{R}^N} (-Lu)u^\star \, dx\right)$$

with

$$\ell^2 = \frac{|p-2|^2}{4(p-1-\delta)} + \frac{(1\wedge a)^2 |c-(N-1)(a-1)|^2}{4(1-\delta)K_n(\delta)}. \qquad \square$$

Remark 3.2. Observe that $\ell = \ell_n$ depends on n and $\ell_n \to \frac{|p-2|}{2\sqrt{p-1-\delta}}$ as $n \to \infty$, for any $0 < \delta < 1 \wedge (p-1)$. This means that in $L^p_{\ge n}$, with n large enough, the angle of complex dissipativity of $L_{p,\min}$ is almost the same as in the simplest case $L = \Delta$.

As a byproduct of the previous computation we can give conditions under which $L_{p,\min}$ is dissipative on the whole $L^p(\mathbb{R}^N)$.

Corollary 3.3. *If* $(1\wedge a)(p-1)\frac{(N-2)^2}{4} + b + \frac{c-(a-1)(N-1)(N-2)}{p} \ge 0$, *then* $L_{p,\min}$ *is dissipative*.

Proof. Since L has real coefficients, we consider only real functions $u \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$ and write (3.2) with $\delta = 0$. Using the Hardy-type inequality of [10, Proposition 8.3], we obtain

$$
\begin{aligned}
-\int_{\mathbb{R}^N} Lu\, u^*\, dx &= (p-1)\int_{\mathbb{R}^N} a|\nabla u|^2|u|^{p-2}\, dx \\
&\quad + \left(b + \frac{(c-(N-1)(a-1))(N-2)}{p}\right)\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^2}\, dx \\
&\geq (1\wedge a)(p-1)\int_{\mathbb{R}^N} |\nabla u|^2|u|^{p-2}\, dx \\
&\quad + \left(b + \frac{(c-(N-1)(a-1))(N-2)}{p}\right)\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^2}\, dx \\
&\geq \left((1\wedge a)(p-1)\frac{(N-2)^2}{4} + b\right. \\
&\quad \left. + \frac{c-(a-1)(N-1)(N-2)}{p}\right)\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^2}\, dx. \qquad \square
\end{aligned}
$$

To prove generation in $L^p_{\geq n}$ we need the following simple lemma.

Lemma 3.4. *Let f, g be continuous functions on $\mathbb{R}$ with $f > 0$ everywhere. If $V \in C^2(\mathbb{R}) \cap L^p(\mathbb{R})$ satisfies*

$$fV - V'' - gV' = 0$$

then $V = 0$.

Proof. Since V belongs to $L^p(\mathbb{R})$, we can find sequences (a_n) and (b_n) such that $a_n \to -\infty$, $b_n \to \infty$ as $n \to \infty$ and $|V(a_n)|, |V(b_n)| \leq 1/n$. Since f is positive, the maximum principle yields $|V(s)| \leq 1/n$ in $[a_n, b_n]$ and the proof is complete. $\square$

Proposition 3.5. *There exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ the closure $L^n_{p,\min}$ of $(L, C_c^\infty(\mathbb{R}^N \setminus \{0\}) \cap L^p_{\geq n})$ generates generates an analytic semigroup of contractions in $L^p_{\geq n}$. Moreover, $L^n_{p,\min}$ coincides with the maximal operator $L^n_{p,\max}$ defined as L on the domain*

$$D\left(L^n_{p,\max}\right) = \left\{u \in L^p_{\geq n} \cap W^{2,p}_{\mathrm{loc}}\big(\mathbb{R}^N \setminus \{0\}\big) : Lu \in L^p_{\geq n}\right\}.$$

Proof. By Proposition 3.1 it follows that $L^n_{p,\min}$ generates an analytic semigroup of contractions in $L^p_{\geq n}$ if $(I-L)(C_c^\infty(\mathbb{R}^N \setminus \{0\}) \cap L^p_{\geq n})$ is dense in $L^p_{\geq n}$. Assume that $v \in L^{p'}_{\geq n} = (L^p_{\geq n})'$, see Lemma 2.6, satisfies

$$\int_{\mathbb{R}^N} (u - Lu)v\, dx = 0, \qquad u \in C_c^\infty(\mathbb{R}^N \setminus \{0\}) \cap L^p_{\geq n}. \tag{3.3}$$

We show that $v = 0$.

Observe that, since $v \in L^{p'}_{\geq n}$, (3.3) holds for every $u \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$. In fact this follows by splitting $u = u_1 + u_2$ using the projection $I \otimes S$ of Lemma 2.4, by noticing that $(I \otimes S)u \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$, see Remark 2.5, and that L commutes with $(I \otimes S)$, by Lemma 2.16. By elliptic regularity, $v \in C^\infty(\mathbb{R}^N \setminus \{0\})$ and $v - L^* v = 0$. Let P be a spherical harmonic of degree at least n and we define

$$w(r) = \int_{S^{N-1}} v(r,\omega) P(\omega)\, d\omega \in L^p_{\text{rad}}$$

and

$$V(s) =: e^{\frac{N}{p}s} w(e^s) = e^{\frac{N}{p}s} \int_{S^{N-1}} v(e^s,\omega) P(\omega)\, d\omega \in C^\infty(\mathbb{R}).$$

Then $V \in L^p(\mathbb{R})$ and moreover

$$V'(s) = e^{(\frac{N}{p}+1)s} \int_{S^{N-1}} D_r v(e^s,\omega) P(\omega)\, d\omega + \frac{N}{p} V(s)$$

and

$$\begin{aligned} V''(s) &= e^{(\frac{N}{p}+2)s} \int_{S^{N-1}} D_{rr} v(e^s,\omega) P(\omega)\, d\omega \\ &\quad + \left(\frac{N}{p}+1\right) e^{(\frac{N}{p}+1)s} \int_{S^{N-1}} D_r v(e^s,\omega) P(\omega)\, d\omega + \frac{N}{p} V'(s) \\ &= a^{-1} e^{(\frac{N}{p}+2)s} \int_{S^{N-1}} \left(a D_{rr} v(e^s,\omega) + \frac{N-1+c^*}{e^s} D_r v(e^s,\omega) \right) P_j(\omega)\, d\omega \\ &\quad + \left(\frac{N}{p} + 1 - \frac{N-1+c^*}{a}\right) \left(V'(s) - \frac{N}{p} V(s) \right) + \frac{N}{p} V'(s). \end{aligned}$$

Here $c^* = 2(N-1)(a-1) - c$ and $b^* = b + (c - (N-1)(a-1))(N-2)$ are the coefficients of L^*, see (2.2). By using

$$v - L^* v = \left(1 - a D_{rr} - \frac{N-1+c^*}{r} D_r + \frac{b^*}{r^2} - \frac{1}{r^2}\Delta_0 \right) v = 0$$

we obtain, as in Proposition 2.15,

$$\begin{aligned} &\int_{S^{N-1}} \left(a D_{rr} v(e^s,\omega) + \frac{N-1+c^*}{e^s} D_r v(e^s,\omega) \right) P(\omega)\, d\omega \\ &= \int_{S^{N-1}} \left(v(e^s,\omega) + \frac{b^*}{e^{2s}} v(e^s,\omega) \right) P(\omega)\, d\omega - \int_{S^{N-1}} \frac{\Delta_0 v(e^s,\omega) P(\omega)}{e^{2s}}\, d\omega \\ &= \int_{S^{N-1}} \left(v(e^s,\omega) + \frac{b^*}{e^{2s}} v(e^s,\omega) \right) P(\omega)\, d\omega - \int_{S^{N-1}} \frac{v(e^s,\omega) \Delta_0 P(\omega)}{e^{2s}}\, d\omega \\ &= \int_{S^{N-1}} \left(v(e^s,\omega) + \frac{b^* + \lambda}{e^{2s}} v(e^s,\omega) \right) P(\omega)\, d\omega, \end{aligned}$$

where $\lambda = j^2 + (N-2)j$, $j = \deg P \geq n$. Then V solves

$$\left(e^{2s} + b^* + \lambda + \left(N - 1 + c^* - \frac{Na}{p} - a\right)\frac{N}{p}\right)V - aV'' + \left(\frac{2Na}{p} + a - (N-1+c^*)\right)V' = 0.$$

If n is sufficiently large, then the coefficient of the zero order term becomes positive, hence Lemma 3.4 applies and therefore $V = 0$. Since this is true for every spherical harmonic of degree at least n, we conclude that $v = 0$ and that $L^n_{p,\min}$ is a generator. The equality $L^n_{p,\min} = L^n_{p,\max}$ follows from the inclusion $L^n_{p,\min} \subset L^n_{p,\max}$ once the injectivity of $I - L^n_{p,\max}$ has been proved. However this follows from the same arguments as before, interchanging the role of L and L^* and taking a larger n, if necessary. $\square$

Precise conditions for the injectivity of $1 - L^*$ in $L^{p'}_{\geq n}$ can be obtained with the results of Subsection 4.3, see the proof of Proposition 3.28. In fact, the function $W(r) = \int_{S^{N-1}} v(r\omega)P(\omega)\,d\omega \in L^p_{\mathrm{rad}}$ satisfies a Bessel equation and its asymptotics near zero is well known. Since, however, n cannot be determined in Proposition 3.1, we prefer to keep the more elementary proof of Proposition 3.5.

A more precise description of the domain follows from Rellich inequalities.

Proposition 3.6. *If n is sufficiently large then both $L^n_{p,\min}$ and $L^n_{p,\max}$ coincide with the operator L defined on the domain*

$$D\left(L^n_{p,\mathrm{reg}}\right) = L^p_{\geq n} \cap W^{2,p}(\mathbb{R}^N) = \left\{u \in L^p_{\geq n} \cap W^{2,p}(\mathbb{R}^N) : |x|^{-1}\nabla u, |x|^{-2}u \in L^p(\mathbb{R}^N)\right\}.$$

Proof. Let W be the space on the right hand side above. The inclusion $D(L^n_{p,\min}) \subset W$ follows for large n from the Rellich inequalities proved in [11, Section 3] and the interpolation inequality [11, Lemma 8.1]. Therefore for large n, using also Proposition 3.5, $D(L^n_{p,\min}) = W = D(L^n_{p,\max})$. We conclude the proof by showing that

$$L^p_{\geq 2} \cap W^{2,p}(\mathbb{R}^N) = \left\{u \in L^p_{\geq 2} \cap W^{2,p}(\mathbb{R}^N) : |x|^{-1}\nabla u, |x|^{-2}u \in L^p(\mathbb{R}^N)\right\}.$$

First of all, let us observe that $L^p_{\geq 2} \cap W^{2,p}(\mathbb{R}^N)$ is contained in $W^{2,p}_0(\Omega)$, the closure of $C_c^\infty(\mathbb{R}^N \setminus \{0\})$ in $W^{2,p}(\mathbb{R}^N)$. This is clear if $1 < p \leq N/2$ since $W^{2,p}(\mathbb{R}^N) = W^{2,p}_0(\Omega)$. If $N/2 < p \leq N$ and $u \in L^p_{\geq 1}$, then

$$\int_{S^{N-1}} u(r\omega)\,d\omega = 0$$

for $r > 0$ and, letting $r \to 0$, $u(0) = 0$ and $u \in W_0^{2,p}(\Omega)$, by [11, Lemma 2.5(ii)]. Finally, if $N < p < \infty$ and $u \in L^p_{\ge 2}$, then $u(0) = 0$, as above, but also

$$0 = \int_{S^{N-1}} u(r\omega)\omega_i \, d\omega = \int_{B_1} D_i(u(rx))\, dx = r^{1-N} \int_{B_r} D_i u(y)\, dy.$$

Dividing it by r and letting $r \to 0$, we obtain that $\nabla u(0) = 0$ and hence $u \in W_0^{2,p}(\Omega)$ by [11, Lemma 2.5(iii)]. The proof is now concluded by observing that Rellich inequality

$$\left\| |x|^{-2} u \right\|_p \le C(N, p) \|\Delta u\|_p$$

and the interpolation inequality

$$\left\| |x|^{-1} \nabla u \right\|_p \le C(N, p) \left(\|\Delta u\|_p + \left\| |x|^{-2} u \right\|_p \right)$$

hold for every $u \in C_c^\infty(\Omega) \cap L^p_{\ge 2}$, $1 < p < \infty$, see [10, Section 7.2] and [11, Lemma 8.1], hence in the closure of the above set, namely in $W_0^{2,p}(\Omega) \cap L^p_{\ge 2}$, by Lemma 2.18. □

Corollary 3.7. *If n is sufficiently large, the semigroups generated in $L^p_{\ge n}$ and in $L^q_{\ge n}$ coincide in $L^p_{\ge n} \cap L^q_{\ge n}$.*

Proof. It is sufficient to prove that the resolvents are consistent for $\lambda > 0$. Assume that $1 < p < q < \infty$ and that $f \in L^p_{\ge n} \cap L^q_{\ge n}$. If $u \in L^p_{\ge n}$ solves $\lambda u - Lu = f$, then $u \in W^{2,p}(\mathbb{R}^N)$ by the previous result and hence $u \in L^{p_1}_{\ge n}$ for a suitable $p < p_1 \le q$, given by Sobolev embedding. Then $Lu \in L^{p_1}$, that is, u belongs to the maximal domain of L in $L^{p_1}_{\ge n}$. Applying Proposition 3.6 again, we see that u belongs to $W^{2,p_1}(\mathbb{R}^N)$ and, iterating the above procedure if necessary, $u, Lu \in L^q_{\ge n}$, hence u is also the solution of $\lambda u - Lu = f$ in $L^q_{\ge n}$. □

3.2. Generation in $L^p_{<n}$: reduction to the 1d case

The space $L^p_{<n}$ can be decomposed as a finite sum

$$L^p_{<n} = \bigoplus_{j \in J} \left(L^p_{\mathrm{rad}} \otimes P_j \right),$$

where $L^p_{\mathrm{rad}} = L^p((0, \infty), r^{N-1}\, dr)$, J is finite and $\{P_j,\ j \in J\}$ is an orthogonal basis of spherical harmonics of degree less than n. Note that we should write $L^p_{\mathrm{rad}} \otimes$ span $\{P_j\}$ since the tensor product is defined for vector spaces. However we keep the above notation because span$\{P_j\}$ is one-dimensional.

Since each term in the above sum is preserved by L we analyze it in each space $L^p_{\mathrm{rad}} \otimes P_j$. Let $v(x) = u(r)P_j(\omega) \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$. By Proposition 2.15 we obtain

$$Lv = \left(a u_{rr} + \frac{N-1+c}{r} u_r - \frac{b + \lambda_k}{r^2} u \right) P_j, \tag{3.4}$$

where $k = \deg \mathrm{P_j} < n$. The results in this section, therefore, depend on a detailed analysis of the following second order differential operator of Bessel-type in L^p_{rad}

$$L_\lambda u = au'' + \frac{N-1+c}{r}u' - \frac{b+\lambda}{r^2}u, \quad \lambda \geq 0.$$

We denote by $L_{\lambda,p,\min}$ the closure in L^p_{rad} of L_λ initially defined on $C_c^\infty((0,\infty))$ and by $L_{\lambda,p,\max}$ the operator L_λ when endowed with the maximal domain

$$D(L_{\lambda,p,\max}) = \left\{u \in L^p_{\mathrm{rad}} \cap W^{2,p}_{\mathrm{loc}}((0,\infty)) : L_\lambda u \in L^p_{\mathrm{rad}}\right\}.$$

We observe that $L^*_{\lambda,p,\min} = L^*_{\lambda,p',\max}$, $L^*_{\lambda,p,\max} = L^*_{\lambda,p',\min}$ where L^* is defined in (2.2) and the duality is referred to the spaces L^p_{rad}.

Note that the equation $L_\lambda u = 0$ has solutions $r^{-s_1^{(\lambda)}}, r^{-s_2^{(\lambda)}}$ where

$$s_1^{(\lambda)} = \frac{N-1+c-a}{2a} - \nu_\lambda, \quad s_2^{(\lambda)} = \frac{N-1+c-a}{2a} + \nu_\lambda$$

and

$$D_\lambda := \frac{b+\lambda}{a} + \left(\frac{N-1+c-a}{2a}\right)^2, \quad \nu_\lambda := \sqrt{D_\lambda},$$

$s_1^{(\lambda)}, s_2^{(\lambda)}$ are real if and only if $D_\lambda \geq 0$ and if $D_\lambda = 0$ we often write $s_0^{(\lambda)} = s_1^{(\lambda)} = s_2^{(\lambda)}$. Note that when $\lambda = 0$, D_λ and $s_i^{(\lambda)}$ reduces to D and s_i defined in (1.2) and (1.1).

3.3. Basic results on Bessel functions

We recall some well-known facts about the modified Bessel functions of first and second kind, I_ν and K_ν, which constitute a basis of solutions of the modified Bessel equation

$$z^2\frac{d^2v}{dz^2}(z) + z\frac{dv}{dz}(z) - \left(z^2+\nu^2\right)v(z) = 0, \quad z \in \mathbb{C}_+. \tag{3.5}$$

We recall that

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{m=0}^\infty \frac{1}{\Gamma(\nu+1+m)}\left(\frac{z}{2}\right)^{2m}, \quad K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin \pi\nu}, \tag{3.6}$$

where limiting values are taken for the definition of K_ν when ν is an integer. The basic properties of these functions we need are collected in the following lemmas.

Lemma 3.8. *For every $\nu \geq 0$, as $|z| \to \infty$, $z \in \mathbb{C}_+$,*

$$|I_\nu(z)| \approx |z|^{-\frac{1}{2}} e^{\operatorname{Re} z}, \qquad |I'_\nu(z)| \approx |z|^{-\frac{1}{2}} e^{\operatorname{Re} z},$$
$$|K_\nu(z)| \approx |z|^{-\frac{1}{2}} e^{-\operatorname{Re} z}, \quad |K'_\nu(z)| \approx |z|^{-\frac{1}{2}} e^{-\operatorname{Re} z}.$$

Moreover, if $\nu > 0$, then as $|z| \to 0$, $z \in \mathbb{C}_+$,

$$|I_\nu(z)| \approx |z|^\nu, \quad |I'_\nu(z)| \approx |z|^{\nu-1}, \quad |K_\nu(z)| \approx |z|^{-\nu}, \quad |K'_\nu(z)| \approx |z|^{-\nu-1},$$

and

$$|I_0(z)| \approx 1, \quad |I'_0(z)| \to 0, \quad |K_0(z)| \approx |\log z|, \quad |K'_0(z)| \approx |z|^{-1}.$$

Proof. See, *e.g.*, [1, 9.6 and 9.7]. □

We also need the precise behavior of the derivatives of I_ν and K_ν.

Lemma 3.9. *If $\nu \in \mathbb{R} \setminus \mathbb{N}$, then for every $z \in \mathbb{C}_+$,*

$$\frac{d}{dz}\Big(z^\nu I_\nu(z)\Big) = z^\nu I_{\nu-1}(z), \quad \frac{d}{dz}\Big(z^\nu I_{-\nu}(z)\Big) = z^\nu I_{-\nu+1}(z). \tag{3.7}$$

In particular, for every $\alpha \in \mathbb{C}$ and $z \in \mathbb{C}_+$,

$$\begin{aligned} &\frac{d}{dz}\Big(z^\nu I_\nu(z) + \alpha z^\nu K_\nu(z)\Big) \\ &= \begin{cases} z^\nu I_{1-\nu}(z) + \left(\dfrac{2\sin(\nu\pi)}{\pi} - \alpha\right) z^\nu K_{1-\nu}(z) & \text{if } \nu \in (0,1), \\ z^\nu I_{\nu-1}(z) - \alpha z^\nu K_{\nu-1}(z) & \text{if } \nu \in (1,\infty) \setminus \mathbb{N}. \end{cases} \end{aligned} \tag{3.8}$$

Proof. The equalities in (3.7) are well-known, see, *e.g.*, [1, Chapter 9, 6.28] or [12, Section 3]. Therefore by (3.6), we have

$$\begin{aligned} \frac{d}{dz}\Big(z^\nu I_\nu(z) + \alpha z^\nu K_\nu(z)\Big) &= \frac{d}{dz}\left(\left(1 - \frac{\pi\alpha}{2\sin(\nu\pi)}\right) z^\nu I_\nu(z) + \frac{\pi\alpha}{2\sin(\nu\pi)} z^\nu I_{-\nu}(z)\right) \\ &= \left(1 - \frac{\pi\alpha}{2\sin(\nu\pi)}\right) z^\nu I_{\nu-1}(z) + \frac{\pi\alpha}{2\sin(\nu\pi)} z^\nu I_{-\nu+1}(z). \end{aligned}$$

If $\nu \in (1,\infty) \setminus \mathbb{N}$, then noting that $\nu - 1 > 0$ and $\sin(\nu\pi) = -\sin((\nu-1)\pi)$, we have the second equality in (3.8). On the other hand, if $\nu \in (0,1)$, then noting that $1 - \nu > 0$ and

$$I_{\nu-1}(z) = I_{1-\nu}(z) + \frac{2\sin(\nu\pi)}{\pi} K_{1-\nu}(z),$$

we deduce the first equality in (3.8). □

We now consider solutions of the equation $\omega^2 u - L_\lambda u = 0, \omega \in \mathbb{C}_+$. Let

$$\begin{aligned} v_{\omega,1}(r) &:= r^{-\frac{N-1+c-a}{2a}} I_{\nu_\lambda}\left(\frac{\omega r}{\sqrt{a}}\right) = r^{-s_1^{(\lambda)}-\nu_\lambda} I_{\nu_\lambda}\left(\frac{\omega r}{\sqrt{a}}\right), \\ v_{\omega,2}(r) &:= r^{-\frac{N-1+c-a}{2a}} K_{\nu_\lambda}\left(\frac{\omega r}{\sqrt{a}}\right) = r^{-s_2^{(\lambda)}+\nu_\lambda} K_{\nu_\lambda}\left(\frac{\omega r}{\sqrt{a}}\right). \end{aligned} \tag{3.9}$$

Then we have

Lemma 3.10. *$v_{\omega,1}$ and $v_{\omega,2}$ solve $\omega^2 u - L_\lambda u = 0$ in $(0,\infty)$. Moreover $v_{\omega,1}(r) \approx r^{-s_1^{(\lambda)}}$, $v_{\omega,2}(r) \approx r^{-s_2^{(\lambda)}}$ as $r \to 0$ when $D_\lambda > 0$ and $v_{\omega,1}(r) \approx r^{-s_0^{(\lambda)}}$, $v_{\omega,2}(r) \approx r^{-s_0^{(\lambda)}}|\log r|$ as $r \to 0$ when $D_\lambda = 0$. More precisely, $r^{s_1^{(\lambda)}} v_{\omega,1}(r) = h(r)$ is an entire function with $h(0) \neq 0$ and, when ν_λ is not an integer, $r^{s_2^{(\lambda)}} v_{\omega,2}(r) = f(r) + r^{2\nu} g(r)$ with and f, g entire functions with $f(0) \neq 0, g(0) \neq 0$.*

Proof. Let $w(r) = r^\beta v(\omega r/\sqrt{a})$ where $\beta = -\frac{N-1+c-a}{2a}$ and v satisfies (3.5) with $\nu = \nu_\lambda$. Then

$$w'(r) = \left(\frac{\omega}{\sqrt{a}}\right) r^\beta v'\left(\frac{\omega r}{\sqrt{a}}\right) + \beta r^{\beta-1} v\left(\frac{\omega r}{\sqrt{a}}\right) = \left(\frac{\omega}{\sqrt{a}}\right) r^\beta v'\left(\frac{\omega r}{\sqrt{a}}\right) + \beta r^{-1} w(r)$$

and therefore

$$\begin{aligned} w''(r) &= \left(\frac{\omega r}{\sqrt{a}}\right)^2 r^{\beta-2} v''\left(\frac{\omega r}{\sqrt{a}}\right) + \beta\left(\frac{\omega r}{\sqrt{a}}\right) r^{\beta-2} v'\left(\frac{\omega r}{\sqrt{a}}\right) + \beta r^{-1} w'(r) - \beta r^{-2} w(r) \\ &= \left(\frac{\omega r}{\sqrt{a}}\right)^2 r^{\beta-2} v\left(\frac{\omega r}{\sqrt{a}}\right) + \nu_\lambda^2 r^{\beta-2} v\left(\frac{\omega r}{\sqrt{a}}\right) + (\beta-1)\left(\frac{\omega r}{\sqrt{a}}\right) r^{\beta-2} v'\left(\frac{\omega r}{\sqrt{a}}\right) \\ &\quad + \beta r^{-1} w'(r) - \beta r^{-2} w(r) \\ &= \frac{\omega^2}{a} w(r) + \frac{2\beta - 1}{r} w'(r) + \frac{\nu_\lambda^2 - \beta^2}{r^2} w(r) \\ &= \frac{1}{a}\left(\omega^2 w(r) - \frac{N-1+c}{r} w'(r) + \frac{b+\lambda}{r^2} w(r)\right). \end{aligned}$$

The last assertions follow the definition of $v_{\omega,1}$, $v_{\omega,2}$, Lemma 3.8 and the power series expansion of $I_\nu, I_{-\nu}$, see (3.6). □

By variation of parameters, all solutions of $\omega^2 u - L_\lambda u = f$, with $\omega \in \mathbb{C}_+$ and $f \in C_c^\infty((0,\infty))$ are given by

$$\begin{aligned} u(r) &= c_1 v_{\omega,1}(r) + c_2 v_{\omega,2}(r) + \frac{v_{\omega,2}(r)}{W(\omega)} \int_0^r t^{\frac{N-1+c}{a}} v_{\omega,1}(t) f(t)\,dt \\ &\quad + \frac{v_{\omega,1}(r)}{W(\omega)} \int_r^\infty t^{\frac{N-1+c}{a}} v_{\omega,2}(t) f(t)\,dt \end{aligned} \tag{3.10}$$

for some $c_1, c_2 \in \mathbb{C}$, where $W(\omega) t^{-\frac{N-1+c}{a}} = v_{\omega,1}' v_{\omega,2} - v_{\omega,1} v_{\omega,2}'$ is the Wronskian of $v_{\omega,1}$ and $v_{\omega,2}$. Note that $W(\omega) > 0$ when ω is real, since $v_{\omega,1}$ is positive and increasing, $v_{\omega,2}$ is positive and decreasing, near infinity.

3.4. The case $D_\lambda < 0$

Here we show that the condition $D_\lambda \geq 0$ is necessary to get positive solutions for all positive data.

Proposition 3.11. *Let* $D_\lambda = \frac{b+\lambda}{a} + \left(\frac{N-1+c-a}{2a}\right)^2 < 0$. *Then for every* $\omega > 0$ *there exists a nonnegative function* $0 \leq \phi \in C_c^\infty((0,\infty))$, $\phi \not\equiv 0$, *such that the problem*

$$\omega^2 v - L_\lambda v = \phi \tag{3.11}$$

does not admit any positive solution in $(0,\infty)$.

Proof. By scaling we may assume that $\omega = 1$. Suppose that there exists $v \geq 0$ satisfying (3.11) in $(0,\infty)$. Setting $w(s) = e^{(\frac{N-1+c-a}{2a})s} v(e^s)$ we get

$$aw''(s) = \left(k + e^{2s}\right) w(s) - e^{(\frac{N-1+c+3a}{2a})s} \phi\left(e^s\right), \quad s \in \mathbb{R},$$

where

$$k = b + \lambda + a\left(\frac{N-1+c-a}{2a}\right)^2 < 0.$$

We choose $m \in \mathbb{R}$ such that $(k + e^{2s}) \leq k/2 < 0$ for $s \leq m$. By the Sturm comparison theorem all non-zero solutions of the homogeneous equation

$$a\zeta''(s) = \left(k + e^{2s}\right)\zeta(s) \tag{3.12}$$

are oscillating for $s \leq m$. By variation of parameters we write

$$w(s) = u_2(s)\int_{-\infty}^{s} u_1(t)g(t)dt + u_1(s)\int_{s}^{\infty} u_2(t)g(t)dt + c_1u_1(s) + c_2u_2(s),$$

where $c_1, c_2 \in \mathbb{C}$, $g(s) = e^{(\frac{N-1+c+3a}{2a})s}\phi(e^s)$ and u_i, $i = 1, 2$ are linearly independent solutions of (3.12) with Wronskian equal to 1. Since g is compactly supported we have for s near $-\infty$

$$w(s) = u_1(s)\int_{\text{supp } g} u_2(t)g(t)dt + c_1u_1(s) + c_2u_2(s).$$

However w is non-negative, because $v \geq 0$, and also oscillating near $-\infty$ since solves (3.12). Hence $w = 0$ near $-\infty$ and therefore

$$c_1 = -\int_{\text{supp } g} u_2(t)g(t)dt, \quad c_2 = 0.$$

This gives

$$\begin{aligned} w(s) &= u_2(s)\int_{-\infty}^{s} u_1(t)g(t)dt + u_1(s)\int_{s}^{\infty} u_2(t)g(t)dt - u_1(s)\int_{\text{supp } g} u_2(t)g(t)dt \\ &= u_2(s)\int_{-\infty}^{s} u_1(t)g(t)dt - u_1(s)\int_{-\infty}^{s} u_2(t)g(t)dt \\ &= \int_{-\infty}^{s} (u_1(t)u_2(s) - u_1(s)u_2(t))g(t)dt. \end{aligned}$$

For fixed s the function $t \mapsto G(s,t) = u_1(t)u_2(s) - u_1(s)u_2(t)$ is also oscillating near $t = -\infty$. Therefore, if we choose $g \neq 0$ such that $G(s,t) < 0$ on supp g, we get $w(s) < 0$ and this contradicts $v \geq 0$. □

3.5. The case $D_\lambda \geq 0$

Here we show that a suitable realization of L_λ generates a semigroup in L^p_{rad} if and only if $s_1^{(\lambda)} < N/p < s_2^{(\lambda)} + 2$. We start with the negative part of the assertion.

Proposition 3.12. *Assume that $D_\lambda \geq 0$. If $\frac{N}{p} \leq s_1^{(\lambda)}$, then $R(\omega^2 - L_{\lambda,p,\max}) \neq L^p_{\text{rad}}$ for every $\omega \in \mathbb{C}_+$ and $\sigma(L_\lambda) = \mathbb{C}$ for every $L_\lambda \subset L_{\lambda,p,\max}$. If $\frac{N}{p} \geq s_2^{(\lambda)} + 2$, then $\sigma(L_\lambda) = \mathbb{C}$ for every $L_\lambda \supset L_{\lambda,p,\min}$.*

Proof. Assume that $\frac{N}{p} \leq s_1^{(\lambda)}$ and let $\omega \in \mathbb{C}_+$. We fix $0 \leq \zeta \in C_c^\infty((0,\infty))$, $\zeta \geq 0$ and consider

$$f(r) := \overline{v_{\omega,2}(r)}\zeta(r) \in C_c^\infty((0,\infty)). \tag{3.13}$$

It follows from (3.10) that every solutions of $(\omega^2 - L_\lambda)u = f$ satisfies near 0,

$$u(r) = \left(c_1 + \frac{1}{W(\omega)}\int_0^\infty t^{\frac{N-1+c}{a}} |v_{\omega,2}(t)|^2 \zeta(t)\,dt\right) v_{\omega,1}(r) + c_2 v_{\omega,2}(r).$$

Since $v_{\omega,1}$ and $v_{\omega,2}$ do not belong to $L^p((0,1), r^{N-1}\,dr)$, by Lemma 3.10, if u belongs to $L^p((0,1), r^{N-1}\,dr)$ then

$$c_1 = -\frac{1}{W(\omega)}\int_0^\infty t^{\frac{N-1+c}{a}} |v_{\omega,2}(t)|^2 \zeta(t)\,dt \neq 0, \quad c_2 = 0.$$

On the other hand, near ∞,

$$u(r) = c_1 v_{\omega,1}(r) + \left(\frac{1}{W(\omega)}\int_0^\infty t^{\frac{N-1+c}{a}} v_{\omega,1}(t) f(t)\,dt\right) v_{\omega,2}(r).$$

Since $v_{\omega,1}$ does not belong to $L^p((1,\infty), r^{N-1}\,dr)$ and $v_{\omega,2}$ belongs to $L^p((1,\infty), r^{N-1}\,dr)$, we conclude that $c_1 = 0$, hence $u = 0$, which is a contradiction. Therefore $R(\omega^2 - L_\lambda) \neq L^p_{\text{rad}}$ for every $\omega \in \mathbb{C}_+$ and $L_\lambda \subset L_{\lambda,p,\max}$. Hence $\sigma(L_\lambda) \supset \mathbb{C} \setminus (-\infty, 0]$ and, since the spectrum is closed, we have $\sigma(L_\lambda) = \mathbb{C}$.

Next we assume $\frac{N}{p} \geq s_2^{(\lambda)} + 2$ and $L_\lambda \supset L_{\lambda,p,\min}$. We consider the adjoint operator (2.2) in $L^{p'}_{\text{rad}}$ and the relative indicial numbers defined in (2.4) (with b replaced by $b + \lambda$)

$$L_\lambda{}^* v = a v'' + \frac{N-1+c^*}{r} v' - \frac{b^*}{r^2} v.$$

Observe that $\frac{N}{p} \geq s_2^{(\lambda)} + 2$ is equivalent to $\frac{N}{p'} \leq s^*{}_1^{(\lambda)}$. Since $L^*_{\lambda,p',\max}$ is the adjoint of $L_{\lambda,p,\min}$, then $L^*_\lambda \subset L^*_{\lambda,p',\max}$ and the previous step yields $\sigma(L_\lambda) = \sigma(L^*_\lambda) = \mathbb{C}$. □

When $\frac{N}{p} \in (s_1^{(\lambda)}, s_2^{(\lambda)} + 2)$ we construct a resolvent using (3.10) with $c_1 = c_2 = 0$. Since $v_{\omega,1}, v_{\omega,2}$ are given by (3.9), we are led to study the following integral operators:

$$\mathcal{R}_1 f(r) := r^{-\frac{N-1+c-a}{2a}} K_{\nu_\lambda}(r) \int_0^r t^{\frac{N-1+c-a}{2a}+1} I_{\nu_\lambda}(t) f(t)\, dt,$$

$$\widetilde{\mathcal{R}}_1 f(r) := r^{-\frac{N-1+c-a}{2a}} K'_{\nu_\lambda}(r) \int_0^r t^{\frac{N-1+c-a}{2a}+1} I_{\nu_\lambda}(t) f(t)\, dt,$$

$$\mathcal{R}_2 f(r) := r^{-\frac{N-1+c-a}{2a}} I_{\nu_\lambda}(r) \int_r^\infty t^{\frac{N-1+c-a}{2a}+1} K_{\nu_\lambda}(t) f(t)\, dt,$$

$$\widetilde{\mathcal{R}}_2 f(r) := r^{-\frac{N-1+c-a}{2a}} I'_{\nu_\lambda}(r) \int_r^\infty t^{\frac{N-1+c-a}{2a}+1} K_{\nu_\lambda}(t) f(t)\, dt.$$

Note that if $u = (\mathcal{R}_1 f + \mathcal{R}_2 f)$, then $u' = C r^{-1}(\mathcal{R}_1 f + \mathcal{R}_2 f) + (\widetilde{\mathcal{R}}_1 f + \widetilde{\mathcal{R}}_2 f)$ for some constant C'.

The main weighted and unweighted estimates of the above operators in L^p_{rad} are contained in the following two lemmas. Note that, when $D_\lambda > 0$ the inequality $s_1^{(\lambda)} < N/p < s_2^{(\lambda)} + 2$ is equivalent to the existence of $\theta \in (0, 1]$ such that $s_1^{(\lambda)} < N/p - 2\theta < s_2^{(\lambda)}$.

Lemma 3.13. *Assume that $D_\lambda > 0$. If $\frac{N}{p} < s_2^{(\lambda)} + 2$, then $(1 \wedge r)^{-2}\mathcal{R}_1$ and $(1\wedge r)^{-1}\widetilde{\mathcal{R}}_1$ are bounded in L^p_{rad}. If $\frac{N}{p} - 2\theta > s_1^{(\lambda)}$ with $\theta \in (0, 1]$, then $(1\wedge r)^{-2\theta}\mathcal{R}_2$ and $(1 \wedge r)^{1-2\theta}\widetilde{\mathcal{R}}_2$ are bounded in L^p_{rad}.*

Proof. Since the arguments for $\widetilde{\mathcal{R}}_1$ and $\widetilde{\mathcal{R}}_2$ are almost the same as those for $\mathcal{R}_1$ and $\mathcal{R}_2$, we give a proof only in the latter case.

(**The estimate for** $\mathcal{R}_1$) Take $\delta > 0$ satisfying $\frac{N}{p} + \delta < s_2^{(\lambda)} + 2$, that is, $s_2^{(\lambda)} + 1 - \frac{N-1}{p} - \delta > -\frac{1}{p'}$. For $r < 1$

$$\begin{aligned}|\mathcal{R}_1 f(r)|^p &\le C r^{-s_2^{(\lambda)} p} \left(\int_0^r t^{s_2^{(\lambda)}+1} |f(t)|\,dt\right)^p \\ &\le C r^{-s_2^{(\lambda)} p} \left(\int_0^r t^{(s_2^{(\lambda)}+1-\frac{N-1}{p}-\delta)p'}\,dt\right)^{p-1} \int_0^r |f(t)|^p t^{N-1+p\delta}\,dt \\ &\le \frac{C}{((s_2^{(\lambda)}+1-\frac{N-1}{p}-\delta)p'+1)^{p-1}} r^{2p-1-(N-1+p\delta)} \int_0^r |f(t)|^p t^{N-1+p\delta}\,dt.\end{aligned}$$

This implies

$$\begin{aligned}\|r^{-2}\mathcal{R}_1 f\|^p_{L^p((0,1),r^{N-1}\,dr)} &\le \tilde{C} \int_0^1 \left(r^{-1-(N-1+p\delta)} \int_0^r |f(t)|^p t^{N-1+p\delta}\,dt\right) r^{N-1}\,dr \\ &\le \tilde{C} \int_0^1 \left(\int_t^1 r^{-1-p\delta}\,dr\right) |f(t)|^p t^{N-1+p\delta}\,dt \\ &\le \frac{\tilde{C}}{p\delta} \int_0^1 |f(t)|^p t^{N-1}\,dt = \frac{\tilde{C}}{p\delta} \|f\|^p_{L^p((0,1),r^{N-1}\,dr)}.\end{aligned}$$

On the other hand, for $r \ge 1$, since $s_2^{(\lambda)} + 2 - N > -N/p' + \delta$,

$$\begin{aligned}|\mathcal{R}_1 f(r)| &\le C r^{-\frac{N-1+c}{2a}} e^{-r} \left(\int_0^1 t^{s_2^{(\lambda)}+1} |f(t)|\,dt + \int_1^r t^{\frac{N-1+c}{2a}} e^t |f(t)|\,dt\right) \\ &\le C r^{-\frac{N-1+c}{2a}} e^{-r} \|r^{s_2^{(\lambda)}+2-N}\|_{L^{p'}((0,1),r^{N-1}\,dr)} \|f\|_{L^p((0,1),r^{N-1}\,dr)} \\ &\quad + C r^{-\frac{N-1}{p}} \int_1^r \exp\left(t - r + \left(\frac{N-1+c}{2a} - \frac{N-1}{p}\right)(\log t - \log r)\right) \\ &\quad \cdot |f(t)| t^{\frac{N-1}{p}}\,dt \\ &\le C r^{-\frac{N-1+c}{2a}} e^{-r} \|r^{s_2+2-N}\|_{L^{p'}((0,1),r^{N-1}\,dr)} \|f\|_{L^p((0,1),r^{N-1}\,dr)} \\ &\quad + C r^{-\frac{N-1}{p}} e^{\alpha(-\log\varepsilon)} \int_1^r e^{(1-\alpha\varepsilon)(t-r)} |f(t)| t^{\frac{N-1}{p}}\,dt \\ &\le C r^{-\frac{N-1+c}{2a}} e^{-r} \|r^{s_2+2-N}\|_{L^{p'}((0,1),r^{N-1}\,dr)} \|f\|_{L^p((0,1),r^{N-1}\,dr)} \\ &\quad + C\varepsilon^{-\alpha} r^{-\frac{N-1}{p}} \left(\int_1^r e^{(1-\alpha\varepsilon)(t-r)} dt\right)^{\frac{1}{p'}} \left(\int_1^r e^{(1-\alpha\varepsilon)(t-r)} |f(t)|^p t^{N-1} dt\right)^{\frac{1}{p}},\end{aligned}$$

where $\alpha = |\frac{N-1+c}{2a} - \frac{N-1}{p}|$. Note that we used the inequality

$$|\log x - \log y| \le (-\log\varepsilon) + \varepsilon|x-y|, \quad \varepsilon > 0,\, x, y \ge 1.$$

If we choose ε satisfying $\varepsilon\alpha < 1$, we have

$$
\begin{aligned}
&\|\mathcal{R}_1 f\|_{L^p((1,\infty),r^{N-1}\,dr)} \\
&\le C \left\| r^{-\frac{N-1+c}{2a}} e^{-r} \right\|_{L^p((1,\infty),r^{N-1}\,dr)} \left\| r^{s_2+2-N} \right\|_{L^{p'}((0,1),r^{N-1}\,dr)} \|f\|_{L^p((0,1),r^{N-1}\,dr)} \\
&\quad + C\varepsilon^{-\alpha} \left((1-\alpha\varepsilon)^{1-p} \int_1^\infty \int_1^r e^{(1-\alpha\varepsilon)(t-r)} |f(t)|^p t^{N-1}\,dt\,dr \right)^{\frac{1}{p}} \\
&\le C \left\| r^{-\frac{N-1+c}{2a}} e^{-r} \right\|_{L^p((1,\infty),r^{N-1}\,dr)} \left\| r^{s_2+2-N} \right\|_{L^{p'}((0,1),r^{N-1}\,dr)} \|f\|_{L^p_{\mathrm{rad}}} \\
&\quad + C\varepsilon^{-\alpha} \left((1-\alpha\varepsilon)^{1-p} \int_1^\infty \left(\int_t^\infty e^{(1-\alpha\varepsilon)(t-r)}\,dr \right) |f(t)|^p t^{N-1}\,dt \right)^{\frac{1}{p}} \\
&\le C \Bigg[\left\| r^{-\frac{N-1+c}{2a}} e^{-r} \right\|_{L^p((1,\infty),r^{N-1}\,dr)} \left\| r^{s_2+2-N} \right\|_{L^{p'}((0,1),r^{N-1}\,dr)} \\
&\qquad + \varepsilon^{-\alpha}(1-\alpha\varepsilon)^{-1} \Bigg] \|f\|_{L^p_{\mathrm{rad}}}.
\end{aligned}
$$

(**The estimate for** $\mathcal{R}_2$) Take $\delta > 0$ satisfying $s_1^{(\lambda)} < \frac{N}{p} - 2\theta - \delta$. For $r < 1$,

$$
\begin{aligned}
|\mathcal{R}_2 f(r)| &\le C r^{-s_1^{(\lambda)}} \int_r^1 t^{s_1^{(\lambda)}+1} |f(t)|\,dt + C r^{-s_1^{(\lambda)}} \int_1^\infty t^{\frac{N-1+c}{2a}} e^{-t} |f(t)|\,dt \\
&\le C r^{-s_1^{(\lambda)}} \left(\int_r^1 t^{(s_1^{(\lambda)}+2\theta+\delta-N)p'+N-1}\,dt \right)^{\frac{1}{p'}} \\
&\quad \cdot \left(\int_r^1 |f(t)|^p t^{(2-2\theta-\delta)p+N-1}\,dt \right)^{\frac{1}{p}} \\
&\quad + C r^{-s_1^{(\lambda)}} \| t^{\frac{N-1+c}{2a}+1-N} e^{-t} \|_{L^{p'}((1,\infty),r^{N-1}\,dr)} \|f\|_{L^p((1,\infty),r^{N-1}\,dr)} \\
&\le \frac{C}{|(s_1^{(\lambda)}+2\theta+\delta-N)p'+N|^{\frac{1}{p'}}} r^{2\theta+\delta-\frac{N}{p}} \\
&\quad \cdot \left(\int_r^1 |f(t)|^p t^{(2-2\theta-\delta)p+N-1}\,dt \right)^{\frac{1}{p}} \\
&\quad + C r^{-s_1^{(\lambda)}} \| t^{\frac{N-1+c}{2a}+1-N} e^{-t} \|_{L^{p'}((1,\infty),r^{N-1}\,dr)} \|f\|_{L^p((1,\infty),r^{N-1}\,dr)}.
\end{aligned}
$$

Therefore

$$\begin{aligned}
&\|r^{-2\theta}\mathcal{R}_2 f\|_{L^p((0,1),r^{N-1}\,dr)}\\
&\le \frac{C}{|(s_1^{(\lambda)}+2\theta+\delta-N)p'+N|^{\frac{1}{p'}}}\left(\int_0^1 r^{-1+p\delta}\int_r^1 |f(t)|^p t^{(2-2\theta-\delta)p+N-1}\,dt\,dr\right)^{\frac{1}{p}}\\
&\quad + C\left\|r^{-s_1^{(\lambda)}-2\theta}\right\|_{L^p((0,1),r^{N-1}\,dr)}\left\|t^{\frac{N-1+c}{2a}+1-N}e^{-t}\right\|_{L^{p'}((1,\infty),r^{N-1}\,dr)}\\
&\quad \cdot \|f\|_{L^p((1,\infty),r^{N-1}\,dr)}\\
&\le \frac{C}{|(s_1^{(\lambda)}+2\theta+\delta-N)p'+N|^{\frac{1}{p'}}(p\delta)^{\frac{1}{p}}}\left(\int_0^1 |f(t)|^p t^{(2-2\theta)p+N-1}\,dt\right)^{\frac{1}{p}}\\
&\quad + C\left\|r^{-s_1^{(\lambda)}-2\theta}\right\|_{L^p((0,1),r^{N-1}\,dr)}\left\|t^{\frac{N-1+c}{2a}+1-N}e^{-t}\right\|_{L^{p'}((1,\infty),r^{N-1}\,dr)}\\
&\quad \|f\|_{L^p((1,\infty),r^{N-1}\,dr)}\\
&\le C\|f\|_{L^p_{\mathrm{rad}}}.
\end{aligned}$$

For $r \ge 1$, proceeding as for $\mathcal{R}_1$ we have

$$\begin{aligned}
|\mathcal{R}_2 f(r)| &\le C r^{-\frac{N-1+c}{2a}} e^r \int_r^\infty t^{\frac{N-1+c}{2a}} e^{-t}|f(t)|\,dt\\
&\le C r^{-\frac{N-1}{p}} e^{\alpha(-\log\varepsilon)} \int_r^\infty e^{(1-\alpha\varepsilon)(r-t)}|f(t)|t^{\frac{N-1}{p}}\,dt.
\end{aligned}$$

Thus $\|\mathcal{R}_2 f\|_{L^p((1,\infty),r^{N-1}\,dr)} \le C\|f\|_{L^p((1,\infty),r^{N-1}\,dr)}$, with similar computations as for $\mathcal{R}_1$, $r \ge 1$. □

Lemma 3.14. *Assume that $D_\lambda = 0$. If $\frac{N}{p} < s_0^{(\lambda)} + 2$, then $(1\wedge r)^{-2\theta}\mathcal{R}_1$ $(1\wedge r)^{1-2\theta}\widetilde{\mathcal{R}}_1$ are bounded in L^p_{rad} for every $\theta \in [0,1)$. If $\frac{N}{p} - 2\theta_0 = s_0^{(\lambda)}$ with $\theta_0 \in (0,1)$, then $\mathcal{R}_2$ and $\widetilde{\mathcal{R}}_2$ are bounded in L^p_{rad} and $\chi_{\{r<\frac{1}{2}\}}|x|^{-2\theta_0}|\log r|^{-\frac{2}{p}}\mathcal{R}_2$ and $\chi_{\{r<\frac{1}{2}\}}|x|^{1-2\theta_0}|\log r|^{-\frac{2}{p}}\widetilde{\mathcal{R}}_2$ are bounded in L^p_{rad}.*

Proof. Since $|\log r| \le c_\varepsilon r^{-\varepsilon}$ for small r, the assertion for $\mathcal{R}_1$ follows by arguments similar to those of Lemma 3.13.

Observing that $r^{s_0^{(\lambda)}+2-N}K_0(r) \in L^{p'}_{\mathrm{rad}}$, we have for $r < \frac{1}{2}$,

$$|\mathcal{R}_2 f(r)| \le r^{-s_0^{(\lambda)}}\int_r^\infty t^{s_0^{(\lambda)}+1}K_0(t)|f(t)|\,dt \le r^{-s_0^{(\lambda)}}\left\|r^{s_0^{(\lambda)}+2-N}K_0(r)\right\|_{L^{p'}_{\mathrm{rad}}}\|f\|_{L^p_{\mathrm{rad}}}.$$

Therefore

$$\left\|r^{-2\theta_0}|\log r|^{-\frac{2}{p}}\mathcal{R}_2 f\right\|_{L^p((0,\frac{1}{2}),r^{N-1}\,dr)} \le (\log 2)^{-\frac{1}{p}}\left\|r^{s_0^{(\lambda)}+2-N}K_0(r)\right\|_{L^{p'}_{\mathrm{rad}}}\|f\|_{L^p_{\mathrm{rad}}}.$$

For $r > \frac{1}{2}$, the computation is as in Lemma 3.13. □

We can now prove the main results of this subsection. In the proposition below, the domain of the operator is defined through a suitable parameter $\theta \in (0, 1]$. The proof shows that the definition is independent of the choice of θ, whenever it satisfies the requirement in the statement.

Proposition 3.15. *Assume that* $D_\lambda > 0$ *and that* $\frac{N}{p} \in (s_1^{(\lambda)}, s_2^{(\lambda)} + 2)$. *Then* $L_{\lambda,p,\mathrm{int}}$ *defined by*

$$D(L_{\lambda,p,\mathrm{int}}) = \left\{u \in D(L_{\lambda,p,\max}) \; ; \; r^{-2\theta} u \in L^p_{\mathrm{rad}} \;\; \forall \theta \in (0,1] \text{ satisfying } \frac{N}{p} - 2\theta \in \left(s_1^{(\lambda)}, s_2^{(\lambda)}\right)\right\}$$

generates a bounded positive analytic semigroup of angle $\frac{\pi}{2}$ *in* L^p_{rad} *and* $\sigma(L_{\lambda,p,\mathrm{int}}) = (-\infty, 0]$. *The domain* $D(L_{\lambda,p,\mathrm{int}})$ *coincide with*

$$D(L_{\lambda,p,\mathrm{reg}}) = \left\{u \in L^p_{\mathrm{rad}} \cap W^{2,p}_{\mathrm{loc}}((0,\infty)) \; ; \; (1 \wedge r)^{2-2\theta} u'', (1 \wedge r)^{1-2\theta} u', (1 \wedge r)^{-2\theta} u \in L^p_{\mathrm{rad}}\right\}$$

for all/one θ *as above.*

Proof. Let $\omega \in \mathbb{C}_+$. If $u \in D(L_{\lambda,p,\mathrm{int}})$ solves $\omega^2 u - Lu = 0$, then $u = c_1 v_{\omega,1} + c_2 v_{\omega,2}$, see (3.9). Since $v_{\omega,1}$ diverges exponentially and $v_{\omega,2}$ tends to 0 at infinity, then $c_1 = 0$. However $v_{\omega,2}(r) \approx r^{-s_2^{(\lambda)}}$ as $r \to 0$, hence $r^{-2\theta} v_{\omega,2} \notin L^p_{\mathrm{rad}}$. Then $c_2 = 0$ and $\omega^2 - L$ is injective on $D(L_{\lambda,p,\mathrm{int}})$. To show the surjectivity we set for every $f \in L^p_{\mathrm{rad}}$,

$$u(r) = \frac{v_{\omega,2}(r)}{W(\omega)} \int_0^r t^{\frac{N-1+c}{a}} v_{\omega,1}(t) f(t)\, dt + \frac{v_{\omega,1}(r)}{W(\omega)} \int_r^\infty t^{\frac{N-1+c}{a}} v_{\omega,2}(t) f(t)\, dt. \tag{3.14}$$

Then $\omega^2 u - L_\lambda u = f$ and, using Lemma 3.8,

$$\begin{aligned} &|W(\omega)u(r)| \\ &\le C^2 r^{-\frac{N-1+c-a}{2a}} K_{\nu_\lambda}\left(\frac{\mathrm{Re}\,\omega}{\sqrt{a}} r\right) \int_0^r t^{\frac{N-1+c-a}{2a}+1} I_{\nu_\lambda}\left(\frac{\mathrm{Re}\,\omega}{\sqrt{a}} t\right) |f(t)|\, dt \\ &\quad + C^2 r^{-\frac{N-1+c-a}{2a}} I_{\nu_\lambda}\left(\frac{\mathrm{Re}\,\omega}{\sqrt{a}} r\right) \int_r^\infty t^{\frac{N-1+c-a}{2a}+1} K_{\nu_\lambda}\left(\frac{\mathrm{Re}\,\omega}{\sqrt{a}} t\right) |f(t)|\, dt \\ &= C^2 \left(\frac{\sqrt{a}}{\mathrm{Re}\,\omega}\right)^2 \left(\frac{\mathrm{Re}\,\omega}{\sqrt{a}} r\right)^{-\frac{N-1+c-a}{2a}} K_{\nu_\lambda}\left(\frac{\mathrm{Re}\,\omega}{\sqrt{a}} r\right) \int_0^{\frac{\mathrm{Re}\,\omega}{\sqrt{a}}} s^{\frac{N-1+c-a}{2a}+1} I_{\nu_\lambda}(s) \left| f\left(\frac{\sqrt{a}}{\mathrm{Re}\,\omega} s\right)\right| ds \\ &\quad + C^2 \left(\frac{\sqrt{a}}{\mathrm{Re}\,\omega}\right)^2 \left(\frac{Re\,\omega}{\sqrt{a}} r\right)^{-\frac{N-1+c-a}{2a}} I_{\nu_\lambda}\left(\frac{\mathrm{Re}\,\omega}{\sqrt{a}} r\right) \int_0^{\frac{Re\,\omega}{\sqrt{a}}} s^{\frac{N-1+c-a}{2a}+1} K_{\nu_\lambda}(s) \left| f\left(\frac{\sqrt{a}}{\mathrm{Re}\,\omega} s\right)\right| ds \\ &\le C^2 \left(\frac{\sqrt{a}}{\mathrm{Re}\,\omega}\right)^2 \left(\mathcal{R}_1 |J_s f| + \mathcal{R}_2 |J_s f|\right) \left(\frac{\mathrm{Re}\omega}{\sqrt{a}} r\right), \end{aligned} \tag{3.15}$$

where $J_s f(r) = f(sr)$ for $s > 0$ and $s = (\sqrt{a})/(\mathrm{Re}\,\omega)$ in (3.15). From Lemma 3.13 we deduce that $u \in L^p_{\mathrm{rad}}$. This yields $\sigma(L_{\lambda,p,\mathrm{int}}) \subset (-\infty, 0]$ and the resolvent estimate $\|(\omega^2 - L_\lambda)^{-1}\| \le C|\omega|^{-2}$ follows via the scaling J_s. Next, let us show that $\sigma(L_{\lambda,p,\mathrm{int}}) = (-\infty, 0]$. Assume that the resolvent set $\rho(L_{\lambda,p,\mathrm{int}})$ contains a point in the negative real axis. By scaling again it contains the point -1, hence the unit circle S^1. Then the resolvent estimate $\|(\omega^2 - L_\lambda)^{-1}\| \le C|\omega|^{-2}$ holds for every $\omega \ne 0$ and yields that $e^{zL_{\lambda,p,\mathrm{int}}}$ is an entire function in $\mathbb{C}$. Then the generator $L_{\lambda,p,\mathrm{int}}$ would be bounded. To prove the domain characterization, it is enough to observe that, by (3.15) and the analogous estimate for u' involving $\widetilde{R}_i$, $i = 1, 2$, Lemma 3.13 yields $(1 \wedge r)^{-2\theta} u, (1 \wedge r)^{1-2\theta} u' \in L^p_{\mathrm{rad}}$. Using the equation, we deduce that $(1 \wedge r)^{2-2\theta} u'' \in L^p_{\mathrm{rad}}$. Finally, the positivity of the generated semigroup follows from that of the resolvent for real ω, since the functions $v_{\omega,1}$, $v_{\omega,2}$ and $W(\omega)$ in (3.14) are positive. □

Proposition 3.16. *If $D_\lambda = 0$ and $\frac{N}{p} \in (s_0^{(\lambda)}, s_0^{(\lambda)} + 2)$, then $L_{\lambda,p,\mathrm{int}}$ defined by*

$$D(L_{\lambda,p,\mathrm{int}}) = \left\{u \in D(L_{\lambda,p,\max})\ ;\ \chi_{\{r<\frac12\}} r^{-2\theta_0} |\log r|^{-\frac{2}{p}} u \in L^p_{\mathrm{rad}}\right\}$$

with $\theta_0 = \frac12(\frac{N}{p} - s_0^{(\lambda)}) \in (0, 1)$, generates a bounded positive analytic semigroup of angle $\frac{\pi}{2}$ in L^p_{rad} and $\sigma(L_{\lambda,p,\mathrm{int}}) = (-\infty, 0]$. The domain $D(L_{\lambda,p,\mathrm{int}})$ coincide with

$$D(L_{\lambda,p,\mathrm{reg}}) = \left\{ u \in L^p_{\mathrm{rad}} \cap W^{2,p}_{\mathrm{loc}}((0,\infty))\ ;\ \begin{array}{l} \chi_{\{r>\frac12\}} u'', \chi_{\{r>\frac12\}} u', \chi_{\{r>\frac12\}} u \in L^p_{\mathrm{rad}}, \\ \chi_{\{r<\frac12\}} r^{2-2\theta_0} |\log r|^{-\frac{2}{p}} u'' \in L^p_{\mathrm{rad}}, \\ \chi_{\{r<\frac12\}} r^{1-2\theta_0} |\log r|^{-\frac{2}{p}} u' \in L^p_{\mathrm{rad}}, \\ \chi_{\{r<\frac12\}} r^{-2\theta_0} |\log r|^{-\frac{2}{p}} u \in L^p_{\mathrm{rad}} \end{array} \right\}.$$

Proof. Similar to that of the above proposition, using Lemma 3.14 instead of Lemma 3.13. □

The cases $s_1^{(\lambda)} = 0, -1$ are special for the domain characterization, since in these cases the operator L_λ has a special form. In fact, $s_1^{(\lambda)} = 0$ if and only if $b + \lambda = 0$ and $(N - 1 + c - a) \ge 0$. In this case $L_\lambda = aD^2 + \frac{N-1+c}{r} D$ has no zero order term. Instead $s_1^{(\lambda)} = -1$ if and only if $b+\lambda = N-1+c$ and $N-1+c+a \ge 0$. In this case $L_\lambda = aD^2 + \frac{N-1+c}{r}(\frac{D}{r} - \frac{1}{r^2})$. We refer to Proposition 2.14 for similar phenomena.

Proposition 3.17. *Assume that $D_\lambda \ge 0$ and that $s_1^{(\lambda)} = 0$ or equivalently $b+\lambda = 0$ and $(N - 1 + c - a) \ge 0$. If $\frac{N}{p} \in (0, s_2^{(\lambda)} + 2)$, then u'', $\frac{u'}{r} \in L^p_{\mathrm{rad}}$ for every $u \in D(L_{\lambda,p,\mathrm{int}})$. It follows that*

$$D(L_{\lambda,p,\mathrm{int}}) = \left\{u \in L^p_{\mathrm{rad}} \cap W^{2,p}_{\mathrm{loc}}((0,\infty))\ ;\ u'', (1 \wedge r)^{-1} u', (1 \wedge r)^{-2} u \in L^p_{\mathrm{rad}}\right\}$$

if $1 < p < \frac{N}{2}$ *and*

$$D(L_{\lambda,p,\mathrm{int}}) = \left\{u \in L^p_{\mathrm{rad}} \cap W^{2,p}_{\mathrm{loc}}((0,\infty))\ ;\ u'', (1 \wedge r)^{-1}u', u \in L^p_{\mathrm{rad}}\right\} \quad (3.16)$$

if $\frac{N}{2} \le p < \infty$.

Proof. If $1 < p < \frac{N}{2}$, then $s_1^{(\lambda)} + 2 = 2 < \frac{N}{p}$ and the assertions follow from Proposition 3.15 since $\theta = 1$. Note that in the critical case $D_\lambda = 0$, then $\frac{N}{p} < s_2^{(\lambda)} + 2 = 2$ implies that $p > \frac{N}{2}$.

Let us therefore assume that $p \ge \frac{N}{2}$. By assumption $b + \lambda = 0$ and $(N - 1 + c - a) \ge 0$. Observe that $s_2^{(\lambda)} = (N - 1 + c - a)/a$ and that for $u \in D(L_{\lambda,p,\mathrm{int}})$, $g = L_\lambda u = au'' + \frac{N-1+c}{r}u'$ belongs to L^p_{rad}. Setting $v = u'$ and $k = \frac{N-1+c}{a} \ge 1$ we obtain $v' + \frac{k}{r}v = f$ with $f = g/a$. Integrating between ε and r we obtain

$$r^k v(r) - \varepsilon^k v(\varepsilon) = \int_\varepsilon^r f(t)t^k\,dt = \int_\varepsilon^r f(t)t^{N-1}t^{1-N+k}\,dt.$$

The integral on the right hand side converges as $\varepsilon \to 0$, by Hölder inequality with respect to the measure $t^{N-1}\,dt$, since $p \ge N/2$, hence $N-1+p'(1-N+k) > -1$ (in the critical case $k = 1$ but $p > N/2$). Then $\varepsilon^k v(\varepsilon)$ has a finite limit ℓ as $\varepsilon \to 0$. If $\ell \ne 0$ and $D_\lambda > 0$ then $k > 1$, $v(r) \approx r^{-k}$ and $u(r) \approx r^{-k+1}$ as $r \to 0$. Choosing $2\theta = N/p - \delta < 2$ in Proposition 3.15 we see that $r^{-2\theta}u$ is not in L^p_{rad} for δ sufficiently small. If $D_\lambda = 0$, then $k = 1$ and $u \approx \log r$ as $r \to 0$. In this case, we pick $\theta_0 = N/(2p)$ in Proposition 3.16 and see again that $r^{-2\theta_0}|\log r|^{-2/p}u \notin L^p_{\mathrm{rad}}$. In both cases this contrasts with $u \in D(L_{\lambda,p,\mathrm{int}})$, hence $\ell = 0$ and therefore

$$\frac{v(r)}{r} = \frac{1}{r^{k+1}}\int_0^r f(t)t^k\,dt = \int_0^1 f(sr)s^k\,ds.$$

Minkowski inequality then yields

$$\left\|r^{-1}v\right\|_{p,\mathrm{rad}} \le \int_0^1 s^k\,\|f(s\cdot)\|_{p,\mathrm{rad}}\,ds \le \int_0^1 s^k\,ds\left(\int_0^\infty |f(t)^p t^{N-1}s^{-N}\,dt\right)^{\frac{1}{p}}$$
$$= \|f\|_{p,\mathrm{rad}}\int_0^1 s^{k-\frac{N}{p}}\,ds = C\|f\|_{p,\mathrm{rad}}$$

since $k - N/p > -1$ both in the critical and in the non critical case. This shows that $u'/r \in L^p_{\mathrm{rad}}$ hence, by difference, also $u'' \in L^p_{\mathrm{rad}}$. To show equality (3.16) let us call W the space on the right hand side. We have just shown that $D(L_{\lambda,p,\mathrm{int}}) \subset W$. Since $W \subset D(L_{\lambda,p,\max})$ we have only to show that $\omega^2 - L_\lambda$ is injective on W for $\omega \in \mathbb{C}_+$. However, this follows from (3.9) and Lemma 3.10 since $v_{\omega,1}$ is unbounded at infinity and $v'_{\omega,2}$ behaves like $r^{-s_2^{(\lambda)}-1}$ when $r \to 0$ in the non critical case and like r^{-1} in critical case (in the critical case $v_{\omega,2}(r) = K_0(\omega r/\sqrt{a})$), hence $r^{-1}v'_{\omega,2}$ does not belong to L^p_{rad}. □

Even though $s_1^{(\lambda)} = -1$ requires only $b+\lambda = N-1+c$ and $N-1+c+a \geq 0$, we deal only with the special case $\lambda = N-1, b = c = 0$.

Proposition 3.18. *Assume that $b = c = 0$ and that $\lambda = N-1$. Then $D_\lambda > 0$ and $s_1^{(\lambda)} = -1$, $s_2^{(\lambda)} = (N-1)/a$. If $\frac{N}{p} \in (0, s_2^{(\lambda)}+2)$, then u'', $\frac{u'}{r} - \frac{u}{r^2} \in L^p_{\text{rad}}$ for every $u \in D(L_{\lambda,p,\text{int}})$. It follows that*

$$D(L_{\lambda,p,\text{int}}) = \left\{u \in L^p_{\text{rad}} \cap W^{2,p}_{\text{loc}}((0,\infty))\ ;\ u'', (1\wedge r)^{-1}u', (1\wedge r)^{-2}u \in L^p_{\text{rad}}\right\}$$

if $1 < p < N$ and

$$D(L_{\lambda,p,\text{int}}) = \left\{u \in L^p_{\text{rad}} \cap W^{2,p}_{\text{loc}}((0,\infty))\ ;\ u', u'', \frac{u'}{r} - \frac{u}{r^2} \in L^p_{\text{rad}}\right\} \tag{3.17}$$

if $N \leq p < \infty$.

Proof. Note that $D_\lambda = (N-1+a)^2/4a^2 \geq 1$. Moreover $s_1^{(\lambda)} = -1$, $s_2^{(\lambda)} = (N-1)/a > 0$ are immediately verified.

If $1 < p < N$, then $s_1^{(\lambda)} + 2 = 1 < \frac{N}{p}$ and the assertions follow from Proposition 3.15 since $\theta = 1$.

Let us therefore assume that $p \geq N$. For $u \in D(L_{\lambda,p,\text{int}})$, $g = L_\lambda u = au'' + \frac{N-1}{r}(u' - u/r)$ belongs to L^p_{rad}. Setting $k = \frac{N-1}{a} > 0$ we obtain $u'' + \frac{k}{r}\left(\frac{u}{r}\right)' = f$ with $f = g/a$. let v=u/r and $w = v' = (u'/r - u/r^2)$. Then

$$w' + \left(\frac{2}{r} + \frac{k}{r^2}\right) w = \frac{f}{r}$$

and integrating between r and s we obtain

$$r^2 e^{-\frac{k}{r}} w(r) - s^2 e^{-\frac{k}{s}} w(s) = \int_s^r f(t) t e^{-\frac{k}{t}}\, dt.$$

The integral on the right hand side converges as $s \to 0$ and then $s^2 e^{-k/s} w(s) \to 0$ as $s \to 0$, as in the proof of the preceding Proposition. Therefore

$$w(r) = r^{-2} e^{\frac{k}{r}} \int_0^r f(t) t e^{-\frac{k}{t}}\, dt.$$

Choose $\delta \in (\frac{N-2}{p}, \frac{2}{p'})$; note that $\frac{N-2}{p} < \frac{2}{p'}$ is equivalent to $p > \frac{N}{2}$. Since $2 - p'\delta > 0$, we have

$$\begin{aligned}
|w(r)|^p &\leq r^{-2p} e^{\frac{kp}{r}} \int_0^r |f(t)| t e^{-\frac{k}{t}}\, dt \\
&\leq r^{-2p} e^{\frac{kp}{r}} \left(\int_0^r t^{1-p'\delta}\, dt\right)^{p-1} \int_0^r |f(t)|^p t^{1+p\delta} e^{-\frac{kp}{t}}\, dt \\
&\leq (2-p'\delta)^{1-p} r^{-2-p\delta} e^{\frac{kp}{r}} \int_0^r |f(t)|^p t^{1+p\delta} e^{-\frac{kp}{t}}\, dt.
\end{aligned}$$

Therefore noting that $p\delta + 2 - N > 0$, we see that

$$\int_0^\infty |w(r)|^p r^{N-1}\,dr \le (2-p'\delta)^{1-p}\int_0^\infty r^{N-3-p\delta} e^{\frac{kp}{r}}\left(\int_0^r |f(t)|^p t^{1+p\delta} e^{-\frac{kp}{t}}\,dt\right)dr$$
$$\le (2-p'\delta)^{1-p}\int_0^\infty \left(\int_t^\infty r^{N-3-p\delta} e^{\frac{kp}{r}}\,dr\right)|f(t)|^p t^{1+p\delta} e^{-\frac{kp}{t}}\,dt$$
$$\le (2-p'\delta)^{1-p}\int_0^\infty \left(\int_t^\infty r^{N-3-p\delta}\,dr\right)|f(t)|^p t^{1+p\delta}\,dt$$
$$\le (2-p'\delta)^{1-p}(p\delta+2-N)^{-1}\int_0^\infty |f(t)|^p t^{N-1}\,dt$$

and hence $w = (u'/r - u/r^2)$ in $L^p_{\rm rad}$. By difference, we obtain that $u'' \in L^p_{\rm rad}$.

To show equality (3.17), let us call W the space on the right hand side. We have just shown that $D(L_{\lambda,p,\rm int}) \subset W$. Since $W \subset D(L_{\lambda,p,\max})$ we have only to show that $\omega^2 - L_\lambda$ is injective on W. However this follows from (3.9) and Lemma 3.10 since $v_{\omega,1}$ is unbounded at infinity and $v_{\omega,2} \approx r^{-s_2^{(\lambda)}}$ as $r \to 0$, hence does not belong to $L^p_{\rm rad}$, since $N/p \le 1 \le s_2^{(\lambda)}$. □

The consistency of the semigroups is proved in the next corollary

Corollary 3.19. *If p, q satisfy the hypotheses of Proposition* 3.15 *or* 3.16*, then the generated semigroups coincide in $L^p_{\rm rad} \cap L^q_{\rm rad}$.*

Proof. This is immediate from the proofs of the above Propositions, since the resolvents coincide, see (3.14). □

3.6. Generation in $L^p_{<n}$

From the results so far obtained we can easily prove generation in $L^p_{<n}$. We recall that $s_i^{(\lambda)}$ coincide with s_i, $i = 1, 2$, defined in (1.1), when $\lambda = 0$ and that we write s_0 for s_1, s_2 when $D = 0$.

Proposition 3.20. *Assume that $D > 0$. If $\frac{N}{p} \in (s_1, s_2+2)$ that is $s_1 < \frac{N}{p} - 2\theta < s_2$ for some $\theta \in (0, 1]$, then $L_{p,\rm int,<n}$ defined through the domain*

$$D(L_{p,\rm int,<n}) = \left\{u \in L^p_{<n} \cap D(L_{p,\max})\ ;\ |x|^{-2\theta} u \in L^p\right\}$$

generates bounded analytic semigroup of angle $\frac{\pi}{2}$ in $L^p_{<n}$. The domain $D(L_{p,\rm int,<n})$ coincides with

$$D(L_{p,\rm reg,<n})$$
$$=\left\{u \in L^p_{<n} \cap W^{2,p}_{\rm loc}\left(\mathbb{R}^N \setminus \{0\}\right); (1\wedge|x|)^{2-2\theta} D^2 u, (1\wedge|x|)^{1-2\theta}\nabla u, |x|^{-2\theta} u \in L^p\left(\mathbb{R}^N\right)\right\}$$

for all/one θ as above. In particular, if $s_1 + 2 < \frac{N}{p} < s_2 + 2$, then one can choose $\theta = 1$ and therefore

$$D\big(L_{p,\mathrm{reg},<n}\big) = \Big\{u \in L^p_{<n} \cap W^{2,p}\big(\mathbb{R}^N\big)\,;\, |x|^{-1}\nabla u,\, |x|^{-2}u \in L^p\big(\mathbb{R}^N\big)\Big\}.$$

Proposition 3.21. *Assume that $D = 0$. If $\frac{N}{p} \in (s_0, s_0 + 2)$, then $L_{p,\mathrm{int},<n}$ defined through the domain*

$$D(L_{p,\mathrm{int},<n}) = \Big\{u \in L^p_{<n} \cap D\big(L_{p,\max}\big)\,;\, |x|^{-2\theta_0}|\log|x||^{-\frac{2}{p}}u \in L^p(B_{\frac{1}{2}})\Big\}$$

with $\theta_0 = \frac{1}{2}(s_0 - \frac{N}{p}) \in (0, 1)$ generates bounded analytic semigroup of angle $\frac{\pi}{2}$ in $L^p_{<n}$. The domain $D(L_{p,\mathrm{int},<n})$ coincides with

$$D(L_{p,\mathrm{reg},<n}) = \left\{ u \in L^p_{<n} \cap W^{2,p}_{\mathrm{loc}}(\mathbb{R}^N\setminus\{0\})\,;\, \begin{array}{l} u \in W^{2,p}(\mathbb{R}^N \setminus B_{\frac{1}{2}}),\\ |x|^{2-2\theta_0}|\log|x||^{-\frac{2}{p}}D^2u \in L^p(B_{\frac{1}{2}}),\\ |x|^{1-2\theta_0}|\log|x||^{-\frac{2}{p}}\nabla u \in L^p(B_{\frac{1}{2}}),\\ |x|^{-2\theta_0}|\log|x||^{-\frac{2}{p}}u \in L^p(B_{\frac{1}{2}}) \end{array} \right\}.$$

Proof. (Propositions 3.20 and 3.21). Observe that $L^p_{<n}$ coincides with L^p_J with a suitable finite J in Lemma 2.4 and that (3.4) holds. Therefore L endowed with domain

$$\bigoplus_{j\in J}\Big(D(L_{\lambda_{n_j},p,\mathrm{int}}) \otimes P_j\Big)$$

generates an analytic semigroup of angle $\frac{\pi}{2}$ in $L^p_{<n}$. Note that we should write $(D(L_{\lambda_{n_j},p,\mathrm{int}}) \otimes \mathrm{span}\,\{P_j\})$ since the tensor product is defined for vector spaces. However we keep the above notation since $\mathrm{span}\{P_j\}$ is one-dimensional. Let us first prove that

$$D(L_{p,\mathrm{int},<n}) = \bigoplus_{j\in J}\Big(D(L_{\lambda_{n_j},p,\mathrm{int}}) \otimes P_j\Big).$$

Consider the case $D > 0$. Let $u \in D(L_{\lambda_{n_j},p,\mathrm{int}})$. Then, since

$$s_1^{(\lambda_{n_j})} \le s_1 < \frac{N}{p} - 2\theta < s_2 \le s_2^{(\lambda_{n_j})},$$

by Proposition 3.15 we have $||x|^{-2\theta}u(r) \otimes P_j(\omega)|^p \le ||x|^{-2\theta}u(r)|^p\|P_j\|^p_\infty \in L^1(\mathbb{R}^N)$.

On the other hand, given $u \in D(L_{p,\mathrm{int},<n})$ we consider the bounded projection T_j defined in Lemma 2.4. Then $r^{-2\theta}T_ju(r) = T_j(|x|^{-2\theta}u)(r) \in L^p_{\mathrm{rad}}$ and $T_jLu = L_{\lambda_{n_j}}T_ju \in L^p_{\mathrm{rad}}$. Therefore $T_ju \in D(L_{\lambda_{n_j},p,\mathrm{int}})$ and

$$u = \sum_{j\in J}(T_ju)P_j \in \bigoplus_{j\in J}\Big(D(L_{\lambda_{n_j},p,\mathrm{int}}) \otimes P_j\Big).$$

To complete the proof we prove that $D(L_{p,\mathrm{reg},<n}) = D(L_{p,\mathrm{int},<n})$, the inclusion $D(L_{p,\mathrm{reg},<n}) \subseteq D(L_{p,\mathrm{int},<n})$ being obvious. To prove the other inclusion, we show that $\bigoplus_{j\in J}\left(D(L_{\lambda_{n_j},p,\mathrm{int}}) \otimes P_j\right) \subseteq D(L_{p,\mathrm{reg},<n})$. Let $u \in D(L_{\lambda_{n_j},p,\mathrm{int}}) \otimes P_j$ for some $j \in J$. Then we may write $u(r\omega) = v(r)P_j(\omega)$ for a suitable radial function $v \in L^p_{\mathrm{rad}}$. Then $|x|^{-2\theta}v(r)P_j(\omega) \in L^p$. Moreover, if $s = \deg P_j$ and $Q_j(x) = r^s P_j(\omega)$ then

$$D_h\left(v(r)P_j(\omega)\right) = D_h\left(\frac{v(r)}{r^s}Q_j(x)\right) = \left(v'(r) - s\frac{v(r)}{r}\right)\frac{x_h}{r}P_j(\omega) + \frac{v(r)}{r}\frac{D_h Q_j(x)}{r^{s-1}}.$$

Since $D_h Q_j$ is homogeneous of degree $(s-1)$, we obtain from above $|\nabla u| \le c\left(|v'(r)| + \frac{|v(r)|}{r}\right)$ with c depending on P_j and then

$$(1 \wedge |x|)^{1-2\theta}|\nabla u| \le c(1 \wedge r)^{1-2\theta}\left(|v'(r)| + \frac{|v(r)|}{r}\right) \in L^p.$$

The estimate for the second order derivatives is similar, see (2.8). □

From Corollary 3.19 we obtain

Corollary 3.22. *If p, q satisfy the hypotheses of Proposition* 3.20 *or* 3.21*, then the generated semigroups coincide in $L^p_{<n} \cap L^q_{<n}$.*

3.7. Generation in L^p and domain characterization

In this section we prove the main results of this paper, summarizing the results of the preceding two sections.

First we observe that the condition $D \ge 0$ is necessary also in the N-dimensional case for the existence of positive solutions for arbitrary data. In fact, if $D < 0$ and $f \ge 0$ is radial, then positive solutions u of $\omega^2 u - Lu = f$ would give positive radial solutions $v(r) = \int_{S^{N-1}} u(r\omega)\, d\omega$ of the same equation, by Proposition 2.15. However, Proposition 3.11 shows that one can find a suitable f for which such a solution does not exist.

Theorem 3.23. *Assume that $D > 0$. If $\frac{N}{p} \in (s_1, s_2 + 2)$ that is $s_1 < \frac{N}{p} - 2\theta < s_2$ for some $\theta \in (0, 1]$, then L endowed with domain*

$$D(L_{p,\mathrm{int}}) = \left\{u \in D(L_{p,\max})\ ;\ |x|^{-2\theta}u \in L^p\right\}$$

generates a bounded positive analytic semigroup on L^p. Moreover,

$$\begin{aligned} D(L_{p,\mathrm{int}}) &= D\big(L_{p,\mathrm{reg}}\big) \\ &:= \left\{u \in D(L_{p,\max})\,;\,(1 \wedge |x|)^{2-2\theta}D^2u,\,(1 \wedge |x|)^{1-2\theta}\nabla u,\,|x|^{-2\theta}u \in L^p\right\} \end{aligned}$$

for all/one θ as above. In particular, if $s_1+2<\frac{N}{p}<s_2+2$, then one can choose $\theta=1$ and therefore

$$D(L_{p,\mathrm{int}})=\left\{u\in W^{2,p}\big(\mathbb{R}^N\big)\,;\,|x|^{-1}\nabla u,\,|x|^{-2}u\in L^p\right\}.$$

When $\frac{N}{p}\notin(s_1,s_2+2)$, then $\sigma(L)=\mathbb{C}$ for every $L_{p,\min}\subset L\subset L_{p,\max}$.

Theorem 3.24. *Assume that $D=0$. If $\frac{N}{p}\in(s_0,s_0+2)$, then L endowed with domain*

$$D(L_{p,\mathrm{int}})=\left\{u\in D(L_{p,\max})\,;\;|x|^{-2\theta_0}|\log|x||^{-\frac{2}{p}}u\in L^p(B_{\frac12})\right\}$$

with $\theta_0=\frac12(s_0-\frac{N}{p})\in(0,1)$ generates a bounded positive analytic semigroup on L^p. Moreover,

$$D(L_{p,\mathrm{int}})=D(L_{p,\mathrm{reg}}):=\left\{u\in D(L_{p,\max})\,;\begin{array}{c}u\in W^{2,p}(\mathbb{R}^N\setminus B_{\frac12}),\\|x|^{2-2\theta_0}|\log|x||^{-\frac{2}{p}}D^2u\in L^p(B_{\frac12}),\\|x|^{1-2\theta_0}|\log|x||^{-\frac{2}{p}}\nabla u\in L^p(B_{\frac12}),\\|x|^{-2\theta_0}|\log|x||^{-\frac{2}{p}}u\in L^p(B_{\frac12})\end{array}\right\}.$$

When $\frac{N}{p}\notin(s_0,s_0+2)$, then $\sigma(L)=\mathbb{C}$ for every $L_{p,\min}\subset L\subset L_{p,\max}$.

Proof. (Theorems 3.23 and 3.24). We fix n sufficiently large so that Propositions 3.5 and 3.6 apply and write $L^p=L^p_{<n}\oplus L^p_{\geq n}$. Using also Propositions 3.15 and 3.16 we see that L with domain

$$D\big(L_{p,\mathrm{reg},<n}\big)\oplus D\left(L^n_{p,\mathrm{reg}}\right)=D\big(L_{p,\mathrm{int},<n}\big)\oplus D\left(L^n_{p,\mathrm{reg}}\right)$$

generates an analytic semigroup in L^p and, moreover, $D(L_{p,\mathrm{reg},<n})\oplus D(L^n_{p,\mathrm{reg}})\subset D(L_{p,\mathrm{reg}})\subset D(L_{p,\mathrm{int}})$.

Conversely, let $u\in D(L_{p,\mathrm{int}})$ and consider the projection $(I\otimes S)u=\sum_{j\in J}T_ju(r)P_j(\omega)\in L^p_{<n}$ of Lemma 2.4. As in the proof of Propositions 3.23 and 3.24 we have $r^{-2\theta}T_ju(r)=T_j(|x|^{-2\theta}u)(r)\in L^p_{\mathrm{rad}}$ and $T_jLu=L_{\lambda_{n_j}}T_ju\in L^p_{\mathrm{rad}}$. Therefore $T_ju\in D(L_{\lambda_{n_j},p,\mathrm{int}})$ and

$$(I\otimes S)u=\sum_{j\in J}(T_ju)P_j\in\bigoplus_{j\in J}\Big(D(L_{\lambda_{n_j},p,\mathrm{int}})\otimes P_j\Big)=D\big(L_{p,\mathrm{reg},<n}\big).$$

Finally, $u-(I\otimes S)u\in D(L^n_{p,\max})=D(L^n_{p,\mathrm{reg}})$, by Proposition 3.6 and this concludes the proof of the generation part and domain characterization.

Let us show that if $\frac{N}{p} \notin (s_1, s_2+2)$, then $\sigma(L_p) = \mathbb{C}$ for every $L_{p,\min} \subset L_p \subset L_{p,\max}$. Assume first that $\frac{N}{p} \le s_1$ and let $\omega \in \mathbb{C}_+$. Let f be defined by (3.13), as in the proof of Proposition 3.12 where we set $\lambda = 0$. We still denote by f the radial function defined in $\mathbb{R}^N$ by $f(|x|)$. If $u \in D(L_{p,\max})$ solves $\omega^2 u - Lu = f$, then

$$v(r) = \frac{1}{|S^{N-1}|} \int_{S^{N-1}} u(r, \omega)\, d\omega \in D(L_{p,\max}), \quad r = |x|$$

is radial and solves $\omega^2 v - Lv = f$, by Proposition 2.15 with $P(\omega) = |S^{N-1}|^{-1}$. However, such a solution v does not exist, according to Proposition 3.12, and hence $R(\omega^2 - L) \subset R(\omega^2 - L_{p,\max}) \neq L^p(\mathbb{R}^N)$. This yields $\sigma(L) = \mathbb{C}$ if $\frac{N}{p} \le s_1$ and, by duality as in the proof of Proposition 3.12, also $\sigma(L) = \mathbb{C}$ if $\frac{N}{p} \ge s_2 + 2$.

The proof of the positivity of the semigroup or, equivalently, of the resolvent for large ω requires some preparation. Let $\overline{a} = (\overline{a}_{ij})$ where

$$\overline{a}_{ij}(x) = \delta_{ij} + (a-1)|x|^{-2} x_i x_j, \quad x \in \mathbb{R}^N \setminus \{0\}, \quad \gamma = \frac{N-1+c}{a} - N + 1.$$

Then

$$Lu = \sum_{i,j=1}^{N} \overline{a}_{ij} D_{ij} u + c\frac{x}{|x|^2} \cdot \nabla u - \frac{b}{|x|^2} u = |x|^{-\gamma} \operatorname{div}(|x|^\gamma \overline{a} \nabla u) - \frac{b}{|x|^2} u.$$

Note that $\gamma = 0$ if and only if $L = L^*$, see (2.2).

Since the matrix $\overline{a}$ is positive definite with eigenvalue a corresponding to the eigenvector $x/|x|$, we have $\langle \overline{a} w, w\rangle \ge \langle \overline{a} w_n, w_n\rangle = a|w_n|^2$, where w_n is the component of a vector w along $x/|x|$. By the weighted Hardy inequalities with best constants, see for example the Appendix in [14], we have for every $u \in W^{1,2}(\mathbb{R}^N)$ with compact support in $\mathbb{R}^N \setminus \{0\}$

$$\begin{aligned}
\int_{\mathbb{R}^N} \langle \overline{a}\nabla u, \nabla u\rangle |x|^\gamma\, dx &\ge \int_{\mathbb{R}^N} \left\langle \overline{a}\left(\frac{\partial u}{\partial r}\frac{x}{|x|}\right), \left(\frac{\partial u}{\partial r}\frac{x}{|x|}\right)\right\rangle |x|^\gamma\, dx \\
&= a \int_{\mathbb{R}^N} \left|\frac{\partial u}{\partial r}\right|^2 |x|^\gamma\, dx \\
&\ge a\left(\frac{N+\gamma-2}{2}\right)^2 \int_{\mathbb{R}^N} |u|^2 |x|^{\gamma-2}\, dx \\
&= a\left(\frac{N-1+c-a}{2a}\right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} |x|^{\gamma-2}\, dx.
\end{aligned} \tag{3.18}$$

Let $\omega > 0$, $0 \le f \in C_c^\infty(\Omega)$. Then $f_0(r) = \int_{S^{N-1}} f(r\omega)\, d\omega$ is nonnegative, smooth and with compact support. We write $f = f_0 + g$ with $g \in C_c^\infty(\Omega) \cap L^p_{\ge 1}$ and, using Lemma 2.18, we approximate g in $W^{k,p}(\mathbb{R}^N)$ with a sequence g_m of the form $g_m = \sum g_j^m(r)P_j(\omega)$ (the sum is finite), $g_j^m \in C_c^\infty((0,\infty))$ and supp $g_m \subset K$, where K is a fixed compact subset of Ω. We choose k such that $kp > N$, so that the convergence is also uniform.

Then f is approximated in $L^p(\mathbb{R}^N)$ and uniformly by the functions $f_m = \sum f_j^m(r)P_j(\omega)$ with $f_0^0 = f_0$ and $f_j^m = g_j^m$ for $m \geq 1$. The solution $u_m \in D(L_{p,\text{int}})$ of the equation $\omega^2 u_m - Lu_m = f_m$ is given, by construction, by $u_m = \sum u_j^m(r)P_j(\omega)$ where $u_j^m \in D(L_{\lambda_n,p,\text{int}})$ solves $\omega^2 u_j^m - L_{\lambda_n} u_j^m = f_j^m$ and n is the order of P_j. From (3.14) we easily obtain $u_0^m(r) \approx cr^{-s_1}$ with $c > 0$. For $j \geq 1$, the same argument yields $|u_j^m(r)| \leq c_j r^{-s_1^{(\lambda_n)}}$, n being the order of P_j and then since $s_1^{(\lambda_n)} < s_1$, then $u_m \geq 0$ in an a small ball B_ε (depending on m). Moreover, near infinity u_j^m is a multiple of the relative $v_{\omega,2}$ and tends to 0 exponentially together with its derivative, hence u_m and ∇u_m tend to 0, exponentially, as $|x| \to \infty$. Let us fix $R > \varepsilon$, multiply the equation $\omega^2 u_m - Lu_m = f_m$ by u_m^- and integrate on $\Omega_{\varepsilon,R} = B_R \setminus B_\varepsilon$ with respect to the measure $|x|^\gamma\,dx$. Since $u_m^- = 0$ for $|x| = \varepsilon$ we obtain

$$\int_{\Omega_{\varepsilon,R}} \left(\omega^2|u_m^-|^2 + \overline{a}(\nabla u_m^-, \nabla u_m^-) + \frac{b}{|x|^2}|u_m^-|^2\right)|x|^\gamma\,dx + \int_{\partial B_R} u_m^- \overline{a} \nabla u_m \cdot \nu |x|^\gamma\,d\sigma$$
$$= -\int_{\Omega_{\varepsilon,R}} f_m u_m^- |x|^\gamma\,dx \leq \int_{\Omega_{\varepsilon,R}} f_m^- u_m^- |x|^\gamma\,dx.$$

Letting $R \to \infty$ all integrals converge, due to the exponential decay of $u_m, \nabla u_m$, and the boundary integral tends to 0. We apply (3.18) to u_m^- (extended to 0 in B_ε), using the exponential decay at ∞, and deduce since $b + a\left(\frac{N-1+c-a}{2a}\right)^2 = aD \geq 0$, that

$$\omega^2 \int_{B_\varepsilon^c} |u_m^-|^2|x|^\gamma\,dx \leq \int_{B_\varepsilon^c} \left(\omega^2|u_m^-|^2 + \overline{a}(\nabla u_m^-, \nabla u_m^-) + \frac{b}{|x|^2}|u_m^-|^2\right)|x|^\gamma\,dx$$
$$\leq \int_{B_\varepsilon^c} f_m^- u_m^- |x|^\gamma\,dx$$

and finally, since $u_m^- = 0$ in B_ε, by Hölder inequality,

$$\omega^4 \int_{\mathbb{R}^N} |u_m^-|^2|x|^\gamma\,dx \leq \int_{R^N} |f_m^-|^2|x|^\gamma\,dx.$$

Since $f_m^- \to 0$ uniformly and the supports are contained in a common bounded set K, the integral on the right hand side tends to 0 as $n \to \infty$. Then

$$u = \left(\omega^2 - L_{p,\text{int}}\right)^{-1} f = \lim_{m\to\infty} \left(\omega^2 - L_{p,\text{int}}\right)^{-1} f_m = \lim_{m\to\infty} u_m$$

satisfies, by Fatou's lemma,

$$\omega^4 \int_{\mathbb{R}^N} |u^-|^2 |x|^\gamma \, dx \le \liminf_{n\to\infty} \int_{\mathbb{R}^N} |u_m^-|^2 |x|^\gamma \, dx \le 0.$$

This yields $u \ge 0$. □

From Corollaries 3.7 and 3.22 we obtain

Corollary 3.25. *If p, q satisfy the hypotheses of Theorems* 3.23 *or* 3.24*, then the generated semigroups coincide in $L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$.*

Remark 3.26. Theorems 3.23 and 3.24 can be stated in $L^p_{\ge n}$ with minor changes, for every $n \in \mathbb{N}_0$, thus extending Proposition 3.5. The conditions $D > 0, s_1 < \frac{N}{p} < s_2 + 2$ have to be replaced by $D_{\lambda_n} > 0$ and $s_1^{(\lambda_n)} < \frac{N}{p} < s_2^{(\lambda_n)} + 2$, and similarly for the others. The description of the domain holds with the same changes. The proof is the same as in the main theorems. Of course the positivity statement makes sense only if $n = 0$.

Example 3.27. Let a be fixed. Fixing $c < 0$ small enough and choosing $b < 0$ such that D is small, one obtains $s_2 + 2 < 0$. Similarly, fixing $c > 0$ large enough and choosing $b < 0$ such that D is small, one obtains $s_1 > N$. In both cases $(s_1, s_2 + 2) \cap (0, N) = \emptyset$ and therefore no operator $L_{p,\min} \subset L \subset L_{p,\max}$ generates a semigroup, for every $1 < p < \infty$. However, if $b \ge 0$ then $s_1 s_2 = -\frac{b}{a} \le 0$ yields $s_1 \le 0, s_2 \ge 0$ and $(s_1, s_2 + 2) \cap (0, N) \ne \emptyset$. Therefore, for some $1 < p < \infty$, the operator $L_{p,\text{int}}$ generates a bounded positive analytic semigroup on $L^p(\mathbb{R}^N)$.

Next we study when $L_{p,\text{int}}$ coincides with $L_{p,\min}$ or $L_{p,\max}$.

Proposition 3.28. *Assume that $D > 0$. Then $L_{p,\text{int}} = L_{p,\max}$ if and only if $\frac{N}{p} \in (s_1, s_2]$ and $L_{p,\text{int}} = L_{p,\min}$ if and only if $\frac{N}{p} \in [s_1+2, s_2+2)$. Therefore, if $s_1 + 2 \le s_2$ or equivalently $D \ge 1$, and if $\frac{N}{p} \in [s_1 + 2, s_2]$, then $L_{p,\text{int}} = L_{p,\min} = L_{p,\max}$.*

Proof. If $\frac{N}{p} \in (s_1, s_2]$, then $\frac{N}{p} \in (s_1^{(\lambda_n)}, s_2^{(\lambda_n)}]$ for every $n \in \mathbb{N}_0$. Let us fix $\omega \in \mathbb{C}_+$ and $u \in D_{p,\max}(L)$ such that $\omega^2 u - Lu = 0$. If P is a spherical harmonic of order n, the function $v(r) = \int_{S^{N-1}} u(r\omega) P(\omega) \, d\omega$ belongs to $D(L_{\lambda_n, p,\max})$ and satisfies $\omega^2 v - L_{\lambda_n} v = 0$. Therefore v is a linear combination of the functions $v_{\omega,1}, v_{\omega,2}$ defined in (3.9). Since $v_{\omega,1}$ is exponentially increasing at infinity and $v_{\omega,2}$ does not to belong to L^p_{rad}, by Lemma 3.10 (since $N/p \le s_2^{(\lambda_n)}$), then $v = 0$. Since this is true for every spherical harmonic P we get $u = 0$ and then $\omega^2 - L_{p,\max}$ is injective, hence $L_{p,\max}$ coincide with $L_{p,\text{int}}$. However if $N/p > s_2$, then the function $v_{\omega,2}$ relative to $\lambda_n = 0$ is a radial eigenfunction of $L_{p,\max}$ for every $\omega \in \mathbb{C}_+$. This proved the part relative to $L_{p,\max}$.

By duality, as in the proof of Proposition 3.12, we obtain $L_{p,\text{int}} = L_{p,\min}$ if and only if $\frac{N}{p} \in [s_1 + 2, s_2 + 2)$. □

The equalities above never occur when $D = 0$.

Proposition 3.29. *Assume that $D = 0$. If $\frac{N}{p} \in (s_0, s_0+2)$, then $L_{p,\min} \subsetneq L_{p,\text{int}} \subsetneq L_{p,\max}$.*

Proof. Since $v_{\omega,2}$ belongs to L^p_{rad}, $\omega^2 - L_{p,\max}$ is never injective and therefore $L_{p,\text{int}} \subsetneq L_{p,\max}$. By duality we get the other part. □

Corollary 3.30. *If p satisfies the hypotheses of Theorems 3.23 or 3.24, then $(L_{p,\text{int}})^* = L^*_{p',\text{int}}$.*

Proof. As in the proof of Theorems 3.23 and 3.24, we write, for large n, $L_{p,\text{int}} = L_{p,\text{int},<n} + L^n_{p,\text{reg}}$ in $L^p(\mathbb{R}^N) = L^p_{<n} \oplus L^p_{\geq n}$. In $L^p_{\geq n}$ we have $L^n_{p,\text{reg}} = L^n_{p,\max}$ hence $(L^n_{p,\text{reg}})^* = (L^n_{p,\max})^* = L^n_{p',\min} = L^n_{p',\text{reg}}$, see Proposition 3.5, and we have to consider only the operator $L_{p,\text{int},<n}$ in $L^p_{<n}$ or, equivalently, the 1-d operators $L_{\lambda_j,p,\text{int}}$ in L^p_{rad}. For $\omega > 0$ and $f \in L^p_{\text{rad}}$ we have

$$\left(\omega^2 - L_{\lambda_j,p,\text{int}}\right)^{-1} f = \int_0^\infty G(r,t) f(t) t^{N-1}\, dt,$$

where the Green function G is given by

$$G(r,t) = C \begin{cases} v_{\omega,t}(r) v_{\omega,2}(t)\, t^\gamma & 0 < r \leq t < \infty \\ v_{\omega,1}(t) v_{\omega,2}(r)\, t^\gamma & 0 < t \leq r < \infty \end{cases} \tag{3.19}$$

and $\gamma = \frac{N-1+c}{a} + 1 - N$, see (3.14). Then $(\omega^2 - (L_{\lambda_j,p,\text{int}})^*)^{-1}$ is given by the Green function $G^*(r,t) = G(t,r)$. To conclude the proof we have to verify that G^* coincides with the Green function of $(\omega^2 - L^*_{\lambda_j,p',\text{int}})^{-1}$. To see this, let us define $v^*_{\omega,1}$, $v^*_{\omega,2}$ as in (3.9) but referred to $L^*_{\lambda_j}$ and $\gamma^* = \frac{N-1+c^*}{a} + 1 - N$. Then $\gamma^* = -\gamma$ and the Green function of $(\omega^2 - (L_{\lambda_j,p,\text{int}})^*)^{-1}$ is given by (3.19) with $v_{\omega,1}$, $v_{\omega,2}$, γ replaced by $v^*_{\omega,1}$, $v^*_{\omega,2}$, $\gamma^* = -\gamma$. Since, using (2.4), $v^*_{\omega,i}(t) = t^\gamma v_{\omega,i}(t)$, $i = 1, 2$, the proof is complete. □

Next we characterize when $D(L_{p,\text{int}})$ is contained in $W^{2,p}(\mathbb{R}^N)$.

Theorem 3.31.

(i) *If $D > 0$ and $s_1 + 2 < \frac{N}{p} < s_2 + 2$ then $D(L_{p,\text{int}}) \subset W^{2,p}(\mathbb{R}^N)$; moreover*

$$D\big(L_{p,\text{int}}\big) = \left\{u \in W^{2,p}\big(\mathbb{R}^N\big)\ ;\ |x|^{-1}\nabla u, |x|^{-2}u \in L^p\big(\mathbb{R}^N\big)\right\};$$

(ii) *If $s_1 \neq 0$ and $s_1 < \frac{N}{p} \leq s_1 + 2$, then $D(L_{p,\text{int}})$ is not contained in $W^{2,p}(\mathbb{R}^N)$;*

(iii) *If $s_1 = 0$ and $s_1^{(\lambda_1)} \neq -1$, then $D(L_{p,\text{int}}) \subset W^{2,p}(\mathbb{R}^N)$ if and only if and $s_1^{(\lambda_1)} + 2 < \frac{N}{p} < s_2 + 2$. In this case*

$$D\big(L_{p,\text{int}}\big) = \left\{u \in W^{2,p}\big(\mathbb{R}^N\big) : |x|^{-1}\nabla u \in L^p\big(\mathbb{R}^N\big)\right\};$$

(iv) *If* $s_1 = 0$ *and* $s_1^{(\lambda_1)} = -1$, *then* $D(L_{p,\text{int}}) \subset W^{2,p}(\mathbb{R}^N)$ *if and only if* $s_1^{(\lambda_2)} + 2 < \frac{N}{p} < s_2 + 2$. *In this case*

$$D\big(L_{p,\text{int}}\big) = W^{2,p}\big(\mathbb{R}^N\big).$$

Proof. (i) This is a special case of Theorem 3.23.

(ii) We assume that $s_1 \neq 0$ and take $f \geq 0$ radial with compact support in the annulus $\alpha \leq |x| \leq \beta$, $\alpha > 0$. By construction the solution $u \in D(L_{p,\text{int}})$ of the equation $u - Lu = f$ is radial and is given by (3.14) with $\omega = 1$. Therefore $u(r) = cv_{1,1}(r)$ for $r < \alpha$, with $c > 0$. Since $r^{s_1} v_{1,1}(r) = h(r)$, with h analytic and $h(0) \neq 0$, see Lemma 3.10, it follows that $\frac{v_{1,1}'(r)}{r} \approx -s_1 h(r) r^{-s_1 - 1}$ near 0 (since $s_1 \neq 0$) and therefore $u'/r \notin L^p_{\text{rad}}$ and $u \notin W^{2,p}(\mathbb{R}^N)$, by Proposition 2.14 (i).

(iii) Assume that $s_1 = 0$, that is $b = 0$ and $N - 1 + c - a \geq 0$. We fix n sufficiently large so that Proposition 3.6 applies and split $f = \sum_{k<n}^{n} f_k + g$ where $f_k \in L^p_k$, $0 \leq k < n$ and $g \in L^p_{\geq n}$, through the projections of Lemma 2.4. Then the solution $u \in D(L_{p,\text{int}})$ of the equation $u - Lu = f$ is given by $u = \sum_{k<n} u_k + v$ with $u_k \in D(L_{p,\lambda_k,\text{int}}) \subset L^p_k$, $0 \leq k < n$ and $v \in D(L^n_{p,\text{reg}})$. By Proposition 3.6, $v \in W^{2,p}(\mathbb{R}^N)$ and also $|x|^{-1}\nabla v$, $|x|^{-2} v \in L^p(\mathbb{R}^N)$. Concerning the functions u_k, we have $f_k(x) = \sum_j f_j^{(k)} P_j^{(k)}$, $u_k(x) = \sum_j u_j^{(k)} P_j^{(k)}$ where the $P_j^{(k)}$ constitute a basis for the spherical harmonics of order k and, by Proposition 2.15,

$$u_j^{(k)} - L_{\lambda_k} u_j^{(k)} = f_j^{(k)}.$$

Since $s_1 = 0$, Proposition 3.17 yields u_0'', $u_0'/r \in L^p_{\text{rad}}$ and, by Proposition 2.14 (i) , $u_0 \in W^{2,p}(\mathbb{R}^N)$. Moreover, $|x|^{-1}|\nabla u_0| = |u_0'|/r \in L^p_{\text{rad}}$ yields $|x|^{-1}\nabla u_0 \in L^p(\mathbb{R}^N)$.

If $s_1^{(\lambda_1)} + 2 < \frac{N}{p}$, then $u_k \in W^{2,p}(\mathbb{R}^N)$ for $1 \leq k < n$, by Propositions 3.15, 2.14 (ii), (iii) and recalling that the numbers $s_1^{(\lambda_k)}$ are decreasing. Moreover $|x|^{-1}\nabla u_k \in L^p(\mathbb{R}^N)$, by Proposition 2.13 (ii), since both u_k'/r and u_k/r^2 are in L^p_{rad}.

This shows that $D(L_{p,\text{int}}) \subset \{u \in W^{2,p}(\mathbb{R}^N) : |x|^{-1}\nabla u \in L^p(\mathbb{R}^N)\} := W$. Since clearly $W \subset D(L_{p,\max})$ to show equality we have only to prove that $1 - L$ is injective on W. Let $u \in W$ be such that $u - Lu = 0$ and P be a spherical harmonic of order k. If $v(r) = \int_{S^{N-1}} u(r\omega) P(\omega)\, d\omega$, then $v - L_{\lambda_k} v = 0$, by Proposition 2.15. Moreover, since $u \in W$, then $v, v', v'', v'/r \in L^p_{\text{rad}}$ and the equation $v - L_{\lambda_k} v = 0$ yields also $v/r^2 \in L^p_{\text{rad}}$ when $k \geq 1$ (since $\lambda_k > 0$) and also when $k = 0$ and $p < N/2$ by the classical Hardy inequality. It follows that $v \in D(L_{\lambda_k,p,\text{int}})$, for $k = 0$ by Proposition 3.17 and for $k \geq 1$ by Proposition 3.15 with $\theta = 1$. Then $v = 0$ and, since this is true for every spherical harmonic P, then $u = 0$.

However, if $N/p \leq s_1^{(\lambda_1)} + 2$, the same argument in the proof of (ii) shows that one can choose $f^{(1)}$ (having only one non-zero component, say the first) such that $u_1^{(1)} \approx r^{-s_1^{(\lambda_1)}}$ as $r \to 0$ and then $u_1'^{(1)}/r - u_1^{(1)}/r^2 \approx -(s_1^{(\lambda_1)} + 1) r^{-s_1^{(\lambda_1)} - 2}$.

It follows that $u_1'^{(1)}/r - u_1^{(1)}/r^2 \notin L^p_{\rm rad}$, hence $u_1 = u_1^{(1)} P_1^{(1)} \notin W^{2,p}(\mathbb{R}^N)$, by Proposition 2.14 (ii).

(iv) Let us assume now that $s_1 = 0$ and $s_1^{(\lambda_1)} = -1$ or, equivalently, that $b = c = 0$ and $0 < a \le N-1$. Then $u_0 \in W^{2,p}(\mathbb{R}^N)$, as in the proof of (iii) but also $u_1 \in W^{2,p}(\mathbb{R}^N)$, by Propositions 3.18, 2.14 (ii) (but this time $|x|^{-1}\nabla u_1$ could be not in $L^p(\mathbb{R}^N)$ for $p \ge N$). If $s_1^{(\lambda_2)} + 2 < \frac{N}{p}$, proceeding as in (ii) for u_k, $2 \le k < n$ and for v we obtain that $D(L_{p,\rm int}) \subset W^{2,p}(\mathbb{R}^N) \subset D(L_{p,\max})$. let us prove that $D(L_{p,\rm int}) = W^{2,p}(\mathbb{R}^N)$ by showing that $1 - L$ is injective on $W^{2,p}(\mathbb{R}^N)$, as in (iii). Let $u \in W^{2,p}(\mathbb{R}^N)$ be such that $u - Lu = 0$ and P be a spherical harmonic of order k. If $v(r) = \int_{S^{N-1}} u(r\omega) P(\omega)\, d\omega$, then $v - L_{\lambda_k} v = 0$, by Proposition 2.15. Since $u \in W^{2,p}(\mathbb{R}^N)$, Proposition 2.15 yields that $v(r)P(\omega) \in W^{2,p}(\mathbb{R}^N)$, too. If $k = 0$, then Proposition 2.14 (i) gives v'', $v'/r \in L^p_{\rm rad}$ (and also $v/r^2 \in L^p_{\rm rad}$ when $p < N/2$, by Hardy inequality) and hence $v \in D(L_{\lambda_0,p,\rm int})$ by Proposition 3.17, hence $v = 0$.

If $k = 1$, then Proposition 2.14 (ii) gives v'', $v'/r - v/r^2 \in L^p_{\rm rad}$ (and also $v'/r \in L^p_{\rm rad}$ when $p < N$, by Hardy inequality and $v/r^2 \in L^p_{\rm rad}$ by difference) and hence $v \in D(L_{\lambda_1,p,\rm int})$ by Proposition 3.18, hence $v = 0$.

If $k \ge 2$, then Proposition 2.14 (iii) gives v'', v'/r, $v/r^2 \in L^p_{\rm rad}$ and hence $v \in D(L_{\lambda_k,p,\rm int})$ by Proposition 3.15 with $\theta = 1$, hence $v = 0$.

Then $v = 0$ in all cases and hence $u = 0$.

However if $N/p \le s_1^{(\lambda_2)} + 2$, then we construct a solution $u_2^{(1)} \approx r^{-s_2^{(\lambda_2)}}$ as $r \to 0$ which does not satisfy $u_2^{(1)}/r^2 \in L^p_{\rm rad}$ and hence $u_2^{(1)} P_2^{(1)} \notin W^{2,p}(\mathbb{R}^N)$, by Proposition 2.14 (iii). □

Observe that, by the classical Hardy inequalities, the conditions $|x|^{-1}\nabla u$, $|x|^{-2}u \in L^p(\mathbb{R}^N)$ are automatically satisfied in $W^{2,p}(\mathbb{R}^N)$ when $1 < p < N/2$ and the first requires only $p < N$.

4. The operator $L = \Delta + (a-1)\sum_{i,j=1}^N \frac{x_i x_j}{|x|^2} D_{ij}$

In this section we specialize our results assuming $b = c = 0$. The pure second order operator has been extensively used as a source of counterexamples but also studied in detail by Manselli [6], concerning existence and uniqueness of the Dirichlet problem $Lu = f$ with u vanishing at the boundary of the unit ball, under the assumption $a \ge 1$. We are not aware of previous results in the parabolic setting or in the case $0 < a < 1$.

The following quantities will be important in the discussion

$$D = \left(\frac{N-1-a}{2a}\right)^2 \ge 0, \quad s_1 = \frac{N-1-a}{2a} - \left|\frac{N-1-a}{2a}\right|,$$
$$s_2 = \frac{N-1-a}{2a} + \left|\frac{N-1-a}{2a}\right|.$$

Moreover $s_1^{(\lambda_1)} = -1$ and

$$s_1^{(\lambda_2)} = \frac{N-1-a}{2a} - \sqrt{\frac{2N}{a} + \left(\frac{N-1-a}{2a}\right)^2} < -1.$$

Observe that $s_1^{(\lambda_2)}$ ranges from $-\infty$ to -1 as a ranges from 0 to ∞. Moreover, $s_1^{(\lambda_2)} \le -2$ if and only if $0 < a \le 1$.

4.1. The case $0 < a \le N - 1$

Here $s_1 = 0, \quad s_2 = \frac{N-1-a}{a}$. Note that s_2 ranges from ∞ to 0 as a ranges from 0 to $N-1$ and that $s_2 = N-2$ when $a = 1$. The critical case $D = 0$ corresponds to $a = N-1$, hence to the Laplacian in dimension 2.

Corollary 4.1. *Let* $0 < a \le N-1$. *Then* $(L, D(L_{p,\mathrm{int}}))$ *generates a bounded positive analytic semigroup in* $L^p(\mathbb{R}^N)$ *if and only if* $0 < \frac{N}{p} < s_2 + 2 = \frac{N-1+a}{a}$. *Moreover,* $D(L_{p,\mathrm{int}}) = W^{2,p}(\mathbb{R}^N)$ *if and only if* $s_1^{(\lambda_2)} + 2 < \frac{N}{p} < s_2 + 2$.

Proof. The generation part follows from Theorems 3.23 and 3.24. and the domain characterization from Theorem 3.31 (iv). □

Remark 4.2. Since $s_1^{(\lambda_2)} + 2 \le 0$ if and only if $0 < a \le 1$, the condition $s_1^{(\lambda_2)} + 2 < \frac{N}{p}$ is always satisfied when $0 < a \le 1$ and $D(L_{p,\mathrm{int}}) = W^{2,p}(\mathbb{R}^N)$ when L is a generator.

We point out that the same results, but only for elliptic solvability in $W^{2,p}(B)$, B being the unit ball, with Dirichlet boundary conditions, have been obtained by Manselli [6], under the restriction $a \ge 1$ and for $s_1^{(\lambda_2)} + 2 < \frac{N}{p} < s_2 + 2$.

4.2. The case $a > N - 1$

Here the situation is different and $s_1 = \frac{N-1-a}{a} \in (-1, 0)$, $s_2 = 0$. The critical case $s_1 = s_2$, or $D = 0$, does not occur.

Corollary 4.3. *Let* $a > N-1$. *Then* $(L, D(L_{p,\mathrm{int}}))$ *generates a bounded positive analytic semigroup in* $L^p(\mathbb{R}^N)$ *if and only if* $\frac{N}{p} < 2$. *When* $s_1 + 2 = \frac{N-1+a}{a} < \frac{N}{p} < 2$, *then*

$$D(L_{p,\mathrm{int}}) = \left\{u \in W^{2,p}(\mathbb{R}^N)\ ;\ |x|^{-1}\nabla u, |x|^{-2}u \in L^p(\mathbb{R}^N)\right\} \subsetneq W^{2,p}(\mathbb{R}^N).$$

On the other hand, if $\frac{N}{p} \le s_1 + 2$, *then* $D(L_{p,\mathrm{int}}) \not\subset W^{2,p}(\mathbb{R}^N)$ *and* $W^{2,p}(\mathbb{R}^N) \not\subset D(L_{p,\mathrm{int}})$.

Proof. The generation part and the domain characterization when $s_1 + 2 < \frac{N}{p} < 2$ follow from Theorem 3.23. Note that $D(L_{p,\mathrm{int}})$ is included in but different from $W^{2,p}(\mathbb{R}^N)$ since $p > N/2$.

Let us now assume that $\frac{N}{p} \le s_1 + 2$. Since $s_1 \neq 0$, then $D(L_{p,\mathrm{int}}) \not\subset W^{2,p}(\mathbb{R}^N)$, by Theorem 3.31 (ii).

On the other hand, let $u(x) = e^{-|x|^2} \in W^{2,p}(\mathbb{R}^N)$. Choosing $\theta \in \left(\frac{N}{2p}, 1\right)$ such that $2\theta < \frac{N}{p} - s_1$, we see that $s_1 < \frac{N}{p} - 2\theta < s_2$ and $|x|^{-2\theta} u \notin L^p(\mathbb{R}^N)$. From Theorem 3.23 we obtain $u \notin D(L_{p,\mathrm{int}})$. □

The results collected in the above corollary are different from those obtained by Manselli in [6], who obtains solvability in $W^{2,p}$ for certain parameters. We explain why this happens and then how Manselli's results can be obtained with our methods.

First of all, let us observe that the interval (s_1, s_2) does not intersect $(0, N)$ (since $s_2 \le 0$) and hence $L_{p,\max}$ is not a generator for every $1 < p < \infty$, by Proposition 3.28. On the other hand, if $s_1 + 2 < N/p < 2$, Corollary 4.3 yields a domain strictly contained in $W^{2,p}(\mathbb{R}^N)$ and $L_{p,\mathrm{int}}$ is equal to $L_{p,\min}$ (even when $N/p = s_1 + 2$), by Proposition 3.28, again. In this last range Manselli does not obtain unique solvability in $W^{2,p}$ and this is clear from Corollary 4.3, since we prove existence and uniqueness in a smaller domain.

Since $0 = s_2 < s_1 + 2$, there is no uniqueness if $\frac{N}{p} \in (0, s_1 + 2)$ and it turns out that, in a smaller subinterval, Manselli's domain is different from $D(L_{p,\mathrm{int}})$ and more regular. In what follows we explain how to obtain his results and consider the operator $L_{|W^{2,p}} = (L, W^{2,p}(\mathbb{R}^N))$.

Lemma 4.4. *Let $a > N - 1$. If $\frac{N}{p} \le s_1^{(\lambda_2)} + 2$, then $(0, \infty) \subset \sigma(L_{|W^{2,p}})$.*

Proof. Let $\omega > 0$ and consider $u(x) = \eta(|x|) v_{\omega,1}(|x|) \omega_1 \omega_2$, where $v_{\omega,1}$ is defined in Lemma 3.10 with $\lambda = \lambda_2$ and satisfies $\omega^2 v_{\omega,1} - L_{\lambda_2} v_{\omega,1} = 0$ and $\eta \in C^\infty((0, \infty))$ satisfies $\eta \equiv 1$ in $(0, 1]$ and $\eta \equiv 0$ in $[2, \infty)$.

The function $f := \omega^2 u - Lu$ is identically zero near $0, \infty$ and belongs to L^p_J, where J is the singleton which identifies the spherical harmonic $\omega_1 \omega_2$. Let us assume, by contradiction, that there exists $w \in W^{2,p}(\mathbb{R}^N)$ such that $\omega^2 w - Lw = f$ and consider its projection $v(|x|)\omega_1\omega_2$ in L^p_J. By Propositions 2.15, 2.14 (iii), $v/r^2 \in L^p_{\mathrm{rad}}$. However $v - \eta v_{\omega,1} \in L^p_{\mathrm{rad}}$ is in the kernel of $\omega^2 - L_{\lambda_2}$, hence

$$v(r) - \eta(r) v_{\omega,1}(r) = c_1 v_{\omega,2}(r)$$

for a suitable $c_1 \in \mathbb{C}$. Using Lemma 3.10 with $\lambda = \lambda_2$, one easily sees that $v/r^2 \notin L^p_{\mathrm{rad}}$ since $s_1^{(\lambda_2)} < s_2^{(\lambda_2)}$ and $\frac{N}{p} \le s_1^{(\lambda_2)} + 2$. □

Lemma 4.5. *Let $a > N - 1$. If $s_1^{(\lambda_2)} + 2 < N/p < s_1 + 2$, then $\rho(L_{p,\mathrm{int}}) \subset \rho(L_{|W^{2,p}})$.*

Proof. Let $\omega^2 \in \rho(L_{p,\mathrm{int}})$. We show that $\omega^2 \in \rho(L_{|W^{2,p}})$. Let $f \in L^p(\mathbb{R}^N)$. To solve $\omega^2 u - Lu = f$ in $W^{2,p}(\mathbb{R}^N)$, we use the decomposition $L^p(\mathbb{R}^N) = L^p_{<n} \oplus L^p_{\geq n} = \big(\bigoplus_{j\in J} L^p_{\{j\}}\big) \oplus L^p_{\geq n}$, where n is given in Proposition 3.6 and J individuates all spherical harmonics of order less than n. Then f can be decomposed as

$$f(x) = \sum_{j\in J} f_j(r) P_j(\omega) + \tilde{f}(x), \quad f_j \in L^p_{\mathrm{rad}},\ j \in J, \quad \tilde{f} \in L^p_{\geq n}.$$

By Proposition 3.6, we have $\tilde{u} := (\omega^2 - L_{p,\mathrm{int}})^{-1}\tilde{f} \in D(L_{p,\mathrm{int}}) \cap L^p_{\geq n} \subset W^{2,p}(\mathbb{R}^N)$.

If $n_j = \deg P_j \geq 2$, we observe that

$$s_1^{(\lambda_{n_j})} \leq s_1^{(\lambda_2)} < \frac{N}{p} - 2 < s_1 < s_2^{(\lambda_{n_j})}.$$

Thus setting $v_j = (\omega^2 - L_{\lambda_j,p,\mathrm{int}})^{-1} f_j$ and applying Proposition 3.15 with $\theta = 1$ we have $v_j, v_j', v_j'', v_j'/r, v_j/r^2 \in L^p_{\mathrm{rad}}$. Therefore from Proposition 2.14 (iii) we deduce that $u_j(x) := v_j(r)P_j(\omega) \in W^{2,p}(\mathbb{R}^N)$.

If $\deg P_j = 1$, then $v_j = (\omega^2 - L_{\lambda_j,p,\mathrm{int}})^{-1} f_j \in D(L_{\lambda_1,p,\mathrm{int}})$. Applying Proposition 3.18 and then Proposition 2.14 (ii) we deduce that $u_j(x) := v_j(r)P_j(\omega) \in W^{2,p}(\mathbb{R}^N)$.

If $\deg P_j = 0$ we look for solutions of the equation $\omega^2 v - av'' + \frac{N-1}{r}v' = f_0$ having the form

$$\begin{aligned} v = c_0 r^{\nu_0} K_{\nu_0}\left(\frac{\omega r}{\sqrt{a}}\right) + (\omega^2 - L_{p,\lambda_0,\mathrm{int}})^{-1} f_0 = \ & c_0 r^{\nu_0} K_{\nu_0}\left(\frac{\omega r}{\sqrt{a}}\right) \\ + C_\omega r^{\nu_0}\Bigg(K_{\nu_0}\left(\frac{\omega r}{\sqrt{a}}\right)\int_0^r t^{1-\nu_0} I_{\nu_0}\left(\frac{\omega t}{\sqrt{a}}\right) f_0(t)\,dt \\ + I_{\nu_0}\left(\frac{\omega r}{\sqrt{a}}\right)\int_r^\infty t^{1-\nu_0} K_{\nu_0}\left(\frac{\omega t}{\sqrt{a}}\right) f_0(t)\,dt\Bigg), \end{aligned}$$

where $\nu_0 = |\frac{N-1-a}{2a}| = -\frac{s_1}{2} \in (0, \frac{1}{2})$, $C_\omega = W^{-1}(\omega)$, see (3.14). Observe that if $g \in C_c^\infty((0,\infty))$ with $g \geq 0$ and $g \not\equiv 0$, then for sufficiently small $r > 0$, we have

$$v(r) = c_0 r^{\nu_0} K_{\nu_0}\left(\frac{\omega r}{\sqrt{a}}\right) + \left(C_\omega \int_0^\infty t^{1-\nu_0} K_{\nu_0}\left(\frac{\omega t}{\sqrt{a}}\right) g(t)\,dt\right) r^{\nu_0} I_{\nu_0}\left(\frac{\omega r}{\sqrt{a}}\right).$$

Therefore by Lemmas 3.9 and 3.8 we see that

$$\frac{v'(r)}{r} \approx \begin{cases} 1 & \text{if } c_0 = c_g := C_\omega \cdot \dfrac{2\sin(\nu_0\pi)}{\pi}\displaystyle\int_0^\infty t^{1-\nu_0} K_{\nu_0}\left(\frac{\omega t}{\sqrt{a}}\right) g(t)\,dt \\ r^{-2(1-\nu_0)} & \text{if } c_0 \neq c_g \end{cases}$$

as $r \to 0$. This means that if $c_0 \neq c_g$, then by $N/p < s_1 + 2$, $\frac{v'}{r} \notin L^p_{\mathrm{rad}}$ and therefore by Proposition 2.14, $v(|x|) \notin W^{2,p}(\mathbb{R}^N)$. Hence we choose v_0 as v with

$c_0 = c_{g_0}$. The linearity and boundedness in $L^p_{\rm rad}$ of the map $f_0 \mapsto v_0$ have been already proved in Lemma 3.13. Moreover, v_0 can be written in the following form:

$$v_0(r) = C\left(r^{\nu_0} K_{\nu_0}\left(\frac{\omega r}{\sqrt{a}}\right)\int_0^r t^{1-\nu_0} U_{\nu_0}\left(\frac{\omega t}{\sqrt{a}}\right) f_0(t)\,dt \right.$$
$$\left. + r^{\nu_0} U_{\nu_0}\left(\frac{\omega r}{\sqrt{a}}\right)\int_r^\infty t^{1-\nu_0} K_{\nu_0}\left(\frac{\omega t}{\sqrt{a}}\right) f_0(t)\,dt\right),$$

with $U_\nu(z) = I_\nu(z) + \frac{2\sin(\nu\pi)}{\pi} K_\nu(z)$. Using Lemma 3.9 and 3.8 again, we have

$$|v_0'| \le C'\left(r^{\nu_0} K_{\nu_0-1}(kr)\int_0^r t^{1-\nu_0} U_{\nu_0}(kt)|f_0(t)|\,dt \right.$$
$$\left. + r^{\nu_0} I_{1-\nu_0}(kr)\int_r^\infty t^{1-\nu_0} K_{\nu_0}(kt)|f_0(t)|\,dt\right)$$

with $k = \frac{\operatorname{Re}\omega}{\sqrt{a}}$. Proceeding as in the proof of Lemma 3.13, we get $\|v_0\|_{L^p_{\rm rad}} + \|v_0'\|_{L^p_{\rm rad}} + \|v_0'/r\|_{L^p_{\rm rad}} \le C\|g_0\|_{L^p_{\rm rad}}$. By difference, we also have $\|v_0''\|_{L^p_{\rm rad}} \le C\|f_0\|_{L^p_{\rm rad}}$. Consequently, by Proposition 2.14 (i) we obtain $u_0(x) := v_0(|x|) \in W^{2,p}(\mathbb{R}^N)$. Taking

$$u = \sum_{j\in J} u_j + \tilde{u} \in W^{2,p}(\mathbb{R}^N),$$

we see that $\omega^2 u - Lu = f$.

Finally, we show $N(\omega^2 - L_{|W^{2,p}}) = \{0\}$. Let $w \in W^{2,p}(\mathbb{R}^N)$ satisfy $\omega^2 w - Lw = 0$. Set $w_j(x) = h_j(r)P_j(\sigma)$ with $h_j(r) = \int_{S^{N-1}} w(r\sigma)P_j(\sigma)\,d\sigma \in L^p_{\rm rad}$ for $j \in J$, and $\tilde{w} = w - \sum_{j\in J} w_j \in L^p_{\ge n}$. We see from Proposition 3.6 that $\tilde{w} \in N(\omega^2 - L^n_{p,\max}) = \{0\}$. Hence it suffices to show $h_j = 0$ for each j. Since h_j satisfies the equation $\omega^2 h_j - a h_j'' - \frac{N-1}{r}h_j' + \frac{\lambda_k}{r^2}h_j = 0$, where k is the order of P_j. Therefore $h_j = c_{1,j}v_{\omega,1} + c_{2,j}v_{\omega,2}$. Noting that $v_{\omega,1}$ and $v_{\omega,2}$ have exponential growth and decay at infinity, we have $c_{1,j} = 0$. On the other hand, since $v_{\omega,2}'/r \notin L^p_{\rm rad}$, it follows from $w_j \in W^{2,p}(\mathbb{R}^N)$ and Proposition 2.14 that $c_{2,j} = 0$. Therefore $w_j = 0$, and hence $w = \tilde{w} + \sum_{j\in J} w_j = 0$. □

Proposition 4.6. *The operator $L_{|W^{2,p}}$ generates a semigroup if and only if $s_1^{(\lambda_2)} + 2 < N/p < s_1 + 2$. In this case the semigroup is positive and analytic.*

Proof. The necessity of the condition follows from Lemma 4.4. Instead, if the above condition is satisfied, the previous lemma implies that the resolvent contains a sector larger than the right half plane and the resolvent estimate $\|(\omega^2 - L_{|W^{2,p}})^{-1}\| \le C|\omega^2|^{-1}$ follows by scaling. Concerning the positivity we note that, by the proof of Lemma 4.5, $(\omega^2 - L_{|W^{2,p}})^{-1}$ coincides with $(\omega^2 - L_{p,{\rm int}})^{-1}$ on $L^p_{\ge 1}$ and, for $\omega > 0$, is larger than it on positive radial functions, by the choice of $c_0 = c_g$ and since $C_\omega = W(\omega)^{-1} > 0$. □

Remark 4.7. The previous proposition shows that the difference of $L_{p,\text{int}}$ and the operator defined by Manselli [6] is only in the subspace of radial functions.

Example 4.8. Let $N = 2$, $p = 2$ and assume that $b = c = 0$. By Remark 4.2 we have that for $a \leq 1$ there is generation and $D(L_{2,\text{int}}) = W^{2,2}(\mathbb{R}^2)$. When $a > 1$, we have $s_2 + 2 = 2$, $s_1^{(\lambda_2)} + 2 < 1$. It follows that $L_{|W^{2,2}}$ generates a bounded analytic semigroup.

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