The parabolic thin obstacle problem for the weighted biLaplacian

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Abstract

In this paper we are concerned with a parabolic thin obstacle problem for the weighted biLaplace operator. Such problems are in connection with various physical applications such as travelling waves phenomenon, pattern formulations in spatially homogeneous systems, phase separation in isothermal binary systems...We establish the well-posedeness of the problem, and we analyze the regularity of the solution depending on the dimension of the domain in consideration. Our methods are based on an implicit time discretization approach.

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1 Introduction

In this work, we are concerned with a thin obstacle problem for the parabolic fourth order differential operator $u_t + \Delta_b^2 u$ for $-1 < b < 1$. Here Δ_b denotes the b-Laplace differential operator

$$\Delta_b u = y^{-b}\nabla \cdot (y^b \nabla u). \tag{1.1}$$

In the particular case where $b = 0$, we recover the parabolic equation for the unweighted biLaplacian $u_t + \Delta^2 u = 0$.

Throughout the paper, B_1 will denote the unit ball in $\mathbb{R}^{n+1}$ centered at the origin, $B_1^+ = \{z = (x, y) \in B_1 : \quad y > 0\}$, $(\partial B_1)^+ = \partial B_1 \cap \{y > 0\}$, and $B_1' = B_1 \cap \{y = 0\}$. We consider the problem of finding a function u, solution of the following time-dependent thin obstacle problem

$$\begin{cases} u_t + \Delta_b^2 u = 0 & \text{in } B_1^+ \times (0,T], \\ u(z,t) = g(z,t), \quad u_\nu(z,t) = 0 & \text{on } (\partial B_1)^+ \times (0,T], \\ u(z,t) \geq h(z,t), u_y(z,t) = 0, \lim_{y\to 0} y^b(\Delta_b u)_y(z,t) \geq f(z,t) & \text{on } B_1' \times (0,T], \\ \lim_{y\to 0} y^b(\Delta_b u)_y(z,t) = f(z,t) & \text{on } (B_1' \times (0,T]) \cap \{u > h\}, \\ u(z,0) = g^0(z) & \text{on } (B_1^+ \cup B_1') \times \{0\}. \end{cases} \tag{1.2}$$

We require the initial data g^0 to obey the compatibility conditions $g^0(z) = g(z,0)$, $g^0_\nu(z) = 0$ on $(\partial B_1)^+$, $g^0(z) \geq h(z,0), g^0_y(z) = 0$ on B_1'. Similarly, we require the lateral boundary data g to be the trace with respect to the spatial variable of a function g satisfying $g(z,t) \geq h(z,t)$, $g_y(z,t) = 0$ on $B_1' \times (0,T]$. We assume that $g \in C^0\left((0,T]; W^{2,2}(B_1^+, y^b)\right)$, $g^0 \in W^{2,2}(B_1^+, y^b)$, $h \in C^0\left((0,T]; C^{1,1}(B_1')\right)$, $f \in C^0\left((0,T]; L^\infty(B_1')\right)$. We also assume that the function f is non-decreasing in the time variable t. See Section 3 for notations and definitions of function spaces.

We observe that the thin parabolic cylindre $B_1' \times (0,T]$ is divided into two regions:

$$\textit{coincidence set}: \Lambda(u) := \left\{u(x,t) = h(x,t)\right\} \cap (B_1' \times (0,T]),$$

and

$$\textit{non-coincidence set}: \Omega(u) := \left\{u(x,t) > h(x,t)\right\} \cap (B_1' \times (0,T]).$$

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The interface between these two regions $\Gamma(u) := \partial_{\mathbb{R}^n}\Omega(u)$ is called the *free boundary* of the solution, and it is a-priori unknown.

Evolution problems associated with fourth order operators appear in various physical applications such as travelling waves phenomenon, pattern formulations in spatially homogeneous systems, phase separation in isothermal binary systems... See Section 2 for more details. Obstacle-type problems for second order elliptic and parabolic PDEs have been extensively studied over the years motivated by the innovative idea of blow-up techniques introduced by Caffarelli in his seminal paper [5], and by the discovery of several powerful monotonicity formulas which provided the necessary tools to establish the optimal regularity of a solution, as well as the regularity and the structure of the free boundary. To mention few papers, we refer the reader to [6], [12], [4], [15], [16],[36], [9]. On the otherhand, much less work has been done on obstacle-type problems associated with fourth order operators.

Elliptic obstacle-type problems for the bi-Laplacian has been considered in few papers over the years. In the two-dimensional case, such problems arise naturally in elastic, homogenous and isotropic plate displacement phenomena in the event of relatively small displacements. The regularity of the solution in such problems has been studied in [20], [21], and [7] for the thick obstacle case; and in [33], and [34] for the thin obstacle case. Moreover, in [7] the authors investigated the structure of the free boundary of a solution, and analyzed its regularity in the two dimensional case. Also, in [2], the author considered the regularity of the free boundary for the thick obstacle problem under one-dimensional assumption on the solution and non-tangential accessibility assumption on the non-coincidence set of the solution.

Very recently, the author investigated with collaborators in a couple of papers some thin obstacle-type problems associated with the fourth order operator Δ_b^2, where Δ_b is the b-Laplace operator defined in (1.1). In [17], we considered a two-phase boundary obstacle problem for the bi-Laplacian, where we investigate the well-posedeness of the problem, the optimal regularity of the solution, and the structural properties of the free boundary. Also, in [18], we analyzed the well-posedeness and the regularity of a solution to a one-phase obstacle problem for the fractional Laplacian $(-\Delta)^s u$, for $1 < s < 2$, in the Euclidean space $\mathbb{R}^n$. As we will see in Section 8, and as first observed by Yang in [37] (see also [11]), such problem is equivalent to a boundary obstacle problem for the b-biLaplacian in $\mathbb{R}^n \times \{y \geq 0\}$ with an obstacle living on $\mathbb{R}^n \times \{0\}$.

Inspired by the work in the litterature concerning the extension of the fractional parabolic operator $(u_t - \Delta)^s$, for $0 < s < 1$, to the upper dimensional space (see for instance [30] and [35]), we expect our problem (1.2) to serve as a localization of the extension to $\mathbb{R}^{n+1}_+ \times (0, T]$ of the following non-local parabolic obstacle-type problem on the lower dimensional space $\mathbb{R}^n \times (0, T]$:

$$\begin{cases} w \geq h & \text{in } \mathbb{R}^n \times (0,T], \\ (w_t - \Delta)^s w \geq f & \text{in } \mathbb{R}^n \times (0,T], \\ (w_t - \Delta)^s w = f & \text{in } (\mathbb{R}^n \times (0,T]) \cap \{w > h\}, \end{cases} \tag{1.3}$$

where $s = (3-b)/2$ which is a value in $(1, 2)$.

While the well-known Caffarelli-Silvestre characterization of the fractional Laplacian $(-\Delta)^s$ for $0 < s < 1$ as a Dirichlet-to-Neumann map (established in [14]) was generalized in [37] (see also [11]) to include the cases $1 < s < 2$; however, to our knowledge, very little is known in the litterature about the extension of the fractional parabolic operator $(u_t - \Delta)^s$ for $1 < s < 2$, which makes it an interesting subject to discover.

Regarding the evolution of obstacle-type problems, Novaga and Okabe considered the global (or thick) obstacle problem for the time dependent bi-harmonic equation $u_t + \Delta^2 u = 0$. They considered the case of one global obstacle in [28], and the case of two global obstacles in [29]. In both papers, the authors follow an implicit time discretization scheme, where some regularity results for the elliptic case studied in [7] (with one global obstacle) and in [8] (for two global obstacles) can be adapted. In this paper, we will take a similar time discretization approach to study problem (1.2). To the best of our knowledge, this paper is the first instance of a work concerning an evolution problem for a thin obstacle-type problem associated with a fourth order operator.

The structure of the paper is as follows. In Section 2, we give an overview of various physical phenomena which can be modeled by evolution problems for fourth order operators, and we mention some related results available in the literature. In Section 3, we state the notations and function spaces which will be used throughout the paper. In Section 4, we start setting up our time discretization approach by defining two sequences $u_m(x,t)$ and $\tilde{u}_m(x,t)$ of approximating functions as piece-wise linear interpolations and piece-wise constant interpolations of minimizers $u_{i,m}(x)$ $(i:1,..,m)$ of some appropriate elliptic functionals. In Section 5, we prove the existence and uniqueness of the minimizers $u_{i,m}$, and we derive the elliptic thin obstacle problems for which these minimizers are solutions. In section 6, we establish the Sobolev convergence of both of u_m and $\tilde{u}_m$ to the same limit function u_0 by means of some compactness arguments, and we prove that the limit function u_0 solves our problem (1.2). In section 7, we show that $\mu(t) := \frac{\partial u_0}{\partial t} + \Delta_b^2 u_0 - f\chi_{\{y=0\}}$ defines a non-negative measure on $B_1^+ \cup B_1'$ for a.e. $t \in (0,T)$. Moreover, we obtain an L^2-estimate of μ as a function of t. This is achieved by first following a penalization approach to obtain uniform estimates of the measures $\mu_{i,m}$ induced by the minimizers $u_{i,m}$, and then establishing the convergence of the piecewise constant interpolation of these measures to the measure μ. At this point, we restrict our dimension to $n=1$ or $n=2$ as an attempt to adapt regularity results for the elliptic version of our problem. To analyze the regularity of the solution u_0, we start with obtaining regularity estimates for the minimizers $u_{i,m}$ uniformly in m. This is the content of Section 8. In particular, adapting regularity results from [18] by the author and collaborators, we derive some uniform $C^{1,1}_{loc}$ and $W^{3,2}_{loc}$ estimates for the minimizers u_{im}. These uniform estimates yield regularity results for the approximating functions u_m and $\tilde{u}_m$, and hence for the solution u_0. In Section 9, we prove the $L^2\left((0,T);W^{2,\infty}(B_\rho^+ \cup B_\rho')\right)$ and $L^2\left((0,T);W^{3,2}(B_\rho^+ \cup B_\rho')\right)$ for all $0<\rho<1$. We also prove $C^{\frac{\beta-\gamma}{4\beta}}\left((0,T);C^\gamma(B_1^+ \cup B_1')\right)$, where $0<\gamma<\beta$, and β satisfies $0<\beta<1$ when $n=1$ and $0<\beta\leq 1/2$ when $n=2$ (see Section 3 for the definitions of the function spaces).

2 Back-ground and physical interpretations

Parabolic fourth-order equations appear when studying the dynamic of travelling waves. Consider the equation

$$\frac{\partial u}{\partial t} = -\gamma\Delta^2 u + \Delta u + f(u), \tag{2.1}$$

where $f(u) = (u-a)(1-u^2)$, $-1 < a \leq 0$, and $\gamma > 0$. A solution of (2.1) is called a travelling wave solution, and it connects the two stable states $u = \pm 1$ of the ordinary differential equation $u' = (u-a)(1-u^2)$. In the case when $\gamma = 0$, equation (2.1) is of second order and has applications in population dynamics. It describes the interaction between dispersal (modeled by the diffusion term) and survival fitness (represented by the function $f(u)$). When $a = 0$, the equation is known as the *Fisher-Kolmogorov equation*. In the case where $\gamma \neq 0$, we see that (2.1) is a fourth order equation. Such an equation appears when studying the dynamics near a critical point in phase transition problems. This happens at the so called *Lifshitz points* where the coefficient of $|\nabla u|^2$ in the Landau free energy functional vanishes, and the lowest order spatial derivatives that appear are of second order (see [38] for more details). When $a = 0$, equation (2.1) is known as the *extended Fisher-Kolmogorov equation*, and it serves as a prototype equation for the study of threshold phenomena, transition layers, and pattern formation in spatially homogeneous systems. In a series of papers starting with [31], Peletier and Troy consider the *extended Fisher-Kolmogorov equation*, and they focus on the existence and uniqueness of odd monotone transition layer solutions which connect the two stable states $u = -1$ and $u = 1$. Moreover, the authors study qualitative properties of the solutions such as the asymptotic behavior as x *to* $\pm\infty$. The analysis is based on the value of γ, and the techniques hinge on reduction of order using some energy identities, and rely on the variational structure of the equation. On the other hand, in the case where $a \neq 0$, the equation neither has a conserved energy nor a variational structure. This case was considered in [1] where the authors observe that, for small γ, the equation can be seen as a perturbation of the problem

with $\gamma = 0$; and hence they use a geometric singular perturbation approach to prove the existence of a travelling wave solution connecting the steady states $u = \pm 1$.

Another type of a time dependent fourth order equation is *Cahn-Hilliard equation*

$$\frac{\partial u}{\partial t} + \gamma \Delta^2 u + \Delta(f(u)), \tag{2.2}$$

which models diffusive phase separation in isothermal binary systems. The positive constant γ is related to the surface tension, and $f(u) = \frac{d}{du}F(u)$ corresponds to the volumetric part of the chemical potential difference between components. The function $F(u)$ is a (volumetric) free energy parametrized by the temperature, and it is naturally non-smooth. The solution u represents the mass concentration, hence it is bounded as $0 \leq u \leq 1$. For more details on this physical application, we refer to [24]. Initial-boundary value problems associated with *Cahn-Hilliard equation* has been studied by many people modeling various physical applications, and aiming mainly to study the well-posedeness of the problem, and the asymptotic behavior of solutions as $t \to \infty$. For instance, in [24], the authors considered the problem for a general form of non-smooth free energy F with some classical boundary conditions. Later, dynamic boundary conditions were considered in several papers to model spinodal decomposition of binary mixtures where the effective interaction between the wall and two mixture components is short-ranged (see for instance [32]). Moreover, people also considered the double-obstacle problem associated with *Cahn-Hilliard equation*. In such a problem, f is no longer a derivative but a certain subdifferential (see the paper [10]).

3 Notations and function spaces

3.1 Notations

For a point $z \in \mathbb{R}^{n+1}$, we write $z = (x, y)$, where $x \in \mathbb{R}^{n+1}$ and $y \in \mathbb{R}$; and we denote by $B_r(z)$ the ball in $\mathbb{R}^{n+1}$ centered at z and of radius r. When $z = 0$, we often write B_r for $B_r(z)$. Also, for a domain $\Omega \subseteq \mathbb{R}^{n+1}$, we let $\Omega^+ = \Omega \cap \{y > 0\}$ and $\Omega' = \Omega \cap \{y = 0\}$. In particular, we identify $\mathbb{R}^{n+1} \cap \{y = 0\}$ with the Euclidian space $\mathbb{R}^n$. Finally, for a function $w(z, t)$ of the space variable z and the time variable t, ∇w (resp. Δw) will denote the gradient (resp. the Laplacian) of w with respect to the spatial variable z.

3.2 Function spaces

For an open domain Ω in the Euclidian space, we let $L^2(\Omega, y^b)$ be the space of functions w defined on Ω such that

$$\|w\|_{L^2(\Omega, y^b)} := \int_\Omega y^b w^2 dx < \infty. \tag{3.1}$$

Also, we let $W^{1,2}(\Omega, y^b)$ be the weighted Sobolev space equipped with the norm

$$\|w\|^2_{W^{1,2}(\Omega, y^b)} := \|y^{\frac{b}{2}} \nabla w\|^2_{L^2(\Omega)} + \|y^{\frac{b}{2}} w\|^2_{L^2(\Omega)}, \tag{3.2}$$

and $W^{2,2}(\Omega, y^b)$ be the space equipped with the norm

$$\|w\|^2_{W^{2,2}(\Omega, y^b)} := \|y^{\frac{b}{2}} \Delta_b w\|^2_{L^2(\Omega)} + \|y^{\frac{b}{2}} \nabla w\|^2_{L^2(\Omega)} + \|y^{\frac{b}{2}} w\|^2_{L^2(\Omega)}, \tag{3.3}$$

where Δ_b is the b-Laplace operator as defined in (1.1). Moreover, for any function space X on Ω, we define

$$\begin{aligned} L^2\left((t_1, t_2); X(\Omega)\right) := \big\{ & w : \Omega \times (t_1, t_2) \to \mathbb{R} : \\ & \|w\|^2_{L^2((t_1,t_2);X(\Omega))} := \left\| \left(\|w(\cdot, t)\|^2_{X(\Omega)} \right) \right\|_{L^2((t_1,t_2))} < \infty \big\}. \end{aligned} \tag{3.4}$$

Similarly, we define

$$L^\infty\left((t_1,t_2);X(\Omega)\right) := \big\{w:\Omega\times(t_1,t_2)\to\mathbb{R}: \\ \|w\|_{L^\infty((t_1,t_2);X(\Omega))} := \big\|\left(\|w(\cdot,t)\|_{X(\Omega)}\right)\big\|_{L^\infty((t_1,t_2))}<\infty\big\}. \tag{3.5}$$

Finally, for $0<\alpha,\beta<1$, we define

$$C^\beta\left((t_1,t_2);X(\Omega)\right) := \big\{w:\Omega\times(t_1,t_2)\to\mathbb{R}: \\ \|w\|_{C^\beta((t_1,t_2);X(\Omega))} := \big\|\left(\|w\|_{X(\Omega)}\right)\big\|_{C^\beta((t_1,t_2))}<\infty\big\}. \tag{3.6}$$

We recall the following definition of a weak solution for the b-biharmonic equation $\Delta_b^2 U=0$.

Definition 3.1 *We say that $U\in W^{2,2}(\Omega,y^b)$ is a weak solution of the b-biharmonic equation $\Delta_b^2 U=0$ in Ω if*

$$\int_\Omega y^b\ \Delta_b U\ \Delta_b\phi = 0 \text{ for all functions } \phi\in W^{2,2}(\Omega,y^b) \text{ satisfying } \phi=\phi_\nu=0 \text{ on } \partial\Omega. \tag{3.7}$$

4 The approximation approach

We start with introducing the notion of a weak solution for problem (1.2).

Definition 4.1 *Suppose u is a function which belongs to the set*

$$\mathcal{B} := \Big\{w: w\in L^2\left((0,T];W^{2,2}(B_1^+,y^b)\right), w_t\in L^2\left((0,T];L^2(B_1^+,y^b)\right) \\ w=g,\ w_\nu=0 \text{ on } (\partial B_1)^+\times(0,T],\quad w_y=0, w\ge h \text{ on } B_1'\times[0,T], \\ \text{and } w=g^0 \text{ on } (B_1^+\cup B_1')\times\{0\}\Big\}. \tag{4.1}$$

We say that u is a weak solution of (1.2) if

$$\int_{B_1^+\times(t_1,t_2]}\left(y^b(w-u)\frac{\partial u}{\partial t}+y^b\Delta_b u\ \Delta_b(w-u)\right)dxdydt-\int_{B_1'\times(t_1,t_2]} f\ (w-u)\ dxdt\ge 0 \tag{4.2}$$

for all $0\le t_1<t_2\le T$ and all functions $w\in\mathcal{B}$.

To prove the existence of a weak solution of (1.2), we will follow an approximation approach based on implicit time discretization techniques. First, we consider the following piecewise linear approximation of the functions f, g, and the obstacle h:

$$f_{i,m}(x)=f(x)\quad \text{for } (x,t)\in B_1'\times[(i-1)\tau_m,i\tau_m], \tag{4.3}$$

$$g_{i,m}(x)=g(x)\quad \text{for } (x,t)\in B_1'\times[(i-1)\tau_m,i\tau_m], \tag{4.4}$$

$$h_{i,m}(x)=h(x)\quad \text{for } (x,t)\in B_1'\times[(i-1)\tau_m,i\tau_m]. \tag{4.5}$$

Now, for $m\in\mathbb{N}$, let $\tau_m=\frac{T}{m}$, and define $\{u_{i,m}\}_{i=0,..,m}$ inductively by setting

$$u_{0,m}=g^0, \tag{4.6}$$

and letting $u_{i,m}$ be the minimizer of the functional

$$J_{i,m}[w]=\frac{1}{2}\int_{(B_1)^+} y^b(\Delta_b w)^2dxdy+\frac{1}{2\tau_m}\int_{(B_1)^+} y^b(w-u_{i-1,m})^2dxdy-\int_{B_1'} f_{i,m}\ w\ dx \tag{4.7}$$

over the set

$$\mathcal{A}_{i,m} := \Big\{ w : \ w \in W^{2,2}(B_1^+, y^b), \quad w = g_{i,m}, \ w_\nu = 0 \text{ on } (\partial B_1)^+, \text{ and } \quad w_y = 0, w \geq h_{i,m} \text{ on } B_1' \Big\}. \tag{4.8}$$

The existence and uniqueness of the minimizers $u_{i,m}$ will be the subject of Section 5.

Now, for a fixed m, and for each $i : 1, .., m$, we define the quotient

$$v_{i,m}(x,y) := \frac{u_{i,m}(x,y) - u_{i-1,m}(x,y)}{\tau_m} \quad \text{for all } (x,y) \in B_1^+ \cup B_1'. \tag{4.9}$$

Then, we consider the piece-wise linear interpolation of $\{u_{i,m}\}$ on $(B_1^+ \cup B_1') \times [0,T]$:

$$u_m(x,y,t) := u_{i-1,m}(x,y) + (t - (i-1)\tau_m)\, v_{i,m}(x,y) \quad \text{for } (x,y,t) \in (B_1^+ \cup B_1') \times [(i-1)\tau_m, i\tau_m], \tag{4.10}$$

and the piecewise constant interpolation

$$\tilde{u}_m(x,y,t) = u_{i,m}(x,y) \quad \text{for } (x,y,t) \in (B_1^+ \cup B_1') \times [(i-1)\tau_m, i\tau_m]. \tag{4.11}$$

We also define

$$v_m(x,y,t) = v_{i,m}(x,y) \quad \text{for } (x,y,t) \in (B_1^+ \cup B_1') \times [(i-1)\tau_m, i\tau_m]. \tag{4.12}$$

Our objective is to show that both sequences $\{u_m\}$ and $\{\tilde{u}_m\}$ converge to a solution u of the problem (1.2). As a first step, we establish the existence and uniqueness of the minimizers $u_{i,m}$.

5 Existence and uniqueness of the minimizers $u_{i,m}$

We will need the following Poincaré type inequality on a half ball.

Lemma 5.1 *For $w \in W^{1,2}(B_r, y^b)$ we have*

$$\frac{n-b}{r^2} \int_{B_r^+} y^b w^2 dxdy \leq \frac{1}{r} \int_{(\partial B_r)^+} y^b w^2 dS + \int_{B_r^+} y^b |\nabla w|^2 dxdy. \tag{5.1}$$

Proof. Letting $z = (x,y)$ we have

$$\begin{aligned}
(n+1) \int_{B_r^+} y^b w^2 dxdy &= \int_{B_r^+} \nabla \cdot (y^b w^2 z) - \int_{B_r^+} \nabla (y^b w^2) \cdot z \\
&= r \int_{(\partial B_r)^+} y^b w^2 - \int_{B_r^+} (2 y^b w \nabla w \cdot z + b w^2 y^b) \\
&\leq r \int_{(\partial B_r)^+} y^b w^2 + \int_{B_r^+} y^b w^2 dxdy + r^2 \int_{B_r^+} y^b |\nabla w|^2 dxdy + b \int_{B_r^+} y^b w^2,
\end{aligned}$$

where in the last line we applied Young's inequality. This gives

$$(n-b) \int_{B_r^+} y^b w^2 dxdy \leq r \int_{(\partial B_r)^+} y^b w^2 dS + r^2 \int_{B_r^+} y^b |\nabla w|^2 dxdy.$$

Dividing by r^2 we obtain the desired inequality. □

Theorem 5.2 *There exists a unique minimizer $u_{i,m}$ of the functional (4.7) over the admissible set $\mathcal{A}_{i,m}$ defined in (4.8).*

Proof. First we note that the admissible set $\mathcal{A}_{i,m} \neq \emptyset$ since $g_{i,m} \in \mathcal{A}_{i,m}$. Next, for $w \in \mathcal{A}_{i,m}$, employing the trace theorem for weighted Sobolev spaces (see for instance [27]), we see that the sum of the first and last terms of the functional $J_{i,m}$ satisfies

$$\begin{aligned}
&\frac{1}{2}\int_{(B_1)^+} y^b(\Delta_b w)^2 dxdy - \int_{B_1'} f_{i,m}\ w\ dx \\
&\geq \frac{1}{2}\int_{(B_1)^+} y^b(\Delta_b w)^2 dxdy - \epsilon\int_{B_1'}\ w^2\ dx - \frac{1}{\epsilon}\int_{B_1'} f_{i,m}^2\ dx \\
&\geq \frac{1}{2}\int_{(B_1)^+} y^b(\Delta_b w)^2 dxdy - C(n)\epsilon\int_{B_1^+}\ y^b(w^2+|\nabla w|^2)\ dx - \frac{1}{\epsilon}\int_{B_1'} f_{i,m}^2\ dx,
\end{aligned} \tag{5.2}$$

where ϵ to be chosen sufficiently small. Moreover, integrating by parts and applying the poincaré inequality (5.1), we have

$$\begin{aligned}
\int_{B_1^+} y^b|\nabla(w-g_{i,m})|^2 dxdy &= -\int_{B_1^+} y^b(w-g_{i,m})\Delta_b(w-g_{i,m})dxdy \\
&\leq \|w-g_{i,m}\|_{L^2(B_1^+,y^b)}\|\Delta_b(w-g_{i,m})\|_{L^2(B_1^+,y^b)} \\
&\leq C(n,b)\|\nabla(w-g_{i,m})\|_{L^2(B_1^+,y^b)}\|\Delta_b(w-g_{i,m})\|_{L^2(B_1^+,y^b)},
\end{aligned}$$

and thus

$$\|\nabla(w-g_{i,m})\|_{L^2(B_1^+,y^b)} \leq C(n,b)\|\Delta_b(w-g_{i,m})\|_{L^2(B_1^+,y^b)}. \tag{5.3}$$

Also, employing (5.3) and the poincaré inequality (5.1), we get

$$\|w-g_{i,m}\|_{L^2(B_1^+,y^b)} \leq C(n,b)\|\nabla(w-g_{i,m})\|_{L^2(B_1^+,y^b)} \leq C(n,b)\|\Delta_b(w-g_{i,m})\|_{L^2(B_1^+,y^b)}. \tag{5.4}$$

Moreover, performing some integration by parts and applying a density argument, one reaches

$$\|w-g_{i,m}\|^2_{W^{2,2}(B_1^+,y^b)} \leq C(n,b)\left(\|w-g_{i,m}\|^2_{L^2(B_1^+,y^b)} + \|\Delta_b(w-g_{i,m})\|^2_{L^2(B_1^+,y^b)}\right). \tag{5.5}$$

Combining (5.5) and (5.4) we reach

$$\|w-g_{i,m}\|_{W^{2,2}(B_1^+,y^b)} \leq C\|\Delta_b(w-g_{i,m})\|_{L^2(B_1^+,y^b)},$$

which yields

$$\|w\|^2_{W^{2,2}(B_1^+,y^b)} \leq C\|\Delta_b w\|^2_{L^2(B_1^+,y^b)} + C\|g_{i,m}\|^2_{W^{2,2}(B_1^+,y^b)}. \tag{5.6}$$

Here, the constant C depending only on b and the dimension n. Combining (5.6) and (5.2), and choosing ϵ in (5.2) sufficiently small, we reach

$$J_{i,m}[w] \geq C\|w\|^2_{W^{2,2}(B_1^+,y^b)} - C\|g_{i,m}\|^2_{W^{2,2}(B_1^+,y^b)} - C\|f_{i,m}\|^2_{L^2(B_1')} + \frac{1}{2\tau_m}\int_{(B_1)^+} y^b(w-u_{i-1,m})^2 dxdy, \tag{5.7}$$

where C depends only on b and the dimension n. In particular, we see that the functional $J_{i,m}$ is bounded from below. Now let $\{u_l\}$ be a minimizing sequence of $J_{i,m}[\cdot]$. From (5.6) and (5.2) we also infer that the sequence $\{u_l\}$ is bounded in $W^{2,2}(B_1^+,y^b)$. By standard compactness arguments, there exists a sub-sequence still denoted by $\{u_l\}$ and a function $u_{i,m} \in W^{2,2}(B_1^+,y^b)$ such that

$$u_l \to u_{i,m} \text{ weakly in } W^{2,2}(B_1^+,y^b) \text{ and strongly in } W^{1,2}(B_1^+,y^b).$$

Since the admissible set $\mathcal{A}_{i,m}$ is a closed subset of $W^{2,2}(B_1^+,y^b)$, we conclude that $u_{i,m} \in \mathcal{A}_{i,m}$. Moreover, from the weak convergence we infer

$$J_{i,m}[u_{i,m}] \leq \liminf J_{i,m}[u_l],$$

which implies that $u_{i,m}$ is a minimizer of the functional $J_{i,m}[\cdot]$.

The uniqueness of the minimizer follows from the convexity of our functional. □

Next we derive the elliptic thin obstacle problem for which the minimizer $u_{i,m}$ is a solution. More specifically, we show that, for a fixed $m \in \mathbb{N}$, and for $i : 1, .., m$, the minimizer $u_{i,m}$ of $J_{i,m}[\cdot]$ is a solution of the thin obstacle problem

$$\begin{cases} \Delta_b^2 w + \frac{1}{\tau_m}(w - u_{i-1,m}) \ = 0 & \text{in } B_1^+, \\ w = g, w_\nu = 0 & \text{on } (\partial B_1)^+, \\ w \geq h_{i,m}, \ w_y = 0, \ \lim_{y\to 0} y^b(\Delta_b w)_y \geq f_{i,m} & \text{on } B_1', \\ \lim_{y\to 0} y^b(\Delta_b w)_y = f_{i,,m} & \text{on } B_1' \cap \{w > h_{i,m}\}, \end{cases} \tag{5.8}$$

in some weak sense.

Lemma 5.3 *For $i : 1, .., m$, the minimizer $u_{i,m}$ of $J_{i,m}[\cdot]$ is a weak solution of (5.8) in the sense that it satisfies the variational inequality*

$$\begin{aligned} &\int_{B_1^+} y^b \Delta_b u_{i,m} \Delta_b (w - u_{i,m}) dxdy + \frac{1}{\tau_m} \int_{B_1^+} y^b \ (u_{i,m} - u_{i-1,m}) \ (w - u_{i,m}) dxdy \\ &- \int_{B_1'} f_{i,m} \ (w - u_{i,m}) \ dx \geq 0, \end{aligned} \tag{5.9}$$

for all functions w in the admissible set $\mathcal{A}_{i,m}$. Moreover, u_{im} satisfies

$$\Delta_b^2 u_{i,m} + \frac{1}{\tau_m}(u_{i,m} - u_{i-1,m}) \ = 0 \ \textit{in} \ B_1^+. \tag{5.10}$$

in a pointwise sense.

Proof. For $w \in \mathcal{A}_{i,m}$, write $w = u_{i,m} + \phi$. Clearly, $u_{i,m} + \tau\phi$ are admissible functions for all small enough $\tau \geq 0$, and by the minimality of $u_{i,m}$ we have $J[u_{i,m} + \tau\phi] \geq J[u_{i,m}]$. Then,

$$\begin{aligned} 0 &\leq \frac{J[u_{i,m} + \tau\phi] - J[u_{i,m}]}{\tau} \\ &= \frac{1}{2} \int_{B_1^+} y^b \left((\Delta_b \phi)^2 \tau + 2\Delta_b u_{i,m} \Delta_b \phi \right) dxdy + \frac{1}{2\tau_m} \int_{B_1^+} y^b \left(2(u_{i,m} - u_{i-1,m}) \ \phi + \phi^2 \tau \right) dxdy \\ &- \int_{B_1'} f_{i,m} \phi dx \end{aligned}$$

Therefore, letting $\tau \searrow 0$ we obtain

$$\int_{B_1^+} y^b \Delta_b u_{i,m} \Delta_b \phi + \frac{1}{\tau_m} \int_{B_1^+} y^b (u_{i,m} - u_{i-1,m}) \ \phi \ dxdy - \int_{B_1'} f_{i,m} \phi dx \geq 0, \tag{5.11}$$

and this shows that (5.9) holds true.

Moreover, since we can make free perturbation away from the thin ball B_1', we can easily see that the inequality (5.11) and its opposite inequality hold true for any smooth test function ϕ compactly supported in B_1^+. From this, (5.10) follows. □

6 Existence of a solution for Problem 1.2

In this section, we study the convergence of the approximating sequence u_m. First, we write the functional (4.7) as

$$J_{i,m}[w] = E_{i,m}[w] + H_{i,m}[w], \tag{6.1}$$

where

$$E_{i,m}[w] = \frac{1}{2}\int_{(B_1)^+} y^b(\Delta_b w)^2 dxdy - \int_{B_1'} f_{i,m}\ w\ dx, \tag{6.2}$$

and

$$H_{i,m}[w] = \frac{1}{2\tau_m}\int_{(B_1)^+} y^b(w - u_{i-1,m})^2 dxdy. \tag{6.3}$$

As a first step, we derive the following uniform estimate for the approximating sequence $\{u_m\}$.

Theorem 6.1 *The sequence $\{u_m\}$ is bounded in $W^{2,2}(B_1^+, y^b)$ uniformly in t and m. Moreover, the sequence $\{v_m\}$ is bounded in $L^2\left((0,T], L^2(B_1^+, y^b)\right)$ uniformly in m.*

Proof. We notice that

$$J_{i,m}[u_{i,m}] \le J_{i,m}[u_{i-1,m}] = E_{i,m}[u_{i-1,m}] \quad \text{for all } i : 1, .., m, \tag{6.4}$$

where the inequality follows from the minimality of $u_{i,m}$. Then,

$$E_{i,m}[u_{i,m}] \le E_{i,m}[u_{i-1,m}], \quad \text{for all } i : 1, .., m. \tag{6.5}$$

From (6.5) we infer

$$E_{i,m}[u_{i,m}] \le E_{i,m}[u_{i-1,m}] \le ... \le E_{i,m}[g^0], \quad \text{for all } i : 1, .., m. \tag{6.6}$$

Combining (6.6) with (5.7), we infer

$$\begin{aligned} \|u_{i,m}\|^2_{W^{2,2}(B_1^+,y^b)} &\le CE_{i,m}[g^0] + C\|g_{i,m}\|^2_{W^{2,2}(B_1^+,y^b)} + C\|f_{i,m}\|^2_{L^2(B_1')} \\ &\le C\left(\|g^0\|^2_{W^{2,2}(B_1^+,y^b)} + \|g\|^2_{C^0\left((0,T);W^{2,2}(B_1^+,y^b)\right)} + \|f\|^2_{C^0\left((0,T);L^2(B_1')\right)}\right), \end{aligned} \tag{6.7}$$

where in the last inequality we applied the trace theorem for Weighted Sobolev spaces on g^0. From (6.7) we obtain

$$\begin{aligned} \|u_m(.,t)\|^2_{W^{2,2}(B_1^+,y^b)} &\le \sup_{1\le i\le m} \|u_{i,m}\|^2_{W^{2,2}(B_1^+,y^b)} \\ &\le C\left(\|g^0\|^2_{W^{2,2}(B_1^+,y^b)} + \|g\|^2_{C^0\left((0,T);W^{2,2}(B_1^+,y^b)\right)} + \|f\|^2_{C^0\left((0,T);L^2(B_1')\right)}\right), \end{aligned} \tag{6.8}$$

for all $t \in [0,T]$. This shows that the sequence $\{u_m\}$ is bounded in $W^{2,2}(B_1^+, y^b)$ uniformly in t and m.

To prove the second part of the theorem, we first observe that (6.4) also gives

$$\begin{aligned} H_{i,m}[u_{i,m}] &\le E_{i,m}[u_{i-1,m}] - E_{i,m}[u_{i,m}] \\ &= \int_{B_1'}(f_{i-1,m} - f_{i,m})(u_{i-1,m} - h_{i-1,m})dx + \int_{B_1'}(f_{i-1,m} - f_{i,m})h_{i-1,m}dx \\ &+ E_{i-1,m}[u_{i-1,m}] - E_{i,m}[u_{i,m}] \\ &\le \|h\|_{C^0((0,T);L^\infty(B_1'))}\int_{B_1'}(f_{i,m} - f_{i-1,m})dx + E_{i-1,m}[u_{i-1,m}] - E_{i,m}[u_{i,m}]. \end{aligned} \tag{6.9}$$

The second inequality follows from the assumption that f nondecreasing in the time variable.

Therefore,

$$
\begin{aligned}
&||v_m||^2_{L^2\left((0,T],L^2(B_1^+,y^b)\right)}\\
&= \Sigma_{i=1}^m \int_{(i-1)\tau_m}^{i\tau_m}\int_{B_1^+} y^b v_{i,m}^2(x,y)dxdydt\\
&= \frac{1}{\tau_m}\Sigma_{i=1}^m \int_{(i-1)\tau_m}^{i\tau_m}\left(\int_{B_1^+} y^b(u_{i,m}-u_{i-1,m})^2dxdy\right)\\
&\le 2\|h\|_{C^0((0,T);L^\infty(B_1'))}\Sigma_{i=1}^m\left(\int_{B_1'}(f_{i,m}-f_{i-1,m})dx\right)+2\Sigma_{i=1}^m\left(E_{i-1,m}[u_{i-1,m}]-E_{i,m}[u_{i,m}]\right)\\
&= 2\|h\|_{C^0((0,T);L^\infty(B_1'))}\int_{B_1'}(f_{m,m}-f_{0,m})dx+2\big(E_{0,m}[g^0]-E_{m,m}[u_{m,m}]\big)\\
&\le C\left(\|h\|^2_{C^0((0,T);L^\infty(B_1'))}+\|f\|_{C^0((0,T);L^2(B_1'))}+\|g^0\|^2_{W^{2,2}(B_1^+,y^b)}\right).
\end{aligned}
\tag{6.10}
$$

where in the second equation we applied Fubini's theorem, and in the first inequality we used the estimate (6.9). □

We now prove the following convergence result.

Theorem 6.2 *Let $\{u_m\}$ be the sequence of functions defined in (4.10). Then, up to a subsequence, u_m converges to a function u_0 weakly in $W^{2,2}(B_1^+,y^b)$ and strongly in $W^{1,2}(B_1^+,y^b)$ as $m\to\infty$ uniformly in t. Moreover, $(u_m)_t$ converges to $(u_0)_t$ weakly in $L^2\left((0,T];L^2(B_1^+,y^b)\right)$. Finally, $\|u_m(\cdot,t)\|_{L^2(B_1^+,y^b)}\to\|u_0(\cdot,t)\|_{L^2(B_1^+,y^b)}$ in $C^\alpha([0,T])$ as $m\to\infty$, for all $\alpha\in(0,1/2]$.*

Proof. From (6.8) and (6.10), we infer that the sequence $\{u_m\}$ is bounded in $W^{2,2}(B_1,y^b)$ uniformly in t and m, and $\{v_m\}$ is bounded in $L^2\left((0,T];L^2(B_1,y^b)\right)$ uniformly in m. By standard compactness arguments, their exists a function u_0 with $u_0\in W^{2,2}(B_1,y^b)$ for all $t\in(0,T]$, and $(u_0)_t\in L^2\left((0,T];L^2(B_1,y^b)\right)$ such that, for a sub-sequence still denoted by $\{u_m\}$,

$$
u_m\to u_0 \text{ weakly in } W^{2,2}(B_1,y^b) \text{ and strongly in } W^{1,2}(B_1,y^b) \text{ uniformly in } t. \tag{6.11}
$$

Moreover,

$$
v_m=\frac{\partial u_m}{\partial t}\to\frac{\partial u_0}{\partial t} \text{ weakly in } L^2((0,T],L^2(B_1^+,y^b)). \tag{6.12}
$$

To prove the last part of the theorem, we observe that for $t_1,t_2\in[0,T]$ we have

$$
\begin{aligned}
\|u_m(.,t_2)-u_m(.,t_1)\|_{L^2(B_1^+,y^b)} &= \left(\int_{B_1^+} y^b\left(\int_{t_1}^{t_2}\frac{\partial u_m}{\partial t}(x,y,t)dt\right)^2dxdy\right)^{\frac12}\\
&\le \left(\int_{B_1^+} y^b(t_2-t_1)\int_{t_1}^{t_2}\left(\frac{\partial u_m}{\partial t}(x,y,t)\right)^2dtdxdy\right)^{\frac12}\\
&= (t_2-t_1)^{\frac12}\|\frac{\partial u_m}{\partial t}\|_{L^2((0,T],L^2(B_1^+,y^b))}\\
&\le CM_1(h,f,g^0)(t_2-t_1)^{\frac12},
\end{aligned}
\tag{6.13}
$$

where

$$
M_1^2(h,f,g^0):=\left(\|h\|^2_{C^0((0,T);L^\infty(B_1'))}+\|f\|_{C^0((0,T);L^2(B_1'))}+\|g^0\|^2_{W^{2,2}(B_1^+,y^b)}\right). \tag{6.14}
$$

The first inequality follows from an application of Hölder inequality and the last inequality follows from (6.10). Combining (6.13) with the a.e. pointwise convergence of $u_m(.,t)$ in $L^2(B_1^+,y^b)$, we can apply Arzela-Ascoli Theorem to conclude that

$$
\|u_m(\cdot,t)\|_{L^2(B_1^+,y^b)}\to\|u_0(\cdot,t)\|_{L^2(B_1^+,y^b)} \text{ in } C^\alpha([0,T]) \text{ as } m\to\infty, \text{ for all } \alpha\in(0,1/2]. \tag{6.15}
$$

□

Next we prove the convergence of the piecewise constant interpolation $\{\tilde{u}_m\}$ defined in (4.11) to the same limit function u_0 found in Theorem 6.2.

Theorem 6.3 *Up to a subsequence, $\{\tilde{u}_m\}$ defined in (4.11) converges weakly in $W^{2,2}(B_1^+, y^b)$ and strongly in $W^{1,2}(B_1^+, y^b)$ as $m \to \infty$ uniformly in t to the limit function u_0 found in Theorem 6.2.*

Proof. Arguing as in the first part of the proof of Theorem 6.2, we can show that, up to a subsequence, $\{\tilde{u}_m\}$ converges weakly in $W^{2,2}(B_1^+, y^b)$ and strongly in $W^{1,2}(B_1^+, y^b)$ as $m \to \infty$ uniformly in t to a function u_1. It remains to show that $u_0 = u_1$. For $(x, y, t) \in (B_1^+ \cup B_1') \times [(i-1)\tau_m, i\tau_m]$,

$$\begin{aligned}
&|u_m(x,y,t) - \tilde{u}_m(x,y,t)| \\
&= \left| u_m(x,y,(i-1)\tau_m) + (t-(i-1)\tau_m)\left(\frac{u_m(x,y,i\tau_m) - u_m(x,y,(i-1)\tau_m)}{\tau_m}\right) - u_m(x,y,i\tau_m)\right| \\
&= \left|\left(\frac{i\tau_m - t}{\tau_m}\right)(u_m(x,y,(i-1)\tau_m) - u_m(x,y,i\tau_m))\right| \\
&\le |u_m(x,y,(i-1)\tau_m) - u_m(x,y,i\tau_m)|\,.
\end{aligned}$$

Moreover, the above estimate still holds if we replace u_m and $\tilde{u}_m$ by their first order and second order partial derivatives respectively. Therefore, recalling the estimate (6.11), we see that $\{u_m - \tilde{u}_m\}$ converges to zero weakly in $W^{2,2}(B_1^+, y^b)$ and strongly in $W^{1,2}(B_1^+, y^b)$ as $m \to \infty$ uniformly in t . This combined with Theorem 6.2 yields the desired result. □

In the following theorem, we show that the limit function u_0 is indeed a weak solution of problem (1.2).

Theorem 6.4 *The limit function u_0 obtained in Theorem 6.2 is a solution of the problem (1.2) in the weak sense given in (4.2).*

Proof. First of all, it is easy to check that the approximating functions u_m, and therefore the limit function u_0, belong to the admissible set $\mathcal{B}$. It remains to show that u_0 satisfies the variational inequality (4.2).

From Lemma 5.3 we know that

$$\int_{B_1^+} y^b \Delta u_{i,m} \Delta(w - u_{i,m}) dxdy + \int_{B_1^+} y^b v_{i,m}(w - u_{i,m}) dxdy - \int_{B_1'} f_{i,m}\ (w - u_{i,m})\ dx \ge 0 \quad (6.16)$$

for all functions w which belong to the admissible set $\mathcal{A}_{i,m}$ defined in (4.8).

Combining (6.16) with the definition (4.11) of $\tilde{u}_m$ and the definition (4.12) of v_m, we infer

$$\int_{t_1}^{t_2}\int_{B_1^+} y^b \Delta \tilde{u}_m \Delta(w - \tilde{u}_m) dxdydt + \int_{t_1}^{t_2}\int_{B_1^+} y^b v_m(w - \tilde{u}_m) dxdydt - \int_{t_1}^{t_2}\int_{B_1'} f_m\ (w - \tilde{u}_m) dxdt \ge 0 \quad (6.17)$$

for all functions w which belong to the admissible set $\mathcal{B}$ defined in (4.1).

Keeping in mind (6.12) and Theorem 6.3, we may pass to the weak limit in (6.17) to obtain

$$\int_{t_1}^{t_2}\int_{B_1^+} y^b \Delta u_0 \Delta(w - u_0) dxdydt + \int_{t_1}^{t_2}\int_{B_1^+} y^b \frac{\partial u_0}{\partial t}(w - u_0) dxdydt - \int_{t_1}^{t_2}\int_{B_1'} f\ (w - u_0) dxdt \ge 0 \quad (6.18)$$

for all functions w in the set (4.1). □

For $0 < t < T$, we define

$$\mu(t) := \frac{\partial u_0}{\partial t}(\cdot, t) + \Delta^2 u_0(\cdot, t) - f\chi_{\{y=0\}}. \tag{6.19}$$

From (6.18), we infer that, for almost every $t \in (0, T)$, $\mu(t)$ is a non-negative measure on $B_1^+ \cup B_1'$. We wish to obtain an estimate of the measures $\mu(t)$.

7 Estimate on the measures $\mu(t)$

We define

$$\mu_m(t) := \mu_{i,m} \quad \text{for} \quad t \in ((i-1)\tau_m, i\tau_m), \tag{7.1}$$

where

$$\mu_{i,m} := \Delta_b^2 u_{i,m} + v_{i,m} - f_{i,m}\chi_{\{y=0\}}. \tag{7.2}$$

From Lemma 5.3, we see that , for all $m \in \mathbb{N}$ and $1 \le i \le m$, $\mu_{i,m}$ defines a non-negative measure on $B_1^+ \cup B_1'$ supported on the thin ball B_1'. As a starting point, we aim to establish the local convergence as $m \to \infty$ of μ_m to μ in the sense

$$\mu_m(\cdot)(K^+ \cup K') \to \mu(K^+ \cup K') \text{ in } L^2((0,T)) \text{ as } m \to \infty, \tag{7.3}$$

for every set $K \subset\subset B_1$.

To obtain a uniform estimate μ_m we observe that, for $K \subset\subset B_1$ we have

$$\begin{aligned} \|\mu_m(\cdot)(K^+ \cup K')\|_{L^2(0,T)} = \int_0^T (\mu_m(t))^2 (K^+ \cup K')dt &= \Sigma_{i=1}^m \int_{(i-1)\tau_m}^{i\tau_m} \mu_{i,m}^2(K^+ \cup K')dt \\ &= \tau_m \sum_{i=1}^m \mu_{i,m}^2(K^+ \cup K'). \end{aligned} \tag{7.4}$$

Thus, our next step will be deriving an estimate for the right hand side of (7.4) uniformly in m. For this purpose, we need to consider the following penalization problems.

7.1 The penalization problem

For every $\epsilon > 0$, we let $u_{i,m}^\epsilon$ be the unique minimizer of the functional

$$\begin{aligned} J_{i,m}^\epsilon[w] =& \frac{1}{2}\int_{(B_1)^+} y^b(\Delta_b w)^2 dxdy + \frac{1}{2\tau_m}\int_{(B_1)^+} y^b(w - u_{i-1,m})^2 dxdy \\ &- \int_{B_1'} f_{i,m}u_{i,m}^\epsilon dx - \int_{B_1'} \gamma_\epsilon(w-h)dx \end{aligned} \tag{7.5}$$

over the set of functions $w \in W^{2,2}(B_1^+, y^b)$ satisfying $w = g_{i,m}$, $w_\nu = 0$ on $(\partial B_1)^+$, and $w_y = 0$ on B_1'. Here, γ_ϵ is a non-positive smooth function such that its smooth derivative β^ϵ satisfies:

$$\beta^\epsilon \ge 0, \quad (\beta^\epsilon)' \le 0, \quad \beta^\epsilon(s) = 0 \text{ for } s \ge 0, \text{ and } \beta^\epsilon(s) = -\epsilon - \frac{s}{\epsilon} \text{ for } s \le -2\epsilon^2. \tag{7.6}$$

It is easy to check that $u_{i,m}^\epsilon$ is a solution of the penalized problem:

$$\begin{cases} \Delta_b^2 w^\epsilon + \frac{1}{\tau_m}(w^\epsilon - u_{i-1,m}) = 0 & \text{in } B_1^+, \\ w^\epsilon = g_{i,m},\ w_\nu^\epsilon = 0 & \text{on } (\partial B_1)^+, \\ w_y^\epsilon = 0 & \text{on } B_1', \\ \lim_{y\to 0} y^b(\Delta_b w^\epsilon)_y = f_{i,m} + \beta^\epsilon(w^\epsilon - h_{i,m}) & \text{on } B_1'. \end{cases} \tag{7.7}$$

We aim to show that a subsequence of $\{u_{i,m}^\epsilon\}$ converges weakly in $W^{2,2}(B_1^+, y^b)$ to the minimizer $u_{i,m}$.

Theorem 7.1 *The sequence $\{u^\epsilon_{i,m}\}$ has a subsequence which converges weakly in $W^{2,2}(B_1^+, y^b)$ and strongly in $W^{1,2}(B_1^+, y^b)$ to $u_{i,m}$, the minimizer of (4.7).*

Proof. By (7.7) we have

$$\int_{B_1^+} y^b \Delta_b u^\epsilon_{i,m} \Delta_b \eta dxdy + \frac{1}{\tau_m}\int_{B_1^+} y^b (u^\epsilon_{i,m} - u_{i-1,m})\ \eta dxdy - \int_{B_1'} f_{i,m}\eta dx = \int_{B_1'} \beta^\epsilon(u^\epsilon_{i,m} - h_{im})\ \eta dx, \tag{7.8}$$

for all functions $\eta \in W^{2,2}(B_1^+, y^b)$ such that $\eta = 0,\ \eta_\nu = 0$ on $(\partial B_1)^+$, and $\eta_y = 0$ on B_1'. In particular, plugging $\eta = u^\epsilon_{i,m} - u_{i,m}$ in (7.8) we obtain

$$\begin{aligned}
&\int_{B_1^+} y^b \Delta_b u^\epsilon_{i,m} \Delta_b (u^\epsilon_{i,m} - u_{i,m}) dxdy + \frac{1}{\tau_m}\int_{B_1^+} y^b (u^\epsilon_{i,m} - u_{i,m})^2 dxdy \\
&+ \frac{1}{\tau_m}\int_{B_1^+} y^b (u_{i,m} - u_{i-1,m})\ (u^\epsilon_{i,m} - u_{i,m}) dxdy - \int_{B_1'} f_{i,m}(u^\epsilon_{i,m} - u_{i,m}) dx \\
&= \int_{B_1'} \beta^\epsilon(u^\epsilon_{i,m} - h_{im})\ (u^\epsilon_{i,m} - u_{i,m}) dx \\
&= \int_{B_1'} \beta^\epsilon(u^\epsilon_{i,m} - h_{im})\ (u^\epsilon_{i,m} - h_{im}) dx + \int_{B_1'} \beta^\epsilon(u^\epsilon_{i,m} - h_{im})\ (h_{im} - u_{i,m}) dx \\
&\le 0,
\end{aligned} \tag{7.9}$$

where the inequality follows since $\beta^\epsilon \ge 0$, $u_{i,m} \ge h_{im}$, and $(u^\epsilon_{i,m} - h_{im})\beta^\epsilon(u^\epsilon_{i,m} - h_{im}) \le 0$. Thus, we have

$$\begin{aligned}
&\int_{B_1^+} y^b (\Delta_b u^\epsilon_{i,m})^2 dxdy + \frac{1}{\tau_m}\int_{B_1^+} y^b (u^\epsilon_{i,m} - u_{i,m})^2 dxdy \\
&\le \int_{B_1^+} y^b \Delta_b u^\epsilon_{i,m} \Delta_b u_{i,m} dxdy - \frac{1}{\tau_m}\int_{B_1^+} y^b (u_{i,m} - u_{i-1,m})\ (u^\epsilon_{i,m} - u_{i,m}) dxdy \\
&+ \int_{B_1'} f_{i,m}(u^\epsilon_{i,m} - u_{i,m}) dx.
\end{aligned} \tag{7.10}$$

Applying Young's inequality on the right hand side of (7.10) we obtain

$$\begin{aligned}
&\int_{B_1^+} y^b (\Delta_b u^\epsilon_{i,m})^2 dxdy + \frac{1}{\tau_m}\int_{B_1^+} y^b (u^\epsilon_{i,m} - u_{i,m})^2 dxdy \\
&\le 2\int_{B_1^+} y^b (\Delta_b u_{i,m})^2 dxdy + \frac{1}{\tau_m}\int_{B_1^+} y^b (u_{i,m} - u_{i-1,m})^2 dxdy + 3\int_{B_1'} f^2_{i,m} dx + \int_{B_1'} u^2_{i,m} dx \\
&\le C\left(\|g\|^2_{C^0\left((0,T);W^{2,2}(B_1^+,y^b)\right)} + \|f\|^2_{C^0\left((0,T);L^2(B_1')\right)} + \|h\|^2_{C^0((0,T);L^\infty(B_1'))} + \|g^0\|^2_{W^{2,2}(B_1^+,y^b)} \right)
\end{aligned} \tag{7.11}$$

for τ_m small enough. The last inequality follows from estimates (6.6), (6.7), (6.10), and the trace theorem for weighted Sobolev spaces. Noticing that (5.6) holds true for $w = u^\epsilon_{i,m}$, we conclude

$$\|u^\epsilon_{i,m}\|^2_{W^{2,2}(B_1^+,y^b)} \le CM(g^0, g, f, h), \tag{7.12}$$

for some constant C independent of ϵ and m. Here,

$$M(g^0, g, f, h) := \|g\|^2_{C^0\left((0,T);W^{2,2}(B_1^+,y^b)\right)} + \|f\|^2_{C^0\left((0,T);L^2(B_1')\right)} + \|h\|^2_{C^0((0,T);L^\infty(B_1'))} + \|g^0\|^2_{W^{2,2}(B_1^+,y^b)}. \tag{7.13}$$

Thus, the sequence $\{u^\epsilon_{i,m}\}$ is uniformly bounded in $W^{2,2}(B_1^+, y^b)$, and up to a subsequence, u^ϵ_{im} converges weakly in $W^{2,2}(B_1^+, y^b)$ and strongly in $W^{1,2}(B_1^+, y^b)$ to a function $\tilde{u}_{i,m}$.

We claim that $\tilde{u}_{i,m} = u_{i,m}$. To prove our claim, we show that $\tilde{u}_{i,m}$ is a minimizer of (4.7), and then the result follows by the uniqueness of the minimizer. Plugging $u^\epsilon_{i,m} - u_{i-1,m}$ for η in (7.8) we obtain

$$\begin{aligned}
&\int_{B_1^+} y^b \Delta_b u^\epsilon_{i,m} \Delta_b (u^\epsilon_{i,m} - u_{i-1,m}) dxdy + \frac{1}{\tau_m} \int_{B_1^+} y^b (u^\epsilon_{i,m} - u_{i-1,m})^2 dxdy - \int_{B_1'} f_{i,m}(u^\epsilon_{i,m} - u_{i-1,m}) dx \\
&= \int_{B_1'} \beta^\epsilon(u^\epsilon_{i,m} - h_{im}) \ (u^\epsilon_{i,m} - u_{i-1,m}) dx. \\
&= \int_{B_1'} \beta^\epsilon(u^\epsilon_{i,m} - h_{im}) \ (u^\epsilon_{i,m} - h_{im}) dx + \int_{B_1'} \beta^\epsilon(u^\epsilon_{i,m} - h_{im}) \ (h_{im} - u_{i-1,m}) dx.
\end{aligned} \tag{7.14}$$

Since $\beta^\epsilon \geq 0$, $u_{i-1,m} \geq h_{im}$, and $(u^\epsilon_{i,m} - h_{im})\beta^\epsilon(u^\epsilon_{i,m} - h_{im}) \leq 0$, we see that both integrals in the last line of (7.14) are non-positive. Therefore,

$$\begin{aligned}
&\left| \int_{B_1'} \beta^\epsilon(u^\epsilon_{i,m} - h_{im}) \ (u^\epsilon_{i,m} - h_{im}) dx \right| \\
&\leq \left| \int_{B_1^+} y^b \Delta_b u^\epsilon_{i,m} \Delta_b (u^\epsilon_{i,m} - u_{i-1,m}) dxdy - \int_{B_1'} f_{i,m}(u^\epsilon_{i,m} - u_{i-1,m}) dx \right| \\
&\leq CM(g^0, g, f, h),
\end{aligned}$$

Where $M(g^0, g, f, h)$ is as defined in (7.13). The last inequality follows from the uniform estimates (6.7) and (7.11), and the trace theorem. Then, for any fixed small $\delta > 0$,

$$\begin{aligned}
\delta \frac{\delta - \epsilon^2}{\epsilon} H^n \left(\{ u^\epsilon_{i,m} < h_{im} - \delta \} \right) &\leq - \int_{B_1'} \beta^\epsilon(u^\epsilon_{i,m} - h_{im}) \ (u^\epsilon_{i,m} - h_{im}) dx \\
&\leq CM(g^0, g, f, h).
\end{aligned}$$

From this we infer that $H^n\big(\{u^\epsilon_{i,m} < h_{im} - \delta\}\big)$ converges to 0 as $\epsilon \to 0$. This immediately implies that $\tilde{u}_{i,m} \geq h_{im}$ a.e. on B_1', and therefore, $\tilde{u}_{i,m}$ belongs to the admissible set $\mathcal{A}_{i,m}$ defined in (4.8). Now we observe that if we apply Young's inequality to the first two terms only on the right hand side of (7.10), then we may reach $J_{i,m}[u^\epsilon_{i,m}] \leq J_{i,m}[u_{i,m}]$. By the property of weak convergence, going to the weak limit we obtain

$$J[\tilde{u}_{i,m}] \leq J[u_{i,m}].$$

□

We denote

$$\mu^\epsilon_{i,m} := \Delta_b^2 u^\epsilon_{i,m} + \frac{1}{\tau_m}(u^\epsilon_{i,m} - u_{i-1,m}) - f_{i,m}\chi_{\{y=0\}}. \tag{7.15}$$

From (7.8) we see that $\mu^\epsilon_{i,m}$ are non-negative finite measures on $B_1^+ \cup B_1'$. Now we prove the following convergence result of the measures $\mu^\epsilon_{i,m}$.

Lemma 7.2 *The sequence of measures $\{\mu^\epsilon_{i,m}\}$ converges to the measure $\mu_{i,m}$ as $\epsilon \to 0$. Moreover, for every $K \subset\subset B_1$, there exists a constant $C(K)$ independent of m such that*

$$\tau_m \sum_{i=1}^m (\mu_{i,m}(K^+ \cup K'))^2 < C(K). \tag{7.16}$$

Proof. For $K \subset\subset B_1$, let ζ be a smooth cut-off function such that $\zeta_y = 0$ on B_1', $0 \le \zeta \le 1$ on B_1, and $\zeta \equiv 1$ on K. We compute

$$\begin{aligned}\mu^\epsilon_{i,m}(K^+ \cup K') &\le \int_{B_1^+} y^b \Delta_b u^\epsilon_{i,m} \Delta_b \zeta dxdy + \frac{1}{\tau_m}\int_{B_1^+} y^b (u^\epsilon_{i,m} - u_{i-1,m})\, \zeta dxdy - \int_{B_1'} f_{i,m}\zeta dx \\ &\le \left(\|\Delta_b u^\epsilon_{i,m}\|_{L^2(B_1^+,y^b)} + \frac{1}{\tau_m}\|u^\epsilon_{i,m} - u_{i-1,m}\|_{L^2(B_1^+,y^b)} + \|f_{i,m}\|_{L^2(B_1')} \right) \|\zeta\|_{W^{2,2}(B_1^+,y^b)} \\ &\le \frac{C}{\sqrt{\tau_m}} M(g^0, g, f, h) \|\zeta\|_{W^{2,2}(B_1^+,y^b)},\end{aligned} \tag{7.17}$$

where $M(g^0, g, f, h)$ is as defined in (7.13). In the last inequality we have used the uniform estimates (6.6), (6.7) and (6.10). The constant C is independent of m and ϵ. This shows that the measures $\mu^\epsilon_{i,m}$ are locally uniformly bounded in ϵ. Therefore, for a subsequence, $\mu^\epsilon_{i,m}$ converges weakly to a measure $\bar{\mu}_{i,m}$ on $K^+ \cup K'$ for every $K \subset\subset B_1$. Now we show that $\bar{\mu}_{i,m} = \mu_{i,m}$. For this, let $K \subset\subset B_1$, and let ζ be a smooth function compactly supported in K. Then,

$$\begin{aligned}\int_K \zeta d\tilde{\mu}_{i,m} &= \lim_{\epsilon\to 0}\left(\int_{B_1^+} y^b \Delta_b u^\epsilon_{i,m} \Delta_b \zeta dxdy + \frac{1}{\tau_m}\int_{B_1^+} y^b (u^\epsilon_{i,m} - u_{i-1,m})\, \zeta dxdy - \int_{B_1'} f_{i,m}\zeta dx \right) \\ &= \int_{B_1^+} y^b \Delta_b u_{i,m} \Delta_b \zeta dxdy + \frac{1}{\tau_m}\int_{B_1^+} y^b (u_{i,m} - u_{i-1,m})\, \zeta dxdy - \int_{B_1'} f_{i,m}\zeta dx \\ &= \int_K \zeta d\mu_{i,m},\end{aligned} \tag{7.18}$$

where the first equation follows from the convergence of the measures $\mu^\epsilon_{i,m}$ to the measure $\tilde{\mu}_{i,m}$, and the second equation follows from Theorem 7.1. This shows that $\tilde{\mu}_{i,m} = \mu_{i,m}$ on $K^+ \cup K'$. The estimate (7.16) follows directly from (7.17). □

Corollary 7.3 *For every $K \subset\subset B_1$, we have $\mu_m(\cdot)(K^+ \cup K')$ are bounded in $L^2(0,T)$ uniformely in m. In particular,*

$$\int_0^T (\mu_m(t))^2 (K^+ \cup K') dt \le C(K). \tag{7.19}$$

Proof. Estimate (7.19) follows immediately after employing the estimate (7.16) in (7.4). □

7.2 The convergence of μ_m to μ

Let $K \subset\subset B_1$. From Corollary 7.3, we infer that, for a subsequence,

$$\mu_m(\cdot)(K^+ \cup K') \rightharpoonup \overline{\mu}(\cdot)(K^+ \cup K') \text{ in } L^2(0,T),$$

for some $\overline{\mu}$ a positive measure on $B_1^+ \cup B_1'$ for a.e. t. In particular,

$$\lim_{m\to\infty} \int_0^T \left(\int_{K^+\cup K'} \phi d\mu_m \right) dt = \int_0^T \left(\int_{K^+\cup K'} \phi d\overline{\mu} \right) dt. \tag{7.20}$$

On the other hand, from the proof of Theorem 6.18, we infer that the limit on the left hand side of (7.20) is equal to $\int_0^T \left(\int_{K^+\cup K'} \phi d\mu(t) \right) dt$. Therefore, $\overline{\mu}(t)(K^+ \cup K') = \mu(t)(K^+ \cup K')$ for a.e. $t \in (0,T)$.

We have derived the following estimate for the measures $\mu(t)$.

Theorem 7.4 *For $0 < t < T$, let $\mu(t)$ be the measure defined in (6.19). For every $K \subset\subset B_1$ we have*

$$\int_0^T \left(\mu(t)(K^+ \cup K')\right)^2 dt \leq C(K), \tag{7.21}$$

for some constant $C(K)$.

We now turn our attention to the regularity analysis of the solution u_0 found in Section 6. Our strategy restricts to the cases of dimension $n = 1$ and $n = 2$, and it incorporates adaptation of some regularity results for the elliptic version of our problem. First, we recall the following compact embedding theorem.

Theorem 7.5 *Let Ω be an open domain in $\mathbb{R}^k$ for $k = 2$ or $k = 3$. Then,*

$$H_0^2(\Omega, y^b) \hookrightarrow C^{0,\gamma}(\Omega) \quad \text{for all } 0 < \gamma < 1, \qquad \text{when } k = 2, \tag{7.22}$$

and

$$H^2(\Omega, y^b) \hookrightarrow C^{0,\gamma}(\Omega) \quad \text{for all } 0 < \gamma \leq 1/2, \qquad \text{when } k = 3. \tag{7.23}$$

We start with studying the regularity of the minimizers $u_{i,m}$.

8 Regularity of the minimizers $u_{i,m}$

We aim to prove the following regularity result for the minimizers $u_{i,m}$ of the functional (4.7).

Theorem 8.1 *The minimizer $u_{i,m}$ of the functional (4.7) is in $C^{1,1}(B_\rho^+ \cup B'_\rho) \cap W^{3,2}(B_\rho^+, y^b)$ for all $0 < \rho < 1$. In particular,*

$$\begin{aligned} \|u_{i,m}\|_{C^{1,1}(B^+_{\rho/2} \cup B'_{\rho/2})} \leq C(\rho, n, b) \Big(&\|\Delta h_{im}\|_{L^\infty(B'_1)} + \|u_{i,m}\|_{W^{1,2}(B^+_{2\rho}, y^b)} \\ &+ \|v_{i,m}\|_{L^2(B^+_{2\rho}, y^b)} + \|f_{i,m}\|_{L^\infty(B'_1)} \Big), \end{aligned} \tag{8.1}$$

and

$$\begin{aligned} \|u_{i,m}\|_{W^{3,2}(B_{\rho/8}, y^b)} \leq C(\rho, n, b) \Big(&\|\Delta h_{im}\|_{L^\infty(B'_1)} + \|u_{i,m}\|_{W^{1,2}(B^+_{2\rho}, y^b)} \\ &+ \|v_{i,m}\|_{L^2(B^+_{2\rho}, y^b)} + \|f_{i,m}\|_{L^\infty(B'_1)} \Big). \end{aligned} \tag{8.2}$$

To prove Theorem 8.1, we first need the following set up. Let $\psi_{i,m}$ be the unique solution of the following problem on the entire unit ball B_1.

$$\begin{cases} \Delta_b^2 \psi_{i,m} = v_{i,m} & \text{in } B_1, \\ \psi_{i,m} = 0, \ \Delta_b \psi_{i,m} = 0 & \text{on } \partial B_1, \end{cases} \tag{8.3}$$

where $v_{i,m}$ is extended to the entire ball B_1 by even symmetry. From the standard elliptic regularity theory (see for instance [23, Corollary 2.21]) we know that

$$\|\psi_{im}\|_{W^{4,2}(B_1, y^b)} \leq C \|v_{i,m}\|_{L^2(B_1^+, y^b)}. \tag{8.4}$$

Moreover, employing the Compact Embedding Theorem 7.5, we conclude that, for $n = 1$ or $n = 2$, $\psi_{im} \in C^{1,1}(B_1)$ and

$$\|\psi_{i,m}\|_{C^{1,1}(B_1)} \leq C \|v_{i,m}\|_{L^2(B_1^+, y^b)}. \tag{8.5}$$

Next, denote

$$W_{i,m} = u_{i,m} + \psi_{i,m}, \tag{8.6}$$

and

$$\Phi_{i,m} = h_{i,m} + \psi_{i,m}. \tag{8.7}$$

Recalling Lemma 5.3, we see that the function $W_{i,m}$ is a solution of the problem

$$\begin{cases} \Delta_b^2 W_{i,m} = 0 & \text{in } B_1^+ \\ W_{i,m} = g_{i,m}, (W_{i,m})_\nu = 0 & \text{on } (\partial B_1)^+ \\ W_{i,m} \geq \Phi_{i,m} & \text{on } B_1' \\ (W_{i,m})_y = 0, \lim_{y\to 0} y^b (\Delta_b W_{i,m})_y \geq f_{i,m} & \text{on } B_1' \\ \lim_{y\to 0} y^b (\Delta_b W_{i,m})_y = f_{i,m} & \text{on } B_1' \cap \{W_{i,m} > \psi_{i,m}\}, \end{cases} \tag{8.8}$$

Recalling (8.5), we observe that obstacle $\Phi_{i,m} = h_{im} + \psi_{i,m}$ associated with the problem (8.8) is in $C^{1,1}(B_1')$. In [18], we considered with collaborators problem (8.8) in the case of a $C^{1,1}$-obstacle. More precisely, problem (8.8) is the localization of the extension to $\mathbb{R}^{n+1}_+$ of the following non-local obstacle-type problem on the lower dimensional space $\mathbb{R}^n$:

$$\begin{cases} w_{i,m} \geq \Phi_{i,m} & \text{in } \mathbb{R}^n, \\ (-\Delta)^s w_{i,m} \geq f_{i,m} & \text{in } \mathbb{R}^n, \\ (-\Delta)^s w_{i,m} = f_{im} & \text{in } \{w_{i,m} > \Phi_{i,m}\}, \end{cases} \tag{8.9}$$

where $s = (3-b)/2$ which is a value in $(1,2)$.

In [18], our objective is to study the well posedeness and the regularity of a solution for problem (8.9). We establish the existence of a solution $w_{i,m}$ by minimizing the corresponding functional, and we show that $(-\Delta)^s w_{i,m} - f_{i,m}$ defines a non-negative measure on $\mathbb{R}^n$. Then, by writing the solution and its first and second order partial derivatives in terms of Riesz potentials of our measure's local restrictions, we use tools from potential theory, such as a *continuity principle* and a *maximum principle* to establish the $C^{1,1}(\mathbb{R}^n) \cap H^{1+s}(\mathbb{R}^n)$-regularity of $w_{i,m}$. Finally, using an extension theorem first proved by Yang in [37] (see also [11]) , we extend our solution $w_{i,m}$ of the lower dimensional non-local problem (8.9) to the function W_{im}, solution for problem (8.8). It turns out that $(-\Delta)^s w_{i,m} - f_{i,m}$ is equal to $\mu_{i,m}$, the measure defined in (7.2). Then, obtaining similar representations in terms of Riesz potential, but this time on the upper half higher dimensional space, we established the following regularity results.

Theorem 8.2 *[18, Theorem 9.2] $W_{i,m}$ is in $C^{1,1}_{\text{loc}}(B_1^+ \cup B_1')$. Moreover,*

$$\|W_{i,m}\|_{C^{1,1}(B^+_{\rho/2} \cup B'_{\rho/2})} \leq C(\rho, n, b) \left(\|\Delta \Phi_{i,m}\|_{L^\infty(B_1')} + \|u_{i,m}\|_{W^{1,2}(B^+_{2\rho}, y^b)} + \|f_{i,m}\|_{L^\infty(B_1')} \right), \tag{8.10}$$

for all $0 < \rho < 1$.

And,

Theorem 8.3 *[18, Corollary 9.4] $W_{i,m}$ is locally in $W^{3,2}(B_1, y^b)$. Moreover,*

$$\|W_{i,m}\|_{W^{3,2}(B_{\rho/8}, y^b)} \leq C(\rho, n, b) \left(\|\Delta \Phi_{i,m}\|_{L^\infty(B_1)} + \|u_{i,m}\|_{W^{1,2}(B_{2\rho}, y^b)} + \|f_{i,m}\|_{L^\infty(B_1')} \right). \tag{8.11}$$

for all $0 < \rho < 1$.

Proof.[*Proof of Theorem 8.1*] The proof follows immediately from the definition (8.6) of $W_{i,m}$, the estimate (8.5), Theorem 8.2, and Theorem 8.3. □

9 Regularity of the solution of problem (1.2)

We are now ready to prove the following regularity result.

Theorem 9.1 *Let $0<\rho<1$. The solution u_0 of problem (1.2) is in $L^2\left((0,T];W^{2,\infty}(B^+_\rho\cup B'_\rho,y^b)\right)$. Moreover, u_0 is in $L^2\left((0,T];W^{3,2}(B^+_\rho,y^b)\right)$. In particular,*

$$\begin{aligned}&\|u_0\|^2_{L^2\left((0,T];W^{2,\infty}(B^+_{\rho/2}\cup B'_{\rho/2},y^b)\right)}\\&\leq C(\rho)\left(\|h\|^2_{C^0\left((0,T];C^{1,1}(B'_1)\right)}+\|g\|^2_{C^0\left((0,T];W^{2,2}(B^+_1,y^b)\right)}+\|f\|^2_{C^0\left((0,T);L^2(B'_1)\right)}+\|g^0\|_{W^{2,2}(B_1,y^b)}\right),\end{aligned}\tag{9.1}$$

and

$$\begin{aligned}&\|u_0\|^2_{L^2\left((0,T];W^{3,2}(B^+_{\rho/8},y^b)\right)}\\&\leq C(\rho)\left(\|h\|^2_{C^0\left((0,T];C^{1,1}(B'_1)\right)}+\|g\|^2_{C^0\left((0,T];W^{2,2}(B^+_1,y^b)\right)}+\|f\|^2_{C^0\left((0,T);L^2(B'_1)\right)}+\|g^0\|_{W^{2,2}(B_1,y^b)}\right),\end{aligned}\tag{9.2}$$

where $C(\rho)$ depends only on ρ, b, T, and the dimension.

Proof. Squaring both sides of the estimate (8.1) and integrating on $(0,T)$ we obtain

$$\begin{aligned}&\|u_m\|^2_{L^2\left((0,T];W^{2,\infty}(B^+_{\rho/2}\cup B'_{\rho/2},y^b)\right)}\\&=\Sigma_{i=1}^m\int_{(i-1)\tau_m}^{i\tau_m}\|u_{i,m}\|^2_{C^{1,1}(B^+_{\rho/2}\cup B'_{\rho/2})}dt\\&\leq C(\rho)\Sigma_{i=1}^m\int_{(i-1)\tau_m}^{i\tau_m}\left(\|\Delta h_{im}\|^2_{L^\infty(B'_1)}+\|u_{i,m}\|^2_{W^{1,2}(B^+_{2\rho},y^b)}+\|v_{i,m}\|^2_{L^2(B^+_{2\rho},y^b)}\right)\\&\leq C(\rho)\left(\|h\|^2_{C^0\left((0,T];C^{1,1}(B'_1)\right)}+\|g\|^2_{C^0\left((0,T];W^{2,2}(B^+_1,y^b)\right)}+\|f\|^2_{C^0\left((0,T);L^2(B'_1)\right)}+\|g^0\|_{W^{2,2}(B_1,y^b)}\right),\end{aligned}\tag{9.3}$$

where $C(\rho)$ depends only on ρ, b, T, and the dimension. In the last inequality, we used estimates (6.8) and (6.10), and the assumed $L^\infty\left((0,T];C^{1,1}(B'_1)\right)$-regularity of the obstacle h. Arguing in a similar way, we reach the second part of the theorem. □

Our last regularity result will be a Hölder continuity estimate for the solution u_0 in both of the space and time variables. We will need the following interpolation inequality.

Lemma 9.2 *Suppose $g\in C^{1,1}_c(B^+_1\cup B'_1)$. Then*

$$\|g\|_{L^\infty(B^+_1\cup B'_1)}\leq C\|g(x,y)\|^{\frac12}_{W^{2,2}(B_1,y^b)}\,\|g(x,y)\|^{\frac12}_{L^2(B_1,y^b)},\tag{9.4}$$

Proof. Applying [13, Theorem 1.6] to the function g we obtain

$$\|g\|_{L^\infty(B^+_1\cup B'_1)}\leq C\|\nabla g\|_{L^2(B_1,y^b)}.\tag{9.5}$$

An integration by parts on the right hand side of (9.5) followed by an application of Holder inequality yield the desired estimate. □

Theorem 9.3 *The solution u_0 is in $C^{\frac{(\beta-\gamma)}{4\beta}}\big((0,T], C^{\gamma}((B_1^+ \cup B_1'))\big)$. Here $0<\gamma<\beta$, and β satisfies $0<\beta<1$ when $n=1$, and $0<\beta\leq 1/2$ when $n=2$.*

Proof. For a fixed $m\in\mathbb{N}$, and for $t_1,t_2\in[0,T]$, we know from (6.13) that

$$\|u_m(\cdot,\cdot,t_2)-u_m(\cdot,\cdot,t_1)\|_{L^2(B_1^+,y^b)}\leq CM_1(h,f,g^0)(t_2-t_1)^{\frac{1}{2}}. \tag{9.6}$$

Also, recalling the estimate (6.8) and applying the Sobolev Embedding Theorem 7.5, we obtain

$$\begin{aligned}\|u_m(\cdot,\cdot,t_2)-u_m(\cdot,\cdot,t_1)\|_{H^{\beta}(B_1^+\cup B_1')} &\leq C(n)\|u_m(\cdot,\cdot,t_2)-u_m(\cdot,\cdot,t_1)\|_{W^{2,2}(B_1^+,y^b)}\\ &\leq CM_2(g^0,g,f),\end{aligned} \tag{9.7}$$

for all $0<\beta<1$ when $n=1$, and for all $0<\beta\leq 1/2$ when $n=2$. Here,

$$M_2^2(g^0,g,f):=\|g^0\|^2_{W^{2,2}(B_1^+,y^b)}+\|g\|^2_{C^0\left((0,T);W^{2,2}(B_1^+,y^b)\right)}+\|f\|^2_{C^0\left((0,T);L^2(B_1')\right)}. \tag{9.8}$$

Then, plugging estimate (9.6) and the second inequality in the estimate (9.7) in the interpolation inequality (9.4), we obtain

$$\begin{aligned}&\|u_m(\cdot,\cdot,t_2)-u_m(\cdot,\cdot,t_1)\|_{L^{\infty}(B_1^+,y^b)}\\ &\leq C\left(M_2(g^0,g,f)\right)^{1/2}\left(M_1(h,f,g^0)(t_2-t_1)^{\frac{1}{2}}\right)^{1/2}\\ &\leq CM(g^0,g,f,h)(t_2-t_1)^{1/4},\end{aligned} \tag{9.9}$$

where $M(g^0,g,f,h)$ is as defined in (7.13). For the simplicity of the writing, we denote

$$g_m(x,y):=u_m(x,y,t_2)-u_m(x,y,t_1). \tag{9.10}$$

Also, for $(x,y)\in\mathbb{R}^n\times[0,\infty)$, we let $z:=(x,y)$. Now, Let β satisfy $0<\beta<1$ when $n=1$, and $0<\beta\leq 1/2$ when $n=2$. For $0<\gamma<\beta$ we write We write

$$\begin{aligned}\frac{|g_m(z_1)-g_m(z_2)|}{|z_1-z_2|^{\gamma}} &= \frac{|g_m(z_1)-g_m(z_2)|^{\gamma/\beta}}{|z_1-z_2|^{\gamma}}|g_m(z_1)-g_m(z_2)|^{1-\gamma/\beta}\\ &= \left(\frac{|g_m(z_1)-g_m(z_2)|}{|z_1-z_2|^{\beta}}\right)^{\gamma/\beta}|g_m(z_1)-g_m(z_2)|^{1-\gamma/\beta}\\ &\leq 2\|g_m\|^{\gamma/\beta}_{H^{0,\beta}(B_1^+)}\,\|g_m\|^{1-\gamma/\beta}_{L^{\infty}(B_1^+,y^b)}\\ &\leq CM(g^0,g,f,h)\,(t_2-t_1)^{\frac{1}{4}(1-\gamma/\beta)},\end{aligned}$$

where in the last inequality we used the estimates (9.7) and (9.9). From this, we infer that the sequence $\{u_m\}$ is bounded in the $C^{\gamma,\frac{(\beta-\gamma)}{4\beta}}$-norm uniformly in m. Applying Arzela-Ascoli's Theorem, we reach the desired result. □

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