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An elliptic equation with indefinite nonlinearities and exponential critical growth in $\mathbb{R}^2$

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Abstract. In this paper we study the existence, nonexistence and multiplicity of positive solutions for a class of semilinear elliptic problems involving indefinite nonlinearities with exponential critical growth of Trudinger-Moser type. The main hypothesis is that the indefinite term is the product of a weight function, having a thick zero set, and a nonlinear function with exponential critical growth satisfying a version of the Ambrosetti-Rabinowitz superlinear condition. Our proofs rely on a variational approach and sub-supersolution methods.

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1. Introduction

In this paper we consider a class of semilinear elliptic problems involving a sign-changing weight function and a nonlinearity with exponential critical growth. More precisely, we study the existence, nonexistence and multiplicity of positive solutions for the Dirichlet problem

$$\begin{cases} -\Delta u = \lambda u + W(x) f(u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{P_λ}$$

where $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain, $W \in C(\overline{\Omega})$ is a weight function that changes sign, $f \in C(\mathbb{R})$ has exponential critical growth and $\lambda \geq \lambda_1$, with λ_1 denoting the first eigenvalue of the operator $-\Delta$ under the Dirichlet boundary conditions.

The existence of positive solutions for indefinite elliptic problems like (P_λ) has already been established in various contexts in the dimension $n \geq 3$. If the domain Ω is a compact manifold of dimension $n \geq 3$, the critical exponent case $f(s) = |s|^{2^*-2}s$, where $2^* = 2n/(n-2)$, arises in the prescribed scalar curvature

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problem (see [23]). For manifolds carrying scalar-flat metrics, sufficient conditions for the existence of positive solutions were given in [18]. In [27] the author studied Problem (P_λ) on a compact manifold with homogeneous nonlinearities of the form $f(s) = |s|^{p-2}s, \quad p > 2$, by using bifurcation arguments. Results for more general nonlinearities were obtained by Alama-Tarantello [4]. After that, many authors have studied indefinite semilinear elliptic problems when the nonlinear term $f(s)$ has polynomial growth (see [3–5,8,9,12–14,17,20,21,31] and references therein). Indefinite problems of type (P_λ) involving critical growth in the Sobolev case were treated by various authors, see for instance [4,12,20,21].

We would like to mention the articles [3–5], where the authors studied Problem (P_λ) in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, under the hypothesis

$$\lim_{s\to+\infty} \frac{f(s)}{s^{p-1}} = 1, \quad 2 < p < 2^*, \tag{1.1}$$

which plays an important role in order to verify the Palais-Smale (henceforth denoted by (PS)) condition. Recently, assuming that the weight function W has a thick zero set, the authors have improved the hypothesis (1.1) by assuming that the nonlinearity $f(s)$ has subcritical growth and satisfies the Ambrosetti-Rabinowitz superlinear condition, see [7].

Our main objective in this work is to establish a version for dimension two of the main result in [4] (see also [25]) when the nonlinearity $f(s)$ has exponential critical growth and satisfies a version of the Ambrosetti-Rabinowitz superlinear condition. We recall that $f(s)$ has *exponential critical growth* at $+\infty$ if it satisfies the following condition:

(f_{α_0}) there exists $\alpha_0 > 0$ such that

$$\lim_{s\to+\infty} \frac{f(s)}{e^{\alpha s^2}} = \begin{cases} 0 & \forall\, \alpha > \alpha_0 \\ +\infty & \forall\, \alpha < \alpha_0. \end{cases}$$

It is worthwhile mentioning that Judovic [22], Pohozaev [29] and Trudinger [33] have extended the classical Sobolev inequalities for bounded domains $\Omega \subset \mathbb{R}^2$ by proving that if $u \in H_0^1(\Omega)$ then

$$\int_\Omega e^{\alpha u^2} < \infty, \quad \text{for every} \quad \alpha > 0. \tag{1.2}$$

Posteriorly, a uniform form of inequality (1.2) has been established by Moser [26], namely

$$\sup_{u\in H_0^1(\Omega),\ \|u\|_{H_0^1(\Omega)}\leq 1} \int_\Omega e^{\alpha u^2} < \infty, \quad \text{for every} \quad \alpha \leq 4\pi. \tag{1.3}$$

Moreover, he proved that the supremum in (1.3) is infinity whenever $\alpha > 4\pi$. For related results in a unbounded domain $\Omega \subset \mathbb{R}^2$, see B. Ruf [30]. The hypothesis (f_{α_0}) has been motivated by the above mentioned results (see also [1,15] and references therein).

Let $F(s) = \int_0^s f(t)dt$ be the primitive of the nonlinear term $f(s)$. In order to establish our main results, we assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying:

(f_1) There exist $q > 2$ and $a \neq 0$ such that $\lim_{s\to 0^+} F(s)/s^q = a$;
(f_2) $\lim_{s\to+\infty} sf(s)/F(s) = +\infty$;
(f_3) $\lim_{s\to+\infty} sf(s)/e^{\alpha_0 s^2} = +\infty$;
(f_4) f is locally Lipschitz continuous.

Denoting by φ_1 the positive normalized eigenfunction in $H_0^1(\Omega)$ associated with the first eigenvalue λ_1, we suppose that the weight function W belongs to $C(\overline{\Omega})$ and satisfies the hypotheses:

(W_1) $l := -a \int_\Omega W(x)\varphi_1^q > 0$;
(W_2) W changes sign and has a thick zero set, *i.e.*, $\Omega^\pm := \{x \in \Omega : \pm W(x) > 0\} \neq \emptyset$ and $\overline{\Omega^+} \cap \overline{\Omega^-} = \emptyset$.

Setting $\Omega^0 = \{x \in \Omega : W(x) = 0\}$, our main result may by summarized as follows:

Theorem 1.1. *Suppose* (f_{α_0}), $(f_1) - (f_4)$, $(W_1) - (W_2)$ *are satisfied. If* $\partial\Omega^0$ *is smooth, then there exists* $\Lambda > \lambda_1$ *such that:*

i) *For* $\lambda = \lambda_1$, *Problem* (P_λ) *has a positive solution;*
ii) *For every* $\lambda \in (\lambda_1, \Lambda)$, *Problem* (P_λ) *has two ordered positive solutions;*
iii) *For every* $\lambda \in (\Lambda, +\infty)$, *Problem* (P_λ) *does not have a positive solution.*

Remark 1.2. As remarked in the paper [4], the case where $\lambda < \lambda_1$ is somewhat standard. A positive solution in this case can be obtained easily by using variational techniques, provided that the Palais-Smale condition is satisfied.

We quote that there are few results in the same line of [4] involving indefinite nonlinearity with exponential critical growth. In [2], the authors consider an indefinite problem having *exponential subcritical growth* in the whole of $\mathbb{R}^2$, but the nonlinearity is of the form $f(s) = \phi(s)e^s$ with $\phi(s)$ between two powers. In their approach they used bifurcation techniques and a priori estimate, which is more delicate when W changes sign. We also emphasize that Alves *et al.* [6] studied Problem (P_λ) in an exterior domain $\Omega \subset \mathbb{R}^2$ with $f(s)$ having exponential critical growth, but the authors do not obtain multiplicity results.

Next, in order to establish a result of existence for (P_Λ), where Λ is given in Theorem 1.1, we assume the following additional hypotheses on f and W:

(f_5) $f(s) = o(s)$ as $s \to 0$;
(W_3) Every connected component of Ω^0 intersects $\overline{\Omega^+}$.

Theorem 1.3. *Suppose* (f_{α_0}), $(f_1)-(f_5)$, $(W_1)-(W_3)$ *are satisfied,* $\partial\Omega^0$ *is smooth and* f *is nonnegative. Then Problem* (P_Λ) *has a positive solution.*

As observed above, Theorem 1.1 provides a version for dimension two from the result established by Alama-Tarantello [4] when the nonlinear term $f(s)$ has exponential critical growth and satisfies a version of the Ambrosetti-Rabinowitz superlinear condition. It is worthwhile mentioning that if $f(s)$ has exponential subcritical growth, *i.e.*, (f_{α_0}) holds for $\alpha_0 = 0$, then Theorems 1.1 and 1.3 hold under the Ambrosetti-Rabinowitz condition (see Remark 3.3). The hypothesis (f_1) in the primitive $F(s)$ was assumed in [25] for studying indefinite problems in dimension $n \geq 3$. We point out that in earlier results, hypothesis of type (f_1) was imposed on the nonlinearity $f(s)$ instead of $F(s)$. We observe that assumption (f_2), which was introduced in [32] (see also [28]), follows from:

$(\widehat{f_2})$ There exist constants $S_0, M_0 > 0$ such that

$$0 < F(s) \leq M_0 f(s) \quad \text{for every} \quad s \geq S_0.$$

The hypothesis $(\widehat{f_2})$ has been assumed in many articles that deal with exponential critical growth (see [15,16]). Furthermore, note that, under the hypothesis (f_{α_0}), (f_2) is equivalent to the assertion that, for every $\theta > 2$ given, there exists $s_1 = s_1(\theta) > 0$ such that

$$0 < \theta F(s) \leq s f(s) \quad \text{for every} \quad s \geq s_1, \tag{1.4}$$

which is a version of the Ambrosetti-Rabinowitz condition.

The hypothesis (f_3), which has been assumed in the articles [28,34], is used to estimate the minimax level in order to prove that the (PS) condition is satisfied.

We emphasize that in Section 2 (see Thereom 2.4), by applying the Ekeland variational principle, we establish an existence result of nonnegative and nontrivial solution for Problem (P_λ) assuming only the hypotheses (f_{α_0}), (f_1) and (W_1).

We conclude this section by noting that as an application of Theorems 1.1, we obtain the existence, nonexistence and multiplicity of positive solutions for Problem (P_λ) with $f(s) = |s|^{q-2}se^{s^2}$, for $q > 1$.

The paper is written as follows: in Section 2, we prove the existence of a nonnegative solution via a minimization argument. In Section 3, we use the mountain pass theorem without the (PS) condition to obtain a second nonnegative solution. In Section 4, we establish an interval on the parameter λ for which Problem (P_λ) admits positive solution. In Section 5, we present the proof of Theorem 1.1 via an argument of sub-super solution. Finally, in Section 6, we prove Theorem 1.3.

Throughout, $H_0^1(\Omega)$ denotes the Sobolev space endowed with the inner product

$$\langle u, v \rangle = \int_\Omega \nabla u \nabla v, \quad u, v \in H_0^1(\Omega),$$

and the associated norm is represented by $\|\cdot\|$. We use $|\cdot|_p$ to denote the norm of the Lebesgue space $L^p(\Omega)$, $1 \leq p \leq \infty$ and $\langle \cdot, \cdot \rangle_2$ to represent the inner product in $L^2(\Omega)$. The symbols C_i, $i = 1, 2, \ldots$ will denote various constants.

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2. Existence of a nonnegative solution

Throughout we deal with the existence of weak solutions for Problem (P_λ), *i.e.*, $u \in H_0^1(\Omega)$ satisfying

$$\langle u, v\rangle - \lambda\langle u, v\rangle_2 - \int_\Omega W(x)f(u)v = 0 \quad \text{for every} \quad v \in H_0^1(\Omega). \tag{2.1}$$

Note that hypothesis (f_1) implies that $f(0) = 0$ and consequently $u = 0$ is a trivial solution for Problem (P_λ). In order to address the existence of nonnegative solutions for Problem (P_λ), we set $f(s) = 0$ for any $s < 0$ and we consider the Euler-Lagrange functional associated to Problem (P_λ) defined by

$$I_\lambda(u) = \frac{\|u\|^2}{2} - \frac{\lambda}{2}|u^+|_2^2 - \int_\Omega W(x)F(u),\ u \in H_0^1(\Omega), \tag{2.2}$$

where $u^+ = \max\{u, 0\}$. By using the Trudinger-Moser inequality (1.2), it follows from hypotheses (f_{α_0}) and (f_1) that the functional I_λ is well-defined. Furthermore, using standard arguments we may prove that $I_\lambda \in C^1(H_0^1(\Omega), \mathbb{R})$ and its derivative is given by

$$\langle I'_\lambda(u), v\rangle = \langle u, v\rangle - \lambda\langle u^+, v\rangle_2 - \int_\Omega W(x)f(u)v, \quad \text{for every} \quad u, v \in H_0^1(\Omega). \tag{2.3}$$

Since $f(s) = 0$ for $s \le 0$, critical points of I_λ are nonnegative weak solutions of Problem (P_λ).

In this section we establish the existence of a nontrivial and nonnegative solution for Problem (P_λ) under the hypotheses (f_{α_0}), (f_1) and (W_1) for $\lambda > \lambda_1$ with $\lambda - \lambda_1$ sufficiently small. To this end, we first establish some basic results. We denote by $\mathcal{X}_B$ the characteristic function of a measurable set $B \subset \Omega$. Given $u \in H_0^1(\Omega)$ and $\delta > 0$, we define $\Omega_\delta(u) := \{x \in \Omega;\ |u(x)| \le \delta\}$. The next basic result will be used to estimate the functional I_λ on a neighborhood of the origin (for a proof see [25]).

Lemma 2.1. *Suppose that* (W_1) *holds. Then, for every* $\delta > 0$,

$$-a\int_{\Omega_\delta(u)} W(x)\varphi_1^q \to l \text{ as } u \to 0 \text{ strongly in } H_0^1(\Omega). \tag{2.4}$$

The next estimate plays a crucial role in order to verify that the origin is a local minimum of I_λ as well as the geometric hypotheses of the mountain pass theorem in the proof of our results. Given $\varepsilon > 0$, we may invoke condition (f_1) to find $0 < \delta < 1$ such that

$$|F(s) - as^q| \le \varepsilon s^q, \quad \text{for every} \quad 0 \le s \le \delta. \tag{2.5}$$

Moreover, using (f_{α_0}), for $\alpha > \alpha_0$ we can find $C_1 > 0$ such that

$$|F(s)| \le C_1|s|e^{\alpha s^2}, \quad \text{for every} \quad s \ge \delta. \tag{2.6}$$

Lemma 2.2. *Suppose (f_{α_0}), (f_1) and (W_1) are satisfied. Then, there exist $\lambda^* > \lambda_1$ and $\sigma, \rho > 0$ such that, for all $\lambda < \lambda^*$,*

$$I_\lambda(u) \ge \sigma \quad \textit{if} \quad \|u\| = \rho.$$

Proof. Without loss of generality, we can assume that $a > 0$. Otherwise, we replace W and f by $-W$ and $-f$, respectively. Writing $u^+ = t\varphi_1 + z$, with $t \ge 0$ and $z \in \langle\varphi_1\rangle^\perp$, we have $\|u\|^2 = \|u^-\|^2 + \lambda_1 t^2 + \|z\|^2$ and $|u^+|_2^2 = t^2 + |z|_2^2$. Hence, using that $\lambda_2|z|_2^2 \le \|z\|^2$, we obtain

$$I_\lambda(u) \ge \frac{\|u^-\|^2}{2} + \frac{(\lambda_2 - \lambda_1)}{2\lambda_2}\|z\|^2 - C_0(\lambda - \lambda_1)\|u\|^2 - \int_\Omega W(x)F(u), \tag{2.7}$$

where λ_2 is the second eigenvalue of the operator $-\Delta$ in $H_0^1(\Omega)$. Let us fix

$$0 < \varepsilon < \frac{l}{2^q 4(1+a)|W|_\infty|\varphi_1|_q^q},$$

where a, q and l are given by (f_1) and (W_1). Moreover, let $0 < \delta < 1$ and $C_1 > 0$ be constants such that (2.5) and (2.6) hold, and consider $0 < \rho_1 < \frac{\delta\lambda_1^{1/2}}{2|\varphi_1|_\infty}$. Since $\|u\| \ge \lambda_1^{1/2} t$, for $u \in B_{\rho_1}(0)$ we have

$$0 \le t\varphi_1(x) \le \frac{\delta}{2}, \quad \text{for every} \quad x \in \Omega. \tag{2.8}$$

Now, setting $\Omega_\delta^+ := \Omega_\delta(u^+) = \{x \in \Omega;\ u^+(x) \le \delta\}$, we may write

$$-\int_\Omega W(x)F(u) = -\int_\Omega W(x)F\big(u^+\big) = -at^q\int_{\Omega_\delta^+} W(x)(\varphi_1(x))^q - (I_1 + I_2 + I_3), \tag{2.9}$$

where

$$I_1 = \int_{\Omega_\delta^+} W(x)\big[F\big(u^+\big) - a\big(u^+\big)^q\big], \quad I_2 = \int_{\Omega_\delta^+} W(x)\big[a\big(u^+\big)^q - at^q(\varphi_1(x))^q\big]$$

and

$$I_3 = \int_{\Omega\setminus\Omega_\delta^+} W(x)F\big(u^+\big).$$

Our next task is to estimate the integrals I_1, I_2 and I_3. In view of (2.5) and (W_1), we have

$$|I_1| \le \varepsilon 2^q |W|_\infty t^q |\varphi_1|_q^q + \varepsilon 2^q |W|_\infty \int_{\Omega_\delta^+} |z|^q. \tag{2.10}$$

From the Sobolev imbedding theorem, we find $C_2 = C_2(\varepsilon) > 0$ such that

$$|I_1| \le \varepsilon 2^q |W|_\infty t^q |\varphi_1|_q^q + C_2\|z\|^q \quad \text{for every} \quad u \in B_{\rho_1}(0). \tag{2.11}$$

Now we shall estimate the integral I_2. Given $x \in \Omega_\delta^+$ and $u \in B_{\rho_1}(0)$, we may apply the mean value theorem to find $C_3 = C_3(\varepsilon) > 0$ such that

$$\begin{aligned}\left|\big(u^+(x)\big)^q - (t\varphi_1(x))^q\right| &\le q2^{q-1}\left[(t\varphi_1(x))^{q-1}|z(x)| + |z(x)|^q\right]\\ &= \frac{q}{q-1}\left[\varepsilon^{(q-1)/q}(t\varphi_1(x))^{q-1}\frac{(q-1)2^{q-1}}{\varepsilon^{(q-1)/q}}|z(x)|\right]\\ &\quad + q2^{q-1}|z(x)|^q \le \varepsilon(t\varphi_1(x))^q + C_3|z(x)|^q,\end{aligned}$$

where above we use the inequality $ab \le \frac{q-1}{q}a^{q/(q-1)} + \frac{1}{q}b^q$ with

$$a = \varepsilon^{(q-1)/q}(t\varphi_1(x))^{q-1}$$

and $b = \frac{(q-1)2^{q-1}}{\varepsilon^{(q-1)/q}}|z(x)|$. From the Sobolev imbedding theorem, we obtain $C_4 = C_4(\varepsilon) > 0$ such that

$$|I_2| \le a\varepsilon|W|_\infty t^q|\varphi_1|_q^q + C_4\|z\|^q \;\text{ for every } u \in B_{\rho_1}(0). \tag{2.12}$$

In what follows, we will provide an estimate for the integral I_3. Taking $u \in B_{\rho_1}(0)$, by (2.8) and the definition of $\Omega_\delta(u^+) \equiv \Omega_\delta^+$, we have $z(x) \ge \delta/2 \ge t\varphi_1(x)$ for almost every $x \in \Omega \setminus \Omega_\delta^+$. Consequently,

$$u^+(x) = t\varphi_1(x) + z(x) \le 2z(x) \;\text{ for every } x \in \Omega \setminus \Omega_\delta^+.$$

Thus, combining the above inequality together with (2.6) we find $C_5 = C_5(\varepsilon) > 0$ such that, for every $u \in B_{\rho_1}(0)$,

$$\begin{aligned}|I_3| &\le |W|_\infty\int_{\Omega\setminus\Omega_\delta^+}\left|F\big(u^+\big)\right| \le C_1|W|_\infty\int_{\Omega\setminus\Omega_\delta^+}|u^+|e^{\alpha(u^+)^2}\\ &\le C_5\|z\|^q\left[\int_{\Omega\setminus\Omega_\delta^+}\exp\left(q'\alpha\left\|u^+\right\|^2\left(\frac{|u^+|}{\|u^+\|}\right)^2\right)\right]^{1/q'},\end{aligned} \tag{2.13}$$

where $q' = q/(q-1)$. Assuming further that $\rho_1 \alpha q' \leq 4\pi$, and since $\|z\| \leq \|u\| \leq \rho_1$, we may invoke (2.11), (2.12), (2.13) and Trudinger-Moser inequality, to find

$$|I_1 + I_2 + I_3| \leq \varepsilon 2^q (1+a)|W|_\infty t^q |\varphi_1|_q^q + C_6 \|z\|^q, \quad \text{for every} \quad u \in B_{\rho_1}(0).$$

The above inequality, (2.7) and (2.9) imply, for every $u \in B_{\rho_1}(0)$,

$$\begin{aligned} I_\lambda(u) \geq{}& \frac{1}{2}\|u^-\|^2 - C_0(\lambda - \lambda_1)\|u\|^2 + \left[\frac{\lambda_2 - \lambda_1}{2\lambda_2} - C_6\|z\|^{q-2}\right]\|z\|^2 \\ &+ t^q \left[-a \int_{\Omega_\delta^+} W(x)(\varphi_1(x))^q - \varepsilon 2^q (1+a)|W|_\infty |\varphi_1|_q^q\right]. \end{aligned}$$

Therefore, from our choice of ε,

$$\begin{aligned} I_\lambda(u) \geq{}& \frac{1}{2}\|u^-\|^2 - C_0(\lambda - \lambda_1)\|u\|^2 + \left[\frac{\lambda_2 - \lambda_1}{2\lambda_2} - C_6\|z\|^{q-2}\right]\|z\|^2 \\ &+ t^q \left[-a \int_{\Omega_\delta^+} W(x)(\varphi_1(x))^q - \frac{l}{4}\right], \end{aligned}$$

whenever $u \in B_{\rho_1}(0)$. Next, we invoke Lemma 2.1 to find $0 < \rho < \rho_1$ such that, for every $u \in B_\rho(0)$ we have $-a \int_{\Omega_\delta(u)} W(x)\varphi_1^q \geq l/2$. Thus,

$$I_\lambda(u) \geq \frac{1}{2}\|u^-\|^2 - C_0(\lambda - \lambda_1)\|u\|^2 + \left[\frac{\lambda_2 - \lambda_1}{2\lambda_2} - C_6\|z\|^{q-2}\right]\|z\|^2 + \frac{l}{4}t^q.$$

Since $q > 2$ and $\|u\|^2 = \|u^-\|^2 + \lambda_1 t^2 + \|z\|^2$, taking $\rho > 0$ smaller if necessary, we find $C_7 > 0$ such that

$$I_\lambda(u) \geq \sigma := C_7\rho^q - C_0(\lambda - \lambda_1)\rho^2, \quad \text{if} \quad \|u\| = \rho. \tag{2.14}$$

If $\lambda < \lambda_1$ the results is immediate. In the case where $\lambda > \lambda_1$ we complete the proof of Lemma 2.2 by taking $\lambda - \lambda_1 \geq 0$ sufficiently small in the above inequality. □

Remark 2.3. As a direct consequence of inequality (2.14), if ρ is sufficiently small then $m_{\lambda_1} := \inf_{u \in \overline{B}_\rho(0)} I_{\lambda_1}(u) = 0$.

Now, we are ready to establish our first existence result for Problem (P_λ).

Theorem 2.4. *Suppose that* (f_{α_0}), (f_1) *and* (W_1) *are satisfied. Then there exists* $\lambda^* > \lambda_1$ *such that Problem* (P_λ) *has a nonnegative solution* $u_0 \in H_0^1(\Omega)$ *such that* $I_\lambda(u_0) < 0$ *for any* $\lambda \in (\lambda_1, \lambda^*)$.

Proof. Let $\rho > 0$ and $\lambda^* > \lambda_1$ obtained in Lemma 2.2. For each $\lambda \in (\lambda_1, \lambda^*)$, we define

$$m_\lambda := \inf_{u \in \overline{B}_\rho(0)} I_\lambda(u). \tag{2.15}$$

If $\varphi_1 > 0$ is the first eigenfunction and $\lambda > \lambda_1$, by (f_1) we infer that

$$\frac{I_\lambda(t\varphi_1)}{t^2} = -\frac{(\lambda - \lambda_1)}{2}|\varphi_1|_2^2 - \int_\Omega W(x)\frac{F(t\varphi_1)}{t^2} < 0, \tag{2.16}$$

for $t > 0$ sufficiently small. This, together with Lemma 2.2, implies

$$m_\lambda = \inf_{u \in \overline{B}_\rho(0)} I_\lambda(u) < 0 < \inf_{\|u\|=\rho} I_\lambda(u).$$

Therefore, by the Ekeland variational principle, there exists a sequence $(u_n) \subset \overline{B}_\rho(0)$ such that

$$\|I'_\lambda(u_n)\| \to 0 \quad \text{and} \quad I_\lambda(u_n) \to m_\lambda. \tag{2.17}$$

Since (u_n) is bounded and $\overline{B}_\rho(0) \subset H_0^1(\Omega)$ is a closed convex set, we may assume that there exists $u_0 \in \overline{B}_\rho(0)$ and a subsequence of (u_n), still denoted by (u_n), such that $u_n \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$, $u_n \to u_0$ strongly in $L^r(\Omega)$ for every $1 \le r < \infty$ and $u_n(x) \to u_0(x)$ for almost every $x \in \Omega$. We claim that, for each $v \in H_0^1(\Omega)$,

$$\int_\Omega W(x)f(u_n)v \to \int_\Omega W(x)f(u_0)v, \quad \text{as} \quad n \to \infty. \tag{2.18}$$

In fact, let $E \subset \Omega$ be an arbitrary measurable subset of Ω. Invoking condition (f_{α_0}), if $\alpha > \alpha_0$ we can find $C > 0$ such that $|f(s)| \le C\exp(\alpha s^2)$ for all $s \in \mathbb{R}$. We may assume that ρ given by Lemma 2.2 satisfies $\alpha_0\rho^2 < 4\pi$. Since $\|u_n\| \le \rho$ we may choose $\alpha > \alpha_0$ close to α_0 and $r > 1$ close to 1 such that $r\alpha\|u_n\|^2 \le 4\pi$, and the Hölder inequality together with the Trudinger-Moser inequality give

$$\begin{aligned}\int_E |W(x)f(u_n)v| &\le C_1\left(\int_E |v|^{r'}\right)^{1/r'}\left(\int_\Omega \exp\left(r\alpha\|u_n\|^2\left(\frac{u_n}{\|u_n\|}\right)^2\right)\right)^{1/r}\\ &\le C_2\left(\int_E |v|^{r'}\right)^{1/r'},\end{aligned}$$

where $r' = r/(r-1)$. Thus, the sequence $(Wf(u_n)v)$ is uniformly integrable and Vitali's theorem implies that $Wf(u_n)v \to Wf(u_0)v$ in $L^1(\Omega)$ as $n \to \infty$, which proves our claim. Consequently, by (2.17) we get

$$\langle u_0, v\rangle - \lambda\left\langle u_0^+, v\right\rangle_2 - \int_\Omega W(x)f(u_0)v = 0, \quad \text{for every} \quad v \in H_0^1(\Omega),$$

and thus u_0 is a critical point of I_λ which is nonnegative. Since $u_n \to u_0$ strongly in $L^r(\Omega)$ for every $1 \le r < \infty$, invoking the convergence (2.18) we obtain

$$\left|\int_\Omega W(x)\big[f\big(u_n\big)u_n - f\big(u_0\big)u_0\big]\right| \le C_1 \int_\Omega |f(u_n)||u_n - u_0| + \left|\int_\Omega W(x)\big[f(u_n) - f(u_0)\big]u_0\right| = o_n(1).$$

In view of $I'_\lambda(u_n)u_n \to 0$ and $I'_\lambda(u_0)u_0 = 0$, we obtain that $\|u_n\| \to \|u_0\|$ as $n \to \infty$. Thus, $u_n \to u_0$ in $H_0^1(\Omega)$ and consequently $I_\lambda(u_0) = m_\lambda < 0$. The proof of Theorem 2.4 is complete. □

3. Existence of two nonnegative solutions

In this section, we shall prove that I_λ has in addition to u_0 obtained in Theorem 2.4 another nontrivial critical point. Before starting, we fix some terminology and notation which will be frequently used in this paper. By hypothesis (W_2) we have that $\text{int}(\Omega^0) \ne \emptyset$. Consider the closed subspace of $H_0^1(\Omega)$ defined by

$$H_D^1\big(\Omega^0\big) = \Big\{v \in H_0^1(\Omega);\ v = 0 \quad \text{a.e. in} \quad \Omega \setminus \Omega^0\Big\}$$

and set

$$\lambda_1^D\big(\Omega^0\big) = \inf\left\{\int_{\Omega^0} |\nabla v|^2;\ v \in H_D^1\big(\Omega^0\big) \ \text{ and } \int_{\Omega^0} v^2 = 1\right\}. \tag{3.1}$$

As a direct consequence of (3.1), we have $\lambda_1^D(\Omega^0) \ge \lambda_1$. Actually, using the compactness of the Sobolev imbedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ and the fact that the eigenfunction φ_1 is positive, a standard argument provides the following result (see, *e.g.*, [25, Lemma 5.1]).

Lemma 3.1. *Suppose that* (W_2) *holds. Then,* $\lambda_1^D(\Omega^0) > \lambda_1$ *and the infimum in* (3.1) *is achieved,* i.e., *there is* $v_0 \in H_D^1(\Omega^0)$ *such that*

$$\lambda_1^D\big(\Omega^0\big) = \int_{\Omega^0} |\nabla v_0|^2 \quad \textit{and} \quad \int_{\Omega^0} v_0^2 = 1.$$

3.1. Boundedness of the (PS) sequence

In this subsection, we state our main result on (PS) sequences for the functional I_λ. Since f is continuous, invoking conditions (f_{α_0}), for $\theta > 2$ we may find $C_1, C_2 > 0$ such that

$$f(s) \ge C_1 s^{\theta-1} - C_2, \quad \text{for every} \quad s \ge 0. \tag{3.2}$$

Lemma 3.2. *Suppose* (f_{α_0}), (f_1) *and* (W_2) *are satisfied. If* $\lambda < \lambda_1^D(\Omega^0)$, *then every* $(PS)_c$ *sequence associated with* I_λ *is bounded.*

Proof. Let $(u_n) \subset H_0^1(\Omega)$ be a $(PS)_c$ sequence associated with I_λ, *i.e.*, for some $c \in \mathbb{R}$,

$$I_\lambda(u_n) \to c \quad \text{and} \quad \left\| I_\lambda'(u_n) \right\| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.3}$$

Arguing by contradiction, we suppose that $\|u_n\| \to \infty$ as $n \to \infty$. First, we may suppose $u_n \geq 0$ almost everywhere in Ω. Indeed, writing $u_n = u_n^+ - u_n^-$, with $u_n^\pm = \max\{\pm u_n, 0\}$, from (3.3) we obtain $\|u_n^-\|^2 = \langle I_\lambda'(u_n), -u_n^-\rangle \leq o_n(1)\|u_n^-\|$. Thus, we have

$$\|u_n^-\| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.4}$$

Combining (3.3) and (3.4), we can conclude that $I_\lambda(u_n^+) \to c$ as $n \to \infty$. Now, given $v \in H_0^1(\Omega)$, we have

$$\left\langle I_\lambda'(u_n), v\right\rangle = -\left\langle u_n^-, v\right\rangle + \left\langle I_\lambda'\left(u_n^+\right), v\right\rangle.$$

Invoking again (3.3) and (3.4) we obtain that $\|I_\lambda'(u_n^+)\| \to 0$ as $n \to \infty$, which implies that (u_n^+) is a $(PS)_c$ sequence. Setting $v_n = u_n/\|u_n\|$, in view of the above claim $v_n \geq 0$ almost everywhere in Ω. Therefore, by the Sobolev imbedding theorem, passing to a subsequence if necessary, we may assume that there exists $v_0 \in H_0^1(\Omega)$ with $v_0(x) \geq 0$ for almost every $x \in \Omega$ such that

$$\begin{aligned} v_n &\rightharpoonup v_0 \quad \text{weakly in} \quad H_0^1(\Omega), \\ v_n &\to v_0 \quad \text{stronlgy in} \quad L^p(\Omega), \quad 1 \leq p < \infty, \\ v_n(x) &\to v_0(x) \quad \text{for a.e. } x \in \Omega, \\ |v_n(x)| &\leq h_p(x) \in L^p(\Omega), 1 \leq p < \infty, \quad \text{for a.e. } x \in \Omega. \end{aligned} \tag{3.5}$$

We claim that $v_0 = 0$. First we observe that $v_0 \in H_D^1(\Omega^0)$. In fact, taking $\varphi \in C_c^\infty(\Omega^+)$ such that $\varphi \geq 0$, from (3.3) and $u_n \geq 0$ in Ω, we have

$$o_n(\|u_n\|) = \langle u_n, u_n\varphi\rangle - \lambda \int_\Omega u_n^2\varphi - \int_\Omega W(x)f(u_n)u_n\varphi. \tag{3.6}$$

Hence, considering $2 < \beta < \theta$, with θ given by (3.2), we obtain

$$\int_\Omega W(x)f(u_n)\frac{u_n\varphi}{\|u_n\|^\beta} \leq o_n(1).$$

Since $\operatorname{supp}\varphi \subset \Omega^+$ is compact, we may use the above inequality and (3.2) to get

$$\|u_n\|^{\theta-\beta} \int_\Omega v_n^\theta \varphi \leq o_n(1).$$

Now taking into account that $\theta - \beta > 0$ and $\|u_n\| \to \infty$, from (3.5) we reach

$$\int_\Omega v_0^\theta \varphi = 0, \quad \text{for every } \varphi \in C_c^\infty(\Omega^+),\ \varphi \geq 0. \tag{3.7}$$

A similar argument implies that (3.7) also holds for every $\varphi \in C_c^\infty(\Omega^-)$, $\varphi \geq 0$ in Ω. Therefore, $v_0 \in H_D^1(\Omega^0)$. From this fact, (W_2), $\|u_n\| \to \infty$, $u_n \geq 0$ in Ω and (3.3), we have

$$o_n(1) = \frac{1}{\|u_n\|}\big\langle I'_\lambda(u_n), v_0\big\rangle = \langle v_n, v_0\rangle - \lambda \int_\Omega v_n v_0.$$

Taking $n \to \infty$ and invoking (3.5), we get $\lambda_1^D(\Omega^0)|v_0|_2^2 \leq \|v_0\|^2 = \lambda |v_0|_2^2$. Since $\lambda < \lambda_1^D(\Omega^0)$, this estimate implies that $v_0 = 0$ almost everywhere in Ω, which proves the claim. Now we consider $\psi \in C_c^\infty(\mathbb{R}^2)$ with $0 \leq \psi \leq 1$, $\psi \equiv 1$ on Ω^+ and $\psi \equiv 0$ on Ω^-, and we use the Hölder inequality to obtain

$$\begin{aligned} I_\lambda(u_n) - \frac{1}{\theta}\big\langle I'_\lambda(u_n), u_n\psi\big\rangle \geq \left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|^2 - \frac{\lambda}{2}\int_\Omega u_n^2 \\ + \int_\Omega W(x)\left[\frac{1}{\theta} f(u_n)u_n\psi - F(u_n)\right] \\ - \frac{1}{\theta}|\nabla\psi|_\infty |u_n|_2 \|u_n\|. \end{aligned} \tag{3.8}$$

Using conditions (f_2) and (W_2), we can find $C_3 > 0$ such that

$$\begin{aligned} &\int_\Omega W(x)\left[\frac{1}{\theta} f(u_n)u_n\psi - F(u_n)\right] \\ &= \int_{\Omega^+} W(x)\left[\frac{1}{\theta} f(u_n)u_n - F(u_n)\right] - \int_{\Omega^-} W(x)F(u_n) \geq -C_3. \end{aligned}$$

Dividing the inequality (3.8) by $\|u_n\|^2$ and using the previous estimate, we get $C_4 > 0$ such that

$$o_n(1) \geq \frac{\theta - 2}{2\theta} - \frac{1}{\theta}|\nabla\psi|_\infty |v_n|_2 - C_4|v_n|_2^2.$$

Taking the limit as $n \to +\infty$ in the above inequality and using (3.5) we conclude that $\theta - 2 \leq 0$, which is a contradiction and the proof of Lemma 3.2 is complete. □

Remark 3.3. If f has exponential subcritical growth, *i.e.*, f satisfies (f_{α_0}) with $\alpha_0 = 0$ then, as a consequence of Lemma 3.2, the functional I_λ satisfies the (PS) condition whenever $\lambda < \lambda_1^D(\Omega^0)$.

As a consequence of Lemma 3.2, we have the following convergence result:

Lemma 3.4. *Assume* (f_{α_0}), $(f_1)-(f_3)$ *and* (W_2) *are satisfied. If* $\lambda < \lambda_1^D(\Omega^0)$, *then any* $(PS)_c$ *sequence* $(u_n) \subset H_0^1(\Omega)$ *associated with* I_λ *has a subsequence still denoted by* (u_n) *converging weakly in* $H_0^1(\Omega)$ *to a solution u of Problem* (P_λ). *Furthermore,*

$$\int_\Omega W(x)F(u_n) \to \int_\Omega W(x)F(u), \quad \textit{as} \quad n \to \infty.$$

Proof. Consider a sequence $(u_n) \subset H_0^1(\Omega)$ such that

$$I_\lambda(u_n) \to c \quad \text{and} \quad \left\| I_\lambda'(u_n) \right\| \to 0, \quad \text{as} \quad n \to \infty. \tag{3.9}$$

By Lemma 3.2, the sequence (u_n) is bounded. Therefore, (u_n) possesses a subsequence, still denoted by (u_n), converging weakly in $H_0^1(\Omega)$ to a function $u \in H_0^1(\Omega)$. By a standard argument (see [28]), we may verify that u is a solution of Problem (P_λ). Furthermore, we may assume that $u_n(x) \to u(x)$ for almost every $x \in \Omega$. Since $f(s) = 0$ for every $s \le 0$, from (3.9) we get that

$$\left\| u_n^- \right\|^2 = \left\langle I_\lambda'(u_n), u_n^- \right\rangle \le \left\| I_\lambda'(u_n) \right\| \left\| u_n^- \right\| \to 0 \quad \text{as} \quad n \to \infty.$$

Consequently, $u_n^+(x) = u_n^-(x) + u_n(x) \to u(x) = u^+(x) \ge 0$, as $n \to \infty$ for almost every $x \in \Omega$. Next, considering a function $\psi \in C_c^\infty(\mathbb{R}^2)$ with $0 \le \psi \le 1$, $\psi \equiv 1$ on Ω^+ and $\psi \equiv 0$ on Ω^-, we obtain

$$\left\langle I_\lambda'(u_n), \psi u_n^+ \right\rangle = \left\langle u_n, \psi u_n^+ \right\rangle - \lambda \int_\Omega \psi \left(u_n^+\right)^2 - \int_{\Omega^+} W(x) f\left(u_n^+\right) u_n^+.$$

Hence, from (3.9) and the boundness of (u_n) in $H_0^1(\Omega)$, we may find $C_1 > 0$ such that

$$\int_{\Omega^+} W(x) f\left(u_n^+\right) u_n^+ \le C_1, \quad \forall\, n \in \mathbb{N}. \tag{3.10}$$

This estimate together with (f_2) and (f_3) implies that given $\varepsilon > 0$, we may find $R > 0$ such that

$$\int_{\Omega^+ \cap [u_n \ge R]} \left| W(x) f\left(u_n^+\right) u_n^+ \right| = \int_{\Omega^+ \cap [u_n \ge R]} W(x) f\left(u_n^+\right) u_n^+ \le \frac{\varepsilon}{2}, \quad \forall\, n \in \mathbb{N}.$$

In fact, by (f_2) and (f_3) given $\hat{\varepsilon} > 0$ there exists $R > 0$ such that $0 < F(s) \le \hat{\varepsilon} f(s)s$ for any $s \ge R$, which implies that

$$\int_{\Omega^+ \cap [u_n \ge R]} \left| W(x) F\left(u_n^+\right) \right| \le \hat{\varepsilon} \int_{\Omega^+ \cap [u_n \ge R]} W(x) f\left(u_n^+\right) u_n^+. \tag{3.11}$$

From (f_3) (or (f_1)) there exists $0 < R_1 < R$ such that $f(s) \geq 0$ for any $s \geq R_1$. Consequently, there exists $C_2 > 0$ (independent of R) such that

$$\begin{aligned}\int_{\Omega^+} W(x) f\big(u_n^+\big)u_n^+ &= \int_{\Omega^+\cap[0\leq u_n<R]} W(x) f\big(u_n^+\big)u_n^+ + \int_{\Omega^+\cap[u_n\geq R]} W(x) f\big(u_n^+\big)u_n^+ \\ &= \int_{\Omega^+\cap[0\leq u_n\leq R_1]} W(x) f\big(u_n^+\big)u_n^+ + \int_{\Omega^+\cap[R_1\leq u_n<R]} W(x) f\big(u_n^+\big)u_n^+ \\ &\quad + \int_{\Omega^+\cap[u_n\geq R]} W(x) f\big(u_n^+\big)u_n^+ \\ &\geq \int_{\Omega^+\cap[u_n\geq R]} W(x) f\big(u_n^+\big)u_n^+ - C_2.\end{aligned}$$

This, together with (3.10), implies that

$$\int_{\Omega^+\cap[u_n\geq R]} W(x) f\big(u_n^+\big)u_n^+ \leq C_1 + C_2.$$

Choosing $\hat{\varepsilon} < \frac{\varepsilon}{2(C_1+C_2)}$, by (3.11) we can find $R > 0$ such that

$$\int_{\Omega^+\cap[u_n\geq R]} \big|W(x)F\big(u_n^+\big)\big| \leq \frac{\varepsilon}{2}. \tag{3.12}$$

Let E be a mensurable subset of Ω^+ with $|E| < \delta$. By (3.12) we have

$$\begin{aligned}\int_E \big|W(x)F\big(u_n^+\big)\big| &= \int_{E\cap[u_n\leq R]} \big|W(x)F\big(u_n^+\big)\big| + \int_{E\cap[u_n\geq R]} \big|W(x)F\big(u_n^+\big)\big| \\ &\leq C_3|E| + \frac{\varepsilon}{2} < \varepsilon,\end{aligned}$$

provided that $\delta < \varepsilon/(2C_3)$. Thus, we conclude that the sequence $(W(x)F(u_n^+)) \subset L^1(\Omega)$ is equi-integrable. Since $W(x)F(u) \in L^1(\Omega)$, $W(x)F(u_n^+(x)) \to W(x)F(u(x))$ for almost every $x \in \Omega^+$ and $(W(x)F(u_n^+))$ is equi-integrable in Ω^+, by the Vitali theorem we conclude that

$$\int_{\Omega^+} W(x)F(u_n^+) \to \int_{\Omega^+} W(x)F(u).$$

Similarly, considering the test function $(1-\psi)u_n^+$, we conclude that

$$\int_{\Omega^-} W(x)F(u_n^+) \to \int_{\Omega^-} W(x)F(u).$$

Using that $F(s) = 0$ for $s \leq 0$, we get

$$\begin{aligned}\int_\Omega W(x)F(u_n) = \int_\Omega W(x)F\big(u_n^+\big) &\to \int_{\Omega^+} W(x)F(u) + \int_{\Omega^-} W(x)F(u) \\ &= \int_\Omega W(x)F(u),\end{aligned}$$

and this completes the proof. $\square$

3.2. Estimate of the minimax level

In this subsection, we present an estimate for the minimax level associated with I_λ. We recall that by Lemma 2.2 there exist $\sigma,\ \rho > 0$ and $\lambda^* > \lambda_1$ such that

$$I_\lambda(u) \geq \sigma, \quad \text{if} \quad \|u\| = \rho \quad \text{and} \quad \lambda < \lambda^*. \tag{3.13}$$

As in [28], in order to estimate the minimax level we define the Moser functions. Let $x_0 \in \Omega^+$ and $R > 0$ be such that $\overline{B}_R(x_0) \subset \Omega^+$. The Moser functions are defined for $0 < r < R$ by

$$M_r(x) = \frac{1}{\sqrt{2\pi \log \frac{R}{r}}} \cdot \begin{cases} \log \frac{R}{r} & \text{if} \quad |x - x_0| \leq r \\ \log \dfrac{R}{|x - x_0|} & \text{if} \quad r \leq |x - x_0| \leq R \\ 0 & \text{if} \quad |x - x_0| \geq R. \end{cases} \tag{3.14}$$

One can see that $M_r \in H_0^1(\Omega)$, $\|M_r\| = 1$ and $\operatorname{supp}(M_r) = \overline{B}_R(x_0) \subset \Omega^+$. Considering $n \in \mathbb{N},\ n > 1$, we define the Moser sequence by $M_n \equiv M_{\frac{R}{n}}$. It follows from inequality (3.2) that, for any $t > 0$,

$$I_\lambda(tM_n) \leq \frac{t^2}{2}\|M_n\|^2 - C_1 t^\theta \int_{\overline{B}_R(x_0)} W(x) M_n^\theta + C_2 t \int_{\overline{B}_R(x_0)} M_n.$$

Since $\theta > 2$ we conclude that $I_\lambda(tM_n) \to -\infty$ as $t \to +\infty$. Thus, $I_\lambda(t_0 M_n) < 0$ for some $t_0 > 0$ sufficiently large. This together with (3.13), implies that I_λ has the mountain pass structure and therefore we can define the minimax level

$$c_\lambda := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)) > 0, \quad \lambda < \lambda^*,$$

where $\Gamma := \{\gamma \in C([0, 1], H_0^1(\Omega)) : \gamma(0) = 0,\ \gamma(1) = t_0 M_n\}$. We have the following estimate:

Lemma 3.5. *If (f_{α_0}) and $(f_1) - (f_3)$ are satisfied then $c_{\lambda_1} < \frac{2\pi}{\alpha_0}$.*

Proof. Since $I_{\lambda_1}(tM_n) \to -\infty$ as $t \to \infty$, we infer from (3.13) that there exists $t_n > 0$ such that

$$c_{\lambda_1} \leq \max_{t > 0} I_{\lambda_1}(tM_n) = I_{\lambda_1}(t_n M_n).$$

We claim that (t_n) is bounded. Indeed, suppose by contradiction that there exists a subsequence such that $t_n \to +\infty$. Using that for $t = t_n$ we have $\frac{d}{dt} I_{\lambda_1}(tM_n) = 0$ and $\|M_n\| = 1$, we get

$$\begin{aligned} t_n^2 &\geq \int_{B_R(x_0)} W(x) f(t_n M_n) t_n M_n \\ &= \int_{|x - x_0| \leq \frac{R}{n}} W(x) f(t_n M_n) t_n M_n + \int_{\frac{R}{n} \leq |x - x_0| \leq R} W(x) f(t_n M_n) t_n M_n \\ &=: I_1(n) + I_2(n). \end{aligned} \tag{3.15}$$

Given $K > 0$, from (f_3) there exists $s_0 = s_0(K) > 0$ such that

$$sf(s) \geq Ke^{\alpha_0 s^2}, \quad \text{for every} \quad s \geq s_0.$$

For n large, we have that $t_n M_n(x) = \frac{t_n}{\sqrt{2\pi}}\sqrt{\log n} \geq s_0$ for all $x \in B_{\frac{R}{n}}(x_0)$. Therefore,

$$I_1(n) \geq K\pi R^2 \left(\min_{\overline{B}_R(x_0)} W \right) \left[\exp\left(\left(\frac{\alpha_0}{4\pi} t_n^2 - 1 \right) 2 \log n \right) \right]. \tag{3.16}$$

On the other hand, using that $f(s) \geq -C_1$ for $s \geq 0$ we obtain

$$f(s)s \geq -C_1 s, \quad \text{for every} \quad s \geq 0.$$

This inequality, together with the fact that $\int_{B_R(x_0)} M_n = o_n(1)$, implies

$$I_2(n) \geq -C_1 \int_{\frac{R}{n} \leq |x - x_0| \leq R} W(x) t_n M_n = t_n o_n(1). \tag{3.17}$$

Combining inequalities (3.15), (3.16) and (3.17), we get

$$t_n^2 \geq K\pi R^2 \left(\min_{\overline{B}_R(x_0)} W \right) \left[\exp\left(\left(\frac{\alpha_0}{4\pi} t_n^2 - 1 \right) 2 \log n \right) \right] + t_n o_n(1), \tag{3.18}$$

which is impossible if $t_n \to +\infty$. To complete the proof it is enough to verify that there exists $n \in \mathbb{N}$ satisfying

$$I_0(t_n M_n) < \frac{2\pi}{\alpha_0}.$$

Suppose by contradiction that this fact does not hold. Since $\|M_n\| = 1$ for every $n \in \mathbb{N}$, we get

$$\frac{t_n^2}{2} \geq \frac{2\pi}{\alpha_0} + \int_{B_R(x_0)} W(x) F(t_n M_n).$$

Invoking hypotheses (f_{α_0}) and (f_1), we can find $d > 0$ such that $F(s) \geq -ds^q$ for all $s \geq 0$ and a direct calculation shows that

$$\int_{B_R(x_0)} M_n^q = O\left(\frac{1}{(\log n)^{q/2}} \right). \tag{3.19}$$

Thus, we obtain the inequality

$$\frac{t_n^2}{2} \geq \frac{2\pi}{\alpha_0} - \frac{C t_n^q}{(\log n)^{q/2}},$$

that is,

$$\frac{\alpha_0}{4\pi}t_n^2 - 1 \geq -\frac{Ct_n^q}{(\log n)^{q/2}}. \tag{3.20}$$

In particular, using that (t_n) is bounded, after take a subsequence, $\lim_{n\to\infty} t_n^2 = t_0 > 0$. Since $q > 2$, using estimate (3.20) and passing to the limit in (3.18) we obtain $t_0^2 \geq K\pi R^2 \min_{\overline{B}_R(x_0)} W$. Choosing K sufficient large we get a contradiction and this completes the proof. □

Let $\lambda \in (\lambda_1, \lambda^*)$ and u_0 be the respective solution obtained in Theorem 2.4. The next lemma will give us an upper bound for c_λ.

Lemma 3.6. *Suppose* (f_{α_0}) *and* $(f_1) - (f_3)$ *are satisfied. Then, there exists* $\widehat{\lambda} \in (\lambda_1, \lambda^*)$ *such that*

$$0 < c_\lambda < I_\lambda(u_0) + 2\pi/\alpha_0, \quad \textit{for every} \quad \lambda \in (\lambda_1, \widehat{\lambda}).$$

Proof. Since $c_\lambda \leq c_{\lambda_1}$ for every $\lambda \geq \lambda_1$, in view of Lemma 3.5 and Remark 2.3, it is sufficient to verify that

$$0 > I_\lambda(u_0) = \inf_{u\in\overline{B}_\rho(0)} I_\lambda(u) = m_\lambda \to m_{\lambda_1} = 0, \quad \text{as} \quad \lambda \to \lambda_1^+.$$

Otherwise, there are sequences $\lambda_n \to \lambda_1^+$ and $(u_n) \subset H_0^1(\Omega)$ with $\|u_n\| \leq \rho$ such that $I_{\lambda_n}(u_n) \to M < 0$. We may assume that $\rho < 4\pi/\alpha_0$ and $u_n \rightharpoonup u$. Since $\|u\| \leq \rho$, we infer by Remark 2.3 that $I_{\lambda_1}(u) \geq 0$. Now, proceeding as in the proof of Theorem 2.4, we obtain $I_{\lambda_n}(u_n) \to I_{\lambda_1}(u) \geq 0$, which is a contradiction. This completes the proof. □

The next auxiliary lemma is an improvement of the Trudinger-Moser inequality due to Lions (see [24, Theorem 1.6]), which will be essential in order to verify that the functional I_λ has a second critical point.

Lemma 3.7. *Let* $\{u_n \in H_0^1(\Omega); \|u_n\| = 1\}$ *be a sequence converging weakly to a nonzero function* u. *Then, for every* $0 < p < 4\pi(1 - \|u\|^2)^{-1}$, *we have*

$$\sup_{n\in\mathbb{N}} \int_\Omega e^{pu_n^2} dx < \infty.$$

Now we are ready to state our first multiplicity result for Problem (P_λ), when $\lambda - \lambda_1 > 0$ is sufficiently small.

Theorem 3.8. *Suppose* (f_{α_0}), $(f_1) - (f_3)$ *and* $(W_1) - (W_2)$ *are satisfied. Then there exists* $\lambda^{**} > \lambda_1$ *such that*

i) *For every* $\lambda \in (\lambda_1, \lambda^{**})$, (P_λ) *has two nonnegative and nontrivial solutions;*
ii) *For* $\lambda = \lambda_1$, (P_λ) *has a nonnegative and nontrivial solution.*

Proof. Let us choose $\lambda^{**} := \min\{\lambda^*, \lambda_1^D(\Omega^0), \widehat{\lambda}\}$. By using the mountain pass theorem without the Palais-Smale condition (see for instance Brezis-Nirenberg [10]), there exists a sequence $(u_n) \subset H_0^1(\Omega)$ such that

$$I'_\lambda(u_n) \to 0 \quad \text{and} \quad I_\lambda(u_n) \to c_\lambda > 0, \quad \lambda \in [\lambda_1, \lambda^{**}). \tag{3.21}$$

According to Lemma 3.2 we have that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$. Invoking Lemma 3.4, we conclude that $I'_\lambda(u) = 0$ and

$$\int_\Omega W(x)F(u_n) \to \int_\Omega W(x)F(u). \tag{3.22}$$

This together with the fact that $u_n^+ \to u^+$ in $L^2(\Omega)$ and (3.21) imply that

$$\lim_{n\to\infty} \|u_n\|^2 = 2(c_\lambda + c_0), \tag{3.23}$$

where $c_0 := \int_\Omega [W(x)F(u) + \frac{\lambda}{2}|u^+|^2]$. If $\lambda \in [\lambda_1, \lambda^{**})$, we claim that $u \neq 0$. Indeed, if $u = 0$ we have that $c_0 = 0$ and invoking Lemmas 3.5 and 3.6 we get

$$\lim_{n\to\infty} \|u_n\|^2 = 2c_\lambda < \frac{4\pi}{\alpha_0}.$$

We claim that

$$\lim_{n\to\infty} \int_\Omega W(x)f(u_n)u_n = 0. \tag{3.24}$$

Indeed, given $\alpha > \alpha_0$, by the hypothesis (f_{α_0}) there exists $C = C(\alpha) > 0$ such that $|f(s)| \leq C\exp(\alpha s^2)$ for all $s \in \mathbb{R}$. Now, we can choose $q > 1$ and $\alpha > \alpha_0$ such that $q\alpha\|u_n\|^2 < 4\pi$ for n large. Thus, using the Trudinger-Moser inequality we get

$$\int_\Omega |W(x)f(u_n)|^q \leq C|W|_\infty^q \int_\Omega \exp\left(q\alpha\|u_n\|^2 \left(\frac{u_n}{\|u_n\|}\right)^2\right) \leq \widetilde{C}. \tag{3.25}$$

Since $u_n \to 0$ strongly in $L^{q'}(\Omega)$, using the Hölder inequality and (3.25) we get

$$\left|\int_\Omega W(x)f(u_n)u_n\right| \leq C|f(u_n)|_q|u_n|_{q'} \to 0.$$

Now, invoking (3.24) and using that $I'_\lambda(u_n)u_n = o_n(1)$ we obtain $u_n \to 0$ strongly in $H_0^1(\Omega)$, which is a contradiction because $I_\lambda(u_n) \to c_\lambda > 0$. If in addition $\lambda \in (\lambda_1, \lambda^{**})$ then $u \neq u_0$. Otherwise, in this case we claim that

$$\lim_{n\to\infty} \int_\Omega W(x)f(u_n)u_n = \int_\Omega W(x)f(u_0)u_0. \tag{3.26}$$

If this is true, using that $u_n^+ \to u_0^+$ strongly in $L^2(\Omega)$ we get

$$\begin{aligned} o_n(1) = I_\lambda'(u_n)u_n &= \|u_n\|^2 - \lambda \left|u_0^+\right|_2^2 - \int_\Omega W(x)f(u_0)u_0 + o_n(1) \\ &= \|u_n\|^2 - \|u_0\|^2 + I_\lambda'(u_0)u_0 + o_n(1). \end{aligned}$$

Since $I_\lambda'(u_0) = 0$ we conclude that $u_n \to u_0$ strongly in $H_0^1(\Omega)$ which is a contradiction because

$$0 < c_\lambda = \lim_{n\to\infty} I_\lambda(u_n) = I_\lambda(u_0) < 0.$$

Thus, it remains to verify (3.26). From (3.23) we may assume that $c_\lambda + c_0 > 0$, otherwise the result is trivial. Define $v_n := u_n/\|u_n\|$ and observe that

$$v_n \rightharpoonup v := \frac{u_0}{\sqrt{2(c_\lambda + c_0)}}, \quad \text{weakly in } H_0^1(\Omega).$$

If $\|v\| = 1$ then $\lim_{n\to\infty} \|u_n\|^2 = 2(c_\lambda + c_0) = \|u_0\|^2$ and the result follows immediately. Suppose that $\|v\| < 1$. Since $\lambda > \lambda_1$, by Lemma 3.6, we can take $\alpha > \alpha_0$ such that $0 < c_\lambda < I_\lambda(u_0) + 2\pi/\alpha$, which implies that

$$2\alpha(c_0 + c_\lambda) < \frac{4\pi}{1 - \|v\|^2}.$$

Thus, we can use (3.23) to obtain $p_0 > 0$ such that $\alpha\|u_n\|^2 < p_0 < 4\pi/(1 - \|v\|^2)$ for n large. We now choose $q > 1$ sufficiently close to 1 in such way that

$$\alpha q\|u_n\|^2 < p < \frac{4\pi}{1 - \|v\|^2},$$

with $p = p_0 q$. It follows from Lemma 3.7 that

$$\sup_{n\in\mathbb{N}} \int_\Omega e^{\alpha q\|u_n\|^2 v_n^2} \le \sup_{n\in\mathbb{N}} \int_\Omega e^{p v_n^2} = C_4 < \infty.$$

Passing to a subsequence if necessary, we may assume that $u_n \to u_0$ strongly in $L^{q'}(\Omega)$. Hence, there exists $\Psi_{q'} \in L^1(\Omega)$ such that $|u_n(x)|^{q'} \le \Psi_{q'}(x)$ for almost every $x \in \Omega$. Since $W \in L^\infty(\Omega)$, for any measurable subset $A \subset \Omega$ and using the Hölder inequality we get

$$\begin{aligned} \left|\int_A W(x)f(u_n)u_n\right| &\le C_5 \left(\int_A |u_n|^{q'}\right)^{1/q'} \left(\int_A \left(e^{\alpha u_n^2}\right)^q\right)^{1/q} \\ &\le C_6 \left(\int_A \Psi_{q'}\right)^{1/q'} \left(\int_A e^{\alpha q\|u_n\|^2 v_n^2}\right)^{1/q} \\ &\le C_7 \left(\int_A \Psi_{q'}\right)^{1/q'}. \end{aligned}$$

Since $\Psi_{q'} \in L^1(\Omega)$ and the set $A \subset \Omega$ is arbitrary, we conclude that the first integral above is uniformly small provided that the measure of A is small. Hence, the set $\{Wf(u_n)u_n\}$ is uniformly integrable and Vitali's Theorem implies that $Wf(u_n)u_n \to Wf(u_0)u_0$ in $L^1(\Omega)$ as claimed. The proof of Theorem 3.8 is complete. □

4. Interval of existence of positive solution

In the next lemma, we prove some regularity properties for critical points of I_λ. As a consequence, we will establish an estimate for the parameter λ in order to Problem (P_λ) admits positive solutions. To this end, we will assume the following hypothesis:

$(\widehat{f_4})$ f is Lipschitz continuous at the origin.

Lemma 4.1. *Suppose (f_{α_0}), (f_1), $(\widehat{f_4})$ and (W_1) are satisfied. If $u \in H_0^1(\Omega)$ is a nontrivial critical point of I_λ, then $u \in C^1(\overline{\Omega})$. Furthermore, if $\lambda \geq 0$ then $u > 0$ in Ω and $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$ (where ν is the exterior unit normal to $\partial\Omega$).*

Proof. For any $\alpha > \alpha_0$ and $p > 1$, condition (f_{α_0}) together with the fact that $W \in C(\overline{\Omega})$ and the inequality (1.2), implies that for each $u \in H_0^1(\Omega)$

$$\int_\Omega |W(x)f(u)|^p \leq C \int_\Omega e^{p\alpha u^2} < \infty.$$

Thus, $u \in C^1(\overline{\Omega})$ by the theory of regularity for semilinear elliptic problems. Next, taking $v = u^-$ in (2.3) and using that $f(s) = 0$ for $s \leq 0$, we obtain $-\int_\Omega |\nabla u^-|^2 = 0$. Hence $u^- = 0$ and, consequently, $u \geq 0$ in Ω. Now, using that u is nonnegative and $\lambda \geq 0$, we have

$$\begin{aligned} h(x) :&= \lambda u(x) + W(x)f(u(x)) \\ &\geq \left[W^+(x)f^+(u) + W^-(x)f^-(u)\right] - \left[W^+(x)f^-(u) + W^-(x)f^+(u)\right]. \end{aligned}$$

Hence, taking $c(x) = [W^+(x)f^-(u(x)) + W^-(x)f^+(u(x))]/u(x) \geq 0$ for $u(x) > 0$ and $c(x) = 0$ otherwise, we have that u is a solution of

$$\begin{cases} -\Delta u + c(x)u \geq 0 & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Since f satisfies $(\widehat{f_4})$ and $u \in L^\infty(\Omega)$, one has $c \in L^\infty(\Omega)$. By the strong maximum principle for weak solutions of elliptic problems (see [33]), we conclude that $u > 0$ in Ω. Finally, observing that $u > 0$ in Ω and $u = 0$ on $\partial\Omega$, we may apply Hopf's lemma for weak solutions of class C^1 of elliptic problems to conclude that $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$ and this completes the proof. □

In order to estimate the parameter λ, we define

$$\Lambda = \sup\left\{\lambda \geq \lambda_1 : (P_\lambda) \quad \text{admits a positive solution } u \in H_0^1(\Omega)\right\}. \tag{4.1}$$

Lemma 4.2. *Assume that* (f_{α_0}), (f_1), $(\widehat{f_4})$ *and* $(W_1) - (W_2)$ *hold. Then* $\Lambda < \infty$.

Proof. Let us choose $r > 0$ such that $B_r \equiv B_r(x_0) \subset \Omega^0$. We take $\varphi_1^r > 0$ the eigenfunction associated with $\lambda_1^r = \lambda_1(B_r)$. If u is a positive solution of (P_λ) by Lemma 4.1 we have that $u \in C^1(\overline{\Omega})$. Applying the divergence theorem for the vector field $u\nabla\varphi_1^r$ and using that $\frac{\partial \varphi_1^r}{\partial \nu} < 0$ on ∂B_r (where ν is the exterior unit normal to B_r) we find

$$\int_{B_r} \nabla u \nabla \varphi_1^r = \int_{B_r} u(-\Delta\varphi_1^r) + \int_{\partial B_r} u \frac{\partial \varphi_1^r}{\partial \nu} d\sigma < \lambda_1^r \int_{B_r} u\varphi_1^r. \tag{4.2}$$

On the other hand, taking into account that u is a positive solution of (P_λ) and $\operatorname{supp}(\varphi_1^r) \subset \Omega^0$, we get

$$\int_{B_r} \nabla u \nabla \varphi_1^r = \int_{\Omega} \nabla u \nabla \varphi_1^r = \lambda \int_{B_r} u\varphi_1^r.$$

This, together with (4.2), implies that $\lambda < \lambda_1^r$ and this completes the proof. □

Remark 4.3. If we assume that $\partial\Omega^0$ is smooth, then we see that $H_0^1(\Omega^0) = H_D^1(\Omega^0)$ and so $\lambda_1(\Omega^0) = \lambda_1^D(\Omega^0)$. Thus, taking $\varphi_1 \in H_0^1(\Omega^0)$ and arguing as in the proof of the above lemma we conclude that $\Lambda \leq \lambda_1^D(\Omega^0)$.

Theorem 4.4. *Suppose* (f_{α_0}), (f_1), $(\widehat{f_4})$ *and* $(W_1) - (W_2)$ *are satisfied. Then* $\lambda_1 < \Lambda < \infty$ *and for every* $\lambda \in (\lambda_1, \Lambda)$, *Problem* (P_λ) *has a nonnegative and nontrivial solution* $u \in H_0^1(\Omega)$ *such that* $I_\lambda(u) < 0$.

Proof. By Theorem 2.4, Lemma 4.1 and 4.2, $\lambda_1 < \Lambda < \infty$. Given $\lambda \in (\lambda_1, \Lambda)$ consider $\overline{\lambda} \in (\lambda, \Lambda)$ and take $\overline{u} \in H_0^1(\Omega)$ a nontrivial critical point of $I_{\overline{\lambda}}$. By Lemma 4.1 we have that $\overline{u} \in C^1(\overline{\Omega})$ and $\frac{\partial \overline{u}}{\partial \nu} < 0$ on $\partial\Omega$. Now consider the continuous function $h_\lambda : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$h_\lambda(x, s) = \lambda s^+ + W(x) f(s), \;\; x \in \Omega, \;\; s \in \mathbb{R}, \tag{4.3}$$

and its truncation

$$\overline{h}_\lambda(x, s) = \begin{cases} h_\lambda(x, \overline{u}(x)) & \text{if} \quad s \geq \overline{u}(x) \\ h_\lambda(x, s) & \text{if} \quad s < \overline{u}(x). \end{cases} \tag{4.4}$$

Next, we consider the semilinear elliptic problem

$$\begin{cases} -\Delta u = \overline{h}_\lambda(x, u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{4.5}$$

and the associated functional $\overline{I}_\lambda : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\overline{I}_\lambda(u) = \frac{1}{2}\|u\|^2 - \int_\Omega \overline{H}_\lambda(x,u), \;\; u \in H_0^1(\Omega), \tag{4.6}$$

where $\overline{H}_\lambda(x,s) = \int_0^s \overline{h}_\lambda(x,t)dt$. Since $\overline{u} \in C^1(\overline{\Omega})$, (4.3), (4.4) and $f(s) = 0$ for $s \le 0$, we find $C_1 > 0$ such that $|\overline{h}_\lambda(x,s)| \le C_1$ for every $s \in \mathbb{R}$ and for almost every $x \in \Omega$. This implies that $\overline{I}_\lambda \in C^1(H_0^1(\Omega), \mathbb{R})$ is coercive and weakly lower semicontinuous. Therefore, there is $u \in H_0^1(\Omega)$ such that

$$-\infty < \overline{m}_\lambda = \overline{I}_\lambda(u) = \inf_{u \in H_0^1(\Omega)} \overline{I}_\lambda(u) \le \overline{I}_\lambda(0) = 0. \tag{4.7}$$

We claim that $\overline{m}_\lambda < 0$. Effectively, since $\overline{u} > 0$ in Ω and $\partial\overline{u}/\partial\nu < 0$ on $\partial\Omega$, there exists $t_1 > 0$ such that $0 < t\varphi_1(x) < \overline{u}(x)$ for every $x \in \Omega$, whenever $0 < t < t_1$. Thus, from (4.3) and (4.4), we have

$$\overline{I}_\lambda(t\varphi_1) = \frac{t^2}{2}(\lambda_1 - \lambda) - \int_\Omega W(x)F(t\varphi_1), \quad \text{for every} \quad 0 < t < t_1.$$

Next, invoking the conditions (f_1), (W_1) we conclude that $\overline{m}_\lambda \le \overline{I}_\lambda(t\varphi_1) < 0$ and the claim is proved. It follows from (4.7) and the above claim that u is a nontrivial solution of (4.5). Since $\lambda < \overline{\lambda}$, from (4.3), (4.4) and the fact that $\overline{u}$ is a solution of $(P_{\overline{\lambda}})$, we get

$$\int_\Omega \nabla u \nabla (u - \overline{u})^+ = \int_\Omega \overline{h}_\lambda(x,u)(u-\overline{u})^+ = \int_\Omega h_\lambda(x,\overline{u})(u-\overline{u})^+ \le \int_\Omega \nabla\overline{u}\nabla(u-\overline{u})^+.$$

Consequently $u \le \overline{u}$ in Ω. Moreover, since $f(s) = 0$ for $s \le 0$, for almost every $x \in \Omega$, we may invoke (4.3), (4.4) and the fact that u is a solution of (4.5) one more time to obtain

$$\|u^-\|^2 = -\int_\Omega \nabla u \nabla u^- = -\int_\Omega \left[\lambda u^+ + W(x)f(u)\right]u^- = 0.$$

This implies that $0 \le u \le \overline{u}$ in Ω and $I'_\lambda(u) = 0$. The proof of Theorem 4.4 is complete. □

Corollary 4.5. *In addition to the hypotheses of Theorem 4.4, suppose that* (f_4) *is satisfied. If* $\overline{u}$ *is a solution of Problem* $(P_{\overline{\lambda}})$ *with* $\lambda < \overline{\lambda} < \Lambda$, *then* (P_λ) *has a positive solution* $u \in C^1(\overline{\Omega})$ *satisfying* $I_\lambda(u) < 0$ *and*

$$0 < u < \overline{u} \quad in \quad \Omega \quad and \quad \frac{\partial\overline{u}}{\partial\nu} < \frac{\partial u}{\partial\nu} < 0 \quad on \quad \partial\Omega.$$

Proof. Let $u \in C^1(\overline{\Omega})$ be the solution of Problem (P_λ), satisfying $0 \le u \le \overline{u}$ in Ω and $I_\lambda(u) < 0$, provided in the proof of Theorem 4.4. Let $M > 0$ be such that $|\overline{u}|_\infty \le M$. Since h_λ, given by (4.3), is locally Lipschitz continuous on $\overline{\Omega} \times [0, M]$, we find $K > 0$ such that $|h_\lambda(x, s_2) - h_\lambda(x, s_1)| \le K|s_2 - s_1|$ for every $x \in \overline{\Omega}$ and $s_1, s_2 \in [0, M]$. In particular, if $0 \le s_1 \le s_2 \le M$, we have

$$h_\lambda(x, s_2) - h_\lambda(x, s_1) \ge -K(s_2 - s_1), \text{ for every } x \in \overline{\Omega}. \tag{4.8}$$

The above inequality, our choice of M and $0 \le u \le \overline{u}$ in Ω imply that

$$\begin{cases} -\Delta(\overline{u} - u) + K(\overline{u} - u) \ge 0 & \text{in} \quad \Omega \\ (\overline{u} - u) \ge 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Therefore, since $\overline{u} \not\equiv u$ in $\overline{\Omega}$, we may apply the strong maximum principle and Hopf's lemma to conclude that $u < \overline{u}$ in Ω and $\frac{\partial \overline{u}}{\partial \nu} < \frac{\partial u}{\partial \nu}$ on $\partial\Omega$. Finally, applying Lemma 4.1, we infer that $u > 0$ in Ω and $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$, which completes the proof of Corollary 4.5. □

5. Proof of Theorem 1.1

In this section, to prove Theorem 1.1, we will first establish some auxiliary results.

5.1. An estimate of the minimax level

The next lemma will be essential to obtain an estimate of the minimax-level c_λ in line with Lemma 3.6 when f is locally Lipschitz continuous. We also mention that similar result was proved in [28]. Let us fix $r > 0$.

Lemma 5.1. *Assume that f satisfies (f_{α_0}) and (f_4), and suppose that (W_2) holds. Given $\lambda, \gamma > 0$ there exist $C > 0$ and an open set $V \subset \Omega^+$ such that, for all $x \in V$, $0 \le s \le r$ and $t \ge 0$,*

$$\frac{\lambda}{2}t^2 + W(x)[F(s+t) - F(s) - f(s)t] \ge -Ct^\gamma. \tag{5.1}$$

Proof. By (W_2) we have that $\partial\Omega^+ \cap \Omega^0 \ne \emptyset$. Since $W \in C(\overline{\Omega})$, there exists $x_1 \in \partial\Omega^+ \cap \Omega^0$ such that $W(x_1) = 0$. Thus, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|W(x)| < \varepsilon, \quad \text{for every} \quad x \in B_\delta(x_1).$$

Since $x_1 \in \partial\Omega^+$, there exists $x_0 \in \Omega^+ \cap B_\delta(x_1)$. Therefore, we can find $R > 0$ such that

$$V \equiv B_R(x_0) \subset B_\delta(x_1) \cap \Omega^+.$$

Consequently,

$$0 < W < \varepsilon \quad \text{in} \quad V \subset \Omega^+.$$

First we consider $0 \le t \le r$. Invoking the mean value theorem and (f_4), we find $0 \le \theta \le 1$ and $L > 0$ such that

$$F(s+t) - F(s) - f(s)t = [f(s+\theta t) - f(s)]t \ge -L\theta t^2.$$

Choosing $\varepsilon < \lambda/2L$, for all $0 \le s, t \le r$ and $x \in V$, we get

$$\frac{\lambda}{2}t^2 + W(x)[F(s+t) - F(s) - f(s)t] \ge \left(\frac{\lambda}{2} - \varepsilon L\right)t^2 \ge 0.$$

Now, invoking (f_{α_0}), we find $T_1 > r$ such that for $0 \le s \le r$ and $t \ge T_1$ we have

$$\frac{\lambda}{2}t^2 + W(x)[F(s+t) - F(s) - f(s)t] \ge 0.$$

Using the above inequalities and the compactness of the set $\overline{V} \times [r, T_1]$, we may find $C > 0$ such that estimate (5.1) follows readily. □

With the aid of Lemma 5.1, we have the following estimate:

Proposition 5.2. *Suppose* (f_{α_0}), $(f_1) - (f_4)$ *and* $(W_1) - (W_2)$ *are satisfied. Then, for all* $\lambda \in (\lambda_1, \lambda^*)$, *there exists* $n \in \mathbb{N}$ *verifying*

$$\max_{t \ge 0} I_\lambda(u_0 + tM_n) < I_\lambda(u_0) + \frac{2\pi}{\alpha_0}, \tag{5.2}$$

where u_0 *is the solution obtained in Theorem* 2.4.

Proof. Considering $V \subset \Omega^+$, the open set given by Lemma 5.1, we take $x_0 \in \Omega$ and $R > 0$ such that $\overline{B}_R(x_0) \subset V$. Let M_n the Moser function defined by (3.14). Since $\text{supp}(M_n) \subset B_R(x_0)$, we have

$$\int_\Omega W(x)F(u_0 + tM_n) = \int_{\Omega \setminus B_R(x_0)} W(x)F(u_0) + \int_{B_{R(x_0)}} W^+(x)F(u_0 + tM_n).$$

Hence, using (3.2), we obtain $t_n > 0$ such that

$$\max_{t \ge 0} I_\lambda(u_0 + tM_n) = I_\lambda(u_0 + t_nM_n).$$

To prove estimate (5.2), it suffices to find $n \in \mathbb{N}$ such that

$$I_\lambda(u_0 + t_nM_n) < I_\lambda(u_0) + \frac{2\pi}{\alpha_0}.$$

Arguing by contradiction, suppose that this estimate does not hold. Since u_0 is a solution of Problem (P_λ) and $\|M_n\| = 1$, for every $n \in \mathbb{N}$, we get

$$\begin{aligned}\frac{2\pi}{\alpha_0} &\le I_\lambda(u_0 + t_nM_n) - I_\lambda(u_0)\\ &= \frac{t_n^2}{2} - \int_{B_R(x_0)} W(x)[F(u_0 + t_nM_n) - F(u_0) - f(u_0)t_nM_n] - \lambda\frac{t_n^2}{2}|M_n|_2^2.\end{aligned}$$

Now, invoking Lemma 5.1 with $\gamma = 3$ and $r = |u_0|_\infty$, we reach

$$\frac{2\pi}{\alpha_0} + \lambda \frac{t_n^2}{2} |M_n|_2^2 \le \frac{t_n^2}{2} + C t_n^3 \int_\Omega M_n^3.$$

Hence, by (3.19) and taking C large if necessary, we obtain

$$\frac{2\pi}{\alpha_0} + \lambda \frac{t_n^2}{2} |M_n|_2^2 \le \frac{t_n^2}{2} + \frac{C t_n^3}{(\log n)^{3/2}}. \tag{5.3}$$

In particular, $t_n \not\to 0$. On the other hand, for $t = t_n$ we have $\frac{d}{dt} I_\lambda(u_0 + t M_n) = 0$, that is,

$$\begin{aligned}\lambda \int_\Omega (u_0 + t_n M_n) t_n M_n + \int_{B_R(x_0)} W(x) f(u_0 + t_n M_n) t_n M_n &= t_n \int_\Omega \nabla(u_0 + t_n M_n) \nabla M_n \\ &\le t_n \|u_0 + t_n M_n\|,\end{aligned}$$

where above we have used that $\|M_n\| = 1$. From (f_3), given $K > 0$, there exists $s_0 > 0$ such that

$$s f(s) \ge K e^{\alpha_0 s^2}, \quad \text{for every} \quad s \ge s_0.$$

Moreover, $u_0(x) + t_n M_n(x) \ge \frac{t_n}{\sqrt{2\pi}} \sqrt{\log n} \ge s_0$ for all $x \in B_{R/n}(x_0)$ and n large. Using that $t_n \not\to 0$, for n large we get

$$\frac{t_n \left(\frac{\log n}{2\pi}\right)^{1/2}}{r + t_n \left(\frac{\log n}{2\pi}\right)^{1/2}} \ge \frac{1}{2}.$$

Since $f(s) \ge -C_1$ for $s \ge 0$ and $t_n M_n/(u_0 + t_n M_n) \le 1$, we conclude that

$$\begin{aligned} t_n \|u_0 + t_n M_n\| &\ge \int_{B_R(x_0)} W(x) f(u_0 + t_n M_n) t_n M_n \\ &\ge K \min_{\overline{B}_R(x_0)} W \int_{B_{R/n}(x_0)} \left(e^{\alpha_0 (u_0 + t_n M_n)^2}\right) \frac{t_n M_n}{u_0 + t_n M_n} - C_R \\ &\ge K \min_{\overline{B}_R(x_0)} W \left[\int_{B_{R/n}(x_0)} e^{\alpha_0 \frac{t_n^2 \log n}{2\pi}} \frac{t_n \left(\frac{\log n}{2\pi}\right)^{1/2}}{r + t_n \left(\frac{\log n}{2\pi}\right)^{1/2}} \right] - C_R \\ &\ge K \min_{\overline{B}_R(x_0)} W \left[\frac{\pi R^2}{2n^2} e^{\alpha_0 \frac{t_n^2 \log n}{2\pi}} \right] - C_R. \end{aligned}$$

Therefore,

$$
\begin{aligned}
t_n(\|u_0\| + t_n) &\geq K \min_{\overline{B}_R(x_0)} W \left[\frac{\pi R^2}{2n^2} e^{\alpha_0 \frac{t_n^2 \log n}{2\pi}} \right] - C_R \\
&= K \min_{\overline{B}_R(x_0)} W \left[\frac{\pi R^2}{2} e^{2\left(\frac{\alpha_0}{4\pi} t_n^2 - 1\right) \log n} \right] - C_R.
\end{aligned} \tag{5.4}
$$

In particular, this inequality implies that (t_n) is bounded. From this and (5.3) we obtain

$$\lim_{n \to \infty} t_n^2 \geq \frac{4\pi}{\alpha_0}.$$

In fact (5.4) implies that $\lim_{n \to \infty} t_n^2 = 4\pi/\alpha_0$ and by (5.3) we have $t_n^2 \geq 4\pi/\alpha_0 - C(\log n)^{-3/2}$. Thus, upon passing to the limit in (5.4), it follows that

$$\frac{4\pi}{\alpha_0} \left(\|u_0\| + \frac{4\pi}{\alpha_0} \right) \geq K \left(\min_{\overline{B}_R(x_0)} W \right) \frac{\pi R^2}{2} - C_R.$$

Choosing K large, we get a contradiction. □

5.2. Local minimum

The next result establishes a global version of Theorem 2.4.

Theorem 5.3. *Suppose* (f_{α_0}), $(f_1) - (f_4)$ *and* $(W_1) - (W_2)$ *are satisfied. Then, for every* $\lambda \in (\lambda_1, \Lambda)$, *Problem* (P_λ) *has a positive solution which is a local minimum of* I_λ.

Before presenting the proof of Theorem 5.3, we need to state a version of the result by Brezis and Nirenberg [11] on $C^1(\overline{\Omega})$ versus $H_0^1(\Omega)$ local minimizers. Since we are considering nonlinearity with exponential critical growth, for the sake of completeness, we present a proof for this setting (see also [19]). We consider the problem

$$
\begin{cases}
-\Delta u = g(x, u) & \text{in} \quad \Omega \\
u = 0 & \text{on} \quad \partial\Omega,
\end{cases} \tag{5.5}
$$

where $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain and $g \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ is a Carathéodory function with exponential critical growth, *i.e.*, there exists $\alpha_0 > 0$ such that

$$
\lim_{|s| \to \infty} \frac{g(x, s)}{e^{\alpha s^2}} =
\begin{cases}
0 & \forall \alpha > \alpha_0, \quad \text{uniformly in} \quad x \in \overline{\Omega} \\
+\infty & \forall \alpha < \alpha_0, \quad \text{uniformly in} \quad x \in \overline{\Omega}.
\end{cases} \tag{g_{α_0}}
$$

Let $J : H_0^1(\Omega) \to \mathbb{R}$ be the functional associated to Problem (5.5) defined by

$$J(u) = \frac{\|u\|^2}{2} - \int_\Omega G(x, u),$$

where $G(x, s) = \int_0^s g(x, t)dt$. Then, we have the following result:

Lemma 5.4. *Assume that g satisfies (g_{α_0}). If $u_0 \in H_0^1(\Omega)$ is a local minimizer of J in the $C^1(\overline{\Omega})$ topology,* i.e., *there exits $\rho > 0$ such that*

$$J(u_0) \leq J(v), \quad \|v - u_0\|_{C^1(\overline{\Omega})} \leq \rho, \tag{5.6}$$

then u_0 is a local minimizer of J in the $H_0^1(\Omega)$ topology, i.e. *there exists $\delta > 0$ such that*

$$J(u_0) \leq J(v), \quad \|v - u_0\| \leq \delta.$$

Proof. Arguing by contradiction, suppose that the conclusion does not hold. Then, there exists a sequence (ρ_n) with $\rho_n > 0$, $\rho_n \to 0$ and $v_n \subset H_0^1(\Omega)$ such $\|v_n - u_0\| \leq \rho_n$ and $J(v_n) < J(u_0)$. Furthermore, there exists a Lagrange multiplier $\mu_n \leq 0$ such that

$$\begin{cases} -(1-\mu_n)\Delta w_n = g(x, w_n + u_0) - g(x, u_0) & \text{in} \quad \Omega \\ w_n = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $w_n = v_n - u_0$. We claim that $w_n \to 0$ in $C^1(\overline{\Omega})$ which contradicts (5.6), because $J(w_n + u_0) < J(u_0)$. To prove this claim, we observe that by condition (g_{α_0}), for any $\alpha > \alpha_0$, there exists $C > 0$ such that, for all $n \in \mathbb{N}$,

$$|g(x, w_n + u_0) - g(x, u_0)| \leq C\left[\exp\left(2\alpha\left(w_n^2 + u_0^2\right)\right) + \exp\left(\alpha u_0^2\right)\right].$$

Thus, fixing $p > 1$ and using the Hölder inequality together with the Trudinger-Moser inequality, we get

$$\begin{aligned} &\int_\Omega |g(x, w_n + u_0) - g(x, u_0)|^p \\ &\leq C_1\left(\int_\Omega \exp\left(2\alpha p^2 w_n^2\right)\right)^{1/p}\left(\int_\Omega \exp\left(2\alpha p p' u_0^2\right)\right)^{1/p'} + \int_\Omega \exp\left(p\alpha u_0^2\right) \\ &\leq C_2\left(\int_\Omega \exp\left(2\alpha p^2\|w_n\|^2\left(\frac{w_n}{\|w_n\|}\right)^2\right)\right)^{1/p} + C_3. \end{aligned}$$

We may assume that $2\alpha p^2\|w_n\|^2 \leq 2\alpha p^2\rho_n \leq 4\pi$ and so we use the Trudinger-Moser inequality to obtain a constant $C_4 > 0$ independent of n such that

$$\int_\Omega |g(x, w_n + u_0) - g(x, u_0)|^p \leq C_4.$$

Choosing $p > 2$ and using the Sobolev imbedding together with classical elliptic estimates, we obtain $\alpha \in (0, 1)$ and $C > 0$ independent of n such that $\|w_n\|_{C^{1,\alpha}(\overline{\Omega})} \leq C\|w_n\|_{W^{2,p}(\Omega)} \leq \widetilde{C}$. It follows from the Ascoli-Arzelà Theorem that $w_n \to w_0$ in $C^1(\overline{\Omega})$. Since $w_n \to 0$ in $H_0^1(\Omega)$, we get $w_0 \equiv 0$ and this concludes the proof. □

Proof of Theorem 5.3. Given $\lambda_1 < \lambda < \Lambda$, we take $\overline{\lambda} \in (\lambda, \Lambda)$ and we invoke Theorem 4.4 and Lemma 4.1 to find a positive solution $\overline{u} \in C^1(\bar{\Omega})$ of $(P_{\overline{\lambda}})$ such that $\partial \overline{u}/\partial \nu < 0$ on $\partial\Omega$. Thus, we may apply Corollary 4.5 to obtain a positive solution u_0 of (P_λ) satisfying

$$0 < u_0 < \overline{u} \quad \text{in} \quad \Omega \quad \text{and} \quad \frac{\partial \overline{u}}{\partial \nu} < \frac{\partial u_0}{\partial \nu} < 0 \quad \text{on} \quad \partial\Omega, \tag{5.7}$$

and

$$\overline{I}_\lambda(u_0) = \inf_{u \in H_0^1(\Omega)} \overline{I}_\lambda(u) < 0, \tag{5.8}$$

where $\overline{I}_\lambda$ is defined by (4.6). In order to conclude that u_0 is a local minimum of I_λ in $H_0^1(\Omega)$, by Lemma 5.4, it suffices to verify that u_0 is a local minimum of I_λ in the C^1 topology. From (5.7), there exists $\delta > 0$ such that

$$\|u_0 - u\|_{C^1(\overline{\Omega})} < \delta \Rightarrow u(x) < \overline{u}(x) \quad \text{for every} \quad x \in \Omega.$$

Hence, from the definition of $\overline{I}_\lambda$ and (5.8), $I_\lambda(u) = \overline{I}_\lambda(u) \geq \overline{I}_\lambda(u_0) \geq I_\lambda(u_0)$, whenever $\|u_0 - u\|_{C^1(\overline{\Omega})} < \delta$. The proof of the theorem is complete. □

Remark 5.5. Note that, as a direct consequence of the proof of Theorem 5.3, Problem (P_λ) has a positive solution with negative energy for every $\lambda_1 < \lambda < \Lambda$.

In order to prove the existence of ordered solutions, we need the following result:

Lemma 5.6. *Let $u_0 \geq 0$ be a local minimum of I_λ. Assume that there are no critical points u of I_λ with $u \geq u_0$ in Ω and $u \neq u_0$. Then, there exist* $0 < \rho_1 < \rho_2$ *and* $\alpha > 0$ *such that*

$$I_\lambda(u_0 + v) - I_\lambda(u_0) \geq \alpha, \quad \textit{for every} \quad \rho_1 \leq \|v\| \leq \rho_2, \quad v \geq 0.$$

Proof. Consider $0 < \rho_1 < \rho_2 < \rho_0$ such that u_0 is a minimum of I_λ on $B_{\rho_0}(u_0)$. Arguing by contradiction, suppose that there exists a sequence $(v_n) \subset H_0^1(\Omega)$, $\rho_1 \leq \|v_n\| \leq \rho_2$ and $v_n \geq 0$ such that $I_\lambda(u_0 + v_n) - I_\lambda(u_0) = o_n(1)$. Passing to a subsequence if necessary, we may assume that $v_n \rightharpoonup v$ weakly in $H_0^1(\Omega)$ and $v_n \to v$ in $L^2(\Omega)$. This, in combination with the fact that $u_0 + v_n \geq 0$, implies that

$$\begin{aligned} I_\lambda(u_0 + v_n) - I_\lambda(u_0) &= \langle u_0, v\rangle - \lambda\langle u_0, v\rangle_2 + \frac{1}{2}\|v_n\|^2 - \frac{\lambda}{2}|v|_2^2 \\ &\quad - \int_\Omega W(x)[F(u_0 + v_n) - F(u_0)] + o_n(1). \end{aligned}$$

Taking ρ_0 smaller if necessary and by applying the Trudinger-Moser inequality we can see that $\int_\Omega W(x)F(u_0 + v_n) \to \int_\Omega W(x)F(u_0 + v)$. Therefore,

$$\begin{aligned} o_n(1) &= \langle u_0, v\rangle - \lambda\langle u_0, v\rangle_2 + \frac{1}{2}\|v_n\|^2 - \frac{\lambda}{2}|v|_2^2 \\ &\quad - \int_\Omega W(x)[F(u_0 + v) - F(u_0)] + o_n(1) \qquad (5.9) \\ &= I_\lambda(u_0 + v) - I_\lambda(u_0) + \frac{1}{2}\|v_n\|^2 - \frac{1}{2}\|v\|^2 + o_n(1). \end{aligned}$$

If $v = 0$, taking the limit in (5.9) and using that $\|v_n\| \geq \rho_1 > 0$ we obtain a contradiction. On the other hand, if $v \neq 0$, taking the limit in (5.9) and using that the norm is weakly lower semicontinuous we get

$$0 \geq I_\lambda(u_0 + v) - I_\lambda(u_0) \geq 0.$$

Thus, $I_\lambda(u_0 + v) = I_\lambda(u_0)$ and consequently $u_0 + v$ is another critical point of I_λ with $v + u_0 \geq u_0$, which contradicts the hypothesis and proof is complete. □

5.3. Proof of Theorem 1.1

Now, we present the proof of Theorem 1.1. In view of Lemma 4.2 and Remark 4.3 we have $\lambda_1 < \Lambda \leq \lambda_1^D(\Omega^0)$. Thus, to prove Theorem 1.1, it suffices to verify that Problem (P_λ) has two ordered positive solutions for every $\lambda_1 < \lambda < \Lambda$. Let u_0 be the positive solution of (P_λ) obtained in Theorem 5.3. In order to find a second solution u_1 of (P_λ) such that $u_1 > u_0$ in Ω, we look for $v = u_1 - u_0$, a positive solution of the semilinear elliptic problem

$$\begin{cases} -\Delta v = -\Delta u - (-\Delta u_0) = h_\lambda\big(x, u_0 + v^+\big) - h_\lambda(x, u_0) & \text{in} \quad \Omega \\ v = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{5.10}$$

with h_λ defined by (4.3). The functional associated with (5.10) is given by

$$\widetilde{I}_\lambda(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega \widetilde{H}_\lambda\big(x, v^+\big), \quad \text{for every} \quad v \in H_0^1(\Omega),$$

where, for $s \in \mathbb{R}$ and $x \in \Omega$,

$$\begin{aligned} \widetilde{H}_\lambda(x, s) &= \int_0^s \big[h_\lambda\big(x, u_0(x) + t^+\big) - h_\lambda(x, u_0(x))\big]\, dt \\ &= H_\lambda\big(x, u_0(x) + s^+\big) - H_\lambda(x, u_0(x)) - h_\lambda(x, u_0(x))s^+, \end{aligned}$$

with $H_\lambda(x, s) = \int_0^s h_\lambda(x, t)dt$. Thus, using that u_0 is a solution of (P_λ), for $v \in H_0^1(\Omega)$, we may write

$$\widetilde{I}_\lambda(v) = \frac{1}{2}\big\|v^-\big\|^2 + \frac{1}{2}\big\|u_0 + v^+\big\|^2 - \int_\Omega H_\lambda\big(x, u_0 + v^+\big) - \frac{1}{2}\|u_0\|^2 + \int_\Omega H_\lambda(x, u_0).$$

From (2.2) we infer that

$$\widetilde{I}_\lambda(v) = \frac{1}{2}\big\|v^-\big\|^2 + I_\lambda\big(u_0 + v^+\big) - I_\lambda(u_0) \quad \text{for every} \quad v \in H_0^1(\Omega). \tag{5.11}$$

Now suppose, by contradiction, that there is not another critical point of I_λ with $u > u_0$ in Ω. Then, using Lemma 5.6 and taking $\rho = (\rho_1 + \rho_2)/2$, we get $\widetilde{\alpha} > 0$ such that

$$\inf_{\|v\|=\rho} \widetilde{I}_\lambda(v) > \widetilde{\alpha}.$$

Next, considering M_n given by Proposition 5.2, there exists $t_0 > 0$ such that $\|t_0 M_n\| > \rho$ and

$$\widetilde{I}_\lambda(t_0 M_n) = I_\lambda(u_0 + t_0 M_n) - I_\lambda(u_0) < 0. \tag{5.12}$$

Let us define the minimax level

$$\widetilde{c}_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \widetilde{I}_\lambda(\gamma(t)) > 0,$$

where $\widetilde{\Gamma} := \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0, \gamma(1) = t_0 M_n\}$. Note that by Proposition 5.2 we have

$$\widetilde{\alpha} \leq \widetilde{c}_\lambda \leq \max_{t \geq 0} I_\lambda(u_0 + t M_n) - I_\lambda(u_0) < \frac{2\pi}{\alpha_0}. \tag{5.13}$$

Now, we can apply a version of the mountain pass theorem without the Palais-Smale condition (see [10]) to obtain a sequence $(v_n) \subset H_0^1(\Omega)$ be a sequence such that

$$\widetilde{I}_\lambda(v_n) \to \widetilde{c}_\lambda \quad \text{and} \quad \left\| \widetilde{I}'_\lambda(v_n) \right\| \to 0. \tag{5.14}$$

Since $\|v_n^-\|^2 = -\langle \widetilde{I}'_\lambda(v_n), v_n^- \rangle$, we have that

$$\|v_n^-\| \to 0. \tag{5.15}$$

Hence, for proving the claim, it suffices to verify that (v_n^+) has a convergent subsequence. Combining (5.11), (5.14) and (5.15), we obtain

$$I_\lambda\Big(u_0 + v_n^+\Big) \to \widetilde{c}_\lambda + I_\lambda(u_0). \tag{5.16}$$

Using that u_0 is a critical point of I_λ, for $v \in H_0^1(\Omega)$, we may write

$$\begin{aligned} \Big\langle \widetilde{I}'_\lambda(v_n), v \Big\rangle &= -\big\langle v_n^-, v \big\rangle + \big\langle v_n^+, v \big\rangle - \int_\Omega \big[h_\lambda\big(x, u_0 + v_n^+\big) - h_\lambda(x, u_0) \big] v \\ &= -\big\langle v_n^-, v \big\rangle + \big\langle I'_\lambda\big(u_0 + v_n^+\big), v \big\rangle. \end{aligned}$$

This, together with (5.14), (5.15) and (5.16) imply that $w_n = u_0 + v_n^+$ satisfies

$$\big\| I'_\lambda(w_n) \big\| \to 0 \quad \text{and} \quad I_\lambda(w_n) \to c_\lambda := \widetilde{c}_\lambda + I_\lambda(u_0). \tag{5.17}$$

By (5.13) we have

$$I_\lambda(u_0) < c_\lambda < I_\lambda(u_0) + \frac{2\pi}{\alpha_0}. \tag{5.18}$$

Since $\lambda_1 < \lambda < \Lambda \leq \lambda_1^D(\Omega^0)$, by Lemma 3.2 we may suppose that $w_n \rightharpoonup w = u_0 + v$ with $v \geq 0$ and $I'(w) = 0$ and so $w = u_0$. Furthermore, by Lemma 3.4 we have

$$\int_\Omega W(x) F(w_n) \to \int_\Omega W(x) F(u_0). \tag{5.19}$$

This together with the fact that $w_n \to u_0$ in $L^2(\Omega)$ and (5.17) imply that

$$\lim_{n\to\infty} \|w_n\|^2 = 2(c_\lambda + c_1) > 0, \tag{5.20}$$

where $c_1 := \int_\Omega [W(x)F(u_0) + \frac{\lambda}{2}|u_0|^2]$. We claim that

$$\lim_{n\to\infty} \int_\Omega W(x)f(w_n)w_n = \int_\Omega W(x)f(u_0)u_0. \tag{5.21}$$

If this is true, using that $w_n^+ \to u_0$ strongly in $L^2(\Omega)$ we get

$$\begin{aligned} o_n(1) = I'_\lambda(w_n)w_n &= \|w_n\|^2 - \lambda|u_0|_2^2 - \int_\Omega W(x)f(u_0)u_0 + o_n(1) \\ &= \|w_n\|^2 - \|u_0\|^2 + I'_\lambda(u_0)u_0 + o_n(1). \end{aligned}$$

Since $I'_\lambda(u_0) = 0$, we obtain that $w_n \to u_0$ strongly in $H_0^1(\Omega)$ and hence

$$I_\lambda(u_0) < c_\lambda = \lim_{n\to\infty} I_\lambda(w_n) = I_\lambda(u_0),$$

which is a contradiction. Thus, it remains to verify (5.21). Define $z_n := w_n/\|w_n\|$ and observe that

$$z_n \rightharpoonup z := \frac{u_0}{\sqrt{2(c_\lambda + c_1)}} \quad \text{weakly in} \quad H_0^1(\Omega).$$

If $\|z\| = 1$ then $\lim_{n\to\infty} \|w_n\|^2 = 2(c_\lambda + c_1) = \|u_0\|^2$ and the result follows immediately. Suppose that $\|z\| < 1$. By (5.18), we can take $\alpha > \alpha_0$ such that $0 < c_\lambda < I_\lambda(u_0) + 2\pi/\alpha$, which implies that

$$2\alpha(c_1 + c_\lambda) < \frac{4\pi}{1 - \|z\|^2}.$$

Thus, we can use (5.20) to obtain $p_0 > 0$ such that $\alpha\|w_n\|^2 < p_0 < 4\pi/(1 - \|z\|^2)$ for n large. We now choose $q > 1$ sufficiently close to 1 in such way that

$$\alpha q\|w_n\|^2 < p < \frac{4\pi}{1 - \|z\|^2},$$

with $p = p_0 q$. It follows from Lemma 3.7 that

$$\sup_{n\in\mathbb{N}} \int_\Omega e^{\alpha q\|w_n\|^2 z_n^2} \le \sup_{n\in\mathbb{N}} \int_\Omega e^{p z_n^2} < \infty.$$

Arguing similarly as in the proof of Theorem 2.4, we conclude that (5.21) holds. The proof of the theorem is complete.

6. Proof of Theorem 1.3

In this section, we present the proof of Theorem 1.3. First, we establish an auxiliary result. Setting $\Omega^* := \Omega \setminus \overline{\Omega^-}$, we have that $\partial\Omega^*$ is smooth since $\partial\Omega^0$ is smooth (see Lemma 7.1 in [25]).

Lemma 6.1. *Suppose* (f_1) *and* (W_2) *are satisfied and* f *is nonnegative. If* $\partial\Omega^0$ *is smooth, then Problem* (P_λ) *does not have a positive solution for every* $\lambda \geq \lambda_1(\Omega^*)$.

Proof. Let $u \in H_0^1(\Omega)$ be a positive solution of (P_λ). Next, take φ_1^* a positive eigenfunction on Ω^* associated with $\lambda_1(\Omega^*)$, the first eigenvalue of the operator $-\Delta$ in the space $H_0^1(\Omega^*)$. By Hopf's lemma, one has $\frac{\partial \varphi_1^*}{\partial \nu} < 0$ on $\partial\Omega^*$, where $\nu(x)$ is the exterior unit normal to Ω^* at $x \in \partial\Omega^*$. Using this estimate, the Divergence Theorem, the fact that $u \in C^1(\overline{\Omega})$ and $u > 0$ in Ω^*, we obtain

$$\int_{\Omega^*} \nabla u \nabla \varphi_1^* = \int_{\partial\Omega^*} u \frac{\partial \varphi_1^*}{\partial \nu} + \int_{\Omega^*} u\big(-\Delta\varphi_1^*\big) < \lambda_1(\Omega^*) \int_{\Omega^*} u\varphi_1^*.$$

On the other hand, using that $\varphi_1^* = 0$ in $\Omega \setminus \Omega^*$ and f is nonnegative, we get

$$\int_{\Omega^*} \nabla u \nabla \varphi_1^* = \lambda \int_{\Omega^*} u\varphi_1^* + \int_{\Omega^*} W(x) f(u) \varphi_1^* \geq \lambda \int_{\Omega^*} u\varphi_1^*,$$

which implies the desired result. □

Proof of Theorem 1.3. By Lemma 6.1, $\Lambda \leq \lambda_1(\Omega^*)$. Since the hypothesis (W_3) implies that $\lambda_1(\Omega^*) < \lambda_1(\Omega^0)$, we may conclude that $\lambda_1 < \Lambda < \lambda_1(\Omega^0)$. Now consider a sequence $\lambda_n \to \Lambda$ $(\lambda_n < \Lambda)$. Invoking Remark 5.5, we find a sequence $(u_n) \subset H_0^1(\Omega)$ of positive solutions of (P_{λ_n}) such that $I_{\lambda_n}(u_n) < 0$. Thus,

$$I_\Lambda(u_n) = I_{\lambda_n}(u_n) - \frac{\Lambda - \lambda_n}{2} |u_n|_2^2 \leq I_{\lambda_n}(u_n) < 0. \tag{6.1}$$

Given $\varphi \in H_0^1(\Omega)$, one has

$$\Big\langle I'_\Lambda(u_n), \varphi \Big\rangle = \Big\langle I'_{\lambda_n}(u_n), \varphi \Big\rangle - (\Lambda - \lambda_n)\langle u_n, \varphi\rangle_2 = -(\Lambda - \lambda_n)\langle u_n, \varphi\rangle_2, \tag{6.2}$$

which implies that $\|I'_\Lambda(u_n)\| = o_n(\|u_n\|)$. From (6.1), (6.2), $\Lambda \leq \lambda_1(\Omega^*) < \lambda_1(\Omega^0)$ and the argument used in the proof of Lemma 3.2, it follows that the sequence $(u_n) \subset H_0^1(\Omega)$ is bounded. Then, passing to a subsequence if necessary, we may assume that $u_n \rightharpoonup u_\Lambda$ weakly in $H_0^1(\Omega)$, $u_n \to u_\Lambda$ in $L^2(\Omega)$ and $u_\Lambda \geq 0$ in Ω. Moreover, by Lemma 3.4, we have that $I'_\Lambda(u_\Lambda) = 0$ and

$$\int_\Omega W(x) F(u_n) \to \int_\Omega W(x) F(u_\Lambda).$$

Next, using that $I_{\lambda_n}(u_n) < 0$ we get

$$\|u_n\|^2 = 2I_{\lambda_n}(u_n) + 2\lambda_n|u_n|_2^2 + 2\int_\Omega W(x)F(u_n) < 2\lambda_n|u_n|_2^2 + 2\int_\Omega W(x)F(u_n).$$

Consequently, if $u_\Lambda = 0$ then $u_n \to 0$ strongly in $H_0^1(\Omega)$. Furthermore, by the same argument used in the proof of Lemma 3.2, we obtain $u_n \to 0$ strongly in $L^\infty(\Omega)$. Now, taking φ_1, the positive eigenfunction in $H_0^1(\Omega)$ associated with the eigenvalue λ_1, and using that u_n is a critical point of I_{λ_n}, for every $n \in \mathbb{N}$, we may write

$$0 = -\left\langle I'_{\lambda_n}(u_n), \varphi_1 \right\rangle = \int_\Omega [(\lambda_n - \lambda_1)u_n + W(x)f(u_n)]\varphi_1. \tag{6.3}$$

On the other hand, since $\lambda_n - \lambda_1 \to \Lambda - \lambda_1 > 0$, $u_n \to 0$ in $L^\infty(\Omega)$, $u_n > 0$ in Ω, we may use the hypothesis (f_5) to get, for n sufficiently large,

$$\int_\Omega [(\lambda_n - \lambda_1)u_n + W(x)f(u_n)]\varphi_1 = \int_\Omega \left[(\lambda_n - \lambda_1) + W(x)\frac{f(u_n)}{u_n}\right] u_n\varphi_1 > 0,$$

which contradicts (6.3). This contradiction implies that $u_\Lambda \neq 0$. Finally, since u_Λ is a nonnegative solution of (P_Λ), by the Maximum Principle, u_Λ is a positive solution of (P_Λ). The proof of Theorem 1.3 is complete. □

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