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Weakly hyperbolic systems by symmetrization

FERRUCCIO COLOMBINI, TATSUO NISHITANI AND JEFFREY RAUCH

Abstract. We prove Gevrey well posedness of the Cauchy problem for general linear systems whose principal symbol is hyperbolic and coefficients are sufficiently Gevrey regular in x and either Lipschitzian or Hölderian in time. Such results date to the seminal paper of Bronshtein. Our proof is by an energy method using a pseudodifferential symmetrizer. The construction of the symmetrizer is based on a Lyapunov function for ordinary differential equations. The method yields new estimates and existence uniformly for spectral truncations and parabolic regularizations.

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1. Introduction

Consider the Cauchy problem for first order systems,

$$Lu \;=\; \partial_t u - \sum_{j=1}^{d} A_j(t,x)\partial_{x_j} u + B(t,x)u \;=\; f, \quad u(0,\cdot) = g\,. \tag{1.1}$$

The coefficients A_j and B take values in the $m \times m$ complex matrices.

Definition 1.1. The *principal symbol* is

$$\tau I \;-\; A(t,x,\xi) \quad \text{with} \quad A(t,x,\xi) \;:=\; \sum_{j=1}^{d} A_j(t,x)\,\xi_j\,.$$

The *characteristic polynomial* is $\det(\tau I - A(t,x,\xi))$. The operator is *hyperbolic*, assumed throughout, when

$$\text{for all } t,x,\xi \in \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d, \qquad \text{Spectrum}\, A(t,x,\xi) \;\subset\; \mathbb{R}\,. \tag{1.2}$$

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Hyperbolicity (1.2) is a necessary condition for the Cauchy problem (1.1) to be well posed for non analytic data (see [15] for Gevrey data and history). It is not sufficient for well posedness for C^∞ data. For most non strictly hyperbolic scalar operators, most lower order terms lead to initial value problems that are ill posed in the C^∞ category. The generic ill posedness holds even if the coefficients are real analytic functions or even constant.

For hyperbolic operators with real analytic coefficients, [8,22] showed that for Gevrey initial data, $\mathcal{G}^s$ with $1 < s < s_0$, there are Gevrey solutions. No condition of E. Levi-type is needed. It came as a surprise to many, including us, when Bronshtein [2] proved that the Cauchy problem for linear hyperbolic partial differential operators, even those with coefficients that are only finitely smooth in time and Gevrey in x, is well posed for Gevrey data. Bronshtein, Ohya-Tarama [19], and Kajitani [13, 14] used parametric constructions either by studying the resolvent close the imaginary axis or by Fourier integral operator constructions. The papers [3, 4, 16, 17] use energy methods of increasing complexity. In this paper we introduce a new energy method that we think is simpler and more natural. Our estimates are proved in spite of ignoring the detailed behavior of the eigenvalue crossings.

Well posedness for Gevrey data for hyperbolic systems is often treated by multiplying the system by the cofactor system to reduce to scalar operators. That approach at least two weaknesses. First applying the cofactor matrix requires that the coefficients have a number of derivatives in time roughly equal to the size of the matrix. Second, this destroys the details of the system structure. For example if a system is merely two copies of a strictly hyperbolic system, the cofactor approach immediately replaces the problem with a scalar problem with double roots, which is much more sensitive to perturbations.

We prove Gevrey *a priori* estimates for first order hyperbolic systems by constructing a pseudodifferential symmetrizer. The symmetrizer is motivated by a special Lyapunov function for asymptotically stable first order systems of linear ordinary differential equations. The proof not only gives *a priori* estimates, but also quantifies some effects coming from the block structure of the system.

This paper discusses only the existence and uniqueness of solutions. The method of [5] gives the natural precise estimate for the influence domain. This allows one to eliminate our hypothesis that the coefficients are independent of x outside a compact subset of space.

In Hypothesis 2.8, we associate, to our systems, an index θ ($0 \le \theta \le m - 1$). The value of θ measures roughly whether the Taylor polynomial of degree $N = \max\{2\theta, m\}$ of the symbol can be uniformly block diagonalized with blocks of size $\theta + 1$. It is always satisfied with $\theta = m - 1$.

The uniformly Gevrey s functions on $\mathbb{R}^d$ (see definition (2.3)) are denoted $\mathcal{G}^s(\mathbb{R}^d)$ and those of compact support by $\mathcal{G}^s_0(\mathbb{R}^d)$. In the results below, the Gevrey index s_0 is usually cruder, that is smaller, than the sharp results of [3,4,21] valid for coefficients only depending on time. An exception is the case of uniformly diagonalisable systems. The Gevrey index in Theorems 1.3 and 1.5 cannot be improved for those systems. The result for Lipschitz coefficients in time is the following.

Hypothesis 1.2. The coefficients A_j and B are independent of x for x outside a fixed compact in $\mathbb{R}^d$. The source terms satisfy $g \in \mathcal{G}_0^s(\mathbb{R}^d)$, and $f \in L^1_{\text{loc}}(\mathbb{R}; \mathcal{G}_0^s(\mathbb{R}^d))$.

Theorem 1.3. *Suppose Hypotheses* 2.8 *is satisfied. Define*

$$s_0 \;:=\; \max\left\{\frac{2+6\theta}{1+6\theta}\,,\,\frac{3+4\theta}{2+4\theta}\right\}.$$

For some $1 < s \le s_0$ *suppose that Hypothesis* 1.2 *is satisfied and that* $A_j(t,x)$ (*respectively* $B(t,x)$) *are Lipschitz* (*respectively continuous*) *in time uniformly on compact sets with values in* $\mathcal{G}^s(\mathbb{R}^d)$. *Then there is a* $T_0 > 0$ *and a unique local solution* $u \in C([0,T_0]\,;\mathcal{G}_0^s(\mathbb{R}^d))$ *to the Cauchy problem* (1.1).

Remark 1.4. The proof shows that for all constants $c > 0$ and $T > 0$ the interval of existence can be chosen uniformly for data satisfying

$$\int |\hat{g}(\xi)|^2 e^{c\langle\xi\rangle^{1/s}} d\xi \;+\; \int_0^T \left(\int |\hat{f}(t,\xi)|^2 e^{c\langle\xi\rangle^{1/s}} d\xi\right)^{1/2} dt \;<\; \infty\,.$$

An analogous remark applies to Theorem 1.5.

The next result concerns coefficients that are Hölder continuous in time.

Theorem 1.5. *Suppose that* $0 < \kappa < 1$ *and that Hypothesis* 2.8 *holds. Define*

$$s_0 \;:=\; \min\left\{\frac{1+\kappa+(2+\kappa)\theta}{1+(2+\kappa)\theta}\,,\;\max\left\{\frac{2+6\theta}{1+6\theta}\,,\,\frac{3+4\theta}{2+4\theta}\right\}\right\}.$$

For some $1 < s \le s_0$ *suppose that Hypothesis* 1.2 *is satisfied and that* $A_j(t,x)$ (*respectively* $B(t,x)$) *are* κ *Hölder continuous* (*respectively continuous*) *in time uniformly on compact sets with values in* $\mathcal{G}^s(\mathbb{R}^d)$. *Then the conclusion of Theorem* 1.3 *holds.*

The idea of the symmetrization is straightforward. We multiply by a positive Hermitian pseudodifferential operator to derive estimates. The change of variables $v = e^{a\langle D\rangle^\rho t}u$ replaces the operator L with $L - a\langle D\rangle^\rho$. Choosing $a \gg 1$ and $0 < \rho < 1$ appropriately, the matrix

$$M(t,x,\xi) \;=\; A(t,x,\xi) \;+\; B(t,x) \;-\; a\langle\xi\rangle^\rho$$

has spectrum with real part $\le -\langle\xi\rangle^\rho$ for all t, x, ξ. For the ordinary differential equation $X' = MX$ with M evaluated at a fixed $\underline{t}, \underline{x}, \underline{\xi}$, the positive definite matrix

$$R(t,x,\xi) \;:=\; \int_0^\infty \left(e^{Ms}\right)^* e^{Ms}\, ds$$

defines a strict Lyapunov function, that is $RM + M^*R < 0$. Our symmetrizer is based on $R(t,x,D)$. This multiplier method has other advantages. For example, it yields uniform estimates in h for the filtered operators

$$\partial_t + \chi(hD)\Big(\sum_j A_j\partial_j + B\Big)\chi(hD) \text{ with } \chi \in \mathcal{S}(\mathbb{R}^d),\quad \chi(0)=1, \text{ and } \quad 0 < h \ll 1.$$

We use these to prove existence. The filtered operators are related to the spectral method analysed in [6]. Similar uniform estimates are valid for the parabolic regularizations,

$$\partial_t \ + \ \sum_j A_j\partial_j \ + \ B \ - \ \epsilon\,\Delta\,.$$

These requires a little work whose details are omitted for sake of brevity. Parabolic regularizations represent dissipative effects neglected in hyperbolic model equations in sciences.

2. Three important preliminary results

2.1. Hyperbolicity and spectral bounds

Hypothesis 2.1. Suppose $\Omega \subset \mathbb{R}^d$ is open, and that $A(t,x) \in C^0\big(\mathbb{R}\,;C^{m+1}(\Omega)\big)$ is an $m \times m$ matrix valued function. Assume that

$$\text{Spectrum}\, A(t,x) \ \subset \ \mathbb{R} \ \text{ for all } \ (t,x) \in \mathbb{R}\times\Omega\,. \tag{2.1}$$

Define

$$H(t,x,y,s) \ := \ \sum_{|\alpha|\le m} \frac{s^{|\alpha|}}{\alpha!}\, y^\alpha\, \partial_x^\alpha A(t,x)\,.$$

The values $H(t,x,y,is)$ for $y \neq 0$ and s real give an extension of A to complex arguments t and $x+isy$. The Taylor polynomial H appears in the Gevrey conjugation Proposition 2.6. The next proposition gives spectral bounds on H.

Proposition 2.2. *If Hypothesis* 2.1 *holds then for any* $T > 0$ *and compact set* $K \subset \Omega$ *there exist* $\delta > 0$ *and* $C > 0$ *so that for all* $x \in K$, $|t| \le T$, $|y| \le 1$ *and* $|s| \le \delta$,

$$\zeta \text{ is an eigenvalue of } H(t,x,y,is) \quad \Longrightarrow \quad |\mathrm{Im}\,\zeta| \le C|s|\,.$$

The long proof of this result is presented in Section 7.

2.2. Gevrey conjugation

Denote

$$\langle \xi \rangle_\ell \ := \ \sqrt{\ell^2 + |\xi|^2} \ = \ \ell \sqrt{1 \ + \ |\xi/\ell|^2} \tag{2.2}$$

where $\ell \geq 1$ is a positive parameter that will be chosen large. Denote $\langle \xi \rangle_1 = \langle \xi \rangle$ and note that $\langle \xi \rangle \leq \langle \xi \rangle_\ell \leq \ell \, \langle \xi \rangle$.

Definition 2.3. If $1 < s < \infty$, the function $a(x) \in C^\infty(\mathbb{R}^d)$ belongs to $\mathcal{G}^s(\mathbb{R}^d)$ if there exist $C > 0$ and $A > 0$ such that

$$\left| \partial_x^\alpha a(x) \right| \ \leq \ C A^{|\alpha|} |\alpha|!^s \quad \text{for all } \ x \in \mathbb{R}^d, \quad \text{for all } \ \alpha \in \mathbb{N}^d \, .$$

Denote $\mathcal{G}_0^s(\mathbb{R}^d) := \mathcal{G}^s(\mathbb{R}^d) \cap C_0^\infty(\mathbb{R}^d)$.

Definition 2.4. For $0 < \delta \leq \rho \leq 1$, the family $a(x, \xi; \ell) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ indexed by ℓ belongs to $\tilde{S}^m_{\rho,\delta}$ if for all $\alpha, \beta \in \mathbb{N}^d$ there is a constant $C_{\alpha\beta}$ independent of $\ell \geq 1, x, \xi$ such that

$$\left| \partial_x^\beta \partial_\xi^\alpha a(x, \xi; \ell) \right| \ \leq \ C_{\alpha\beta} \ \langle \xi \rangle_\ell^{m - \rho|\alpha| + \delta|\beta|} .$$

Denote $\tilde{S}^m := \tilde{S}^m_{1,0}$.

Definition 2.5. For $1 < s$, and $m \in \mathbb{R}$, the family $a(x, \xi; \ell) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ belongs to $\tilde{S}^m_{(s)}$ if there exist constants $C > 0$ and $A > 0$ independent of $\ell \geq 1, x, \xi$ such that for all $\alpha, \beta \in \mathbb{N}^d$,

$$\left| \partial_x^\beta \partial_\xi^\alpha a(x, \xi; \ell) \right| \ \leq \ C \ A^{|\alpha + \beta|} \ |\alpha + \beta|!^s \ \langle \xi \rangle_\ell^{m - |\alpha|} \, .$$

We often write $a(x, \xi)$ for $a(x, \xi; \ell)$ dropping the ℓ. If $a(x, \xi)$ is the symbol of a differential operator of order m with coefficients $a_\alpha(x) \in \mathcal{G}^s(\mathbb{R}^d)$ then $a(x, \xi) \in \tilde{S}^m_{(s)}$ because $|\partial_\xi^\beta \xi^\alpha| \leq C A^{|\beta|} |\beta|! \langle \xi \rangle_\ell^{|\alpha| - |\beta|}$ and $|\partial_x^\beta a_\alpha(x)| \leq C_\alpha A_\alpha^{|\beta|} |\beta|!^s$ for any $\beta \in \mathbb{N}^d$.

Proposition 2.6. *Suppose $1/2 \leq \rho < 1$ and $s = 1/\rho$, and let $a(x, \xi)$ be $m \times m$ matrix valued with entries in $\tilde{S}^1_{(s)}$ and $\partial_x^\alpha a(x, \xi) = 0$ outside $|x| \leq R$ for some $R > 0$ if $|\alpha| > 0$. Then the operator $b(x, D) = e^{\tau \langle D \rangle_\ell^\rho} a(x, D) e^{-\tau \langle D \rangle_\ell^\rho}$ is for small $|\tau|$, a pseudodifferential operator with symbol given by*

$$b(x, \xi) \ = \ \sum_{|\alpha| \leq k} \frac{1}{\alpha!} \, D_x^\alpha a(x, \xi) \left(\tau \nabla_\xi \langle \xi \rangle_\ell^\rho \right)^\alpha \ + \ R(x, \xi) \quad \text{and} \quad D_{x_j} := -i \frac{\partial}{\partial x_j} \, ,$$

with $R(x, \xi) \in \tilde{S}^{\max\{\rho - k(1-\rho), \, -1+\rho\}}$.

For completeness a proof is given in Section 6.

If a were real analytic in x then the sum on the right would be

$$\sum_{|\alpha|\le k} \frac{\partial_x^\alpha a}{\alpha!}\,(-iy)^\alpha \;=\; a(x-iy\,,\,\xi)\;+\;O\big(|y|^{k+1}\big)\,, \qquad y=\tau\nabla_\xi\langle\xi\rangle_\ell^\rho\,.$$

When $|\xi|\to\infty$, then y tends to zero because $\rho<1$. Therefore, this is a very small displacement in the complex direction. In [8,22] the coefficients were analytic and one could make such complex displacements. For our problems, the coefficients are not analytic and the replacement for complex displacement is to put complex arguments into Taylor polynomials. An alternative method is to take an almost analytic extension of a that satisfies the Cauchy-Riemann equations with error $O(|y|^\infty)$ at $y=0$.

Corollary 2.7. *If $a(x,\xi)\in\tilde S^0_{(s)}$ then $e^{\tau\langle D\rangle_\ell^\rho}a(x,D)e^{-\tau\langle D\rangle_\ell^\rho}\in \mathrm{Op}\,\tilde S^0$ for small $|\tau|$.*

Proof. Choose k so that $\rho-k(1-\rho)\le 0$. Then $D_x^\alpha a\,(x,\xi)\,(\tau\nabla_\xi\langle\xi\rangle_\ell^\rho)^\alpha\in\tilde S^{-(1-\rho)|\alpha|}\subset\tilde S^0$ for $a\in\tilde S^0$. The assertion follows from Proposition 2.6. □

2.3. The block size barometer θ

We introduce an integer valued parameter θ $(0\le\theta\le m-1)$ that measures the extent to which the principal symbol can be block diagonalized by matrices bounded with bounded inverse. For example in the strictly hyperbolic case, blocks of size 1 are attainable. The easy case of the Kreiss Matrix Theorem[1] asserts that block size one is equivalent to $e^{iA(t,x,\xi)}\in L^\infty(\mathbb{R}^d_\xi)$ locally uniformly in t,x. By convention θ is one smaller than the block size. Block size m and therefore $\theta=m-1$ is always possible. The definition of θ is not directly given in these terms. The relation to block size is discussed in the examples.

Assume that $A_j(t,x)\in C^0(\mathbb{R}\,;C^\infty(\mathbb{R}^d))$ and (1.2) is satisfied. Proposition 2.2 implies that for any $T>0$ and compact set $K\subset\mathbb{R}^d$ there exist $\delta>0$ and $c>0$ such that if ζ is an eigenvalue of

$$\sum_{|\alpha+\beta|\le m}\frac{(is)^{|\alpha+\beta|}}{\alpha!\beta!}\,\partial_x^\alpha\partial_\xi^\beta A(t,x,\xi)\;y^\alpha\eta^\beta \tag{2.3}$$

then $|\mathrm{Im}\,\zeta|\le c\,|s|$ for any $|(y,\eta)|\le 1$, $x\in K$, $|\xi|\le 1$, $|t|\le T$. Define

$$\mathcal{H}_r(t,x,\xi;\epsilon)\;:=\;\sum_{|\alpha|\le r}\frac{\epsilon^{|\alpha|}}{\alpha!}\,D_x^\alpha A(t,x,\xi)\,\xi^\alpha.$$

Choosing $(y,\eta)=(\xi,0)$ in (2.3) we see that there is $\epsilon_0>0$, $c>0$ such that

$$\zeta \text{ is an eigenvalue of } \mathcal{H}_m(t,x,\xi;\epsilon)\;\Longrightarrow\;|\mathrm{Im}\,\zeta|\le c\,|\epsilon| \tag{2.4}$$

for any $x\in K$, $|\xi|\le 1$, $|\epsilon|\le\epsilon_0$, $|t|\le T$. Introduce θ defined as follows.

[1] e^{itA} bounded forward and backward in time, see [20].

Hypothesis 2.8. Assume the system is θ*-regular* with integer $0 \le \theta \le m-1$ in the sense that for any $T > 0$ and any compact $K \subset \mathbb{R}^d$ there exist $C > 0$ and $c > 0$ and $\epsilon_0 > 0$ such that with $N = \max\{2\theta, m\}$

$$\frac{\epsilon^\theta}{C\, e^{cs\epsilon}} \;\le\; \left\| e^{is\mathcal{H}_N(t,x,\xi;\epsilon)} \right\| \;\le\; \frac{C\, e^{cs\epsilon}}{\epsilon^\theta}\,, \tag{2.5}$$

for all $s \ge 0, 0 < \epsilon \le \epsilon_0, |\xi| = 1, x \in K, |t| \le T$.

A system that is θ-regular is ϕ-regular for all $\theta < \phi \le m-1$.

Denote

$$H_N(\rho, \ell, \tau, t, x, \xi) \;:=\; \sum_{|\alpha| \le N} \frac{1}{\alpha!} D_x^\alpha A(t,x,\xi) \left(\tau \nabla_\xi \langle \xi \rangle_\ell^\rho\right)^\alpha .$$

The definition of $\mathcal{H}_N$ implies that

$$H_N(\rho, \ell, \tau, t, x, \xi) \;=\; \langle \xi \rangle_\ell\, \mathcal{H}_N\Big(t\,,\; x\,,\; \xi/\langle \xi \rangle_\ell\,;\; \tau\rho \langle \xi \rangle_\ell^{\rho-1}\Big).$$

Choosing $s\langle \xi \rangle_\ell, \tau\rho\langle \xi \rangle_\ell^{\rho-1}$ $(\tau > 0), \xi/\langle \xi \rangle_\ell$ for s, ϵ, ξ in (2.5) yields

$$\frac{\tau^\theta}{C\, \langle \xi \rangle_\ell^{\theta(1-\rho)}\, e^{cs\tau\langle \xi \rangle_\ell^\rho}} \;\le\; \left\| e^{isH_N(\ell,\tau,t,x,\xi)} \right\| \;\le\; \frac{C\, \langle \xi \rangle_\ell^{\theta(1-\rho)}\, e^{cs\tau\langle \xi \rangle_\ell^\rho}}{\tau^\theta} \tag{2.6}$$

for $|t| \le T, \ell \ge \ell_0$ where τ, ℓ_0 are constrained to satisfy

$$0 \;<\; \tau \ell_0^{\rho-1} \;\le\; \epsilon_0\,. \tag{2.7}$$

Example 2.9. Estimate (2.5) always holds with $\theta = m-1$. Indeed write $\mathcal{H}_N = \mathcal{H}_m + L_N$ where $\|L_N\| \le C\epsilon^{m+1}$. Take an orthogonal matrix T such that $T\mathcal{H}_m T^{-1}$ is upper triangular. Let $S = \operatorname{diag}(1, \epsilon, \ldots, \epsilon^{m-1})$ then $ST\mathcal{H}_m(ST)^{-1} = \operatorname{diag}(\lambda_1, \ldots, \lambda_m) + K$ with $\|K\| \le C\epsilon$. From (2.4) we have $|\operatorname{Im}\lambda_j| \le c_1|\epsilon|$. This proves that $|\operatorname{Re} iST\mathcal{H}_m(ST)^{-1}X, X)| \le C|\epsilon||X|^2$ for any $X \in \mathbb{C}^d$. Therefore $e^{-cs\epsilon} \le \|(ST)e^{is\mathcal{H}_N}(ST)^{-1}\| \le e^{cs\epsilon}$ for $0 \le \epsilon \le \epsilon_0$ with some $c > 0, \epsilon_0 > 0$ because $\|STL_N(ST)^{-1}\| \le C\epsilon$. Since $\|S^{-1}\| \le C\epsilon^{-(m-1)}$ and $\|S\| \le C$ this implies the desired bounds,

$$\epsilon^{m-1}e^{-cs\epsilon}/C \le \left\| e^{is\mathcal{H}_N} \right\| \le C\epsilon^{-(m-1)}e^{cs\epsilon}.$$

Example 2.10. If $A(t,x,\xi)$ is uniformly diagonalizable then (2.5) holds with $\theta = 0$. Indeed, choose $T = T(t,x,\xi)$ with uniform bounds of $\|T\|$ and $\|T^{-1}\|$ independent of (t,x,ξ) such that $T^{-1}A(t,x,\xi)T = \operatorname{diag}\{i\lambda_j\}$ is diagonal with real λ_j. Then

$$T^{-1}e^{is\mathcal{H}_m}T = e^{isT^{-1}\mathcal{H}_m T}, \quad \text{and} \quad T^{-1}\mathcal{H}_m T = \operatorname{diag}\{i\lambda_j\} + A_1, \quad \text{with} \quad \|A_1\| \le C\epsilon.$$

This implies the desired bounds,

$$e^{-cs\epsilon}/C \le \|e^{is\mathcal{H}_m}\| \le C\, e^{cs\epsilon}.$$

Example 2.11. Suppose that there exists $T = T(t,x,\xi;\epsilon)$ with bounds on $\|T\|$ and $\|T^{-1}\|$ independent of $(t,x,\xi;\epsilon)$ such that $T^{-1}\mathcal{H}_m T$ is a direct sum $\oplus A_j$ where the size of A_j is at most μ. Then (2.5) holds with $\theta = \mu - 1$. This follows by arguments as in Example 2.9. Our results take account of this purely system behavior. Roots of high multiplicity but small blocks behave according to the size of the blocks and not the multiplicity. In particular if each $A_j(t,x)$ is block diagonal with r blocks $a_{jk}(t,x)$ of the same size $\ell \times \ell$ ($m = \ell r$) and $a_k(t,x,\xi) = \sum_{j=1}^{d} a_{jk}(t,x)\xi_j$ has a same eigenvalue $\underline{\tau}$ of multiplicity ℓ at $(\underline{t}, \underline{x}, \underline{\xi})$ ($\underline{\xi} \neq 0$) then Theorem 1.3 holds with $s_0 = (4\ell-1)/(4\ell-2)$ (one can choose $\theta = \ell - 1$) while previous results assert no more than $s_0 = \ell r/(\ell r - 1)$ which clearly is smaller than $(4\ell-1)/(4\ell-2)$ if $r \geq 4$.

Example 2.12. Suppose that there is $r \in \mathbb{N}$ such that for any (t,x,ξ,ϵ) we can find $c(t,x,\xi,\epsilon) \in \mathbb{C}$ such that

$$\mathrm{Rank}\Big(\mathcal{H}_m(t,x,\xi,\epsilon) - c(t,x,\xi,\epsilon)I\Big) \leq r\,.$$

Then hypothesis (2.5) holds with $\theta = r$.

3. The symmetrizer construction

3.1. Lyapunov function for linear ODE

Suppose that M is a matrix whose eigenvalues all lie in the open left half plane $\{\mathrm{Re}\, z < 0\}$. The solutions $X(t)$ of the ordinary differential equation $X' = MX$ tend exponentially to zero as $t \nearrow \infty$. Lyapunov proved that there are positive definite symmetric matrices R so that the scalar product (RX, X) is strictly decreasing on orbits. For differential equations the quantity $(R\,\cdot\,,\ \cdot)$ is called a Lyapunov function. In the partial differential equations context, R is often called a symmetrizer.

There is a remarkable explicit choice

$$R = \int_0^\infty \big(e^{sM}\big)^* e^{sM}\, ds\,. \tag{3.1}$$

For that R,

$$\begin{aligned} RM + M^*R &= \int_0^\infty e^{sM^*} e^{sM} M\, ds + \int_0^\infty M^* e^{sM^*} e^{sM}\, ds \\ &= \int_0^\infty e^{sM^*} \frac{d}{ds}\Big(e^{sM}\Big)\, ds + \int_0^\infty \frac{d}{ds}\Big(e^{sM^*}\Big) e^{sM}\, ds \\ &= \int_0^\infty \frac{d}{ds}\Big(e^{sM^*} e^{sM}\Big)\, ds = -I. \end{aligned}$$

Therefore

$$\begin{aligned}\frac{d}{dt}\big(RX(t), X(t)\big) &= \big(RX'(t), X(t)\big) + \big(RX(t), X'(t)\big)\\ &= \big(RMX(t), X(t)\big) + \big(RX(t), MX(t)\big)\\ &= \Big(\big(RM + M^*R\big)X(t)\,,\ X(t)\Big) \ = \ -\big(X(t), X(t)\big),\end{aligned}$$

proving that $(R\,\cdot\,,\ \cdot)$ is a strict Lyapunov function.

The last identity is easily understood. With $X(t) = e^{tM}X(0)$, the definition of R yields for $s > 0$,

$$\big(RX(0), X(0)\big) = \int_0^\infty \|X(t)\|^2\, dt \quad \text{and} \quad \big(RX(s), X(s)\big) = \int_s^\infty \|X(t)\|^2\, dt\,.$$

Differentiating the last with respect to s yields the formula for $(RX, X)'$.

For applications to partial differential equations one has matrices M that depend smoothly on parameters and it is important that the symmetrizers also depend smoothly. The standard constructions of Lyapunov functions depending either on Schur's unitary upper triangularization or Jordan's canonical upper triangularization do not have smooth dependence. Formula (3.1) in contrast does depend smoothly on parameters. It pays no attention to the spectral details of M. Where eigenvalues cross and the associated spectral projections usually misbehave, the formula for R does not.

The inequality $RM+M^*R < 0$ is important. It implies a negativity of symbols that translates, using the sharp Gårding inequality, to a negativity of operators.

3.2. The symmetrizer *R* and its derivatives

Assume (2.5) and hence (2.6). Define

$$M(a, \ell, \tau, \rho, t, x, \xi) \ := \ iH_N(\ell, \tau, t, x, \xi) \ - \ a\,\langle\xi\rangle_\ell^\rho$$

with

$$0 < \rho < 1 \le \min\{a, \ell\}\,. \tag{3.2}$$

Proposition 2.2 implies that there is an $a_0 \ge 1, c > 0$ so that

$$\text{Spectrum}\, M \ \subset \ \Big\{z \ : \ \operatorname{Re} z \ \le \ c(a_0 - a)\,\langle\xi\rangle_\ell^\rho\Big\}\,.$$

We suppose that

$$a \ \ge \ a_0 + 1\,. \tag{3.3}$$

The parameters τ, a, T are constrained to satisfy

$$c_1 \ \le \ c\,\tau \ \le \ T, \quad \text{and} \quad 2\,c\,T \le a \tag{3.4}$$

for some $T > c_1 > 0$. For ease of reading, the ℓ, τ, a, ρ dependence of M and R is often omitted. Introduce the candidate symmetrizer

$$R(a, \ell, \tau, \rho, t, x, \xi) \ := \ a \int_0^\infty \langle\xi\rangle_\ell^\rho \ \big(e^{sM(t,x,\xi)}\big)^* \big(e^{sM(t,x,\xi)}\big)\, ds.$$

We need lower bounds on R so that it yields good estimates and need to verify that R defines a classical symbol. Interestingly, we do not need that R is a Gevrey symbol.

The parameters ℓ, ρ, a are constrained by

$$1 \ \le \ a \ \le \ \ell^{1-\rho}. \tag{3.5}$$

Since $\|e^{sM}\| = e^{-as\langle\xi\rangle_\ell^\rho}\|e^{isH_N}\|$, (3.4) implies

$$\tau^\theta \langle\xi\rangle_\ell^{-\theta(1-\rho)} e^{-c_1 as\langle\xi\rangle_\ell^\rho}/C \ \le \ \big\|e^{sM}\big\| \ \le \ C\tau^{-\theta}\langle\xi\rangle_\ell^{\theta(1-\rho)} \ e^{-c_2 as\langle\xi\rangle_\ell^\rho}$$

with $c_1, c_2 > 0$ and $C > 0$ independent of $\ell, \tau, a, t, x, \xi, s$. This yields

$$\begin{aligned}(Rv, v) &= a\int_0^\infty \langle\xi\rangle_h^\rho \big\|e^{sM}v\big\|^2\, ds\\ &\ge C^{-2}\tau^{2\theta}\|v\|^2\langle\xi\rangle_\ell^{-2\theta(1-\rho)} \int_0^\infty a\langle\xi\rangle_\ell^\rho e^{-2c_1 as\langle\xi\rangle_\ell^\rho}\, ds\\ &\ge c'\,\tau^{2\theta}\langle\xi\rangle_\ell^{-2\theta(1-\rho)}\|v\|^2 .\end{aligned}$$

This is equivalent to the important lower bound

$$R \ge c'\,\tau^{2\theta}\langle\xi\rangle_\ell^{-2\theta(1-\rho)}. \tag{3.6}$$

Theorem 3.1. *Assume* (2.5), (3.4) *with* $K = \mathbb{R}^d$ *and* $0 \le \theta \le m-1$. *Denote* $\nu := \theta(1-\rho)$. *Suppose that* $A(t, x, \xi)$ *is Lipschitz in time uniformly on compact sets with values in the* $\tilde{S}^1(\mathbb{R}^d \times \mathbb{R}^d)$. *Then* $R(t, x, \xi)$ *(resp.* $\partial_t R$*) is bounded in time uniformly on compacts with values in* $\tilde{S}^{2\nu}_{\rho-\nu,1-\rho+\nu}(\mathbb{R}^d \times \mathbb{R}^d)$ *(resp.* $\tilde{S}^{1-\rho+3\nu}_{\rho-\nu,1-\rho+\nu}(\mathbb{R}^d \times \mathbb{R}^d)$*). That is for all* α, β *one has*

$$\begin{aligned}\big|\partial_x^\beta\partial_\xi^\alpha R(t, x, \xi)\big| &\le C_{\alpha\beta}\, a^{-|\alpha+\beta|}\,\langle\xi\rangle_\ell^{2\nu+(1-\rho+\nu)|\beta|-(\rho-\nu)|\alpha|},\\ \big|\partial_x^\beta\partial_\xi^\alpha\partial_t R(t, x, \xi)\big| &\le C_{\alpha\beta}\, a^{-|\alpha+\beta|-1}\,\langle\xi\rangle_\ell^{1-\rho+3\nu+(1-\rho+\nu)|\beta|-(\rho-\nu)|\alpha|}\end{aligned} \tag{3.7}$$

with $C_{\alpha\beta}$ *independent of* $a, \rho, \ell, \tau, t, x, \xi$.

Remark 3.2. The estimate for $\partial_x^\beta\partial_\xi^\alpha\partial_t R$ is exactly the same as the estimate for a derivative $\partial_x^\gamma\partial_\xi^\alpha R$ with $|\gamma| = |\beta| + 1$. The time derivative is like an extra space derivative.

Proof. Denote

$$X(s;t,x,\xi) := e^{sM(t,x,\xi)}v \quad \text{and} \quad X^\alpha_\beta(s;t,x,\xi) := \partial_x^\beta \partial_\xi^\alpha X(s;t,x,\xi)\,.$$

Step I. Estimates for X^α_β. We prove, by induction on $|\alpha+\beta|$, that

$$\left|X^\alpha_\beta(s)\right| \;\le\; C_{\alpha\beta}\big(s+\langle\xi\rangle_\ell^{-1}\big)^{|\alpha|}\,\big(1+s\langle\xi\rangle_\ell\big)^{|\beta|}\,\langle\xi\rangle_\ell^{\nu(|\alpha+\beta|+1)}\,e^{-cas\langle\xi\rangle_\ell^\rho}\,. \tag{3.8}$$

The constraint (3.5) implies that

$$a\langle\xi\rangle_\ell^{\rho-1}\le 1\,, \quad \text{that is} \quad 1\le a^{-1}\langle\xi\rangle_\ell^{1-\rho}. \tag{3.9}$$

The identity $\langle\xi\rangle_\ell = \ell\langle\xi/\ell\rangle$ from (2.2) implies

$$\left|\partial_\xi^\alpha\langle\xi\rangle_\ell^s\right| \lesssim \langle\xi\rangle_\ell^{s-|\alpha|} \quad \text{so} \quad \partial_\xi^\alpha\langle\xi\rangle_\ell \;=\; \ell\,\ell^{-|\alpha|}\,(\partial/\partial\zeta)^\alpha\langle\zeta\rangle\big|_{\zeta=\xi/\ell}\,.$$

Introduce

$$E(s) \;:=\; \langle\xi\rangle_\ell^\nu\,e^{-cs\,a\langle\xi\rangle_\ell^\rho}$$

so that $|X|\lesssim E(s)$ and $E(s)E(\tilde s) = \langle\xi\rangle_\ell^\nu E(s+\tilde s)$. The desired estimate (3.8) is equivalent to

$$\left|X^\alpha_\beta(s)\right| \le C_{\alpha\beta}\,\big(s+\langle\xi\rangle_\ell^{-1}\big)^{|\alpha|}\,\big(1+s\langle\xi\rangle_\ell\big)^{|\beta|}\,\langle\xi\rangle_\ell^{\nu|\alpha+\beta|}\,E(s)\,. \tag{3.10}$$

The constraints (3.4) and (3.9) imply

$$\left|M^{(\alpha)}_{(\beta)}\right| \le C_{\alpha\beta}\langle\xi\rangle_\ell^{1-|\alpha|}\,. \tag{3.11}$$

For $|\alpha|=1$ differentiate the equation for X to find,

$$\dot X^\alpha = MX^\alpha + M^{(\alpha)}X \quad \text{with} \quad X^\alpha(0)=0\,. \tag{3.12}$$

Then (3.11) and Duhamel's representation yield

$$\begin{aligned}|X^\alpha(s)| = \left|\int_0^s e^{(s-\tilde s)M}M^{(\alpha)}X\,d\tilde s\right| &\lesssim \int_0^s E(s-\tilde s)\,E(\tilde s)\,d\tilde s\\ = s\,\langle\xi\rangle_\ell^\nu\,E(s) &\le \big(s+\langle\xi\rangle_\ell^{-1}\big)\,\langle\xi\rangle_\ell^\nu\,E(s).\end{aligned}$$

Similarly for $|\beta|=1$, one has $\dot X_\beta = MX_\beta + M_{(\beta)}X$ with $X_\beta(0)=0$ so

$$|X_\beta| \le \int_0^s E(s-\tilde s)\langle\xi\rangle_\ell E(\tilde s)d\tilde s \;\le\; s\langle\xi\rangle_\ell\langle\xi\rangle_\ell^\nu E(s) \le (1+s\langle\xi\rangle_\ell)\langle\xi\rangle_\ell^\nu E(s).$$

This proves (3.10) for $|\alpha+\beta|=1$.

Assume $k \geq 1$ and that (3.10) holds for $|\alpha + \beta| \leq k$. It suffices to prove (3.10) X^γ_δ with $|\gamma + \delta| = k + 1$. Differentiation of the equation for X yields

$$\dot{X}^\alpha_\beta \;=\; M\, X^\alpha_\beta \;+ \sum_{\substack{\alpha_1+\alpha_2=\alpha\,,\,\beta_1+\beta_2=\beta \\ \alpha_1+\beta_1\neq 0}} C_{\alpha_1,\beta_1}\, M^{(\alpha_1)}_{(\beta_1)}\, X^{\alpha_2}_{\beta_2} \quad \text{with} \quad X^\alpha_\beta(0) = 0\,. \tag{3.13}$$

Duhamel's formula yields

$$\big|X^\alpha_\beta(s)\big| \lesssim \sum_{\alpha_1+\beta_1\neq 0} \int_0^s \Big| e^{(s-\tilde{s})M}\, M^{(\alpha_1)}_{(\beta_1)}\, X^{\alpha_2}_{\beta_2} \Big|\, d\tilde{s}\,.$$

The inductive hypothesis estimates the right hand side by

$$\begin{aligned} &\lesssim \sum_{\alpha_1+\beta_1\neq 0} \int_0^s E(s-\tilde{s}) \langle\xi\rangle_\ell^{1-|\alpha_1|} (\tilde{s} + \langle\xi\rangle_\ell^{-1})^{|\alpha_2|} (1 + \tilde{s}\langle\xi\rangle_\ell)^{|\beta_2|} \\ &\quad \cdot \langle\xi\rangle_\ell^{\nu|\alpha_2+\beta_2|} E(\tilde{s}) d\tilde{s} \\ &\lesssim \sum_{\alpha_1+\beta_1\neq 0} s\, \langle\xi\rangle_\ell^{1-|\alpha_1|}\, (s + \langle\xi\rangle_\ell^{-1})^{|\alpha_2|}\, (1 + s\langle\xi\rangle_\ell)^{|\beta_2|} \langle\xi\rangle_\ell^{\nu(|\alpha_2+\beta_2|+1)}\, E(s)\,. \end{aligned} \tag{3.14}$$

If $|\beta_1| \geq 1$ so that $|\beta_2| < |\beta|$, then

$$s\, \langle\xi\rangle_\ell\, \langle\xi\rangle_\ell^{\nu(|\alpha_2+\beta_2|+1)}\, (1 + s\langle\xi\rangle_\ell)^{|\beta_2|} \;\leq\; \langle\xi\rangle_\ell^{\nu|\alpha+\beta|}\, (1 + s\langle\xi\rangle_\ell)^{|\beta|}$$

and the right-hand side of (3.14) is bounded by (3.10). If $|\beta_1| = 0$ so that $\beta_2 = \beta$ and $|\alpha_1| \geq 1$, one has

$$\begin{aligned} &s\, \langle\xi\rangle_\ell^{1-|\alpha_1|} \big(s + \langle\xi\rangle_\ell^{-1}\big)^{|\alpha_2|}\, \langle\xi\rangle_\ell^{\nu(|\alpha_2+\beta|+1)} \\ &\leq \langle\xi\rangle_\ell^{-\nu(|\alpha_1|-1)}\, \langle\xi\rangle_\ell^{\nu|\alpha+\beta|}\, s\, \langle\xi\rangle_\ell^{-(|\alpha_1|-1)}\, \big(s + \langle\xi\rangle_\ell^{-1}\big)^{|\alpha_2|} \\ &\leq \langle\xi\rangle_\ell^{\nu|\alpha+\beta|}\, \big(s + \langle\xi\rangle_\ell^{-1}\big)^{|\alpha|} \end{aligned} \tag{3.15}$$

implying the same conclusion. This completes the inductive proof of (3.10).

Step II. Estimates for $\partial_t X^\alpha_\beta$. Differentiating the equation $\dot{X} = MX$ with respect to t or with respect to x are entirely parallel. With the exception that one can only take one temporal derivative because M is only Lipschitz continuous in t. The result is a bound for $\dot{X}^\alpha_\beta$ that is the same as the bound for X with one more x derivative, that is

$$\big|\dot{X}^\alpha_\beta\big| \lesssim \big(s + \langle\xi\rangle_\ell^{-1}\big)^{|\alpha|} \big(1 + s\langle\xi\rangle_\ell\big)^{|\beta|+1} \langle\xi\rangle_\ell^{\nu(|\alpha+\beta|+1)}\, E(s)\,. \tag{3.16}$$

Step III. Estimates for $\partial_x^\beta \partial_\xi^\alpha R$. Begin with the estimate from Leibniz' rule,

$$|\partial_x^\beta \partial_\xi^\alpha R| \lesssim \sum_{\substack{\beta_1+\beta_2=\beta \\ \alpha_1+\alpha_2+\alpha_3=\alpha}} \int_0^\infty a \Big| \big(\langle\xi\rangle_\ell^\rho\big)^{(\alpha_1)} \big(e^{sM^*}\big)^{(\alpha_2)}_{(\beta_1)} \big(e^{sM}\big)^{(\alpha_3)}_{(\beta_2)} \Big| \, ds \,. \tag{3.17}$$

Thanks to (3.8), the integrand in (3.17) is less than or equal to

$$\begin{aligned} &a\langle\xi\rangle_\ell^{\rho-|\alpha_1|}\big(s+\langle\xi\rangle_\ell^{-1}\big)^{|\alpha_2+\alpha_3|}(1+s\langle\xi\rangle_\ell)^{|\beta_1+\beta_2|} \\ &\cdot \langle\xi\rangle_\ell^{\nu(|\beta+\alpha-\alpha_1|+2)} e^{-cas\langle\xi\rangle_\ell^\rho} e^{-cas\langle\xi\rangle_\ell^\rho} \,. \end{aligned}$$

The pair of estimates

$$\begin{aligned} s+\langle\xi\rangle_\ell^{-1} &= \big(as\langle\xi\rangle_\ell^\rho + a\langle\xi\rangle_\ell^{-1+\rho}\big)a^{-1}\langle\xi\rangle_\ell^{-\rho} \le \big(as\langle\xi\rangle_\ell^\rho+1\big)a^{-1}\langle\xi\rangle_\ell^{-\rho}, \\ 1+s\langle\xi\rangle_\ell &= as\langle\xi\rangle_\ell^\rho\big(a^{-1}\langle\xi\rangle_\ell^{1-\rho}\big)+1 \le \big(as\langle\xi\rangle_\ell^\rho+1\big)a^{-1}\langle\xi\rangle_\ell^{1-\rho}, \end{aligned}$$

yield

$$\big|\partial_x^\beta \partial_\xi^\alpha R\big| \lesssim a^{-|\beta+\alpha|}\, \langle\xi\rangle_\ell^q \int_0^\infty a\,\big(1+as\langle\xi\rangle_\ell^\rho\big)^{|\beta+\alpha|}\, \langle\xi\rangle_\ell^\rho\, e^{-2cas\langle\xi\rangle_\ell^\rho}\, ds\,,$$

with

$$q := (1-\rho)|\beta| - \rho|\alpha| + \nu(|\beta+\alpha|+2) = 2\nu + (1-\rho+\nu)|\beta| - (\rho-\nu)|\alpha|\,.$$

Use $(1+as\langle\xi\rangle_\ell^\rho)^{|\beta+\alpha|}\, e^{-cas\langle\xi\rangle_\ell^\rho} \lesssim 1$ to find

$$\big|\partial_x^\beta \partial_\xi^\alpha R\big| \lesssim a^{-|\beta+\alpha|}\, \langle\xi\rangle_\ell^q \int_0^\infty a\,\langle\xi\rangle_\ell^\rho\, e^{-cas\langle\xi\rangle_\ell^\rho}\, ds \;\lesssim\; a^{-|\beta+\alpha|}\, \langle\xi\rangle_\ell^q\,.$$

This is the first estimate of (3.7).

Step IV. Estimates for $\partial_t R^\alpha_\beta$. As in Remark 3.2, the estimate for the time derivative is the same as taking one additional space derivative. The details are left to the reader.

This completes the proof of Theorem 3.1. □

4. Theorem 1.3, examples and proof

We begin with some examples illustrating the conclusion.

Example 4.1. If $A(t,x,\xi) = \sum_{j=1}^d A_j(t,x)\xi_j$ is uniformly diagonalizable then one can take $\theta = 0$ so that Theorem 1.3 holds with $1 < s \le 2$. In [13] Kajitani has proved that the Cauchy problem for uniformly diagonalizable system is well posed in $\mathcal{G}^2(\mathbb{R}^d)$ when the coefficients are smooth enough in time.

Example 4.2. We can always take $\theta = m - 1$ and Theorem 1.3 holds with $1 < s \le (4m-1)/(4m-2)$.

Example 4.3. If (2.5) holds with $\theta = \mu - 1, 2 \le \mu \le m - 1$, then one can choose $1 < s \le (4\mu - 1)/(4\mu - 2)$ in Theorem 1.3.

Proof of Theorem 1.3.

Step I. Compact support in x. Choose $R > 0$ so that the support of f, g, and $\nabla_{t,x} A_j$, $\nabla_{t,x} B$ are all contained in $\{|x| \le R\}$. Denote by $c_{\max}$ an upper bound for the propagation speed for the constant coefficient hyperbolic operator L on $|x| \ge R$.

Finite speed for that constant coefficient operator implies that u vanishes for $|x| \ge R + c_{\max} t$ with $t \ge 0$.

Step II. First *a priori* estimate. In this section standard notation for the Weyl calculus of pseudodifferential operators from [7] is used. Consider (1.1). Set $v = e^{\langle D\rangle_\ell^\rho (T-at)} u$ with small T to be chosen below. Define

$$\tilde{A} := e^{\langle D\rangle_\ell^\rho (T-at)} A e^{-\langle D\rangle_\ell^\rho (T-at)} \quad \text{and} \quad \tilde{B} := e^{\langle D\rangle_\ell^\rho (T-at)} B e^{-\langle D\rangle_\ell^\rho (T-at)}$$

and $\tilde{f} := e^{\langle D\rangle_\ell^\rho (T-at)} f$. Compute

$$\begin{aligned} \frac{d}{dt}\Big(R\, e^{\langle D\rangle_\ell^\rho (T-at)} u, e^{\langle D\rangle_\ell^\rho (T-at)} u\Big) = {} & (\partial_t R\, v, v) \\ & + \big(R(i\tilde{A} + \tilde{B} - a\langle D\rangle_\ell^\rho)v, v\big) \\ & + \big(Rv, (i\tilde{A} + \tilde{B} - a\langle D\rangle_\ell^\rho)v\big) \\ & + \big(R\tilde{f}, v\big) + \big(Rv, \tilde{f}\big). \end{aligned} \tag{4.1}$$

For any $0 < c_1 < T$, one has $c_1 \le T - at \le T$ for $0 \le t \le (T - c_1)/a$. If $T > 0$ is small, Proposition 2.6 implies that $\tilde{A} = H_N + K$ with

$$K \in \tilde{S}^{\max\{\rho - N(1-\rho),\, -1+\rho\}}. \tag{4.2}$$

Corollary 2.7 implies $\tilde{B} \in \tilde{S}^0$.

Since $i\tilde{A} - a\langle D\rangle_\ell^\rho = M + iK$ the right-hand side of (4.1) is equal to

$$\begin{aligned} (\partial_t R\, v, v) &+ \big((RM + M^* R)v, v\big) + (R\tilde{f}, v) \\ &+ \big((R(iK + \tilde{B}) + (iK + \tilde{B})^* R)v, v\big) + (Rv, \tilde{f}). \end{aligned}$$

Recall that $M \in \tilde{S}^1$ and R satisfies (3.7). Therefore

$$R\# M + M^* \# R = RM + M^* R + K_1 = -a\langle \xi\rangle_\ell^\rho + K_1$$

where $aK_1 \in \tilde{S}^{1-\rho+3\nu}_{\rho-\nu\,,\,1-\rho+\nu}$ with bound independent of a which is large (see, *e.g.*, [7, Proposition 18.5.7]). Choose $a_1 \geq a_0$ so that if $a > a_1$ one has $Ca^{-1} \leq 3a/4$. Then,

$$-a\big(\langle D\rangle_\ell^\rho u, u\big) + \mathrm{Re}\,(K_1 u, u) \leq -a\big\|\langle D\rangle_\ell^{\rho/2} u\big\|^2 + Ca^{-1}\Big\|\langle D\rangle_\ell^{(1-\rho+3\nu)/2} u\Big\|^2$$
$$\leq -(a/4)\Big\|\langle D\rangle_\ell^{\rho/2} u\Big\|^2$$

provided

$$\rho \geq 1-\rho+3\nu\,, \quad \text{equivalently} \quad \rho \geq (1+3\theta)/(2+3\theta). \tag{4.3}$$

Note that $a\,\partial_t R \in \tilde{S}^{1-\rho+3\nu}_{\rho-\nu\,,\,1-\rho+\nu}$ with a-independent bound so

$$\mathrm{Re}\,\big(\partial_t R u, u\big) \;\leq\; Ca^{-1}\Big\|\langle D\rangle_\ell^{\rho/2} u\Big\|^2$$

if $2\rho \geq 1+3\nu$, that is $\rho \geq (1+3\theta)/(2+3\theta)$.

Using (4.2), $R \in \tilde{S}^{2\nu}_{\rho-\nu,1-\rho+\nu}$, and $\tilde{B} \in \tilde{S}^0$ yield the pair of estimates

$$\big|\big((R\tilde{B}+\tilde{B}^*R)v, v\big)\big| \leq C\big\|\langle D\rangle_\ell^\nu v\big\|^2 \leq C\ell^{-(\rho-2\nu)}\Big\|\langle D\rangle_\ell^{\rho/2} v\Big\|^2,$$
$$\big|\big(i(RK-K^*R)v, v\big)\big| \leq C\Big\|\langle D\rangle_\ell^{\nu+\rho/2-N(1-\rho)/2} v\Big\|^2 \leq C'\Big\|\langle D\rangle_\ell^{\rho/2} v\Big\|^2$$

because $2\nu+\rho-N(1-\rho) = (2\theta-N)(1-\rho)+\rho \leq \rho$ and $1-\rho+\nu \leq \rho-\nu$ when $2\rho \geq 1+3\nu$. In addition,

$$\big|(R\tilde{f}v, v)\big| + \big|(Rv, \tilde{f})\big| \leq 2\big\|\langle D\rangle_\ell^{-3\nu} Rv\big\|\,\big\|\langle D\rangle_\ell^{3\nu}\tilde{f}\big\| \leq C\big\|\langle D\rangle_\ell^{-\nu} v\big\|\,\big\|\langle D\rangle_\ell^{3\nu}\tilde{f}\big\|.$$

Thus there exist $c, C > 0$ so that

$$\frac{d}{dt}(R\,v, v) \;+\; ca\Big\|\langle D\rangle_\ell^{\rho/2} v\Big\|^2 \;\leq\; C\|\langle D\rangle_\ell^{-\nu} v\|\,\big\|\langle D\rangle_\ell^{3\nu}\tilde{f}\big\|\,. \tag{4.4}$$

The definition of R together with (3.6) and (3.4) show that if $T_1 < T$ and $0 \leq t \leq T_1/a$, then $R = R^* \geq c\,\langle\xi\rangle_\ell^{-2\nu}$.

Introduce the metric

$$G := a^{-2}\Big(\langle\xi\rangle_\ell^{2(1-\rho+\nu)}|dx|^2 + \langle\xi\rangle_\ell^{-2(\rho-\nu)}|d\xi|^2\Big)\,. \tag{4.5}$$

Then $G/G^\sigma = a^{-4}\langle\xi\rangle_\ell^{2(1-2\rho+2\nu)}$. Use the Weyl calculus with $H = a^2\langle\xi\rangle_\ell^{2\rho-1-4\nu}(R - c\,\langle\xi\rangle_\ell^{-2\nu}) \in S((G/G^\sigma)^{-1/2}, G)$ satisfying $H \geq 0$. The sharp Gårding inequality [7, Theorem 18.6.7] yields

$$(Hv, v) \;\geq\; -C\|v\|^2.$$

Write

$$H \;=\; a^2\langle\xi\rangle_\ell^{\rho-1/2-2\nu} \,\#\, \big(R - c\,\langle\xi\rangle_\ell^{-2\nu}\big) \,\#\, \langle\xi\rangle_\ell^{\rho-1/2-2\nu} \;+\; K$$

where $K \in S(1, G)$. Introduce $u := \langle D\rangle_\ell^{\rho-1/2-2\nu} v$ to find

$$\begin{aligned}(Hv, v) &= a^2\big((R - c\,\langle D\rangle_\ell^{-2\nu})u, u\big) + (Kv, v)\\ &\ge -C\|v\|^2 = -C\left\|\langle D\rangle_\ell^{-\rho+1/2+2\nu} u\right\|^2.\end{aligned}$$

Since $|(Kv, v)| \le C\|v\|^2 = C\|\langle D\rangle_\ell^{-\rho+1/2+2\nu} u\|^2$ it follows that

$$(Ru, u) - c\left\|\langle D\rangle_\ell^{-\nu} u\right\|^2 \;\ge\; -Ca^{-2}\left\|\langle D\rangle_\ell^{-\rho+1/2+2\nu} u\right\|^2. \tag{4.6}$$

If $-\nu \ge 2\nu + 1/2 - \rho$, that is

$$\rho \ge \frac{1+6\theta}{2+6\theta}$$

then there exist constants $c' > 0$ and $\ell_0 > 0$, so that for $\ell \ge \ell_0$ one has

$$(Ru, u) \;\ge\; c'\left\|\langle D\rangle_\ell^{-\nu} u\right\|^2.$$

Integrating (4.4) yields

$$\left\|\langle D\rangle_\ell^{-\nu} v(t)\right\|^2 \le c\left\|\langle D\rangle_\ell^{\nu} v(0)\right\|^2 + 2C\mathcal{M}\int_{/}^{\sqcup} \left\|\langle\mathcal{D}\rangle_\ell^{\ni\nu}\tilde{\{}\right\| \lceil\tau\,,$$

where $\mathcal{M} := \sup_{/\le\tau\le\sqcup} \|\langle\mathcal{D}\rangle_\ell^{-\nu}\sqsubseteq(\tau)\|$. Therefore

$$\left(\mathcal{M} - \mathcal{C}\int_{/}^{\sqcup} \left\|\langle\mathcal{D}\rangle_\ell^{\ni\nu}\tilde{\{}\right\| \lceil\tau\right)^2 \le \left(\sqrt{c}\|\langle D\rangle_\ell^{\nu} v(0)\| + C\int_0^t \left\|\langle D\rangle_\ell^{3\nu}\tilde{f}\right\| d\tau\right)^2$$

which gives

$$\left\|\langle D\rangle_\ell^{-\nu} v(t)\right\| \le 2\sqrt{c}\left\|\langle D\rangle_\ell^{\nu} v(0)\right\| + 2C\int_0^t \left\|\langle D\rangle_\ell^{3\nu}\tilde{f}\right\| d\tau.$$

This proves the following important *a priori* estimate.

Proposition 4.4. *If $\rho \ge (1+6\theta)/(2+6\theta)$ then there exist $T > 0$ and $a > 0$ and $\ell_0 > 0$ such that for any $T_1 < T$ one can find $C > 0$ such that for all u so that $e^{\langle D\rangle_\ell^\rho (T-at)}\partial_{t,x}^\gamma u \in L^1([0,T];\; H^{3\nu}(\mathbb{R}^d))$ for $|\gamma| \le 1$, one has*

$$\|\langle D\rangle_\ell^{-\nu} e^{\langle D\rangle_\ell^\rho (T-at)} u\| \le C\|\langle D\rangle_\ell^{\nu} e^{T\langle D\rangle_\ell^\rho} u(0)\| + C\int_0^t \|\langle D\rangle_\ell^{3\nu} e^{\langle D\rangle_\ell^\rho (T-a\tau)} Lu\|\, d\tau$$

for $0 \le t \le T_1/a$ and $\ell \ge \ell_0$, where $\nu = \theta(1-\rho)$.

Step III. Second *a priori* estimate. For some values of ρ and θ, one can improve the estimate for the left hand side $\|\langle D\rangle_\ell^{-\nu} e^{\langle D\rangle_\ell^\rho (T-at)} u\|$ in Proposition 4.4. Recall that $\partial_t u = iA(t,x,D)u + B(t,x)u + f$ and $v = e^{\langle D\rangle_\ell^\rho (T-at)} u$. Then,

$$\frac{d}{dt}\big\|e^{\langle D\rangle_\ell^\rho (T-at)}u\big\|^2 = -2a\big\|\langle D\rangle_\ell^{\rho/2} v\big\|^2 + \big((i\tilde{A}+\tilde{B})v, v\big) + \big(v, (i\tilde{A}+\tilde{B})v\big) + \big(\tilde{f}, v\big) + \big(v, \tilde{f}\big). \tag{4.7}$$

Since $i\tilde{A}+\tilde{B} \in \tilde{S}^1$ one has $|((i\tilde{A}+\tilde{B})\,v, v) + (v, (i\tilde{A}+\tilde{B})\,v)| \le C\|\langle D\rangle_\ell^{1/2} v\|^2$ so

$$\|v(t)\|^2 \;\le\; \|v(0)\|^2 \;+\; C\int_0^t \Big\|\langle D\rangle_\ell^{1/2} v\Big\|^2 ds \;+\; 2\int_0^t \|v\|\|\tilde{f}\| d\tau.$$

Replacing v by $\langle D\rangle_\ell^{(\rho-1)/2} v$ yields

$$\Big\|\langle D\rangle_\ell^{(\rho-1)/2} v(t)\Big\|^2 \le \Big\|\langle D\rangle_\ell^{(\rho-1)/2} v(0)\Big\|^2 + C\int_0^t \Big\|\langle D\rangle_\ell^{\rho/2} v\Big\|^2 d\tau + 2\int_0^t \Big\|\langle D\rangle_\ell^{(\rho-1)/2} v\Big\| \|\tilde{f}\|^2 d\tau.$$

On the other hand, the reasoning leading to (4.4) yields

$$\frac{d}{dt}\big(R\,v, v\big) + ca\Big\|\langle D\rangle_\ell^{\rho/2} v\Big\|^2 \le C\Big\|\langle D\rangle_\ell^{(\rho-1)/2} v\Big\|\Big\|\langle D\rangle_\ell^{2\nu-(\rho-1)/2}\tilde{f}\Big\|.$$

If

$$(\rho-1)/2 \ge -\rho + 1/2 + 2\nu\,, \quad \text{equivalently,} \quad \rho \ge (2+4\theta)/(3+4\theta) \tag{4.8}$$

then we can control (Rv, v) taking (4.6) into account. Since $(2+4\theta)/(3+4\theta) \ge (1+3\theta)/(2+3\theta)$ and $2\nu - (\rho-1)/2 \ge 0$ if (4.8) is verified then

$$\Big\|\langle D\rangle_\ell^{(\rho-1)/2} v(t)\Big\|^2 \le c\big\|\langle D\rangle_\ell^{\nu} v(0)\big\|^2 + 2C\mathcal{M}\int_/^{\sqcup} \Big\|\langle \mathcal{D}\rangle_\ell^{\in\nu-(\rho-\infty)/\in}\tilde{\{}\Big\| \lceil\tau$$

where $\mathcal{M} := \sup_{/\le\tau\le\sqcup} \|\langle\mathcal{D}\rangle_\ell^{(\rho-\infty)/\in} \sqsubseteq(\tau)\|$. Repeating the same arguments proving Proposition 4.4 yields the following alternative *a priori* estimate.

Proposition 4.5. *If $\rho \ge (2+4\theta)/(3+4\theta)$, then there exist $T > 0$, $a > 0$, and $\ell_0 > 0$, so that for any $T_1 < T$ there is $C > 0$ such that for all u so that $e^{\langle D\rangle_\ell^\rho (T-at)}\partial_{t,x}^\gamma u \in L^1([0,T];\, H^{2\nu-(\rho-1)/2}(\mathbb{R}^d))$ for $|\gamma| \le 1$, one has*

$$\Big\|\langle D\rangle_\ell^{(\rho-1)/2} e^{\langle D\rangle_\ell^\rho (T-at)} u\Big\| \le C\Big\|\langle D\rangle_\ell^{\nu} e^{T\langle D\rangle_\ell^\rho} u(0)\Big\| + C\int_0^t \Big\|\langle D\rangle_\ell^{2\nu-(\rho-1)/2} e^{\langle D\rangle_\ell^\rho (T-a\tau)} Lu\Big\|\, d\tau$$

for $0 \le t \le T_1/a$ and $\ell \ge \ell_0$, where $\nu = \theta(\rho-1)$.

Step IV. Uniform estimates for regularized equations. Choose $\chi(x) \in C_0^\infty(\mathbb{R}^d)$ that is equal to 1 on a neighborhood of $x = 0$ and such that $|\chi(x)| \le 1$. Consider the regularized operator

$$L^h \;:=\; \partial_t - \chi(hD)\big(iA(t,x,D) \;+\; B(t,x)\big)\chi(hD) \;:=\; \partial_t - iA^h - B^h.$$

Denote

$$\tilde{L}^h := e^{\langle D\rangle_\ell^\rho (T-at)} L^h e^{-\langle D\rangle_\ell^\rho (T-at)} := \partial_t - i\tilde{A}^h - \tilde{B}^h\,, \quad \text{and} \quad \tilde{L}^0 := \tilde{L}\,.$$

Denote $\chi_h(D) := \chi(hD)$ so

$$\tilde{A}^h = e^{\langle D\rangle_\ell^\rho (T-at)}\; \chi_h\; A\; \chi_h\; e^{-\langle D\rangle_\ell^\rho (T-at)} = \chi_h\; \tilde{A}\; \chi_h\,,$$
$$\tilde{B}^h = e^{\langle D\rangle_\ell^\rho (T-at)}\; \chi_h\; B\; \chi_h\; e^{-\langle D\rangle_\ell^\rho (T-at)} \;=\; \chi_h\; \tilde{B}\; \chi_h\,.$$

Note that $\chi_h(\xi) \in \tilde{S}^0$ uniformly in $0 < h \le \ell^{-1}$ because $\langle\xi\rangle_\ell \le C|\xi|$ on the support of $\nabla_\xi \chi(h\xi)$.

Recall that $\tilde{A} = H_N + K$ with K in (4.2). Since $H_N \in \tilde{S}^1$ it follows that

$$\chi(h\xi)\,\#\,H_N\,\#\;\chi(h\xi) \;=\; \chi^2(h\xi)H_N \;+\; K_1^h$$

where $K_1^h \in \tilde{S}^0$, uniformly in $0 < h \le \ell^{-1}$. It is clear that $\chi_h \# \tilde{B} \# \chi_h \in \tilde{S}^0$ and $\chi_h \# K \# \chi_h \in \tilde{S}^{\max\{\rho - N(1-\rho)\,,\,-1+\rho\}}$ uniformly in $0 < h \le \ell^{-1}$.

From here on the pseudodifferential calculus is understood to be uniform in $0 < h \le \ell^{-1}$. Denote $H_N^h = \chi^2(h\xi)H_N$ so that

$$H_N^h(\ell,\tau,t,x,\xi) = \chi_h^2(h\xi)\;\langle\xi\rangle_\ell\; \mathcal{H}_N\Big(t\,,\;x\,,\;\xi/\langle\xi\rangle_\ell\,;\;\tau\rho\langle\xi\rangle_\ell^{\rho-1}\Big).$$

Inserting $s\chi_h^2\langle\xi\rangle_\ell$ and $\tau\rho\langle\xi\rangle_\ell^{\rho-1}$ $(\tau > 0)$ and $\xi/\langle\xi\rangle_\ell$ for s, ϵ, ξ in (2.5) yields

$$\frac{\tau^\theta}{C\;\langle\xi\rangle_\ell^{\theta(1-\rho)}\; e^{cs\tau\chi_h^2\langle\xi\rangle_\ell^\rho}} \;\le\; \big\|e^{isH_N^h(\ell,\tau,t,x,\xi)}\big\| \;\le\; \frac{C\;\langle\xi\rangle_\ell^{\theta(1-\rho)}\; e^{cs\tau\chi_h^2\langle\xi\rangle_\ell^\rho}}{\tau^\theta}\,.$$

Define

$$M^h \;:= iH_N^h(\ell,\tau,t,x,\xi) - a\langle\xi\rangle_\ell^\rho \quad \text{with} \quad \tau = T - at$$

and the corresponding symmetrizer

$$R^h(t,x,\xi) \;:=\; a\int_0^\infty \langle\xi\rangle_\ell^\rho\;\big(e^{sM^h(\ell,\tau,t,x,\xi)}\big)^*\;\big(e^{sM^h(\ell,\tau,t,x,\xi)}\big)\,ds\,.$$

Since $\|e^{sM^h}\| = e^{-as\langle\xi\rangle_\ell^\rho}\|e^{sM^h}\|$ and $0 \le \chi_h^2 \le 1$ one has

$$\tau^\theta\langle\xi\rangle_\ell^{-\theta(1-\rho)} e^{-c_1 as\langle\xi\rangle_\ell^\rho}/C \;\le\; \big\|e^{sM^h}\big\| \;\le\; C\tau^{-\theta}\langle\xi\rangle_\ell^{\theta(1-\rho)}\; e^{-c_2 as\langle\xi\rangle_\ell^\rho}$$

with $c_1, c_2 > 0$ and $C > 0$ independent of $0 < h \le \ell^{-1}$, ℓ and a. Since

$$|\partial_\xi^\alpha \partial_x^\beta M^h| \ \le\ C_{\alpha\beta}\ \langle\xi\rangle_\ell^{1-|\alpha|}$$

uniformly in $0 < h \le \ell^{-1}$, the estimates for R^h are the same as those for R, so one has (3.7) with $C_{\alpha\beta}$ independent of $0 < h \le \ell^{-1}$, ℓ, x, ξ and a. Repeating the same arguments proving Proposition 4.4 proves the following.

Proposition 4.6. *If $\rho \ge (1+6\theta)/(2+6\theta)$ then there exist $T > 0$ and $a > 0$ and $\ell_0 > 0$ such that for any $T_1 < T$ one can find $C > 0$ such that for all v so that $v \in C^1\big([0,T]; H^{3\nu}(\mathbb{R}^d)\big)$, one has*

$$\big\|\langle D\rangle_\ell^{-\nu}(t)v\big\| \le C\big\|\langle D\rangle_\ell^{\nu} v(0)\big\| + C\int_0^t \big\|\langle D\rangle_\ell^{3\nu}\tilde{L}^h v(\tau)\big\|\,d\tau \tag{4.9}$$

for $0 \le t \le T_1/a$ and $\ell \ge \ell_0$ where C is independent of ℓ and $0 < h \le \ell^{-1}$.

Step V. Construction of a solution. Next solve

$$\tilde{L}^h v^h = \big(\partial_t - i\tilde{A}^h - \tilde{B}^h\big)v^h = \tilde{f} \quad \text{with} \quad v^h(0) = \tilde{g}. \tag{4.10}$$

Since $i\tilde{A} + \tilde{B} \in C(\mathbb{R}; \tilde{S}^1)$ and $\chi_h \in S^{-1}$ with h-dependent bound, it follows that $i\tilde{A}^h + \tilde{B}^h \in C(\mathbb{R}; \tilde{S}^0)$ so it is bounded from $H^k(\mathbb{R}^d)$ to $H^k(\mathbb{R}^d)$ for any $k \in \mathbb{R}$. Therefore for any $\tilde{g} \in H^k(\mathbb{R}^d)$ and $\tilde{f} \in L^1_{\text{loc}}(\mathbb{R}; H^k(\mathbb{R}^d))$ there exists a unique solution $v^h \in C^1(\mathbb{R}; H^k(\mathbb{R}^d))$ to the linear ordinary differential equation (4.10).

Assume

$$\tilde{f} = e^{\langle D\rangle_\ell^\rho (T-at)} f \in L^1\big([0,T']; H^{3\nu}(\mathbb{R}^d)\big) \quad \text{and} \quad \tilde{g} = e^{T\langle D\rangle_\ell^\rho} g \in H^{3\nu}\big(\mathbb{R}^d\big).$$

Denote $T' := T_1/a$ and the corresponding solutions to (4.10) by $v^h \in C^1([0,T']; H^{3\nu}(\mathbb{R}^d))$. Then (4.9) yields

$$\big\|\langle D\rangle_\ell^{-\nu} v^h(t)\big\| \le C\,\big\|\langle D\rangle_\ell^{\nu}\tilde{g}\big\| + C\int_0^t \big\|\langle D\rangle_\ell^{3\nu}\tilde{f}\big\|\,d\tau$$

for $0 \le t \le T'$. Therefore $\{v^h\}$ is bounded in $L^\infty([0,T']; H^{-\nu})$. Since $L^\infty([0,T']; H^{-\nu}(\mathbb{R}^d))$ is the dual of $L^1([0,T']; H^{\nu}(\mathbb{R}^d))$, one can choose a subsequence (still denoted by $\{v^h\}$) weak* convergent in $L^\infty([0,T']; H^{-\nu}(\mathbb{R}^d))$ to v. It is easy to see that $\chi(hD)v^h$ converges to v weakly in $L^\infty([0,T'], H^{-\nu}(\mathbb{R}^d))$. Since $i\tilde{A}+\tilde{B}$ maps $L^\infty([0,T'], H^{-\nu}(\mathbb{R}^d))$ into $L^\infty([0,T']; H^{-\nu-1})$ then $\chi(hD)(i\tilde{A}+\tilde{B})\chi(hD)v^h$ converges to $(i\tilde{A}+\tilde{B})v$ weakly* in $L^\infty([0,T']; H^{-\nu-1}(\mathbb{R}^d))$. Since it is clear that

$$\int_0^{T'} \big(\partial_t v^h, \phi\big)dt \to -\int_0^{T'} (v, \partial_t\phi)dt$$

for any $\phi \in C_0^\infty((0,T')\times\mathbb{R}^d)$ it follows that v satisfies $\tilde{L}v = \tilde{f}$ and $v(0) = \tilde{g}$, that is

$$e^{\langle D\rangle_\ell^\rho (T-at)} L e^{-\langle D\rangle_\ell^\rho (T-at)} v \;=\; \tilde{f} \;=\; e^{\langle D\rangle_\ell^\rho (T-at)} f\,.$$

The equation $\tilde{L}v = \tilde{f}$ yields $\partial_t v \in L^\infty([0,T']; H^{-\nu-1}(\mathbb{R}^d))$ which implies $v \in C([0,T']; H^{-\nu-1}(\mathbb{R}^d))$. With $u = e^{-\langle D\rangle_\ell^\rho (T-at)} v$ we conclude that

$$Lu = f \quad \text{for} \quad (t,x) \in (0,T')\times\mathbb{R}^d, \quad \text{and} \quad u(0) = g\,.$$

This completes the proof of existence of a solution u with $e^{\langle D\rangle_\ell^\rho (T-at)} u = v \in L^\infty([0,T'], H^{-\nu}(\mathbb{R}^d))$.

Step VI. Proof of uniqueness. Suppose that u is a solution with vanishing data f, g. Define $0 \le t_1 \le T_0$ so that u vanishes on $[0,t_1]\times\mathbb{R}^d$ but does not vanish on $[0, t_1+\epsilon)\times\mathbb{R}^d$ for any $\epsilon > 0$. We need to show that $t_1 = T_0$. Suppose that $t_1 < T_0$.

Using Remark 1.4 applied to the adjoint operator with time reversed, choose $0 < \underline{t} \le T_0 - t_1$ and $C \gg 1$ so that for $F(t,x)$ compactly supported in x and satisfying

$$\int_0^{T_0} \left(\int_{\mathbb{R}^d} \left|\hat{F}(t,\xi)\right|^2 e^{C\langle\xi\rangle^{1/s}}\, d\xi \right)^{1/2} dt \;<\; \infty \tag{4.11}$$

the adjoint problem

$$L^* w \;=\; F \quad \text{on} \quad (t_1, t_1+\underline{t})\times\mathbb{R}^d \quad \text{with} \quad w\big|_{t=t_1+\underline{t}} \;=\; 0$$

has a solution in $C\big([t_1, t_1+\underline{t}]\,;\, \mathcal{G}_0^s(\mathbb{R}^d)\big)$.

Both u and w being compactly supported in x belong to $H^1((t_1, t_1+\underline{t})\times\mathbb{R}^d)$ so integration by parts shows that with integrals over $(t_1, t_1+\underline{t})\times\mathbb{R}^d$,

$$\iint (u, F)\, dxdt \;=\; \iint (u, L^* w)\, dxdt \;=\; \iint (Lu, w)\, dxdt \;=\; 0\,,$$

where the initial conditions $u(t_1) = w(t_1+\underline{t}) = 0$ cancels the boundary contributions from $t = t_1$, and $t = t_1 + \underline{t}$.

Since the set of F satisfying (4.11) is dense in $L^2([t_1, t_1+\underline{t}]\times\mathbb{R}^d)$ it follows that $u = 0$ on $[t_1, t_1+\underline{t}]\times\mathbb{R}^d$. Therefore u vanishes on $[0, t_1+\underline{t}]\times\mathbb{R}^d$ violating the choice of t_1. Thus one must have $t_1 = T_0$ proving uniqueness.

Step VII. Proof of continuity in time. Compute $\partial_t u = e^{-\langle D\rangle_\ell^\rho (T-at)}(a\langle D\rangle_\ell^\rho v + \partial_t v)$. Since $a\langle D\rangle_\ell^\rho v + \partial_t v \in L^\infty([0,T']; H^{-\nu-1}(\mathbb{R}^d))$ it follows that for any $0 < c < T - T_1$

$$\int \Big(|\hat{u}(t,\xi)|^2 \;+\; |\partial_t \hat{u}(t,\xi)|^2 \Big)\, e^{2c\langle\xi\rangle^{1/s}}\, d\xi \;\in\; L^\infty([0,T'])\,.$$

This implies that u is continuous with values in $\mathcal{G}_0^s(\mathbb{R}^d)$.

This completes the proof of Theorem 1.3. □

5. Theorem 1.5, examples and proof

Begin with two examples that illustrate the conclusion of Theorem 1.5.

Example 5.1. If $A(t,x,\xi)$ is uniformly diagonalizable, Hypothesis 2.8 is satisfied with $\theta=0$. Theorem 1.5 holds with $1<s\le 1+\kappa$. The examples of [21] show that the $\mathcal{G}^{1+\kappa}$ regularity cannot be improved.

Example 5.2. If (2.5) holds with $\theta=\mu-1, 2\le\mu\le m$ then the constraints on s read

$$1<s\le \min\Big\{(2\mu-1+\kappa\mu)/(2\mu-1+\kappa(\mu-1))\ ,\ (4\mu-1)/(4\mu-2)\Big\}\,.$$

Proof of Theorem 1.5. We present only the *a priori* estimate. Existence and uniqueness follow as in the preceding section. The proof follow the arguments in [9,16]. By hypothesis,

$$|\partial_x^\alpha(A_j(t,x)-A_j(\tau,x))|\ \le\ CA^{|\alpha|}|\alpha|!^s|t-\tau|^\kappa \quad \text{for} \quad 0<\kappa\le 1\,.$$

Choose $\chi(s)\in C_0^\infty(\mathbb{R})$ such that $\chi(s)=\chi(-s)$ with $\int\chi(s)ds=1$. Define, with $0<\delta$ to be chosen later,

$$\tilde R(t,x,\xi)\ :=\ \langle\xi\rangle_\ell^\delta\int R(\tau,x,\xi)\ \chi\big((t-\tau)\,\langle\xi\rangle_\ell^\delta\big)\,d\tau.$$

Since $|\partial_\xi^\alpha\chi((t-\tau)\langle\xi\rangle_\ell^\delta)|\le C_\alpha\langle\xi\rangle_\ell^{-|\alpha|}$, Theorem 3.1 implies that $\tilde R\in S\big(\langle\xi\rangle_\ell^{2\nu},G\big)$ with G from (4.5) . It is clear that $\tilde R\ge c\,\langle\xi\rangle_\ell^{-2\nu}$.

Lemma 5.3. *One has* $R(t)-R(\tau)\in S\big(|t-\tau|^\kappa\langle\xi\rangle_\ell^{3\nu+1-\rho},G\big)$ *uniformly in* t,τ. *That is, for all* α,β,

$$\Big|\partial_x^\beta\partial_\xi^\alpha(R(t)-R(\tau))\Big|\le C_{\alpha\beta}a^{-|\alpha+\beta|}|t-\tau|^\kappa\langle\xi\rangle_\ell^{3\nu+1-\rho}\langle\xi\rangle_\ell^{(1-\rho+\nu)|\beta|-(\rho-\nu)|\alpha|}.$$

Idea of proof of Lemma. It suffices to repeat arguments similar to those proving Theorem 3.1. □

Since

$$\tilde R(t)-R(t)=\langle\xi\rangle_\ell^\delta\int\big(R(\tau)-R(t)\big)\ \chi\big((t-\tau)\ \langle\xi\rangle_\ell^\delta\big)\,d\tau,$$

Lemma 5.3 implies that

$$\tilde R(t)-R(t)\ \in\ S\big(\langle\xi\rangle_\ell^{3\nu+1-\rho-\kappa\delta}\,,G\big).$$

Similarly,

$$\begin{aligned}\partial_t \tilde{R}(t) &= \langle \xi \rangle_\ell^{2\delta} \int R(\tau, x, \xi)\, \chi'\big((t-\tau)\langle \xi \rangle_\ell^{\delta}\big)\, d\tau \\ &= \langle \xi \rangle_\ell^{2\delta} \int \big(R(\tau) - R(t)\big)\, \chi'\big((t-\tau)\langle \xi \rangle_\ell^{\delta}\big)\, d\tau\end{aligned}$$

implies that

$$\partial_t \tilde{R}(t) \in S\big(\langle \xi \rangle_\ell^{3\nu+1-\rho+\delta-\kappa\delta},\ G\big). \tag{5.1}$$

Computing

$$\frac{d}{dt}\Big(\tilde{R}\, e^{\langle D \rangle_\ell^\rho (T-at)} u\,,\ e^{\langle D \rangle_\ell^\rho (T-at)} u\Big)$$

yields (4.1) with R replaced by $\tilde{R}$. It follows that

$$\begin{aligned}&\frac{d}{dt}\Big(\tilde{R}\, e^{\langle D \rangle_\ell^\rho (T-at)} u,\ e^{\langle D \rangle_\ell^\rho (T-at)} u\Big) \\ &= \big(\partial_t \tilde{R}\, v, v\big) + \big((\tilde{R}M + M^*\tilde{R})v, v\big) + \big(\tilde{R}\tilde{f}, v\big) \\ &\quad + \big((\tilde{R}(iK+\tilde{B}) + (iK+\tilde{B})^*\tilde{R})v, v\big) + \big(\tilde{R}v, \tilde{f}\big)\end{aligned}$$

where $i\tilde{A} - a\langle D \rangle_\ell^\rho = M + iK$, and K verifies (4.2). In what follows we assume (4.3) and hence $\max\{\rho - N(1-\rho)\,,\ -1+\rho\} \le \rho - 2\nu$. Thus

$$\tilde{R}\# M \ +\ M^*\#\tilde{R} \ =\ \tilde{R}M \ +\ M^*\tilde{R} \ +\ K_1 \qquad \text{with} \quad aK_1 \in \tilde{S}^{1-\rho+3\nu}_{\rho-\nu\,,\,1-\rho+\nu}\,.$$

Write

$$\begin{aligned}&\tilde{R}(t)M(t) + M^*(t)\tilde{R}(t) \\ &= \langle \xi \rangle_\ell^{\delta} \int R(\tau)\big(M(t) - M(\tau)\big)\chi\big((t-\tau)\langle \xi \rangle_\ell^{\delta}\big)d\tau \\ &\quad + \langle \xi \rangle_\ell^{\delta} \int \big(M^*(t) - M^*(\tau)\big)R(\tau)\chi\big((t-\tau)\langle \xi \rangle_\ell^{\delta}\big)d\tau \\ &\quad + \langle \xi \rangle_\ell^{\delta} \int \big(R(\tau)M(\tau) + M^*(\tau)R(\tau)\big)\chi\big((t-\tau)\langle \xi \rangle_\ell^{\delta}\big)d\tau.\end{aligned} \tag{5.2}$$

Since $R(\tau)M(\tau) + M^*(\tau)R(\tau) = -a\langle \xi \rangle_\ell^\rho$, the last term on the right is equal to $-a\langle \xi \rangle_\ell^\rho$. Proposition 2.6 together with the Hölder continuity hypothesis imply that

$$\big|\partial_x^\beta \partial_\xi^\alpha (M(t) - M(\tau))\big| \le C_{\alpha\beta}\, |t-\tau|^\kappa \langle \xi \rangle_\ell^{1-|\alpha|}.$$

The same estimate follows for the adjoint, $|\partial_x^\beta \partial_\xi^\alpha (M^*(t) - M^*(\tau))|$. It follows that

$$\langle \xi \rangle_\ell^{\delta} \int R(\tau)\big(M(t) - M(\tau)\big)\chi\big((t-\tau)\langle \xi \rangle_\ell^{\delta}\big)d\tau \ \in\ \tilde{S}^{2\nu+1-\kappa\delta}_{\rho-\nu,1-\rho+\nu}$$

modulo a term in $\tilde{S}^{\rho}_{\rho-\nu,1-\rho+\nu}$. The same holds for the second term on the right hand side of (5.2). Therefore,

$$\tilde{R}(t)M(t) + M^*(t)\tilde{R}(t) \;=\; -a\langle\xi\rangle_\ell^\rho + T_1 + T_2 \tag{5.3}$$

where $T_1 \in \tilde{S}^{2\nu+1-\kappa\delta}_{\rho-\nu,1-\rho+\nu}$ and $T_2 \in \tilde{S}^{\rho}_{\rho-\nu,1-\rho+\nu}$.

For the terms

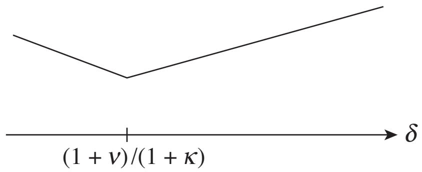
$$\big((\tilde{R}(iK+\tilde{B}) + (iK+\tilde{B})^*\tilde{R})v, v\big) \quad \text{and} \quad \big(\tilde{R}\tilde{f}, v\big) + \big(\tilde{R}v, \tilde{f}\big),$$

use the same estimates as in Section 4. For the other terms use (5.1) and (5.3) to find the pair of estimates

$$\big((\tilde{R}M + M^*\tilde{R})v, v\big) \le -(a/4)\Big\|\langle D\rangle_\ell^{\rho/2} v\Big\|^2 + C\Big\|\langle D\rangle_\ell^{(2\nu+1-\kappa\delta)/2} v\Big\|^2,$$

$$\big|(\partial_t\tilde{R}v, v)\big| \le C\Big\|\langle D\rangle_\ell^{(3\nu+1-\rho+(1-\kappa)\delta)/2} v\Big\|^2.$$

If $2\nu + 1 - \kappa\delta \le \rho$ and $3\nu + 1 - \rho + (1-\kappa)\delta \le \rho$ (which implies (4.3)), then both terms are bounded by $\|\langle D\rangle_\ell^{\rho/2} v\|^2$ and can be absorbed in a Gronwall estimate. With κ and ν fixed, the region in the δ, ρ plane described by the two constraints is bounded below by a pair of lines as in the figure.



The minimal value of ρ satisfying the constraints occurs at $\delta = (1+\nu)/(1+\kappa)$ and yields

$$\rho \;\ge\; \frac{1 + (2+\kappa)\theta}{1 + \kappa + (2+\kappa)\theta}\,.$$

The desired *a priori* estimate follows.

6. Proof of the conjugation Proposition 2.6

Lemma 6.1. *Let* $a(x,\xi) \in \tilde{S}^m_{(s)}$ *and assume* $\partial_x^\alpha a(x,\xi) = 0$ *outside* $|x| \le R$ *with some* $R > 0$ *if* $|\alpha| > 0$. *Set*

$$e^{\tau\langle D\rangle_\ell^\rho}\, a(x, D)\, e^{-\tau\langle D\rangle_\ell^\rho} \;=\; b(x, D)$$

where $\tau \in \mathbb{R}$, *then* $b(x,\xi)$ *is given by*

$$b(x,\xi) \;=\; \int e^{-iy\eta}\, e^{\tau\langle\xi+\frac{\eta}{2}\rangle_\ell^\rho - \tau\langle\xi-\frac{\eta}{2}\rangle_\ell^\rho}\, a(x+y,\xi)\, dy d\eta. \tag{6.1}$$

Proof. Write $\phi(\xi) = \tau\langle\xi\rangle_\ell^\rho$ and insert $v = e^{-\phi(D)}u(y) = \int e^{iy\zeta-\phi(\zeta)}\hat{u}(\zeta)d\zeta$ into

$$e^{\phi(D)}\,a(x,D)\,v \;=\; \int e^{i(x\xi - z\xi + (z-y)\eta)}\,e^{\phi(\xi)}\,a\left(\frac{z+y}{2},\eta\right)\,v(y)\,dyd\eta dz d\xi$$

to get

$$e^{\phi(D)}\,a(x,D)\,e^{-\phi(D)}u = \int e^{ix\zeta}\,I(x,\zeta,\mu)\,\hat{u}(\zeta)\,d\zeta$$

where

$$I \;=\; \int e^{i(x\xi - z\xi + (z-y)\eta + y\zeta - x\zeta)}\,e^{\phi(\xi)}\,a\left(\frac{z+y}{2},\eta\right)\,e^{-\phi(\zeta)}\,dyd\eta dz d\xi.$$

The change of variables $\tilde{z} = (y+z)/2$, $\tilde{y} = (y-z)/2$ yields

$$\begin{aligned} I &= 2^n \int e^{i\tilde{y}(\xi - 2\eta + \zeta)}\,d\tilde{y}\;\int e^{-i(\tilde{z}-x)(\xi-\zeta)}\,e^{\phi(\xi)}\,a(\tilde{z},\eta)\,e^{-\phi(\zeta)}\,d\eta d\tilde{z}d\xi \\ &= 2^n \int e^{-2i(\tilde{z}-x)(\eta-\zeta)}e^{\phi(2\eta-\zeta)}\,a(\tilde{z},\eta,\mu)\,e^{-\phi(\zeta)}\,d\eta d\tilde{z} \\ &= \int e^{-i\tilde{z}\eta}\,e^{\phi(\sqrt{2}\eta+\zeta)-\phi(\zeta)}\,a\left(x+\frac{\tilde{z}}{\sqrt{2}},\zeta+\frac{\eta}{\sqrt{2}}\right)\,d\eta d\tilde{z} \end{aligned}$$

and then

$$e^{\phi(D)}\,a(x,D)\,e^{-\phi(D)}\,u = \int e^{i(x-y)\xi}\;p(x,\xi)\,u(y)\,dyd\xi = \mathrm{Op}^0(p)u$$

with

$$p(x,\xi) \;=\; \int e^{-iy\eta}\,e^{\phi(\xi+\sqrt{2}\eta)-\phi(\xi)}\,a\left(x+\frac{y}{\sqrt{2}},\xi+\frac{\eta}{\sqrt{2}}\right)\,dyd\eta. \tag{6.2}$$

Here we remark $\mathrm{Op}^0(p) = b(x,D)$ with $b(x,\xi)$ given by

$$b(x,\xi) \;=\; \int e^{iz\zeta}\;p\left(x+\frac{z}{\sqrt{2}},\xi+\frac{\zeta}{\sqrt{2}}\right)\,dzd\zeta. \tag{6.3}$$

Indeed we see

$$\begin{aligned} b(x,D)u \;&=\; \int e^{i(x-y)\xi}\,b\left(\frac{x+y}{2},\xi\right)\,u(y)\,dyd\xi \\ &= \int e^{i(x\xi - y\xi + z\zeta)}\;p\left(\frac{x+y}{2}+\frac{z}{\sqrt{2}},\xi+\frac{\zeta}{\sqrt{2}}\right)\,u(y)\,dyd\xi dz d\zeta \\ &= \int e^{i((x-y-z)\xi + z\zeta)}\;p\left(\frac{x+y+z}{2},\zeta\right)\,u(y)\,dyd\xi dz d\zeta \\ &= \int e^{iz\zeta}\;p(x,\zeta)\,u(x-z)\,dzd\zeta \;=\; \mathrm{Op}^0(p)u. \end{aligned}$$

Insert (6.2) into (6.3) to get

$$b(x,\xi)=\int e^{i(z\zeta-y\eta)}\,e^{\phi(\sqrt{2}\eta+\xi+\frac{\zeta}{\sqrt{2}})-\phi(\xi+\frac{\zeta}{\sqrt{2}})}\,a\,dyd\eta dzd\zeta,$$
$$a=a\left(x+\frac{z+y}{\sqrt{2}},\xi+\frac{\eta+\zeta}{\sqrt{2}}\right).$$

The change of variables

$$\tilde{z}=\frac{z+y}{\sqrt{2}},\quad \tilde{y}=\frac{y-z}{\sqrt{2}},\quad \tilde{\zeta}=\frac{\zeta+\eta}{\sqrt{2}},\quad \tilde{\eta}=\frac{\eta-\zeta}{\sqrt{2}}$$

gives

$$\begin{aligned}b(x,\xi)&=\int e^{-i(\tilde{z}\tilde{\eta}+\tilde{y}\tilde{\zeta})}\,e^{\phi(\frac{3\tilde{\zeta}}{2}+\xi+\frac{\tilde{\eta}}{2})-\phi(\xi+\frac{\tilde{\zeta}}{2}-\frac{\tilde{\eta}}{2})}\,a(x+\tilde{z},\xi+\tilde{\zeta})\,d\tilde{y}d\tilde{\eta}d\tilde{z}d\tilde{\zeta}\\&=\int e^{-i\tilde{z}\tilde{\eta}}\,e^{\phi(\xi+\frac{\tilde{\eta}}{2})-\phi(\xi-\frac{\tilde{\eta}}{2})}\,a(x+\tilde{z},\xi)\,d\tilde{z}d\tilde{\eta},\end{aligned}$$

proving (6.1). □

Proof of Proposition 2.6. Insert

$$\begin{aligned}a(x+y,\xi)&=\sum_{|\alpha|\le k}\frac{1}{\alpha!}D_x^\alpha a(x,\xi)(iy)^\alpha\\&\quad+\sum_{|\alpha|=k+1}\frac{k+1}{\alpha!}(iy)^\alpha\int_0^1(1-\theta)^kD_x^\alpha a(x+\theta y,\xi)d\theta\end{aligned}$$

into (6.1) to get

$$\begin{aligned}b(x,\xi)&=\sum_{|\alpha|\le k}\frac{1}{\alpha!}\int e^{-iy\eta}e^{\phi(\xi+\frac{\eta}{2})-\phi(\xi-\frac{\eta}{2})}D_x^\alpha a(x,\xi)(iy)^\alpha dyd\eta\\&\quad+\sum_{|\alpha|=k+1}\frac{k+1}{\alpha!}\int e^{-iy\eta}e^{\phi(\xi+\frac{\eta}{2})-\phi(\xi-\frac{\eta}{2})}(iy)^\alpha dyd\eta\\&\quad\cdot\int_0^1(1-\theta)^kD_x^\alpha a(x+\theta y,\xi)d\theta.\end{aligned}\tag{6.4}$$

Since $e^{-iy\eta}(iy)^\alpha=(-\partial_\eta)^\alpha e^{-iy\eta}$ the first term on the right-hand side of (6.4) is

$$\sum_{|\alpha|\le k}\frac{1}{\alpha!}\partial_\eta^\alpha e^{\phi(\xi+\frac{\eta}{2})-\phi(\xi-\frac{\eta}{2})}\Big|_{\eta=0}D_x^\alpha a(x,\xi).\tag{6.5}$$

Note that $\partial_\eta^\alpha e^{\phi(\xi+\frac{\eta}{2})-\phi(\xi-\frac{\eta}{2})}\Big|_{\eta=0}$ is a linear combination of

$$\partial_\xi^{\alpha_1}\phi(\xi)\cdots\partial_\xi^{\alpha_s}\phi(\xi) \quad \text{with} \quad \sum_{j=1}^{s}\alpha_j=\alpha \quad \text{and} \quad |\alpha_j|\geq 1.$$

Divide the linear combination into two parts; the sum over $\sum\alpha_j=1$ and $|\alpha_j|=1$ and the remaining sum called r. If $|\alpha_j|\geq 2$ for some j then $s\leq|\alpha|-1$ and hence $s\rho-|\alpha|\leq-(1-\rho)|\alpha|-\rho\leq-2+\rho$ so that

$$\partial_\xi^{\alpha_1}\phi(\xi)\cdots\partial_\xi^{\alpha_s}\phi(\xi)\in\tilde{S}^{-2+\rho}.$$

Then (6.5) yields

$$\sum_{|\alpha|\leq k}\frac{1}{\alpha!}D_x^\alpha a(x,\xi)(\tau\nabla_\xi\langle\xi\rangle_\ell^\rho)^\alpha+r\,,\qquad\text{for}\quad r\in\tilde{S}^{-1+\rho}.$$

Define

$$\begin{aligned}H_\alpha(\xi,\eta,\mu)&=\frac{1}{\alpha!}\,\partial_\eta^\alpha e^{\phi(\xi+\frac{\eta}{2})-\phi(\xi-\frac{\eta}{2})}\\&=2^{-|\alpha|}\sum_{\beta+\gamma=\alpha}\frac{1}{\beta!\gamma!}\,\partial_\xi^\beta e^{\phi(\xi+\frac{\eta}{2})}\,(-\partial_\xi)^\gamma e^{-\phi(\xi-\frac{\eta}{2})}\end{aligned}$$

where the second term on the right-hand side of (6.4) is, up to a multiplicative constant,

$$\begin{aligned}&\sum_{|\alpha|=k+1}\int e^{-iy\eta}H_\alpha(\xi,\eta)dyd\eta\int_0^1(1-\theta)^kD_x^\alpha a(x+\theta y,\xi)d\theta\\&=\sum_{|\alpha|=k+1}\iint_0^1 e^{ix\eta}(1-\theta)^kH_\alpha(\xi,\theta\eta)d\eta d\theta\int e^{-iy\eta}D_x^\alpha a(y,\xi)dy.\end{aligned}$$

Define $E_\alpha(\eta,\xi):=\int e^{-iy\eta}D_x^\alpha a(y,\xi)dy$ and

$$R_k:=\sum_{|\alpha|=k+1}\iint_0^1 e^{ix\eta}\,(1-\theta)^k\,H_\alpha(\xi,\theta\eta)\,E_\alpha(\eta,\xi)\,d\eta d\theta\,. \tag{6.6}$$

Lemma 6.2. *There is $c>0$ such that for any $\delta\in\mathbb{N}^n$,*

$$\left|\partial_\xi^\delta E_\alpha(\eta,\xi)\right|\leq C_{\alpha\delta}\,\langle\xi\rangle_\ell^{1-|\delta|}\,e^{-c\langle\eta\rangle^\rho}.$$

Proof. Integration by parts gives

$$\eta^\nu \partial_\xi^\delta E_\alpha(\eta,\xi) = \int e^{-iy\eta}\, \partial_\xi^\delta D_x^{\alpha+\nu} a(y,\xi)\, dy\,.$$

Then there exist constants $A > 0$ and C_δ such that

$$\left|\partial_\xi^\delta E_\alpha(\eta,\xi)\right| \le C_\delta \langle\xi\rangle_\ell^{1-|\delta|} A^{|\alpha+\nu|}|\alpha+\nu|!^s\langle\eta\rangle^{-|\nu|} \le C_{\alpha\delta}\langle\xi\rangle_\ell^{1-|\delta|} A^{|\nu|}|\nu|!^s\langle\eta\rangle^{-|\nu|}.$$

Choose ν minimizing $A^{|\nu|}\,|\nu|!^s\,\langle\eta\rangle^{-|\nu|}$, that is $|\nu| \sim e^{-1}A^{-1/s}\langle\eta\rangle^{1/s}$ so that $A^{|\nu|}|\nu|!^s\langle\eta\rangle^{-|\nu|} \lesssim e^{-s^{-1}A^{-1/s}\langle\eta\rangle^{1/s}} = e^{-c\langle\eta\rangle^\rho}$. □

Returning to the proof of Proposition 2.6, note that $H_\alpha(\xi,\eta)$ is a linear combination of terms

$$\partial_\xi^{\beta_1}\phi\left(\xi+\frac{\eta}{2}\right)\cdots\partial_\xi^{\beta_s}\phi\left(\xi+\frac{\eta}{2}\right)\partial_\xi^{\gamma_1}\phi\left(\xi-\frac{\eta}{2}\right)\cdots\partial_\xi^{\gamma_t}\phi\left(\xi-\frac{\eta}{2}\right)e^{\phi(\xi+\frac{\eta}{2})-\phi(\xi-\frac{\eta}{2})}$$
$$:= k_{\beta_1,\dots,\beta_s,\gamma_1,\dots,\gamma_t}(\xi,\eta)\, e^{\phi(\xi+\frac{\eta}{2})-\phi(\xi-\frac{\eta}{2})}$$

where $\sum\beta_j = \beta$ and $\sum\gamma_j = \gamma$ with $|\beta_j| \ge 1$ while $|\gamma_j| \ge 1$, $\beta+\gamma = \alpha$. Since $\langle\xi\pm\eta/2\rangle_\ell^r \le C_r\langle\xi\rangle_\ell^r\langle\eta\rangle^{|r|}$ one sees that

$$\left|\partial_\xi^\delta k_{\beta_1,\dots,\beta_s,\gamma_1,\dots,\gamma_t}(\xi,\eta)\right| \le C_\delta\,\langle\xi\rangle_\ell^{-|\alpha|(1-\rho)-|\delta|}\,\langle\eta\rangle^{|\alpha|+|\delta|}. \tag{6.7}$$

For some $0 < \theta < 1$ one has

$$\partial_\xi^\alpha\left(\phi\left(\xi+\frac{\eta}{2}\right)-\phi\left(\xi-\frac{\eta}{2}\right)\right)=\sum_{j=1}^n\frac{1}{2}\eta_j\left(\partial_\xi^\alpha\partial_{\xi_j}\phi\left(\xi+\frac{\theta\eta}{2}\right)+\partial_\xi^\alpha\partial_{\xi_j}\phi\left(\xi-\frac{\theta\eta}{2}\right)\right). \tag{6.8}$$

Then $\langle\xi\pm\theta\eta/2\rangle_\ell^{\rho-1-|\alpha|} \le \langle\xi\pm\theta\eta/2\rangle_\ell^{-|\alpha|} \le C_\alpha\langle\xi\rangle_\ell^{-|\alpha|}\langle\eta\rangle^{|\alpha|}$ and

$$\left|\partial_\xi^{\alpha_1}\left[\phi\left(\xi+\frac{\eta}{2}\right)-\phi\left(\xi-\frac{\eta}{2}\right)\right]\cdots\partial_\xi^{\alpha_t}\left[\phi\left(\xi+\frac{\eta}{2}\right)-\phi\left(\xi-\frac{\eta}{2}\right)\right]\right|$$
$$\le C_\alpha\,\langle\xi\rangle_\ell^{-|\alpha|}\langle\eta\rangle^{2|\alpha|}\quad\text{for}\quad\alpha=\alpha_1+\cdots+\alpha_t$$

yield

$$\left|\partial_\xi^\delta e^{\phi(\xi+\frac{\eta}{2})-\phi(\xi-\frac{\eta}{2})}\right| \le C_\delta\,\langle\xi\rangle_\ell^{-|\delta|}\,\langle\eta\rangle^{2|\delta|}\,e^{\phi(\xi+\eta/2)-\phi(\xi-\eta/2)}. \tag{6.9}$$

Next prove that with some $c_1 > 0$,

$$\left|\phi(\xi+\eta/2)-\phi(\xi-\eta/2)\right| \;\le\; c_1|\tau|\,\langle\eta\rangle^\rho\,.$$

Indeed if $\ell+|\xi| \ge |\eta|$ then $\langle\xi\rangle_\ell \approx \langle\xi\pm\theta\eta/2\rangle_\ell$ for $|\theta| \le 1$ hence (6.8) gives

$$|\phi(\xi+\eta/2)-\phi(\xi-\eta/2)| \le C|\tau|\,\langle\eta\rangle\,\langle\xi\pm\theta\eta/2\rangle_\ell^{\rho-1}$$
$$\le C'\,|\tau|\,\langle\eta\rangle\,\langle\xi\rangle_\ell^{\rho-1} \le C''\,|\tau|\,\langle\eta\rangle^\rho.$$

While if $\ell+|\xi| \le |\eta|$ then $\langle \xi \pm \eta/2\rangle_\ell \le C\langle\eta\rangle$ and the assertion is clear. From (6.7) and (6.9) we have

$$|\partial_\xi^\delta H_\alpha(\xi,\eta)| \;\le\; C_{\alpha\delta}\,\langle\xi\rangle_\ell^{-|\alpha|(1-\rho)}\,\langle\xi\rangle_\ell^{-|\delta|}\,\langle\eta\rangle^{|\alpha|+2|\delta|}\,e^{c_1|\tau|\langle\eta\rangle^\rho}. \tag{6.10}$$

From Lemma 6.2 and (6.10) one has

$$\left|\partial_\xi^\delta(H_\alpha(\xi,\eta)E_\alpha(\eta,\xi))\right| \le C_{\alpha\delta}\,\langle\xi\rangle_\ell^{1-|\delta|-|\alpha|(1-\rho)}\,\langle\eta\rangle^{|\alpha|+2|\delta|}\,e^{-(c-c_1|\tau|)\langle\eta\rangle^\rho}$$

where $c>0$ is the constant in Lemma 6.2. If $c-c_1|\tau|>0$ then

$$\left|\partial_x^\beta\partial_\xi^\delta R_k(x,\xi)\right| \le C_{\delta\beta}\,\langle\xi\rangle_\ell^{1-|\delta|-(k+1)(1-\rho)}.$$

Since $1-(k+1)(1-\rho)=\rho-k(1-\rho)$, the assertion follows. □

7. Proof of the spectral bound, Proposition 2.2

7.1. Quantitative Nuij

The first step in the proof of Proposition 2.2 is to prove a quantitative version of Nuij's root splitter [18] due to Wakabayashi [23] (see also [5, Lemma 3.1]).

Lemma 7.1 (Nuij). *A monic polynomial $P(\zeta)$ in ζ of degree m whose roots are all real, defines for $s\in\mathbb{R}$, real $\lambda_1(s)<\lambda_2(s)<\cdots<\lambda_m(s)$ so that $(1+sd/d\zeta)^m P(\zeta)=\prod_{j=1}^m(\zeta-\lambda_j(s))$. Then there exists $c=c(m)>0$ so that for $s\in\mathbb{R}$,*

$$\lambda_{j+1}(s)-\lambda_j(s) \;\ge\; c\,|s|, \qquad j=1,\dots,m-1.$$

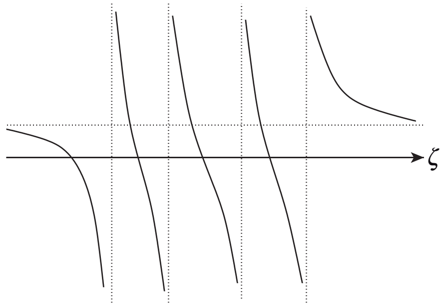
Proof. Let $P(\zeta)=\prod_{j=1}^m(\zeta-\lambda_j)$ with $\lambda_1\le\cdots\le\lambda_m$ and consider for $l=1,\dots,m+1$, the successive Nuij splittings for $s>0$ (the case $s<0$ is similar),

$$(1+sd/d\zeta)^{l-1}P(\zeta)=\prod_{j=1}^m\big(\zeta-\lambda_j^l(s)\big),$$

where $\lambda_1^l(s)<\cdots<\lambda_{l-1}^l(s)\le\lambda_l^l(s)\le\cdots\le\lambda_m^l(s)$. Compute

$$h_l(\zeta,s)=\frac{(1+sd/d\zeta)^l P(\zeta)}{(1+sd/d\zeta)^{l-1}P(\zeta)}=1+s\sum_{j=1}^m\frac{1}{\zeta-\lambda_j^l(s)}. \tag{7.1}$$

Consider the passage from the roots of the denominator called mother roots to the roots of the numerator called daughters. The derivative $dh_l/d\zeta$ is strictly negative on each interval not including a mother root, and $\lim_{|\zeta|\to\infty}h=1$. The graph of h below has four mother roots where the dotted verticals cross the horizontal axis. The mother roots toward the right may have high multiplicity.



There is a simple daughter root to the left of the mother roots and a new simple daughter root between each of the mother roots. Each multiple mother root becomes a daughter root with multiplicity reduced by one and gives rise to a daughter root to the left. Each simple mother root yields a daughter to left. The $\{\lambda_k^{l+1}(s)\}$ are all real, separate the $\{\lambda_k^l(s)\}$, and the first ones are simple. That is,

$$\lambda_1^{l+1}(s) \le \lambda_1^l(s) \le \lambda_2^{l+1}(s) \le \lambda_2^\ell(s) \le \cdots \le \lambda_m^{l+1}(s) \le \lambda_m^l(s),$$
$$\lambda_1^l(s) < \lambda_2^l(s) < \cdots < \lambda_{l-1}^l(s) < \lambda_l^l(s) \le \cdots \le \lambda_m^l(s)\,.$$

We prove by induction on $l \ge 2$, that there exists $c_l > 0$ such that

$$\lambda_k^l(s) \;-\; \lambda_{k-1}^l(s) \;\ge\; c_l\, s \quad \text{for} \quad k = 2, \ldots, l. \tag{7.2}$$

The summands $s/(\zeta - \lambda_j^l(s))$ in (7.1) are all negative to the left of the mother roots. For $l = 1$ the first is equal to -1 when $\zeta = \lambda_1^1(s) - s$. Therefore $h_1(\lambda_1^1 - s, s) < 0$. The root $\lambda_1^2(s)$ lies where the graph of h_1 crosses the axis and therefore to the left of $\lambda_1^1 - s$, so $\lambda_1^2(s) \le \lambda_1^1 - s$.

From $\lambda_2^2(s) \ge \lambda_1^1$ it follows that $\lambda_2^2(s) - \lambda_1^2(s) \ge s$. Therefore (7.2) holds with $c_2 = 1$ when $l = 2$.

Suppose (7.2) holds for $2 \le k \le l$. Prove the case $l + 1$. In (7.1) with $\zeta = \lambda_k^l(s) - \delta s$ the last $m - k + 1$ terms are negative and the first $k - 1$ terms do not exceed $1/(\lambda_k^l(s) - \delta s - \lambda_{k-1}^l(s))$. Therefore by (7.2),

$$h_l\left(\lambda_k^l(s) - \delta s, s\right) \;\le\; 1 + \frac{s(k-1)}{\lambda_k^l(s) - \delta s - \lambda_{k-1}^l(s)} - \frac{1}{\delta} \;\le\; 1 + \frac{k-1}{c_l - \delta} - \frac{1}{\delta}.$$

The right hand side vanishes when $\delta = \left(k + c_l - \sqrt{(k+c_l)^2 - 4c_l}\right)/2 > 0$. We have $h_l(\lambda_k^l(s) - \delta s, s) \le 0$. Therefore $\lambda_k^{l+1}(s) \le \lambda_k^l(s) - \delta s$. Define

$$c_{l+1} := \min_{2 \le k \le l} \left(k + c_l - \sqrt{(k+c_l)^2 - 4c_l}\right)/2 > 0\,.$$

Then

$$\lambda_{k+1}^{l+1}(s) - \lambda_k^{l+1}(s) = \lambda_{k+1}^{l+1}(s) \; - \; \lambda_k^l(s) \; + \; \lambda_k^l(s) \; - \; \lambda_k^{l+1}(s)$$
$$\geq \lambda_k^l(s) - \lambda_k^{l+1}(s) \; \geq \; c_{l+1}\, s$$

for $k = 1, \ldots, l$. This completes the inductive step, so yields (7.2) for $l = m+1$. □

7.2. Three lemmas

This subsection presents three lemmas needed in the proof of Proposition 2.2. Define

$$Q(\zeta, t, x, y, s) \; := \; \det\big(\zeta I - H(t, x, y, s)\big)\,.$$

Then $Q(\zeta, t, x, 0, s) = \det(\zeta - A(t, x))$ and for real t, x, y, s, it holds

$$q(\zeta, t, x, y, s) = \det(\zeta I - A(t, x + sy)) = \det(\zeta I - H + R_{m+1})$$
$$= Q(\zeta, t, x, y, s) + R(\zeta, t, x, y, s)$$

where $R(\zeta,t,x,y,s)$ is a polynomial in ζ of degree $m-1$ with coefficients $O(|s|^{m+1})$.

The next lemma examines what happens when the Taylor expansion and root splitter are applied simultaneously. Apply Nuij's root splitter to obtain polynomials with distinct roots denoted with a tilde,

$$\begin{aligned} \tilde{q}(\zeta, t, x, y, s) &:= (1 + s\partial/\partial\zeta)^m q(\zeta, t, x, y, s) = \prod_{j=1}^{m} (\zeta - \tilde{\lambda}_j(t, x, y, s)), \\ \tilde{Q}(\zeta, t, x, y, s) &:= (1 + s\partial/\partial\zeta)^m Q(\zeta, t, x, y, s) = \prod_{j=1}^{m} (\zeta - \tilde{\Lambda}_j(t, x, y, s)). \end{aligned} \tag{7.3}$$

Lemma 7.2. *If $I \times K \subset \mathbb{R} \times \Omega$ is compact, there is $s_0 > 0$ so that for $(t, x, s) \in I \times K \times [-s_0, s_0]$ and $|y| \leq 1$, all roots ζ of $\tilde{Q} = 0$ are real.*

Proof. We may assume that $x + sy \in \Omega$ when $(t, x) \in I \times K$ with $|y| \leq 1$ and $|s| \leq s_0$. The definitions (7.3) imply that $\tilde{q}(\zeta, t, x, y, s) - \tilde{Q}(\zeta, t, x, y, s) = \tilde{R}$ where $\tilde{R}(\zeta, t, x, y, s)$ is a polynomial in ζ of degree $m - 1$ with coefficients $O(|s|^{m+1})$ uniformly in $(t, x) \in I \times K$ and $|y| \leq 1$. Lemma 7.1 implies

$$|\tilde{\lambda}_{j+1}(t, x, y, s) - \tilde{\lambda}_j(t, x, y, s)| \; \geq \; c(m)|s|\,.$$

Let C_j be the circle of radius $c(m)\,|s|\,/\,2$ with center $\tilde{\lambda}_j(t, x, y, s)$ so that $|\tilde{q}(\zeta, t, x, y, s)| \geq (c(m)/2)^m |s|^m$ if $\zeta \in C_j$. Since $|\tilde{q}(\zeta,t,x,y,s) - \tilde{Q}(\zeta, t, x, y, s)| \leq C|s|^{m+1}$, Rouché's theorem implies that there exists $s_1 > 0$ such that there is exactly one root of $\tilde{Q}(\zeta, t, x, y, s)$ inside C_j for $|s| \leq s_1$. Since $\tilde{Q}(\zeta, t, x, y, s)$ is a real polynomial, the root must be real. □

Lemma 7.3. *Suppose that* $\tilde{Q}(\bar{\lambda}, \bar{t}, \bar{x}, 0, 0) = \det(\bar{\lambda} I - A(\bar{t}, \bar{x})) = 0$. *Then there exists* $\delta > 0$ *such that when* $|\zeta - \bar{\lambda}| < \delta$, $|t - \bar{t}| < \delta$, $|x - \bar{x}| < \delta$, $|y| < \delta$, $|s| < \delta$, *one has* $\tilde{Q}(\zeta, t, x, y, s) \neq 0$ *if* $\operatorname{Im} \zeta \leq 0$ *and* $\operatorname{Im} s < 0$ (*or* $\operatorname{Im} \zeta \geq 0$ *and* $\operatorname{Im} s > 0$).

Proof. Define $p(\zeta, t, x) := \det(\zeta - A(t, x)) = \prod_{j=1}^{m} (\zeta - \lambda_j(t, x))$. If $\operatorname{Im} \zeta < 0$, $\operatorname{Im} s \leq 0$ then

$$\tilde{Q}(\zeta, t, x, 0, s) = (1 + s\partial/\partial\zeta)^m p(\zeta, t, x) \neq 0.$$

Indeed,

$$\frac{(1 + s\partial/\partial\zeta) p(\zeta, t, x)}{p(\zeta, t, x)} = 1 + s \sum_{k=1}^{m} 1/(\zeta - \lambda_j(t, x)) = 0$$

implies that $\sum_{k=1}^{m} 1/(\zeta - \lambda_j(t, x)) = -1/s$ so that $\operatorname{Im} \sum_{k=1}^{m} 1/(\zeta - \lambda_j(t, x)) > 0$ provided that $\operatorname{Im} \lambda_j(t, x) \geq 0$ for all j, which is a contradiction.

That is $(1 + s\partial/\partial\zeta) p(\zeta, t, x) = 0$ implies $\operatorname{Im} \zeta \geq 0$. It is enough to repeat this argument. Since $\tilde{Q}(\bar{\lambda}, \bar{t}, \bar{x}, 0, 0) = 0$ and $\tilde{Q}(\zeta, t, x, 0, s)$ is a polynomial in s of degree m with leading term ms^m, we can find $\delta_1 > 0$ so that the roots s of

$$\tilde{Q}(\zeta, t, x, y, s) = 0$$

with $|s| < s_0$ are continuous in (ζ, t, x, y) for $|\zeta - \bar{\lambda}| < \delta_1$, $|t - \bar{t}| < \delta_1$, $|x - \bar{x}| < \delta_1$, $|y| < \delta_1$.

Suppose that $\tilde{Q}(\hat{\zeta}, \hat{t}, \hat{x}, \hat{y}, \hat{s}) = 0$ with $\operatorname{Im} \hat{\zeta} \leq 0$, $\operatorname{Im} \hat{s} < 0$, $|\hat{s}| \leq s_0$, $|\hat{\zeta} - \bar{\lambda}| < \delta_1$, $|\hat{t} - \bar{t}| < \delta_1$, $|\hat{x} - \bar{x}| < \delta_1$, $|\hat{y}| < \delta_1$. Moving $\hat{\zeta}$ a little bit if necessary, we may assume that $\operatorname{Im} \hat{\zeta} < 0$. Consider $F(\theta) = \min_{|s(\theta)| \leq s_0} \operatorname{Im} s(\theta)$ where the minimum is taken over all roots $s(\theta)$ of $\tilde{Q}(\hat{\zeta}, \hat{t}, \hat{x}, \theta\hat{y}, s) = 0$ with $|s(\theta)| \leq s_0$. Since $F(1) < 0$, $F(0) \geq 0$ there exist $\hat{\theta}$ and $s(\hat{\theta})$ such that $\operatorname{Im} s(\hat{\theta}) = 0$ which contradicts Lemma 7.2.

The proof for the case $\operatorname{Im} \zeta \geq 0$ and $\operatorname{Im} s > 0$ is similar. □

Lemma 7.4. *Assume* (2.1). *Let* $(\bar{t}, \bar{x}) \in \mathbb{R} \times \Omega$ *and let* $\bar{\lambda}$ *be an eigenvalue of* $A(\bar{t}, \bar{x})$ *with multiplicity* r *so that* $\det(\bar{\lambda} - A(\bar{t}, \bar{x})) = 0$. *Then there exists* $\delta > 0$ *so that for all* $|\lambda - \bar{\lambda}| \leq \delta$, $|t - \bar{t}| < \delta$, $|x - \bar{x}| \leq \delta$, $|y| \leq \delta$ *and* $|s| \leq \delta$,

$$|Q(\lambda + is, t, x, y, is)| \geq |s|^r. \tag{7.4}$$

Proof. Define $I := \{i \mid \tilde{\Lambda}_i(\bar{t}, \bar{x}, 0, 0) = \bar{\lambda}\}$ and $I^c := \{i \mid \tilde{\Lambda}_i(\bar{t}, \bar{x}, 0, 0) \neq \bar{\lambda}\}$. Then for $|t - \bar{t}| \leq \delta$, $t \in I$, $|x - \bar{x}| \leq \delta$, $|y| \leq \delta$, $|s| < \delta$, one has

$$\begin{aligned} \tilde{Q}(\zeta, t, x, y, is) &= \prod_{j \in I} \Big(\zeta - \tilde{\Lambda}_j(t, x, y, is)\Big) \prod_{j \in I^c} \Big(\zeta - \tilde{\Lambda}_j(t, x, y, is)\Big) \\ &:= \tilde{Q}_1(\zeta, t, x, y, is)\; \tilde{Q}_2(\zeta, t, x, y, is). \end{aligned}$$

Lemma 7.3 implies that $\pm \operatorname{Im} \tilde{\Lambda}_j(t, x, y, is) \geq 0$ if $\pm s < 0$ and $j \in I$. This shows that if $M > 0$, then

$$|\tilde{Q}_1(\lambda + iMs, t, x, y, is)| \geq 2^{-r/2} \prod_{j \in I} \Big(|\lambda - \operatorname{Re}\tilde{\Lambda}_j| + M|s| + |\operatorname{Im}\tilde{\Lambda}_j| \Big)$$

for small $s \in \mathbb{R}$. The right-hand side is bounded from below by

$$c(M|s|)^k \sum \prod_{j_p \in I, j_1 < \cdots < j_{r-k}} \big(|\lambda - \operatorname{Re}\tilde{\Lambda}_j| + M|s| + |\operatorname{Im}\tilde{\Lambda}_j| \big) \tag{7.5}$$

for all $1 \leq k \leq r$. We prove that there are c_k such that

$$Q(\zeta, t, x, y, s) = \tilde{Q}(\zeta, t, x, y, s) + \sum_{l=1}^{m} c_l (s\partial/\partial\zeta)^l \tilde{Q}(\zeta, t, x, y, s). \tag{7.6}$$

The definition of $\tilde{Q}$ implies

$$(1 - s\partial/\partial\zeta)^m \tilde{Q} = (1 - s\partial/\partial\zeta)^m (1 + s\partial/\partial\zeta)^m Q = \big(1 - s^2\partial^2/\partial\zeta^2\big)^m Q\,.$$

Repeating this argument yields

$$\big(1 + s^{2l}\partial^{2l}/\partial\zeta^{2l}\big)^m \cdots \big(1 + s^2\partial^2/\partial\zeta^2\big)\big(1 - s\partial/\partial\zeta\big)^m \tilde{Q} = \big(1 - s^{4l}\partial^{4l}/\partial\zeta^{4l}\big)^m Q$$

where the right-hand side coincides with Q if $4l \geq m + 1$.

For $|s|M \leq 1$, note that

$$\begin{aligned} &\Big| \big((s\partial/\partial\zeta)^k \tilde{Q}_1\big)(\lambda + iMs, t, x, y, is) \Big| \\ &\lesssim \sum |s|^k \prod_{j_p \in I,\ j_1 < \cdots < j_{r-k}} \Big(\Big|\lambda - \operatorname{Re} \tilde{\Lambda}_j\Big| + M|s| + \Big|\operatorname{Im} \tilde{\Lambda}_j\Big| \Big) \\ &\lesssim M^{-k} \Big| \tilde{Q}_1(\lambda + iMs, t, x, y, is) \Big| \end{aligned}$$

by (7.5) and

$$\Big| \big((s\partial/\partial\zeta)^k \tilde{Q}_2\big)(\lambda + iMs, t, x, y, is) \Big| \ \leq\ C|s|^k.$$

Because $M|s| \leq 1$, Leibniz' rule yields

$$\begin{aligned} &\big| (s\partial/\partial\zeta)^l \big(\tilde{Q}_1 \tilde{Q}_2\big)(\lambda + iMs, t, x, y, is) \big| \\ &\lesssim M^{-l} \sum_{j=0}^{l} (M|s|)^{l-j} \Big| \tilde{Q}_1(\lambda + iMs, t, x, y, is) \Big| \\ &\lesssim\ M^{-l} \Big| \tilde{Q}_1(\lambda + iMs, t, x, y, is) \Big|. \end{aligned}$$

Therefore using (7.6), $|Q(\lambda + iMs, t, x, y, is)|$ is bounded from below by

$$\left|\tilde{Q}_2(\lambda + iMs, t, x, y, is)\right| \Big\{ \left|\tilde{Q}_1(\lambda + iMs, t, x, y, is)\right| - C \sum_{l=1}^{m} M^{-l} \left|\tilde{Q}_1(\lambda + iMs, t, x, y, is)\right| \cdot \left|\tilde{Q}_2(\lambda + iMs, t, x, y, is)\right|^{-1} \Big\}.$$

Choosing $M > 0$ large yields

$$|Q(\lambda + iMs, t, x, y, is)| \ \geq\ c\left|\tilde{Q}_1(\lambda + iMs, t, x, y, is)\right| \ \geq\ cM^r|s|^r$$

because

$$\left|\tilde{Q}_2(\lambda + iMs, t, x, y, is)\right| = \prod_{j \in I^c} \left|\lambda + iMs - \tilde{\Lambda}_j(t, x, y, is)\right| \ \geq\ c_1 \ >\ 0.$$

Since

$$Q(\lambda + is, t, x, y, is) = Q\left(\lambda + i\widetilde{M}(\widetilde{M}^{-1}s), t, x, \widetilde{M}y, i\widetilde{M}^{-1}s\right) \geq |s|^r$$

with $\widetilde{M} := c^{1/r}M$, the desired conclusion follows. □

7.3. Proof of Proposition 2.2

Proof. Suppose that $(\bar{t}, \bar{x}) \in \{|t| \leq T\} \times K$ and $\bar{\lambda}_j$ are the distinct eigenvalues of $A(\bar{t}, \bar{x}) = H(\bar{t}, \bar{x}, 0, 0)$, possibly with multiplicity greater than one. Then there is $\delta > 0$ such that Lemma 7.4 holds for any j. Taking $0 < \delta_1 \leq \delta$ small one can assume that $|\mathrm{Re}\,\zeta - \bar{\lambda}_\mu| < \delta$ for some μ if $Q(\zeta, t, x, y, is) = 0$ and $|t - \bar{t}| \leq \delta_1$, $|x - \bar{x}| \leq \delta_1$, $|y| \leq \delta_1$, $|s| < \delta_1$.

Suppose that there were $|\hat{t} - \bar{t}| \leq \delta_1$, $|\hat{x} - \bar{x}| \leq \delta_1$, $|\hat{y}| \leq \delta_1$, $|\hat{s}| < \delta_1$ and ζ_j such that

$$\left|\mathrm{Im}\,\zeta_j(\hat{t}, \hat{x}, \hat{y}, \hat{s})\right| \ >\ |\hat{s}|.$$

Clearly $\hat{y} \neq 0$ and $\hat{s} \neq 0$. First suppose that $\mathrm{Im}\,\zeta_j(\hat{t}, \hat{x}, \hat{y}, \hat{s}) > |\hat{s}|$. Introduce

$$\Lambda(\theta) := \max\Big\{\mathrm{Im}\,\zeta_j(\hat{t}, \hat{x}, \theta\hat{y}, \hat{s}) \ :\ |\mathrm{Re}\,\zeta_j \ -\ \bar{\lambda}_\mu| < \delta\Big\}. \tag{7.7}$$

Note that $\Lambda(0) = 0$ and $\Lambda(1) > |\hat{s}|$. Since $\Lambda(\theta)$ is continuous there exist l and $\hat{\theta}$ such that $\Lambda(\hat{\theta}) = |\hat{s}|$ so that $\zeta_l(\hat{t}, \hat{x}, \hat{\theta}\hat{y}, \hat{s}) = \alpha + i|\hat{s}|$ with $\alpha \in \mathbb{R}$ and $Q(\alpha + i|\hat{s}|, \hat{t}, \hat{x}, \hat{\theta}\hat{y}, i\hat{s}) = 0$. This contradicts Lemma 7.4 if $\hat{s} > 0$.

If $\hat{s} < 0$ then $H(\hat{t}, \hat{x}, \hat{\theta}\hat{y}, i\hat{s}) = H(\hat{t}, \hat{x}, -\hat{\theta}\hat{y}, -i\hat{s})$ yields

$$Q(\alpha - i\hat{s}, \hat{t}, \hat{x}, -\hat{\theta}\hat{y}, -i\hat{s}) = 0.$$

This contradicts Lemma 7.4.

If $\operatorname{Im} \zeta_j(\hat{t}, \hat{x}, \hat{y}, \hat{s}) < -|\hat{s}|$ it is enough to consider the minimum in (7.7). Thus we conclude that if $Q(\zeta, t, x, y, is) = 0$ with $|t - \bar{t}| \le \delta_1$, $|x - \bar{x}| \le \delta_1$, $|y| \le \delta_1$, $|s| < \delta_1$ then $|\operatorname{Im} \zeta| \le |s|$. Since $\{|t| \le T\} \times K$ is compact there is $\delta_2 > 0$ such that $|\operatorname{Im} \zeta| \le |s|$ if $Q(\zeta, t, x, y, is) = 0$ and $|t - \bar{t}| \le \delta_2$, $|x - \bar{x}| \le \delta_2$, $|y| \le \delta_2$, $|s| < \delta_2$. The identity

$$H(t, x, y, is) = H\left(t, x, \delta_2 y, i\delta_2^{-1} s\right)$$

yields the desired conclusion. □

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Dipartimento di Matematica
Università di Pisa
Pisa, Italia
ferruccio.colombini@unipi.it

Department of Mathematics
Graduate School of Science
Osaka University, Osaka, Japan
nishitani@math.sci.osaka-u.ac.jp

Department of Mathematics
University of Michigan
Ann Arbor, Michigan, USA
rauch@umich.edu