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Euler characteristics and q-difference equations

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Abstract. This paper is concerned with linear q-difference equations. Our main result is an explicit formula for the Euler characteristic of the sheaf of analytic solutions attached to any linear algebraic q-difference equation. This formula involves certain invariants attached to the so-called intermediate singularities. As an application, we interpret the index of rigidity recently introduced by Sakai and Yamaguchi in cohomological terms.

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1. Introduction

This work grew out of an attempt by the first author to find a cohomological interpretation of the index of rigidity for q-difference equations defined by Sakai and Yamaguchi in [15, Section 3]; and of an attempt by the second author to understand the role of the so-called "intermediate singularities" (those other than $0, \infty$, see further below) in the global behaviour of rational q-difference equations. Only the former problem will be tackled here, we intend to pursue the latter one in a future work.

The approach developed in the present paper relies on a sheaf $\mathcal{F}_A$ of analytic solutions attached to any q-difference system

$$Y(qz) = A(z)Y(z) \tag{1.1}$$

with $q \in \mathbf{C}^\times$, $|q| \neq 1$, and $A(z) \in \mathrm{GL}_n(\mathbf{C}(z))$. This is a sheaf over the Riemann surface $\mathbf{E}_q^{an} = \mathbf{C}^\times/q^{\mathbf{Z}}$. It turns out that $\mathcal{F}_A$ is a locally free $\mathcal{O}_{\mathbf{E}_q^{an}}$-module of rank n and, hence, defines a vector bundle over $\mathbf{E}_q^{an}$. One of our main results is an explicit formula for the Euler characteristic $\chi(\mathcal{F}_A)$ of $\mathcal{F}_A$: we prove that it is the sum of *local* invariants of (1.1) attached to the intermediate singularities; see Theorem 3.21. By intermediate singularities, we mean the poles of A or A^{-1} on $\mathbb{P}^1(\mathbf{C}) \setminus \{0, \infty\}$.

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When it is applied to the “internal End” of (1.1), this formula essentially gives Sakai and Yamaguchi’s index of rigidity attached to (1.1); see Paragraph 3.5. This is parallel to Katz’s [7, Theorem 1.1.2].

This paper also includes a cohomological study of natural extensions of $\mathcal{F}_A$ to “completions” of $\mathbf{E}_q^{an}$. We refer to Paragraphs 4 and 5 for the details.

Let us now explain the origin of our approach. The celebrated formula of Grothendieck-Ogg-Shafarevitch [14] was transposed by Deligne to the case of differential equations (see “théorème de comparaison” in [3]), then used by Bertrand in [1,2]. This is the same formula that Katz uses in his study of rigidity. A long time ago, Bertrand asked one of us if it could be transposed to q-difference equations. Our answer, which roughly follows the lines of [3, Chapter 6], is contained in the present paper.

2. General notation and basic definitions

2.1. Sheaves and rings of functions

Consider a Riemann surface X. We let $\mathcal{O}_X$ (respectively $\mathcal{M}_X$) be the sheaf of holomorphic (respectively meromorphic) functions over X. We will use the shorthand notation $\mathcal{O}(X) := \mathcal{O}_X(X)$ (respectively $\mathcal{M}(X) := \mathcal{M}_X(X)$) for the corresponding ring (respectively field) of global sections. As usual, the stalk of $\mathcal{O}_X$ (respectively $\mathcal{M}_X$) at $x \in X$ will be denoted by $\mathcal{O}_{X,x}$ (respectively $\mathcal{M}_{X,x}$). We will let u_x be a local coordinate centered at $x \in X$. Hence, u_x is a uniformizer of the discrete valuation ring $\mathcal{O}_{X,x}$. We denote by $v_x : \mathcal{M}_{X,x} \to \mathbf{Z} \cup \{+\infty\}$ the corresponding u_x-adic valuation (it depends on x only, not on a particular choice of the local coordinate u_x).

We will denote by

$$\mathbb{P}^1(\mathbf{C})^{an} = \mathbf{C} \cup \{\infty\}$$

the complex projective line, endowed with its structure of Riemann surface. We will use the following classical notations for the stalks of $\mathcal{O}_{\mathbb{P}^1(\mathbf{C})^{an}}$ and $\mathcal{M}_{\mathbb{P}^1(\mathbf{C})^{an}}$ at 0 and ∞:

$$\begin{aligned} \mathbf{C}\{z\} &:= \mathcal{O}_{\mathbb{P}^1(\mathbf{C})^{an},0}, & \mathbf{C}(\{z\}) &:= \mathcal{M}_{\mathbb{P}^1(\mathbf{C})^{an},0}, \\ \mathbf{C}\{1/z\} &:= \mathcal{O}_{\mathbb{P}^1(\mathbf{C})^{an},\infty}, & \mathbf{C}(\{1/z\}) &:= \mathcal{M}_{\mathbb{P}^1(\mathbf{C})^{an},\infty}. \end{aligned}$$

2.2. The q-dilatation operator σ_q and the complex torus $\mathbf{E}_q^{an}$

In the whole paper, q is a nonzero complex number such that $|q| > 1$. We let σ_q be the q-dilatation operator acting on a function $f(z)$ of the complex variable z as follows:

$$\sigma_q f(z) := f(qz).$$

We can and will identify the field $\mathcal{M}(\mathbf{C}^\times)^{\sigma_q}$ of σ_q-invariant meromorphic functions over $\mathbf{C}^\times$ with the field $\mathcal{M}(\mathbf{E}_q^{an})$ of meromorphic functions over the complex torus

$$\mathbf{E}_q^{an} := \mathbf{C}^\times/q^{\mathbf{Z}},$$

i.e.,

$$\mathcal{M}(\mathbf{C}^\times)^{\sigma_q} = \mathcal{M}\big(\mathbf{E}_q^{an}\big).$$

We will denote by $\pi : \mathbf{C}^\times \to \mathbf{E}_q^{an}$ the canonical covering and will use the following notations :

$$\overline{a} := \pi(a), \quad [a; q] := \pi^{-1}(\pi(a)) = aq^{\mathbf{Z}}.$$

2.3. q-difference systems

Until the end of the paper, we consider

$$A \in \mathrm{GL}_n(\mathbf{C}(z)).$$

The associated q-difference system is:

$$\sigma_q X = AX. \tag{2.1}$$

2.3.1. *Intermediate singularities*

The singular locus of equation (2.1) is defined as:

$$\begin{aligned}\mathrm{Sing}(A) &:= \{\text{poles of } A \text{ on } \mathbf{C}^\times\} \cup \{\text{poles of } A^{-1} \text{ on } \mathbf{C}^\times\} \\ &= \{\text{poles of } A \text{ on } \mathbf{C}^\times\} \cup \{\text{zeroes of } \det A \text{ on } \mathbf{C}^\times\}.\end{aligned} \tag{2.2}$$

Thus, if $U \subset \mathbf{C}^\times$ is an open subset which does not meet $\mathrm{Sing}(A)$, then A is *regular* on U, meaning that it is holomorphic over U as well as its inverse A^{-1}.

Remark 2.1. In what follows, the relative positions of the elements of $\mathrm{Sing}(A)$ will be of particular importance. More precisely, we will have to be careful when there are $\lambda, \mu \in \mathrm{Sing}(A)$ such that $\lambda \neq \mu$ but $\lambda/\mu \in q^{\mathbf{Z}}$. This is the reason why we consider the poles of A and A^{-1} instead of just considering their q-orbits (*i.e.*, their images by π).

2.3.2. *Gauge transform*

Let (K, ϕ) be a difference field extension of $(\mathbf{C}(z), \sigma_q)$. This means that K is a field extension of $\mathbf{C}(x)$ endowed with a field automorphism $\phi : K \to K$ extending $\sigma_q : \mathbf{C}(z) \to \mathbf{C}(z)$. The gauge transform $F[A]$ of A by $F \in \mathrm{GL}_n(K)$ is:

$$F[A] := (\phi F) A F^{-1} \in \mathrm{GL}_n(K).$$

Let us consider $A, B \in \mathrm{GL}_n(\mathbf{C}(z))$. We say that A and B are (K, ϕ)-equivalent if there exists $F \in \mathrm{GL}_n(K)$ such that $B = F[A]$. If A and B are (K, ϕ)-equivalent for $(K, \phi) = (\mathbf{C}(z), \sigma_q)$ (respectively $(\mathbf{C}(\{z\}), \sigma_q)$, $(\mathbf{C}(\{1/z\}), \sigma_q)$), we say that A and B are rationally equivalent (respectively analytically equivalent at 0, analytically equivalent at ∞). Note that if A and B are analytically equivalent at 0 (respectively ∞) and if $F \in \mathrm{GL}_n(\mathbf{C}(\{z\}))$ (respectively $F \in \mathrm{GL}_n(\mathbf{C}(\{1/z\}))$) is such that $B = F[A]$ then one has automatically $F \in \mathrm{GL}_n(\mathcal{M}(\mathbf{C}))$ (respectively $F \in \mathrm{GL}_n(\mathcal{M}(\mathbf{C}^\times \cup \{\infty\}))$) (indeed, this meromorphic continuation property of F follows from the functional equation $\phi F = BFA^{-1}$ and the fact that $|q| \neq 0, 1$).

2.3.3. *Solutions*

Let (R, ϕ) be a difference algebra over the difference field $(\mathbf{C}(z), \sigma_q)$. This means that R is a $\mathbf{C}(z)$-algebra endowed with a ring automorphism $\phi : R \to R$ such that, for all $a \in \mathbf{C}(z)$ and $r \in R$, $\phi(ar) = \sigma_q(a)\phi(r)$. The solutions of (2.1) in (R, ϕ) form the $\mathbf{C}$-vector space:

$$\mathrm{Sol}(A, R) := \big\{X \in R^n \mid \phi X = AX\big\},$$

where we consistently identify R^n with the space of column vectors $\mathrm{Mat}_{n,1}(R)$. A fundamental matricial solution of (2.1) in R is a matrix $\mathcal{X} \in \mathrm{GL}_n(R)$ such that $\phi\mathcal{X} = A\mathcal{X}$.

In what follows, we will frequently consider $\mathrm{Sol}(A, R)$ for R a $\mathbf{C}(z)$-algebra of meromorphic functions stable by σ_q; it will be implicit that $\phi = \sigma_q$ for such a R.

3. Euler characteristics of some sheaves of modules over $\mathbf{E}_q^{an}$

We recall that, until the end of the paper, we consider

$$A \in \mathrm{GL}_n(\mathbf{C}(z)).$$

3.1. Some sheaves of functions and of solutions

3.1.1. *Some sheaves of functions*

Note that σ_q operates on the direct image sheaves $\pi_*\mathcal{O}_{\mathbf{C}^\times}$ and $\pi_*\mathcal{M}_{\mathbf{C}^\times}$ and that we have the following obvious identifications for the fixed subsheaves:

$$(\pi_*\mathcal{O}_{\mathbf{C}^\times})^{\sigma_q} = \mathcal{O}_{\mathbf{E}_q^{an}}, \quad (\pi_*\mathcal{M}_{\mathbf{C}^\times})^{\sigma_q} = \mathcal{M}_{\mathbf{E}_q^{an}}.$$

We shall now introduce various subsheaves of $\pi_*\mathcal{M}_{\mathbf{C}^\times}$. We will denote by $D(0,r) \subset \mathbf{C}$ the open disk with center 0 and radius r and by $D(0, r)^c$ its complement in $\mathbf{C}$. Let V be an open subset of $\mathbf{E}_q^{an}$ and let $U := \pi^{-1}(V)$, so that $\pi_*\mathcal{M}_{\mathbf{C}^\times}(V) = \mathcal{M}_{\mathbf{C}^\times}(U)$.

We define the subsheaves $\mathcal{A}^{(0)}$, $\mathcal{A}^{(\infty)}$, $\mathcal{A}$, $\mathcal{A}'$ of $\pi_*\mathcal{M}_{\mathbf{C}^\times}$ by:

$$\mathcal{A}^{(0)}(V) := \big\{f \in \mathcal{M}_{\mathbf{C}^\times}(U) \mid f \text{ is holomorphic over } U \cap D(0,r) \text{ for some } r > 0\big\}, \tag{3.1}$$

$$\mathcal{A}^{(\infty)}(V) := \big\{f \in \mathcal{M}_{\mathbf{C}^\times}(U) \mid f \text{ is holomorphic over } U \cap D(0,R)^c \text{ for some } R > 0\big\}, \tag{3.2}$$

$$\mathcal{A}(V) := \big\{f \in \mathcal{M}_{\mathbf{C}^\times}(U) \mid f \text{ is holomorphic over } U\big\} \text{ (thus } \mathcal{A} = \pi_*\mathcal{O}_{\mathbf{C}^\times}), \tag{3.3}$$

$$\mathcal{A}'(V) := \big\{f \in \mathcal{M}_{\mathbf{C}^\times}(U) \mid f \text{ has at worst a finite number of poles in any } q\text{-spiral } [a;q] \subset U\big\}. \tag{3.4}$$

It is easily seen that $\mathcal{A} \subset \mathcal{A}^{(0)} \cap \mathcal{A}^{(\infty)} \subset \mathcal{A}'$.

3.1.2. *Some sheaves of solutions*

To any subsheaf $\mathcal{B}$ of $\pi_*\mathcal{M}_{\mathbf{C}^\times}$, we shall associate a sheaf of solutions $\mathrm{Sol}(A,\mathcal{B})$ on $\mathbf{E}_q^{an}$ for which the sections over an open subset $V \subset \mathbf{E}_q^{an}$ are the solutions $X \in \mathcal{B}(V)^n$ of (2.1), *i.e.*,

$$\mathrm{Sol}(A,\mathcal{B})(V) = \big\{X \in \mathcal{B}(V)^n \mid \sigma_q X = AX\big\}.$$

Taking successively for $\mathcal{B}$ the sheaves $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{A}^{(0)}$, $\mathcal{A}^{(\infty)}$, we obtain the sheaves of solutions on $\mathbf{E}_q^{an}$ respectively denoted by

$$\mathcal{F}_A = \mathrm{Sol}(A,\mathcal{A}),\ \mathcal{F}'_A = \mathrm{Sol}\big(A,\mathcal{A}'\big),\ \mathcal{F}_A^{(0)} = \mathrm{Sol}\big(A,\mathcal{A}^{(0)}\big),\ \mathcal{F}_A^{(\infty)} = \mathrm{Sol}\big(A,\mathcal{A}^{(\infty)}\big).$$

We check easily that $\mathcal{F}_A \subset \mathcal{F}_A^{(0)} \cap \mathcal{F}_A^{(\infty)} = \mathcal{F}'_A$. In the course of what follows, we shall find out that all these sheaves are locally free of rank n over $\mathcal{O}_{\mathbf{E}_q^{an}}$, whence define holomorphic vector bundles over $\mathbf{E}_q^{an}$.

Remark 3.1. Taking $\mathcal{B} := \pi_*\mathcal{M}_{\mathbf{C}^\times}$ yields the sheaf $\mathrm{Sol}(A,\pi_*\mathcal{M}_{\mathbf{C}^\times})$ of all meromorphic solutions, plainly a $(\pi_*\mathcal{M}_{\mathbf{C}^\times})^{\sigma_q} = \mathcal{M}_{\mathbf{E}_q^{an}}$-module. It was proved by Praagman in [12] that this is a free $\mathcal{M}_{\mathbf{E}_q^{an}}$-module of rank n. Said otherwise, there exists a fundamental matricial solution $\mathcal{X} \in \mathrm{GL}_n(\mathcal{M}(\mathbf{C}^\times))$ of (2.1). The proof relies on the fact that it is a meromorphic vector bundle on the compact Riemann surface $\mathbf{E}_q^{an}$ and that such a bundle is free.

The proof of the following is immediate:

Lemma 3.2. *Let $F \in \mathrm{GL}_n(\mathbf{C}(z))$ (respectively $F \in \mathrm{GL}_n(\mathcal{M}(\mathbf{C}))$, $F \in \mathrm{GL}_n(\mathcal{M}(\mathbf{C}^\times \cup \{\infty\}))$) such that $B := F[A] \in \mathrm{GL}_n(\mathbf{C}(z))$. Then the automorphism $X \mapsto FX$ of $(\pi_*\mathcal{M}_{\mathbf{C}^\times})^n$ induces identifications:*

$$F\mathcal{F}'_A = \mathcal{F}'_B \ \left(\text{respectively } F\mathcal{F}_A^{(0)} = \mathcal{F}_B^{(0)},\ F\mathcal{F}_A^{(\infty)} = \mathcal{F}_B^{(\infty)}\right).$$

Remark 3.3. Note however that it is not generally true that $F\mathcal{F}_A = \mathcal{F}_B$. For instance, take $A := 1$ and $F := 1/(z-1)$ so that $B := F[A] = \frac{z-1}{qz-1}$. Then, $\mathcal{F}_B$ is isomorphic to $\mathcal{O}_{\mathbf{E}_q^{an}}(-[\overline{1}])$, whereas $\mathcal{F}_A = \mathcal{O}_{\mathbf{E}_q^{an}}$.

3.2. Reminders on the Euler characteristic

The Euler characteristic $\chi(\mathcal{F})$ of a sheaf $\mathcal{F}$ of $\mathbf{C}$-vector spaces over a topological space X is defined as

$$\chi(\mathcal{F}) := \sum_{i \geq 0} (-1)^i \dim_{\mathbf{C}} H^i(X, \mathcal{F}),$$

whenever this makes sense, *i.e.*, whenever all cohomology spaces $H^i(X, \mathcal{F})$ are finite dimensional and almost all of them vanish. For coherent sheaves over a compact Riemann surface, these conditions are satisfied and only the first two terms $H^0(X, \mathcal{F})$ and $H^1(X, \mathcal{F})$ do not vanish. So, in this case, we have

$$\chi(\mathcal{F}) = \dim_{\mathbf{C}} H^0(X, \mathcal{F}) - \dim_{\mathbf{C}} H^1(X, \mathcal{F}).$$

Because of the long exact sequence of cohomology associated to an exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0,$$

one has additivity:

$$\chi(\mathcal{F}) = \chi\big(\mathcal{F}'\big) + \chi\big(\mathcal{F}''\big).$$

It follows that, if $\mathcal{F}$ is a successive extension of $\mathcal{F}_1, \ldots, \mathcal{F}_n$, then

$$\chi(\mathcal{F}) = \chi(\mathcal{F}_1) + \cdots + \chi(\mathcal{F}_n).$$

For line bundles (*i.e.*, locally free rank one sheaves), the value of $\chi(\mathcal{F})$ can be computed using Riemann-Roch formula.

For skyscraper sheaves (over a curve, this is the same as torsion sheaves), the cohomology spaces $H^i(X, \mathcal{F})$ are trivial for $i \geq 1$ and $\chi(\mathcal{F}) = \dim_{\mathbf{C}} H^0(X, \mathcal{F}) = \sum_{x \in X} \dim_{\mathbf{C}} \mathcal{F}_x$, this sum actually involving only a finite number of non zero terms. These are the only facts we shall need.

Since they are not so easy to find in this compact form in the litterature, here are some references. The basic theory is in Serre's famous [16] and (for the analytic viewpoint) in [5]. However in another celebrated article, Serre shows that, for compact Riemann surfaces, the analytic and algebraic point of view are equivalent [17]. So one is entitled to refer to the following more modern and more detailed books [6,10,11].

3.3. Euler characteristics of the sheaves $\mathcal{F}_A^{(0)}$ and $\mathcal{F}_A^{(\infty)}$

The aim of this section is to compute the Euler characteristics of the sheaves $\mathcal{F}_A^{(0)}$ and $\mathcal{F}_A^{(\infty)}$. We start with the following lemma.

Lemma 3.4. *The sheaves $\mathcal{F}_A^{(0)}$ and $\mathcal{F}_A^{(\infty)}$ are locally free $\mathcal{O}_{\mathbf{E}_q^{an}}$-modules of rank n.*

Proof. Let $\overset{\circ}{\mathrm{D}}(0,r)$, $r > 0$, any punctured disk on which A is regular. Let V be a trivializing open subset of $\mathbf{E}_q^{an}$ for the covering π so that $\pi^{-1}(V)$ is the disjoint union of the $q^k W$, for $k \in \mathbf{Z}$, where $\pi_{|W}$ is a homeomorphism onto V. Up to shrinking V, we can assume that $W \subset \overset{\circ}{\mathrm{D}}(0,r)$. Let $V' \subset V$ be any open subset and set $W' = \pi^{-1}(V') \cap W$ so that $\pi^{-1}(V')$ is the disjoint union of the $q^k W'$ for $k \in \mathbf{Z}$. Then any $X \in \mathcal{O}_{\mathbf{C}^\times}(W')^n$ extends successively holomorphically to $q^{-1}W'$, $q^{-2}W', \dots$ through (2.1) used as a recursive definition $X(z) := A(z)^{-1}X(qz)$; and meromorphically to qW', q^2W', $\dots$ through (2.1) used as a recursive definition $X(qz) := A(z)X(z)$. Thus, X extends uniquely to an element of $\mathcal{F}_A^{(0)}(V')$. This continuation procedure shows that $(\mathcal{O}_{\mathbf{C}^\times})^n_{|W} = (\mathcal{O}_{\mathbf{E}_q^{an}})^n_{|V}$ is isomorphic to $(\mathcal{F}_A^{(0)})_{|V}$. The proof at ∞ is similar. □

The sheaves $\mathcal{F}_A^{(0)}$ and $\mathcal{F}_A^{(\infty)}$ being locally free (and, hence, coherent), they have finite dimensional cohomology spaces concentrated in degree 0 and 1 (see Section 3.2). In particular, their Euler characteristics are well-defined.

Proposition 3.5. *The Euler characteristics of the sheaves $\mathcal{F}_A^{(0)}$ and $\mathcal{F}_A^{(\infty)}$ are given by the following formulas:*

$$\chi\left(\mathcal{F}_A^{(0)}\right) = v_0(\det A) \text{ and } \chi\left(\mathcal{F}_A^{(\infty)}\right) = v_\infty(\det A), \tag{3.5}$$

where we recall that $v_0 : \mathbf{C}(\{z\}) \to \mathbf{Z} \cup \{+\infty\}$ is the z-adic valuation and $v_\infty : \mathbf{C}(\{1/z\}) \to \mathbf{Z} \cup \{+\infty\}$ is the z^{-1}-adic valuation.

The proofs at 0 and ∞ are entirely similar, so we shall concentrate on the first case, which occupies this whole section.

3.3.1. *Newton polygons and plan of the proof of Proposition* 3.5

One classically associates to the q-difference system $\sigma_q X = AX$ a Newton polygon at 0 (for the sake of conciseness, this Newton polygon will be called the Newton polygon of A at 0); see for instance [13]. This Newton polygon has a certain number of slopes $\mu_1 < \cdots < \mu_k$ in $\mathbf{Q}$. These slopes come with certain multiplicities $r_1, \dots, r_k \in \mathbf{N}^*$ such that $r_1 + \cdots + r_k = n$ and $r_i\mu_i \in \mathbf{Z}$ for $i \in \{1, \dots, k\}$.

According to [13], A is analytically equivalent at 0 to (*i.e*. there exists $F \in \mathrm{GL}_n(\mathcal{M}(\mathbf{C}))$ such that $F[A]$ is equal to) a upper-triangular by block matrix

$$\begin{pmatrix} A_1 & \dots & \dots & \dots \\ 0 & \ddots & U_{i,j} & \vdots \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & A_k \end{pmatrix} \tag{3.6}$$

where each $A_i \in \mathrm{GL}_{r_i}(\mathbf{C}(z))$ has unique slope μ_i and where each $U_{i,j}$ belongs to $\mathrm{Mat}_{r_i,r_j}(\mathbf{C}[z, z^{-1}])$. From Lemma 3.2, we see that $\chi(\mathcal{F}_A^{(0)}) = \chi(\mathcal{F}_{A'}^{(0)})$. So, we can and will assume for the proof of Proposition 3.5 that A is in the form (3.6).

The proof of Proposition 3.5 proceeds in three steps, corresponding to the following properties of the slopes of the Newton polygon of A at 0:

Step 1. first, reduction to the case of a pure isoclinic system, *i.e*., to the case $A = A_1$.

Step 2. then, reduction to the case of a pure isoclinic system with integral slope, *i.e*., to the case $A = A_1$ and $\mu_1 \in \mathbf{Z}$.

Step 3. last, proof in the case of pure isoclinic systems with integral slope.

We shall now outline the strategies of proof of these three steps, which rely on well-known properties of q-difference equations proved in [13] for instance.

Step 1 is a direct consequence of the exactness of the functor $A \rightsquigarrow \mathcal{F}_A^{(0)}$ proved in Lemma 3.7 below and of the additivity of the Euler characteristic.

Step 2 relies on the fact that, if A is pure isoclinic with slope $\mu = d/r$, then the change of variable (ramification) $z = z'^r$, $q = q'^r$ yields a q'-difference system with matrix $A'(z') := A(z)$ which is pure isoclinic with slope $\mu' = r\mu = d \in \mathbf{Z}$. Section 3.3.3 is then devoted to the study of the effect of this change of variable on the Euler characteristic $\chi(\mathcal{F}_A^{(0)})$.

Step 3 relies on the fact that we have an explicit description of the pure isoclinic systems with integral slope. Namely, if $A = A_1$ and $\mu := \mu_1 \in \mathbf{Z}$, then A is rationally equivalent to $z^\mu C$ for some $C \in \mathrm{GL}_r(\mathbf{C})$. Section 3.3.4 is devoted to the calculation of the Euler characteristic $\chi(\mathcal{F}_A^{(0)})$ when A has this specific form.

Remark 3.6. Normal forms for pure isoclinic systems with non integral slopes were obtained by van der Put and Reversat [18], but we will not used them, as they are rather complicated and ramification gives us a shorter way to prove Proposition 3.5.

These steps correspond to the following three sections.

3.3.2. *Reduction to the case of a pure isoclinic system*

A useful model when studying the linear properties of q-difference systems is the following. Call *q-difference module* a pair (V, Φ), where V is a $\mathbf{C}(z)$-vector space of finite rank and Φ a σ_q*-linear* automorphism of V, *i.e.*, Φ is a group automorphism satisfying the following q-analogue of Leibniz relation:

$$\forall a \in \mathbf{C}(z)\ ,\ \forall x \in V\ ,\ \Phi(ax) = \sigma_q(a)\Phi(x).$$

A morphism from the q-difference module (V, Φ) to the q-difference module (W, Ψ) is by definition a $\mathbf{C}(z)$-linear map $f : V \to W$ which moreover intertwines Φ and Ψ, *i.e.*, such that $f \circ \Phi = \Psi \circ f$. We thus obtain an Abelian $\mathbf{C}$-linear category, the category DiffMod of q-difference modules.

On the other hand, one can consider the category DiffSyst of q-difference systems whose objects are the matrices $A \in \mathrm{GL}_n(\mathbf{C}(z))$ and whose morphisms from an object $A \in \mathrm{GL}_n(\mathbf{C}(z))$ to an object $B \in \mathrm{GL}_p(\mathbf{C}(z))$ are the matrices $F \in \mathrm{Mat}_{p,n}(\mathbf{C}(z))$ such that $(\sigma_q F)A = BF$. In particular, isomorphisms are just gauge transformations. This is also an Abelian $\mathbf{C}$-linear category.

There is a close relationship between DiffMod and DiffSyst. Indeed, every object of DiffMod is isomorphic to one of the form $(\mathbf{C}(z)^n, \Phi_A)$ where $A \in \mathrm{GL}_n(\mathbf{C}(z))$ and $\Phi_A : \mathbf{C}(z)^n \to \mathbf{C}(z)^n, X \mapsto A^{-1}\sigma_q(X)$: this is done by choosing a basis $\mathcal{B}$ of V over $\mathbf{C}(z)$ and by considering the matrix $A \in \mathrm{GL}_n(\mathbf{C}(z))$ such that $\mathcal{B} = \Phi(\mathcal{B})A$. Thus, the functor from DiffSyst to DiffMod acting on the objects by $A \rightsquigarrow (\mathbf{C}(z)^n, \Phi_A)$ and on the morphisms by $F \rightsquigarrow (\mathbf{C}(z)^n \to \mathbf{C}(z)^p, X \mapsto FX)$ defines an equivalence of Abelian $\mathbf{C}$-linear categories.

For details on what precedes, we refer to [13].

The following fact is stated without proof in [13] (and other places).

Lemma 3.7. *The functor* $A \rightsquigarrow \mathcal{F}_A^{(0)}$*, from the Abelian category of q-difference systems* DiffSyst *to the Abelian category of coherent sheaves over* $\mathbf{E}_q^{an}$*, is exact.*

Proof. In the Abelian category of q-difference systems, an exact sequence

$$0 \to A' \to A \to A'' \to 0$$

takes (up to isomorphism) the form:

$$A = \begin{pmatrix} A' & N \\ 0 & A'' \end{pmatrix}.$$

In the associated sequence

$$0 \to \mathcal{F}_{A'}^{(0)} \to \mathcal{F}_A^{(0)} \to \mathcal{F}_{A''}^{(0)} \to 0 \tag{3.7}$$

the morphisms $\mathcal{F}_{A'}^{(0)} \to \mathcal{F}_A^{(0)}$ and $\mathcal{F}_A^{(0)} \to \mathcal{F}_{A''}^{(0)}$ take, over any fixed $V \subset \mathbf{E}_q^{an}$, the form $X' \mapsto (X', 0)$ and $(X', X'') \mapsto X''$ respectively. Then exactness of the sequence (3.7) is obvious, except from right exactness $\mathcal{F}_A^{(0)} \to \mathcal{F}_{A''}^{(0)} \to 0$ which we now proceed to prove.

So let $x \in \mathbf{E}_q^{an}$ and let $X'' \in (\mathcal{F}_{A''}^{(0)})_x$, which we may represent by some $X'' \in \mathcal{F}_{A''}^{(0)}(V)$ for some open neighborhood $V \subset \mathbf{E}_q^{an}$ of x; and we can as well assume that V is a trivializing neighborhood for the covering π so that $\pi^{-1}(V)$ is the disjoint union of the $q^k U$, for $k \in \mathbf{Z}$, where $\pi_{|U}$ is a homeomorphism onto V. Let $\overset{\circ}{\mathrm{D}}(0,r)$, $r > 0$, a punctured disk centered at 0 on which A is regular, so the same holds for A', A''. We choose the above U such that $U \subset \overset{\circ}{\mathrm{D}}(0,r)$.

To lift X'' to $\mathcal{F}_A^{(0)}(V)$, we have to find X' such that $X := (X', X'')$ is a solution of (2.1), which is equivalent (since we already know that $\sigma_q X'' = A'' X''$) to $\sigma_q X' = A' X' + C$, where $C := N X''$. Moreover, we want X to be holomorphic over $\overset{\circ}{\mathrm{D}}(0,r') \cap \pi^{-1}(V)$, for some $r' > 0$. So we choose $r' \leq r$ such that X'' is holomorphic over $\overset{\circ}{\mathrm{D}}(0,r') \cap \pi^{-1}(V)$ and proceed to solve the equation in X'. Let $U' := q^k U$ where $k \in \mathbf{Z}$ is chosen in such a way that $U' \subset \overset{\circ}{\mathrm{D}}(0,r')$. We set the value of X' on U' *arbitrarily* (we only require it to be holomorphic) and then use the functional equation $X' = A'^{-1}(\sigma_q X' - C)$ to extend it successively to $q^{-1}U'$, $q^{-2}U', \ldots$: all these are holomorphic; and the equation $\sigma_q X' = A' X' + C$ to extend X' successively to qU', $q^2 U', \ldots$: those ones are meromorphic. This X' and the corresponding X have the expected properties. □

In the Abelian category of q-difference systems DiffSyst, we have that A is a successive extension of $A_1, \ldots, A_k$. We deduce from Lemma 3.7 that $\mathcal{F}_A^{(0)}$ is a successive extension of $\mathcal{F}_{A_1}^{(0)}, \ldots, \mathcal{F}_{A_k}^{(0)}$ and hence that

$$\chi\left(\mathcal{F}_A^{(0)}\right) = \chi\left(\mathcal{F}_{A_1}^{(0)}\right) + \cdots + \chi\left(\mathcal{F}_{A_k}^{(0)}\right).$$

Since obviously $v_0(\det A) = v_0(\det A_1) + \cdots + v_0(\det A_k)$, we see that we just have to prove formula (3.5) in the case of a pure isoclinic system, *i.e.*, for $k = 1$.

3.3.3. *Reduction to the case of integral slopes*

So, we assume that A is pure isoclinic, *i.e.*, that $A = A_1$ and we set $\mu := \mu_1 = d/r$. From [13] we know that the change of variable (ramification) $z = z'^r$, $q = q'^r$ yields a q'-difference system with matrix $A'(z') := A(z)$ which is pure isoclinic with slope $\mu' = r\mu = d$. For this system, formula (3.5) must be interpreted with v_0 meaning the z'-adic valuation, *i.e.* $v_0(\det A') = r v_0(\det A)$.

Let $\rho : \mathbf{C}^\times \to \mathbf{C}^\times, z' \mapsto z := z'^r$. This induces a commutative diagram:

$$\begin{array}{ccc} \mathbf{C}^\times & \xrightarrow{\rho} & \mathbf{C}^\times \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ \mathbf{E}_{q'}^{an} & \xrightarrow{\overline{\rho}} & \mathbf{E}_q^{an}, \end{array}$$

where $\pi' : \mathbf{C}^\times \to \mathbf{E}_{q'}^{an}$ denotes the canonical projection.

Lemma 3.8. *With these notations,* $\overline{\rho}^*\mathcal{F}_A^{(0)} = \mathcal{F}_{A'}^{(0)}$.

Proof. Let $V \subset \mathbf{E}_q^{an}$ an arbitrary open subset, $V' := \overline{\rho}^{-1}(V)$ its preimage in $\mathbf{E}_{q'}^{an}$, and $U := \pi^{-1}(V)$, $U' := \pi'^{-1}(V') = \rho^{-1}(U)$ their respective preimages in $\mathbf{C}^\times$. Any solution of $\sigma_q Y = AY$ analytic over U near 0 gives rise, by the changes of variables $A'(z') = A(z)$, $Y'(z') = Y(z)$, to a solution of $\sigma_{q'} Y' = A'Y'$ analytic over U' near 0. The maps $\mathcal{F}_A^{(0)}(V) \to \mathcal{F}_{A'}^{(0)}(V') = \overline{\rho}_*\mathcal{F}_{A'}^{(0)}(V)$ thus defined make up a morphism of sheaves of linear spaces $\mathcal{F}_A^{(0)} \to \overline{\rho}_*\mathcal{F}_{A'}^{(0)}$, whence, by adjunction, a morphism of sheaves of linear spaces $\overline{\rho}^{-1}\mathcal{F}_A^{(0)} \to \mathcal{F}_{A'}^{(0)}$ (the source here is the topological inverse image sheaf) and then a morphism of sheaves of modules $\overline{\rho}^*\mathcal{F}_A^{(0)} \to \mathcal{F}_{A'}^{(0)}$. We now show that this is an isomorphism. It is enough to do so by restriction to a basis of open subsets.

So let $V \subset \mathbf{E}_q$ be a trivializing open subset for the covering $\overline{\rho}$ and let $W' \subset V' := \overline{\rho}^{-1}(V)$ such that $W' \to V = \overline{\rho}(W')$ is a homeomorphism. Then a solution of $\sigma_q X = AX$ over $\pi^{-1}(V)$ gives rise to a solution $X'(z') := X(z)$ of $\sigma_{q'} X' = A'X'$ over $\pi'^{-1}(W')$. In this way, we get a $\mathbf{C}$-linear isomorphism from $\overline{\rho}^{-1}\mathcal{F}_A^{(0)}(W')$ to $\mathcal{F}_{A'}^{(0)}(W')$. The isomorphism of modules

$$\overline{\rho}^*\mathcal{F}_A^{(0)}(W') := \overline{\rho}^{-1}\mathcal{F}_A^{(0)}(W') \otimes_{\mathcal{O}_{\mathbf{E}_q^{an}}(V)} \mathcal{O}_{\mathbf{E}_{q'}^{an}}(W') \simeq \mathcal{F}_{A'}^{(0)}(W')$$

follows, because here $\mathcal{O}_{\mathbf{E}_{q'}^{an}}(W') = \mathcal{O}_{\mathbf{E}_q^{an}}(V)$. □

The following statement is obviously a particular case of much more general facts, but, for lack of a convenient reference, we give a direct proof.

Lemma 3.9. *Let $p : X' \to X$ an isogeny of degree r between two complex tori and let $\mathcal{F}$ a locally free sheaf on X. Then:*

$$\chi(p^*\mathcal{F}) = r\chi(\mathcal{F}).$$

Proof. Since the inverse image functor is exact and since χ is additive for exact sequences, the triangularisation of holomorphic vector bundles over compact Riemann surfaces [5, corollary of Theorem 10, page 63] allow us to assume that $\mathcal{F}$ has rank 1, $\mathcal{F} = \mathcal{O}_X(D)$ for some divisor D. But then $p^*\mathcal{F} = \mathcal{O}_{X'}(D')$, where $D' := p^*D$. Writing $d := \deg D$, so that $\deg D' = rd$, we have, by Riemann-Roch theorem for line bundles (with here $g = 1$), $\chi(\mathcal{F}) = \deg D = d$ and $\chi(p^*\mathcal{F}) = \deg D' = rd$. □

Remark 3.10. For any finite morphism $p : X' \to X$ and any coherent sheaf $\mathcal{F}'$ on X', we have equality of the cohomology groups: $H^i(X', \mathcal{F}') = H^i(X, p_*\mathcal{F}')$ [4, page 63], thus in the case of our lemma $\chi(p_*p^*\mathcal{F}) = \chi(p^*\mathcal{F}) = r\,\chi(\mathcal{F})$. However, even in the case of an etale covering, it is not true that $p_*p^*\mathcal{F} \simeq \mathcal{F}^r$. For instance, taking $\mathcal{F} := \mathcal{O}_X$, we see that $p_*p^*\mathcal{O}_X$ is locally free of rank r but its global sections has dimension 1: indeed, $p^*\mathcal{O}_X = \mathcal{O}_{X'}$.

Combining Lemmas 3.8 and 3.9, we get the equality:

$$\chi\left(\mathcal{F}_{A'}^{(0)}\right) = r\chi\left(\mathcal{F}_{A}^{(0)}\right).$$

Since we found that $v_0(\det A') = r v_0(\det A)$, we see that it is enough to prove formula (3.5) for A', *i.e.* for a pure isoclinic system with integral slope.

3.3.4. *Proof in the case of a pure system with integral slopes*

So, we are now reduced to prove Proposition 3.5 in the case of a pure system with integral slopes, *i.e.*, in the case $A = A_1$ and $\mu := \mu_1 \in \mathbf{Z}$. From [13], we know that A is rationally equivalent to $z^\mu C$ for some $C \in \mathrm{GL}_r(\mathbf{C})$. On the one hand, from Lemma 3.2, we get that $\chi(\mathcal{F}_A^{(0)}) = \chi(\mathcal{F}_{z^\mu C}^{(0)})$, and, on the other hand, we have $v_0(\det A) = v_0(\det z^\mu C)$. So, we can and will assume that

$$A = z^\mu C.$$

Then:

$$\mathcal{F}_A^{(0)} \simeq \mathcal{O}_{\mathbf{E}_q^{an}}(\mu) \otimes \mathcal{F}_C,$$

and $\mathcal{F}_C$ is a flat vector bundle of rank r. Again from general facts, it follows that $\chi(\mathcal{F}_A^{(0)}) = r\mu$, but for the lack of convenient reference, we give a direct argument.

Lemma 3.11. *We have* $\chi(\mathcal{F}_A^{(0)}) = r\mu$.

Proof. We can assume that C is triangular (conjugacy by an element of $\mathrm{GL}_n(\mathbf{C})$ is a particular case of rational gauge equivalence), so that $\mathcal{F}_A^{(0)}$ is an iterated extension of r sheaves of the form $\mathcal{F}_{cz^\mu}^{(0)}$ with $c \in \mathbf{C}^\times$. By additivity of the Euler characteristics, we are drawn to prove that $\chi(\mathcal{F}_{cz^\mu}^{(0)}) = \mu$. But a nontrivial meromorphic section of $\mathcal{F}_{cz^\mu}^{(0)}$ can be obtained as $s := {\theta_q}^{\mu-1}(z)\theta_q(cz)$, where the theta function $\theta_q \in \mathcal{O}(\mathbf{C}^\times)$ satisfies $\sigma_q\theta_q = z\theta_q$ and $\mathrm{div}_{\mathbf{C}^\times}(\theta_q) = \sum_{a\in[-1;q]}[a]$ (see [13]), so that the degree of the section s is μ, and we can apply Riemann-Roch theorem again. ☐

Since $\det A = z^{r\mu}\det C$, we have $v_0(\det A) = r\mu$ and the expected formula follows. This terminates the proof of Proposition 3.5.

Remark 3.12. A somewhat different proof is possible along the following lines. Using the results of this section, one can prove (using the notations of Proposition 3.5) that:

$$\det \mathcal{F}_A^{(0)} = \mathcal{F}_{\det A}^{(0)}. \tag{3.8}$$

Indeed, the equality is easy when A has integral slopes, using the existence of a triangular form with diagonal terms cz^μ and Lemma 3.7; and one can reduce to this case by extension of the base just as in 3.3.3. Once equality (3.8) is proved, the theorem of Riemann-Roch for vector bundles over compact Riemann surfaces [5] allows one to conclude immediately.

3.4. Sheaves of solutions related to intermediate singularities

In this section, we intend to compute $\chi(\mathcal{F}'_A)$ as a sum of local terms defined at $0, \infty$ and at the "intermediate singularities", *i.e.* points in $\mathrm{Sing}(A)$. The reason for using $\mathcal{F}'_A$ instead of $\mathcal{F}_A$ is the fact, mentioned at the end of Subsection 3.1 (Lemma 3.2 and Remark 3.3), that the former is in some sense intrinsic (up to rational isomorphisms) while the latter is not. This is related to so-called "resonancies" and we shall first show how to deal with them.

3.4.1. *Resonancies*

Definition 3.13. A singularity $a \in \mathrm{Sing}(A)$ is called *resonant*[1] if $q^k a \in \mathrm{Sing}(A)$ for some $k \in \mathbf{Z}, k \neq 0$. The system A is said to be *nonresonant* if it has no resonant singularities, *i.e.* if $\mathrm{Sing}(A) \cap q^{\mathbf{N}^*}\mathrm{Sing}(A) = \emptyset$.

Lemma 3.14. *If A is nonresonant, then $\mathcal{F}_A = \mathcal{F}'_A$.*

Proof. Let V be an open subset of $\mathbf{E}_q^{an}$, let $U = \pi^{-1}(V)$ and let X be a solution of (2.1) meromorphic over U. Consider $a \in U$. In order to prove the lemma, it is sufficient to prove that X is either holomorphic over $[a;q]$ or has infinitely many poles over $[a;q]$. If $a \notin q^{\mathbf{Z}}\mathrm{Sing}(A)$, the relation $X(qz) = A(z)X(z)$ and the fact that A is regular over $[a;q]$ imply that X either has no poles over $[a;q]$ or has infinitely many poles over $[a;q]$. It remains to consider the case $a \in q^{\mathbf{Z}}\mathrm{Sing}(A)$. Up to replacing a by aq^j for some $j \in \mathbf{Z}$, we can assume that $a \in \mathrm{Sing}(A)$. Then, no $q^k a$ with $k \neq 0$ belongs to $\mathrm{Sing}(A)$, so we deduce that the same dichotomy as above holds separately over both half q-spirals $aq^{-\mathbf{N}}$ and $q^{\mathbf{N}^*}$. In any case, the conclusion follows. □

Lemma 3.15. *For every $A \in \mathrm{GL}_n(\mathbf{C}(z))$, there exists a rational gauge transformation $F \in \mathrm{GL}_n(\mathbf{C}(z))$ such that $F[A]$ has all its singularities within the fundamental annulus $\mathcal{C}(1, |q|)$:*

$$\forall a \in \mathrm{Sing}(F[A]) \ , \ 1 \leq |a| < |q| \, .$$

In particular, $F[A]$ is nonresonant.

Proof. Note that $A = uA_0$ where $u \in \mathbf{C}(z)^{\times}$ and $A_0 \in \mathrm{GL}_n(\mathbf{C}(z)) \cap \mathrm{Mat}_n(\mathbf{C}[z])$. We may write in the same way $F = fF_0$, and then clearly $F[A] = f[u]F_0[A_0]$. We shall deal separately with the scalar components f, u and with the polynomial components F_0, A_0.

Write $u = c\prod(z - a_i)^{r_i}, c \in \mathbf{C}^{\times}$. Then, if for instance $|a_j| \geq |q|$, the gauge transform $(z - a_j)^{r_j}[u]$ has singularity a_j replaced by a_j/q, so, iterating, we can move it to the fundamental annulus. The case where $|a_j| < 1$ is tackled similarly. In this way we get $\mathrm{Sing}\, f[u] \subset \mathcal{C}(1, |q|)$.

[1] The notion of resonancy comes from the local study of q-different systems, where one has to get rid of resonant *exponents* (eigenvalues of the fuchsian components of the system), [13]. Be careful, there is a difference between a resonant singularity and a singularity with resonant exponents. The two concepts have not much in common.

Since A_0 is polynomial, its singularities are the zeroes of $\det A_0$. So let a such that $\det(A_0(a)) = (\det A_0)(a) = 0$ and let $X_0 \in \mathbf{C}^n$ non trivial such that $A_0(a)X_0 = 0$. Complete X_0 to a basis of $\mathbf{C}^n$, thus yielding $P \in \mathrm{GL}_n(\mathbf{C})$ such that P has first column X_0. Then the first column of $P^{-1}AP$ vanishes at a, so it is a multiple of $z-a$ in $\mathbf{C}[z]^n$. Now assume for instance that $|a| \geq |q|$ and use the gauge transformation $S := \mathrm{Diag}(z-a, 1, \dots, 1)$: we see that $S[P^{-1}AP] = (SP^{-1})[A]$ has the same singularities as A except that one zero a of the determinant has been replaced by a/q. Iterating, we may move it to the fundamental annulus. The case where $|a| < 1$ is tackled similarly. In this way we get the wanted property. □

Note that, writing $B := F[A]$ we then have:

$$\left(F\mathcal{F}_A^{(0)} = \mathcal{F}_B^{(0)} \text{ and } F\mathcal{F}'_A = \mathcal{F}'_B\right) \Longrightarrow \mathcal{F}_A^{(0)}/\mathcal{F}'_A \simeq \mathcal{F}_B^{(0)}/\mathcal{F}'_B = \mathcal{F}_B^{(0)}/\mathcal{F}_B.$$

3.4.2. *Computation of $\chi(\mathcal{F}'_A)$ for a nonresonant system*

The proof of the following result is left to the reader.

Lemma 3.16. *Assume that $A \in \mathrm{GL}_n(\mathbf{C}(z))$ is nonresonant. Let V be an open subset of $\mathbf{E}_q^{an}$, let $U = \pi^{-1}(V)$ and let X be a solution of* (2.1) *meromorphic over U. If some element of $q^{-\mathbf{N}}a$ is a pole of X then any element of $q^{-\mathbf{N}}a$ is a pole of X. Similarly, if some element of $q^{\mathbf{Z}_{>0}}a$ is a pole of X then any element of $q^{\mathbf{Z}_{>0}}a$ is a pole of X.*

Lemma 3.17. *Assume that $A \in \mathrm{GL}_n(\mathbf{C}(z))$ is nonresonant. Consider $a \in \mathbf{C}^\times$ and let $x = \pi(a)$. Let $R := \mathcal{O}_{\mathbf{C}^\times, a} = \mathcal{O}_{\mathbf{E}_q^{an}, x}$ (thus a discrete valuation ring). Then we have isomorphisms of R-modules:*

$$\left(\mathcal{F}_A^{(0)}/\mathcal{F}_A\right)_x \simeq \frac{R^n}{R^n \cap A^{-1}R^n}, \quad \left(\mathcal{F}_A^{(\infty)}/\mathcal{F}_A\right)_x \simeq \frac{R^n}{R^n \cap AR^n}.$$

Proof. If $a \notin \mathrm{Sing}(A)$, it follows from Lemma 3.16 that $(\mathcal{F}_A^{(0)})_x = (\mathcal{F}_A)_x$ and hence $\left(\mathcal{F}_A^{(0)}/\mathcal{F}_A\right)_x$ is trivial. Since $A \in \mathrm{GL}_n(R)$, the modules $R^n/(R^n \cap A^{-1}R^n)$ and $R^n/(R^n \cap AR^n)$ are trivial as well, so the isomorphisms are valid.

We assume that $a \in \mathrm{Sing}(A)$. Let U a connected neighborhood of a small enough that π induces a homeomorphism $U \to V := \pi(U)$ and that $U \cap \mathrm{Sing}(A) = \{a\}$. Then the restriction maps $\mathcal{F}_A(V) \to (\mathcal{F}_A)_x$ and $\mathcal{F}_A^{(0)}(V) \to (\mathcal{F}_A^{(0)})_x$ are bijective.

As we already saw in the proof of Lemma 3.4, a solution $X \in \mathcal{F}_A^{(0)}(V)$ can be taken arbitrarily in $\mathcal{O}_{\mathbf{C}^\times}(U)^n$ and then uniquely extended to V using the functional equation $X(z) = A(z)^{-1}X(qz)$. This yields an identification of $(\mathcal{F}_A^{(0)})_x$ with R^n. Under this identification, the condition that X belongs to $\mathcal{F}_A(V)$ is that all $X(q^k z)$, $k \geq 1$ be holomorphic on U; according to Lemma 3.16, it is enough to check it for $X(qz) = A(z)X(z)$, *i.e.* it is enough to require that $AX \in R^n$, whence the identification of $(\mathcal{F}_A)_x$ with $R^n \cap A^{-1}R^n$ and, in the end, of $\left(\mathcal{F}_A^{(0)}/\mathcal{F}_A\right)_x$ with $R^n/(R^n \cap A^{-1}R^n)$.

The isomorphism at ∞ is proven in the same way. □

Lemma 3.18. *Let R a discrete valuation ring, u a uniformizer, K the fraction field of R and $A \in \mathrm{GL}_n(K)$. There exist $P, Q \in \mathrm{GL}_n(R)$ and $D = \mathrm{Diag}(u^{d_1}, \dots, u^{d_n})$ for some integers $d_1 \leq \cdots \leq d_n$ such that $A = PDQ$. We have isomorphisms of R-modules:*

$$\frac{R^n}{R^n \cap AR^n} \simeq \prod_{d_i > 0} \frac{R}{u^{d_i}R}, \quad \frac{R^n}{R^n \cap A^{-1}R^n} \simeq \prod_{d_i < 0} \frac{R}{u^{-d_i}R}.$$

Proof. The given decomposition $A = PDQ$ comes from the theorem of invariant factors for finitely generated modules over principal ideal rings. We shall only prove the second formula, the first one being similar (and simpler). Since $P, Q \in \mathrm{GL}_n(R)$, one has $QR^n = P^{-1}R^n = R^n$ and:

$$\begin{aligned}\frac{R^n}{R^n \cap A^{-1}R^n} &= \frac{R^n}{R^n \cap Q^{-1}D^{-1}P^{-1}R^n} \simeq \frac{QR^n}{Q(R^n \cap Q^{-1}D^{-1}P^{-1}R^n)} \\ &= \frac{QR^n}{QR^n \cap D^{-1}P^{-1}R^n} = \frac{R^n}{R^n \cap D^{-1}R^n},\end{aligned}$$

and $R^n \cap D^{-1}R^n = \prod_{i=1}^n (R \cap u^{-d_i}R)$. Last:

$$R \cap u^k R = \begin{cases} R \text{ if } k \leq 0, \\ u^k R \text{ if } k > 0. \end{cases}$$ □

We introduce the following notations. Let $a \in \mathbf{C}^\times$ and $x = \pi(a)$. If $a \in \mathrm{Sing}(A)$ and $R := \mathcal{O}_{\mathbf{E}_q^{an},x}$, we write $\Delta_a(A) = \Delta_x(A)$ the multiset of all the d_i appearing as exponents of the diagonal part D of A in the two lemmas above, and Δ^+, respectively Δ^- the submultisets of positive, respectively negative exponents. For a nonsingular a, we can consider that $\Delta_a(A)$ consists in n times 0, and that Δ^+, Δ^- are empty. To summarize:

Proposition 3.19. *Assume that $A \in \mathrm{GL}_n(\mathbf{C}(z))$ is nonresonant. The coherent sheaves $\mathcal{F}_A^{(0)}/\mathcal{F}_A$ and $\mathcal{F}_A^{(\infty)}/\mathcal{F}_A$ are supported at $\pi(\mathrm{Sing}(A))$. They are skyscraper sheaves with stalks:*

$$\left(\mathcal{F}_A^{(0)}/\mathcal{F}_A\right)_x \simeq \prod_{d \in \Delta_x^-(A)} \frac{\mathcal{O}_{\mathbf{E}_q^{an},x}}{u_x^{-d}\mathcal{O}_{\mathbf{E}_q^{an},x}}, \quad \left(\mathcal{F}_A^{(\infty)}/\mathcal{F}_A\right)_x \simeq \prod_{d \in \Delta_x^+(A)} \frac{\mathcal{O}_{\mathbf{E}_q^{an},x}}{u_x^{d}\mathcal{O}_{\mathbf{E}_q^{an},x}}.$$

Corollary 3.20. *For a nonresonant $A \in \mathrm{GL}_n(\mathbf{C}(z))$, the Euler characteristic of $\mathcal{F}_A$ is given by*

$$\chi(\mathcal{F}_A) = v_0(\det A) - \sum_{x \in \pi(\mathrm{Sing}(A))} \dim_{\mathbf{C}} \left(\mathcal{F}_A^{(0)}/\mathcal{F}_A\right)_x = v_0(\det A) + \sum_{a \in \mathrm{Sing}(A)} \sum_{d \in \Delta_a^-(A)} d,$$

and also by

$$\chi(\mathcal{F}_A) = v_\infty(\det A) - \sum_{x\in\pi(\mathrm{Sing}(A))} \dim_{\mathbf{C}}\left(\mathcal{F}_A^{(\infty)}/\mathcal{F}_A\right)_x = v_\infty(\det A) - \sum_{a\in\mathrm{Sing}(A)} \sum_{d\in\Delta_a^+(A)} d.$$

Proof. Since the cohomology in degree ≥ 1 of skyscraper sheaves is trivial, we obtain:

$$\chi\left(\mathcal{F}_A^{(0)}/\mathcal{F}_A\right) = \sum_{x\in\pi(\mathrm{Sing}(A))} \dim_{\mathbf{C}}\left(\mathcal{F}_A^{(0)}/\mathcal{F}_A\right)_x.$$

Using the additivity of the Euler characteristic, we get:

$$\chi\left(\mathcal{F}_A\right) = \chi\left(\mathcal{F}_A^{(0)}\right) - \sum_{x\in\pi(\mathrm{Sing}(A))} \dim_{\mathbf{C}}\left(\mathcal{F}_A^{(0)}/\mathcal{F}_A\right)_x.$$

But, we have already seen that $\chi(\mathcal{F}_A^{(0)}) = v_0(\det A)$. Moreover, Proposition 3.19 ensures that, for all $x \in \pi(\mathrm{Sing}(A))$, $\dim_{\mathbf{C}}(\mathcal{F}_A^{(0)}/\mathcal{F}_A)_x = \sum_{d\in\Delta_a^-(A)}(-d)$. Whence the first formula. The proof of the second formula is similar. □

3.4.3. *Computation of $\chi(\mathcal{F}'_A)$ in the general case*

We now release all resonancy conditions: we consider an arbitrary $A \in \mathrm{GL}_n(\mathbf{C}(z))$. We introduce the following notations for every $x \in \mathbf{E}_q^{an}$:

$$\ell_x^-(A) := \dim_{\mathbf{C}}\left(\mathcal{F}_A^{(0)}/\mathcal{F}'_A\right)_x,$$
$$\ell_x^+(A) := \dim_{\mathbf{C}}\left(\mathcal{F}_A^{(\infty)}/\mathcal{F}'_A\right)_x,$$
$$\ell_x(A) := \ell_x^-(A) + \ell_x^+(A).$$

Note that the dimensions over $\mathbf{C}$ are as well lengths of $\mathcal{O}_{\mathbf{E}_q^{an},x}$-modules. Also, if $x \notin \pi(\mathrm{Sing}(A))$, we have $\ell_x^-(A) = \ell_x^+(A) = \ell_x(A) = 0$.

Theorem 3.21.

(i) *For an arbitrary $A \in \mathrm{GL}_n(\mathbf{C}(z))$:*

$$\chi\left(\mathcal{F}'_A\right) = v_0(\det A) - \sum_{x\in\mathbf{E}_q^{an}} \ell_x^-(A),$$
$$\chi\left(\mathcal{F}'_A\right) = v_\infty(\det A) - \sum_{x\in\mathbf{E}_q^{an}} \ell_x^+(A),$$
$$2\chi\left(\mathcal{F}'_A\right) = v_0(\det A) + v_\infty(\det A) - \sum_{x\in\mathbf{E}_q^{an}} \ell_x(A);$$

(ii) *For a nonresonant* $A \in \mathrm{GL}_n(\mathbf{C}(z))$*:*

$$\chi\left(\mathcal{F}_A\right) = \chi\left(\mathcal{F}'_A\right),$$
$$\ell_x^+(A) = \sum_{d \in \Delta_x^+(A)} d,$$
$$\ell_x^-(A) = \sum_{d \in \Delta_x^-(A)} (-d).$$

Proof. We know form Lemma 3.15 that there exists $F \in \mathrm{GL}_n(\mathbf{C}(z))$ such that $B := F[A]$ is nonresonant. Since $F\mathcal{F}'_A = \mathcal{F}_B$ and $F\mathcal{F}_A^{(0)} = \mathcal{F}_B^{(0)}$, we see that $\mathcal{F}'_A \cong \mathcal{F}_B$ and $\mathcal{F}_A^{(0)}/\mathcal{F}'_A \cong \mathcal{F}_B^{(0)}/\mathcal{F}'_B$. Therefore, using Proposition 3.20, we obtain:

$$\chi\left(\mathcal{F}'_A\right) = \chi\left(\mathcal{F}'_B\right) = v_0(\det B) - \sum_{x \in \mathbf{E}_q^{an}} \dim_{\mathbf{C}} \left(\mathcal{F}_B^{(0)}/\mathcal{F}_B\right)_x$$
$$= v_0(\det A) - \sum_{x \in \mathbf{E}_q^{an}} \dim_{\mathbf{C}} \left(\mathcal{F}_A^{(0)}/\mathcal{F}_A\right)_x = v_0(\det A) - \sum_{x \in \mathbf{E}_q^{an}} \ell_x^-(A).$$

The proof of the second formula is similar and the third formula is an obvious consequence of the first and second ones.

The last part of the result, about the nonresonant case, is a direct consequence of the Paragraph 3.4.2. □

3.4.4. *Relation with "Riemann-Hilbert correspondence on the quantum torus"*

In various talks, Kontsevich and Soibelman stated an equivalence of categories between rational q-difference systems and data made up of a coherent sheaf over $\mathbf{E}_q^{an}$ and two so-called "anti-Harder-Narasimhan filtrations" over this sheaf. They consider this as an extension of the results in [13]. There seems to be no written published version of this theory (of which we heard after submitting the present work), but see for instance the videos [8,9].

Although they give no indication of the construction of these data, it is clear that their sheaf $\mathcal{F}$ should contain our sheaves $\mathcal{F}_A^{(0)}$ and $\mathcal{F}_A^{(\infty)}$, that these are endowed with their respective slope filtrations, and that the quotients $\mathcal{F}/\mathcal{F}_A^{(0)}$ and $\mathcal{F}/\mathcal{F}_A^{(\infty)}$ should be torsion sheaves, thus skyscraper sheaves, and that they should be concentrated at intermediate singularities.

From the above description of $\mathcal{F}_A^{(0)}/(\mathcal{F}_A^{(0)} \cap \mathcal{F}_A^{(\infty)})$ and $\mathcal{F}_A^{(\infty)}/(\mathcal{F}_A^{(0)} \cap \mathcal{F}_A^{(\infty)})$, it is clear, using standard isomorphism theorems of the form $(M+N)/N \simeq M/(M \cap N)$, that the sheaf $\mathcal{F} := \mathcal{F}_A^{(0)} + \mathcal{F}_A^{(\infty)}$ would do the job.

3.5. Application to a formula of Sakai and Yamaguchi

3.5.1. *Euler characteristics and index of rigidity*

Recall from [13] that the "internal End" of a q-difference system $A \in \mathrm{GL}_n(\mathbf{C}(z))$ is defined as:

$$B := \underline{\mathrm{End}}(A) := A^\vee \otimes A \in \mathrm{GL}_{n^2}(\mathbf{C}(z)),$$

where the "dual" $A^\vee$ of A is the contragredient ${}^tA^{-1}$ and where $\otimes$ denotes the Kronecker product. We intend here to compute $\chi(\mathcal{F}'_B)$ and to compare it to the *rigidity index* introduced by Sakai and Yamaguchi in [15, Section 3]. An important preliminary fact is that, for any scalar $f \in \mathbf{C}(z)^\times$:

$$\underline{\mathrm{End}}(fA) = \underline{\mathrm{End}}(A).$$

Therefore, we may and will assume that A is polynomial and that its coefficients have no common factor. From general linear algebra, $\det B = (\det {}^tA^{-1})^n(\det A)^n = 1$. Obviously, $\mathrm{Sing}B = \mathrm{Sing}A$ (the inclusion might have been strict if the coefficients of A had a common factor); this singular set is the set of zeroes of $\det A$ over $\mathbf{C}^\times$. We write N the number of these zeroes, counted with multiplicities. Also, from now on, we assume A (and therefore B) to be nonresonant.

Let $a \in \mathrm{Sing}A$ and $A = PDQ$ the corresponding decomposition as in 3.4.2. Then $A^\vee = P^\vee D^\vee Q^\vee$ whence $B = (P^\vee \otimes P)(D^\vee \otimes D)(Q^\vee \otimes Q)$. Since $P^\vee \otimes P$ and $Q^\vee \otimes Q$ are regular at a, we see that $\Delta_a(B) = \Delta_a(A) - \Delta_a(A)$, meaning that if $\Delta_a(A)$ is the multiset $d_1 \leq \cdots \leq d_n$, then $\Delta_a(B)$ is the multiset of all $d_i - d_j$, $i, j = 1, \ldots, n$. Thus, writing $x := \pi(a)$:

$$\begin{aligned}
\ell_x^\pm(B) &= \sum_{1\leq i<j\leq n}(d_j - d_i) = \sum_{1\leq i<j\leq n} d_j - \sum_{1\leq i<j\leq n} d_i \\
&= \sum_{1\leq j\leq n}(j-1)d_j - \sum_{1\leq n}(n-i)d_i = \sum_{i=1}^{n}(2i-1-n)d_i.
\end{aligned}$$

We deduce:

$$\begin{aligned}
nv_a(\det A) - \frac{1}{2}\ell_x(B) &= \sum_{i=1}^{n}(2n-2i+1)d_i \\
&= \sum_{i=1}^{n}(2i-1)d_{n-i+1} = d_n + 3d_{n-1} + \cdots + (2n-1)d_1 \\
&= d_1(1+3+\cdots+(2n-1)) \\
&\quad + (d_2-d_1)(1+3+\cdots+(2n-3)) + \cdots \\
&= n^2d_1 + (n-1)^2(d_2-d_1) + \cdots + 1^2(d_n - d_{n-1}),
\end{aligned}$$

that is $e_1^2 + \cdots + e_p^2$, where $p := d_n$ and $e_1, \ldots, e_p$ is the *dual Young tableau* of the *Young tableau* $d_1, \ldots, d_n$. Summing these equalities for all $a \in \mathrm{Sing}(A)$, we get, with obvious notations:

$$\chi(\mathcal{F}'_B) = \sum_{a \in \mathrm{Sing}(A)} \sum e_i^2(a) - nN.$$

This is the part of Sakai-Yamaguchi's index of rigidity [15, Section 3] that depends only on intermediate singularities $a \in \mathrm{Sing}(A) \subset \mathbf{C}^\times$ and not on 0 and ∞.

3.5.2. *Taking into account* 0 *and* ∞

We shall now introduce a topological space $\widetilde{\mathbf{E}_q^{an}}$ and a sheaf on it in order to take into account 0 and ∞. We consider the set

$$\widetilde{\mathbf{E}_q^{an}} = \{0\} \sqcup \mathbf{E}_q^{an} \sqcup \{\infty\}.$$

We endow this set with the following topology: a basis of open sets is given by the open sets of $\mathbf{E}_q^{an}$, and by the subsets $\{0\}$ and $\{\infty\}$, so that $\widetilde{\mathbf{E}_q^{an}}$ has three connected components $\mathbf{E}_q^{an}$, $\{0\}$ and $\{\infty\}$. For any $A \in \mathrm{GL}_n(\mathbf{C}(z))$, we let $\widetilde{\mathcal{F}'_A}$ be the sheaf on $\widetilde{\mathbf{E}_q^{an}}$ such that $(\widetilde{\mathcal{F}'_A})_{|\mathbf{E}_q^{an}} = \mathcal{F}'_A$ and with stalks at 0 and ∞ given by $(\widetilde{\mathcal{F}'_A})_0 = \mathrm{Sol}(A, \mathbf{C}(\{z\}))$ and $(\widetilde{\mathcal{F}'_A})_\infty = \mathrm{Sol}(A, \mathbf{C}(\{z^{-1}\}))$. Then, we obviously have

$$\chi\left(\widetilde{\mathcal{F}'_A}\right) = \chi\left(\mathcal{F}'_A\right) + \dim_{\mathbf{C}} \mathrm{Sol}(A, \mathbf{C}(\{z\})) + \dim_{\mathbf{C}} \mathrm{Sol}\left(A, \mathbf{C}\left(\left\{z^{-1}\right\}\right)\right).$$

We shall now apply this formula to the above $B \in \mathrm{GL}_{n^2}(\mathbf{C}(z))$ when A is regular singular at 0 and ∞, *i.e.*, we assume that there exist $A^{(0)}, A^{(\infty)} \in \mathrm{GL}_n(\mathbf{C})$, $F^{(0)}(z) \in \mathrm{GL}_n(\mathbf{C}(\{z\}))$ and $F^{(\infty)}(z) \in \mathrm{GL}_n(\mathbf{C}(\{z^{-1}\}))$ such that

$$F^{(0)}(qz)A^{(0)} = A(z)F^{(0)}(z) \text{ and } F^{(\infty)}(qz)A^{(\infty)} = A(z)F^{(\infty)}(z).$$

We can and will assume that both $A^{(0)}$ and $A^{(\infty)}$ are nonresonant, *i.e.* that, for any eigenvalue λ, μ of $A^{(0)}$ (respectively $A^{(\infty)}$), we have $\lambda/\mu \notin q^{\mathbf{Z}^*}$. Then, we have

$$\dim_{\mathbf{C}} \mathrm{Sol}(B, \mathbf{C}(\{z\})) = \dim_{\mathbf{C}} Z\left(A^{(0)}\right) \text{ and } \dim_{\mathbf{C}} \mathrm{Sol}\left(B, \mathbf{C}\left(\left\{z^{-1}\right\}\right)\right) = \dim_{\mathbf{C}} Z\left(A^{(\infty)}\right)$$

where $Z(\cdot)$ denotes the centralizer in $\mathrm{Mat}_n(\mathbf{C})$. So,

$$\chi\left(\widetilde{\mathcal{F}'_B}\right) = \sum_{a \in \mathrm{Sing}(A)} \sum e_i^2(a) + \dim_{\mathbf{C}} Z\left(A^{(0)}\right) + \dim_{\mathbf{C}} Z\left(A^{(\infty)}\right) - nN.$$

This is Sakai-Yamaguchi's index of rigidity [15, Section 3].

It would have been more natural to look for a connected topological space X (instead of the non connected $\widetilde{\mathbf{E}_q^{an}}$) and for a sheaf $\mathcal{F}$ on X such that $\chi(\mathcal{F})$ is the index of rigidity of A. Unfortunately, we were not able to find such a topological space. However, this led us to compute the Euler characteristics of natural "extensions" of $\mathcal{F}_A$ of independent interest; this is the content of the rest of the paper.

4. A natural extension of $\mathcal{F}'_A$ and its Euler characteristic

We consider the set

$$\overline{\mathbf{E}_q^{an}} = \{0\} \sqcup \mathbf{E}_q^{an} \sqcup \{\infty\}.$$

We endow this set with the following topology: the open sets of $\overline{\mathbf{E}_q^{an}}$ are the open subsets of $\mathbf{E}_q^{an}$, and the subsets $\{0\} \sqcup \mathbf{E}_q$, $\mathbf{E}_q \sqcup \{\infty\}$ and $\overline{\mathbf{E}_q^{an}}$.

We denote by $\overline{\varpi} : \mathbb{P}^1(\mathbf{C})^{an} \to \overline{\mathbf{E}_q^{an}}$ the natural continuous map.

Let V an open subset of $\overline{\mathbf{E}_q^{an}}$ and let $U := \overline{\varpi}^{-1}(V)$. We consider the subsheaf $\overline{\mathcal{A}'}$ of $\overline{\varpi}_* \mathcal{M}_{\mathbb{P}^1(\mathbf{C})^{an}}$ given by

$$\overline{\mathcal{A}'}(V) := \mathcal{A}'\Big(V \cap \mathbf{E}_q^{an}\Big) \cap \mathcal{M}_{\mathbb{P}^1(\mathbf{C})^{an}}(U). \tag{4.1}$$

Note that

$$\overline{\mathcal{A}'}_{|\mathbf{E}_q^{an}} = \mathcal{A}'.$$

As in Paragraph 3.1.2, we associate to the sheaf $\overline{\mathcal{A}'}$ the sheaf of solutions on $\overline{\mathbf{E}_q^{an}}$ denoted by $\overline{\mathcal{F}'_A}$. The sections of this sheaf on an open subset V of $\overline{\mathbf{E}_q^{an}}$ are given by

$$\overline{\mathcal{F}'_A}(V) = \Big\{F \in (\overline{\mathcal{A}'}(V))^n \mid \forall z \in \overline{\varpi}^{-1}(V), \sigma_q(F)(z) = A(z)F(z)\Big\}.$$

Note that

$$\left(\overline{\mathcal{F}'_A}\right)_{|\mathbf{E}_q^{an}} = \mathcal{F}'_A.$$

In order to compute the Euler characteristic of this sheaf, we will need the following lemmas.

Lemma 4.1. *Any sheaf of Abelian groups on the topological subspace* $\{0\} \sqcup \mathbf{E}_q^{an}$ *(respectively* $\mathbf{E}_q^{an} \sqcup \{\infty\}$*) of* $\overline{\mathbf{E}_q^{an}}$ *is acyclic.*

Proof. Let $\mathcal{F}$ be a sheaf of Abelian groups on $\{0\} \sqcup \mathbf{E}_q^{an}$. Let $0 \to \mathcal{F} \to \mathcal{I}^\cdot$ be an injective resolution of $\mathcal{F}$. Taking the stalks at 0, we get the exact sequence $0 \to \mathcal{F}_0 \to \mathcal{I}^\cdot_0$. Since $\{0\} \sqcup \mathbf{E}_q^{an}$ is the only open subset of $\{0\} \sqcup \mathbf{E}_q^{an}$ containing 0, we see that the stalks at 0 coincide with the global sections. So the sequence of global sections obtained from $0 \to \mathcal{F} \to \mathcal{I}^\cdot$ is exact, whence the result. □

Lemma 4.2. *Let* $\mathcal{F}$ *be a sheaf of* $\mathbf{C}$*-vector spaces on* $\overline{\mathbf{E}_q^{an}}$. *Assume that* $H^0(\{0\} \sqcup \mathbf{E}_q^{an}, \mathcal{F})$, $H^0(\mathbf{E}_q^{an} \sqcup \{\infty\}, \mathcal{F})$ *and any* $H^k(\mathbf{E}_q^{an}, \mathcal{F})$ *are finite dimensional. Then, the* $H^k(\overline{\mathbf{E}_q^{an}}, \mathcal{F})$ *are finite dimensional and we have*

$$\chi\big(\overline{\mathbf{E}_q^{an}}, \mathcal{F}\big) = -\chi\big(\mathbf{E}_q^{an}, \mathcal{F}_{|\mathbf{E}_q^{an}}\big) + h^0\big(\{0\} \sqcup \mathbf{E}_q^{an}, \mathcal{F}\big) + h^0\big(\mathbf{E}_q^{an} \sqcup \{\infty\}, \mathcal{F}\big).$$

Proof. The Mayer-Vietoris long exact sequence for $\mathcal{F}$ with respect the open covering $\{\{0\}\sqcup \mathbf{E}_q^{an}, \mathbf{E}_q^{an}\sqcup\{\infty\}\}$ of $\overline{\mathbf{E}_q^{an}}$ reads as follow:

$$\cdots \to H^{k-1}\big(\mathbf{E}_q^{an},\mathcal{F}\big)\to H^k\big(\overline{\mathbf{E}_q^{an}},\mathcal{F}\big)\to H^k\big(\{0\}\sqcup\mathbf{E}_q^{an},\mathcal{F}\big)\oplus H^k\big(\mathbf{E}_q^{an}\sqcup\{\infty\},\mathcal{F}\big)$$
$$\to H^k\big(\mathbf{E}_q^{an},\mathcal{F}\big)\to H^{k+1}\big(\overline{\mathbf{E}_q^{an}},\mathcal{F}\big)\to\cdots$$

Using Lemma 4.1, we see that, for all $k\geq 1$, $H^j(\{0\}\sqcup\mathbf{E}_q^{an},\mathcal{F})\oplus H^j(\mathbf{E}_q^{an}\sqcup\{\infty\},\mathcal{F})$ is trivial for $j\in\{k,k+1\}$ and, hence, that $H^k(\mathbf{E}_q^{an},\mathcal{F})$ and $H^{k+1}(\overline{\mathbf{E}_q^{an}},\mathcal{F})$ are isomorphic $\mathbf{C}$-vector spaces. Moreover, the first terms of the Mayer-Vietoris sequence above give the exact sequence:

$$0\to H^0\big(\overline{\mathbf{E}_q^{an}},\mathcal{F}\big)\to H^0\big(\{0\}\sqcup\mathbf{E}_q^{an},\mathcal{F}\big)\oplus H^0\big(\mathbf{E}_q^{an}\sqcup\{\infty\},\mathcal{F}\big)$$
$$\to H^0\big(\mathbf{E}_q^{an},\mathcal{F}\big)\to H^1\big(\overline{\mathbf{E}_q^{an}},\mathcal{F}\big)\to 0.$$

So that,

$$\begin{aligned}&h^1\big(\overline{\mathbf{E}_q^{an}},\mathcal{F}\big)\\ &=h^0\big(\mathbf{E}_q^{an},\mathcal{F}\big)-\big(h^0(\{0\}\sqcup\mathbf{E}_q,\mathcal{F}\big)+h^0\big(\mathbf{E}_q\sqcup\{\infty\},\mathcal{F}\big)\big)+h^0\big(\overline{\mathbf{E}_q^{an}},\mathcal{F}\big)\\ &=h^0\big(\mathbf{E}_q^{an},\mathcal{F}\big)-\big(h^0\big(\{0\}\sqcup\mathbf{E}_q^{an},\mathcal{F}\big)+h^0\big(\mathbf{E}_q^{an}\sqcup\{\infty\},\mathcal{F}\big)\big)+h^0\big(\overline{\mathbf{E}_q^{an}},\mathcal{F}\big).\qquad\square\end{aligned}$$

Applying this lemma to $\overline{\mathcal{F}'_A}$ and using Theorem 3.21, we get the following result:

Theorem 4.3. *For an arbitrary* $A\in \mathrm{GL}_n(\mathbf{C}(z))$*:*

$$\chi\big(\overline{\mathcal{F}'_A}\big)=-v_0(\det A)+\sum_{x\in\mathbf{E}_q^{an}}\ell_x^-(A)+\dim_{\mathbf{C}}\mathrm{Sol}(A,R_0)+\dim_{\mathbf{C}}\mathrm{Sol}(A,R_\infty),$$
$$\chi\big(\overline{\mathcal{F}'_A}\big)=-v_\infty(\det A)+\sum_{x\in\mathbf{E}_q^{an}}\ell_x^+(A)+\dim_{\mathbf{C}}\mathrm{Sol}(A,R_0)+\dim_{\mathbf{C}}\mathrm{Sol}(A,R_\infty),$$
$$\begin{aligned}2\chi\big(\overline{\mathcal{F}'_A}\big)=&-v_0(\det A)-v_\infty(\det A)+\sum_{x\in\mathbf{E}_q^{an}}\ell_x(A)\\ &+2\dim_{\mathbf{C}}\mathrm{Sol}(A,R_0)+2\dim_{\mathbf{C}}\mathrm{Sol}(A,R_\infty),\end{aligned}$$

where R_0 *(respectively* R_∞*) is the* $\mathbf{C}$*-vector space of meromorphic functions over* $\mathbf{C}$ *(respectively* $\mathbf{C}^\times\sqcup\{\infty\}$*) with at most finitely many poles on any* q*-spiral* $[a;q]\subset\mathbf{C}^\times$.

5. Another natural extension of $\mathcal{F}'_A$

Let $\mathbf{E}_q^{an'}$ be a copy of $\mathbf{E}_q^{an}$. We consider the set

$$X=\{0\}\sqcup\mathbf{E}_q^{an}\sqcup\{\infty\}\sqcup\mathbf{E}_q^{an'}.$$

We endow X with the following topology: the open subsets of X are of the form

- $U \sqcup U'$ where U (respectively U') is a subset of $\mathbf{E}_q^{an}$ (respectively $\mathbf{E}_q^{an'}$) such that $U' \subset U$;
- $\{0\} \sqcup \mathbf{E}_q^{an} \sqcup U'$ where U' is an open subset of $\mathbf{E}_q^{an'}$;
- $\mathbf{E}_q^{an} \sqcup \{\infty\} \sqcup U'$ where U' is an open subset of $\mathbf{E}_q^{an'}$;
- $\{0\} \sqcup \mathbf{E}_q^{an} \sqcup \{\infty\} \sqcup U'$ where U' is an open subset of $\mathbf{E}_q^{an'}$.

The reader will easily check that this defines a topology on X.

We let $\mathcal{B}$ be the sheaf on X whose sections on an open subset V of X are the meromorphic functions $f(z)$ on $\pi^{-1}(V \cap \mathbf{E}_q^{an})$ (where $\pi : \mathbf{C}^\times \to \mathbf{E}_q^{an}$ is the natural projection) such that

- $f(z)$ is meromorphic at 0 if $0 \in V$;
- $f(z)$ is meromorphic at ∞ if $\infty \in V$;
- $f(z)$ is meromorphic on $\pi'^{-1}(V \cap \mathbf{E}_q^{an'})$ with at most finitely many poles on any q-spiral $[a; q] \subset \pi'^{-1}(V \cap \mathbf{E}'_q)$ (where $\pi' : \mathbf{C}^\times \to \mathbf{E}_q^{an'}$ is the natural projection).

The restriction maps are the natural ones (restriction of functions).

The corresponding sheaf of solutions on X is given, for any open subset V of X, by

$$\mathcal{H}'_A(V) = \Big\{F \in (\mathcal{B}(V))^n \mid \forall z \in \pi^{-1}\big(V \cap \mathbf{E}_q^{an}\big), \sigma_q(F)(z) = A(z)F(z)\Big\}.$$

This section is devoted to the proof of the following result.

Theorem 5.1. *We have:*

- *For all* $k \geq 2$, $H^k(X, \mathcal{H}'_A) = 0$;
- $\dim_{\mathbf{C}} H^1(X, \mathcal{H}'_A) = \infty$;
- $H^0(X, \mathcal{H}'_A) = \mathrm{Sol}(A, \mathbf{C}(z))$.

In order to prove this result, we state and prove some lemmas.

5.1. Lemmas

We let Y be the topological subspace of X given by

$$Y = \mathbf{E}_q^{an} \sqcup \mathbf{E}_q^{an'}.$$

We denote by

$$i : \mathbf{E}_q^{an} \to Y \text{ and } j : \mathbf{E}_q^{an'} \to Y$$

the natural (continuous) inclusions.

Let $\mathcal{F}$ be a sheaf of Abelian groups on $\mathbf{E}_q^{an}$. We set

$$\widetilde{\mathcal{F}} = i_*\mathcal{F}.$$

Lemma 5.2. *Let $\mathcal{F}$ be a sheaf of Abelian groups on $\mathbf{E}_q^{an}$. We have, for all $k \geq 0$,*

$$H^k\big(Y, \widetilde{\mathcal{F}}\big) = H^k\big(\mathbf{E}_q^{an}, \mathcal{F}\big).$$

Proof. This follows from the facts that the direct image functor i_* is exact (be careful, since i is the inclusion of an open subset, the exactness of i_* is not a general fact but is true in our special case) and sends flasque sheaves on flasque sheaves (this is a general fact for direct images). □

Lemma 5.3. *Let $\mathcal{F}$ be a sheaf of Abelian groups on Y. Assume that $i^{-1}\mathcal{F}$ is acyclic. Then, for all $k \geq 0$, we have*

$$H^k(Y, \mathcal{F}) \cong H^k\big(\mathbf{E}_q^{an'}, j^{-1}\mathcal{F}\big).$$

Proof. We start with the exact sequence [6, II, Exercise 1.19]

$$0 \to i_! i^{-1}\widetilde{i^{-1}\mathcal{F}} = i_! i^{-1}\mathcal{F} \to \widetilde{i^{-1}\mathcal{F}} \to j_* j^{-1}\widetilde{i^{-1}\mathcal{F}} \to 0, \tag{5.1}$$

where $i_!$ is the extension by zero outside $\mathbf{E}_q^{an}$ functor. Note that

$$H^k\Big(Y, j_* j^{-1}\widetilde{i^{-1}\mathcal{F}}\Big) = H^k\Big(\mathbf{E}_q^{an'}, j^{-1}\widetilde{i^{-1}\mathcal{F}}\Big) = H^k\big(\mathbf{E}_q^{an}, i^{-1}\mathcal{F}\big).$$

(The first equality is general [6, III, Lemma 2.10] because $\mathbf{E}_q^{an'}$ is closed in Y; the second one follows from the fact that there is an obvious identification of the topological space $\mathbf{E}_q^{an'}$ with $\mathbf{E}_q^{an}$, and that $j^{-1}\widetilde{i^{-1}\mathcal{F}}$ corresponds to $i^{-1}\mathcal{F}$ under this identification.) Since $i^{-1}\mathcal{F}$ is acyclic, we get that $j_* j^{-1}\widetilde{i^{-1}\mathcal{F}}$ is acyclic. Considering the long exact sequence of cohomology group obtained from (5.1), we obtain that, for all $k \geq 2$,

$$H^k\big(Y, i_! i^{-1}\mathcal{F}\big) = H^k\Big(Y, \widetilde{i^{-1}\mathcal{F}}\Big) = 0$$

(the last equality follows from Lemma 5.2 and from the fact that $i^{-1}\mathcal{F}$ is acyclic by assumption) and we also obtain the exact sequence

$$\begin{aligned} 0 &\to H^0(Y, i_! i^{-1}\mathcal{F}) \to H^0\Big(Y, \widetilde{i^{-1}\mathcal{F}}\Big) \to H^0\Big(Y, j_* j^{-1}\widetilde{i^{-1}\mathcal{F}}\Big) \\ &\to H^1\big(Y, i_! i^{-1}\mathcal{F}\big) \to H^1\Big(Y, \widetilde{i^{-1}\mathcal{F}}\Big) \to \cdots \end{aligned}$$

But, we have

$$H^0\Big(Y, j_* j^{-1}\widetilde{i^{-1}\mathcal{F}}\Big) = H^0\Big(\mathbf{E}_q^{an'}, j^{-1}\widetilde{i^{-1}\mathcal{F}}\Big) = H^0\Big(Y, \widetilde{i^{-1}\mathcal{F}}\Big)$$

and the map $H^0(Y, \widetilde{i^{-1}\mathcal{F}}) \to H^0(Y, \widetilde{j_* j^{-1} i^{-1}\mathcal{F}})$ is actually the identity. Moreover, Lemma 5.2 ensures that $H^1(Y, \widetilde{i^{-1}\mathcal{F}}) = H^1(\mathbf{E}_q^{an}, i^{-1}\mathcal{F}) = 0$. It follows that

$$H^1\big(Y, i_! i^{-1}\mathcal{F}\big) = 0.$$

Using the long exact sequence of cohomology groups obtained from the exact sequence

$$0 \to i_! i^{-1}\mathcal{F} \to \mathcal{F} \to j_* j^{-1}\mathcal{F} \to 0,$$

we get, for all $k \geq 1$,

$$H^k(Y, \mathcal{F}) = H^k\big(Y, j_* j^{-1}\mathcal{F}\big) = H^k\big(\mathbf{E}_q^{an'}, j^{-1}\mathcal{F}\big).$$

This equality is obviously true for $k = 0$. □

5.2. Proof of Theorem 5.1

In order to compute the cohomology of $\mathcal{H}'_{\mathcal{A}}$ on X, we first use the Mayer-Vietoris long exact sequence for the open covering $\{\{0\} \sqcup \mathbf{E}_q^{an}, \mathbf{E}_q^{an} \sqcup \{\infty\} \sqcup \mathbf{E}_q^{an'}\}$ of X:

$$\begin{aligned} \cdots \to H^{k-1}\big(\mathbf{E}_q^{an}, \mathcal{H}'_A\big) &\to H^k\big(X, \mathcal{H}'_A\big) \\ &\to H^k\big(\{0\} \sqcup \mathbf{E}_q^{an}, \mathcal{H}'_A\big) \oplus H^k\big(\mathbf{E}_q^{an} \sqcup \{\infty\} \sqcup \mathbf{E}_q^{an'}, \mathcal{H}'_A\big) \\ &\to H^k\big(\mathbf{E}_q^{an}, \mathcal{H}'_A\big) \to H^{k+1}\big(X, \mathcal{H}'_A\big) \to \cdots \end{aligned}$$

But, for $k \geq 2$, we have $H^{k-1}(\mathbf{E}_q^{an}, \mathcal{H}'_A) = H^k(\mathbf{E}_q^{an}, \mathcal{H}'_A) = 0$ (because the restriction of $\mathcal{H}'_A$ to $\mathbf{E}_q^{an}$ is a meromorphic fiber bundle) and $H^k(\{0\} \sqcup \mathbf{E}_q^{an}, \mathcal{H}'_A) = 0$ (follows from Lemma 4.1), so

$$H^k\big(X, \mathcal{H}'_A\big) = H^k\big(\mathbf{E}_q^{an} \sqcup \{\infty\} \sqcup \mathbf{E}_q^{an'}, \mathcal{H}'_A\big).$$

Now, we use the Mayer-Vietoris long exact sequence for the open covering $\{\mathbf{E}_q^{an} \sqcup \{\infty\}, Y\}$ of $\mathbf{E}_q^{an} \sqcup \{\infty\} \sqcup \mathbf{E}_q^{an'}$:

$$\begin{aligned} \cdots \to H^{k-1}\big(\mathbf{E}_q^{an}, \mathcal{H}'_A\big) &\to H^k\big(\mathbf{E}_q^{an} \sqcup \{\infty\} \sqcup \mathbf{E}_q^{an'}, \mathcal{H}'_A\big) \\ &\to H^k\big(\mathbf{E}_q^{an} \sqcup \{\infty\}, \mathcal{H}'_A\big) \oplus H^k\big(Y, \mathcal{H}'_A\big) \\ &\to H^k\big(\mathbf{E}_q^{an}, \mathcal{H}'_A\big) \to H^{k+1}\big(\mathbf{E}_q^{an} \sqcup \{\infty\} \sqcup \mathbf{E}_q^{an'}, \mathcal{H}'_A\big) \to \cdots \end{aligned}$$

Arguing as above, we get that, for $k \geq 2$,

$$H^k\big(\mathbf{E}_q^{an} \sqcup \{\infty\} \sqcup \mathbf{E}_q^{an'}, \mathcal{H}'_A\big) = H^k\big(Y, \mathcal{H}'_A\big).$$

But, $i^{-1}\mathcal{H}'_A$ is acyclic (it is a meromorphic vector bundle on $\mathbf{E}_q^{an}$), so Lemma 5.3 ensures that, for $i \geq 0$,

$$H^k\big(Y, \mathcal{H}'_A\big) = H^k\big(\mathbf{E}_q^{an'}, j^{-1}\mathcal{H}'_{\mathcal{A}}\big).$$

Therefore, we have proved that, for $k \geq 2$,

$$H^k\big(X, \mathcal{H}'_A\big) = H^k\big(Y, \mathcal{H}'_A\big) = H^k\big(\mathbf{E}_q^{an'}, j^{-1}\mathcal{H}'_A\big) = 0. \tag{5.2}$$

Moreover, the first terms of the first Mayer-Vietoris sequence above gives:

$$\begin{aligned} 0 &\to H^0\big(X, \mathcal{H}'_A\big) \to H^0\big(\{0\} \sqcup \mathbf{E}_q^{an}, \mathcal{H}'_A\big) \oplus H^0\big(\mathbf{E}_q^{an} \sqcup \{\infty\} \sqcup \mathbf{E}_q^{an'}, \mathcal{H}'_A\big) \\ &\to H^0\big(\mathbf{E}_q^{an}, \mathcal{H}'_A\big) \to H^1\big(X, \mathcal{H}'_A\big) \to \cdots \end{aligned}$$

But $H^0(\{0\} \sqcup \mathbf{E}_q^{an}, \mathcal{H}'_A)$ and $H^0(\mathbf{E}_q^{an} \sqcup \{\infty\} \sqcup \mathbf{E}_q^{an'}, \mathcal{H}'_A)$ are finite dimensional $\mathbf{C}$-vector spaces, whereas $H^0(\mathbf{E}_q^{an}, \mathcal{H}'_A)$ is infinite dimensional. Therefore, $H^1(X, \mathcal{H}'_A)$ is infinite dimensional.

The last assertion of the theorem is obvious.

References

[1] D. BERTRAND, *Extensions de $\mathcal{D}$-modules et groupes de Galois differentiels*, (Extensions of *$\mathcal{D}$-modules and differential Galois groups)*, In: "p-adic analysis" (Trento, 1989), Lecture Notes in Math., Vol. 1454, 1990, 125–141.

[2] D. BERTRAND, *Un analogue différentiel de la théorie de Kummer*, In: "Approximations Diophantiennes et Nombres Transcendants", Comptes-rendus du colloque tenu au C.I.R.M. de Luminy, France, 18-22 Juin, 1990, Berlin, de Gruyter, 1992, 39–49.

[3] P. DELIGNE, "Equations Différentielles à Points Singuliers Réguliers", Lecture Notes in Math., Vol. 163, Berlin-Heidelberg-New York, Springer-Verlag, 1970, 133.

[4] G. FISCHER, "Complex Analytic Geometry", Lecture Notes in Math., Vol. 538, Springer-Verlag, Berlin-New York, 1976.

[5] R. C. GUNNING, "Lectures on Vector Bundles over Riemann Surfaces. Preliminary Informal Notes of University Courses and Seminars in Mathematics", Mathematical Notes. Princeton: Princeton University Press and the University of Tokyo Press, 1967.

[6] R. HARTSHORNE "Algebraic Geometry", Graduate Texts in Mathematics, Vol. 52, Springer-Verlag, New York-Heidelberg, 1977.

[7] N. M. KATZ, "Rigid Local Systems", Annals of Mathematics Studies, Vol. 139, Princeton University Press, Princeton, NJ, 1996.

[8] M. KONTSEVICH and Y. SOIBELMAN, https://www.youtube.com/watch?v=4j_kltlpwqk.

[9] M. KONTSEVICH and Y. SOIBELMAN, https://www.youtube.com/watch?v=fewubifuibk.

[10] Q. LIU, "Algebraic Geometry and Arithmetic Curves", translated from the French by Reine Erné, Oxford Graduate Texts in Mathematics, 6, Oxford Science Publications, Oxford University Press, Oxford, 2002.

[11] R. MIRANDA, "Algebraic Curves and Riemann Surfaces", Graduate Studies Mathematics, Vol. 5, American Mathematical Society, Providence, RI, 1995.

[12] C. PRAAGMAN, *Fundamental solutions for meromorphic linear difference equations in the complex plane, and related problems*, J. Reine Angew. Math. **369** (1985), 101–109.

[13] J.-P. RAMIS, J. SAULOY and C. ZHANG, "Local Analytic Classification of q-Difference Equations", Astérisque, Vol. 355, 2013.

[14] M. RAYNAUD, *Caractéristique d'Euler-Poincaré d'un faisceau et cohomologie des variétés abéliennes*, In: "Dix Exposés sur la Cohomologie des Schémas", Adv. Stud. Pure Math., Vol. 3, North-Holland, Amsterdam, 1968, 12–30.

[15] H. SAKAI and M. YAMAGUCHI, *Spectral types of linear q-difference equations and q-analog of middle convolution*, Int. Math. Res. Not. IMRN **2017** (2017), 1975–2013.

[16] J.-P. SERRE, *Faisceaux algébriques cohérents*, Ann. of Math. (2) **61** (1955), 197–278.
[17] J.-P. SERRE, *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier (Grenoble) **6** (1956), 1–42.
[18] M. VAN DER PUT and M. REVERSAT, *Galois theory of q-difference equations*, Ann. Fac. Sci. Toulouse Math. (6) **16** (2007), 665–718.

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