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A transcendental approach to injectivity theorem for log canonical pairs

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Abstract. We study a transcendental approach to the cohomology groups of adjoint bundles of log canonical pairs, aiming to establish an analytic theory for log canonical singularities. As a result, in the case of purely log terminal pairs, we give an analytic proof for the injectivity theorem originally proved by Hodge theory. Our method is based on the theory of harmonic integrals and the L^2-method for the $\overline{\partial}$-equation, and it enables us to generalize the injectivity theorem to the complex analytic setting.

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1. Introduction

The following result is a generalization of Kollár's injectivity theorem [21] to log canonical pairs (*lc*, for short), whose proof heavily depends on Hodge theory (see [1, 2, 8, 11 Section 6, 13, 15]). In [7], Enoki gave an analytic proof for Kollár's injectivity theorem (the special case of $D = 0$ in Theorem 1.1) under the weaker assumption that F is a semi-positive line bundle on a compact Kähler manifold X. Therefore, in the same direction as Enoki, it is natural to ask whether we can give an analytic proof for Theorem 1.1 in the complex analytic setting. This question (Conjecture 1.2) was posed in [17]. In this paper, we affirmatively solve Conjecture 1.2 in the case of purely log terminal pairs (*plt*, for short), by developing an analytic approach to lc singularities instead of Hodge theory, which depends on the theory of harmonic integrals and the L^2-method for the $\overline{\partial}$-equation.

Theorem 1.1. *Let D be a simple normal crossing divisor on a smooth projective variety X and F be a semi-ample line bundle on X. Let s be a* (*holomorphic*)

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section of a positive multiple F^m such that the zero locus $s^{-1}(0)$ contains no lc centers of the lc pair (X, D). Then, the multiplication map induced by the tensor product with s,

$$H^q(X, K_X \otimes D \otimes F) \xrightarrow{\otimes s} H^q(X, K_X \otimes D \otimes F^{m+1})$$

is injective for every q. Here K_X denotes the canonical bundle of X.

Conjecture 1.2 ([17, Conjecture 2.21], *cf.* [14, Problem 1.8]). Let D be a simple normal crossing divisor on a compact Kähler manifold X and F be a semi-positive line bundle on X (that is, it admits a smooth Hermitian metric with semi-positive curvature). Let s be a section of a positive multiple F^m such that the zero locus $s^{-1}(0)$ contains no lc centers of the lc pair (X, D). Then we obtain the same conclusion as in Theorem 1.1.

For a simple normal crossing divisor D on a complex manifold X with the irreducible decomposition $D = \sum_{i \in I} D_i$, an irreducible component of $D_{i_1} \cap \cdots \cap D_{i_k} (\neq \phi)$ is called an lc center of the pair (X, D). Note that we interchangeably use the words "(Cartier) divisors", "(holomorphic) line bundles", and "invertible sheaves" throughout this paper.

The following theorem, which is one of the main results of this paper, gives an affirmative answer for Conjecture 1.2 in the case of *plt* pairs.

Theorem 1.3 (Main theorem). *Let D be a simple normal crossing divisor on a compact Kähler manifold X. Let F (respectively M) be a (holomorphic) line bundle on X with a smooth Hermitian metric h_F (respectively h_M) such that*

$$\sqrt{-1}\Theta_{h_M}(M) \geq 0 \text{ and } \sqrt{-1}\big(\Theta_{h_F}(F) - t\Theta_{h_M}(M)\big) \geq 0 \text{ for some } t > 0.$$

We assume that the pair (X, D) is a plt *pair. Let s be a section of M such that the zero locus $s^{-1}(0)$ contains no lc centers of the lc pair (X, D). Then, the multiplication map induced by the tensor product with s,*

$$H^q(X, K_X \otimes D \otimes F) \xrightarrow{\otimes s} H^q(X, K_X \otimes D \otimes F \otimes M)$$

is injective for every q.

Corollary 1.4. *Under the same situation as in Conjecture* 1.2*, we assume that the pair (X, D) is a* plt *pair. Then, the same conclusion as in Conjecture* 1.2 *holds. In particular, Conjecture* 1.2 *is affirmatively solved for a* plt *pair (X, D).*

Theorem 1.3 can be reduced to the following theorem. In this reduction step, we use the assumption that (X, D) is a *plt* pair. However, we emphasize that we do not need this assumption in Theorem 1.5. The proof of Theorem 1.5 provides an analytic method to study lc singularities. (See [3,20] for another approach.)

Theorem 1.5 (Key result). *Let D be a simple normal crossing divisor on a compact Kähler manifold X. Let F (respectively M) be a (holomorphic) line bundle on X with a smooth Hermitian metric h_F (respectively h_M) satisfying the same assumptions as in Theorem* 1.3. *We consider the map*

$$\Phi_D : H^q(X, K_X \otimes F) \longrightarrow H^q(X, K_X \otimes D \otimes F)$$

induced by the natural inclusion $\mathcal{O}_X \hookrightarrow \mathcal{O}_X(D)$. Then, the multiplication map on the image $\operatorname{Im} \Phi_D$ *induced by the tensor product with s*

$$\operatorname{Im} \Phi_D \xrightarrow{\otimes s} H^q(X, K_X \otimes D \otimes F \otimes M)$$

is injective for every q.

The main idea of the proof of Theorem 1.5 is as follows: when we study kawamata log terminal singularities (*klt*, for short), multiplier ideal sheaves (which can be seen as a "non-klt" ideal) play an important role. Since multiplier ideals can be analytically described by the L^2-integrability of holomorphic functions, we can obtain various injectivity theorems for klt singularities by using the L^2-method (for example see [12, 14, 18, 19, 25–30]). However we can not (at least directly) apply the L^2-method for lc pairs since lc singularities are worse than klt singularities. This is one of the difficulties of Conjecture 1.2. To overcome this difficulty, in the proof of Theorem 1.3, we approximate lc singularities with klt singularities and we carefully estimate the order of divergence of suitable L^2-norms (that is, how far from klt singularities). In this step, we need a refinement (Theorem 1.6) of the hard Lefschetz theorem with multiplier ideals proved in [6], which is independently of interest.

Theorem 1.6. *Let ω be a Kähler form on a compact Kähler manifold X and (G, h) be a singular Hermitian line bundle with semi-positive curvature. Assume that the singular Hermitian metric h is smooth on a non-empty Zariski open set in X. Then, for a harmonic G-valued (n, q)-form $u \in \mathcal{H}^{n,q}_{h,\omega}(G)$ with respect to h and ω, we have*

$$*u \in H^0\big(X, \Omega_X^{n-q} \otimes G \otimes \mathcal{I}(h)\big),$$

where $*$ is the Hodge star operator with respect to ω.

This paper is organized as follows: in Section 2, we summarize the fundamental results needed later. We give a proof of Theorem 1.6 (respectively Theorem 1.5, Theorem 1.3) in Subsection 3.1 (respectively Subsection 3.2, Subsection 3.3). In Subsection 3.4, we discuss open problems related to the contents of this paper.

Conjecture 1.2 was recently proved in the two-dimensional case using Theorem 1.3 after this paper had been accepted (see [24]).

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2. Preliminaries

2.1. Singularities of pairs and multiplier ideal sheaves

We treat only log smooth pairs in this paper, and thus we shortly recall the notion of singularities for log smooth pairs (see [22] for more details).

Definition 2.1. Let (X, D) be a log smooth pair (that is, a pair of a smooth variety X and an effective $\mathbb{Q}$-divisor D on X with simple normal crossing support). Let $D = \sum b_i D_i$ be the irreducible decomposition.

- The pair (X, D) is *plt* if and only if $\lfloor D \rfloor$ is a sum of disjoint prime divisors;
- The pair (X, D) is lc if and only if $b_i \leq 1$ for every i.

Here $\lfloor D \rfloor$ denotes the divisor defined by the round-downs of the coefficients of D.

We give the definition and an example of multiplier ideal sheaves (see [5] for singular Hermitian metrics and curvatures).

Definition 2.2 (Multiplier ideal sheaves). Let G be a (holomorphic) line bundle on a complex manifold X and h be a singular Hermitian metric on G such that $\sqrt{-1}\Theta_h(G) \geq \gamma$ for some smooth $(1, 1)$-form γ on X. Then the *multiplier ideal sheaf* $\mathcal{I}(h)$ of h is defined to be

$$\mathcal{I}(h)(B) := \left\{ f \in \mathcal{O}_X(B) \;\middle|\; |f|e^{-\varphi} \in L^2_{\mathrm{loc}}(B) \right\}$$

for every open set $B \subset X$, where φ is a local weight of h.

Example 2.3. For an effective divisor D on a complex manifold X, let g be a smooth Hermitian metric on the line bundle D and t be the natural section of the effective divisor D. Then the singular Hermitian metric h_D on the line bundle D can be defined by

$$\varphi := \frac{1}{2}\log\left(|t|_g^2\right) \quad \text{and} \quad h_D := g e^{-2\varphi} = \frac{1}{|t|^2},$$

where $|t|_g$ is the point-wise norm of t with respect to g (see Subsection 2.2). Note that the singular Hermitian metric h_D does not depend on the choice of g. Then it is easy to see that $\mathcal{I}(h_D) = \mathcal{O}_X(-\lfloor D \rfloor)$ when the support of D is normal crossing.

2.2. L^2-spaces and differential operators

From now on, throughout Section 2, let X be a (not necessarily compact) complex manifold of dimension n and G be a (holomorphic) line bundle on X. Further let ω be a positive $(1, 1)$-form on X and h be a singular Hermitian metric on G. We always assume that the curvature $\sqrt{-1}\Theta_h(G)$ of h satisfies $\sqrt{-1}\Theta_h(G) \geq \gamma$ for some smooth $(1, 1)$-form γ.

For G-valued (p, q)-forms u and v, the notation $\langle u, v\rangle_{h,\omega}$ denotes the pointwise inner product with respect to h and ω, and $\langle\!\langle u, v\rangle\!\rangle_{h,\omega}$ denotes the inner product defined by

$$\langle\!\langle u, v\rangle\!\rangle_{h,\omega} := \int_X \langle u, v\rangle_{h,\omega}\, dV_\omega,$$

where dV_ω is the volume form defined by $dV_\omega := \omega^n/n!$. The L^2-space of G-valued (p, q)-forms with respect to h and ω is defined by

$$\begin{aligned} L^{p,q}_{(2)}(G)_{h,\omega} &:= L^{p,q}_{(2)}(X, G)_{h,\omega} \\ &:= \{u \mid u \text{ is a } G\text{-valued } (p, q)\text{-form with } \|u\|_{h,\omega} < \infty\}. \end{aligned}$$

Then the maximal closed extension of the $\overline{\partial}$-operator determines a densely defined closed operator $\overline{\partial} : L^{p,q}_{(2)}(G)_{h,\omega} \to L^{p,q+1}_{(2)}(G)_{h,\omega}$ with the domain

$$\operatorname{Dom} \overline{\partial} := \bigl\{u \in L^{p,q}_{(2)}(G)_{h,\omega} \mid \overline{\partial}u \in L^{p,q+1}_{(2)}(G)_{h,\omega}\bigr\}.$$

Strictly speaking, the closed operator $\overline{\partial}$ depends on h and ω since the domain and the range depend on them, but we often omit the subscript (for example, we simply write $\overline{\partial}_{h,\omega}$ as $\overline{\partial}$). In general, we have the orthogonal decomposition

$$L^{n,q}_{(2)}(G)_{h,\omega} = \overline{\operatorname{Im} \overline{\partial}} \oplus \mathcal{H}^{n,q}_{h,\omega}(G) \oplus \overline{\operatorname{Im} \overline{\partial}^*_{h,\omega}},$$

where $\overline{\partial}^*_{h,\omega}$ is the Hilbert space adjoint of $\overline{\partial}$, the subspace $\operatorname{Im} \overline{\partial}$ (respectively $\operatorname{Im} \overline{\partial}^*_{h,\omega}$) is the range of $\overline{\partial}$ (respectively $\overline{\partial}^*_{h,\omega}$), and the subspace $\mathcal{H}^{n,q}_{h,\omega}(G)$ is the set of harmonic forms with respect to h and ω, that is,

$$\mathcal{H}^{n,q}_{h,\omega}(G) := \bigl\{u \in L^{n,q}_{(2)}(G)_{h,\omega} \mid \overline{\partial}u = 0 \text{ and } \overline{\partial}^*_{h,\omega}u = 0\bigr\}.$$

For example, see [5, (1.2) Theorem] for the above orthogonal decomposition.

When h is smooth on X, the Chern connection $D = D_{(G,h)}$ can be determined by the holomorphic structure of G and the smooth Hermitian metric h, which can be written as $D = D'_h + \overline{\partial}$ with the $(1, 0)$-connection D'_h and the $\overline{\partial}$-operator. The maximal closed extension of the $(1, 0)$-connection D'_h is also a densely defined closed operator $D'_h : L^{p,q}_{(2)}(G)_{h,\omega} \to L^{p+1,q}_{(2)}(G)_{h,\omega}$, whose domain is

$$\operatorname{Dom} D'_h := \bigl\{u \in L^{p,q}_{(2)}(G)_{h,\omega} \mid D'_h u \in L^{p+1,q}_{(2)}(G)_{h,\omega}\bigr\}.$$

We consider the Hodge star operator $*$ with respect to ω

$$* = *_\omega : C^{p,q}_\infty(G) \to C^{n-q,n-p}_\infty(G),$$

where $C^{p,q}_\infty(G)$ is the set of smooth G-valued (p, q)-forms on X. By the definition, we have $\langle u, v\rangle_{h,\omega}\, dV_\omega = u \wedge H\overline{*v}$ and $**u = (-1)^{\deg u}u$, where H is a local function representing h. In this paper, the notations $D'^*_{h,\omega}$ and $\overline{\partial}^*_{h,\omega}$ denote the Hilbert space adjoint of D'_h and $\overline{\partial}$. If ω is complete, the Hilbert space adjoint coincides with the maximal closed extension of the formal adjoint (for example, see [4, (8.2) Lemma]). In particular, when ω is complete, we have

$$D'^{*}_{h,\omega} = - * \overline{\partial} * \quad \text{and} \quad \overline{\partial}^{*}_{h,\omega} = - * D'_{h,\omega} * .$$

The following proposition is obtained from the Bochner-Kodaira-Nakano identity and the density lemma (for example see [6] and [4, (1.2) Theorem]).

Proposition 2.4. *Under the same situation as in the first part of Subsection* 2.2*, we assume that ω is a complete Kähler form and h is smooth on X. Then we have the following identity:*

$$\left[\overline{\partial}, \overline{\partial}^{*}_{h,\omega}\right] = \left[D'_h, D'^{*}_{h,\omega}\right] + \left[\sqrt{-1}\Theta_h(G), \Lambda_\omega\right],$$

where Λ_ω is the adjoint operator of the wedge product $\omega \wedge \bullet$, and $[\bullet, \bullet]$ is the graded bracket defined by $[A, B] = A - (-1)^{\deg A \deg B} B$.

Moreover, for every $u \in \mathrm{Dom}\, \overline{\partial} \cap \mathrm{Dom}\, \overline{\partial}^{*}_{h,\omega} \subset L^{p,q}_{(2)}(G)_{h,\omega}$, we have

$$\|\overline{\partial} u\|^2_{h,\omega} + \|\overline{\partial}^{*}_{h,\omega} u\|^2_{h,\omega} = \|D'_h u\|^2_{h,\omega} + \|D'^{*}_{h,\omega} u\|^2_{h,\omega} + \left\langle\!\left\langle \sqrt{-1}\Theta_h(G)\Lambda_\omega u, u \right\rangle\!\right\rangle_{h,\omega}.$$

For the proof of our results, it is important to use special characteristics of canonical bundles (differential (n, q)-forms). By the following lemma, we can compare the norms of (n, q)-forms and $(p, 0)$-forms with respect to different positive $(1, 1)$-forms. Lemma 2.5 is obtained from straightforward computations, and thus we omit the proof.

Lemma 2.5. *Let ω and $\widetilde{\omega}$ be positive $(1, 1)$-forms such that $\omega \le \widetilde{\omega}$. Then we have the following:*

- *There exists $C > 0$ such that $|a \wedge b|_\omega \le C|a|_\omega |b|_\omega$ for differential forms a, b;*
- *The inequality $|a|^2_{\widetilde{\omega}} \le |a|^2_\omega$ holds for a differential form a;*
- *The inequality $|a|^2_{\widetilde{\omega}}\, dV_{\widetilde{\omega}} \le |a|^2_\omega\, dV_\omega$ holds for an (n, q)-form a;*
- *The inequality $|a|^2_{\widetilde{\omega}}\, dV_{\widetilde{\omega}} \ge |a|^2_\omega\, dV_\omega$ holds for a $(p, 0)$-form a;*
- *The equality $|a|^2_{\widetilde{\omega}}\, dV_{\widetilde{\omega}} = |a|^2_\omega\, dV_\omega$ holds for an $(n, 0)$-form a.*

2.3. De Rham-Weil isomorphisms

In this subsection, we explain facts on the De Rham-Weil isomorphism from the $\overline{\partial}$-cohomology to the Čech cohomology. The contents in this subsection may be known for specialists, but we will summarize them for the reader's convenience.

Let ω be a Kähler form on a compact Kähler manifold X and h be a singular Hermitian metric on a (holomorphic) line bundle G such that $\sqrt{-1}\Theta_h(G) \geq -\omega$. Further let Z be a proper subvariety on X and let $\widetilde{\omega}$ be a Kähler form on the Zariski open set $Y := X \setminus Z$ with the following properties:

(B) $\widetilde{\omega} \geq \omega$ on $Y = X \setminus Z$;
(C) For every point p in X, there exists a "bounded" function Φ on an open neighborhood of p in X such that $\widetilde{\omega} = \sqrt{-1}\partial\overline{\partial}\Phi$.

As explained in Subsection 2.2, for the L^2-space of G-valued (n, q)-forms on Y with respect to h and $\widetilde{\omega}$

$$L^{n,q}_{(2)}(G)_{h,\widetilde{\omega}} := L^{n,q}_{(2)}(Y, G)_{h,\widetilde{\omega}} := \big\{u \,|\, u \text{ is a } G\text{-valued } (n, q)\text{-form with } \|u\|_{h,\widetilde{\omega}} < \infty\big\},$$

we have the orthogonal decomposition

$$L^{n,q}_{(2)}(G)_{h,\widetilde{\omega}} = \overline{\operatorname{Im}\overline{\partial}} \oplus \mathcal{H}^{n,q}_{h,\widetilde{\omega}}(G) \oplus \overline{\operatorname{Im}\overline{\partial}^{*}_{h,\widetilde{\omega}}}.$$

The following proposition is proved by the observation on the De Rham-Weil isomorphism (for example see [25, Proposition 5.8] for the precise proof.)

Proposition 2.6 (*cf.* [25, Proposition 5.8]). *Consider the same situation as above. That is, we consider a Kähler form ω on a compact Kähler manifold X, a singular Hermitian metric h on a (holomorphic) line bundle G such that $\sqrt{-1}\Theta_h(G) \geq -\omega$, and a Kähler form $\widetilde{\omega}$ on a Zariski open set Y with properties (B), (C). Then the ranges* $\operatorname{Im}\overline{\partial}$ *and* $\operatorname{Im}\overline{\partial}^{*}_{h,\widetilde{\omega}}$ *are closed subspaces in $L^{n,q}_{(2)}(G)_{h,\widetilde{\omega}}$. In particular, we have the orthogonal decomposition*

$$L^{n,q}_{(2)}(G)_{h,\widetilde{\omega}} = \operatorname{Im}\overline{\partial} \oplus \mathcal{H}^{n,q}_{h,\widetilde{\omega}}(G) \oplus \operatorname{Im}\overline{\partial}^{*}_{h,\widetilde{\omega}}.$$

We fix a finite open cover $\mathcal{U} := \{B_i\}_{i\in I}$ of X by sufficiently small Stein open sets B_i. We consider the set of q-cochains $C^q(\mathcal{U}, K_X \otimes G \otimes \mathcal{I}(h))$ with coefficients in $K_X \otimes G \otimes \mathcal{I}(h)$ calculated by $\mathcal{U}$ and the coboundary operator

$$\delta : C^q(\mathcal{U}, K_X \otimes G \otimes \mathcal{I}(h)) \to C^{q+1}(\mathcal{U}, K_X \otimes G \otimes \mathcal{I}(h)).$$

Then we have the isomorphism

$$\frac{\operatorname{Ker}\delta}{\operatorname{Im}\delta} \text{ of } C^q(\mathcal{U}, K_X \otimes G \otimes \mathcal{I}(h)) \cong \check{H}^q(X, K_X \otimes G \otimes \mathcal{I}(h)),$$

since the open cover $\mathcal{U}$ is a Stein cover. By using suitable local solutions of the $\overline{\partial}$-equation, we can construct the De Rham-Weil isomorphism

$$\overline{f_{h,\widetilde{\omega}}} : \frac{\operatorname{Ker}\overline{\partial}}{\operatorname{Im}\overline{\partial}} \text{ of } L^{n,q}_{(2)}(G)_{h,\widetilde{\omega}} \xrightarrow{\cong} \frac{\operatorname{Ker}\delta}{\operatorname{Im}\delta} \text{ of } C^q(\mathcal{U}, K_X \otimes G \otimes \mathcal{I}(h)).$$

Then, by the construction of $\overline{f_{h,\widetilde{\omega}}}$ (for example see [25, Proposition 5.5]), we can easily check the following proposition:

Proposition 2.7. *Consider the same situation as in Proposition* 2.6.

(1) *The following diagram is commutative:*

$$\begin{array}{ccc} \check{H}^q(X, K_X \otimes G \otimes \mathcal{I}(h)) & = & \check{H}^q(X, K_X \otimes G \otimes \mathcal{I}(h)) \\ \cong \Big\uparrow \overline{f_{h,\omega}} & & \cong \Big\uparrow \overline{f_{h,\widetilde{\omega}}} \\ \dfrac{\operatorname{Ker}\overline{\partial}}{\operatorname{Im}\overline{\partial}} \text{ of } L^{n,q}_{(2)}(G)_{h,\omega} & \xrightarrow{\ j_1\ } & \dfrac{\operatorname{Ker}\overline{\partial}}{\operatorname{Im}\overline{\partial}} \text{ of } L^{n,q}_{(2)}(G)_{h,\widetilde{\omega}}, \end{array}$$

where j_1 is the map induced by the natural map $L^{n,q}_{(2)}(G)_{h,\omega} \to L^{n,q}_{(2)}(G)_{h,\widetilde{\omega}}$;

(2) *Let h' be a singular Hermitian metric on G such that $\sqrt{-1}\Theta_{h'}(G) \geq -\omega$ and $h' \geq h$. Then the following diagram is commutative:*

$$\begin{array}{ccc} \check{H}^q(X, K_X \otimes G \otimes \mathcal{I}(h')) & \xrightarrow{\ j\ } & \check{H}^q(X, K_X \otimes G \otimes \mathcal{I}(h)) \\ \cong \Big\uparrow \overline{f_{h',\omega}} & & \cong \Big\uparrow \overline{f_{h,\widetilde{\omega}}} \\ \dfrac{\operatorname{Ker}\overline{\partial}}{\operatorname{Im}\overline{\partial}} \text{ of } L^{n,q}_{(2)}(G)_{h',\widetilde{\omega}} & \xrightarrow{\ j_2\ } & \dfrac{\operatorname{Ker}\overline{\partial}}{\operatorname{Im}\overline{\partial}} \text{ of } L^{n,q}_{(2)}(G)_{h,\widetilde{\omega}}, \end{array}$$

where j_2 is the map induced by the natural map $L^{n,q}_{(2)}(G)_{h',\widetilde{\omega}} \to L^{n,q}_{(2)}(G)_{h,\widetilde{\omega}}$ and j is the map induced by $\mathcal{I}(h') \hookrightarrow \mathcal{I}(h)$.

Remark 2.8. By property (B) and the third claim of Lemma 2.5, we have $\|u\|_{h,\widetilde{\omega}} \leq \|u\|_{h,\omega}$ for an arbitrary G-valued (n, q)-form u. Therefore the natural map j_1 is well-defined. In the same way, we can easily check that j_2 is well-defined from $\|u\|_{h,\widetilde{\omega}} \leq \|u\|_{h',\widetilde{\omega}}$.

2.4. Weak convergence in Hilbert spaces

In this subsection, we summarize Lemma 2.9 and Lemma 2.10. See [18, Section 2] for the proof.

Lemma 2.9. *Let L be a closed subspace in a Hilbert space $\mathcal{H}$. Then L is closed with respect to the weak topology of $\mathcal{H}$, that is, if a sequence $\{w_k\}_{k=1}^{\infty}$ in L weakly converges to w, then the weak limit w belongs to L.*

Lemma 2.10. *Let $\varphi : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded operator* (*continuous linear map*) *between Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. If $\{w_k\}_{k=1}^{\infty}$ weakly converges to w in $\mathcal{H}_1$, then $\{\varphi(w_k)\}_{k=1}^{\infty}$ weakly converges to $\varphi(w)$ in $\mathcal{H}_2$.*

3. Proof of the main results

3.1. Proof of Theorem 1.6

In this subsection, we prove Theorem 1.6. To this end, we first show the following proposition.

Proposition 3.1. *Let ω be a Kähler form on a compact Kähler manifold X and (G, h) be a singular Hermitian line bundle with semi-positive curvature. Let $\widetilde{\omega}$ be a Kähler form on a non-empty Zariski open set Y with the following properties:*

(B) *$\widetilde{\omega} \geq \omega$ on Y;*

(C) *For every point $p \in X$, there exists a bounded function Φ on an open neighborhood of p in X such that $\widetilde{\omega} = \sqrt{-1}\partial\overline{\partial}\Phi$.*

Then, we have $\langle\!\langle u, w \rangle\!\rangle_{h,\omega} = 0$ for any $u \in \mathcal{H}^{n,q}_{h,\omega}(G)$ and $w \in L^{n,q}_{(2)}(G)_{h,\omega}$ such that $w \in \operatorname{Im} \overline{\partial} \subset L^{n,q}_{(2)}(G)_{h,\widetilde{\omega}}$.

Proof. Note that we have $w \in \operatorname{Ker} \overline{\partial} \subset L^{n,q}_{(2)}(G)_{h,\omega}$ by the assumption $w \in \operatorname{Im} \overline{\partial} \subset L^{n,q}_{(2)}(G)_{h,\widetilde{\omega}}$. It follows that $\overline{\partial} w = 0$ on X from $\overline{\partial} w = 0$ on Y and $\|w\|_{h,\omega} < \infty$ (see [4, (7.3) Lemma, Chapter VIII]). By applying Proposition 2.6 for ω, we obtain the orthogonal decomposition

$$L^{n,q}_{(2)}(G)_{h,\omega} \supset \operatorname{Ker} \overline{\partial} = \operatorname{Im} \overline{\partial} \oplus \mathcal{H}^{n,q}_{h,\omega}(G).$$

By this orthogonal decomposition, w can be decomposed as follows:

$$w = w_1 + w_2 \ \text{ for some } w_1 \in \operatorname{Im} \overline{\partial} \ \text{ and } \ w_2 \in \mathcal{H}^{n,q}_{h,\omega}(G) \ \text{ in } L^{n,q}_{(2)}(G)_{h,\omega}.$$

We will show that w_2 is actually zero by the assumption $w \in \operatorname{Im} \overline{\partial} \subset L^{n,q}_{(2)}(G)_{h,\widetilde{\omega}}$. Then we obtain the conclusion $\langle\!\langle u, w \rangle\!\rangle_{h,\omega} = 0$ since we have

$$\langle\!\langle u, w \rangle\!\rangle_{h,\omega} = \langle\!\langle u, w_2 \rangle\!\rangle_{h,\omega} = 0 \text{ by } u \in \mathcal{H}^{n,q}_{h,\omega}(G) \text{ and } w_1 \in \operatorname{Im} \overline{\partial} \subset L^{n,q}_{(2)}(G)_{h,\omega}.$$

To prove that $w_2 = 0$, we consider the following composite map:

$$\phi : \mathcal{H}^{n,q}_{h,\omega}(G) \to \frac{\operatorname{Ker} \overline{\partial}}{\operatorname{Im} \overline{\partial}} \text{ of } L^{n,q}_{(2)}(G)_{h,\omega} \xrightarrow{j_1} \frac{\operatorname{Ker} \overline{\partial}}{\operatorname{Im} \overline{\partial}} \text{ of } L^{n,q}_{(2)}(G)_{h,\widetilde{\omega}},$$

where j_1 is the map induced by the natural map $L^{n,q}_{(2)}(G)_{h,\omega} \to L^{n,q}_{(2)}(G)_{h,\widetilde{\omega}}$. The map ϕ is a (well-defined) isomorphism by Proposition 2.6 and Proposition 2.7. It follows that

$$w_1 \in \Big(\operatorname{Im} \overline{\partial} \ \text{ in } \ L^{n,q}_{(2)}(G)_{h,\omega}\Big) \subset \Big(\operatorname{Im} \overline{\partial} \ \text{ in } \ L^{n,q}_{(2)}(G)_{h,\widetilde{\omega}}\Big)$$

from the third claim of Lemma 2.5 and property (B) of $\widetilde{\omega}$. Hence $w_2 = w - w_1$ also belongs to $\operatorname{Im} \overline{\partial} \subset L^{n,q}_{(2)}(G)_{h,\widetilde{\omega}}$ by the assumption $w \in \operatorname{Im} \overline{\partial} \subset L^{n,q}_{(2)}(G)_{h,\widetilde{\omega}}$. In particular, this implies that $\phi(w_2) = 0$. We obtain $w_2 = 0$ since the map ϕ is an isomorphism. □

In the remander of this subsection, we prove Theorem 1.6.

Proof of Theorem 1.6. Let Y be a non-empty Zariski open set in X such that h is smooth on Y. We first take a complete Kähler form $\widetilde{\omega}$ on Y with the following properties:

- $\widetilde{\omega}$ is a complete Kähler form on Y;
- $\widetilde{\omega} \geq \omega$ on Y;
- For every point $p \in X$, there exists a bounded function Φ on an open neighborhood of p in X such that $\omega = \sqrt{-1}\partial\overline{\partial}\Phi$.

See [12, Section 3] for the construction of $\widetilde{\omega}$. For the Kähler form ω_δ on Y defined by

$$\omega_\delta := \omega + \delta\widetilde{\omega} \ \text{ for } \ \delta > 0,$$

it is easy to check the following properties:

(A) ω_δ is a complete Kähler form on Y for every $\delta > 0$;
(B) $\omega_{\delta_2} \geq \omega_{\delta_1} \geq \omega$ on Y for $\delta_2 \geq \delta_1 > 0$;
(C) For every point $p \in X$, there exists a bounded function Φ_δ on an open neighborhood of p in X such that $\omega_\delta = \sqrt{-1}\partial\overline{\partial}\Phi_\delta$.

Note that we can apply Proposition 2.4 for ω_δ thanks to property (A). In the proof of Theorem 1.6, we will omit the subscription h of the norm, the L^2-space, and so on. For example, we will use the notation

$$\| \bullet \|_{\omega} := \| \bullet \|_{h,\omega}, \ \| \bullet \|_{\omega_\delta} := \| \bullet \|_{h,\omega_\delta}, \ \text{ and } L^{n,q}_{(2)}(G)_{\omega_\delta} := L^{n,q}_{(2)}(G)_{h,\omega_\delta}.$$

It follows that

$$\|u\|_{\omega_\delta} \leq \|u\|_{\omega} < \infty \tag{3.1}$$

from Lemma 2.5 and property (B). In particular u belongs to $L^{n,q}_{(2)}(G)_{\omega_\delta}$ for every $\delta > 0$. By the orthogonal decomposition (see Proposition 3.1)

$$L^{n,q}_{(2)}(G)_{\omega_\delta} = \operatorname{Im}\overline{\partial} \oplus \mathcal{H}^{n,q}_{\omega_\delta}(G) \oplus \operatorname{Im}\overline{\partial}^{*}_{\omega_\delta},$$

the G-valued (n, q)-form u can be decomposed as follows:

$$u = w_\delta + u_\delta \ \text{ for some } w_\delta \in \operatorname{Im}\overline{\partial} \ \text{ and } \ u_\delta \in \mathcal{H}^{n,q}_{\omega_\delta}(G) \ \text{ in } L^{n,q}_{(2)}(G)_{\omega_\delta}.$$

The strategy of the proof is the following: in the first step, we check that u_δ weakly converges to some u_0 in suitable L^2-spaces. In the second step, we show that the limit u_0 actually coincides with u by Proposition 3.1. In the third step, we prove that $*_\delta u_\delta \in H^0(X, \Omega_X^{n-q} \otimes G \otimes \mathcal{I}(h))$ by the theory of harmonic integrals and $*_\delta u_\delta$ converges to $*u_0 = *u$, where $*_\delta$ (respectively $*$) is the Hodge star operator with respect to ω_δ (respectively ω).

We first check that u_δ has a a suitable weak limit by the following proposition. Since we use Cantor's diagonal argument in the proof of Proposition 3.2, we need to handle only a countable sequence $\{\delta'\}_{\delta'>0}$.

Proposition 3.2. *For a countable sequence* $\{\delta'\}_{\delta'>0}$ *converging to zero, there exist a subsequence* $\{\delta_\nu\}_{\nu=1}^{\infty}$ *of* $\{\delta\}_{\delta>0}$ *and* $u_0 \in L^{n,q}_{(2)}(G)_\omega$ *with the following properties:*

- *For every* $\delta' > 0$*, as* δ_ν *goes to* 0*,*

  $$u_{\delta_\nu} \text{ converges to } u_0 \text{ with respect to the weak topology in } L^{n,q}_{(2)}(G)_{\omega_{\delta'}};$$

- $\|u_0\|_\omega \le \|u\|_\omega$.

Proof. For a given $\delta' > 0$, the sequence $\{u_\delta\}_{\delta' \ge \delta > 0}$ is bounded in $L^{n,q}_{(2)}(G)_{\omega_{\delta'}}$. Indeed, for $\delta' \ge \delta > 0$, we obtain

$$\|u_\delta\|_{\omega_{\delta'}} \le \|u_\delta\|_{\omega_\delta} \le \|u\|_{\omega_\delta} \le \|u\|_\omega < \infty. \tag{3.2}$$

The first inequality follows from Lemma 2.5 and $\omega_{\delta'} \ge \omega_\delta$, the second inequality follows since u_δ is the orthogonal projection of u in $L^{n,q}_{(2)}(G)_{\omega_\delta}$, and the third inequality follows from inequality (3.1). Hence there exists a subsequence $\{\delta_\nu\}_{\nu=1}^{\infty}$ of $\{\delta\}_{\delta>0}$ such that u_{δ_ν} weakly converges to some $u_{0,\delta'}$ in $L^{n,q}_{(2)}(G)_{\omega_{\delta'}}$, which may depend on δ'. We can choose a suitable subsequence independent of δ' by Cantor's diagonal argument, and thus we can assume that this subsequence $\{\delta_\nu\}_{\nu=1}^{\infty}$ is independent of δ'.

Now we show that the weak limit $u_{0,\delta'}$ is also independent of δ'. For any $\delta_1 \ge \delta_2$, the natural inclusion $L^{n,q}_{(2)}(G)_{\omega_{\delta_2}} \to L^{n,q}_{(2)}(G)_{\omega_{\delta_1}}$ is a bounded operator (continuous linear map) by Lemma 2.5 and $\omega_{\delta_1} \ge \omega_{\delta_2}$. By Lemma 2.10, we can see that u_{δ_ν} weakly converges to u_{0,δ_2} not only in $L^{n,q}_{(2)}(G)_{\omega_{\delta_2}}$ but also in $L^{n,q}_{(2)}(G)_{\omega_{\delta_1}}$. Therefore it follows that $u_{0,\delta_1} = u_{0,\delta_2}$ since u_{δ_ν} weakly converges to u_{0,δ_1} in $L^{n,q}_{(2)}(G)_{\omega_{\delta_1}}$ and the weak limit is uniquely determined.

Finally we estimate the L^2-norm of the weak limit u_0. Fatou's lemma yields

$$\|u_0\|^2_\omega = \int_Y |u_0|^2_\omega \, dV_\omega \le \liminf_{\delta' \to 0} \int_Y |u_0|^2_{\omega_{\delta'}} \, dV_{\omega_{\delta'}} = \liminf_{\delta' \to 0} \|u_0\|^2_{\omega_{\delta'}}.$$

On the other hand, it is easy to see that

$$\|u_0\|_{\omega_{\delta'}} \le \liminf_{\delta_\nu \to 0} \|u_{\delta_\nu}\|_{\omega_{\delta'}} \le \liminf_{\delta_\nu \to 0} \|u_{\delta_\nu}\|_{\omega_{\delta_\nu}} \le \|u\|_\omega < \infty.$$

The first inequality follows from lower semi-continuity with respect to the weak convergence, the second inequality follows from Lemma 2.5 and $\omega_{\delta_\nu} \le \omega_{\delta'}$, the third inequality follows from inequality (3.2). Therefore we obtain the desired inequality $\|u_0\|_\omega \le \|u\|_\omega$. □

For simplicity, we use the same notation $\{u_\delta\}_{\delta>0}$ for the subsequence $\{u_{\delta_\nu}\}_{\delta_\nu>0}$ chosen in Proposition 3.2. The following proposition is obtained from Proposition 3.1.

Proposition 3.3. *The weak limit u_0 coincides with u.*

Proof. We fix $\delta_0 > 0$ in the proof of Proposition 3.3. By Lemma 2.5, we can see that

$$\operatorname{Im}\overline{\partial} \text{ in } L^{n,q}_{(2)}(G)_{\omega_\delta} \subset \operatorname{Im}\overline{\partial} \text{ in } L^{n,q}_{(2)}(G)_{\omega_{\delta_0}}$$

for an arbitrary δ with $\delta_0 \geq \delta > 0$. Hence, it follows that

$$u - u_\delta = w_\delta \in \operatorname{Im}\overline{\partial} \text{ in } L^{n,q}_{(2)}(G)_{\omega_{\delta_0}}$$

from the construction of u_δ and w_δ. The subspace $\operatorname{Im}\overline{\partial}$ is closed not only with respect to the L^2-topology but also with respect to the weak topology (see Proposition 2.6 and Lemma 2.9). By taking the weak limit, we can conclude that

$$w_0 := u - u_0 = \operatorname*{w\text{-}lim}_{\delta \to 0} w_\delta \in \operatorname{Im}\overline{\partial} \text{ in } L^{n,q}_{(2)}(G)_{\omega_{\delta_0}}.$$

Since the Kähler form ω_{δ_0} on Y satisfies properties (B) and (C), we have $\langle\!\langle u, w_0 \rangle\!\rangle_\omega = 0$ by Proposition 3.1, where w_0 is the weak limit of $w_\delta = u - u_\delta$. Hence we obtain $\|u_0\|^2_\omega = \|u\|^2_\omega + \|w_0\|^2_\omega$. This is a contradiction to the inequality $\|u_0\|_\omega \leq \|u\|_\omega$ in Proposition 3.2 if w_0 is not zero. Therefore w_0 is actually zero. We obtain the desired conclusion $u = u_0$. □

From now on, we consider the Hodge star operator $*_\delta$ with respect to ω_δ and the G-valued $(n-q, 0)$-form $*_\delta u_\delta$. Note that $*_\delta u_\delta$ is a G-valued $(n-q, 0)$-form on Y (not X) since the Kähler form ω_δ is defined only on Y. However, by the following proposition, we can regard $*_\delta u_\delta$ as a holomorphic G-valued $(n-q, 0)$-form on X.

Proposition 3.4. *The G-valued $(n-q, 0)$-form $*_\delta u_\delta$ can be extended to a holomorphic G-valued $(n-q, 0)$-form on X (that is, $\overline{\partial} *_\delta u_\delta = 0$ on X). Moreover we have*

$$\| *_\delta u_\delta \|_\omega \leq \|u\|_\omega < \infty.$$

In particular, we have $*_\delta u_\delta \in H^0(X, \Omega^{n-q}_X \otimes G \otimes \mathcal{I}(h))$.

Proof. Let $*_\delta u_\delta = \sum_J f_J dz_J$ be a local expression in terms of a local coordinate $(z_1, z_2, \ldots, z_n)$, where J is an ordered multi-index with degree $(n-q)$. We will show that every coefficient f_J is holomorphic on Y and can be extended to a holomorphic function on X.

Since ω_δ is a complete Kähler form on Y, we can apply Proposition 2.4 to u_δ. Proposition 2.4 yields

$$0 = \|\overline{\partial} u_\delta\|^2_{\omega_\delta} + \|\overline{\partial}^*_\delta u_\delta\|^2_{\omega_\delta} = \|D'^*_\delta u_\delta\|^2_{\omega_\delta} + \left\langle\!\!\left\langle \sqrt{-1}\Theta_h(G)\Lambda_{\omega_\delta} u_\delta, u_\delta \right\rangle\!\!\right\rangle_{\omega_\delta}. \tag{3.3}$$

The first equality follows since u_δ is harmonic with respect to ω_δ. Here D'^*_δ denotes the Hilbert space adjoint of the $(1,0)$-part of the Chern connection $D_h = D'_h + \overline{\partial}$ and Λ_{ω_δ} denotes the adjoint operator of the wedge product $\omega_\delta \wedge \bullet$.

The second term of the right hand side is non-negative by the assumption $\sqrt{-1}\Theta_h(G) \geq 0$, and thus the first term and the second term must be zero. In particular we obtain $|D_\delta'^* u_\delta|_{\omega_\delta} = 0$. The Hilbert space adjoint coincides with the formal adjoint since ω_δ is complete (see, for example, [4, (3.2) Theorem in Chapter VIII]). Hence we have $D_\delta'^* = - *_\delta \overline{\partial} *_\delta$. It follows that $0 \equiv |D_\delta'^* u_\delta|_{\omega_\delta} = |- *_\delta \overline{\partial} *_\delta u_\delta|_{\omega_\delta} = |\overline{\partial} *_\delta u_\delta|_{\omega_\delta}$ since the Hodge star operator $*_\delta$ preserves the point-wise norm $| \bullet |_{\omega_\delta}$. Therefore the G-valued $(n-q, 0)$-form $*_\delta u_\delta$ is $\overline{\partial}$-closed on Y, that is, the coefficient f_J is a holomorphic function on Y.

Now we show that the L^2-norm of the coefficient f_J with respect to h is uniformly bounded (that is, $\int |f_J|_h^2 \, dV_\omega < C$ for some $C > 0$). The key point here is the following inequality:

$$\| *_\delta u_\delta \|_\omega \leq \| *_\delta u_\delta \|_{\omega_\delta} = \| u_\delta \|_{\omega_\delta} \leq \| u \|_\omega < \infty. \tag{3.4}$$

The first inequality follows from the fourth claim of Lemma 2.5 and $\omega \leq \omega_\delta$, the second inequality follows since $*_\delta$ preserves the point-wise norm $| \bullet |_{\omega_\delta}$, the third inequality follows from inequality (3.2). On the other hand, there is a constant C' (independent of δ) such that $|f_J|_h^2 \leq C' | *_\delta u_\delta|_\omega^2$. Indeed, by the first claim of Lemma 2.5, we can easily check that

$$\begin{aligned} |f_J|_h \inf\left(|dz_J \wedge dz_{\hat{J}} \wedge d\overline{z}|_\omega\right) &\leq \left|f_J dz_J \wedge dz_{\hat{J}} \wedge d\overline{z}\right|_{h,\omega} \\ &= \left| *_\delta u_\delta \wedge dz_{\hat{J}} \wedge d\overline{z}\right|_{h,\omega} \\ &\leq C'' | *_\delta u_\delta|_\omega \sup\left(|dz_{\hat{J}} \wedge d\overline{z}|_\omega\right) \end{aligned}$$

for some positive constant C'' (independent of δ), where $\hat{J}$ is the complementary index of J. By combining with inequality (3.4), we obtain

$$\int |f_J|_h^2 \, dV_\omega \leq C' \| *_\delta u_\delta \|_\omega^2 \leq C' \|u\|_\omega^2.$$

Therefore, by the Riemann extension theorem, the coefficient f_J can be extended as a holomorphic function. □

We put $f_\delta := *_\delta u_\delta$ and consider a local expression $f_\delta = *_\delta u_\delta = \sum_J f_{\delta,J} dz_J$ again. By the proof of Proposition 3.4, we can see that the L^2-norm of the coefficient $f_{\delta,J}$ is uniformly bounded with respect to δ. Hence, by Montel's theorem, there exists a subsequence $\{\delta_\nu\}_{\nu=1}^\infty$ of $\{\delta\}_{\delta>0}$ such that $f_{\delta_\nu} = *_{\delta_\nu} u_{\delta_\nu}$ uniformly converges to some f_0, that is, the local sup-norm $\sup |f_{\delta_\nu,J} - f_{0,J}|$ converges to zero, where $f_{0,J}$ is the coefficient of $f_0 = \sum_J f_{0,J} dz_J$. Then the L^2-norm $\| f_{\delta_\nu} - f_0 \|_{h,\omega}$ also converges to zero (for example see [25, Lemma 5.2]). In particular, the limit f_0 satisfies $f_0 \in H^0(X, \Omega_X^{n-q} \otimes G \otimes \mathcal{I}(h))$. For simplicity we use the same notation $\{f_\delta\}_{\delta>0}$ for this subsequence. Then we show that u_0 (which is the weak limit obtained in Proposition 3.2) coincides with $(-1)^{n+q} * f_0$.

Proposition 3.5. *The weak limit u_0 coincides with $(-1)^{n+q} * f_0$. In particular, we can see that $u = (-1)^{n+q} * f_0$ by Proposition* 3.3.

Proof. For a contradiction, we assume that $u_0 \neq (-1)^{n+q} * f_0$ in $L^{n,q}_{(2)}(G)_{\omega_{\delta'}}$. Since the smooth G-valued (n,q)-forms with compact support in Y is dense in $L^{n,q}_{(2)}(G)_{\omega_{\delta'}}$, there exists a smooth G-valued (n,q)-form η with compact support in Y such that $\langle\langle u_0, \eta\rangle\rangle_{\omega_{\delta'}} \neq \langle\langle (-1)^{n+q} * f_0, \eta\rangle\rangle_{\omega_{\delta'}}$. Since u_δ weakly converges to u_0 in $L^{n,q}_{(2)}(G)_{\omega_{\delta'}}$, we have $\langle\langle u_0, \eta\rangle\rangle_{\omega_{\delta'}} = \lim_{\delta\to 0} \langle\langle u_\delta, \eta\rangle\rangle_{\omega_{\delta'}}$. On the other hand, it follows that $*_\delta f_\delta$ uniformly converges $*f_0$ on every relatively compact set in Y since f_δ uniformly converges f_0 and ω_δ uniformly converges ω on every relative compact set in Y. Indeed, it is sufficient to consider $(*_\delta f_\delta - *f_\delta)$ since we have

$$*_\delta f_\delta - *f_0 = \big(*_\delta f_\delta - *f_\delta\big) + \big(* f_\delta - *f_0\big),$$
$$\sup_X \big| * f_\delta - *f_0\big|_\omega = \sup_X \big|f_\delta - f_0\big|_\omega \to 0.$$

For a relatively compact set K in Y and a given point $x \in K$, we take a local coordinate $(z_1, z_2, \dots, z_n)$ centered at $x \in K$ such that

$$\omega = \frac{\sqrt{-1}}{2}\sum_{i=1}^{n} dz_i \wedge d\overline{z_i} \quad \text{and} \quad \widetilde{\omega} = \frac{\sqrt{-1}}{2}\sum_{i=1}^{n} \lambda_i dz_i \wedge d\overline{z_i} \quad \text{at } x.$$

By $K \Subset Y$, there exists a positive constant C such that $0 \leq \widetilde{\omega} \leq C\omega$ on K. In particular we have $0 \leq \lambda_i \leq C$. Note that the eigenvalues of ω_δ with respect to ω are $\{(1+\delta\lambda_i)\}_{i=1}^{n}$. When f_δ is locally written as $f_\delta = \sum_J f_{\delta,J} dz_J$, we can easily see that

$$\begin{aligned} | *_\delta f_\delta - *f_\delta|_\omega &= \Big|\sum_J f_{\delta,J}(*_\delta dz_J - *dz_J)\Big|_\omega \\ &= \Big|\sum_J f_{\delta,J}\ \mathrm{sign}(J\hat{J})\ \big\{\Pi_{i\in\hat{J}}(1+\delta\lambda_i) - 1\big\} dz_{(1,2,\dots,n)} \wedge d\overline{z_{\hat{J}}}\Big|_\omega \\ &\leq \delta C' \sum_J \sup_K |f_{\delta,J}| \big|dz_{(1,2,\dots,n)} \wedge d\overline{z_{\hat{J}}}\big|_\omega \end{aligned}$$

for some constant C'. The coefficient $f_{\delta,J}$ is a holomorphic function, and thus the (local) sup-norm $\sup_K |f_{\delta,J}|$ of $f_{\delta,J}$ can be bounded by the L^2-norm. Further the L^2-norm of $f_{\delta,J}$ is uniformly bounded with respect to δ (see Proposition 3.4). Therefore $(*_\delta f_\delta - *f_\delta)$ uniformly converges to zero on $K \Subset Y$. Hence, by the definition of $f_\delta = *_\delta u_\delta$, we obtain

$$\begin{aligned} \langle\langle (-1)^{n+q} * f_0, \eta\rangle\rangle_{\omega_{\delta'}} &= \lim_{\delta\to 0} \langle\langle (-1)^{n+q} *_\delta f_\delta, \eta\rangle\rangle_{\omega_{\delta'}} \\ &= \lim_{\delta\to 0} \langle\langle (-1)^{n+q} *_\delta *_\delta u_\delta, \eta\rangle\rangle_{\omega_{\delta'}} = \lim_{\delta\to 0} \langle\langle u_\delta, \eta\rangle\rangle_{\omega_{\delta'}}. \end{aligned}$$

This is a contradiction to $\langle\langle u_0, \eta \rangle\rangle_{\omega_{\delta'}} \neq \langle\langle (-1)^{n+q} * f_0, \eta \rangle\rangle_{\omega_{\delta'}}$. Therefore we can conclude that $u_0 = (-1)^{n+q} * f_0$ in $L^{n,q}_{(2)}(G)_{\omega_{\delta'}}$ for every $\delta' > 0$. Then, by Fatou's lemma, we can easily see that

$$\|u_0 - (-1)^{n+q} * f_0\|_{\omega} \leq \liminf_{\delta' \to 0} \|u_0 - (-1)^{n+q} * f_0\|_{\omega_{\delta'}} = 0. \qquad \square$$

By $f_0 \in H^0(X, \Omega_X^{n-q} \otimes G \otimes \mathcal{I}(h))$, we obtain the desired conclusion

$$*u = (-1)^{n+q} ** f_0 = f_0 \in H^0\big(X, \Omega_X^{n-q} \otimes G \otimes \mathcal{I}(h)\big)$$

in Proposition 1.6. This completes the proof. $\square$

3.2. Proof of Theorem 1.5

In this subsection, we prove Theorem 1.5.

Proof of Theorem 1.5. Let g be a smooth Hermitian metric on the line bundle D and t be the natural section of the effective divisor D. Then we define the smooth Hermitian metric g_ε on the line bundle D by

$$\varphi_\varepsilon := \frac{1}{2} \log\left(|t|_g^2 + \varepsilon\right) \quad \text{and} \quad g_\varepsilon := g e^{-2\varphi_\varepsilon} = g \cdot \left(\frac{1}{|t|_g^2 + \varepsilon}\right).$$

It is easy to see that

- $g_{\varepsilon_2} \leq g_{\varepsilon_1}$ for $\varepsilon_1 \leq \varepsilon_2$;
- g_ε converges to $g_0 = h_D$ in the point-wise sense as ε tends to zero;

where h_D is the singular metric defined by the effective divisor D (see Example 2.3). We have $\mathcal{I}(g_0) = \mathcal{I}(h_D) = \mathcal{O}_X(-D)$ since D is a simple normal crossing divisor. Let ω be a Kähler form on X, and let h_F and h_M be smooth Hermitian metrics satisfying the assumptions in Theorem 1.5. We often omit the subscripts ω, h_F, and h_M of the norm, the L^2-space, and so on. For example, we use the notation

$$L^{n,q}_{(2)}(D \otimes F)_{g_\varepsilon} := L^{n,q}_{(2)}(D \otimes F)_{g_\varepsilon h_F, \omega} \text{ and } \mathcal{H}^{n,q}_{g_0}(D \otimes F) := \mathcal{H}^{n,q}_{g_0 h_F, \omega}(D \otimes F).$$

We first consider the following commutative diagram:

$$\begin{array}{ccc}
H^q(X, K_X \otimes D \otimes F \otimes \mathcal{I}(g_0)) = H^q(X, K_X \otimes F) & \xrightarrow{\Phi_D} & H^q(X, K_X \otimes D \otimes F) \\
\cong \big\uparrow \overline{f_0} & & \cong \big\uparrow \overline{f_\varepsilon} \\
\dfrac{\operatorname{Ker} \bar\partial}{\operatorname{Im} \bar\partial} \text{ of } L^{n,q}_{(2)}(D \otimes F)_{g_0} & & \dfrac{\operatorname{Ker} \bar\partial}{\operatorname{Im} \bar\partial} \text{ of } L^{n,q}_{(2)}(D \otimes F)_{g_\varepsilon} \\
\cong \big\uparrow j & \overset{\phi}{\nearrow} & \\
\mathcal{H}^{n,q}_{g_0}(D \otimes F). & &
\end{array}$$

Here $\overline{f_0}$ and $\overline{f_\varepsilon}$ are the De Rham-Weil isomorphisms given in Subsection 2.3 and j (respectively ϕ) is the map induced by the natural inclusion $\mathcal{H}^{n,q}_{g_0}(D\otimes F)\hookrightarrow \operatorname{Ker}\overline{\partial}\subset L^{n,q}_{(2)}(D\otimes F)_{g_0}$ (respectively $\mathcal{H}^{n,q}_{g_0}(D\otimes F)\hookrightarrow \operatorname{Ker}\overline{\partial}\subset L^{n,q}_{(2)}(D\otimes F)_{g_\varepsilon}$). For a cohomology class α such that $\alpha\in\operatorname{Im}\Phi_D\subset H^q(X,K_X\otimes D\otimes F)$, we assume that $s\alpha=0\in H^q(X,K_X\otimes D\otimes F\otimes M)$. Our goal is to show that the cohomology class α is actually zero under this assumption. By $\alpha\in\operatorname{Im}\Phi_D$, there exists a cohomology class $\beta\in H^q(X,K_X\otimes F)$ such that $\Phi_D(\beta)=\alpha$. By the above isomorphisms, the cohomology class β can be represented by the harmonic form $u_1\in\mathcal{H}^{n,q}_{g_0}(D\otimes F)$ (that is, $\beta=\{u_1\}$). Since $\mathcal{H}^{n,q}_{g_0}(D\otimes F)$ is a finite dimensional vector space with the inner product $⟪\bullet,\bullet⟫_{g_0}:=⟪\bullet,\bullet⟫_{g_0h_F,\omega}$, we have the orthogonal decomposition

$$\mathcal{H}^{n,q}_{g_0}(D\otimes F)=\operatorname{Ker}\phi\oplus(\operatorname{Ker}\phi)^{\perp}. \tag{3.5}$$

From this orthogonal decomposition, the harmonic form u_1 can be decomposed as follows:

$$u_1=u_2+u \text{ for some } u_2\in\operatorname{Ker}\phi \text{ and } u\in(\operatorname{Ker}\phi)^{\perp}.$$

Then it is easy to see that $\Phi_D(\{u\})=\Phi_D(\{u_2+u\})=\Phi_D(\beta)=\alpha$. Note that $\{u_2+u\}$ is equal to β, but it is not necessarily equal to $\{u\}$. We can see that if we can prove $u=0$, we obtain $\alpha=0$ (the desired conclusion of Theorem 1.5). Hence our goal is to show $u=0$.

By the assumption $\sqrt{-1}\Theta_{h_F}(F)\geq 0$, the line bundle $G=D\otimes F$ and the singular Hermitian metric $h=g_0h_F$ satisfy the assumptions in Theorem 1.6. By applying Theorem 1.6 for u, we obtain

$$*u\in H^0\big(X,\Omega_X^{n-q}\otimes D\otimes F\otimes\mathcal{I}(g_0)\big). \tag{3.6}$$

In particular $*u$ is smooth on X. Although u is a priori $D\otimes F$-valued (n,q)-form on $Y:=X\setminus\operatorname{Supp}D$ (not X), it follows that $u=(-1)^{n+q}**u$ is smooth on X from (3.6).

Remark 3.6. (1) It seems to be difficult to show that u is smooth on X without using Theorem 1.6, since g_0 is a singular Hermitian metric and ω is not complete on Y.

(2) Note that we have $\mathcal{I}(g_0)=\mathcal{O}(-D)$ since D is a simple normal crossing divisor. Therefore $*u/t$ is a holomorphic F-valued $(n-q,0)$-form. In particular $*u/t$ is still smooth on X, which plays a crucial role later.

By the standard De Rham-Weil isomorphism, we have

$$\Phi_D(\{su\})=s\alpha=0\in H^q(X,K_X\otimes D\otimes F\otimes M)\cong\frac{\operatorname{Ker}\overline{\partial}}{\operatorname{Im}\overline{\partial}} \text{ of } C^{n,q}_{\infty}(D\otimes F\otimes M),$$

where $C^{n,q}_{\infty}(D\otimes F\otimes M)$ is the set of smooth $D\otimes F\otimes M$-valued (n,q)-forms on X. Hence, by the assumption $s\alpha=0$, we can take a smooth $D\otimes F\otimes M$-valued

$(n, q-1)$-form v such that $su = \overline{\partial} v$. Lebesgue's dominated convergence theorem yields

$$\|su\|^2_{g_0} = \lim_{\varepsilon \to 0} \int_Y |su|^2_{g_\varepsilon} \, dV_\omega = \lim_{\varepsilon \to 0} \langle\!\langle su, su \rangle\!\rangle_{g_\varepsilon},$$

since $|su|^2_{g_\varepsilon} \leq |su|^2_{g_0}$ and $|su|^2_{g_0}$ is integrable. Therefore, from Cauchy-Schwartz inequality, we obtain

$$\|su\|^2_{g_0} = \lim_{\varepsilon \to 0} \langle\!\langle su, su \rangle\!\rangle_{g_\varepsilon} = \lim_{\varepsilon \to 0} \langle\!\langle su, \overline{\partial} v \rangle\!\rangle_{g_\varepsilon} \leq \lim_{\varepsilon \to 0} \|\overline{\partial}^*_{g_\varepsilon} su\|_{g_\varepsilon} \|v\|_{g_\varepsilon}. \tag{3.7}$$

The strategy of the proof of Theorem 1.5 is as follows: we will show that $\|v\|_{g_\varepsilon} = O(-\log \varepsilon)$ and $\|\overline{\partial}^*_{g_\varepsilon} su\|_{g_\varepsilon} = O(\varepsilon(-\log \varepsilon))$. Then, from inequality (3.7), we obtain $\|su\|^2_{g_0} = 0$ (that is, $su = 0$). This completes the proof. We first check the following lemma.

Lemma 3.7. *Let $(z_1, z_2, \dots, z_n)$ be the standard coordinate of $\mathbb{C}^n$ and B be an open ball containing the origin. Then, for every $1 \leq k \leq n$, we have*

$$\int_B \frac{1}{\varepsilon + |z_1 z_2 \cdots z_k|^2} = O(-\log \varepsilon).$$

Proof. By the variable change $z_i = r_i e^{\sqrt{-1}\theta_i}$, the problem can be reduced to showing

$$\int_{0 \leq r_1 \leq 1} \int_{0 \leq r_2 \leq 1} \cdots \int_{0 \leq r_k \leq 1} \frac{r_1 r_2 \cdots r_k}{\varepsilon + |r_1 r_2 \cdots r_k|^2} dr_1 dr_2 \cdots dr_k = O(-\log \varepsilon).$$

Further, by using the polar coordinate, we can obtain the conclusion from the following computation:

$$\int_{0 \leq R \leq 1} \frac{R^{2k-1}}{\varepsilon + R^{2k}} dR = \frac{1}{2k}(\log(\varepsilon + 1) - \log \varepsilon).$$

□

By Lemma 3.7, we can easily obtain the following proposition. In the proof of the following proposition, we essentially use the fact that v is smooth on X.

Proposition 3.8. $\|v\|_{g_\varepsilon} = O(-\log \varepsilon)$.

Proof. By the definition of g_ε, we can see that

$$\|v\|^2_{g_\varepsilon} = \int_X |v|^2_g \frac{1}{\varepsilon + |t|^2_g} dV_\omega \leq \sup_X |v|^2_g \int_X \frac{1}{\varepsilon + |t|^2_g} dV_\omega.$$

It follows that $\sup_X |v|^2_g$ is finite since v and g are smooth on X. Since $D = \operatorname{div} t$ is a simple normal crossing divisor, we can obtain the conclusion by Lemma 3.7. □

It remains to show that

$$\left\| \overline{\partial}^{*}_{g_\varepsilon} su \right\|_{g_\varepsilon} = O(\varepsilon(-\log\varepsilon)).$$

By applying Proposition 2.4 for su, g_ε, and ω, we obtain

$$\left\| \overline{\partial}^{*}_{g_\varepsilon} su \right\|^2_{g_\varepsilon} = \left\| D'^{*}_{g_\varepsilon} su \right\|^2_{g_\varepsilon} + \left\langle\!\!\left\langle \sqrt{-1}\Theta_{g_\varepsilon h_F h_M}(D\otimes F\otimes M)\Lambda su, su \right\rangle\!\!\right\rangle_{g_\varepsilon}, \tag{3.8}$$

where $D'^{*}_{g_\varepsilon}$ (respectively $\overline{\partial}^{*}_{g_\varepsilon}$) is the Hilbert space adjoint of the $(1,0)$-part D'_{g_ε} (respectively the $(0,1)$-part $\overline{\partial}$) of the Chern connection $D_{g_\varepsilon} = D'_{g_\varepsilon} + \overline{\partial}$, and Λ is the adjoint operator of the wedge product $\omega \wedge \bullet$. Here we used that $\overline{\partial} su = s\overline{\partial} u = 0$.

We consider the first term $\| D'^{*}_{g_\varepsilon} su \|_{g_\varepsilon}$ of the right hand side of (3.8). It follows that $D'^{*}_{g_\varepsilon} = - * \overline{\partial} *$ since X is compact and ω is defined on X. We have $\overline{\partial} * u = 0$ by (3.6) (see Theorem 1.6), and thus we obtain

$$D'^{*}_{g_\varepsilon} su = - * \overline{\partial} * su = - * \overline{\partial} s * u = - * s \overline{\partial} * u = 0. \tag{3.9}$$

In particular we can see $\| D'^{*}_{g_\varepsilon} su \|_{g_\varepsilon} = 0$.

The problem is the second term of the right hand side of (3.8). From simple computations, we can obtain

$$\sqrt{-1}\Theta_{g_\varepsilon}(D) = \varepsilon \frac{1}{|t|^2_g + \varepsilon}\sqrt{-1}\Theta_g(D) + \varepsilon \frac{D'_g t \wedge \overline{D'_g t}}{\left(|t|^2_g + \varepsilon\right)^2},$$

where D'_g is the $(1,0)$-part of the Chern connection D_g. Now we compute the negativity of the curvature $\sqrt{-1}\Theta_{g_\varepsilon}(D)$. By the above equality, we have

$$\sqrt{-1}\Theta_{g_\varepsilon}(D) \geq \varepsilon \frac{1}{|t|^2_g + \varepsilon}\sqrt{-1}\Theta_g(D).$$

On the other hand, there exists a positive constant C such that $\sqrt{-1}\Theta_g(D) \geq -C\omega$ on X since X is compact and g is smooth on X. Therefore we have

$$\sqrt{-1}\Theta_{g_\varepsilon}(D) + A_\varepsilon \geq 0, \text{ where } A_\varepsilon := \varepsilon \frac{C}{|t|^2_g + \varepsilon}\omega \geq 0.$$

Then we can see that

$$\begin{aligned}
&\left\langle\!\!\left\langle \sqrt{-1}\Theta_{g_\varepsilon h_F h_M}(D\otimes F\otimes M)\Lambda su, su \right\rangle\!\!\right\rangle_{g_\varepsilon} \\
&\leq \left\langle\!\!\left\langle \left(\sqrt{-1}\Theta_{g_\varepsilon h_F h_M}(D\otimes F\otimes M) + A_\varepsilon\right)\Lambda su, su \right\rangle\!\!\right\rangle_{g_\varepsilon} \\
&\leq \sup_X |s|^2_{h_M} \left\langle\!\!\left\langle \left(\sqrt{-1}\Theta_{g_\varepsilon h_F h_M}(D\otimes F\otimes M) + A_\varepsilon\right)\Lambda u, u \right\rangle\!\!\right\rangle_{g_\varepsilon} \\
&\leq \sup_X |s|^2_{h_M} \left\langle\!\!\left\langle \left(\sqrt{-1}\Theta_{g_\varepsilon h_F h_M}(D\otimes F\otimes M) + A_\varepsilon\right)\Lambda u, u \right\rangle\!\!\right\rangle_{g_0}.
\end{aligned}$$

The first inequality is obtained from $A_\varepsilon \geq 0$, the second inequality is obtained from $\sqrt{-1}\Theta_{g_\varepsilon}(D) + A_\varepsilon \geq 0$, and the third inequality is obtained from $g_\varepsilon \leq g_0$. Further, by the assumption $\sqrt{-1}\Theta_{h_F}(F) \geq t\sqrt{-1}\Theta_{h_M}(M)$, we can see that

$$\sqrt{-1}\Theta_{g_\varepsilon h_F h_M}(D\otimes F\otimes M)+A_\varepsilon \leq \sqrt{-1}\Theta_{g_\varepsilon}(D) + A_\varepsilon + \left(1+\frac{1}{t}\right)\sqrt{-1}\Theta_{h_F}(F)$$
$$\leq \left(1+\frac{1}{t}\right)\left(\sqrt{-1}\Theta_{g_\varepsilon}(D)+A_\varepsilon+\sqrt{-1}\Theta_{h_F}(F)\right).$$

Here we used $\sqrt{-1}\Theta_{g_\varepsilon}(D) + A_\varepsilon \geq 0$ to obtain the second inequality. In summary, we have

$$\left\langle\!\!\left\langle \sqrt{-1}\Theta_{g_\varepsilon h_F h_M}(D \otimes F \otimes M)\Lambda su, su\right\rangle\!\!\right\rangle_{g_\varepsilon}$$
$$\leq \sup_X |s|^2_{h_M} \left(1+\frac{1}{t}\right) \left\langle\!\!\left\langle \left(\sqrt{-1}\Theta_{g_\varepsilon}(D) + A_\varepsilon + \sqrt{-1}\Theta_{h_F}(F)\right)\Lambda u, u\right\rangle\!\!\right\rangle_{g_0}.$$

For the proof of Theorem 1.5, it is sufficient to estimate the order of the right hand side.

Proposition 3.9 (*cf.* [30, Proposition 3.8]). *Under the above situation, we have*

$$\left\langle\!\!\left\langle \left(\sqrt{-1}\Theta_{g_\varepsilon}(D) + \sqrt{-1}\Theta_{h_F}(F)\right)\Lambda u, u\right\rangle\!\!\right\rangle_{g_0} = 0.$$

Proof. For simplicity, we put

$$w := \sqrt{-1}\Theta_{g_\varepsilon h_F}(D \otimes F)\Lambda u = \left(\sqrt{-1}\Theta_{g_\varepsilon}(D) + \sqrt{-1}\Theta_{h_F}(F)\right)\Lambda u.$$

Then it follows that $w \in L^{n,q}_{(2)}(D \otimes F)_{g_0}$ since the metric $g_\varepsilon h_F$ is smooth on X and $u \in L^{n,q}_{(2)}(D \otimes F)_{g_0}$. Indeed, there is a positive constant C such that $-C\omega \leq \sqrt{-1}\Theta_{g_\varepsilon h_F}(D \otimes F) \leq C\omega$. Then we have $|w|_{g_0} \leq Cq|u|_{g_0}$, and thus we can see that $w \in L^{n,q}_{(2)}(D\otimes F)_{g_0}$ by $u \in L^{n,q}_{(2)}(D\otimes F)_{g_0}$. Further, by $u \in \mathcal{H}^{n,q}_{g_0}(D\otimes F)$ and (3.9), we have $\overline{\partial}u = 0$ and $D'^{*}_{g_\varepsilon}u = 0$. Therefore we obtain

$$\overline{\partial}\,\overline{\partial}^*_{g_\varepsilon}u = \sqrt{-1}\Theta_{g_\varepsilon h_F}(D \otimes F)\Lambda u = w$$

from Proposition 2.4. In particular, we can see that $w \in \operatorname{Ker}\overline{\partial} \subset L^{n,q}_{(2)}(D \otimes F)_{g_0}$.

By (3.5), we have the orthogonal decomposition

$$\operatorname{Ker}\overline{\partial} = \operatorname{Im}\overline{\partial} \oplus \operatorname{Ker}\phi \oplus (\operatorname{Ker}\phi)^\perp \text{ in } L^{n,q}_{(2)}(D \otimes F)_{g_0},$$

and thus w can be decomposed as follows:

$$w = w_1 + w_2 + w_3 \text{ for some } w_1 \in \operatorname{Im}\overline{\partial},\ w_2 \in \operatorname{Ker}\phi, \text{ and } w_3 \in (\operatorname{Ker}\phi)^\perp.$$

Since we have $u \in (\operatorname{Ker}\phi)^{\perp}$ by the construction of u, we obtain $\langle\!\langle w, u\rangle\!\rangle_{g_0} = \langle\!\langle w_3, u\rangle\!\rangle_{g_0}$. It is sufficient for the proof to show that w_3 is zero. It follows that $\overline{\partial}^*_{g_\varepsilon} u \in L^{n,q}_{(2)}(D \otimes F)_{g_\varepsilon}$ since $\overline{\partial}^*_{g_\varepsilon} u$ is smooth on X. (Note that we do not know whether $\overline{\partial}^*_{g_\varepsilon} u \in L^{n,q}_{(2)}(D \otimes F)_{g_0}$.) By combining with $\overline{\partial}\,\overline{\partial}^*_{g_\varepsilon} u = w$, we can conclude that

$$w_2 + w_3 = w - w_1 \in \operatorname{Im}\overline{\partial} \subset L^{n,q}_{(2)}(D \otimes F)_{g_\varepsilon},$$

and thus we obtain $w_2+w_3=w-w_1 \in \operatorname{Ker}\phi$. In particular we can see $w_3=0$. □

Finally we prove the following proposition.

Proposition 3.10. *Under the above situation, we have*

$$\langle\!\langle A_\varepsilon \Lambda u, u\rangle\!\rangle_{g_0} = O(\varepsilon(-\log\varepsilon)).$$

Proof. By Remark 3.6 (which is obtained from Theorem 1.6), we see that $|u|_{g_0}$ is a bounded function on X. By the definition of A_ε, we can easily see that

$$\langle\!\langle A_\varepsilon \Lambda u, u\rangle\!\rangle_{g_0} = \varepsilon \int_Y \frac{Cq}{|t|_g^2 + \varepsilon} |u|^2_{g_0}\, dV_\omega \leq \varepsilon \sup_X |u|^2_{g_0} \int_Y \frac{Cq}{|t|_g^2 + \varepsilon}\, dV_\omega.$$

By Lemma 3.7, we obtain the conclusion. □

Remark 3.11. The integral in Lemma 3.7 naturally appears when we prove Proposition 3.8 and Proposition 3.10, but the reasons why the integral appears are different. The integral in Proposition 3.8 comes from the definition of g_ε. On the other hand, the same integral comes from the curvature of g_ε when we prove Proposition 3.10.

By Proposition 3.8, Proposition 3.10, and inequality (3.7), we complete the proof of Theorem 1.5. □

3.3. Proof of Theorem 1.3

In this subsection, we show that Theorem 1.5 leads to Theorem 1.3. In particular, Conjecture 1.2 is affirmatively solved for *plt* pairs (see Corollary 1.4). Corollary 1.4 is easily obtained from Theorem 1.3. Indeed, the Hermitian line bundle $(M, h_M) := (F^m, h_F^m)$ satisfies the assumption $\sqrt{-1}\Theta_{h_F}(F) \geq (1/m)\sqrt{-1}\Theta_{h_M}(M)$ in Theorem 1.3.

Proof of Theorem 1.3. Let $D = \sum_{i \in I} D_i$ be the irreducible decomposition of D. We remark that $D_i \cap D_j = \emptyset$ for $i \neq j$ since (X, D) is a *plt* pair. For every $i \in I$, we consider the long exact sequence induced by the standard short exact sequence:

$$\begin{array}{ccc}
\downarrow & & \downarrow \\
H^q(X, \mathcal{O}_X(K_X \otimes F \otimes \hat{D}_i)) & \xrightarrow{\otimes s} & H^q(X, \mathcal{O}_X(K_X \otimes F \otimes \hat{D}_i \otimes M)) \\
\downarrow \Phi_{D_i} & & \downarrow \\
H^q(X, \mathcal{O}_X(K_X \otimes D \otimes F)) & \xrightarrow{\otimes s} & H^q(X, \mathcal{O}_X(K_X \otimes D \otimes F \otimes M)) \\
\downarrow r_i & & \downarrow \\
H^q(D_i, \mathcal{O}_{D_i}(K_{D_i} \otimes F \otimes \hat{D}_i)) & \xrightarrow{f_i := \otimes s|_{D_i}} & H^q(D_i, \mathcal{O}_{D_i}(K_{D_i} \otimes F \otimes \hat{D}_i \otimes M)). \\
\downarrow & & \downarrow
\end{array} \tag{3.10}$$

Here $\hat{D}_i$ is the divisor defined by $\hat{D}_i := \sum_{k\in I, k\neq i} D_k$ and f_i is the multiplication map induced by the tensor product with the restriction $s|_{D_i}$ of s to D_i. Further Φ_{D_i} is the map induced by the natural inclusion $\mathcal{O}_X \hookrightarrow \mathcal{O}_X(D_i)$ and r_i is the map induced by the restriction map $\mathcal{O}_X \to \mathcal{O}_{D_i}$. Note that we used the adjunction formula $\mathcal{O}_{D_i}(K_X \otimes D_i) = \mathcal{O}_{D_i}(K_{D_i})$.

Remark 3.12. By the assumption $D_i \cap D_j = \emptyset$, we actually have $\mathcal{O}_{D_i}(K_{D_i} \otimes F \otimes \hat{D}_i) = \mathcal{O}_{D_i}(K_{D_i} \otimes F)$, but we used the notation $\mathcal{O}_{D_i}(K_{D_i} \otimes F \otimes \hat{D}_i)$ for Observation 3.15.

Let α be a cohomology class in $H^q(X, \mathcal{O}_X(K_X \otimes D \otimes F))$ such that

$$s\alpha = 0 \in H^q(X, \mathcal{O}_X(K_X \otimes D \otimes F \otimes M)).$$

The above commutative diagram implies that $f_i(r_i(\alpha)) = 0$. Note that we have $\mathcal{O}_{D_i}(K_{D_i} \otimes F \otimes \hat{D}_i) = \mathcal{O}_{D_i}(K_{D_i} \otimes F)$ by the assumption $D_i \cap D_j = \emptyset$. The restriction $\mathcal{O}_{D_i}(F)$ is a semi-positive line bundle on D_i since F is semi-positive, and further the restriction $s|_{D_i}$ is non-zero since the zero locus $s^{-1}(0)$ does not contain D_i by the assumption. In particular $\mathcal{O}_{D_i}(F)$ and $s|_{D_i}$ satisfy the assumptions of Enoki's injectivity theorem, and thus the multiplication map f_i is injective. Therefore we obtain $r_i(\alpha) = 0$ for every $i \in I$.

We have the following exact sequence:

$$\cdots \to H^q(D, K_X \otimes F) \xrightarrow{\Phi_D} H^q(X, K_X \otimes D \otimes F) \xrightarrow{r_D} H^q(X, \mathcal{O}_D(K_X \otimes D \otimes F)) \to \cdots,$$

where r_D is the map induced by the restriction map $\mathcal{O}_X \to \mathcal{O}_D$. On the other hand, we have

$$H^q(D, \mathcal{O}_D(K_X \otimes D \otimes F)) = \bigoplus_{i\in I} H^q(D_i, \mathcal{O}_{D_i}(K_{D_i} \otimes F))$$

by the assumption $D_i \cap D_j = \emptyset$. Then we can easily check $r_D(\alpha) = 0$ by the above exact sequence since we have $r_i(\alpha) = 0$ for every $i \in I$. Therefore Theorem 1.5 leads to the desired conclusion $\alpha = 0$ of Theorem 1.3. □

3.4. Open problems related to Conjecture 1.2

In this subsection, we give several open problems related to Conjecture 1.2.

We first consider a generalization of Theorem 1.6. For Conjecture 1.2, our formulation of Theorem 1.6 is enough, but it is an interesting problem to remove the technical assumption in Theorem 1.6. We remark that Problem 3.13 can be seen as a refinement of [6, Theorem 0.1].

Problem 3.13. Consider the same situation as in Theorem 1.6. Can we remove the assumption that h is smooth on a non-empty Zariski open set?

The following problem may give a strategy to solve Conjecture 1.2. By Theorem 1.5 and the proof of Theorem 1.3, we can see that if Problem 3.14 is affirmatively solved, we can prove Conjecture 3.14.

Problem 3.14. Let D be a simple normal crossing divisor on a compact Kähler manifold X and F be a semi-positive line bundle on X. Let s be a (holomorphic) section of $\mathcal{O}_D(F^m)$ restricted to the (possibly non-irreducible) variety D. Then, is the following multiplication map injective?

$$H^q(D, \mathcal{O}_D(K_X \otimes D \otimes F)) \xrightarrow{\otimes s} H^q\big(D, \mathcal{O}_D\big(K_X \otimes D \otimes F^{m+1}\big)\big).$$

Finally, in order to clarify what is needed for Conjecture 1.2, we attempt to prove Conjecture 1.2 by the induction on $n = \dim X$.

Observation 3.15 (Observation for Conjecture 1.2). In the case $D = 0$, Conjecture 1.2 is the same as Enoki's injectivity theorem, and thus we may assume that $D \neq 0$. When n is one, the conclusion of Conjecture 1.2 is obvious since $D \otimes F$ is ample. Hence we may assume that Conjecture 1.2 holds for compact Kähler manifolds of dimension $(n-1)$.

We consider the commutative diagram (3.10) in the proof of Theorem 1.3. We remark that the pair $(D_i, \hat{D}_i)$ is an lc pair. Since the zero locus $s^{-1}(0)$ contains no lc centers of (X, D), we can show that the restriction $s|_{D_i}$ contains no lc centers of $(D_i, \hat{D}_i)$. Further the restriction $\mathcal{O}_{D_i}(F)$ is a semi-positive line bundle on D_i. Therefore the multiplication map f_i in (3.10) is injective by the induction hypothesis.

For a cohomology class α in $H^q(X, \mathcal{O}_X(K_X \otimes D \otimes F))$ such that $s\alpha = 0 \in H^q(X, \mathcal{O}_X(K_X \otimes D \otimes F \otimes M))$, we have $f_i(r_i(\alpha)) = 0$. Then it follows that $r_i(\alpha) = 0$ for every $i \in I$ since f_i is injective. In the case of *plt* pairs, we have $D_i \cap D_j = \emptyset$ for $i \neq j$. Then we can obtain $r_D(\alpha) = 0$ from $r_i(\alpha) = 0$ (see the proof of Theorem 1.3). If we can show that $r_D(\alpha) = 0$ in the case of lc pairs, Conjecture 1.2 is affirmatively solved by Theorem 1.5. However we do not know whether we can conclude $r_D(\alpha) = 0$ from $r_i(\alpha) = 0$ in this case.

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