GROMOV HYPERBOLIC JOHN IS QUASIHYPERBOLIC JOHN I

QINGSHAN ZHOU* AND SAMINATHAN PONNUSAMY

Abstract. In this paper, we introduce the concept of quasihyperbolic John spaces and provide a necessary and sufficient condition for a space to be quasihyperbolic John. Using this criterion, we exhibit a simple proof to show that a John space with a Gromov hyperbolic quasihyperbolization is quasihyperbolic John, quantitatively. This answers in the affirmative to an open question proposed by Heinonen (Rev. Mat. Iberoam., 1989) and studied by Gehring et al. (Math. Scand., 1989). As a tool, we study the connection between quasihyperbolic John spaces and several geometric conditions, such as the uniformity, the linear local connectedness, the Gehring–Hayman condition, and the ball separation condition.

1. Introduction and Main Results

This paper focuses on geometric properties of quasihyperbolic geodesics in John spaces. In what follows, (X, d) is a locally compact, incomplete, and rectifiably connected metric space, $\partial X = \overline{X} \setminus X$ with $\overline{X}$ the completion of X is the metric boundary of X, and the identity map $(X, d) \to (X, \ell)$ is continuous, where ℓ is the length metric of X with respect to d. Following [1, 18, 24], we call such a space *minimally nice*. Note that every proper domain in a Euclidean space $\mathbb{R}^n$ is a minimally nice metric space.

A minimally nice space (X, d) is called *a-John* for a constant $a \geq 1$ if every pair of two points x, y in X can be joined by an arc α satisfying for each $z \in \alpha$ the condition that

$$\min\{\ell(\alpha[x, z]), \ell(\alpha[z, y])\} \leq a\, d(z),$$

where $d(z) = \operatorname{dist}(z, \partial X)$ and $\alpha[x, z]$ and $\alpha[z, y]$ denote the two subarcs of α between x and z, and z and y, respectively. The arc α is called a *double a-cone* arc. The class of Euclidean John domains was first considered by John [22] in the study of elasticity theory.

In the literature, a minimally nice space (X, d) is called *a-John with center x_0* if $a \geq 1$ is a constant and $x_0 \in X$ a distinguished point such that for all $x \in X$, we may join x to x_0 by an arc γ in X such that for all $z \in \gamma$, $\ell(\gamma[x, z]) \leq a\, d(z)$; the arc

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* Corresponding author.

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γ is called an *a-cone* arc. Note that the above definition ensures that X is bounded. Our version of definition of John spaces works well even if X is unbounded. Indeed, if X is bounded, then these two definitions are equivalent to each other not only for domains of Euclidean spaces, see [15, 17, 26], but also in the setting of metric spaces as follows.

(1) Suppose first that X is a-John with a center $x_0 \in X$. Fix a pair of points $x_1, x_2 \in X$. Choose an a-cone arc α_i joining x_i to x_0 in X for $i = 1, 2$. Let x_3 be the first point in α_1 from x_1 with $x_3 \in \alpha_2$. Then $\alpha = \alpha_1[x_1, x_3] \cup \alpha_2[x_3, x_2]$ is the desired arc.

(2) Next we suppose that X is a bounded a-John space with $a > 1$. Let $\lambda = (a+1)/2a$; then $0 < \lambda < 1$. Since X is bounded, we know that $0 < \kappa = \sup\{d(x) : x \in X\} < \infty$. Choose $x_0 \in X$ with $d(x_0) \geq \lambda\kappa$. Let $x \in X$ be a given point. Pick a double a-cone arc β in X connecting x and x_0. Let u be the arclength midpoint of β. For every $z \in \beta$, if $z \in \beta[x, u]$, then we see from the double cone condition that $\ell(\beta[x, z]) \leq ad(z)$. Suppose that $z \in \beta[x_0, u]$. Now $d(u) \leq \kappa \leq d(x_0)/\lambda \leq (a+1)d(z)/\lambda = 2ad(z)$, and thus $\ell(\beta) \leq 2ad(u) \leq 4a^2 d(z)$, as desired.

The quasihyperbolic metric was introduced by Gehring and Palka [14] in Euclidean setting and we refer to [13] for more geometric properties. In [25], Martio and Sarvas introduced the concept of uniform domains, that is, every pair of points can be connected by a uniform arc, i.e., a double cone arc satisfying the quasiconvexity condition. The importance of the quasihyperbolic metric and uniform domains in quasiconformal mapping theory is well understood, see for example [6, 8, 9, 11, 15, 16, 18, 20].

In [3], Bonk et al. investigated negative curvature of uniform metric spaces and demonstrated many phenomena in function theory from the point of view of Gromov hyperbolicity of the quasihyperbolic metric. It was shown in [3, Theorem 2.10] that every quasihyperbolic geodesic in a uniform space is a uniform arc. Note that the existence of a quasihyperbolic geodesic between any pair of points in a minimally space follows from [3, Proposition 2.8]. Since the uniformity of spaces implies the John property, there raises the natural problem of determining sufficient and/or necessary conditions (in John spaces) for quasihyperbolic geodesics to be double cone arcs.

This problem has been studied by several authors. For background and more information see [28]. For example, in the case of planar subdomains, Gehring et al. [12, Theorem 4.1] proved that simple connectedness is a sufficient condition for a John domain to have every quasihyperbolic geodesic ending at the center to be a cone arc. In the case of dimensions $n \geq 2$, Gromov hyperbolicity of the quasihyperbolic metric is also sufficient to assert that quasihyperbolic geodesics are double cone arcs. Using conformal moduli of path families and Ahlfors n-regularity of n-Lebesgue measure, this result can be obtained by [3, Proposition 7.12] because John domains in $\mathbb{R}^n$, $n \geq 2$, satisfy the linear local connectedness condition LLC_2; see the proof of [16, Lemma 3.5] or Proposition 3.21. For this result, a new proof may be obtained from [16, Proposition 3.7] and also from [28, Theorem 1.3], which uses geometric characterization of Gromov hyperbolicity that appeared in [1, 23, 24].

In this paper, we focus on this problem and introduce the following concept.

Definition 1.1. Let $a \geq 1$. A minimally nice space (X, d) is called *quasihyperbolic a-John*, if every quasihyperbolic geodesic α in X is a double a-cone arc. Moreover, X is called *quasihyperbolic John* if there is a constant $a \geq 1$ such that X is quasihyperbolic a-John.

Obviously, quasihyperbolic John spaces are John. There are John spaces which are not quasihyperbolic John, see [12, Examples 5]. It follows from [3, Theorem 2.10] that every uniform space is quasihyperbolic John. Indeed, we shall see that the class of quasihyperbolic John spaces is very wide.

In the following, we first present a complete solution to the above problem by providing a necessary and sufficient condition for spaces to be quasihyperbolic John.

Theorem 1.2. *A minimally nice space X is quasihyperbolic a-John if and only if there is a constant $A > 0$ such that for each quasihyperbolic geodesic $[x, y]_k$ satisfying $d(u) \leq 2\min\{d(x), d(y)\}$ for all $u \in [x, y]_k$, we have $k(x, y) \leq A$, where k is the quasihyperbolic metric of X. The constants a and A can be chosen to depend only on each other.*

Next we consider whether Theorem 1.2 can tell us that Gromov hyperbolic John spaces are quasihyperbolic John. Note that, in a minimally nice space (X, d), X is called Gromov δ-hyperbolic if it has a *Gromov hyperbolic quasihyperbolization.* That is to say, (X, k) is δ-hyperbolic for some constant $\delta \geq 0$, where k is the quasihyperbolic metric of X. The class of Gromov hyperbolic John spaces includes uniform domains and inner uniform domains in $\mathbb{R}^n$, simply connected John domains in the plane, etc.

In particular, it was asked by Heinonen in [17, Question 2] whether a John domain in $\mathbb{R}^n$ quasiconformally equivalent to the unit ball is quasihyperbolic John. Recently, the authors in [28, Theorem 1.1] obtained a dimension-free answer to this question by showing that every quasihyperbolic geodesic in a John space which admits a roughly starlike Gromov hyperbolic quasihyperbolization is a double cone arc. The approach in [28] is elementary and makes use of the uniformization theory of Gromov hyperbolic spaces established in [3]. With the aid of Theorem 1.2, we obtain the following:

Theorem 1.3. *Let $a \geq 1$ and $\delta \geq 0$. Every quasihyperbolic geodesic in a minimally nice a-John Gromov δ-hyperbolic space X is a double M-cone arc with $M = M(a, \delta)$. In particular, a John space with a Gromov hyperbolic quasihyperbolization is quasihyperbolic John.*

Remark 1.4. It follows from Theorem 1.3 that quasihyperbolic geodesics in John hyperbolic spaces are double cone arcs, quantitatively. Moreover, Theorem 1.3 is an improvement of [3, Proposition 7.12], [12, Theorem 4.1], and [16, Remark 3.10]. We remark that the related results of [3, 12, 16] were considered in bounded domains of Euclidean spaces $\mathbb{R}^n$ and the parameters depended on n. Indeed, their proofs may not hold without assuming the spaces to be Ahlfors regular. This is because we do not know whether the spaces/domains satisfy the ball separation condition in this setting.

On the other hand, the rough starlikeness is not required in Theorem 1.3 and so it gives an improvement of [28, Theorem 1.1]. Indeed, our proof is not only more direct but is also simpler than that of [28, Theorem 1.1]. An important observation is that every double cone arc is a union of two quasihyperbolic quasigeodesics. Using this observation and the stability property of quasigeodesics in Gromov hyperbolic spaces, one can easily verify the criterion for quasihyperbolic John spaces given in Theorem 1.2. See Lemma 3.11.

In view of the above discussions, Theorem 1.3 asserts that simply connected planar John domains and Gromov hyperbolic John domains in $\mathbb{R}^n$ are quasihyperbolic John. There are many applications of these domains in quasiconformal mapping theory and potential analysis. For example, Guo [16] proved that quasihyperbolic geodesics in Gromov hyperbolic generalized φ-John domains are φ-inner uniform curves in order to study the uniform continuity of quasiconformal mappings onto Gromov hyperbolic φ-John domains. Recently, Chen and Ponnusamy [11] established certain relationships between K-quasiconformal harmonic mappings and John disks.

Moreover, we study the connection between quasihyperbolic John spaces and other geometric conditions, such as the linear local connectedness, the Gehring–Hayman condition, and the ball separation condition. We begin with some definitions. Following [1] and [5], we say that (X, d) is a *GHS space*, if it is a minimally nice length space and it satisfies both the Gehring–Hayman condition and the ball separation condition, see Definitions 3.16 and 3.17. A minimally nice space (X, d) is *c-LLC_2*, if $c \geq 1$ and points in $X \setminus \overline{B}(x, r)$ can be joined by a curve in $X \setminus \overline{B}(x, r/c)$. In the following, we demonstrate that a GHS space satisfying the John or LLC_2 condition is quasihyperbolic John.

Theorem 1.5. *Let (X, d) be a minimally nice length space. If X is a* GHS *space satisfying the John or* LLC_2 *condition, then X is quasihyperbolic John.*

As applications of the above results, we investigate geometric characterizations of uniform metric spaces. Here is an immediate corollary to Theorem 1.5.

Corollary 1.6. *A minimally nice length space is uniform if and only if it is John and* GHS.

As a corollary to Theorem 1.3, we investigate the connection between uniform spaces and quasihyperbolic John spaces under the assumption of Ahlfors regularity.

Corollary 1.7. *Let (X, d, μ) be a minimally nice, Ahlfors Q-regular metric measure space with $Q > 1$. Then X is uniform if and only if it is quasiconvex, John, and has a roughly starlike Gromov hyperbolic quasihyperbolization.*

Using a similar argument as in the proof of [16, Proposition 3.7], one may get the sufficiency part of Corollary 1.7 provided the underlying space is locally externally connected (cf. [5, p. 202]); see Remark 4.1. Since not all of uniform spaces are locally externally connected, this assumption is unnecessary. Under the absence of this extra condition, the sufficiency part of Corollary 1.7 follows from Theorem 1.3 and the Gehring–Hayman condition (cf. [24, Theorem 5.1]); see Section 4.1.

In [1], Balogh and Buckley proved the equivalence of three different geometric properties of metric measure spaces, that is, roughly, Gromov hyperbolicity of the quasihyperbolic metric, the slice condition (for a precise definition see [1, p. 264]), and a combination of the Gehring–Hayman and ball separation conditions. So one may characterize uniform spaces in terms of the slice condition; for related discussions see for example [5, 6, 9, 10] and the references therein. Along this way, we now present a second application of Theorem 1.3.

Corollary 1.8. *A quasiconvex, John minimally nice metric space* (X, d) *satisfying the slice condition is uniform.*

Finally, as indicated in Theorem 1.3, every John space with a Gromov hyperbolic quasihyperbolization is quasihyperbolic John. For the converse, it is natural to consider the following question:

Question 1.9. Does every quasihyperbolic John length space have a Gromov hyperbolic quasihyperbolization?

Note that if a quasihyperbolic John length space satisfies the Gehring–Hayman condition, then it is uniform, and therefore admits a Gromov hyperbolic quasihyperbolization by [3, Theorem 3.6]. In other words, if there is a quasihyperbolic John length space that does not satisfy the Gehring–Hayman condition, then one obtains a counterexample to Question 1.9.

On the other hand, it is not hard to observe that quasihyperbolic John spaces satisfy the ball separation condition, see Proposition 3.21. In view of this result, Question 1.9 is related to a question pointed out by Balogh and Buckley in [1, p. 272]: For a minimally nice length space, we do not know whether the ball separation condition implies the Gehring–Hayman condition.

Thanks to [1, Theorem 6.1], if the answer to the above question arisen by Balogh and Buckley is positive, then so is to Question 1.9. Thus, if there is a counterexample to Question 1.9, then it answers negatively to the above question arisen by Balogh and Buckley. We also remark that the answers to the above two questions are unclear even in the situation of Euclidean domains.

This paper is organized as follows. Section 2 contains notations and the basic definitions. After establishing certain useful lemmas, we prove Theorems 1.2, 1.3, and 1.5 in Section 3. Section 4 is devoted to the proofs of Corollaries 1.7 and 1.8. Finally, we show two facts in metric geometry in Section 5, the Appendix.

2. Preliminaries

Let (X, d) denote a metric space with its metric completion $\overline{X}$ and its metric boundary $\partial X = \overline{X} \setminus X$. The space X is incomplete if ∂X is non-empty. For all $x \in X$, $d(x) = \mathrm{dist}(x, \partial X)$ if $\partial X \neq \emptyset$. The open (resp. closed) metric ball with center $x \in X$ and radius $r > 0$ is denoted by

$$B(x, r) = \{z \in X : \ d(z, x) < r\} \ \ (\text{resp. } \overline{B}(x, r) = \{z \in X : \ d(z, x) \leq r\}),$$

and the metric sphere by $S(x, r) = \{z \in X : \ d(z, x) = r\}$.

A *curve* in X is a continuous map $\gamma\colon\ I \to X$ of an interval $I \subset \mathbb{R}$ to X. If γ is an embedding of I, it is also called an *arc*. We also denote the image set $\gamma(I)$ of γ by γ. The *length* $\ell(\gamma)$ of γ with respect to the metric d is defined in an obvious way. Here the parameter interval I is allowed to be open or half-open. Suppose γ is a curve in X with endpoints x and y. We say that z is a *midpoint* of γ if $\ell(\gamma[x,z]) = \ell(\gamma[y,z])$.

A *geodesic* arc γ joining $x \in X$ to $y \in X$ is a continuous map γ from an interval $I = [0,l] \subset \mathbb{R}$ into X such that $\gamma(0) = x$, $\gamma(l) = y$ and

$$d(\gamma(t), \gamma(t')) = |t - t'| \quad \text{for all } t, t' \in I.$$

A metric space X is said to be *geodesic* if every pair of points can be joined by a geodesic arc. We use $[x,y]$ to denote a geodesic between two points x and y in X. Let $\lambda \geq 1$ and $\mu \geq 0$. A (λ,μ)-*quasigeodesic* curve in X is a (λ,μ)-quasi-isometric embedding $\gamma\colon I \to X$ of an interval $I \subset \mathbb{R}$. More explicitly,

$$\lambda^{-1}|t - t'| - \mu \leq d(\gamma(t), \gamma(t')) \leq \lambda|t - t'| + \mu \quad \text{for all } t, t' \in I.$$

A metric space (X,d) is called *rectifiably connected* if every pair of points in X can be joined with a curve γ in X with $\ell(\gamma) < \infty$. In a minimally nice space (X,d), the *quasihyperbolic metric* k in X is defined by

$$k(x,y) = \inf\left\{\int_\gamma \frac{ds}{d(z)}\right\}$$

where the infimum is taken over all rectifiable curves γ in X with endpoints x and y and ds denotes the arc length element with respect to the metric d. We remark that the resulting space (X,k) is complete, proper, and geodesic, and that k is topologically compatible (cf. [3]). Moreover, we need the following well-known inequalities (cf. [3, 5]): for all $x, y \in X$,

$$\left|\log \frac{d(x)}{d(y)}\right| \leq k(x,y) \tag{2.1}$$

and

$$\log\left(1 + \frac{\ell(\gamma)}{\min\{d(x), d(y)\}}\right) \leq \ell_k(\gamma) = \int_\gamma \frac{ds}{d(z)}, \tag{2.2}$$

where γ is a curve in X with endpoints $x, y \in \gamma$ and $\ell_k(\gamma)$ is the quasihyperbolic length of γ.

Let $\delta \geq 0$. A geodesic space (X,d) is called (*Gromov*) δ-*hyperbolic* if for each triplet of geodesics arcs $[x,y], [y,z], [z,x]$ in (X,d), every point in $[x,y]$ is within distance δ from $[y,z] \cup [z,x]$. Let X be a proper geodesic δ-hyperbolic space, $w \in X$ and $K \geq 0$. We say that X is K-*roughly starlike* with respect to w if for each $x \in X$ there is a point ξ on the Gromov boundary $\partial_\infty X$ of the δ-hyperbolic space X such that for the geodesic ray $\gamma = [w,\xi]$ from w ending at ξ (see [3, p. 18]), we have $\operatorname{dist}(x,\gamma) \leq K$.

We now recall the stability property of quasigeodesics in a δ-hyperbolic space; see [4, Chapter III.H Theorem 1.7].

Theorem 2.3. *For all $\delta \geq 0, \lambda \geq 1, \mu \geq 0$, there is a number $R = R(\delta,\lambda,\mu)$ with the following property: If X is a δ-hyperbolic space, and if γ is a (λ,μ)-quasigeodesic* *in X and $[p,q]$ is a geodesic joining the endpoints of γ, then the Hausdorff distance between $[p,q]$ and γ is less than R.*

Let $a \geq 1$. A metric space X is called *a-quasiconvex* if each pair of points $x, y \in X$ can be joined by an a-quasiconvex curve α, that is, for which $\ell(\alpha) \leq a\, d(x,y)$. A minimally nice space X is called *a-uniform* if each pair of points in X can be joined by a double a-cone and a-quasiconvex arc.

3. Proofs of Main results

3.1. **A criterion for quasihyperbolic John spaces.** In this subsection, we study geometric properties of quasihyperbolic geodesics in quasihyperbolic John spaces. The aim of this part is to show Theorem 1.2, which provides a criterion for quasihyperbolic John spaces. Before the proof, we need an auxiliary lemma which asserts that every double cone arc is a union of two quasihyperbolic quasigeodesics.

Lemma 3.1. *Let (X,d) be a minimally nice a-John space, let γ be a double a-cone arc with endpoints x_1 and x_2 in X, and let x_0 be the midpoint of γ. Then for any $y \in \gamma[x_1, x_0]$ and $z \in \gamma[y, x_0]$, we have*

$$k(y,z) \leq \ell_k(\gamma[y,z]) \leq 3a \log\left(1 + \frac{ad(z)}{d(y)}\right) \leq 3ak(y,z) + 3a\log 3a. \tag{3.2}$$

Moreover, the subarcs $\gamma[x_1, x_0]$ and $\gamma[x_0, x_2]$, when parametrized by the quasihyperbolic arc length, are both quasihyperbolic (λ, μ)- quasigeodesics with $\lambda = 3a$ and $\mu = \log 3a$.

Proof. We only need to verify (3.2), because it implies that the subarc $\gamma[x_1, x_0]$ is evidently a quasihyperbolic quasigeodesic; and by using a symmetric argument, one easily sees that the other arc $\gamma[x_0, x_2]$ shares the same property.

Towards this end, since γ is a double a-cone arc and x_0 is the midpoint of γ, for any $u \in \gamma[y,z]$, we have

$$\ell(\gamma[y,u]) \leq ad(u). \tag{3.3}$$

Also, we claim that

$$d(y) \leq 2ad(u). \tag{3.4}$$

Indeed, if $u \in B(y, d(y)/2)$, then we get $d(u) \geq d(y) - d(y,u) \geq d(y)/2$. Otherwise, by (3.3) we have

$$ad(u) \geq \ell(\gamma[y,u]) \geq d(y,u) \geq d(y)/2,$$

as required.

Now, by (3.3) and (3.4), we obtain $\ell(\gamma[y,u]) + d(y) \le 3ad(u)$. Therefore,

$$\begin{aligned} k(y,z) &\le \ell_k(\gamma[y,z]) = \int_{\gamma[y,z]} \frac{|du|}{d(u)} \\ &\le \int_0^{\ell(\gamma[y,z])} \frac{3adt}{t+d(y)} = 3a\log\left(1+\frac{\ell(\gamma[y,z])}{d(y)}\right) \\ &\le 3a\log\left(1+\frac{ad(z)}{d(y)}\right) \\ &\le 3a\log\frac{d(z)}{d(y)} + 3a\log 3a \\ &\le 3ak(y,z) + 3a\log 3a, \end{aligned}$$

and the lemma follows. □

3.2. **Proof of Theorem 1.2.** We first prove the necessity. Suppose that X is quasihyperbolic a-John with constant $a \ge 1$. Fix $x, y \in X$ and let $[x,y]_k = \alpha$ be a quasihyperbolic geodesic connecting x and y and satisfying $d(u) \le 2\min\{d(x), d(y)\}$ for all $u \in [x,y]_k$. Let x_0 be the midpoint of α. By Lemma 3.1, we have

$$k(x,x_0) \le 3a\log\left(1+\frac{ad(x_0)}{d(x)}\right) \le 3a\log(1+2a)$$

and similarly, $k(x_0,y) \le 3a\log(1+2a)$. These two estimates imply that $k(x,y) \le 6a\log(1+2a) =: A$, as desired.

We are thus left to show the sufficiency. Let $[p,q]_k$ be a quasihyperbolic geodesic in X with endpoints p and q. We show that $[p,q]_k$ is a double M-cone arc with $M = M(A)$. Let $w_0 \in [p,q]_k$ be a point such that

$$d(w_0) = \max\{d(z): \ z \in [p,q]_k\} =: T.$$

There is a unique nonnegative integer n such that

$$2^n d(p) \le T < 2^{n+1} d(p).$$

For each $i = 0, \ldots, n$, let p_i be the first point on $[p,q]_k$ with

$$d(p_i) = 2^i d(p), \tag{3.5}$$

when travelling from p to q.

Similarly, we define q_j to be the first point on $[p,q]_k$ with

$$d(q_j) = 2^j d(q)$$

for $j = 0, \ldots, m$, when travelling from q to p. Here m is the unique nonnegative integer such that

$$2^m d(q) \le T < 2^{m+1} d(q).$$

Then $p = p_0$ and $q = q_0$. Thus we note that the curve $[p,q]_k$ has been divided into $n + m + 1$ non-overlapping (modulo end points) subcurves:

$$[p_0,p_1]_k, \ldots, [p_{n-1},p_n]_k,\ [p_n,q_m]_k, [q_m,q_{m-1}]_k, \ldots, [q_1,q_0]_k.$$

Moreover, all subcurves in the above are also quasihyperbolic geodesics between their respective endpoints, and for any one of the above, denoted by $[x,y]_k$, we have

$$d(u) \leq 2\min\{d(x), d(y)\} \quad \text{for all } u \in [x,y]_k. \tag{3.6}$$

Then for every $0 \leq i \leq n-1$, by (3.6), we may apply the assumption to the subarc $[p_i, p_{i+1}]_k$ and obtain by (2.2) that

$$\log\left(1+\frac{\ell([p_i,p_{i+1}]_k)}{d(p_i)}\right) \leq k(p_i,p_{i+1}) \leq A, \tag{3.7}$$

which ensures that

$$\ell([p_i,p_{i+1}]_k) \leq e^A d(p_i). \tag{3.8}$$

Similarly, for each $0 \leq j \leq m-1$, we get

$$\ell([q_{j+1},q_j]_k) \leq e^A d(q_j) \quad \text{and} \quad \ell([p_n,q_m]_k) \leq e^A d(p_n). \tag{3.9}$$

We are now in a position to complete the proof of Theorem 1.2. For each $v \in [p,q]_k$, we have three cases: $v \in [p_i,p_{i+1}]_k$ or $[p_n,q_m]_k$ or $[q_{j+1},q_j]_k$ for some $0 \leq i \leq n-1$ and $0 \leq j \leq m-1$. It is enough to consider the former two cases because the last one follows from a similar argument as the first case.

On the one hand, if $v \in [p_i,p_{i+1}]_k$ for some $0 \leq i \leq n-1$, then by (2.1) and (3.7) we have

$$\left|\log\frac{d(v)}{d(p_i)}\right| \leq k(p_i,v) \leq k(p_i,p_{i+1}) \leq A$$

and thus,

$$d(p_i) \leq e^A d(v). \tag{3.10}$$

This, together with (3.5) and (3.8), shows that

$$\ell([p,v]_k) \leq \sum_{s=0}^{i} \ell([p_s,p_{s+1}]_k) \leq e^A \sum_{s=0}^{i} d(p_s) \leq 2e^A d(p_i) \leq 2e^{2A} d(v),$$

as desired.

On the other hand, if $v \in [p_n,q_m]_k$, then we easily see from a similar argument as (3.10) that $d(p_n) \leq e^A d(v)$. Using this, (3.8) and (3.9), we obtain

$$\begin{aligned}
\ell([p,v]_k) &\leq \sum_{s=0}^{n-1} \ell([p_s,p_{s+1}]_k) + \ell([p_n,q_m]_k)\\
&\leq e^A d(p_n) + e^A d(p_n)\\
&\leq 2e^{2A} d(v).
\end{aligned}$$

Therefore, we show that the quasihyperbolic geodesic $[p,q]_k$ is a double M-cone arc with $M = 2e^{2A}$. This completes the proof of Theorem 1.2. □

3.3. **Gromov hyperbolic John spaces are quasihyperbolic John.** In this part, we prove Theorem 1.3 with the aid of Theorem 1.2. Together with Lemma 3.1 and the stability property of quasigeodesics in Gromov hyperbolic spaces, we verify the criterion for quasihyperbolic John spaces.

Lemma 3.11. *Let (X, d) be a minimally nice a-John Gromov δ-hyperbolic space, and let $[x, y]_k$ be a quasihyperbolic geodesic in X connecting x and y. If $d(u) \le 2\min\{d(x), d(y)\}$ for all $u \in [x, y]_k$, then there is a positive number A such that $k(x, y) \le A$ with A depending only on a and δ.*

Proof. First, take a double a-cone arc γ connecting x and y in X and let x_0 be the midpoint of γ. Then pick another two quasihyperbolic geodesics $[x, x_0]_k$ and $[y, x_0]_k$ joining x_0 to x and y, respectively. Consider the quasihyperbolic geodesic triangle

$$\Delta = [x_0, x]_k \cup [x, y]_k \cup [y, x_0]_k.$$

There is a tripod map $f\colon \Delta \to T$ where T is a tripod consisting of three line segments in $\mathbb{R}^2$ satisfying the following two properties: f is an isometry on each of the sides of Δ and $f(u) = f(v)$ implies $k(u, v) \le 4\delta$, see [2, Lemma 1.3]. Let $u \in [x, y]_k$, $v \in [x, x_0]_k$, and $w \in [x_0, y]_k$ be the points of the quasihyperbolic geodesic triangle Δ whose image under f is the origin of T. Thus we have

$$\max\{k(u, v), k(u, w)\} \le 4\delta. \tag{3.12}$$

Without loss of generality, we may assume that $k(x, u) \ge k(u, y)$. Thus

$$k(x, y) \le 2k(x, u). \tag{3.13}$$

Next, by Lemma 3.1, one immediately sees that $\gamma[x, x_0]$ is a quasihyperbolic (λ, μ)-quasigeodesic with $\lambda = 3a$ and $\mu = \log 3a$. Moreover, according to Theorem 2.3, there is a point $v_0 \in \gamma[x, x_0]$ and a number $R = R(\lambda, \mu, \delta) = R(a, \delta)$ such that

$$k(v_0, v) \le R.$$

This, together with (2.1) and (3.12), guarantees that

$$\left|\log \frac{d(v_0)}{d(u)}\right| \le k(v_0, u) \le k(v_0, v) + k(v, u) \le R + 4\delta, \tag{3.14}$$

and so

$$d(v_0) \le e^{R+4\delta} d(u) \le 2e^{R+4\delta} d(x),$$

where the last inequality follows from the assumption.

Moreover, by (3.13) and (3.14), a second application of Lemma 3.1 gives

$$\begin{aligned} k(x, y) &\le 2k(x, u) \le 2k(x, v_0) + 2k(v_0, u) \\ &\le 6a \log\left(1 + \frac{a d(v_0)}{d(x)}\right) + 2R + 8\delta \\ &\le 6a \log\left(1 + 2ae^{R+4\delta}\right) + 2R + 8\delta =: A, \end{aligned}$$

and we complete the proof of Lemma 3.11. □

3.4. **Proof of Theorem 1.3.** By Lemma 3.11 and Theorem 1.2, Theorem 1.3 follows immediately and one finds that every Gromov hyperbolic John space is quasihyperbolic John. □

3.5. **Quasihyperbolic John and GHS spaces.** In this subsection, we investigate the relationships between GHS spaces, linearly locally connected spaces, and quasihyperbolic John spaces. Our goal is to prove Theorem 1.5. We start with several definitions.

Definition 3.15. Let (X, d, μ) be a metric measure space and $Q > 0$. We say that X is *Ahlfors Q-regular* if there is a constant $C_\mu \geq 1$ such that for each $x \in X$ and $0 < r \leq \operatorname{diam}(X)$, we have $C_\mu^{-1} r^Q \leq \mu(B(x, r)) \leq C_\mu r^Q$.

Definition 3.16. Let (X, d) be a minimally nice space and $C_{gh} \geq 1$ be a constant. We say that X satisfies the *C_{gh}-Gehring–Hayman condition*, if for all x, y in X and for each quasihyperbolic geodesic γ joining x and y, we have $\ell(\gamma) \leq C_{gh}\ell(\beta_{x,y})$ whenever $\beta_{x,y}$ is a curve joining x and y in X.

Definition 3.17. Let (X, d) be a minimally nice space and $C_{bs} \geq 1$ be a constant. We say that X satisfies the *C_{bs}-ball separation condition*, if for all x, y in X, for each quasihyperbolic geodesic γ joining x and y, and for every $z \in \gamma$,

$$B(z, C_{bs}d(z)) \cap \beta_{x,y} \neq \emptyset$$

whenever $\beta_{x,y}$ is a curve joining x and y in X.

Next, we introduce some facts about metric geometry of minimally nice spaces, see [3, 19]. Let (X, d) be a minimally nice space. The *length metric* $\ell(x, y)$ of (X, d) is defined as the infimum of the lengths $\ell(\gamma)$ of all rectifiable curves γ in X with endpoints x and y. The space (X, d) is called a *length space* if $d = \ell$.

Recall that a GHS space is a length space satisfying both the Gehring–Hayman and ball separation conditions defined as above. We first prove that a GHS space satisfying the LLC_2 condition is quasihyperbolic John.

Proposition 3.18. *Let (X, d) be a minimally nice length space. If (X, d) is* GHS *and C_0-*LLC_2*, then X is quasihyperbolic John.*

Proof. For each pair of points $x, y \in X$, for any quasihyperbolic geodesic $[x, y]_k = \alpha$ joining x to y, and for each $z \in \alpha$, we need to find an upper bound for the constant C such that

$$\min\{\ell(\alpha[x, z]), \ell(\alpha[z, y])\} = Cd(z).$$

Since (X, d) satisfies the C_{gh}-Gehring–Hayman condition, we have

$$\ell(\alpha[x, z]) \leq C_{gh}d(x, z) \quad \text{and} \quad \ell(\alpha[y, z]) \leq C_{gh}d(y, z),$$

because both the subarcs $\alpha[x, z]$ and $\alpha[z, y]$ are quasihyperbolic geodesics. This in turn implies that

$$\min\{d(x, z), d(z, y)\} \geq \frac{C}{C_{gh}}d(z)$$

and so, if $\lambda \in (0,1)$,

$$x, y \in X \setminus \overline{B}\Big(z, \frac{\lambda C}{C_{gh}} d(z)\Big).$$

Now the C_0-LLC$_2$ property of (X,d) ensures that there is a curve γ joining x to y such that

$$\gamma \subset X \setminus \overline{B}\Big(z, \frac{\lambda C}{C_0 C_{gh}} d(z)\Big).$$

Moreover, since α is a quasihyperbolic geodesic and (X,d) satisfies the C_{bs}-ball separation condition, it follows that

$$\gamma \cap B(z, C_{bs} d(z)) \neq \emptyset.$$

This, together with the above fact, shows that $\lambda C < C_0 C_{gh} C_{bs}$, and thus

$$C \le C_0 C_{gh} C_{bs},$$

as required. The proof of Proposition 3.18 is complete. $\square$

We now derive a couple of corollaries to the above result. The first is an analogue of [5, Theorem 4.2]. Recall that (X,d) is *c-locally externally connected*, abbreviated c-LEC, provided $c \ge 1$ and the c-LLC$_2$ property holds for all points $x \in X$ and for all $r \in (0, d(x)/c)$. Recall also that a domain in $\mathbb{R}^n$, $n \ge 2$, is c-LEC for all $c > 1$. See [5, p. 202].

Corollary 3.19. *A minimally nice space (X,d) is uniform and* LEC *if and only if (X,d) is quasiconvex and (X,ℓ) is* LLC$_2$ *and* GHS, *where ℓ denotes the length metric of X.*

Proof. As (X,d) is a minimally nice space, (X,d) is locally compact, incomplete, and rectifiably connected; moreover, if $\ell = \ell_d$ is the length metric of (X,d), then the identity map $\mathrm{id}\colon (X,d) \to (X,\ell)$ is continuous and thus a homeomorphism. Hence, (X,ℓ) is locally compact. By [3, Proposition A.7], $\ell_d(\gamma) = \ell_{\ell_d}(\gamma)$ for the lengths of a curve γ in (X,d) and in (X,ℓ), respectively. It follows that $\ell_d(x,y) = \ell_{\ell_d}(x,y)$ for all $x, y \in X$. Thus, (X,ℓ) is a rectifiably connected length space.

We may always assume that (X,d) is a-quasiconvex for some $a \ge 1$, because this is the case if (X,d) is a-uniform. Then $d \le \ell \le ad$. Thus, $\mathrm{id}\colon (X,d) \to (X,\ell)$ is a-bi-Lipschitz. Hence, (X,ℓ) is incomplete. It follows, as $\mathrm{id}\colon (X,\ell_d) \to (X,\ell_{\ell_d})$ is isometric, that (X,ℓ) is minimally nice.

Considering the a-bi-Lipschitz homeomorphism $\mathrm{id}\colon \overline{(X,d)} \to \overline{(X,\ell)}$ we have $d(x) \le \ell(x) \le ad(x)$ for $x \in X$, $d(x) = \mathrm{dist}_d(x, \overline{X} \setminus X)$, and $\ell(x) = \mathrm{dist}_\ell(x, \overline{X} \setminus X)$.

Proof for "if": Suppose that (X,ℓ) is c-LLC$_2$ for some $c \ge 1$, that (X,ℓ) satisfies the C_{gh}-Gehring–Hayman condition for some $C_{gh} \ge 1$, and that (X,ℓ) satisfies the C_{bs}-ball separation condition for some $C_{bs} \ge 1$. Proposition 3.18 implies that (X,ℓ) is quasihyperbolic A-John for some $A \ge 1$.

To show that (X,d) is uniform, let $x, y \in X$. Now there is a quasihyperbolic geodesic α in (X,ℓ) joining x and y. As (X,ℓ) is quasihyperbolic A-John, α is a double A-cone arc in (X,ℓ). Hence, $\min\{\ell(\alpha[x,z]), \ell(\alpha[z,y])\} \le A\ell(z) \le Aa\, d(z)$ if $z \in \alpha$. Choose a curve β in X joining x and y such that $\ell(\beta) \le ad(x,y)$. Then

$\ell(\alpha) \leq C_{gh}\ell(\beta) \leq C_{gh}ad(x,y)$. Therefore, α is a B-uniform arc in (X,d) with $B = \max\{Aa, C_{gh}a\}$. Thus (X,d) is B-uniform.

To show that (X,d) is LEC, let $x \in X$ and $r > 0$. Consider $y,z \in X \setminus \overline{B}_d(x,r)$. Then $y,z \in X \setminus \overline{B}_\ell(x,r)$. As (X,ℓ) is c-LLC$_2$, there is a curve $\sigma \subset X \setminus \overline{B}_\ell(x,r/c)$ joining y and z. Now $\sigma \subset X \setminus \overline{B}_d(x,r/ac)$. Thus, (X,d) is ac-LLC$_2$ and hence ac-LEC.

Proof for "only if": Suppose that (X,d) is a-uniform for some $a \geq 1$ and b-LEC for some $b \geq 1$. Then (X,d) is a-quasiconvex.

It is shown on [5, p. 204] that (X,d) is c-LLC$_2$ with $c = 1 + 2ab^2$ (in $\overline{B}(z;s)$ replace z by u). If $x \in X$ and $r > 0$, then $X \setminus \overline{B}_\ell(x,r) \subset X \setminus \overline{B}_d(x,r/a)$ and $X \setminus \overline{B}_d(x,r/ac) \subset X \setminus \overline{B}_\ell(x,r/ac)$. It follows that (X,ℓ) is ac-LLC$_2$.

To establish that (X,ℓ) is a GHS space, let $x,y \in X$, let γ be a quasihyperbolic geodesic in (X,ℓ) joining x and y, let $z \in \gamma$, and let β be a curve in X joining x and y. One first observes from Lemma 5.1 in the Appendix that the completion $(\overline{X},d)$ of X is also a-quasiconvex. The completion of the length metric ℓ of (X,d), also denoted by ℓ, is defined on $\overline{X}$ with $d \leq \ell \leq ad$ on $\overline{X}$. As then $d(u) < \ell(u) < ad(u)$ for $u \in X$, we thus know from [21, Lemma 3.10] that the identity map $\mathrm{id}\colon (X,k_d) \to (X,k_\ell)$ is a-bi-Lipschitz; see also [28, Lemma 4.8]. This ensures that γ is actually a quasihyperbolic $(a,0)$-quasigeodesic in (X,d) joining x and y. Since (X,d) is a-uniform, performing a slight modification to the proof of [3, Theorem 2.10] we know that there is $A \geq 1$ depending only on a such that γ is an A-uniform arc in (X,d); see Lemma 5.2 in the Appendix or [5, Remark 3.2(c)]. Thus

$$\ell(\gamma) \leq Ad(x,y) \leq A\ell(\beta).$$

Hence (X,ℓ) satisfies the A-Gehring–Hayman condition. Choose $B > A$ depending only on A. We may assume that $\ell(\gamma[x,z]) \leq \ell(\gamma[z,y])$. Then $\ell(x,z) \leq \ell(\gamma[x,z]) \leq Ad(z) \leq A\ell(z) < B\ell(z)$. Thus $x \in B_\ell(z, B\ell(z)) \cap \beta$. Hence (X,ℓ) satisfies the B-ball separation condition. □

The second is an analogue of [3, Proposition 7.12] in the setting of metric spaces.

Corollary 3.20. *Let (X,d,μ) be a minimally nice Ahlfors Q-regular length metric measure space with $Q > 1$. If (X,d) is* LLC$_2$ *and if (X,k) is a roughly starlike, δ-hyperbolic space, then (X,d) is uniform.*

Proof. Applying [24, Theorem 5.1], one finds that (X,d) is a GHS space. Moreover, by Proposition 3.18, we know that every quasihyperbolic geodesic in X is a double cone arc. This last fact and the fact that (X,d) satisfies the Gehring–Hayman condition yield that (X,d) is uniform, as required. □

The following result which may be of independent interest.

Proposition 3.21. *Let (X,d) be a minimally nice space.*

(1) If X is quasihyperbolic a-John for some $a \geq 1$, then it satisfies the $2a$-ball separation condition.

*(2) If X is a-John and b-*LEC *for some $a,b \geq 1$, then it is c-*LLC$_2$ *with $c = ab^2 + 2ab + 2b$.*

Proof. (1): For each pair of points $x, y \in X$, for every quasihyperbolic geodesic $\alpha = [x, y]_k$ joining x and y, and for all $z \in \alpha$, we have

$$\min\{d(x,z), d(z,y)\} \le \min\{\ell(\alpha[x,z]), \ell(\alpha[z,y])\} \le ad(z) < 2ad(z),$$

because α is a double a-cone arc. This implies that

$$\{x, y\} \cap B(z, 2ad(z)) \neq \emptyset,$$

which gives the first assertion.

(2): To show that (X, d) is c-LLC$_2$, let $x_0 \in X$ and $y, z \in X \setminus \overline{B}(x_0, r)$. Pick a double a-cone arc α joining y and z in X. Then for all $x \in \alpha$, we have

$$\min\{d(x,y), d(x,z)\} \le \min\{\ell(\alpha[x,y]), \ell(\alpha[x,z])\} \le ad(x). \tag{3.22}$$

We consider two cases. If $\alpha \cap S(x_0, 2br/c) = \emptyset$, then there is nothing to prove as $2br/c < r$. If $\alpha \cap S(x_0, 2br/c) \neq \emptyset$, then there are two points $y_1, z_1 \in \alpha \cap S(x_0, 2br/c)$ (perhaps $y_1 = z_1$) such that both $\alpha[y, y_1]$ and $\alpha[z_1, z]$ lie outside of $B(x_0, 2br/c)$. Thus, by (3.22) we get

$$ad(y_1) \ge \min\{d(y_1, y), d(y_1, z)\} \ge \min\{d(x_0, y), d(x_0, z)\} - d(x_0, y_1) > r - 2br/c,$$

which implies that

$$d(x_0) \ge d(y_1) - d(x_0, y_1) > \frac{r - 2br/c}{a} - 2br/c = b^2 r/c,$$

where the last inequality follows from our choice of the constant c. Thus $br/c < d(x_0)/b$.

Applying the LEC condition, there is a curve $\beta \subset X \setminus \overline{B}(x_0, r/c)$ joining y_1 and z_1. Therefore, we get a curve

$$\gamma := \alpha[y, y_1] \cup \beta \cup \alpha[z_1, z],$$

which connects y and z in $X \setminus \overline{B}(x_0, r/c)$. □

Finally, we shall show that GHS John spaces are also quasihyperbolic John. By [1, Theorem 6.1], we see that a GHS space is Gromov hyperbolic with respect to its quasihyperbolic metric. From this fact and Theorem 1.3, it follows that GHS John spaces are quasihyperbolic John. In the following, we give a direct proof for this conclusion.

Proposition 3.23. *Let (X, d) be a minimally nice length space. If (X, d) is* GHS *and a-John, then X is quasihyperbolic John.*

Proof. Fix $x, y \in X$ and take a quasihyperbolic geodesic $\alpha = [x, y]_k$ connecting x to y. Since X is a-John, there is a double a-cone arc γ joining x and y. Because (X, d) satisfies the C_{bs}-ball separation condition, for each $z \in \alpha$, there is a point $z_0 \in \gamma$ such that

$$d(z, z_0) \le C_{bs} d(z), \tag{3.24}$$

which implies

$$d(z_0) \le d(z, z_0) + d(z) \le (1 + C_{bs}) d(z).$$

Moreover, since γ is a double a-cone arc, we have

$$\min\{\ell(\gamma[x,z_0]),\ell(\gamma[z_0,y])\}\leq ad(z_0)\leq a(1+C_{bs})d(z).$$

This combined with (3.24), ensures that

$$\begin{aligned}\min\{d(x,z),d(z,y)\} &\leq d(z,z_0)+\min\{\ell(\gamma[x,z_0]),\ell(\gamma[z_0,y])\}\\ &\leq (a+1)(1+C_{bs})d(z).\end{aligned}\tag{3.25}$$

On the other hand, since (X,d) satisfies the C_{gh}-Gehring–Hayman inequality, it follows that

$$\ell(\alpha[x,z])\leq C_{gh}d(x,z)\quad\text{and}\quad \ell(\alpha[y,z])\leq C_{gh}d(y,z),$$

because both the subarcs $\alpha[x,z]$ and $\alpha[z,y]$ are quasihyperbolic geodesics. Therefore, by (3.25) we obtain

$$\min\{\ell(\alpha[x,z]),\ell(\alpha[z,y])\}\leq C_{gh}(a+1)(1+C_{bs})d(z).$$

Proposition 3.23 follows. □

3.6. **Proof of Theorem 1.5.** Theorem 1.5 follows from Propositions 3.18 and 3.23. □

4. Proofs of Corollaries 1.7 and 1.8

4.1. **Proof of Corollary 1.7.** Let (X,d,μ) be a minimally nice Ahlfors Q-regular metric measure space with $Q>1$. To prove the necessity, we assume that X is an a-uniform space. By definition, the quasiconvexity and John properties follow immediately. Thanks to [3, Theorem 3.6], (X,k) is a proper geodesic Gromov hyperbolic space, where k is the quasihyperbolic metric of X. It remains to show the roughly starlikeness of (X,k). If X is bounded, then this again follows from [3, Theorem 3.6].

We next consider the case that X is unbounded. Take a base point $p\in\partial X$. Let $(X,\widehat{d_p})$ be the sphericalized space of (X,d) with respect to p; then $\mathrm{id}\colon (X,d)\to(X,\widehat{d_p})$ is a homeomorphism. For the definition see [7, Section 3.B]. According to [7, Theorem 5.5], we know that $(X,\widehat{d_p})$ is a bounded B-uniform metric space with $B=B(a)$ because (X,d) is unbounded. Arguing as in the former case, we see that $(X,\widehat{k_p})$ is a proper geodesic Gromov hyperbolic metric space and K-roughly starlike with respect to a point $w_0\in X$ for some $K=K(a)\geq 0$, where $\widehat{k_p}$ is the quasihyperbolic metric of X with respect to $\widehat{d_p}$.

For any $x\in X$, there is a $\widehat{\xi}$ on the Gromov boundary $\partial_\infty(X,\widehat{k_p})$ of the hyperbolic space $(X,\widehat{k_p})$ and a $\widehat{k_p}$-quasihyperbolic geodesic ray $[w_0,\widehat{\xi}]_{\widehat{k_p}}$ emanating from w_0 to $\widehat{\xi}$ such that

$$\widehat{k_p}(x,[w_0,\widehat{\xi}]_{\widehat{k_p}})\leq K.$$

Moreover, [7, Theorem 4.11] ensures that the identity map $(X,k)\to(X,\widehat{k_p})$ is $80a$-bi-Lipschitz. It follows from [27, Theorem 5.35] that the identity map $(X,k)\to(X,\widehat{k_p})$ induces a homeomorphism $\varphi\colon\partial_\infty(X,k)\to\partial_\infty(X,\widehat{k_p})$ between the Gromov boundaries. Let ξ be a point on the Gromov hyperbolic boundary $\partial_\infty(X,k)$ of the hyperbolic space (X,k) with $\varphi(\xi)=\widehat{\xi}$. This guarantees that $[w_0,\widehat{\xi}]_{\widehat{k_p}}$ is a $(80a,0)$-quasigeodesic ray in the space (X,k) from w_0 ending at ξ. Then take the quasihyperbolic geodesic ray $[w_0,\xi]_k$ in the space (X,k) connecting w_0 and ξ. By the extended stability theorem (cf. [27, Theorem 6.32]) of Gromov hyperbolic spaces, there is a positive number $R=R(a)$ such that the Hausdorff distance between the two curves $[w_0,\xi]_k$ and $[w_0,\widehat{\xi}]_{\widehat{k_p}}$ with respect to the quasihyperbolic metric k is bounded above by R. Therefore, we obtain

$$k(x,[w_0,\xi]_k)\le 80aK+R.$$

Hence we are done and the necessity follows.

For the sufficiency, we assume that (X,d) is c-quasiconvex and a-John and that (X,k) is K-roughly starlike Gromov δ-hyperbolic. To show that (X,d) is uniform, let $x,y\in X$ be a pair of points. Choose a quasihyperbolic geodesic γ joining x and y. On one hand, as (X,d) is a-John and (X,k) is δ-hyperbolic, Theorem 1.3 ensures that the quasihyperbolic geodesic γ is a double M-cone arc with M depending only on a and δ.

On the other hand, because (X,d,μ) is a c-quasiconvex and Ahlfors Q-regular metric measure space with $Q>1$ and (X,k) is K-roughly starlike Gromov δ-hyperbolic, it follows from [24, Theorem 5.1] that there is a constant $C_{gh}\ge 1$ such that (X,d) satisfies the C_{gh}-Gehring–Hayman condition, where we can choose C_{gh} to depend only on c,K,δ,Q, the coefficient C_μ for Ahlfors regularity, and (possibly) the chosen base point w of X. As (X,d) is c-quasiconvex, there is a curve β in X joining x and y such that $\ell(\beta)\le cd(x,y)$. Therefore, we obtain

$$\ell(\gamma)\le C_{gh}\ell(\beta)\le cC_{gh}d(x,y).$$

Letting $A=\max\{M,cC_{gh}\}$, we know that γ is an A-uniform arc. Hence (X,d) is A-uniform, completing the proof. □

Remark 4.1. We also give a proof not based on Theorem 1.3 for the sufficiency in Corollary 1.7 under the LEC assumption. Choose $a,b,c\ge 1$ such that (X,d) is a-John, b-LEC, and c-quasiconvex. Then (X,d) is b'-LLC$_2$ with $b'=1+2ab^2$ as shown in [5, p. 204] (note that there $x,y\notin\overline{B}(u;s)$). By [24, Theorem 5.1] choose $C_{gh},C_{bs}\ge 1$ such that (X,d) satisfies the C_{gh}-Gehring–Hayman condition and the C_{bs}-ball separation condition. Let $x,y\in X$, and let γ be a quasihyperbolic geodesic in X joining x and y. We show that γ is an A-uniform arc with $A=cC_{gh}b'C_{bs}$. First, $\ell(\gamma)\le cC_{gh}d(x,y)\le Ad(x,y)$ as shown in the above. Now let $z\in\gamma$. If $x,y\in X\backslash\overline{B}(z,b'C_{bs}d(z))$, then there is a curve β joining x and y in $X\backslash\overline{B}(z,C_{bs}d(z))$, a contradiction. Thus $\min\{d(x,z),d(z,y)\}\le b'C_{bs}d(z)$ and hence $\min\{\ell(\gamma[x,z]),\ell(\gamma[z,y])\}\le cC_{gh}\min\{d(x,z),d(z,y)\}\le Ad(z)$.

4.2. **Proof of Corollary 1.8.** First, thanks to [1, Theorem 2.3], we know that (X,ℓ) satisfies the Gehring–Hayman and ball separation conditions, where ℓ is the length metric on X induced by d. Thus (X,ℓ) is a GHS space. Now it follows from Proposition 3.23 that (X,ℓ) is quasihyperbolic John. Since (X,d) is quasiconvex and (X, ℓ) satisfies the Gehring–Hayman condition, we find that (X, d) is uniform. Hence, Corollary 1.8 follows. $\square$

5. Appendix

In this section, we first prove an elementary fact regarding quasiconvex metric spaces.

Lemma 5.1. *Let $a \geq 1, (X, d)$ a metric space, and Y an a-quasiconvex dense subspace of X. Then X is b-quasiconvex for every $b > a$. Let ℓ be the length metric of X. Then $d(x, y) \leq \ell(x, y) \leq ad(x, y)$ for all $x, y \in X$.*

Proof. The latter claim follows from the former claim, and to prove the former claim, let $b > a$ and $x, y \in X$. We must find a curve σ in X joining x and y such that $\ell(\sigma) \leq bd(x, y)$.

Consider first the case $x, y \notin Y$. We may assume that $x \neq y$. Let $\lambda = b/a$, and choose $k \in \mathbb{N}$ with $2^{-k} \leq (\lambda - 1)/8$. For each $j \geq k$ choose points $x_j, y_j \in Y$ with $d(x_j, x) \leq 2^{-j} d(x, y)$ and $d(y_j, y) \leq 2^{-j} d(x, y)$. For each $j \geq k$ choose an arc α_j in Y joining x_j and x_{j+1} such that $\ell(\alpha_j) \leq ad(x_j, x_{j+1})$. Here $d(x_j, x_{j+i}) \leq d(x_j, x) + d(x_{j+1}, x) \leq 3 \cdot 2^{-j-1} d(x, y)$. Now $\alpha = (\bigcup_{j \geq k} \alpha_j) \cup \{x\}$ is a curve in X joining x_k and x such that $\ell(\alpha) \leq 3 \cdot 2^{-k} ad(x, y)$. Choose similarly a curve β in X joining y_k and y such that $\ell(\beta) \leq 3 \cdot 2^{-k} ad(x, y)$. Choose a curve γ in Y joining x_k and y_k such that $\ell(\gamma) \leq ad(x_k, y_k)$. Here $d(x_k, y_k) \leq d(x, y) + d(x_k, x) + d(y_k, y) \leq (1 + 2 \cdot 2^{-k}) d(x, y)$. Now $\sigma = \overline{\alpha} \cup \gamma \cup \beta$ is a curve in X joining x and y such that $\ell(\sigma) \leq (1 + 8 \cdot 2^{-k}) ad(a, y) \leq \lambda ad(x, y) = bd(x, y)$.

Consider then the case $x \notin Y$ and $y \in Y$. Choose $k \in \mathbb{N}$ with $2^{-k} \leq (\lambda - 1)/4$. Choose the points x_j, the arcs α_j, and the curve α as above. Choose a curve γ in Y joining x_k and y such that $\ell(\gamma) \leq ad(x_k, y)$. Here $d(x_k, y) \leq d(x, y) + d(x_k, x) \leq (1 + 2^{-k}) d(x, y)$. Now $\sigma = \overline{\alpha} \cup \gamma$ is a curve in X joining x and y such that $\ell(\sigma) \leq (1 + 4 \cdot 2^{-k}) ad(x, y) \leq \lambda ad(x, y) = bd(x, y)$. The case $x \in Y$ and $y \notin Y$ is similar. The case $x, y \in Y$ is trivial. $\square$

Next, we establish a result similar to [3, Theorem 2.10] as follows.

Lemma 5.2. *Let (X, d) be an A-uniform metric space. Then every quasihyperbolic geodesic in (X, ℓ) is a B_1-uniform arc in (X, d).*

Proof. In the proof of [3, Theorem 2.10] in [3, p. 12] assume that γ is a quasihyperbolic geodesic in (X, ℓ) joining points $x, y \in X$. Let $L = \ell_{k_\ell}(\gamma)$, and let $\gamma \colon [0, L] \to (X, k_\ell)$ be the isometric parametrization of γ by its arc length with $\gamma(0) = x$ and $\gamma(L) = y$. As $\mathrm{id} \colon (X, k_d) \to (X, k_\ell)$ is A-bi-Lipschitz, $\gamma \colon [0, L] \to (X, k_d)$ is A-bi-Lipschitz, that is, γ is a quasihyperbolic $(A, 0)$-quasigeodesic in (X, d). Note that the quasihyperbolic metric k in the proof in [3] means k_d. We are allowed to use the interval $[0, L]$ in place of $[0, 1]$ as in [3].

In [3, Lemma 2.13] the parametrization $\gamma \colon [0, 1] \to \Omega$ of the curve γ in Ω from x to y can well be replaced by the parametrization $\gamma \colon [0, L] \to \Omega$ of γ by its arc length with $L = \ell_d(\gamma)$ as is done, in fact, just in the proof of [3, Lemma 2.13].

We next establish a modification of [3, (2.18)]. By the first formula in [3, (2.17)] we get as earlier

$$\ell_k(\gamma_\nu) = \int_{\gamma_\nu} \frac{|dz|}{d(z)} \geq \frac{2^{|\nu|-1}}{D} \int_{\gamma_\nu} |dz| = \frac{2^{|\nu|-1}}{D} \ell_d(\gamma_\nu).$$

Now let $t_\nu^1 \leq t_\nu^2$ be the points in $[0, L]$ such that $z_\nu^1 = \gamma(t_\nu^1)$ and $z_\nu^2 = \gamma(t_\nu^2)$ are the endpoints of γ_ν; then $\gamma_\nu = \gamma[t_\nu^1, t_\nu^2]$ and thus

$$\begin{aligned} \ell_k(\gamma_\nu) &\leq \leq A|t_\nu^1 - t_\nu^2| \leq A^2 k(z_\nu^1, z_\nu^2) \leq 4A^4 \log\left(1 + \frac{d(z_\nu^1, z_\nu^2)}{d(z_\nu^1) \wedge d(z_\nu^2)}\right) \\ &\leq 4A^4 \log\left(1 + \frac{2^{|\nu|}}{D} \ell_d(\gamma_v)\right) \end{aligned}$$

by [3, (2.16)] and the second formula in [3, (2.17)]. Thus, the only difference is the multiplication of the right hand side by A^2 (not by A).

Then $(2^{|\nu|}/D)\ell_d(\gamma_\nu) \leq 64A^8$ and consequently $\ell_k(\gamma_\nu) \leq 32A^8$ and thus

$$d(z) \geq (D/2^{|\nu|}) \exp(-32A^8)$$

for $z \in \gamma_\nu$. Now for $z = \gamma(t) \in \gamma_\nu$ we get $\ell_d(\gamma|[0, t]) \wedge \ell_d(\gamma|[t, L]) \leq B_1(A)d(z)$ with $B_1(A) = 128A^8 \exp(32A^8)$, that is, [3, (1.8)].

In the proof of [3, (2.19)] (i.e., of [3, (1.7)]), the only change is contained in the above change of the constant $B_1(A)$. □

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Qingshan Zhou, School of Mathematics and Big Data, Foshan university, Foshan, Guangdong 528000, People's Republic of China

E-mail address: qszhou1989@163.com; q476308142@qq.com

Saminathan Ponnusamy, Department of Mathematics, Indian Institute of Technology Madras, Chennai 600036, India

Lomonosov Moscow State University, Moscow Center of Fundamental and Applied Mathematics, Moscow, Russia.

E-mail address: samy@iitm.ac.in