# ON SINGULAR STRICTLY CONVEX SOLUTIONS TO THE MONGE-AMPÈRE EQUATION

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Abstract. We show the existence of a strictly convex function $u : B_1 \to \mathbb{R}$ with associated Monge-Ampère measure represented by a function $f$ with $0 < f < 1$ a.e. whose Hessian has a singular part. This extends the work [13] and answers an open question of [14, Sec. 6.2(1)].

## 1. Introduction

This paper is concerned with (low) regularity of convex solutions $u : B_1 \to \mathbb{R}$, $B_1 \subset \mathbb{R}^n$, $n \geq 2$, to the Monge-Ampère equation

$$\begin{cases} \det D^2 u = f, & \text{in } B_1, \\ u = 0, & \text{on } \partial B_1. \end{cases} \tag{1.1}$$

Here solutions are intended in the Alexandrov sense, so that

$$\det D^2 u = f$$

shall be interpreted as

$$\mu_u = f dx$$

where $\mu_u$ is the Monge-Ampère measure associated to $u$, see [10]. When $0 < \delta \leq f \leq 1$ one can show that solutions of (1.1) belongs to $W^{2,1}$ (meaning that the Sym(2)-valued measure representing the Hessian of $u$ has no singular part) and indeed $u \in W^{2,1+\varepsilon}_{\text{loc}}(B_1)$, for some $\varepsilon(\delta) > 0$,[4, 5, 15].

When $n = 2$ and $0 \leq f \leq 1$, it is a well known theorem of Alexandrov that strictly convex solutions of (1.1) are indeed $C^1$, see for instance [10, Theorem 2.17]. In view of applications to the semigeostrophic equations it was asked in [14, Sec. 6.2(1)] whether solution of (1.1) with $f \leq 1$ can have a singular part of the Hessian measure. In [13], Mooney constructed a convex solution to (1.1) whose Hessian has a nontrivial Cantor part, via a sophisticated extension procedure of a convenient one-dimensional convex function. The graph of solutions constructed however contains open segments, leaving open the question whether an example can be constructed with $u$ strictly convex. The main result of this note shows that it is indeed possible.

**Theorem 1.1.** *There exists a strictly convex $u \in C^1(\overline{B_1})$ with $Du \in BV \cap C^0(\overline{B_1})$ and $u = 0$ on $\partial B_1$ such that*

$$\mu_u = f dx \quad 0 < f \leq 1 \text{ a.e.},$$

*but*

$$u \notin W^{2,1}(B_1).$$

With the same method we may also construct $u \in C^{1,\alpha}$, having fixed any $\alpha \in (0, 1)$.

The construction is performed by exploiting the convex integration method introduced by D. Faraco in [7], based on objects known in the literature as *staircase laminates*. This is an extremely versatile method

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for creating pathological examples of *concentration*, and has already been used in many contexts, see [1–3, 9]. In particular, it is used in [8] in connection with the degenerate Monge-Ampère equation

$$\det(D^2u) = 0 \quad \text{a.e. in } B_1.$$

Similar constructions have also been considered in [11, 12].

1.1. **Notation.** We let $\mathrm{Sym}(2)$ be the space of $2\times 2$ symmetric matrices. For any set $E \subset \mathbb{R}^2$, we denote by $\overline{E}$ its closure, by $\partial E$ its topological boundary and by $\mathrm{int}\, E$ its interior. We denote by $|E|$ the 2-dimensional Lebesgue measure of a measurable set $E \subset \mathbb{R}^2$.

For an open set $\Omega \subset \mathbb{R}^2$ we say that a function $u : \Omega \to \mathbb{R}$ is piecewise quadratic if it is continuous and there exist countably many pairwise disjoint open sets $\Omega_n \subset \Omega$ with

$$\left|\Omega \setminus \bigcup_n \Omega_n\right| = 0$$

and $u|_{\Omega_n}$ is a polynomial of second order. For a couple $(x, y) \in \mathbb{R}^2$, we set

$$\mathrm{diag}(x, y) \doteq \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

We denote by $a \cdot b$ the standard scalar product of $a, b \in \mathbb{R}^2$.

## 2. Laminates of finite order

We wish to use the convex integration methods of [1, 3]. We will say that $B, C \in \mathrm{Sym}(2)$ are rank-one connected if

$$\mathrm{rank}(B - C) = 1.$$

The building block of these methods is given by the following:

**Lemma 2.1.** *Let $\Omega \subset \mathbb{R}^2$ be an open and bounded domain. Consider $B, C \in \mathrm{Sym}(2)$ with $\mathrm{rank}(B-C) = 1$. For any $\lambda \in [0, 1]$, let*

$$A \doteq \lambda B + (1 - \lambda)C$$

*Fix any $b \in \mathbb{R}^2$, $c \in \mathbb{R}$ and let $u_0(x) \doteq \frac{1}{2}Ax \cdot x + b \cdot x + c$. Then, for all $\varepsilon > 0$, there exists a piecewise quadratic function $u_\varepsilon \in W^{2,\infty}(\Omega)$ with the following properties:*

*(1) $u_\varepsilon(x) = \frac{1}{2}Ax \cdot x + b \cdot x + c$, $Du_\varepsilon(x) = Ax + b$ on $\partial\Omega$ and $\|u_\varepsilon - u_0\|_{C^1(\overline{\Omega})} \le \varepsilon$;*
*(2) $D^2u_\varepsilon(x) \in B_\varepsilon(B) \cup B_\varepsilon(C)$ for a.e. $x \in \Omega$;*
*(3) $|\{x \in \Omega : D^2u_\varepsilon(x) = B\}| \ge (1 - \varepsilon)\lambda|\Omega|$, $|\{x \in \Omega : D^2u_\varepsilon(x) = C\}| \ge (1 - \varepsilon)(1 - \lambda)|\Omega|$;*
*(4) $|\{x \in \Omega : D^2u_\varepsilon(x) \in B_\varepsilon(B)\}| \le (1 + \varepsilon)\lambda|\Omega|$, $|\{x \in \Omega : D^2u_\varepsilon(x) \in B_\varepsilon(C)\}| \le (1 + \varepsilon)(1 - \lambda)|\Omega|$;*

*Proof.* A map having the first three properties was constructed in the proof of [1, Lemma 2.1], see in particular [1, (4.1),(4.2),(4.3),(4.4)]. Let us denote the family of functions obtained in this way by $v_\varepsilon$. In order to show that there is a map that in addition fulfills (4), we fix $\varepsilon$ and $\lambda$ and assume without loss of generality that $\lambda \le 1 - \lambda$. We can also assume that $\varepsilon > 0$ is so small that $B_\varepsilon(B) \cap B_\varepsilon(C) = \emptyset$. Thus, having defined $\varepsilon' \doteq \frac{\varepsilon\lambda}{1-\lambda}$ and the map $v_{\varepsilon'}$, we set $u_\varepsilon \doteq v_{\varepsilon'}$ and claim that this function has the four required properties. Since $\varepsilon' \le \varepsilon$, the first three properties are straightforward from the analogous properties of $v_{\varepsilon'}$. To obtain (4), simply estimate

$$\begin{aligned}|\{x \in \Omega : D^2u_\varepsilon(x) \in B_\varepsilon(B)\}| &= |\{x \in \Omega : D^2v_{\varepsilon'}(x) \in B_{\varepsilon'}(B)\}| \\ &\le |\Omega| - |\{x \in \Omega : D^2v_{\varepsilon'}(x) = C\}| \overset{(3)}{\le} |1 - (1 - \varepsilon')(1 - \lambda)||\Omega|.\end{aligned}$$

Analogously,

$$|\{x \in \Omega : D^2u_\varepsilon(x) \in B_\varepsilon(C)\}| \le |1 - (1 - \varepsilon')\lambda||\Omega|.$$

The fact that $\lambda \le 1-\lambda$ and our choice of $\varepsilon' = \varepsilon\frac{\lambda}{1-\lambda}$ implies that

$$|1-(1-\varepsilon')(1-\lambda)| \le (1+\varepsilon)\lambda \text{ and } |1-(1-\varepsilon')\lambda| \le (1+\varepsilon)(1-\lambda)$$

and allows us to conclude the proof. □

This result can be easily iterated, but before stating the next proposition, we start with a definition.

**Definition 2.2.** Let $\nu, \mu$ be probability atomic measures on $\mathrm{Sym}(2)$. Let $\nu = \sum_{i=1}^N \lambda_i \delta_{A_i}$. We say that $\mu$ can be obtained via *elementary splitting from* $\nu$ if for some $i \in \{1, \dots, N\}$, there exist matrices $B, C \in \mathrm{Sym}(2)$, $\lambda \in [0,1]$ such that

$$\mathrm{rank}(B-C) = 1, \quad A_i = sB + (1-s)C,$$

for some $s \in (0,1)$ and

$$\mu = \nu + \lambda\lambda_i(-\delta_{A_i} + s\delta_B + (1-s)\delta_C).$$

A probability measure $\nu = \sum_{i=1}^r \lambda_i \delta_{A_i}$ is called a *laminate of finite order* if there exists a finite number of measures $\nu_1, \dots, \nu_r \in$ such that

$$\nu_1 = \delta_X, \quad \nu_r = \nu$$

and $\nu_{j+1}$ can be obtained via elementary splitting from $\nu_j$, for every $j \in \{1, \dots, N-1\}$.

A simple inductive procedure starting from Lemma 2.1 yields then the next result, see also [1, Proposition 2.3] or [16, Proposition 4.6].

**Lemma 2.3.** *Let $\Omega \subset \mathbb{R}^2$ be an open domain. Consider a laminate of finite order $\nu = \sum_{i=1}^r \lambda_i \delta_{A_i}$ with barycenter $A \in \mathrm{Sym}(2)$, i.e.:*

$$A = \int_{\mathbb{R}^{2\times 2}} X d\nu(X).$$

*Fix any $b \in \mathbb{R}^2$, $c \in \mathbb{R}$ and let $u_0(x) \doteq \frac{1}{2}Ax\cdot x + b\cdot x + c$. Then, for all $\varepsilon > 0$, there exists a piecewise quadratic function $u_\varepsilon \in W^{2,\infty}(\Omega)$ with the following properties:*

1. *$u_\varepsilon(x) = \frac{1}{2}Ax\cdot x + b\cdot x + c$, $Du_\varepsilon(x) = Ax + b$ on $\partial\Omega$ and $\|u_\varepsilon - u_0\|_{C^1(\overline{\Omega})} \le \varepsilon$;*
2. *$D^2u_\varepsilon(x) \in \bigcup_{i=1}^r B_\varepsilon(A_i)$ for a.e. $x \in \Omega$;*
3. *$|\{x \in \Omega : D^2u_\varepsilon(x) = A_i\}| \ge (1-\varepsilon)\lambda_i|\Omega|, \forall i$;*
4. *$|\{x \in \Omega : D^2u_\varepsilon(x) \in B_\varepsilon(A_i)\}| \le (1+\varepsilon)\lambda_i|\Omega|, \forall i$.*

Operatively, what one usually does is the following. Starting with a matrix $A \in \mathrm{Sym}(2)$, one *splits* $A$ into $B$ and $C$ with weight $\lambda$ if

$$\mathrm{rank}(B-C) = 1 \text{ and } A = \lambda B + (1-\lambda)C.$$

We will use the short-hand notation:

$$\delta_A \mapsto \nu_1 = \lambda\delta_B + (1-\lambda)\delta_C. \tag{2.1}$$

By definition, $\nu_1$ is a laminate of finite order. One can then reiterate this reasoning by splitting $C$ into $D$ and $E$ with weight $\mu$, provided

$$\mathrm{rank}(D-E) = 1 \text{ and } C = \mu D + (1-\mu)E,$$

so that

$$\delta_C \mapsto \mu\delta_D + (1-\mu)\delta_E.$$

By construction, the measure

$$\nu_2 \doteq \lambda\delta_B + (1-\lambda)\mu\delta_D + (1-\lambda)(1-\mu)\delta_E.$$

is again a laminate of finite order. Before concluding this section we show one last result, which will be useful in the proof of Theorem 1.1.

**Lemma 2.4.** *Let $\Omega \subset \mathbb{R}^2$ be open and let $u : \Omega \to \mathbb{R}$ be a $W^{2,1}(\Omega)$ piecewise quadratic function. Let $\{\Omega_n\}$ be the collection of pairwise disjoint open sets such that*

$$\left|\Omega \setminus \bigcup_n \Omega_n\right| = 0$$

*and $u|_{\Omega_n}$ is a polynomial of second order. Define, for any Borel $F \subset \mathrm{Sym}(2)$, the set*

$$E \doteq \{x \in \Omega : x \text{ is a Lebesgue point for } D^2u(x) \text{ and } D^2u(x) \in F\},$$

*Then,*

$$\left|E \setminus \bigcup_{n:\Omega_n \cap E \neq \emptyset} \Omega_n\right| = 0 \tag{2.2}$$

*and*

$$|\partial E \cap \Omega| = 0. \tag{2.3}$$

*Proof.* Notice that, since $u|_{\Omega_n}$ is a polynomial of second order, $E$ enjoys the property

$$E \cap \Omega_n \neq \emptyset \Rightarrow \Omega_n \subset E, \quad \forall n \in \mathbb{N}. \tag{2.4}$$

Thus, either $|E| = 0$, in which case (2.2) is obvious, or $|E| > 0$. In this case, (2.4) shows

$$\bigcup_{n:\Omega_n \cap E \neq \emptyset} \Omega_n \subset \mathrm{int}\, E \subset E$$

and (2.2) is immediate. To show (2.3), suppose by contradiction that for some $F$,

$$|\partial E \cap \Omega| > 0.$$

In this case we must have that, for some $m \in \mathbb{N}$,

$$\Omega_m \cap \partial E \cap \Omega \neq \emptyset.$$

Let $x \in \Omega_m \cap \partial E \cap \Omega$. On one hand, we must have that $x \notin E$, since otherwise (2.4) yields $\Omega_m \subset E$ and hence $x \notin \partial E$. On the other hand, since $\Omega_m$ intersects $\partial E$, then $\Omega_m$ must intersect $E$ and hence $x \in E$ by (2.4), which yields the required contradiction. □

## 3. Proof of Theorem 1.1

We divide the proof into steps.

Construction of the laminates. We will work in the space of diagonal matrices, where we use coordinates

$$(x, y) \in \mathbb{R}^2 \mapsto \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

Notice that in the space of diagonal matrices two matrices $(x, y)$ and $(a, b)$ are rank-one connected if and only if $x = a$ or $y = b$. We define the laminate of finite order $\nu_n$ in the following way. First, we let for all $n \geq 1$:

$$x_n = 4^{-n}, \quad y_n = 2^n, \quad b_n = \frac{3}{4}2^{-n}, \quad z_n = 4^{-n}. \tag{3.1}$$

Moreover, set

$$x_0 \doteq \frac{1}{2}, \quad y_0 \doteq \frac{1}{2}, \quad b_0 \doteq \frac{3}{4}, \quad z_0 = \frac{1}{8}.$$

Observe that $0 < x_{n+1} < x_n < b_n$ for all $n \in \mathbb{N}$. Thus, we can first split horizontally

$$\delta_{(x_n,y_n)} \mapsto \frac{b_n - x_n}{b_n - x_{n+1}}\delta_{(x_{n+1},y_n)} + \frac{x_n - x_{n+1}}{b_n - x_{n+1}}\delta_{(b_n,y_n)}.$$

Next, since $0 < z_n < y_n < y_{n+1}$ for all $n \in \mathbb{N}$, we are allowed to split vertically

$$\delta_{(x_{n+1},y_n)} \mapsto \frac{y_n - z_n}{y_{n+1} - z_n}\delta_{(x_{n+1},y_{n+1})} + \frac{y_{n+1} - y_n}{y_{n+1} - z_n}\delta_{(x_{n+1},z_n)}.$$

By construction, the probability measure

$$\nu_n \doteq \frac{b_n - x_n}{b_n - x_{n+1}}\frac{y_n - z_n}{y_{n+1} - z_n}\delta_{(x_{n+1},y_{n+1})} + \frac{b_n - x_n}{b_n - x_{n+1}}\frac{y_{n+1} - y_n}{y_{n+1} - z_n}\delta_{(x_{n+1},z_n)} + \frac{x_n - x_{n+1}}{b_n - x_{n+1}}\delta_{(b_n,y_n)}$$

is a laminate of finite order. Let for all $n \in \mathbb{N}$

$$A_n = \operatorname{diag}(x_n, y_n), \quad B_n = \operatorname{diag}(x_{n+1}, z_n), \quad C_n = \operatorname{diag}(b_n, y_n)$$

and

$$\alpha_n \doteq \frac{b_n - x_n}{b_n - x_{n+1}}\frac{y_n - z_n}{y_{n+1} - z_n}, \quad \beta_n \doteq \frac{b_n - x_n}{b_n - x_{n+1}}\frac{y_{n+1} - y_n}{y_{n+1} - z_n}, \quad \gamma_n \doteq \frac{x_n - x_{n+1}}{b_n - x_{n+1}}. \tag{3.2}$$

The base step. The reasoning to construct the function $u$ of the statement of the present Theorem is inductive. We start with

$$u_0(z) \doteq \frac{1}{2}A_0 z \cdot z, \quad \forall z \in \mathbb{R}^2.$$

Next we define:

$$\varepsilon_n \doteq 10^{-n}. \tag{3.3}$$

We can consider $u_1 : B_1 \to \mathbb{R}$, which is obtained employing Lemma 2.3 with $\nu_0$ and $\varepsilon_1$ in place of $\varepsilon$. In other words, $u_1 \in W^{2,\infty}$ is a piecewise quadratic function which satisfies:

- $u_1 = u_0$, $Du_1 = Du_0$ on $\partial B_1$ and $\|u_1 - u_0\|_{C^1(\overline{B}_1)} \le \varepsilon_1$;
- $D^2u_1 \in B_{\varepsilon_1}(A_1) \cup B_{\varepsilon_1}(B_0) \cup B_{\varepsilon_1}(C_0)$ a.e. in $B_1$;
- $|\{z \in B_1 : D^2u_1(z) = A_1\}| \ge \alpha_0(1 - \varepsilon_1)|B_1|$.
- $|\{z \in B_1 : D^2u_1(z) \in B_{\varepsilon_1}(A_1)\}| \le (1 + \varepsilon_1)\alpha_0|B_1|$.

Consider the set $E_1 = \{z \in \Omega_0 \doteq B_1 : D^2u_1(z) = A_1\}$. Since $|E_1| > 0$, Lemma 2.4 allows us to choose another open set $\Omega_1 \subset E_1$ which is a finite union of balls whose closures are pairwise disjoint and

$$|E_1 \setminus \Omega_1| \le \varepsilon_1|E_1|.$$

Since $u_1$ is piecewise quadratic, we can subdivide $\Omega_1$ into countably many smaller open sets $\{F_n\}_{n\in\mathbb{N}}$ on which $u_1$ is quadratic. Let $u_1(z) = \frac{1}{2}A_1 z \cdot z + b \cdot z + c$ on $F_n$ for some $b \in \mathbb{R}^2, c \in \mathbb{R}$. Then, we can replace $u_1$ on $F_n$ using a function $w_{2,n}$ given again by Lemma 2.3 with $\nu_1$ and $\varepsilon_2$ in place of $\nu$ and $\varepsilon$. We then define

$$u_2 \doteq \begin{cases} u_1 \text{ on } B_1 \setminus \bigcup_n \overline{F_n} \\ w_{2,n} \text{ on } \overline{F_n}. \end{cases}$$

By Proposition 2.3

$$w_{2,n} = \frac{1}{2}A_1 z \cdot z + b \cdot z + c \text{ and } Dw_{2,n} = A_1 z + b \text{ on } \partial F_n,$$

and hence $u_2$ is still in $W^{2,\infty}(B_1)$. We will now set up the inductive procedure.

The inductive step. Suppose we are given an open, non-empty set $\Omega_{j-1} \subset B_1$ and a piecewise quadratic function $u_j \in W^{2,\infty}(B_1)$ satisfying

$$D^2u_j \in \bigcup_{k=1}^{j} B_{\varepsilon_k}(A_k) \cup \bigcup_{k=0}^{j-1} B_{\varepsilon_{k+1}}(B_k) \cup B_{\varepsilon_{k+1}}(C_k) \text{ a.e. in } B_1.$$

Let

$$E_j = \{z \in \Omega_{j-1} : D^2u_j(z) = A_j\}, \tag{3.4}$$

which we assume to have positive measure. By Lemma 2.4, we can choose $\Omega_j = \bigcup_{i=1}^{N_j} B_{r_i}(x_i)$ with $r_i \le \frac{1}{j}$ for all $i$, $\overline{B_{r_i}(x_i)} \subset E_j$, and

$$\overline{B_{r_i}(x_i)} \cap \overline{B_{r_k}(x_k)} = \emptyset, \quad \forall i \neq k,$$

so that

$$\overline{\Omega_j} \text{ does not contain segments of length larger than } \frac{2}{j}. \tag{3.5}$$

The balls forming $\Omega_j$ are chosen in such a way that

$$|E_j \setminus \Omega_j| \le \varepsilon_j |E_j|. \tag{3.6}$$

Finally notice that

$$|\partial \Omega_j| = 0. \tag{3.7}$$

We can split $\Omega_j = \bigcup_n \Omega_{j,n}$ such that on each open set $\Omega_{j,n}$, $u_j$ is quadratic. In particular, we have

$$u_j(z) = \frac{1}{2} A_j z \cdot z + b_{j,n} \cdot z + c_{j,n} \text{ on } \Omega_{j,n}, \text{ where } A_j = \operatorname{diag}(x_j, y_j), b_{j,n} \in \mathbb{R}^2, c_{j,n} \in \mathbb{R}.$$

On each $\Omega_{j,n}$, we can consider as boundary datum $u_j$ and use the laminate of finite order $\nu_j$ in conjunction with Lemma 2.3 to construct a map $w_{j,n} \in W^{2,\infty}$ with the following properties:

(a) $w_{j,n} = u_j$, $Dw_{j,n} = Du_j$ on $\partial\Omega_{j,n}$ and $\|w_{j,n} - u_j\|_{C^1(\overline{\Omega_{j,n}})} \le \varepsilon_{j+1}$;
(b) $D^2 w_{j,n} \in B_{\varepsilon_{j+1}}(A_{j+1}) \cup B_{\varepsilon_{j+1}}(B_j) \cup B_{\varepsilon_{j+1}}(C_j)$ a.e. in $\Omega_{j,n}$;
(c) furthermore

$$|\{x \in \Omega_{j,n} : D^2 w_{j,n}(x) = A_{j+1}\}| \ge (1 - \varepsilon_{j+1})\alpha_j |\Omega_{j,n}|$$

and

$$|\{x \in \Omega_{j,n} : D^2 w_{j,n}(x) \in B_{\varepsilon_{j+1}}(A_{j+1})\}| \le (1 + \varepsilon_{j+1})\alpha_j |\Omega_{j,n}|.$$

Thus, if we define

$$u_{j+1}(x) \doteq \begin{cases} u_j(x), & \text{if } x \in B_1 \setminus \overline{\Omega_j} \\ w_{j,n}(x), & \text{if } x \in \overline{\Omega_{j,n}}, \end{cases} \tag{3.8}$$

we see that $u_{j+1} \in W^{2,\infty}(B_1)$ enjoys the following properties:

(i) $u_{j+1} = u_j$, $Du_{j+1} = Du_j$ on $\partial B_1$ and $\|u_{j+1} - u_j\|_{C^1(\overline{B_1})} \le \varepsilon_{j+1}$;
(ii) $D^2 u_{j+1} \in \bigcup_{k=1}^{j+1} B_{\varepsilon_k}(A_k) \cup \bigcup_{k=0}^{j} B_{\varepsilon_{k+1}}(B_k) \cup B_{\varepsilon_{k+1}}(C_k)$ a.e. in $B_1$;
(iii) furthermore

$$|\{x \in \Omega_j : D^2 u_{j+1}(x) = A_{j+1}\}| \ge (1 - \varepsilon_{j+1})\alpha_j |\Omega_j|$$

and

$$|\{x \in \Omega_j : D^2 u_{j+1}(x) \in B_{\varepsilon_{j+1}}(A_{j+1})\}| \le (1 + \varepsilon_{j+1})\alpha_j |\Omega_j|.$$

Definition of the required map and first properties. The inductive scheme gives us a sequence $\{u_j\}_{j\in\mathbb{N}}$. Extend each function $u_j$ to $B_3$ by taking

$$\bar{u}_j \doteq \begin{cases} u_j, & \text{in } \overline{B_1} \\ u_0, & \text{in } B_3 \setminus \overline{B_1}. \end{cases}$$

First, by (i) we have that

$$\sum_{j=1}^{\infty} \|\overline{u}_{j+1} - \overline{u}_j\|_{C^1(\overline{B_2})} < +\infty.$$

Thus we can define $\overline{u}_\infty = \lim_j \overline{u}_j$, where the limit is taken in the $C^1(\overline{B}_2)$ topology. As for $u_j$, we will denote $u_\infty$ the restriction of $\overline{u}_\infty$ to $\overline{B}_1$. Moreover,

$$u_\infty = u_0 = \frac{1}{4}(x^2 + y^2) = \frac{1}{4} \text{ on } \partial B_1$$

again by (i). We claim that $u \doteq u_\infty - \frac{1}{4}$ gives the required counterexample. We will always work with $u_\infty$ in the following, since anyway the subtraction of the constant $\frac{1}{4}$ does not change the properties we are interested in. Due to our choice of $\{\varepsilon_j\}_{j\in\mathbb{N}}$, $\{x_j\}, \{y_j\}, \{b_j\}$ and $\{z_j\}$, see (3.1)-(3.3), and exploiting (ii), we also find $\rho_j > 0$ such that

$$D^2\overline{u}_j(x) \geq \rho_j \,\mathrm{id} \tag{3.9}$$

a.e. in $B_3$ in the sense of quadratic forms. In particular, this yields that $\overline{u}_j$ is convex for all $j$ and thus $\overline{u}_\infty$ is convex too. Moreover in the sense of measures

$$D^2\overline{u}_j \overset{*}{\rightharpoonup} D^2\overline{u}_\infty, \text{ in } B_2. \tag{3.10}$$

Since $u_j \in W^{2,\infty}(B_1)$ for all $j$ we have

$$\mu_{u_j} = \det(D^2u_j)dz$$

and once again our choice of $\{\varepsilon_j\}_j, \{x_j\}, \{y_j\}, \{b_j\}$ and $\{z_j\}$ in conjunction with (ii) yield

$$0 < \det(D^2u_j)(x) < 1 \text{ a.e. in } B_1.$$

By [10, Proposition 2.6], it follows that $\mu_{u_\infty}$ is the weak-$*$ limit of $\mu_{u_j}$ and it is therefore represented as

$$\mu_{u_\infty} = f dz, \text{ and } 0 \leq f \leq 1.$$

Up to now, we have showed that $u_\infty$ enjoys the following properties:

- $u_\infty$ is convex;
- $u_\infty \in C^1(\overline{B_1})$ and $Du \in BV \cap C^0(\overline{B_1})$;
- $u_\infty = \frac{1}{4}$ on $\partial B_1$, so that $u = 0$ on $\partial B_1$;
- $\mu_{u_\infty} = f dz$ and $0 \leq f \leq 1$ a.e..

To conclude, we still need to show that

- $0 < f$ a.e.;
- $u_\infty$ is strictly convex;
- $D^2u_\infty$ is singular with respect to the Lebesgue measure.

In order to so, we define the closed set

$$\Omega_\infty \doteq \bigcap_{j=1}^{\infty} \overline{\Omega}_j.$$

We claim that

$$|\Omega_j| \overset{(3.7)}{=} |\overline{\Omega}_j| \to 0 \text{ as } j \to \infty \tag{Claim 1}$$

and that there exists $c > 0$ such that

$$c \leq \int_{\Omega_j} \partial_{22}u_j \text{ for all } j. \tag{Claim 2}$$

First assume these claims and let us show how they allow us to conclude the proof.

$\boxed{f > 0 \text{ a.e..}}$ (Claim 1) implies that $|\Omega_\infty| = 0$. Thus it suffices to prove that $f > 0$ a.e. on $B_1 \setminus \Omega_\infty = \bigcup_j (B_1 \setminus \overline{\Omega_j})$. On the open set $B_1 \setminus \overline{\Omega_j}$ we have, by (3.8) and the fact that $\Omega_{j+1} \subset \Omega_j$ for all $j \in \mathbb{N}$,

$$u_\infty = u_j,$$

and hence, since $u_j \in W^{2,\infty}$,

$$\mu_{u_\infty} = f dz = \mu_{u_j} = \det(D^2u_j)dz \overset{(3.9)}{>} 0, \text{ a.e. on } B_1 \setminus \overline{\Omega_j}.$$

$\boxed{u_\infty \text{ is stricly convex.}}$ Suppose by contradiction this does not hold. Then there exists a segment $\sigma \subset B_1$ such that $u_\infty$ is affine on $\sigma$. First, we note that

$$\sigma \not\subset \Omega_\infty.$$

Indeed, if by contradiction $\sigma \subset \Omega_\infty$, then by definition $\sigma \subset \overline{\Omega}_j$ for all $j \in \mathbb{N}$. This is in contradiction with our construction of $\Omega_j$, see (3.5), for $j$ large enough. Since $\Omega_\infty$ is closed, we find another (nontrivial) segment

$$\sigma' \subset \sigma$$

with

$$\sigma' \cap \Omega_\infty = \emptyset.$$

Thus

$$\sigma' = \sigma' \cap \bigcup_j (B_1 \setminus \overline{\Omega}_j).$$

Let $x_0 \in \sigma'$ and $j_0 \in \mathbb{N}$ such that $x_0 \in B_1 \setminus \overline{\Omega}_{j_0}$. By (3.8), we find a small ball $B_\varepsilon(x_0) \subset B_1$ such that $u_\infty = u_{j_0}$ on $B_\varepsilon(x_0)$. Thus,

$$u_\infty = u_{j_0} \text{ on } B_\varepsilon(x_0) \cap \sigma' \neq \emptyset.$$

By (3.9), $u_{j_0}$ is strictly convex on $B_\varepsilon(x_0)$, which is in contradiction with the fact that $u_\infty$ is affine on $\sigma$. It follows that $u_\infty$ is strictly convex in $B_1$.

$\boxed{D^2 u_\infty \text{ is singular.}}$ It suffices to prove that $\partial_{22} u_\infty$ is singular. Notice that by convexity, $\partial_{22} u_j, \partial_{22} u_\infty \geq 0$ in the sense of measures. Now, by construction, $\Omega_{j+1} \subset \Omega_j$ for all $j \in \mathbb{N}$. Fix $j$ and take any $\ell > j$. Then:

$$c \overset{\text{(Claim 2)}}{\leq} \int_{\Omega_\ell} \partial_{22} u_\ell dz \leq \int_{\Omega_j} \partial_{22} u_\ell dz \overset{(3.7)}{=} \int_{\overline{\Omega}_j} \partial_{22} u_\ell dz.$$

By (3.10), $\partial_{22} u_\ell$ weakly-$*$ converges in the sense of measures to $\partial_{22} u_\infty$ and hence by [6, Theorem 1.40(ii)], we may use the previous chain of inequalities to conclude that, for all $j \in \mathbb{N}$,

$$c \leq \limsup_{\ell \to \infty} \int_{\overline{\Omega}_j} \partial_{22} u_\ell dz \leq \partial_{22} u_\infty(\overline{\Omega}_j).$$

Exploiting once more (Claim 1), we conclude that $\partial_{22} u_\infty$ and hence $D^2 u_\infty$ is singular with respect to the Lebesgue measure.

We now turn to the proof of the claims.

$\boxed{\text{Proof of (Claim 1).}}$ We wish to show that there exist constants $c_1, c_2 > 0$ such that

$$c_1 \leq \liminf_{j \to \infty} |\Omega_j| y_j \leq \limsup_{j \to \infty} |\Omega_j| y_j \leq c_2. \tag{3.11}$$

This would imply (Claim 1). We start by noticing that

$$k_0 = \prod_{i=0}^{\infty} (1 - \varepsilon_{i+1})^2 \in (0, \infty), \quad k_0' = \prod_{i=1}^{\infty} (1 + \varepsilon_i) \in (0, \infty), \tag{3.12}$$

$$k_1 \doteq \prod_{i=1}^{\infty} \frac{b_i - x_i}{b_i - x_{i+1}} = \prod_{i=1}^{\infty} \frac{1 - \frac{x_i}{b_i}}{1 - \frac{x_{i+1}}{b_i}} \in (0, \infty), \quad k_2 \doteq \prod_{i=1}^{\infty} \frac{1 - \frac{z_i}{y_i}}{1 - \frac{z_i}{y_{i+1}}} \in (0, \infty). \tag{3.13}$$

To prove these statements recall the following elementary fact: if $\{a_i\}_i$ is a sequence of real numbers, with $1 + a_i > 0$ for all $i \in \mathbb{N}$, then in order for

$$\lim_{j \to \infty} \prod_{i=1}^{j} (1 + a_i) \in (0, +\infty) \tag{3.14}$$

it is sufficient that

$$\sum_{i=1}^{\infty}|a_i| < +\infty. \tag{3.15}$$

By (3.1)-(3.3) we immediately obtain (3.15) in the cases considered in (3.12)-(3.13), so these hold. Having shown (3.12)-(3.13), we can move to (3.11). From the first line of (iii) and by (3.4), we find

$$(1-\varepsilon_{j+1})\alpha_j|\Omega_j| \le |E_{j+1}|.$$

Using (3.6), we further get

$$(1-\varepsilon_{j+1})\alpha_j|\Omega_j| \le (1-\varepsilon_{j+1})^{-1}|\Omega_{j+1}|, \quad \forall j \in \mathbb{N}. \tag{3.16}$$

Inductively we can use (3.16) to write for all $j \in \mathbb{N}$

$$k_0\left(\prod_{i=1}^{j-1}\alpha_i\right)|\Omega_1| = \left(\prod_{i=0}^{\infty}(1-\varepsilon_{i+1})^2\right)\left(\prod_{i=1}^{j-1}\alpha_i\right)|\Omega_1| \le |\Omega_j|.$$

Moreover, using the fact that $\Omega_j \subset E_j$ for all $j \in \mathbb{N}$ we find:

$$|\Omega_{j+1}| \le |E_{j+1}| \overset{(3.4)}{\le} |\{x \in \Omega_j : D^2u_{j+1}(x) \in B_{\varepsilon_{j+1}}(A_{j+1})\}| \overset{(iii)}{\le} \alpha_j(1+\varepsilon_{j+1})|\Omega_j|, \quad \forall j \in \mathbb{N}. \tag{3.17}$$

Using (3.17) inductively, we obtain

$$|\Omega_j| \le |\Omega_1|\prod_{i=1}^{j-1}\alpha_i\prod_{i=2}^{j}(1+\varepsilon_i) \le |\Omega_1|\prod_{i=1}^{j-1}\alpha_i\prod_{i=2}^{\infty}(1+\varepsilon_i) \le k_0'|\Omega_1|\prod_{i=1}^{j-1}\alpha_i, \quad \forall j \in \mathbb{N}.$$

As $|\Omega_1| > 0$, in order to conclude (3.11) we only need to study the asymptotic behaviour of $\prod_{i=1}^{j}\alpha_i$. We have:

$$\lim_{j\to\infty} y_{j+1}\prod_{i=1}^{j}\alpha_i \overset{(3.2)}{=} \lim_{j\to\infty} y_{j+1}\prod_{i=1}^{j}\frac{b_i-x_i}{b_i-x_{i+1}}\frac{y_i-z_i}{y_{i+1}-z_i} \overset{(3.13)}{=} y_1k_1k_2 = 2k_1k_2,$$

where we used that $y_1 = 2$. This shows (3.11) and concludes the proof of this claim.

Proof of (Claim 2). By (3.4) and since $\Omega_j \subset E_j$, we have

$$\int_{\Omega_j}\partial_{22}u_j dz = y_j|\Omega_j|$$

and hence (Claim 2) follows from (3.11).

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