On unexpected curves of type $(d+k,d)$.

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Abstract

We present a construction explaining the existence of (unexpected) curves of degree $d+k$, passing through a finite set Z of points on $\mathbb{P}^2$, and having a generic point P of multiplicity d. The construction is based on the syzygies of the k-th powers of the Jacobian of the product of lines dual to the points of Z. These syzygies give rise to vector bundle of rank $k+1$. We prove a result giving a sufficient condition for the unexpectedness of curves via the splitting type of the bundle (restricted to the line dual to P) providing a generalization of the theory initiated by Faenzi and Valles and by Cook II, Harbourne, Migliore and Nagel.

Keywords unexpected curves, syzygies, Jacobian ideal

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1 Introduction

Studying the dimension of a given linear system of divisors is one of the most classical problems in algebraic geometry. Typical examples of interesting linear systems arise when imposing vanishing conditions on divisors in a complete linear system. Determining the dimension of such a system amounts to deciding if the imposed vanishing conditions are independent or not. If the underlying variety is a projective space and vanishing (to order one) is imposed in *general points* then the resulting system is either empty or its dimension is determined by the number of points. Imposing vanishing to order two in general points of a projective space is also well understood due to a highly non-trivial result of Alexander and Hirschowitz, [1]. For points of higher multiplicity, even a conjectural picture in projective spaces of arbitrary dimension is, in general, missing. However, in the case of the complex projective plane the so-called SHGH conjecture [4], promises an understanding of what can possibly happen. Despite intensive investigations over the last 40 years, the SHGH conjecture remains widely open. Based on the results due to Faenzi and Valles [13, 25], and results by Cook II, Harbourne, Migliore and Nagel in their paper [3], started a new direction of research. They considered curves of degree $d+1$ passing through a set $Z \subset \mathbb{P}^2$ of non-general points and having multiplicity d in a general point P. It is crucial here that the point P is general. If the existence of such curves does not follow from the naïve dimension count, then the authors of [3] call them unexpected of type $(d+1, d)$. Their work was motivated by findings for $d = 3$ in [6] by DiGennaro, Ilardi and Vallès. Unexpected curves, and more generally, unexpected hypersurfaces, attracted a lot of attention, see e.g., [2, 7, 8, 10, 11, 12, 14, 15, 17, 18, 23, 24]. In particular, the authors of [20] provide an infinite family of unexpected curves of type $(2m+1, 3)$.

The present work was motivated by an attempt to explain the existence of this family. Then we wanted to create a more general theory, along the lines of [3], explaining the existence of unexpected curves with the difference between the degree and the multiplicity in the general point greater than one. In [3] the existence of unexpected cures of type $(d+1, d)$ is explained via degree d syzygies of the Jacobian ideal J of an arrangement $\mathcal{A}_Z$ of lines dual to the points of the set Z, defined by the equation $f = 0$. The purpose of our work is to investigate unexpected curves of type $(d+k, d)$ from this point of view. More specifically, we present two main results. The first is Theorem 3.1, where we describe a construction of curves of type $(d+k, d)$ based on syzygies of the k-th power of a suitable twist of the Jacobian ideal of the arrangement $\mathcal{A}_Z$. This construction generalizes that of Cook II, Harbourne, Migliore, Nagel from [3], where they dealt with $k = 1$ case. Theorem 5.5 provides, in turn, a sufficient condition for the curves resulting from Theorem 3.1 to be unexpected. Section 4 and Lemma 5.3 are the most technical part of the work. In Section 4 we show that the syzygy bundle of $(J/fS)^k$ may be treated as a subbundle of the k-th symmetric powers of the tangent bundle. A generalization of logarithmic derivation was also studied in [26], with additional assumption of $(r+1)$-genericity od points. In Lemma 5.3 we prove the connection between the splitting type of the above bundle and the dimension of the ideal $I_Z + dP$ in degree $d+k$. In the last section of our paper, we present examples of (various types of) unexpected curves, some of them may be a starting point for new problems.

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2 Basic facts and the syzygies construction

In this section, we establish the notation and recall basic facts and definitions needed in the sequel.

Let $S = \mathbb{C}[x, y, z]$ be the graded ring of complex polynomials. Let $\check{\mathbb{P}}^2$ be the space dual to $\mathbb{P}^2 = \mathbb{P}(S)$ and denote by Z a set of reduced points in $\check{\mathbb{P}}^2$. We associate with the set $Z \subset \check{\mathbb{P}}^2$ the line arrangement $\mathcal{A}_Z = \mathcal{A} = \{H_1, \ldots, H_{|Z|}\}$, where $H_i's$ are lines dual to the points of Z. We denote by ℓ_i a linear form defining H_i. In this paper, f always stands for $f = \ell_1 \cdot \ldots \cdot \ell_{|Z|}$, which we call the defining polynomial of $\mathcal{A}$. For a given point $Q = (a, b, c)$ in $\mathbb{P}^2$, we use L_Q to denote the line in $\check{\mathbb{P}}^2$ dual to this point. Similarly, if $L \subset \mathbb{P}^2$ is a generic line with the equation $\alpha x + \beta y + \gamma z = 0$, then $P_L = (\alpha, \beta, \gamma)$ indicates the dual point of L.

Let $\mathrm{Der}(S)$ be the module of $\mathbb{C}$-derivations of S and denote by $\theta_E = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \in \mathrm{Der}(S)$ the Euler derivation. For a given homogeneous element f we define

$$\mathrm{D}(f) = \{\theta \in \mathrm{Der}(S) \mid \theta(f) \in fS\}.$$

It is known that $\mathrm{D}(f) = S\theta_E \oplus \mathrm{D}_0(f)$, where $\mathrm{D}_0(f)$ is the kernel of the map $\partial \mapsto \partial(f)$ and it is called the derivation module, giving rise to the derivation bundle, see [3]. We also use the notation $\mathrm{D}(\mathcal{A})$ if $\mathcal{A}$ is the arrangement defined by f.

In the same spirit, we define $\mathrm{Der}^k(S)$ to be the module

$$\mathrm{Der}^k(S) = \left\{\theta \mid \theta = h_{k,0,0}\Big(\frac{\partial}{\partial x}\Big)^k + h_{k-1,1,0}\Big(\frac{\partial}{\partial x}\Big)^{k-1}\frac{\partial}{\partial y} + \ldots + h_{0,0,k}\Big(\frac{\partial}{\partial z}\Big)^k\right\} \cong S^{\binom{k+2}{2}},$$

where $h_{i_1,i_2,i_3} \in S$. Please note that here $\left(\frac{\partial}{\partial x}\right)^{i_1}\left(\frac{\partial}{\partial y}\right)^{i_2}\left(\frac{\partial}{\partial z}\right)^{i_3}$ is a product of partial derivatives of first order, not a derivative of higher order.

In what follows $J = (f_x, f_y, f_z)$ always denotes the Jacobian ideal of f.

Let us now recall the definition of unexpected curves. Let I_Z be the saturated homogeneous ideal of a finite set Z of pairwise different points in $\check{\mathbb{P}}^2$ and let P be a generic point in $\check{\mathbb{P}}^2$. Given a homogeneous ideal $I \subseteq S$ we denote by $[I]_t$ the vector space of all forms in I of degree t. Let k be a positive integer.

Definition 2.1. We say that a curve C given as a zero set of a form in $[I_Z]_{d+k}$, having a point of multiplicity d at P, is unexpected of type $(d + k, d)$ if

$$\dim[I_{P\cup Z}]_{d+k} > \max\left(0, \dim[I_Z]_{d+k} - \binom{d+1}{2}\right);$$

i.e., C is unexpected if vanishing to order d at P imposes on $[I_Z]_{d+k}$ fewer than the expected number of conditions. Moreover, we assume (as some authors do and some do not) that vanishing in Z imposes independent conditions on the forms of degree $d + k$.

As we mentioned above, the existence of unexpected curves of type $(d+1, d)$ is explained in [3] by considering the curve as an image of a certain map from a line $\mathbb{P}^1$ to $\mathbb{P}^2$, where the map is defined with the help of the syzygies of the Jacobian ideal of f.

This construction is the starting point for our paper. We describe it from a slightly modified point of view.

Construction 2.2. [3]

- Take a set Z of pairwise different points in a projective plane. Treat this plane as a dual projective plane $\check{\mathbb{P}}^2$.
- The points of Z give dual lines in $\mathbb{P}^2$ with the equations given by the forms $l_1, \ldots, l_{|Z|}$. Let $f = \ell_1 \cdots \ell_{|Z|}$.
- Take a generic line $L \in \mathbb{P}^2$, with its dual point $P_L = (\alpha, \beta, \gamma)$. Take a point $Q = (a, b, c)$ in L.
- Let J denote the Jacobian ideal of f with fixed generators. Take a syzygy of $J + (L)$ (of minimal degree), say (s_0, s_1, s_2, s_3). Thus, for any point, $Q = (a, b, c) \in \check{\mathbb{P}}^2$ we have $0 = s_0(Q)f_a(Q) + s_1(Q)f_b(Q) + s_2(Q)f_c(Q) + s_3(Q)L(Q)$. If $Q \in L$, then $L(Q) = 0$ and we have $0 = s_0(Q)f_a(Q) + s_1(Q)f_b(Q) + s_2(Q)f_c(Q)$, where s_0, s_1, s_2 are of the same degree, say d. In such a case we will denote $(s_0(Q), s_1(Q), s_2(Q))$ as $s(Q)$.
- Then take two lines in the dual plane: ℓ_Q (dual to Q) and $\ell_{s(Q)}$, dual to $s(Q)$. The lines (in general) intersect in a point P, so we have a map: $L \ni Q \to P \in \check{\mathbb{P}}^2$.
- The map is not defined if $\ell_{s(Q)} = \ell_Q$. It may happen only when Q is a point of intersection of L and f.
- In [3] it is proved that when a point Q moves along the generic line L, the image P moves along a curve C of degree $d + 1$.
- The curve C constructed this way passes through all such points of Z that the map is defined there, and has a point of multiplicity d in $P_L = \check{L} = (\alpha, \beta, \gamma)$.

3 Syzygies-based construction

The following theorem and its proof describe a construction, generalizing Construction 2.2. By means of this generalized construction, we will get curves of type $(d + k, d)$, for $k \geqslant 1$.

Theorem 3.1. *Let Z be a set of $|Z|$ points in $\check{\mathbb{P}}^2$ and let L be a generic line on $\mathbb{P}^2$ with the equation $\alpha a + \beta b + \gamma c = 0$. Denote by $(g_{k,0,0}, \ldots, g_{0,0,k}, g)$ a (reduced) syzygy of $J^k + (L)$ where g_{i_1,i_2,i_3} are all of degree d, i.e., for any $Q = (a : b : c) \in \mathbb{P}^2$ we have*

$$g_{k,0,0}(Q) f_x(Q)^k + g_{k-1,1,0}(Q) f_x(Q)^{k-1} f_y(Q) + \cdots + g_{0,0,k}(Q) f_z(Q)^k + g(Q)L(Q) = 0.$$

Take any $Q = (a, b, c) \in L$ and define S_Q as the curve of degree k in $\check{\mathbb{P}}^2$ given by the equation

$$S_Q(x, y, z) := g_{k,0,0}(Q)x^k + g_{k-1,1,0}(Q)x^{k-1}y + \cdots + g_{0,0,k}(Q)z^k = 0.$$

Observe that, in fact, we have a one-dimensional family of such curves. Take again $Q = (a, b, c) \in L$. Consider in $\check{\mathbb{P}}^2$ the following system of equations

$$\begin{cases} \alpha a + \beta b + \gamma c = 0 \\ ax + by + cz = 0 \\ g_{k,0,0}(a,b,c)x^k + g_{k-1,1,0}(a,b,c)x^{k-1}y + \cdots + g_{0,0,k}(a,b,c)z^k = 0 \end{cases} \qquad (\star)$$

in variables $a, b, c, \alpha, \beta, \gamma, x, y, z$ (from which we will eliminate a, b, c). We will say that this system is not determined *in $Q = (a, b, c) \in L$ if for all (x, y, z) we have*

$$g_{k,0,0}(a,b,c)x^k + g_{k-1,1,0}(a,b,c)x^{k-1}y + \cdots + g_{0,0,k}(a,b,c)z^k = (ax + by + cz)^k.$$

Let $P_L = \check{L} = (\alpha, \beta, \gamma)$. Then:

1. *The system $(\star)$ is not determined, only for points Q on $\mathcal{A} \cap L$.*
2. *All the solutions (x, y, z) to the system $(\star)$ lie on a curve in $\check{P}^2$, of degree (at most) $d + k$. Denote this curve by C_L.*
3. *C_L passes through Z.*
4. *C_L has a point of multiplicity at least d in P_L.*
5. *The curve C_L may be treated as $C_L(x, y, z)$ with parameters (α, β, γ) and "dually" as $C_L(\alpha, \beta, \gamma)$ with parameters (x, y, z). The partial derivatives computed in point (α, β, γ) with respect to (x, y, z) and computed in point (x, y, z) with respect to (α, β, γ) are the same up to order d.*

Proof.

Ad 1) If the system $(\star)$ is not determined in a point $Q = (a, b, c)$ on the line L then, in particular, for $(x, y, z) := (f_a(Q), f_b(Q), f_c(Q))$ we have

$$0 = g_{k,0,0}(Q) f_a(Q)^k + g_{k-1,1,0}(Q) f_a(Q)^{k-1} f_b(Q) + \cdots + g_{0,0,k}(Q) f_c(Q)^k = \\ = (a f_a(Q) + b f_b(Q) + c f_c(Q))^k.$$

From Euler's Homogeneous Function Theorem $a f_a(Q) + b f_b(Q) + c f_c(Q) = \deg(f) f(Q)$, so $f(Q) = 0$

Ad 2) Here we show that the curve C_L passes through all such points of Z where ($\star$) is determined. We have that C_L passes through $z_j \in Z$ if $z_j \in L_Q$ and if also $z_j \in S_Q$, for a point $Q \in L$. Let then $z_j = (z_{j0}, z_{j1}, z_{j2})$. Observe that from duality $z_j \in L_Q$ if and only if $Q \in \ell_j = \check{z}_j$, so Q must be the intersection point of ℓ_j and L. For such an intersection point Q and from the fact that g_{i_1,i_2,i_3} are syzygies of J^k, we have:

$$g_{k,0,0}(Q)f_x(Q)^k + g_{k-1,1,0}(Q)f_x(Q)^{k-1}f_y(Q) + \cdots + g_{0,0,k}(Q)f_z(Q)^k = 0.$$

As $f = \ell_1 \cdots \ell_j \cdots \ell_{|Z|}$, we have

$$(f_x(Q), f_y(Q), f_z(Q)) = \nabla_Q f = h(Q)\nabla_Q \ell_j = h(Q)(z_{j0}, z_{j1}, z_{j2}),$$

with a suitable polynomial h satisfying $h(Q) \neq 0$. Thus,

$$g_{k,0,0}(Q)z_{j0}^k + g_{k-1,1,0}(Q)z_{j0}^{k-1}z_{j1} + \cdots + g_{0,0,k}(Q)z_{j2}^k = 0.$$

This gives that $z_j \in S_Q$, so C_L passes through z_j.

Ad 3) If ($\star$) is not determined at $Q_i = L \cap l_{z_i}$, then L_{Q_i} is contained in the set of solutions of ($\star$). Of course, $z_i \in L_{Q_i}$ (as $Q_i \in l_{z_i}$).

From 2) and 3) we see that C_L passes through all $z \in Z$.

Ad 4) The curve C_L has a point of multiplicity d in $P_L = \check{L} = (\alpha, \beta, \gamma)$. Indeed, observe that as the point P_L belongs to every L_Q, where $Q \in L$, we need only to check that P_L belongs to d of curves S_Q. Let H_L be a hyperplane in $\mathbb{P}^r$, where $r = \binom{k+2}{2} - 1$, defined as

$$H_L = \left\{(x_0, \ldots, x_r) : \alpha^k x_0 + \alpha^{k-1}\beta x_1 + \cdots + \gamma^k x_r = 0\right\}.$$

Then $\{G(Q) := (g_{k,0,0}(Q), \ldots, g_{0,0,k}(Q)), Q \in L\}$ is a curve of degree d in $\mathbb{P}^r$. From the genericity of L (and so of H_L) it cuts the hyperplane H_L in d pairwise different points, say $Q_1, \ldots, Q_d$. For any such point Q_i, with $i = 1, \ldots d$, we have

$$g_{k,0,0}(Q_i)\alpha^k + g_{k-1,1,0}(Q_i)\alpha^{k-1}\beta + \cdots + g_{0,0,k}(Q_i)\gamma^k = 0,$$

so $P_L \in S_{Q_i}$.

Ad 5) Here we describe a very explicit construction of the curve C_L. This part of the proof proves 5 and is an alternative proof of 4.

For $(a, b, c) \in L$, where $L = \alpha a + \beta b + \gamma c = 0$ is a generic line on $\mathbb{P}^2$, the solutions of the two equations:

$$ax + by + cz = 0$$

and

$$g_{k,0,0}(a,b,c)x^k + g_{k-1,1,0}(a,b,c)x^{k-1}y + \cdots + g_{0,0,k}(a,b,c)z^k = 0$$

lie on a curve of degree $d+k$. Indeed, take a point $(a,b,c) \in L$. As the syzygy is reduced, we may assume, without loss of generality, that a^d appears in at least one g_{i_1,i_2,i_3}. We may also assume that $c = 1$, and that $\alpha, \beta \neq 0$, so

$$b = \frac{-\alpha a - \gamma}{\beta}.$$

Thus

$$g_{k,0,0}(a,b,c)x^k + g_{k-1,1,0}(a,b,c)x^{k-1}y + \cdots + g_{0,0,k}(a,b,c)z^k = 0$$

is equivalent to

$$g_{k,0,0}\left(a, \frac{-\alpha a - \gamma}{\beta}, 1\right) x^k + g_{k-1,1,0}\left(a, \frac{-\alpha a - \gamma}{\beta}, 1\right) x^{k-1}y + \cdots + g_{0,0,k}\left(a, \frac{-\alpha a - \gamma}{\beta}, 1\right) z^k = 0.$$

As g_i are of degree d we may reorder the last equation to get:

$$a^d h_1(x,y,z) + a^{d-1}h_2(x,y,z) + \cdots + h_d(x,y,z) = 0 \tag{1}$$

with some homogeneous polynomials $h_j(x,y,z)$ of degree k (and in general depending also on α, β, γ). On the other hand, putting $b = \frac{-\alpha a - \gamma}{\beta}$ and $c = 1$ into the equation $\alpha a + \beta b + \gamma c = 0$, we get

$$\beta a x + (-\alpha a - \gamma)y + \beta z = 0.$$

Thus for all points (x,y,z) except (α,β,γ) we have

$$a = \frac{\gamma y - \beta z}{\beta x - \alpha y}. \tag{2}$$

Then we substitute a by (2) into the equation (1) and multiply by $(\beta x - \alpha y)^d$. We get:

$$(\gamma y - \beta z)^d h_1(x,y,z) + (\gamma y - \beta z)^{d-1}(\beta x - \alpha y)h_2(x,y,z) + \cdots + h_d(x,y,z)(\beta x - \alpha y)^d = 0. \tag{3}$$

This way we obtain an equation (3) of curve C_L of degree $d+k$ in variables x,y,z, with a d-fold point at $(\alpha,\beta,\gamma) = \check{L}$. Moreover, the partial derivatives of the left-hand side of **(3)** computed with respect to (x,y,z) in the point $(x,y,z) := \check{L}$ and computed with respect to (α,β,γ) in the point $(\alpha,\beta,\gamma) := \check{L}$ are the same up to order d.

□

Remark 3.2. We also provide an alternative more direct proof for the assertion 3) in Theorem 3.1, that if the system $(\star)$ is not determined in a point Q_1 on L, then the line L_{Q_1} is a component of C_L. Indeed, in the above construction, we take the curve

$$g_{k,0,0}(a,b,c)x^k + g_{k-1,1,0}(a,b,c)x^{k-1}y + \cdots + g_{0,0,k}(a,b,c)z^k = 0,$$

replace a, b by $\frac{\gamma y-\beta z}{\beta x-\alpha y}$ and $\frac{\alpha z-\gamma x}{\beta x-\alpha y}$ and multiply by $(\beta x-\alpha y)^d$. We get

$$g_{k,0,0}\left(\frac{\gamma y-\beta z}{\beta x-\alpha y}, \frac{\alpha z-\gamma x}{\beta x-\alpha y}, 1\right) x^k + g_{k-1,1,0}\left(\frac{\gamma y-\beta z}{\beta x-\alpha y}, \frac{\alpha z-\gamma x}{\beta x-\alpha y}, 1\right) x^{k-1}y + \dots$$

$$\dots + g_{0,0,k}\left(\frac{\gamma y-\beta z}{\beta x-\alpha y}, \frac{\alpha z-\gamma x}{\beta x-\alpha y}, 1\right) z^k = 0. \qquad (\triangle)$$

Assume that a point, say $(a_1, b_1, 1) = Q_1 = L \cap l_{z_1}$, is such that the system $(\star)$ is not determined. Then

$$\begin{cases} g_{k,0,0}(a_1, b_1, 1) = a_1^k \\ g_{k-1,1,0}(a_1, b_1, 1) = k a_1^{k-1} b_1 \\ g_{k-2,2,0}(a_1, b_1, 1) = \binom{k}{2} a_1^{k-2} b_1^2 \\ \dots \\ g_{0,0,k}(a_1, b_1, 1) = 1. \end{cases} \tag{4}$$

Put $z = -a_1x - b_1y$ into $(\triangle)$. Then we get

$$\frac{\gamma y - \beta(-a_1x - b_1y)}{\beta x - \alpha y} = a_1, \qquad \frac{\alpha(-a_1x - b_1y) - \gamma x}{\beta x - \alpha y} = b_1.$$

So in $(\triangle)$ we get

$$g_{k,0,0}(a_1, b_1, 1)x^k + g_{k-1,1,0}(a_1, b_1, 1)x^{k-1}y + \dots + g_{0,0,k}(a_1, b_1, 1)(-a_1x - b_1y)^k,$$

thus L_{Q_1} divides C_L.

4 Derivations

Motivated by relations between syzygies of J/fS and the module of derivatives $\mathrm{D}_0(\mathcal{A})$, we exhibit in this section relations between syzygies of $(J/fS)^k$ and the module of higher order derivatives $\mathrm{D}_0^k(\mathcal{A})$. Our main result in this section is Proposition 4.4. The connection between the syzygies of $(J/fS)^k$ and a bundle of derivations $\mathrm{D}_0^k(\mathcal{A})$ is analogous to the connection between $\mathrm{syz}(J/fS)$ and $\mathrm{D}_0(\mathcal{A})$. The generalization of logarithmic derivations was already studied in [26] (with some additional assumptions). We present here detailed proofs, to make the paper self-contained, to establish the notation and to expose the connection with the syzygies. The reader familiar with the subject may skip the proofs in this section, whereas the reader not familiar with the relation between $\mathrm{syz}(J/fS)$ and $\mathrm{D}_0(\mathcal{A})$ may want to see Appendix to [3] for a detailed introduction to the subject, as well as to the papers [13, 22, 26] for articles also related to this subject.

Having Proposition 4.4 we will in Section 5 establish a relation between the degree of a curve which can be obtained from the construction described in Theorem 3.1 and the exponents in the splitting type of the bundle given by $\mathrm{D}_0^k(\mathcal{A})$ restricted to a general line L, giving a sufficient condition for such a curve to be unexpected, see Theorem 5.5.

Let $J = (f_x, f_y, f_z)$. We have an exact sequence

$$0 \longrightarrow \mathrm{D}^k(\mathcal{A}) \longrightarrow S^{\binom{k+2}{2}} \xrightarrow{\ \phi\ } ((J/fS)(|Z|-1))^k \longrightarrow 0,$$

where, for an element, $(g_{k,0,0}, g_{k-1,1,0}, \ldots, g_{0,0,k}) \in S^{\binom{k+2}{2}}$ there is

$$\phi\left(\begin{bmatrix} g_{k,0,0} \\ \vdots \\ g_{0,0,k} \end{bmatrix}\right) = g_{k,0,0} \cdot \left(\frac{\partial f}{\partial x}\right)^k + g_{k-1,1,0} \cdot \left(\frac{\partial f}{\partial x}\right)^{k-1} \frac{\partial f}{\partial y} + \ldots + g_{0,0,k} \cdot \left(\frac{\partial f}{\partial z}\right)^k \pmod f,$$

and $\mathrm{D}^k(\mathcal{A}) \subset \mathrm{Der}^k(S)$ is the set of such derivations δ that $\delta(f) \in J^{k-1}(f)$. Let us remind, that here $\left(\frac{\partial}{\partial x}\right)^{i_1}\left(\frac{\partial}{\partial y}\right)^{i_2}\left(\frac{\partial}{\partial z}\right)^{i_3}$ is a product of partial derivatives of the first order, not a derivative of a higher order.

In order to define our main object of this section, the module $\mathrm{D}_0^k(\mathcal{A})$, we need the following.

Definition 4.1. Let $i_1 + i_2 + i_3 = k - 1$. For all $j \in \left\{1, 2, \ldots, \binom{k+1}{2}\right\}$ we define a derivation $E_j \in D^k(\mathcal{A})$:

$$E_j := (dx)^{i_1}(dy)^{i_2}(dz)^{i_3}\Big(M(k)\Big) \star P(k),$$

where $M(k)$ and $P(k)$ are $\underbrace{k+1 \times \cdots \times k+1}_{k}$ dimensional matrices which consist of all monomials and derivatives of degree and rank k respectively. The symbol $\star$ denotes the Hadamard product of matrices.

Example 4.2. For $k = 2$ we have that,

$$E_1 = dx(M(2)) \star P(2) = dx\left(\begin{bmatrix} x^2 & xy & xz \\ yx & y^2 & yz \\ zx & zy & z^2 \end{bmatrix}\right) \star \begin{bmatrix} \partial_x \cdot \partial_x & \partial_x \cdot \partial_y & \partial_x \cdot \partial_z \\ \partial_y \cdot \partial_x & \partial_y \cdot \partial_y & \partial_y \cdot \partial_z \\ \partial_z \cdot \partial_x & \partial_z \cdot \partial_y & \partial_z \cdot \partial_z \end{bmatrix} =$$

$$= \begin{bmatrix} 2x & y & z \\ y & 0 & 0 \\ z & 0 & 0 \end{bmatrix} \star \begin{bmatrix} \partial_x \cdot \partial_x & \partial_x \cdot \partial_y & \partial_x \cdot \partial_z \\ \partial_y \cdot \partial_x & \partial_y \cdot \partial_y & \partial_y \cdot \partial_z \\ \partial_z \cdot \partial_x & \partial_z \cdot \partial_y & \partial_z \cdot \partial_z \end{bmatrix} = \begin{bmatrix} 2x\partial_x \cdot \partial_x & y\partial_x \cdot \partial_y & z\partial_x \cdot \partial_z \\ y\partial_y \cdot \partial_x & 0 & 0 \\ z\partial_z \cdot \partial_x & 0 & 0 \end{bmatrix}.$$

Let us define a basis of symmetric 3×3 matrices:

$$e_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, e_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, e_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, e_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Thus E_1 can be identified with a vector $E_1 = [x\partial_x \cdot \partial_x, y\partial_x \cdot \partial_y, z\partial_x \cdot \partial_z, 0, 0, 0]$. Using this basis we also obtain

$$E_2 = dy(M(2)) \star P(2) = [0, x\partial_x \cdot \partial_y, 0, y\partial_y \cdot \partial_y, z\partial_y \cdot \partial_z, 0],$$
$$E_3 = dz(M(2)) \star P(2) = [0, 0, x\partial_x \cdot \partial_z, 0, y\partial_y \cdot \partial_z, z\partial_z \cdot \partial_z].$$

Observe that the action of ϕ on E_j gives an element from $J \cdot (f)$. Indeed, $\phi(E_1) = f_x \cdot f$ and so on. More generally, when $E_j \in D^k(\mathcal{A})$, then $\phi(E_j) = (f_x)^{i_1}(f_y)^{i_2}(f_z)^{i_3} f \in J^{k-1} \cdot (f)$.

Definition 4.3. We define the module

$$\mathrm{D}_0^k(\mathcal{A}) = \mathrm{D}^k(\mathcal{A}) \Big/ SE_1 \oplus \cdots \oplus SE_{\binom{k+1}{2}}.$$

The following result will be used in Section 5.

Proposition 4.4. *The sheafification of the module $D^k(\mathcal{A})$ is a vector bundle of rank $\binom{k+2}{2}$. The sheafification of $D_0^k(\mathcal{A})$ is a vector bundle of rank $k+1$, with the first Chern class equal $\frac{k(k+1)}{2} - |Z|$.*

Proof. We use the exact sequence below.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & & \mathrm{syz}((J(|Z|-1))^k) & & D^k(\mathcal{A}) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J^{k-1}fS(-1) & \longrightarrow & J^{k-1}fS(-1) \oplus S^{\binom{k+2}{2}} & \longrightarrow & S^{\binom{k+2}{2}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \phi & & \\
0 & \longrightarrow & J^{k-1}fS(-1) & \xrightarrow{\times f} & (J(|Z|-1))^k & \longrightarrow & ((J/fS)(|Z|-1))^k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The middle column has a free module in the middle. The sheafification of J^k is torsion-free. Thus, $\mathrm{syz}(J^k)$ is after sheafification a reflexive sheaf $(\widetilde{\mathrm{syz}(J^k)})$. On a surface, a reflexive sheaf is locally free (see [21]), so $(\widetilde{\mathrm{syz}(J^k)})$ is a vector bundle of rank $\binom{k+2}{2}$, and so is $\widetilde{D^k(\mathcal{A})}$.

We may, analogously as it was done in the case $k = 1$ at the end of Appendix in [3], represent $D^k(\mathcal{A})$ as a direct sum, where one summand is $SE_1 \oplus \cdots \oplus SE_{\binom{k+1}{2}}$ and the other is the module of such derivations δ that $\delta(f) = 0$. Taking the quotient of $D^k(\mathcal{A})$ by this first summand we get $D_0^k(\mathcal{A})$.

Observe that $\widetilde{D_0^k(\mathcal{A})}$ is a vector bundle of rank $k+1$ as it arises as a division of $\widetilde{D^k(\mathcal{A})}$ by the sum of the sheafification of SE_j for $j = 1, \dots, \binom{k+1}{2}$, and $\bigoplus \widetilde{SE_j}$ corresponds to a global non-vanishing section of $\widetilde{D^k(\mathcal{A})}$.

To get the Chern class of $\widetilde{D_0^k(\mathcal{A})}$, consider the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & SE_1 \oplus \cdots \oplus SE_{\binom{k+1}{2}} & \longrightarrow & SE_1 \oplus \cdots \oplus SE_{\binom{k+1}{2}} & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{D}^k(\mathcal{A}) & \longrightarrow & S^{\binom{k+2}{2}} & \xrightarrow{\ \phi\ } & ((J/fS)(|Z|-1))^k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{D}_0^k(\mathcal{A}) & \longrightarrow & S^{\binom{k+2}{2}} \big/ SE_1 \oplus \cdots \oplus SE_{\binom{k+1}{2}} & \longrightarrow & ((J/fS)(|Z|-1))^k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

It is known that the sheafification of S^3/E is $T_{\mathbb{P}^2}(-1)$, so we prove that have an isomorphism $\widetilde{\left(S^{\binom{k+2}{2}} \Big/ \bigoplus_{j=1}^{\binom{k+1}{2}} SE_j\right)} \cong \mathrm{Sym}^k(T_{\mathbb{P}^2}(-1))$. Indeed, denote by $\sim$ the permutation action of the symmetric group Σ_n. Then we have the consecutive isomorphisms

$$\mathrm{Sym}^k(S^3/E) \quad \cong \quad \bigotimes_{i=1}^k S^3 \Big/ \bigoplus_{j=1}^k S^3 \otimes S^3 \otimes \cdots \otimes \overset{(j)}{E} \otimes \cdots \otimes S^3 \Big/ \sim \quad \cong$$

$$\bigotimes_{i=1}^k S^3 /\sim \Big/ \bigoplus_{j=1}^k S^3 \otimes S^3 \otimes \cdots \otimes \overset{(j)}{E} \otimes \cdots \otimes S^3 /\sim \quad \cong \quad S^{\binom{k+2}{2}} \Big/ \bigoplus_{j=1}^{\binom{k+1}{2}} SE_j \ ,$$

and after taking the sheafification we get the assertion.

To get $c_1(\mathrm{Sym}^k(T_{\mathbb{P}^2}(-1))$ we use the results on symmetric powers of vector bundles and their Chern classes, see e.g. [**?**, 19]. We obtain that

$$c_1(\mathrm{Sym}^k(T_{\mathbb{P}^2}(-1)) = \binom{k+1}{2} \cdot c_1(T_{\mathbb{P}^2}(-1)) = \binom{k+1}{2}.$$

We have also that

$$c_1\big(((\mathcal{J}/f\mathcal{O}_{\mathbb{P}^2})(|Z|-1))^k\big) = |Z|,$$

where $\mathcal{J}$ is the sheafification of J. Indeed, this is done in [13, Theorem 2]. The authors prove there that the kernel of the exact sequence

$$0 \to K \to \mathrm{Sym}^k(T_{\mathbb{P}^2}(-1)) \to ((J/fS)(|Z|-1))^k \to 0,$$

has $c_1(K) = \binom{k+1}{2} - |Z|$. They prove also that this kernel is unique (we repeat this proof below, in Lemma 5.3), so K is isomorphic to $\widetilde{D_0^k(\mathcal{A})}$ (cf. the proof of Lemma 5.3). □

5 Unexpectedness

In the last section, we have seen that the syzygies of $((J/fS)(|Z|-1))^k$ form (after sheafification) a vector bundle of rank $k+1$. Thus, this bundle splits as L to $\mathcal{O}_L(-a_1) \oplus \cdots \oplus \mathcal{O}_L(-a_{k+1})$, with $0 \leqslant a_1 \leqslant \cdots \leqslant a_{k+1}$.

Let us remind, that the construction presented in Section 2 gives a curve $C = C_L$ of degree $a_i + k$ passing through a generic point P_L with multiplicity a_i, so this is a curve of type $(a_i + k, a_i)$. The next result says when such a curve is unexpected. This result is related to Theorem 1.5 from [3].

Proposition 5.1. *Take the syzygies of $((J/fS)(|Z|-1))^k$ of degree a_i. The curve C of type $(a_i + k, a_i)$ (constructed as in Section 3) is unexpected if:*

1. *Z imposes independent conditions on curves of degree $a_i + k$ and*
2. *$(a_i + 1)(k + 1) \leqslant \sum_{j=1}^{k+1} a_j$.*

Proof. Indeed, under our assumptions, C is unexpected when

$$\binom{a_i + k + 2}{2} - |Z| - \binom{a_i + 1}{2} \leqslant 0.$$

This is equivalent to

$$ka_i + a_i + \frac{k(k+3)}{2} + 1 \leqslant |Z|.$$

Let $\mathcal{S}$ be the (rank $k+1$) bundle of syzygies of $((J/fS)(|Z|-1))^k$. Remember that $\mathcal{S}$ splits over L to $\mathcal{O}_L(-a_1) \oplus \cdots \oplus \mathcal{O}_L(-a_{k+1})$ with $a_1 \leqslant \cdots \leqslant a_{k+1}$, and this gives

$$a_1 + \cdots + a_{k+1} = -\frac{k(k+1)}{2} + |Z|.$$

On the other hand, if $a_1 + \cdots + a_{k+1} = -\frac{k(k+1)}{2} + |Z|$, we have, by Assumption 2,

$$|Z| = \frac{k(k+1)}{2} + a_1 + \cdots + a_{k+1} \geqslant \frac{k(k+1)}{2} + (k+1)(a_i+1) = ka_i + a_i + \frac{k(k+3)}{2} + 1.$$

□

The above Proposition explains, for example, the unexpectedness of the curve of type $(9,7)$ for DF_5 (see Example 6.3 below), or of type $(7,4)$ for DF_5-without two points $(1,e,e^2),(1,e^2,1)$. However, it does not explain the unexpectedness of the curve of type $(8,5)$ for DF_5. To say something about the unexpectedness of a curve of type $(d+k,d)$ with positive expected dimension, we have to prove a lemma, generalizing Lemma 3.3 from [3]. Let us quote:

Lemma 5.2 (Lemma 3.3 of [3]). *Let Z be a set of points on $\check{\mathbb{P}}^2$ and let P be a general point on $\mathbb{P}^2$. Let f denote, as above, the product of lines dual to the points of Z. Let $\mathcal{S}$ be the (rank 2) bundle of syzygies of $(J/fS)(|Z|-1)$. This bundle splits on a generic line L (dual to P), with the splitting type (a,b).*

Then, for each integer j

$$\dim[I_Z + jP]_{j+1} = \max\{0, j-a+1\} + \max\{0, j-b+1\}.$$

The generalization is the following:

Lemma 5.3. *Let Z, P, f and L be as above. Let $\mathcal{S}$ be the (rank $k+1$) bundle of syzygies of $((J/fS)(|Z|-1))^k$. This bundle splits on a generic line L (dual to P), with the splitting type $(a_1, a_2, \dots a_{k+1})$.*

Then, for each integer j

$$\dim[I_Z + jP]_{j+k} = \max\{0, j-a_1+1\} + \cdots + \max\{0, j-a_{k+1}+1\}.$$

Proof. For the proof of this Lemma we need the construction described by Faenzi and Vallès in [13]. They consider the flag variety $\mathbb{F} = \{(Q,l) \in \mathbb{P}^2 \times \check{\mathbb{P}}^2 \mid Q \in l\}$. By p,q they denote the projections to the first and the second factor, respectively. Then they consider the sheaf $p_*q^*I_Z(1)$ and they prove that this sheaf is isomorphic to the logarithmic derivation bundle $\widetilde{D_0^1(\mathcal{A}_Z)}$, so also it is isomorphic with the syzygies of $(J/fS)(|Z|-1)$.

We want to prove an extension of this result to $k>1$, namely the following claim.

Claim:

$$p_*q^*I_Z(k) \cong \mathcal{S},$$

where $\mathcal{S}$ is, as above, the rank $k+1$ syzygy bundle of $((J/fS)(|Z|-1))^k$.

Proof of the claim:

The first part of the proof concerns the kernel K of a map ϕ:

$$0 \to K \to \mathrm{Sym}^k(T_{\mathbb{P}^2}(-1)) \xrightarrow{\phi} \bigoplus_{z\in Z} \mathcal{O}_{l_z}.$$

We prove that this kernel is unique up to an isomorphism, repeating the reasoning from [13]. We repeat the arguments in order to make the proof explicit and the paper self-contained. We underline explicitly whenever we wander from the original proof, i.e., from the case of $k = 1$ to the case $k > 1$.

First, observe that $\mathrm{Hom}(T_{\mathbb{P}^2}(-1), \mathcal{O}_{L_z}) = \mathbb{C}$. This follows from the fact that $H^0(\mathrm{Hom}(T_{\mathbb{P}^2}(-1), \mathcal{O}_{L_z})) = H^0(T_{\mathbb{P}^2}(-1)^\vee \otimes \mathcal{O}_{L_z}) = H^0(\Omega_{\mathbb{P}^2}(1) \otimes \mathcal{O}_{L_z})$. Take the sequence

$$0 \to \mathcal{O}_{L_z} \to \Omega_{L_z}(1) \oplus \mathcal{O}_{L_z} \to \Omega_{L_z}(1) \to 0.$$

Using the fact that $\Omega_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1} = \Omega_{\mathbb{P}^2}(1) \otimes \mathcal{O}_{\mathbb{P}^1}$ and that $\Omega_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2)$, we get

$$0 \to \mathcal{O}_{L_z} \to \Omega_{\mathbb{P}^2}(1) \otimes \mathcal{O}_{L_z} \to \mathcal{O}_{L_z}(-1) \to 0.$$

Taking the long sequence of cohomologies, we get $H^0(\Omega_{\mathbb{P}^2}(1) \otimes \mathcal{O}_{L_z}) = H^0(\mathcal{O}_{\mathbb{P}^1}) = \mathbb{C}$.

Now we have to show that, for $k \geqslant 2$, there is $\mathrm{Hom}(\mathrm{Sym}^k(T_{\mathbb{P}^2}(-1)), \mathcal{O}_{L_z}) = \mathbb{C}$. This part of the proof is not covered by [13], as it concerns $k \geqslant 2$ case. Here we use the facts that the dual of a symmetric power is the symmetric power of the dual space and that the symmetric power of a tensor product is given by the following formula:

$$\mathrm{Sym}^k(V \otimes W) = \bigoplus_{\lambda \vdash k} \mathbb{S}^\lambda V \otimes \mathbb{S}^\lambda W,$$

where λ is a partition of k giving Young tableau with at most minimum of $\dim V$, $\dim W$ rows and $\mathbb{S}$ is the Schur functor, see [16, Exercise 6.11]. In our case, we will apply this formula to $\mathrm{Sym}^k(\Omega(1) \otimes \mathcal{O}_{L_z})$. The only possible partition gives one row in Young tableau, and we obtain

$$\mathrm{Sym}^k(\Omega(1) \otimes \mathcal{O}_{L_z}) = \mathrm{Sym}^k(\Omega(1)) \otimes \mathrm{Sym}^k(\mathcal{O}_{L_z}) = \mathrm{Sym}^k(\Omega(1)) \otimes \mathcal{O}_{L_z}.$$

Using the above, we have

$$\mathrm{Hom}(\mathrm{Sym}^k(T_{\mathbb{P}^2}(-1)), \mathcal{O}_{L_z}) = H^0((\mathrm{Sym}^k(T_{\mathbb{P}^2}(-1))^\vee \otimes \mathcal{O}_{L_z})) = H^0(\mathrm{Sym}^k(\Omega_{\mathbb{P}^2}(1) \otimes \mathcal{O}_{L_z})).$$

Now we proceed by induction. For $k = 1$ we have $H^0(\Omega_{\mathbb{P}^2}(1) \otimes \mathcal{O}_{L_z})$ equal to $\mathbb{C}$. Assume that $H^0(\mathrm{Sym}^j(\Omega_{\mathbb{P}^2}(1) \otimes \mathcal{O}_{L_z})) = \mathbb{C}$ for $j < k$ and take the k-th symmetric power of the sequence

$$0 \to \mathcal{O}_{L_z} \to \Omega_{\mathbb{P}^2}(1) \otimes \mathcal{O}_{L_z} \to \mathcal{O}_{L_z}(-1) \to 0,$$

obtaining

$$0 \to \mathcal{O}_{L_z} \otimes \mathrm{Sym}^{k-1} \Omega_{\mathbb{P}^2}(1) \to \mathrm{Sym}^k(\Omega_{\mathbb{P}^2}(1)) \otimes \mathcal{O}_{L_z} \to \mathcal{O}_{L_z}(-k) \to 0.$$

As the global sections of $\mathcal{O}_{L_z}(-k)$ are 0, from the inductive assumption, we are done.

The next paragraph of the proof is done in [13] (proof of Theorem 1) for $k = 1$ case; we extend it to $k \geqslant 2$ case using the above. Thus, we know that all the maps from $\mathrm{Sym}^k(T_{\mathbb{P}^2}(-1))$ to $\bigoplus_{z \in Z} \mathcal{O}_{L_z}$, where $k \geqslant 1$ are given by a choice of constants $(\alpha_z)_{z \in Z}$. Now assume that we chose two sets of such constants, $(\alpha_z)_{z\in Z}$ and $(\beta_z)_{z\in Z}$. Assume additionally that all the constants α_z and β_z are nonzero. For two choices of such nonzero constants, consider the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_1 & \longrightarrow & \mathrm{Sym}^k(T_{\mathbb{P}^2}(-1)) & \xrightarrow{\alpha} & \bigoplus_{z\in Z}\mathcal{O}_{L_z} \\ & & \downarrow & & \downarrow{=} & & \downarrow{\gamma=\frac{\alpha}{\beta}} \\ 0 & \longrightarrow & K_2 & \longrightarrow & \mathrm{Sym}^k(T_{\mathbb{P}^2}(-1)) & \xrightarrow{\beta} & \bigoplus_{z\in Z}\mathcal{O}_{L_z} \end{array}$$

From this diagram (and the fact that the map γ has an inverse, as α_z and β_z are nonzero) we see that K_1 and K_2 are isomorphic.

Now take any $z \in Z$ and the sequence

$$0 \to p_*q^*I_z(k) \to \mathrm{Sym}^k(T_{\mathbb{P}^2}(-1)) \xrightarrow{\alpha} \mathcal{O}_{L_z}.$$

If $\alpha = 0$, then $p_*q^*I_z(k) \cong \mathrm{Sym}^k(T_{\mathbb{P}^2}(-1))$. On the other hand, from [13, Theorem 2], we have that $c_1(p_*q^*I_z(k)) = \binom{k+1}{2} - 1$, and we also know that $c_1(\mathrm{Sym}^k(T_{\mathbb{P}^2}(-1))) = \binom{k+1}{2}$ so we get a contradiction.

To get the claim for $p_*q^*I_Z(k)$ we apply p_*q^* to $0 \to I_Z(k) \to \mathcal{O}_{\mathbb{P}^2}(k) \to \mathcal{O}_Z(k) \to 0$, obtaining, as in [13],

$$0 \to p_*q^*I_Z(k) \to \mathrm{Sym}^k(T_{\mathbb{P}^2}(-1)) \to \bigoplus_{z\in Z}\mathcal{O}_{l_z}. \tag{5}$$

Note that, as in [13] and in contrary to [26], we do not need in (5) the right exactness of the sequence.

On the other hand, we may also apply Sym^k to the sequence:

$$0 \to \mathrm{syz}\ ((J/fS)(|Z|-1)) \to T_{\mathbb{P}^2}(-1) \to (J/fS)(|Z|-1) \to 0$$

obtaining

$$0 \to K \to \mathrm{Sym}^k(T_{\mathbb{P}^2}(-1)) \to ((J/fS)(|Z|-1))^k \to 0,$$

where K denotes the kernel. Thus, this kernel consists of syzygies of $((J/fS)(|Z|-1))^k$ and $((J/fS)(|Z|-1))^k \subset \bigoplus_{z\in Z}\mathcal{O}_\ell$ (see e.g. [9]). As this kernel is unique up to isomorphism, we have

$$K \cong p_*q^*I_Z(k),$$

what proves the claim.

Having the claim, we proceed exactly as it is done in Lemma 3.3 of [3]. Let us, for the reader's convenience, go through this part of the proof. Observe, as it is done in [3], that q restricted to the set $Y = \{(Q,\ell) : Q \in L = L_P\} \subset \mathbb{F}$, where P is the generic point, may be treated as a blowup of $\check{\mathbb{P}}^2$ in P. Thus, $q^*(I_Z(k))$ may be treated as a sheaf on Y given by $I_Z \otimes \mathcal{O}_Y(kH)$, with H being a pullback of a line, so

$$p_*(I_Z \otimes \mathcal{O}_Y((j+k)H - jE) = p_*(I_Z \otimes \mathcal{O}_Y((kH) \otimes p^*(\mathcal{O}_L(j)) \cong p_*(I_Z \otimes \mathcal{O}_Y((kH) \otimes \mathcal{O}_L(j))),$$

by the projection formula. On the other hand, the projection p maps Y onto L as $\mathbb{P}^1$ bundle. According to Theorem 2 from [13], $p_*q^*I_Z(k)$ is a vector bundle of rank $k+1$ and as such decomposes, after restriction to L, as a sum of line bundles, say $\sum_{i=1}^{k+1}\mathcal{O}_L(-a_i)$. Thus, we get that $p_*q^*I_Z(j+k)$ restricted to L is $\sum_{i=1}^{k+1}\mathcal{O}_L(j-a_i)$. Computing the appropriate dimensions we have

$$\dim[I_{Z+jP}]_{j+k} = h^0(\mathbb{P}^2, I_{Z+jP}\otimes\mathcal{O}_{\mathbb{P}^2}(j+k)) = h^0(\mathbb{P}^2, I_Z\otimes I_{jP}\otimes\mathcal{O}_{\mathbb{P}^2}(j+k)) = h^0(Y, I_Z\otimes\mathcal{O}_Y((j+k)H-jE)).$$

In our case p_* preserves global sections, so we have

$$h^0(L, p_*(I_Z\otimes\mathcal{O}_Y((j+k)H-jE)) = h^0(L, p_*(I_Z(k)\otimes\mathcal{O}_Y((jH-jE)) = h^0(L, p_*(I_Z(k)\otimes p^*\mathcal{O}_L(j))),$$

and from the projection formula it is equal to $h^0(L, p_*(I_Z(k))\otimes\mathcal{O}_L(j)) = h^0(L, \oplus_i\mathcal{O}_L(j-a_i))$. □

Remark 5.4. To get a more specific description of this kernel, we use the formula for a kernel of the symmetric power, obtaining

$$\widetilde{\text{syz}}(((J/fS)(|Z|-1))^k) = \widetilde{\text{syz}}\ ((J/fS)(|Z|-1))\otimes\text{Sym}^{k-1}(T_{\mathbb{P}^2}(-1))$$

and so

$$p_*q^*I_Z(k) = p_*q^*I_Z(1)\otimes\text{Sym}^{k-1}(T_{\mathbb{P}^2}(-1)).$$

Now we are in the position to prove a result describing when a curve C_L, constructed as in Section 2 is unexpected.

Theorem 5.5. *Let Z, P, f be as above. Take the (rank $k+1$) bundle of syzygies of $((J/fS)(|Z|-1))^k$. This bundle splits on a generic line L (dual to P), with the splitting type $(a_1, a_2, \ldots a_{k+1})$. Let us introduce the following notation:*

$$(a_1, a_2, \ldots a_{k+1}) = (a+\epsilon_0, \ldots, a+\epsilon_0, a+\epsilon_1, \ldots, a+\epsilon_1, a+\epsilon_2, \ldots, a+\epsilon_2, a+\epsilon_3, \ldots, a+\epsilon_s)$$

where $\epsilon_0 = 0, 1 \leqslant \epsilon_1$ and $\epsilon_i < \epsilon_{i+1}$, and $a+\epsilon_i$, for $i = 0, 1, \ldots, s$, appears in the sequence t_i times, $t_0 + \cdots + t_s = k+1$. Take syzygies of $((J/fS)(|Z|-1))^k$, of degree $a+\epsilon_j$, for a given $j \in \{0, 1, \ldots, s\}$. The curve C_L of type $(a+\epsilon_j+k, a+\epsilon_j)$ is unexpected if:

1) Z imposes independent conditions on curves of degree $a+\epsilon_j+k$ and

2) $0 < \sum_{i=j+1}^{s} t_i(\epsilon_i - \epsilon_j - 1)$.

Proof. Before we start the proof, note that we prove only a sufficient (and not necessary and sufficient, as in [3]) condition for the existence of the unexpected curve, neither we prove its irreducibility.

From Lemma 5.3 it follows, that

$$\dim[I_Z + (a+\epsilon_j)P]_{a+\epsilon_j+k} = \max\{0, a+\epsilon_j - a_1 + 1\} + \cdots + \max\{0, a+\epsilon_j - a_{k+1} + 1\} = \\ = (\epsilon_j + 1)t_0 + (\epsilon_j + 1 - \epsilon_1)t_1 + \cdots + (\epsilon_j + 1 - \epsilon_j)t_j.$$

On the other hand, the expected dimension is

$$\binom{a+\epsilon_j+k+2}{2} - |Z| - \binom{a+\epsilon_j+1}{2}.$$

We also know that $a_1 + \cdots + a_{k+1} = |Z| - \binom{k+1}{2}$. Thus, the expected dimension is less than the actual dimension iff

$$\binom{a+\epsilon_j+k+2}{2} - \left(\binom{k+1}{2} + (k+1)a + \sum_{i=1}^{s} t_i\epsilon_i\right) - \binom{a+\epsilon_j+1}{2} < \\ < (t_0 + \cdots + t_j)(\epsilon_j+1) - \sum_{i=1}^{s} t_i\epsilon_i,$$

what is equivalent to

$$(k+1)(\epsilon_j+1)\sum_{i=1}^{s} t_i\epsilon_i < (t_0 + \cdots + t_j)(\epsilon_j+1) - \sum_{i=1}^{s} t_i\epsilon_i.$$

So, as $t_0 + \cdots + t_s = k+1$ and $\epsilon_0 = 0$ we have equivalently

$$\sum_{i=j+1}^{s} t_i(\epsilon_j+1) < \sum_{i=j+1}^{s} t_i\epsilon_i,$$

and thus

$$0 < \sum_{i=j+1}^{s} t_i(\epsilon_i - \epsilon_j - 1).$$

□

Remark 5.6. It might happen that the dimension of a system of curves of type $(d+k, d)$ passing once through Z is equal to the expected dimension, but there is an unexpected curve of this type, with multiplicity greater than one in some points of Z.

In Example 6.3 there are three linearly independent curves of type $(7,5)$ for DF_4 arrangement. As far as Singular [5] can check, they are irreducible. Moreover, one of them passes doubly through two points of Z, so the expected dimension count should take this under consideration.

6 Examples

This section presents some examples which were the starting point for the considerations.

Example 6.1. Here we show how the construction of the unexpected curve works in case of B_3 configuration and for $k = 1$.

Take the syzygy of the Jacobian ideal of $f = abc(a^2 - b^2)(a^2 - c^2)(b^2 - c^2)$ given by

$$\begin{aligned} g_0(a,b,c) &= 4a^3 - 5ab^2 - 5ac^2 \\ g_1(a,b,c) &= -5a^2b + 4b^3 - 5bc^2 \\ g_2(a,b,c) &= -5a^2c - 5b^2c + 4c^3 \end{aligned}$$

so that $g_0(a,b,c)f_a(a,b,c) + g_1(a,b,c)f_b(a,b,c) + g_2(a,b,c)f_c(a,b,c) = 0$. Take a generic line L in the plane $\alpha a + \beta b + \gamma c = 0$. Then take the line

$$L_G : g_0(a,b,c)x + g_1(a,b,c)y + g_2(a,b,c)z = 0$$

in the dual projective plane, and, for any point $Q = (a,b,c) \in L$, the dual line

$$L_Q : ax + by + cz = 0.$$

Assume that $c = 1$. Then compute

$$b = \frac{-\alpha a - \gamma}{\beta},$$

substitute into the equation of L_G and multiply by β^3. We get:

$$\begin{gathered} x(-5a^3\alpha^2\beta - 5a\beta^3 + 4a^3\beta^3 - 10a^2\alpha\beta\gamma - 5a\beta\gamma^2)+ \\ y(-4a^3\alpha^3 + 5a\alpha\beta^2 + 5a^3\alpha\beta^2 - 12a^2\alpha^2 S + 5\beta^2\gamma + 5a^2\beta^2\gamma - 12a\alpha\gamma^2 - 4\gamma^3)+ \\ z(-4a^3\alpha^3 + 5a\alpha\beta^2 + 5a^3\alpha\beta^2 - 12a^2\alpha^2\gamma + 5\beta^2\gamma + 5a^2\beta^2\gamma - 12a\alpha\gamma^2 - 4\gamma^3) = 0, \end{gathered}$$

or equivalently

$$\begin{aligned} a^3(-5\alpha^2\beta x + 4\beta^3 x - 4\alpha^3 y + 5\alpha\beta^2 y) + a^2(-10\alpha\beta\gamma x - 12\alpha^2\gamma y + 5\beta^2\gamma y - 5\alpha^2\beta z - 5\beta^3 z)+ \\ +a(-5\beta^3 x - 5\beta\gamma^2 x + 5\alpha\beta^2 y - 12\alpha\gamma^2 y - 10\alpha\beta\gamma z) + 5\beta^2\gamma y - 4\gamma^3 y + 4\beta^3 z - 5\beta\gamma^2 z = 0. \end{aligned} \tag{6}$$

Then, for any point (x,y,z) different from (α,β,γ), we get from the equations of L and L_Q:

$$a = \frac{\gamma y - \beta z}{\beta x - \alpha y}.$$

Substituting this for a in (6) and multiplying by the denominator in the third power we get:

$$\begin{gathered} 9\beta^3(-\gamma^3x^3y + \gamma^3xy^3 + \beta^3x^3z - 3\alpha\beta^2x^2yz + 3\alpha\gamma^2x^2yz + 3\alpha^2\beta xy^2z \\ -3\beta\gamma^2xy^2z - \alpha^3y^3z - 3\alpha^2xyz^2 + 3\beta^2\gamma xyz^2 - \beta^3xz^3 + \alpha^3yz^3) = 0. \end{gathered}$$

The expression in parenthesess is the (equation of the) unexpected quartic with a generic triple point described in [3] and in [2].

Example 6.2. The theory developed in Section 4 allows us to compute the module $\mathrm{syz}(J^2+L)$, in case of B_3 configuration, which is generated by three elements

$$\big[[0,0,0,0,y^2-z^2,0],\ [0,y^2,0,xy,xz,0],\ [0,0,z^2,0,xy,xz]\big]=(\sigma_1,\sigma_2,\sigma_3).$$

Thus

$$D_0^2(B_3)\otimes \mathcal{O}_L=\mathcal{O}_L(-2)\otimes\mathcal{O}_L(-2)\otimes\mathcal{O}_L(-2).$$

If we take as a general line, the line with equation $L=-12x+10y+7z$ and syzygy σ_2, then Theorem 3.1 gives the equation of curve to be

$$49x^3y-49xy^3+168x^2yz+140xy^2z+44xyz^2=0.$$

Example 6.3. Let e be the n-th primitive root from unity. Denote by $DF_n=xyz\prod_{i,j=0}^{n-1}(x+e^iy+e^jz)$ configuration of lines dual to the points cut by the so-called Fermat configuration of lines $(x^n-y^n)(x^n-z^n)(y^n-z^n)$. Fermat configurations contain exactly n^2+3 points, and we set Z to be the set of those points. The following tables give the exponents a_i in the splitting type, i.e.

$$D_0^k(DF_n)\otimes\mathcal{O}_L=\mathcal{O}_L(-a_1)\otimes\cdots\otimes\mathcal{O}_L(-a_{k+1}),$$

for $n=3,4,5$ and all k which fulfil inequality $n^2+3>\binom{k+1}{2}$, together with the values of ϵ_i and t_i described in Theorem 5.5. The last column contains all unexpected curves of type $(d+k,d)$ which can be obtained by this theorem. For the readers convenience, we adopt here the convention that we give the exact number of values of ϵ_i and t_i. Therefore, if some values do not exist, we put 0 instead of omitting.

$n = 3$

k	$a_1, \ldots, a_{k+1}$	ϵ_1	t_0, t_1	$(d+k, d)$
1	4,7	3	1,1	(5,4)
2	3,3,3	0	0,0	—
3	1,1,2,2	1	2,2	—
4	0,0,0,1,1	1	3,2	—
5	0,0,0,0,0,1	1	5,1	—

$n = 4$

k	$a_1, \ldots, a_{k+1}$	ϵ_1, ϵ_2	t_0, t_1, t_2	$(d+k, d)$
1	9,9	—	—	—
2	4,5,7	1,3	1,1,1	(6,4),(7,5)
3	3,3,3,4	1,0	3,1,0	—
4	1,1,2,2,3	1,2	2,2,1	—
5	0,0,0,1,1,2	1,2	3,2,1	—
6	0,0,0,0,0,0,1	1,0	6,1,0	—

$n = 5$

k	$a_1, \ldots, a_{k+1}$	$\epsilon_1, \epsilon_2, \epsilon_3$	t_0, t_1, t_2, t_3	$(d+k, d)$
1	13,14	1,0,0	1,1,0,0	—
2	7,9,9	2,0,0	1,2,0,0	(9,7)
3	4,5,6,7	1,2,3	1,1,1,1	(7,4)*,(8,5)
4	3,3,3,4,5	1,2,0	3,1,1,0	(7,3)*
5	1,1,2,2,3,4	1,2,3	2,2,1,1	(6,1)*, (7,2)*
6	0,0,0,1,1,2,3	1,2,3	3,2,1,1	—
7	0,0,0,0,0,0,0,2	2,0,0	7,1,0,0	—
8	0,0,0,0,0,0,0,0,1	1,0,0	8,1,0,0	—

* means a case when the conditions imposed by Z are dependent.

Some interesting examples can be found among all given cases. Consider for instance the curve $(7,5)$ for $n = 4$. As is computed, the curve constructed by Theorem 3.1 for this case has an unusual property. Namely, the curve passes through all points of the set Z once, except points $(0,1,0)$ and $(0,0,1)$, which are double. The equation of this curve, where the general point has coordinates (a,b,c), is

$$\begin{aligned}\mathcal{C}_{4,7,5} = {} & (5b^4c+3c^5)x^6y + (-20ab^3c)x^5y^2 + (30a^2b^2c)x^4y^3 + (-20a^3bc)x^3y^4 + (5a^4c-3c^5)x^2y^5 + (-3b^5-5bc^4)x^6z + \\ & (10ab^4-10ac^4)x^5yz + (-10a^2b^3)x^4y^2z + (5a^4b+5bc^4)x^2y^4z + (-2a^5+10ac^4)xy^5z + (20abc^3)x^5z^2 + (10a^2c^3)x^4yz^2 + (-20abc^3)xy^4z^2 + \\ & (-10a^2c^3)y^5z^2 + (-30a^2bc^2)x^4z^3 + (30a^2bc^2)y^4z^3 + (20a^3bc)x^3z^4 + (-5a^4c-5b^4c)x^2yz^4 + (20ab^3c)xy^2z^4 + (-30a^2b^2c)y^3z^4 \\ & + (-5a^4b+3b^5)x^2z^5 + (2a^5-10ab^4)xyz^5 + (10a^2b^3)y^2z^5 = 0.\end{aligned}$$

On the other hand, if we consider the system $\mathcal{L}$ of all curves which pass once through all points dual to DF_4 and which pass through a general point with multiplicity 5, then we can compute that $\dim[\mathcal{L}]_7 = 3$, while the expected dimension is

$$\binom{9}{2} - |Z| - \binom{6}{2} = 36 - 19 - 15 = 2.$$

Therefore, there exists an unexpected curve of type $(7, 5)$ different from what we got previously from Theorem 3.1. By using computer algebra software, it can be computed that the equation of such a curve is

$$\begin{aligned}\mathcal{C}'_{4,7,5} = {} & (50ab^6c^2 + 90ab^2c^6)x^4y^3 + (-150a^2b^5c^2 - 90a^2bc^6)x^3y^4 + (150a^3b^4c^2 + 30a^3c^6)x^2y^5 + (-50a^4b^3c^2 - 30b^3c^6)xy^6 + \\ & (-30b^8c - 60b^4c^5 - 6c^9)x^5yz + (50ab^7c - 110ab^3c^5)x^4y^2z + (60a^3bc^5)x^2y^4z + (-50a^4b^4c - 30a^4c^5 + 90b^4c^5 + 6c^9)xy^5z + \\ & (30a^5b^3c + 50ab^3c^5)y^6z + (15b^9 + 66b^5c^4 + 15bc^8)x^5z^2 + (-25ab^8 + 190ab^4c^4 + 15ac^8)x^4yz^2 + (25a^4b^5 - 15a^4bc^4 - 75b^5c^4 - 15bc^8)xy^4z^2 + \\ & (-15a^5b^4 + 9a^5c^4 - 175ab^4c^4 - 15ac^8)y^5z^2 + (-200ab^5c^3 - 60abc^7)x^4z^3 + (200ab^5c^3 + 60abc^7)y^4z^3 + (150a^2b^5c^2 + 90a^2bc^6)x^3z^4 + \\ & (-150a^3b^4c^2 - 30a^3c^6)x^2yz^4 + (50a^4b^3c^2 + 30b^3c^6)xy^2z^4 + (-50ab^6c^2 - 90ab^2c^6)y^3z^4 + (-60a^3bc^5)x^2z^5 + \\ & (50a^4b^4c + 30a^4c^5 + 30b^8c - 30b^4c^5)xyz^5 + (-30a^5b^3c - 50ab^7c + 60ab^3c^5)y^2z^5 + (-25a^4b^5 + 15a^4bc^4 - 15b^9 + 9b^5c^4)xz^6 + \\ & (15a^5b^4 - 9a^5c^4 + 25ab^8 - 15ab^4c^4)yz^6 = 0,\end{aligned}$$

where the general point has coordinates (a, b, c).

This example suggests that perhaps not all unexpected curves of type $(d + k, d)$ can be derived from syzygies or a different construction should be used.

A similar situation takes place for $n = 5$ and the curve of type $(8, 5)$. The construction of Theorem 3.1 leads to a curve with 2 double points among the set of points dual to DF_5. The explicit equation of this curve with general point (a, b, c) is

$$\begin{aligned}\mathcal{C}_{5,8,5} = {} & (3b^5c + 2c^6)x^7y + (-10ab^4c)x^6y^2 + (10a^2b^3c)x^5y^3 + (-5a^4bc)x^3y^5 + (2a^5c - 2c^6)x^2y^6 + (-2b^6 - 3bc^5)x^7z + (6ab^5 - 6ac^5)x^6yz + \\ & (-5a^2b^4)x^5y^2z + (2a^5b + 3bc^5)x^2y^5z + (-a^6 + 6ac^5)xy^6z + (10abc^4)x^6z^2 + (5a^2c^4)x^5yz^2 + (-10abc^4)xy^5z^2 + (-5a^2c^4)y^6z^2 + \\ & (-10a^2bc^3)x^5z^3 + (10a^2bc^3)y^5z^3 + (5a^4bc)x^3z^5 + (-2a^5c - 3b^5c)x^2yz^5 + (10ab^4c)xy^2z^5 + (-10a^2b^3c)y^3z^5 + (-2a^5b + 2b^6)x^2z^6 + \\ & (a^6 - 6ab^5)xyz^6 + (5a^2b^4)y^2z^6 = 0,\end{aligned}$$

whereas we can again find a curve passing simply through Z and through a general point with multiplicity 5, and such a curve has a different equation.

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