Weak and strong L^p-limits of vector fields with finitely many integer singularities in dimension n

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Abstract

For every given $p \in [1, +\infty)$ and $n \in \mathbb{N}$ with $n \geqslant 1$, the authors identify the strong L^p-closure $L^p_{\mathbb{Z}}(D)$ of the class of vector fields having finitely many integer topological singularities on a domain D which is either bi-Lipschitz equivalent to the open unit n-dimensional cube or to the boundary of the unit $(n+1)$-dimensional cube. Moreover, for any $n \in \mathbb{N}$ with $n \geqslant 2$ the authors prove that $L^p_{\mathbb{Z}}(D)$ is weakly sequentially closed for every $p \in (1, +\infty)$ whenever D is an open domain in $\mathbb{R}^n$ which is bi-Lipschitz equivalent to the open unit cube. As a byproduct of the previous analysis, a useful characterisation of such class of objects is obtained in terms of existence of a (minimal) connection for their singular set.

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Contents

1. Introduction

Throughout the following paper we will always use the notations listed in Section 1.4.

1.1. Statement of the results and motivation

Given any smooth map $u : X \to Y$ between closed, oriented and connected $(n-1)$-dimensional manifolds, the *degree* of u is a measure of how many times X wraps around Y under the action of u. It can be defined as follows[1]:

$$\deg(u) = \int_X u^*\omega,$$

where by $u^* : \Omega^\bullet(Y) \to \Omega^\bullet(X)$ we denote the standard pull-back operation through the map u acting on smooth differential forms and $\omega \in \Omega^n(Y)$ is a renormalized volume form on Y, i.e. ω is a smooth, nowhere vanishing n-form on Y such that

$$\int_Y \omega = 1.$$

Now let $D \subset \mathbb{R}^n$ be an open and bounded Lipschitz domain. By Nash theorem, there exists an isometric embedding $Y \hookrightarrow \mathbb{R}^k$ for some $k \in \mathbb{N}$ large enough. Recalling that "$\mathcal{L}^n$" stands for the Lebesgue measure on $\mathbb{R}^n$, for every $p \in [1, +\infty]$ we let

$$W^{1,p}(D,Y) := \{u = (u_1, ..., u_k) : D \to \mathbb{R}^k \text{ s.t. } u_i \in W^{1,p}(D) \forall\, i = 1, ..., k \\ \text{and } u(x) \in Y \text{ for } \mathcal{L}^n\text{-a.e. } x \in D\}.$$

Consider $u \in W^{1,p}(D,Y)$ being *smooth up to finitely many point singularities*, which simply means that $u \in C^\infty(D \smallsetminus S_u, Y)$ for some finite set $S_u \subset D$. In this case we write $u \in R^{1,p}(D,Y)$. We define the *degree of u at some singular point $x \in S_u$* as

$$\deg(u,x) := \deg(u|_{\partial D'}) = \int_{\partial D'} u^*\omega \in \mathbb{Z}, \tag{1.1}$$

where $D' \subset\subset D$ is any open, piecewise smooth domain in D such that

$$\overline{D'} \cap S_u = D' \cap S_u = \{x\}.$$

Notice that Definition (1.1) is independent from the choice of the set D'.
If $\deg(u,x) \neq 0$ for some $x \in S_u$, then we say that x is a *topological singularity* of u and we refer to the subset of S_u made of the topological singularities of u as the *topological singular set of* u, which we denote by S_u^{top}.
Notice that if $u \in R^{1,n-1}(D)$ then $u^*\omega \in \Omega_1^{n-1}(D)$ (recall that $\Omega_1^{n-1}(D)$ stands for the space of the $(n-1)$-forms on D having L^1-integrable coefficients with respect to the Lebesgue measure $\mathcal{L}^n$ on D). Moreover by (1.1) we see that the $u^*\omega$ "detects" the topological singularities of u, in the sense that

$$\int_{\partial D'} u^*\omega = \sum_{x \in S_u^{top} \cap D'} \deg(u,x) \tag{1.2}$$

[1] An alternative definition can be given in terms of the orientations of the preimages of regular points of u, see for instance [3, Chapter 7].

for every open, piecewise smooth domain $D' \subset\subset D$ such that $\partial D' \cap S_u = \emptyset$. From (1.2) one can deduce that

$$*d(u^*\omega) = \sum_{x \in S_u^{top}} \deg(u,x)\delta_x \quad \text{in } \mathcal{D}'(D), \tag{1.3}$$

where by "$*$" we denote the standard Hodge-$*$ operator associated with euclidean metric on D. For the reader convenience, we recall that the distributional meaning of equation (1.3) is the following:

$$-\int_D d\varphi \wedge u^*\omega = \sum_{x \in S_u^{top}} \deg(u,x)\varphi(x), \qquad \forall\, \varphi \in C_c^\infty(D).$$

Remark 1.1. In the following, we denote the open unit ball in $\mathbb{R}^n$ by $B^n \subset \mathbb{R}^n$, for every $n \in \mathbb{N}$. Sobolev maps that are smooth up to a finite set of topological singularities arise frequently as solutions of variational problems in critical or supercritical dimension. For example, this is the best regularity which is possible to guarantee for energy minimizing harmonic maps in $W^{1,2}(B^3, \mathbb{S}^2)$ (see [29, Theorem II]). Again, quite recently the second author and T. Rivière considered the following "weak $\mathbb{S}^1$-harmonic map equation"

$$\operatorname{div}(u \wedge \nabla u) = 0 \qquad \text{in } \mathcal{D}'(B^2)$$

and gave a completely variational characterization of the solutions in $R^{1,p}(B^2, \mathbb{S}^1)$ with finite "renormalized Dirichlet energy" for $p > 1$ (see [14, Theorem I.3] for further details).

We also remark that the presence of topological singularities is deeply linked to fundamental questions concerning the strong $W^{1,p}$-approximability through smooth maps of elements in $W^{1,p}(D, Y)$ (see [5], [17]).

The previous discussion motivates the following general definition. Recall that we denote by $\Omega_p^{n-1}(D)$ denotes the space of the $(n-1)$-forms on D having L^p-integrable coefficients with respect to the Lebesgue measure $\mathcal{L}^n$ on D (see Section 1.4 for a precise definition).

Definition 1.1. Let $p \in [1, \infty]$. Let $F \in \Omega_p^{n-1}(D)$. We say that F has *finitely many integer singularities* if there exists a finite set of points $S \subset D$ such that $F \in \Omega^{n-1}(D \smallsetminus S)$ and

$$*dF = \sum_{x \in S} a_x \delta_x \qquad \text{in } \mathcal{D}'(D),$$

where $a_x \in \mathbb{Z}$ for every $x \in S$. The class of L^p integrable $(n-1)$-forms on D having finitely many integer singularities will be denoted by $\Omega_{p,R}^{n-1}(D)$.

As we have seen above, $u^*\omega \in \Omega_{1,R}^{n-1}(D)$ for every $u \in R^{1,n-1}(D, Y)$, for any closed, oriented and connected $n-1$-manifold Y. Other simple examples of elements of $\Omega_{p,R}^{n-1}(D)$ can be constructed as follows. Let $\sigma : D \to \mathbb{R}$ denote the fundamental solution of the Laplace equation, i.e.

$$\sigma(x) = \begin{cases} -|x| & \text{if } n = 1, \\ -\frac{1}{2\pi}\log|x| & \text{if } n = 2 \\ \frac{1}{n(n-2)\alpha(n)}\frac{1}{|x|^{n-2}} & \text{if } n \geqslant 3, \end{cases}$$

where $\alpha(n)$ denotes the volume of the unit ball in $\mathbb{R}^n$. Then $*d\sigma \in \Omega^{n-1}_{p,R}(D)$ for any $p \in [1, \frac{n}{n-1})$. In fact $*d(*d\sigma) = \Delta\sigma = \delta_0$.
Clearly any finite linear combination with integer coefficients of translations of $*d\sigma$ also belongs to $\Omega^{n-1}_{p,R}(D)$. In fact one can show that any element F of $\Omega^{n-1}_{p,R}(D)$ can be decomposed as such a linear combination plus some $\tilde{F} \in \Omega^{n-1}_p(D)$ with $*d\tilde{F} = 0$. In particular, if $p \geqslant \frac{n}{n-1}$ then $*dF = 0$. Thus the class $\Omega^{n-1}_{p,R}(D)$ is relatively simple from an analytical point of view and so it is natural to ask which forms in $\Omega^{n-1}_p(D)$ can be approximated by elements in $\Omega^{n-1}_{p,R}(D)$. The main purpose of the present paper consists in giving a description of the strong and weak closure of the class $\Omega^{n-1}_{p,R}(D)$ for any open domain in $\mathbb{R}^n$ which is bi-Lipschitz equivalent to the open unit n-cube $Q_1(0) \subset \mathbb{R}^n$.
First we will address the strong closure in the case of the open n-dimensional cube $Q_1(0)$ centered at the origin of $\mathbb{R}^n$ and having edge-length 1. To this end we introduce the class of $(n-1)$-forms with integer-valued fluxes.
Recall that by $d_\infty : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ we denote the L^∞-distance between points in $\mathbb{R}^n$ (see Section 1.4 for a precise definition).
For any $F \in \Omega^{n-1}_{L^p_{loc}}(Q_1(0))$ (meaning that F is $(n-1)$-form on $Q_1(0)$ with L^p_{loc} coefficients with respect to the Lebesgue measure $\mathcal{L}^n$ on $Q_1(0)$), given $x_0 \in Q_1(0)$ let $r_0 := \mathrm{dist}_\infty(x_0, \partial Q^n_1(0))$ and define $\tilde{R}_{F,x_0} \subset (0, r_0)$ to be the set of radii $\rho \in (0, r_0)$ such that

1. the hypersurface $\partial Q_\rho(x_0)$ consists $\mathscr{H}^{n-1}$-a.e. of Lebesgue points of F,
2. there holds $|F| \in L^p\big(\partial Q_\rho(x_0), \mathscr{H}^{n-1}\big)$.

Notice that $\mathcal{L}^1\big((0, r_{x_0}) \setminus \tilde{R}_{F,x_0}\big) = 0$.

Definition 1.2. Let $p \in [1, \infty]$, let $F \in \Omega^{n-1}_{L^p_{loc}} \cap \Omega^{n-1}_p(Q_1(0), \mu)$ for some Radon measure μ, for any $x_0 \in Q_1(0)$ let $\tilde{R}_{F,x_0}$ be defined as above. We say that F has *integer-valued fluxes* if for any $x_0 \in Q_1(0)$, for $\mathcal{L}^1-$a.e. $\rho \in \tilde{R}_{F,x_0}$ there holds[2]

$$\int_{\partial Q_\rho(x_0)} i^*_{\partial Q_\rho(x_0)} F \in \mathbb{Z}, \tag{1.4}$$

where by $i_{\partial Q_r(x_0)}$ we denote the inclusion $\partial Q_r(x_0) \hookrightarrow \mathbb{R}^n$. The space of $L^p(\mu)$-integrable $(n-1)$-forms with integer valued fluxes will be denoted by $\Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0), \mu)$. The set of radii $\rho \in \tilde{R}_{F,x_0}$ for which (1.4) holds will be denoted by R_{F,x_0}

We will always write $\Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$ for $\Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0), \mathcal{L}^n)$, where $\mathcal{L}^n$ denotes the n-dimensional Lebesgue measure.
First of all we observe that (1.2) implies that $\Omega^{n-1}_{p,R}(Q_1(0)) \subset \Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$. More general examples of forms in $\Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$ can be constructed as follows. Let again Y be a smooth, closed, oriented and connected $n-1$-dimensional manifold. Let $u \in W^{1,n-1}(Q_1(0), Y)$. Then for any $x_0 \in Q_1(0)$, for a.e. $\rho \in (0, 2\,\mathrm{dist}_\infty(x_0, \partial Q_1(0)))$, $u\big|_{\partial Q_\rho(0)} \in W^{1,n-1}(\partial Q_\rho(0), Y)$.

[2]Notice that for the associated vector field $V = (*F)^\flat$ condition (1.4) reads

$$\int_{\partial Q_\rho(x_0)} V \cdot \nu_{\partial Q_\rho(x_0)} d\mathscr{H}^{n-1} \in \mathbb{Z}.$$

Therefore for any such ρ

$$\int_{\partial Q_\rho(0)} i^*_{\partial Q_\rho(x_0)}(u^*\omega) = \deg\left(u\big|_{\partial Q_\rho(x_0)}\right) \in \mathbb{Z} \tag{1.5}$$

Notice that $\deg\left(u\big|_{\partial Q_\rho(x_0)}\right)$ is well defined (by means of approximation by functions in $W^{1,\infty}(\partial Q_\rho(x_0), Y)$, see [12], Section I.3).

Notice that, since strong $L^p(\mu)$-convergence implies subsequently μ-a.e. convergence, the space $\Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$ is strongly closed with respect to the $L^p(\mu)$-norm. Thus, since we have already observed that $\Omega^{n-1}_{p,R}(Q_1(0)) \subset \Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$, we have that the closure of $\Omega^{n-1}_{p,R}(Q_1(0))$ in $\Omega^{n-1}_p(Q_1(0))$ is contained in $\Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$. We will show that in fact the closure of $\Omega^{n-1}_{p,R}(Q_1(0))$ in $\Omega^{n-1}_p(Q_1(0))$ is exactly $\Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$. More precisely we have

Theorem 1.1. *Let $n \in \mathbb{N}$ and let $p \in [1, \infty)$. Let $\mu = f\mathcal{L}^n$, where $\mathcal{L}^n$ is the n-dimensional Lebesgue measure and*

$$f = \left(\frac{1}{2} - ||\cdot||_\infty\right)^q$$

for some $q \in (-\infty, 1]$. Let $F \in \Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0), \mu)$. Then we have

1. *if $q \in [0,1]$ and $p \in \left[1, \frac{n}{n-1}\right)$, then there exists a sequence $\{F_k\}_{k\in\mathbb{N}}$ in $\Omega^{n-1}_{p,R}(Q_1(0))$ such that $F_k \to F$ in $\Omega^{n-1}_p(Q_1(0))$ as $k \to \infty$.*
2. *if $q \in (-\infty, 0]$ and $p \in \left[\frac{n}{n-1}, +\infty\right)$, then $*dF = 0$.*

Remark 1.2. Notice that Theorem 1.1 and Stokes theorem imply immediately that when $p \in \left[\frac{n}{n-1}, 1\right)$ and $q \in (-\infty, 0]$ we have

$$\Omega^{n-1}_{p,\mathbb{Z}}(Q^n_1(0)) = \{F \in \Omega^{n-1}_p(Q^n_1(0)) \text{ s.t. } *dF = 0\}.$$

The reason why we have introduced the weighted measures $\mu = f\,\mathcal{L}^n$ for $q \neq 0$ is that forms belonging to $\Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0), \mu)$ will appear naturally when we will generalize the statement of Theorem 1.1 to the case of the Lipschitz manifold $\partial Q^n_1(0)$ (see Corollary 2.2). Nevertheless, we advise the reader to assume $q = 0$ (i.e. $\mu = \mathcal{L}^n$) throughout Section 2 at a first reading of the present paper. This allows to skip many technicalities without losing formality, since all the results of this paper (apart from Theorem 2.3) are independent on Corollary 2.2.

With the help of Theorem 1.1 we will get another characterization of the L^p-closure of $\Omega^{n-1}_{p,R}(Q_1(0))$. For this we recall the following definition (compare with [9, Section II]):

Definition 1.3 (Connection and minimal connection)**.** Let $M \subset \mathbb{R}^n$ be any embedded Lipschitz m-dimensional submanifold of $\mathbb{R}^n$ (with or without boundary) such that $\overline{M}$ is compact as a subset of $\mathbb{R}^n$.

A 1-dimensional integer-multiplicity rectifiable current I is said to be a *connection* for (the singular set of) F if $\mathbb{M}(I) < +\infty$ and $\partial I = *dF$ in $(W^{1,\infty}_0(M))^*$, where "$\mathbb{M}$" and "∂" denote respectively the mass and the boundary of a current (see Section 1.4 for further details).

A 1-dimensional integer-multiplicity rectifiable current L is said to be a *minimal connection* for (the singular set of) F if it is a connection for F and

$$\mathbb{M}(L) = \inf_{\substack{T\in\mathcal{D}_1(M)\\ \partial T = *dF}} \mathbb{M}(T),$$

where by $\mathcal{D}_1(M)$ we denote the set of 1-dimensional currents on the manifold M (see Section 1.4 for further details). In Corollary 2.1 we will show that F admits a connection if and only if it admits a minimal connection. This fact is known and follows by standard compactness results for integer-multiplicity rectifiable currents (see e.g.) but we will give a proof which is largely independent of the classical argument in geometric measure theory.
Here is the characterization of $\Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$ in terms of minimal connections:

Theorem 1.2. *Let $n\in\mathbb{N}$ be such that $n>0$, let $p\in[1,+\infty)$. Let $F\in\Omega^{n-1}_p(Q^n_1(0))$. Then, the following are equivalent:*

1. *there exists and integer-multiplicity rectifiable 1-current L such that $\partial L = *dF$ in $(W^{1,\infty}_0(Q_1(0)))^*$.*
2. *for every Lipschitz function $f:\overline{Q_1(0)}\to[a,b]\subset\mathbb{R}$ such that $f|_{\partial Q_1(0)}\equiv b$, we have*
$$\int_{f^{-1}(t)} i^*_{f^{-1}(t)}F\in\mathbb{Z},\qquad \text{for } \mathcal{L}^1\text{-a.e. } t\in[a,b],$$
where $i_{f^{-1}(t)}$ denotes the the inclusion $f^{-1}(t)\hookrightarrow Q_1(0)$.
3. *$F\in\Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$.*

In other words, $F\in\Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$ if and only if F admits a (minimal) connection. This characterization allows to generalize the definition of the class $\Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$ to general Lipschitz domains:

Definition 1.4. Let $M\subset\mathbb{R}^n$ be any embedded Lipschitz m-dimensional submanifold of $\mathbb{R}^n$ (with or without boundary). We define

$$\Omega^{n-1}_{p,\mathbb{Z}}(M) := \{F\in\Omega^{n-1}_p(M) \text{ s.t. } \exists \text{ a connection } L \text{ for } F\}.$$

Notice that if $M=Q_1(0)$, Definition 1.2 and Definition 1.4 coincide by Theorem 1.2.
We will deduce from the previous results that the approximation result can be extended to any open domain which is bi-Lipschitz equivalent to $Q_1(0)$ or $\partial Q_1(0)$ (see Theorem 2.3).
We mention here two other corollaries of Theorem 1.1.

Corollary 1.1. *Let $n\in\mathbb{N}$. Let I be an integer-multiplicity rectifiable 1-current. Then there exists a 1-form $\omega\in\Omega^{n-1}_{1,\mathbb{Z}}(Q^n_1(0))$ such that $*d\omega=\partial I$ and ∂I can be approximated in $(W^{1,\infty}_0(Q^n_1(0)))^*$ by finite sums of Dirac-deltas with integer coefficients. More precisely, there exist sequences $(p_i)_{i\in\mathbb{N}}$ and $(n_i)_{i\in\mathbb{N}}$ of points in $Q^n_1(0)$ such that*

$$\partial I = \sum_{i\in\mathbb{N}}(\delta_{p_i}-\delta_{n_i}) \text{ in } (W^{1,\infty}_0(Q^n_1(0)))^* \text{ and } \sum_{i\in\mathbb{N}}|p_i-n_i|<\infty.$$

Moreover if I is supported on a Lipschitz submanifold M of $\mathbb{R}^n$ compactly contained in $Q_1(0)$, the points in the sequences $(p_i)_{i\in\mathbb{N}}$ and $(n_i)_{i\in\mathbb{N}}$ can be chosen to belong to M.

The next corollary was obtained first by R. Schoen and K. Uhlenbeck ([30], Section 4) and F. Bethuel and X. Zheng ([6, Theorem 4]).

Corollary 1.2. *Let $Q_1(0) \subset \mathbb{R}^2$ be the unit cube in $\mathbb{R}^2$. Let $u \in W^{1,p}(Q_1(0), \mathbb{S}^1)$ for some $p \in (1, \infty)$.*
If $p < 2$, then u can be approximated in $W^{1,p}$ by a sequence of functions in

$$\mathcal{R} := \left\{ v \in W^{1,p}(Q_1(0), \mathbb{S}^1); v \in C^\infty(Q_1(0) \smallsetminus A, \mathbb{S}^1), \text{ where } A \text{ is some finite set} \right\}.$$

If $p \geqslant 2$, then u can be approximated in $W^{1,p}$ by a sequence of functions in $C^\infty(Q_1(0), \mathbb{S}^1)$.

In the second part of the paper we turn our attention to the weak closure of the space $\Omega^{n-1}_{p,R}(D)$ for a domain $D \subset \mathbb{R}^n$ which is bi-Lipschitz equivalent to $Q_1(0)$ (or equivalently of $\Omega^{n-1}_{p,\mathbb{Z}}(D)$). In particular, we will show the following.

Theorem 1.3 (Weak closure)**.** *Let $n \in \mathbb{N}$ be such that $n \geqslant 2$, $p \in (1, +\infty)$ and $D \subset \mathbb{R}^n$ be any open and bounded domain in $\mathbb{R}^n$ which is bi-Lipschitz equivalent to the n-dimensional unit cube $Q^n_1(0)$. Then, the space $\Omega^{n-1}_{p,\mathbb{Z}}(D)$ is weakly sequentially closed in $\Omega^{n-1}_p(D)$.*

Notice that, by Remark 1.2 (which generalizes trivially to the case of general Lipschitz domains through Theorem 2.3), the statement of Theorem 1.3 is trivial for $p \in [n/(n-1), +\infty)$. Thus, we just need to provide a proof in case $p \in (1, n/(n-1))$.
We will treat first the case of the open unit n-cube $Q_1(0) \subset \mathbb{R}^n$ (Theorem 3.1). To this end we will work with the characterization of $\Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$ given by Theorem 1.2. The main step in the proof consists in obtaining uniform estimates on the modulus of continuity of an appropriately defined slice function (see Proposition 3.3), an approach which is inspired by the works of Jerrard [18], Ambrosio-Kirchheim [2, §7-8], and Hardt-Rivière [16]. Finally we extend the result to the general case by standard arguments (see Remark 3.8).
We remark that the case $n = 1$ is different. In fact for any interval $I \subset \mathbb{R}$ there holds $\overline{\Omega^0_{p,R}(I)}^{\Omega^0_p(I)} = \Omega^0_p(I)$ (see Lemma 3.3).

Our main motivation to look at forms (instead of maps) with finitely many integer topological singularities is the need of developing geometric measure theory for principal bundles in order to face the still deeply open questions arising in the study of p-Yang-Mills lagrangians.
Let $p \in [1, +\infty)$ and G be any compact matrix Lie group. Consider the trivial G-principal bundle on B^n given by $\mathrm{pr}_1 : P := B^n \times G \to B^n$, where pr_1 is the canonical projection on the first factor. The *p-Yang-Mills lagrangian* on P is given by

$$\mathrm{YM}_p(A) := \int_{B^n} |F_A|^p \, d\mathcal{L}^n = \int_{B^n} |dA + A \wedge A|^p \, d\mathcal{L}^n, \qquad \forall\, A \in \Omega^1(B^n, \mathfrak{g}),$$

where $\mathfrak{g}$ denotes the Lie algebra of G.
As it is described in [19], the reason why we aim to extend the set of the by now classical Sobolev connections is purely analytic and justified by issues arising in the application of the direct method of calculus of variations to p-Yang-Mills Lagrangians. On the other hand, the need to extend the notion of bundles in order to allow more and more singularities to appear has already been faced in many geometric applications, which brought to the introduction

of coherent and reflexive sheaves in gauge theory (see [20],[21]).

Notice that all results mentioned above can be formulated in terms of vector fields: for any $F \in \Omega_p^{n-1}(D)$ we can consider the associated vector field $V_F := (*F)^\sharp$. In fact for the proof of some of the results we preferred to work with vector fields instead of $n-1$-forms.

1.2. Related literature and open problems

Theorem 1.1 was firstly announced to hold for a 3-dimensional domain in [19] and a full proof of the 3-dimensional case was eventually given by the first author in [13]. Some form of the 2-dimensional case was treated in [25], where M. Petrache proved that a strong approximability result holds for 1-forms admitting a connection both on the 2-dimensional disk and on the 2-sphere. In both cases, the proof that we give here is more general and simple. The 3-dimensional version of Theorem 1.3 was already treated by M. Petrache and T. Rivière in [27]. Nevertheless, here we took the opportunity to present the arguments in a more detailed and complete way. Both Theorem 1.1 and Theorem 1.3 in dimension $n \neq 2, 3$ are instead completely new.

The first open problems that relate directly to our results are linked to the celebrated Yang-Mills Plateau problem. Indeed, the weak sequential closedness of the class $\Omega^2_{p,\mathbb{Z}}(B^3)$ implies that such forms behave well-enough to be considered as suitable "very weak" curvatures for the resolution of the p-Yang-Mills Plateau problem for $U(1)$-bundles on B^3 (see the introduction of [27] for further details). The question to address would be if and how we can exploit the same kind of techniques in order to face the existence and regularity issues linked to the so called "non abelian case" (i.e. the case of bundles having a non abelian structure group) in supercritical dimension. An interesting proposal in this sense is due to M. Petrache and T. Rivière and can be found in [26], where a suitable class of weak connections in the supercritical dimension 5 is introduced and studied.

One could also hope that the technique presented in this paper could be adapted to study other classes of differential forms exhibiting "integer fluxes" properties similar to the one described in Definition 1.2. As an example we define here the class $\Omega^n_{p,H}(Q_1^{2n}(0))$ of differential forms with "Hopf singularities".

Recall that for any $n \in \mathbb{N}$ such that $n \geqslant 1$, for any smooth map $f : \mathbb{S}^{2n-1} \to \mathbb{S}^n$ the *Hopf invariant* of f is defined as follows: let ω be the standard volume form on $\mathbb{S}^n$. Let $\alpha \in \Omega^{n-1}(\mathbb{S}^{2n-1})$ be such that $f^*\omega = d\alpha$. Then the Hopf invariant of f is given by

$$H(f) := \int_{\mathbb{S}^{2n-1}} \alpha \wedge d\alpha.$$

One can show that $H(f) \in \mathbb{Z}$ and that it is independent of the choice of α (see [7], Proposition 17.22). In the spirit of Definition 1.2 we say that a form $F \in \Omega^n_p(Q_1^{2n}(0))$ belongs to $\Omega^n_{p,H}(Q_1^{2n}(0))$ for some $p \geqslant 2 - \dfrac{1}{n}$ if there exists $A \in \Omega^{n-1}_{W^{1,p}}(Q_1^{2n}(0))$ such that $dA = F$ and if for every $x_0 \in Q_1^{2n}(0)$ there exists a set $R_{F,x_0} \subset (0, r_{x_0})$, with $r_{x_0} := 2\,\mathrm{dist}_\infty(x_0, \partial Q_1^{2n}(0))$ such that:

1. $\mathcal{L}^1\big((0, r_{x_0}) \smallsetminus R_{F,x_0}\big) = 0$;

2. for every $\rho \in R_{F,x_0}$, the hypersurface $\partial Q^{2n}_\rho(x_0)$ consists $\mathscr{H}^{2n-1}$-a.e. of Lebesgue points of F, A and ∇A (the matrix of all the partial derivatives of the components of A);
3. for every $\rho \in R_{F,x_0}$ we have $|F|, |A|, |\nabla A| \in L^p\big(\partial Q^{2n}_\rho(x_0), \mathscr{H}^{n-1}\big)$;
4. for every $\rho \in R_{F,x_0}$ it holds that

$$\int_{\partial Q^{2n}_\rho(x_0)} i^*_{\partial Q^{2n}_\rho(x_0)}(A \wedge F) \in \mathbb{Z}.$$

Notice that if $u \in W^{1,2n-1}(Q^{2n}_1(0), \mathbb{S}^n)$, then $u^*\omega \in \Omega^n_{p,H}(Q^{2n}_1(0))$. As an interesting open problem, the authors propose to investigate the strong L^p-approximability and the weak L^p-closure of the space $\Omega^n_{p,H}(Q^{2n}_1(0))$.

1.3. Organization of the paper

The paper is organized as follows. Section 2 is dedicated to the strong L^p-closure of $\Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0), \mu)$ for a Radon measure μ as in the statement of Theorem 1.1. First we present some preliminary and rather technical lemmata (Sections 2.1-2.3), then we give a proof of Theorem 1.1 (Section 2.4). In Section 2.5 we show the characterization of $\Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$ in terms of (minimal) connections (Theorem 1.2). In Section 2.6 we exploit this result to extend the approximation result to other Lipschitz manifolds, and in particular to $\partial Q^n_1(0)$. Finally in Section 2.7 we prove Corollary 1.1 and Corollary 1.2.
In Section 3 we discuss the weak L^p closure of $\Omega^{n-1}_{p,R}(Q_1(0))$. First we will introduce a slice distance, first on spheres (Section 3.1) and then on cubes (Section 3.2). In Section 3.3 we discuss some of the properties of the slice distance and in Section 3.4 we use it to obtain a proof of Theorem 1.3. We will also discuss briefly the special case $n = 1$.

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1.5. Notation

Let $M^m \subset \mathbb{R}^n$ be any m-dimensional, embedded Lipschitz submanifold of $\mathbb{R}^n$ (with or without boundary) such that $\overline{M}$ is compact as a subset of $\mathbb{R}^n$.

- We denote by $i_M : M \hookrightarrow \mathbb{R}^n$ the usual inclusion map.
- We always assume that M is endowed with the L^∞-Riemannian metric given by $g_M := i^*_M g_e$, where g_e denotes the standard euclidean metric on $\mathbb{R}^n$.
- For every $k = 1, ..., m$, we define the following spaces of *smooth k-forms* on M:

$$\Omega^k(M) := \{i^*_M\omega \text{ s.t. } \omega \text{ is a smooth } k\text{-form on } \mathbb{R}^n\},$$
$$\mathcal{D}^k(M) := \{\omega \in \Omega^k(M) \text{ s.t. } \operatorname{spt}(\omega) \subset\subset M\}.$$

For every $p \in [1, +\infty]$, we denote by $\Omega^k_p(M)$ and $\Omega^k_{W^{1,p}}(M)$ the completions of $\Omega^k(M)$ with respect to the usual L^p-norm and $W^{1,p}$-norm respectively. We call $\Omega^k_p(M)$ the space of *L^p k-forms* on M and $\Omega^k_{W^{1,p}}(M)$ the space of *$W^{1,p}$ k-forms* on M.

- By the symbols "$*$" and d^*, we denote respectively the Hodge star operator and the codifferential associated with the metric g_M on M. By "$\flat$" and "$\sharp$" we denote the usual musical isomorphisms associated with the metric g_M. Recall that, under this notation, the map

$$\Omega^{n-1}_p(M) \ni \omega \mapsto (*\omega)^\sharp \in L^p(M, \mathbb{R}^n) \tag{1.6}$$

gives an isomorphism onto its image. Exploiting this fact, we frequently identify $(n-1)$-forms with vector fields on M.

Let $x_0 \in \mathbb{R}^n$ and $\rho > 0$.

- We denote by $\|\cdot\|_\infty : \mathbb{R}^n \to [0, +\infty)$ the following norm on $\mathbb{R}^n$:

$$\|x\|_\infty := \max_{j=1,\dots,n} |x_j|.$$

We denote by $d_\infty : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ the distance associated to $\|\cdot\|_\infty$.

- We let

$$Q^n_\rho(x_0) := \left\{x \in \mathbb{R}^n \text{ s.t. } \|x - x_0\|_\infty < \frac{\rho}{2}\right\}$$

be the open cube in $\mathbb{R}^n$ centered at x_0 and having edge-length ρ. We will sometimes omit the n when the dimension is clear from the context.

We denote by $B^n_\rho(x_0)$ the open ball in $\mathbb{R}^n$ centered at x_0 with radius ρ (here again we will sometime omit the n).

- Given any $E \subset \mathbb{R}^n$, we denote by $\mathbb{1}_E$ the indicator function of E.
- We define

$$\begin{aligned}
\mathcal{D}_k(M) &:= \{k\text{-dimensional currents on } M\},\\
\mathcal{M}_k(M) &:= \{T \in \mathcal{D}_k(M) \text{ s.t. } \mathbb{M}(T) < +\infty\},\\
\mathcal{N}_k(M) &:= \{T \in \mathcal{D}_k(M) \text{ s.t. } \mathbb{M}(T), \mathbb{M}(\partial T) < +\infty\},\\
\mathcal{R}_k(M) &:= \{T \in \mathcal{D}_k(M) \text{ s.t. } T \text{ is integer-multiplicity rectifiable}\},
\end{aligned}$$

where "$\mathbb{M}$" and "∂" denote respectively the mass and the boundary of a current (see [22] or [24] for further details on the definitions of such objects). Moreover, given any $F \in \Omega^k_1(M)$ we define the *$(m-k)$-current associated to F* by

$$\langle T_F, \omega\rangle := \int_M F \wedge \omega, \qquad \forall \omega \in \mathcal{D}^{m-k}(M).$$

2. The strong L^p-approximation theorem

In this section we provide a proof of Theorem 1.1. In an attempt to make the proof more accessible, we reformulate the Theorem in terms of vector fields. We start by defining the function spaces in this setting (compare with Definitions 1.1 and 1.2).

Definition 2.1. For any Radon measure $\mu := f\mathcal{L}^n$ with $f = \left(\frac{1}{2} - \|\cdot\|_\infty\right)^q$, with $q \in (-\infty, 1]$ let

$$L^p_R(Q_1(0), \mu) := \{V \in L^p(Q_1(0), \mu) \text{ vector field s.t. } *V^\flat \in \Omega^{n-1}_{p,R}(Q_1(0))\}$$

and let

$$L^p_{\mathbb{Z}}(Q_1(0), \mu) := \{V \in L^p(Q_1(0), \mu) \text{ vector field s.t. } *V^\flat \in \Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))\}.$$

We will sometime write $L^p_{\mathbb{Z}}(Q_1(0))$ for $L^p_{\mathbb{Z}}(Q_1(0), \mathcal{L}^n)$, where $\mathcal{L}^n$ denotes the n-dimensional Lebesgue measure.

In terms of vector fields, Theorem 1.1 can be restated as follows.

Theorem 2.1. *Let $V \in L^p_{\mathbb{Z}}(Q^n_1(0), \mu)$. The following facts hold:*

1. *if $q \in [0,1]$ and $p \in \big[1, n/(n-1)\big)$, then there exists a sequence $\{V_k\}_{k\in\mathbb{N}} \subset L^p_R(Q_1(0), \mu)$ such that $V_k \to V$ strongly in $L^p(Q^n_1(0), \mu)$;*
2. *if $q \in (-\infty, 0]$ and $p \in \big[n/(n-1), +\infty\big)$, then $\operatorname{div}(V) = 0$ distributionally on $Q^n_1(0)$.*

The case $n = 1$ is particularly easy and is treated in Lemma 2.6. For the proof in the case $n \geqslant 2$ we follow the ideas of [25] and [13]. We present here a plan of the proof, reducing to the case $q = 0$ for simplicity.

Let $V \in L^p_{\mathbb{Z}}(Q^n_1(0))$. First of all we show that for any $\varepsilon > 0$ it is possible to decompose $Q_1(0)$ into cubes Q of size ε (plus a negligible remainder) so that

$$\int_{\partial Q} V \cdot \nu_{\partial Q}\, d\mathscr{H}^{n-1} \in \mathbb{Z} \tag{2.1}$$

and so that the number of cubes where the integral is different from zero is controlled (Section 2.1). We will call *good* the cubes for which the integral (2.1) vanishes and we will say that all the other cubes are *bad*, following the same approach and conventions that F. Bethuel used in [4]. We will then show that V can be approximated on the boundaries of the small cubes Q by smooth vector fields $(V_\varepsilon)_{\varepsilon>0}$ with similar properties (Section 2.2). In Section 2.3 we show that the vector fields V_ε can be extended inside the cubes Q in such a way that the extension $\tilde{V}_\varepsilon$ has a finite number of singularities in Q (more precisely $\tilde{V}_\varepsilon|_Q \in L^p_R(Q)$) and is close to V in $L^p(Q)$. In Section 2.4 we will combine the previous elements to show that the approximating fields constructed above (up to some shifting and smoothing) satisfy the claim of the theorem.

2.1. Choice of a suitable cubic decomposition

Throughout the following section, given a cube $Q \subset \mathbb{R}^n$ we will denote by c_Q its center. Fix any $\varepsilon \in (0, \frac{1}{6})$ and $a \in Q_\varepsilon(0)$. Let

$$\begin{aligned} q_\varepsilon &:= \max\{q \in \mathbb{N} \text{ s.t. } \varepsilon q \leqslant 1 - 3\varepsilon\}, \\ C_\varepsilon &:= \left\{\left(j + \frac{1}{2}\right)\varepsilon - \frac{1}{2}, \text{ with } j = 1, ..., q_\varepsilon - 1\right\}^n, \\ \mathscr{C}_{\varepsilon,a} &:= \big\{Q_\varepsilon(x) + a, \text{ with } x \in C_\varepsilon\big\}. \end{aligned}$$

Notice that the constant $\frac{1}{6}$ is chosen in such a way that

$$q_\varepsilon - 1 \geqslant 2$$

for every $\varepsilon \in (0, \frac{1}{6})$, implying that there are always at least 2 cubes in $\mathscr{C}_{\varepsilon,a}$ for every $\varepsilon \in (0, \frac{1}{6})$. We say that $\mathscr{C}_{\varepsilon,a}$ is the *cubic decomposition* of $Q_1(0)$ with origin in a and mesh thickness ε. Let

$$\begin{aligned} \mathscr{F}_{\varepsilon,a} &:= \big\{F \,|\, F \text{ is an } (n-1)\text{-dimensional face of } \partial Q, \text{ for some open cube } Q \in \mathscr{C}_{\varepsilon,a}\big\}, \\ S_{\varepsilon,a} &:= \bigcup_{F \in \mathscr{F}_{\varepsilon,a}} F. \end{aligned}$$

We say that $S_{\varepsilon,a}$ is the $(n-1)$-*skeleton* of the cubic decomposition $\mathscr{C}_{\varepsilon,a}$.

Lemma 2.1 (Choice of the cubic decomposition). *Let $n \in \mathbb{N}$ such that $n > 0$. Let $V \in L^p_{\mathbb{Z}}(Q_1(0), \mu)$ where $\mu := f\mathcal{L}^n$ with*

$$f(x) := \left(\frac{1}{2} - \|x\|_\infty\right)^q$$

for some $q \in (-\infty, 1]$. Then, there exists a subset $E_V \subset (0, \frac{1}{6})$ satisfying the following properties:

1. $\mathcal{L}^1\big((0, \frac{1}{6}) \smallsetminus E_V\big) = 0$;
2. *for every $\varepsilon \in E_V$, there exists $a_\varepsilon \in Q_\varepsilon(0)$ such that $\varepsilon \in R_{V,c_Q} := R_{V^\flat,c_Q}$* [3] *for every $Q \in \mathscr{C}_{\varepsilon,a_\varepsilon}$ and*

$$\lim_{\substack{\varepsilon \in E_V \\ \varepsilon \to 0^+}} \varepsilon \Bigg(\sum_{Q \in \mathscr{C}_{\varepsilon,a_\varepsilon}} \int_{\partial Q} |V - (V)_Q|^p f(c_Q) \, d\mathscr{H}^{n-1} \Bigg) = 0, \tag{2.2}$$

where $(V)_Q = \fint_Q V \, d\mathcal{L}^n$.

Proof. For any $x \in Q_1(0)$ let $R_{V,x} := R_{V^\flat,x}$. By assumption

$$\begin{aligned} \int_0^{\frac{1}{6}} \int_{Q_{1-3\rho}(0)} \mathbb{1}_{\rho \in R_{V,x}} \, d\mathcal{L}^n \, d\rho &= \int_{Q_1(0)} \mathcal{L}^1(R_{V,x}) \, d\mathcal{L}^n = \int_{Q_1(0)} 2 \operatorname{dist}(x, \partial Q_1(0)) \, d\mathcal{L}^n \\ &= \int_0^{\frac{1}{6}} \mathcal{L}^n(Q_{1-3\rho}(0)) \, d\rho. \end{aligned} \tag{2.3}$$

For any $\rho \in (0, \frac{1}{6})$ let

$$X_\rho = \{x \in Q_{1-3\rho}(0) \,:\, \rho \notin R_{V,x}\},$$

then by (2.3) $\mathcal{L}^n(X_\rho) = 0$ for a.e. $\rho \in (0, \frac{1}{6})$. Now notice that for any $\rho \in (0, \frac{1}{6})$

$$\mathcal{L}^n(X_\rho) \geqslant \sum_{c \in C_\rho} \int_{Q_\rho(0)} \mathbb{1}_{X_\rho}(x + c) \, d\mathcal{L}^n = \int_{Q_\rho(0)} \sum_{c \in C_\rho} \mathbb{1}_{X_\rho}(x + c) \, d\mathcal{L}^n,$$

[3] see Definition 1.2.

thus for a.e. $\rho \in (0, \frac{1}{6})$ we have that for a.e. $a_\rho \in Q_\rho(0)$ there holds $\rho \in R_{V,c_Q}$ for any $Q \in \mathscr{C}_{\rho,a_\rho}$. Let E_V be the set of all such $\rho \in (0, \frac{1}{6})$.
Now let $\varepsilon \in E_V$. We claim that

$$I_\varepsilon := \int_{Q_\varepsilon(0)} \sum_{Q \in \mathscr{C}_{\varepsilon,a}} \int_{\partial Q} |V(x) - (V)_Q|^p f(c_Q)\, d\mathscr{H}^{n-1}(x)\, d\mathcal{L}^n(a) = o\big(\varepsilon^{n-1}\big), \varepsilon \to \tag{2.4}$$

as 0^+ in E_V. Indeed, let $\mathscr{F}$ be the set of the faces of the cube $Q_1(0) \subset \mathbb{R}^n$ and notice that

$$\begin{aligned}
I_\varepsilon &= \int_{Q_\varepsilon(0)} \sum_{F_0 \in \mathscr{F}} \sum_{c \in C_\varepsilon} \int_{\varepsilon F_0} \left| V(x+c+a) - \fint_{Q_\varepsilon(c+a)} V \right|^p f(c+a)\, d\mathscr{H}^{n-1}(x)\, d\mathcal{L}^n(a) \\
&= \sum_{F_0 \in \mathscr{F}} \sum_{c \in C_\varepsilon} \int_{\varepsilon F_0} \int_{Q_\varepsilon(0)} \left| V(x+c+a) - \fint_{Q_\varepsilon(c+a)} V \right|^p f(c+a)\, d\mathcal{L}^n(a)\, d\mathscr{H}^{n-1}(x) \\
&= \sum_{F_0 \in \mathscr{F}} \sum_{c \in C_\varepsilon} \int_{\varepsilon F_0} \int_{Q_\varepsilon(c)} \left| V(x+y) - \fint_{Q_\varepsilon(y)} V \right|^p f(y)\, d\mathcal{L}^n(y)\, d\mathscr{H}^{n-1}(x).
\end{aligned}$$

Observe that for any $c \in C_\varepsilon$, $x \in \partial Q_\varepsilon(0)$

$$\begin{aligned}
&\int_{Q_\varepsilon(c)} \left| V(x+y) - \fint_{Q_\varepsilon(y)} V(z) \right|^p f(y) d\mathcal{L}^n(y) \\
&\leqslant \fint_{Q_\varepsilon(0)} \int_{Q_\varepsilon(c)} |V(x+y) - V(z+y)|^p f(y) d\mathcal{L}^n(y) d\mathcal{L}^n(z).
\end{aligned}$$

Thus for any $F_0 \in \mathscr{F}$

$$\begin{aligned}
&\sum_{c \in C_\varepsilon} \int_{\varepsilon F_0} \int_{Q_\varepsilon(c)} \left| V(x+y) - \fint_{Q_\varepsilon(y)} V \right|^p f(y)\, d\mathcal{L}^n(y)\, d\mathscr{H}^{n-1}(x) \\
&\leqslant \int_{\varepsilon F_0} \fint_{Q_\varepsilon(0)} \int_{Q_{1-2\varepsilon}(0)} |V(x+y) - V(z+y)|^p f(y) d\mathcal{L}^n(y) d\mathcal{L}^n(z) d\mathscr{H}^{n-1}(x) \\
&\leqslant 2^{p-1} \int_{\varepsilon F_0} \int_{Q_{1-2\varepsilon}(0)} |V(x+y) - V(y)|^p f(y) d\mathcal{L}^n(y) d\mathscr{H}^{n-1}(x) \\
&\quad + 2^{p-1} \varepsilon^{n-1} \fint_{Q_\varepsilon(0)} \int_{Q_{1-2\varepsilon}(0)} |V(z+y) - V(y)|^p f(y) d\mathcal{L}^n(y) d\mathcal{L}^n(z) \\
&\leqslant 2^p \varepsilon^{n-1} \sup_{\alpha \in Q_\varepsilon(0)} \|V - V(\cdot - \alpha)\|^p_{L^p(Q_{1-2\varepsilon}, \mu)}.
\end{aligned}$$

Since $C^0_c(Q_1(0))$ is dense in $L^p(Q_1(0), \mu)$ (see [23, Theorem 4.3]), given any $\delta > 0$ we can find $\tilde{V} \in C^0_c(Q_1(0))$ such that

$$\int_{Q_1(0)} |V - \tilde{V}|^p\, d\mu \leqslant \delta.$$

Notice that by Taylor's Theorem

$$\left| \frac{f(x+\alpha) - f(x)}{f(x)} \right| = \left| \frac{(\frac{1}{2} - \|x+\alpha\|_\infty)^q - (\frac{1}{2} - \|x\|_\infty)^q}{(\frac{1}{2} - \|x\|)^q} \right| \leqslant q \|\alpha\|_\infty \frac{(\frac{1}{2} - \|x\|_\infty - \frac{\varepsilon}{2})^{q-1}}{(\frac{1}{2} - \|x\|_\infty)^q} \tag{2.5}$$

$$\leqslant q\varepsilon 2^{1-q}\left(\frac{1}{2}-\|x\|_\infty\right)^{-1}\leqslant C$$

for every $x\in Q_{1-2\varepsilon}(0)$, $\alpha\in Q_\varepsilon(0)$ and for some constant $C>0$ depending only on q. Thus

$$\begin{aligned}\|V-V(\cdot-\alpha)\|^p_{L^p(Q_{1-\varepsilon}(0),\mu)}&\leqslant 4^{p-1}\bigg(\int_{Q_{1-2\varepsilon}(0)}|V-\tilde V|^p\,d\mu+\int_{Q_{1-\varepsilon}(0)}|\tilde V-\tilde V(\cdot-\alpha)|^p\,d\mu\\&\quad+\int_{Q_{1-2\varepsilon}(0)}|\tilde V(\cdot-\alpha)-V(\cdot-\alpha)|^p\,d\mu\bigg)\\&=4^{p-1}\bigg(2\int_{Q_1(0)}|V-\tilde V|^p\,d\mu+\int_{Q_{1-2\varepsilon}(0)}|\tilde V-\tilde V(\cdot-\alpha)|^p\,d\mu\\&\quad+\int_{Q_{1-2\varepsilon}(-\alpha)}|\tilde V-V|^p\,\frac{f(\cdot+\alpha)-f}{f}\,d\mu\bigg)\\&\leqslant 4^{p-1}(2+C)\delta+4^{p-1}\int_{Q_{1-2\varepsilon}(0)}|\tilde V-\tilde V(\cdot-\alpha)|^p\,d\mu.\end{aligned}$$

As $\tilde V\in C^0_c(Q_1(0))$

$$\sup_{\alpha\in Q_\varepsilon(0)}\|\tilde V-\tilde V(\cdot-\alpha)\|^p_{L^p(Q_{1-2\varepsilon}(0),\mu)}\to 0 \text{ as } \varepsilon\to 0^+,$$

we have

$$\limsup_{\varepsilon\to 0^+}\sup_{\alpha\in Q_\varepsilon(0)}\|V-V(\cdot-\alpha)\|^p_{L^p(Q_{1-2\varepsilon}(0),\mu)}\leqslant 4^{p-1}(2+C)\delta.$$

By letting $\delta\to 0^+$ in the previous inequality we get

$$\sup_{\alpha\in Q_\varepsilon(0)}\|V-V(\cdot-\alpha)\|^p_{L^p(Q_{1-2\varepsilon}(0),\mu)}\to 0 \text{ as } \varepsilon\to 0^+, \tag{2.6}$$

and claim (2.4) follows.

By Fubini's theorem, for every fixed $\varepsilon\in E_V$ there exists some non-negligible subset $T_\varepsilon\subset Q_\varepsilon(0)$ such that for any $a\in T_\varepsilon$ $\varepsilon\in R_{V,c_Q}$ for any $Q\in\mathscr{C}_{\varepsilon,a}$ and

$$\begin{aligned}\sum_{Q\in\mathscr{C}_{\varepsilon,a}}\int_{\partial Q}|V-(V)_Q|^p f(c_Q)\,d\mathscr{H}^{n-1}&\leqslant\frac{1}{\varepsilon^n}\int_{Q_\varepsilon(0)}\sum_{Q\in\mathscr{C}_{\varepsilon,a}}\int_{\partial Q}|V-(V)_Q|^p\,f(c_Q)d\mathscr{H}^{n-1}\,d\mathcal{L}^n(a)\\&=\frac{1}{\varepsilon^n}I_\varepsilon.\end{aligned}$$

By (2.4), for every $a\in T_\varepsilon$ we have

$$\sum_{Q\in\mathscr{C}_{\varepsilon,a}}\int_{\partial Q}|V-(V)_Q|^p f(c_Q)\,d\mathscr{H}^{n-1}=o(\varepsilon^{-1})$$

as $\varepsilon\to 0^+$ in E_V. The statement follows. □

Fix any $V\in L^p_{\mathbb{Z}}(Q_1(0),\mu)$ and $\varepsilon\in E_V$. From now on, we will denote simply by $\mathscr{C}_\varepsilon$ the cubic decomposition $\mathscr{C}_{\varepsilon,a_\varepsilon}$ provided by Lemma 2.1. Accordingly, the subscript "a_ε" will be omitted in any writing referring to such a cubic decomposition.

Given any $Q \in \mathscr{C}_\varepsilon$, we say that Q is a *good cube* if

$$\int_{\partial Q} V \cdot \nu_{\partial Q} = 0$$

and that Q is a *bad cube* otherwise. We denote by $\mathscr{C}_\varepsilon^g$ the subfamily of $\mathscr{C}_\varepsilon$ made of all the good cubes and by $\mathscr{C}_\varepsilon^b$ the one made of all the bad cubes. Moreover, we let

$$\Omega_\varepsilon := \bigcup_{Q \in \mathscr{C}_\varepsilon} Q, \quad \Omega_\varepsilon^g := \bigcup_{Q \in \mathscr{C}_\varepsilon^g} Q, \quad \Omega_\varepsilon^b := \bigcup_{Q \in \mathscr{C}_\varepsilon^b} Q,$$
$$S_\varepsilon := \bigcup_{Q \in \mathscr{C}_\varepsilon} \partial Q, \quad S_\varepsilon^g := \bigcup_{Q \in \mathscr{C}_\varepsilon^g} \partial Q, \quad S_\varepsilon^b := \bigcup_{Q \in \mathscr{C}_\varepsilon^b} \partial Q.$$

Lemma 2.2. *Assume that $n \geqslant 2$. Then, we have*

$$\lim_{\substack{\varepsilon \in E_V \\ \varepsilon \to 0^+}} \varepsilon^n \sum_{Q \in \mathscr{C}_\varepsilon} f(c_Q) = 0.$$

In particular if $q = 0$ (and thus $\mu = \mathcal{L}^n$) we have

$$\lim_{\substack{\varepsilon \in E_V \\ \varepsilon \to 0^+}} \mathcal{L}^n(\Omega_\varepsilon^b) = 0.$$

Proof. Notice that by estimate (2.5)

$$\frac{|f - f(c_Q)|}{f} \leqslant C \qquad \text{on } Q \text{ for every } Q \in \mathscr{C}_\varepsilon \tag{2.7}$$

for some universal constant $C > 0$. For every bad cube $Q \in \mathscr{C}_\varepsilon^b$, it holds that

$$1 \leqslant \left| \int_{\partial Q} V \cdot \nu_{\partial Q} \, d\mathscr{H}^{n-1} \right| \leqslant \int_{\partial Q} |V| \, d\mathscr{H}^{n-1}.$$

Since $V \in L^p_{\mathbb{Z}}(Q_1^5(0), \mu)$, multiplying the previous inequality by $f(c_Q)$ and summing over all the bad cubes we get

$$\begin{aligned}
\sum_{Q \in \mathscr{C}_\varepsilon^b} f(c_Q) &\leqslant \sum_{Q \in \mathscr{C}_\varepsilon^b} \int_{\partial Q} |V| f(c_Q) \, d\mathscr{H}^{n-1} \\
&\leqslant \Bigg(\sum_{Q \in \mathscr{C}_\varepsilon^b} \int_{\partial Q} |V|^p f(c_Q) \, d\mathscr{H}^{n-1} \Bigg)^{\frac{1}{p}} \Bigg(\sum_{Q \in \mathscr{C}_\varepsilon^b} \int_{\partial Q} f(c_Q) \, d\mathscr{H}^{n-1} \Bigg)^{\frac{1}{p'}} \\
&= (2n)^{\frac{1}{p'}} \varepsilon^{\frac{n-1}{p'}} \Bigg(\sum_{Q \in \mathscr{C}_\varepsilon^b} \int_{\partial Q} |V|^p f(c_Q) \, d\mathscr{H}^{n-1} \Bigg)^{\frac{1}{p}} \Bigg(\sum_{Q \in \mathscr{C}_\varepsilon^b} f(c_Q) \Bigg)^{\frac{1}{p'}},
\end{aligned}$$

which is equivalent to

$$\sum_{Q \in \mathscr{C}_\varepsilon^b} f(c_Q) \leqslant (2n)^{p-1} \varepsilon^{(p-1)(n-1)} \sum_{Q \in \mathscr{C}_\varepsilon^b} \int_{\partial Q} |V|^p f(c_Q) \, d\mathscr{H}^{n-1}.$$

Hence, by the triangle inequality, we get

$$\begin{aligned}\sum_{Q\in\mathscr{C}_\varepsilon^b} f(c_Q) &\leqslant (4n)^{p-1}\varepsilon^{(p-1)(n-1)-1}\Bigg(\varepsilon\sum_{Q\in\mathscr{C}_\varepsilon^b}\int_{\partial Q}|V-(V)_Q|^p f(c_Q)\,d\mathscr{H}^{n-1}\\ &\qquad + 2n\sum_{Q\in\mathscr{C}_\varepsilon^b}\int_Q |V|^p f(c_Q)\,d\mathcal{L}^n\Bigg)\\ &\leqslant (4n)^{p-1}\varepsilon^{(p-1)(n-1)-1}\Bigg(\varepsilon\sum_{Q\in\mathscr{C}_\varepsilon}\int_{\partial Q}|V-(V)_Q|^p f(c_Q)\,d\mathscr{H}^{n-1}\\ &\qquad + 2n\sum_{Q\in\mathscr{C}_\varepsilon^b}\int_Q |V|^p f\,d\mathcal{L}^n + 2n\sum_{Q\in\mathscr{C}_\varepsilon^b}\int_Q |V|^p\frac{f(c_Q)-f}{f} f\,d\mathcal{L}^n\Bigg)\\ &\leqslant (4n)^{p-1}\varepsilon^{(p-1)(n-1)-1}\Bigg(\varepsilon\sum_{Q\in\mathscr{C}_\varepsilon}\int_{\partial Q}|V-(V)_Q|^p f(c_Q)\,d\mathscr{H}^{n-1} \qquad (2.8)\\ &\qquad + 2n(1+C)\int_{\Omega_\varepsilon^b}|V|^p f\,d\mathcal{L}^n\Bigg).\end{aligned}$$

Therefore

$$\begin{aligned}\varepsilon^n\sum_{Q\in\mathscr{C}_\varepsilon^b} f(c_Q) &\leqslant (4n)^{p-1}\varepsilon^{p(n-1)}\Bigg(\varepsilon\sum_{Q\in\mathscr{C}_\varepsilon}\int_{\partial Q}|V-(V)_Q|^p f(c_Q)\,d\mathscr{H}^{n-1}\\ &\qquad + 2n(1+C)\int_{Q_1(0)}|V|^p f\,d\mathcal{L}^n\Bigg)\end{aligned}$$

and the statement follows from (2.2) (here we need the assumption $n>1$). □

Remark 2.1. Assume that $p\in\big[n/(n-1),+\infty\big)$. In this case $\varepsilon^{(p-1)(n-1)-1}$ remains bounded as $\varepsilon\to 0^+$. Now by Lemma 2.1

$$\varepsilon\sum_{Q\in\mathscr{C}_\varepsilon}\int_{\partial Q}|V-(V)_Q|^p f(c_Q)\,d\mathscr{H}^{n-1}\to 0 \quad \text{as } \varepsilon\to 0^+ \text{ in } E_V.$$

Moreover, by Lemma 2.2, we have

$$\int_{\Omega_\varepsilon^b} f\,d\mathcal{L}^n = (1+C)\varepsilon^n\sum_{Q\in\mathscr{C}_\varepsilon^b} f(c_Q)\to 0 \quad \text{as } \varepsilon\to 0^+ \text{ in } E_V.$$

This implies $\mathcal{L}^n(\Omega_\varepsilon^b)\to 0$ as $\varepsilon\to 0^+$ in E_V, therefore

$$\int_{\Omega_\varepsilon^b}|V|^p f\,d\mathcal{L}^n\to 0 \quad \text{as } \varepsilon\to 0^+ \text{ in } E_V$$

by absolute continuity of the integral. Thus it follows from (2.8) that

$$\sum_{Q\in\mathscr{C}_\varepsilon^b} f(c_Q)\to 0^+ \quad \text{as } \varepsilon\to 0^+ \text{ in } E_V.$$

Let N^b_ε be the number of bad cubes in $\mathscr{C}_\varepsilon$. Notice that for $q \leqslant 0$, we have $f \geqslant 2^{-q}$ on $Q_1(0)$. This implies

$$N^b_\varepsilon \leqslant 2^q \sum_{Q \in \mathscr{C}^b_\varepsilon} f(c_Q) \to 0^+ \quad \text{as } \varepsilon \to 0^+ \text{ in } E_V.$$

Since $N^b_\varepsilon \in \mathbb{Z}$ for any $\varepsilon \in E_V$, $N^b_\varepsilon = 0$ for every $\varepsilon \in E_V$ small enough. Hence, whenever $p \in \big[n/(n-1), +\infty\big)$ and $q \leqslant 0$ we will assume, without losing generality, that there are no bad cubes in our chosen decomposition.

2.2. Smoothing on the $(n-1)$-skeleton of the cubic decomposition

The following smoothing lemma shows that for any k-dimensional cube $Q \subset \mathbb{R}^n$ and a vector field $V \in L^p(Q, \mathbb{R}^n)$ we can approximate V strongly in $L^p(Q)$ by a smooth and compactly supported vector field $V' \in C^\infty_c(Q, \mathbb{R}^n)$ having the same integral as V on Q.

Lemma 2.3. *Let $Q \subset \mathbb{R}^k$ be a k-dimensional cube with side length R. Let $V \in L^p(Q, \mathbb{R}^n)$. Let $\delta > 0$. There exists $V_\delta \in C^\infty_c(Q, \mathbb{R}^n)$ such that*

$$\int_Q V_\delta \, d\mathcal{L}^k = \int_Q V \, d\mathcal{L}^k$$

and

$$\|V_\delta - V\|_{L^p(Q)} < \delta.$$

Proof. Without loss of generality, we will assume that Q is centered in the origin of $\mathbb{R}^k$. Let $\psi \in C^\infty_c(\frac{1}{2}Q)$ and $r_0 \in (1/2, 1)$ such that

$$\int_Q \psi \, d\mathcal{L}^k = 1 \quad \text{and } R^k(1 - r_0^k) < \frac{\delta}{\|\psi\|_{L^p(Q)}}.$$

Let $r \in (r_0, 1)$ be such that

$$\|V\|_{L^p(Q \smallsetminus rQ)} \leqslant \min\left\{ \left(\frac{\delta}{\|\psi\|_{L^p(Q)}} \right)^{\frac{1}{p}}, \delta \right\}.$$

Set

$$s := \int_{Q \smallsetminus rQ} V \, d\mathcal{L}^k, \quad \tilde{V} := \mathbb{1}_{rQ} V + s\psi \in L^p(Q, \mathbb{R}^n).$$

Then

$$\int_Q \tilde{V} \, d\mathcal{L}^k = \int_{rQ} V \, d\mathcal{L}^k + \left(\int_{Q \smallsetminus rQ} V \, d\mathcal{L}^k \right) \int_Q \psi \, d\mathcal{L}^k = \int_Q V \, d\mathcal{L}^k.$$

Moreover

$$|s| = \left| \int_{Q \smallsetminus rQ} V \, d\mathcal{L}^k \right| \leqslant |Q \smallsetminus rQ|^{\frac{1}{p'}} \|V\|_{L^p(Q \smallsetminus rQ)}$$

$$\leqslant (R^k(1-r^k))^{\frac{1}{p'}} \left(\frac{\delta}{\|\psi\|_{L^p(Q)}}\right)^{\frac{1}{p}} \leqslant \frac{\delta}{\|\psi\|_{L^p(Q)}}.$$

Therefore

$$\|s\psi\|_{L^p(Q)} = \|\psi\|_{L^p(Q)}|s| \leqslant \delta$$

and, by choice of $\tilde{V}$,

$$\|V - \tilde{V}\|_{L^p(Q)} \leqslant \|s\psi\|_{L^p(Q)} + \|V\|_{L^p(Q \setminus rQ)} \leqslant 2\delta.$$

Notice that $\tilde{V}\big|_{Q\setminus rQ} \equiv 0$.
Let $\eta \in C_c^\infty(B_1(0))$ with $\int_{B_1(0)} \eta \, d\mathcal{L}^k = 1$. For any $\delta > 0$ let

$$\eta_\delta(x) := \frac{1}{\delta^k}\eta\left(\frac{x}{\delta}\right) \qquad \forall\, x \in \mathbb{R}^k.$$

Choose $\delta_0 > 0$ such that

$$2\delta_0 < \operatorname{dist}\big(\partial Q, \partial(rQ)\big) \qquad \text{and} \qquad \|\tilde{V} - \tilde{V} * \eta_{\delta_0}\|_{L^p(Q)} \leqslant \delta.$$

Set $V_\delta := \tilde{V} * \eta_{\delta_0}$. Then $V_\delta \in C_c^\infty(Q)$,

$$\int_Q V_\delta \, d\mathcal{L}^k = \int_{\mathbb{R}^k} \eta_{\delta_0} \, d\mathcal{L}^k \int_Q \tilde{V} \, d\mathcal{L}^k = \int_Q \tilde{V} \, d\mathcal{L}^k = \int_Q V \, d\mathcal{L}^k$$

and

$$\|V - V_\delta\|_{L^p(Q)} \leqslant \|V - \tilde{V}\|_{L^p(Q)} + \|\tilde{V} - V_\delta\|_{L^p(Q)} \leqslant 3\delta.$$

□

2.3. Extensions on good and bad cubes

In the following lemma we will show that, given $p \in (1, +\infty)$ and some bounded and connected Lipschitz domain $\Omega \subset \mathbb{R}^n$, for every mean-vanishing boundary datum $f \in L^p(\partial\Omega)$ we can find a divergence-free vector field $V_f \in L^p(\Omega)$ whose projection in the direction of the outer normal to $\partial\Omega$ is exactly f and whose L^p-norm is bounded by the L^p-norm of f. The proof of this statement is based on energy methods.

Lemma 2.4 (Extension on the good cubes)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected Lipschitz domain and $p \in [1, \infty)$. Let $f \in L^p(\partial\Omega)$ with*

$$\int_{\partial\Omega} f \, d\mathscr{H}^{n-1} = 0 \tag{2.9}$$

If $p \in (1, +\infty)$, there exists a vector field $V \in L^p(\Omega)$ such that

$$\int_\Omega V \cdot \nabla\varphi \, d\mathcal{L}^n = \int_{\partial\Omega} f\varphi \, d\mathscr{H}^{n-1} \qquad \forall\, \varphi \in C^\infty(\mathbb{R}^n) \tag{2.10}$$

and

$$\int_{\Omega}|V|^p \, d\mathcal{L}^n \leqslant C(p,\Omega)\int_{\partial\Omega}|f|^p \, d\mathscr{H}^{n-1} \tag{2.11}$$

for some constant $C(p,\Omega)$ depending only on p and Ω.
If $p=1$, then for every $q\in\left[1,\frac{n}{n-1}\right)$ there exists $V_q\in L^q(\Omega)$ (depending on q) satisfying (2.10) and (2.11) with $p=1$.

Remark 2.2. Observe that (2.10) implies that V is a distributional solution of the following Neumann problem

$$\begin{cases}\operatorname{div}(V)=0 & \text{in } \Omega\\ V\cdot\nu_{\partial\Omega}=f & \text{on } \partial\Omega,\end{cases}$$

where $\nu_\Omega:\partial\Omega\to\mathbb{R}^n$ is the outer unit-normal to $\partial\Omega$.

Proof.
Step 1: First we consider the case $p\in(1,\infty)$.
Let $p':=\dfrac{p}{p-1}$. For any $u\in W^{1,p'}(\Omega)$ let

$$E_p(u)=\frac{1}{p'}\int_\Omega|\nabla u|^{p'}\,d\mathcal{L}^n-\int_{\partial\Omega}fu\,d\mathscr{H}^{n-1}.$$

Recall that any function $u\in W^{1,p'}(\Omega)$ has a trace in $L^{p'}(\partial\Omega)$, and that the trace operator is continuous. Thus for any $v\in W^{1,p'}(\Omega)$ with $\int_\Omega v=0$ by Poincaré Lemma there holds

$$\left|\int_{\partial\Omega}fv\,d\mathscr{H}^{n-1}\right|\leqslant\|f\|_{L^p(\partial\Omega)}\|v\|_{L^{p'}(\partial\Omega)}\leqslant C(p,\Omega)\|f\|_{L^p(\partial\Omega)}\|\nabla v\|_{L^{p'}(\Omega)}$$

for some constant $C(p,\Omega)$ depending only on p and Ω. In particular the energy E_p is well defined on $W^{1,p'}(\Omega)$.
Let

$$\dot{W}^{1,p'}(\Omega):=\left\{v\in W^{1,p'}(\Omega),\int_\Omega v\,d\mathcal{L}^n=0\right\}$$

and observe that E_p is strictly convex on $\dot{W}^{1,p'}(\Omega)$. Let u be the unique minimizer of E_p in $\dot{W}^{1,p'}(\Omega)$. Then[4]

$$\int_\Omega|\nabla u|^{p'-2}\nabla u\cdot\nabla\varphi\,d\mathcal{L}^n=\int_{\partial\Omega}f\varphi\,d\mathscr{H}^{n-1},\qquad\forall\varphi\in C^\infty(\mathbb{R}^n). \tag{2.12}$$

Moreover, as u is a minimizer of E_p, $E_p(u)\leqslant E_p(0)=0$. It follows that

$$\frac{1}{p'}\int_\Omega|\nabla u|^{p'}\,d\mathcal{L}^n\leqslant\int_{\partial\Omega}fu\,d\mathscr{H}^{n-1}\leqslant\|f\|_{L^p(\partial\Omega)}\|u\|_{L^{p'}(\partial\Omega)}\leqslant C(p,\Omega)\|f\|_{L^p(\partial\Omega)}\|\nabla u\|_{L^{p'}(\Omega)}.$$

[4] The argument above shows that (2.12) holds for any $\varphi\in C^\infty(\mathbb{R}^n)$ with $\int_\Omega\varphi d\mathcal{L}^n=0$, but assumption (2.9) implies that (2.12) remains valid for any $\varphi\in C^\infty(\mathbb{R}^n)$.

Thus

$$\int_\Omega |\nabla u|^{p'}\,d\mathcal{L}^n \leqslant \big(p' C(p,\Omega)\big)^p \int_{\partial\Omega} |f|^p\,d\mathscr{H}^{n-1}.$$

Set $V := |\nabla u|^{p'-2}\nabla u \quad \text{in } \Omega$. Then by (2.12)

$$\int_\Omega V\cdot\nabla\varphi\,d\mathcal{L}^n = \int_{\partial\Omega} f\varphi\,d\mathscr{H}^{n-1} \quad \forall\varphi\in C^\infty(\mathbb{R}^n)$$

and

$$\int_\Omega |V|^p\,d\mathcal{L}^n = \int_\Omega |\nabla u|^{p'}\,d\mathcal{L}^n \leqslant \big(p' C(p,\Omega)\big)^p \int_{\partial\Omega} |f|^p\,d\mathscr{H}^{n-1}.$$

Step 2: Next we consider the case $p=1$.
Let $s>n$. For any $u\in W^{1,s}(\Omega)$ let

$$E_s(u) = \frac{1}{s}\int_\Omega |\nabla u|^s\,d\mathcal{L}^n - \int_{\partial\Omega} fu\,d\mathscr{H}^{n-1}.$$

Notice that E_s is well defined and strictly convex in $\dot{W}^{1,s}(\Omega)$.
Recall the Sobolev embedding

$$W^{1,s}(\Omega) \hookrightarrow C^{0,\alpha}(\overline{\Omega})$$

for $\alpha = 1-\frac{n}{s}$. Then for any $u\in W^{1,s}(\Omega)$ the trace of u on $\partial\Omega$ lies in $C^{0,\alpha}(\partial\Omega)$ and if $\int_\Omega u\,d\mathcal{L}^n = 0$. Poincaré inequality implies

$$\|u\|_{L^\infty(\partial\Omega)} \leqslant C(s,\Omega)\|\nabla u\|_{L^s(\Omega)}$$

for some constant $C(s,\Omega)$ depending only on s and Ω.
Let u be the unique minimizer of E_s in $\dot{W}^{1,s}(\Omega)$. Then since $E_s(u)\leqslant E_s(0)=0$ there holds

$$\begin{aligned}\frac{1}{s}\int_\Omega |\nabla u|^s\,d\mathcal{L}^n &\leqslant \int_{\partial\Omega} fu\,d\mathscr{H}^{n-1}\\ &\leqslant \|f\|_{L^1(\partial\Omega)}\|u\|_{L^\infty(\partial\Omega)} \leqslant C(s,\Omega)\|f\|_{L^1(\partial\Omega)}\|\nabla u\|_{L^s(\Omega)}.\end{aligned}$$

Therefore

$$\left(\int_\Omega |\nabla u|^s\,d\mathcal{L}^n\right)^{\frac{s-1}{s}} \leqslant sC(s,\Omega)\int_{\partial\Omega}|f|\,d\mathscr{H}^{n-1}.$$

Moreover, since u is a minimizer of E_s,

$$\int_\Omega |\nabla u|^{s-2}\nabla u\cdot\nabla\varphi\,d\mathcal{L}^n = \int_{\partial\Omega} f\varphi\,d\mathscr{H}^{n-1} \quad \forall\varphi\in C^\infty(\mathbb{R}^n).$$

Similarly as before set $V := |\nabla u|^{s-2}\nabla u \quad \text{in } \Omega$. Then

$$\int_\Omega |V|\,d\mathcal{L}^n = \int_\Omega |\nabla u|^{s-1}\,d\mathcal{L}^n \leqslant \mathcal{L}^n(\Omega)^{\frac{1}{s}}\left(\int_\Omega |\nabla u|^s\,d\mathcal{L}^n\right)^{\frac{s-1}{s}} \leqslant sC(s,\Omega)\mathcal{L}^n(\Omega)^{\frac{1}{s}}\int_{\partial\Omega}|f|\,d\mathscr{H}^{n-1}.$$

Moreover

$$\int_\Omega |V|^{\frac{s}{s-1}}\,d\mathcal{L}^n = \int_\Omega |\nabla u|^s\,d\mathcal{L}^n < \infty$$

□

Remark 2.3. Let $Q \subset \mathbb{R}^n$ be the unit cube and let $C(p,Q)$ be the corresponding constant in (2.11). By an easy scaling argument one sees that for any $\varepsilon > 0$ one can choose $C(p,\varepsilon Q) = \varepsilon C(p,Q)$.

Given any n-dimensional cube $Q \subset \mathbb{R}^n$, the following lemma illustrates the properties of the so called "radial extension" of a boundary datum $f \in L^\infty(\partial Q)$ to the interior of the cube Q, given by

$$V(x) := \frac{1}{2^{n-1}} f\left(\frac{\varepsilon}{2}\frac{x-c_Q}{\|x-c_Q\|_\infty} + c_Q\right)\frac{x-c_Q}{\|x-c_Q\|_\infty^n} \qquad \forall\, x \in Q. \tag{2.13}$$

Lemma 2.5 (Extension on the bad cubes). *Let $Q := Q_\varepsilon(c_Q) \subset \mathbb{R}^n$ and $f \in L^\infty(\partial Q)$. Consider the radial extension $V : Q \to \mathbb{R}^n$ of f given by (2.13). Then, the following facts hold:*

1. *$V \in L^p(Q)$ for every $p \in \big[1, n/(n-1)\big)$;*
2. *for some constant $C(n,p) > 0$ depending only on n and p we have*

$$\int_Q |V|^p \, d\mathcal{L}^n \leqslant \varepsilon C(n,p) \int_{\partial Q} |f|^p \, d\mathscr{H}^{n-1}, \tag{2.14}$$

3. *for every $\varphi \in C^\infty(\mathbb{R}^n)$ we have*

$$\int_Q V \cdot \nabla\varphi \, d\mathcal{L}^n = \int_{\partial Q} f\varphi \, d\mathscr{H}^{n-1} - \left(\int_{\partial Q} f \, d\mathscr{H}^{n-1}\right)\varphi(c_Q). \tag{2.15}$$

Proof. Without losing generality, we assume that $\varepsilon = 1$ and $c_Q = 0$. Let $r(x) := \|x\|_\infty$, for every $x \in \mathbb{R}^n$. First, notice that $r : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz map such that $|\nabla r(x)| = 1$, for a.e. $x \in \mathbb{R}^n$. Moreover, since all the norms are equivalent on $\mathbb{R}^n$ there exists a constant $\tilde{C}(n) > 0$ depending only on n such that $|x| \leqslant \tilde{C} r(x)$, for a.e. $x \in \mathbb{R}^n$. Now choose any $p \in \big[1, n/(n-1)\big)$. By coarea formula we have

$$\begin{aligned}
\int_Q |V|^p \, d\mathcal{L}^n &\leqslant \frac{\tilde{C}^p}{2^{(n-1)p}} \int_0^{\frac{1}{2}} \frac{1}{\rho^{(n-1)p}} \int_{\partial Q_{2\rho}(0)} \left| f\left(\frac{x}{2\rho}\right)\right|^p d\mathscr{H}^{n-1}(x)\, d\rho \\
&= \frac{\tilde{C}^p}{2^{(n-1)(p-1)}} \left(\int_0^{\frac{1}{2}} \frac{1}{\rho^{(n-1)(p-1)}}\, d\rho\right)\left(\int_{\partial Q} |f(y)|^p \, d\mathscr{H}^{n-1}(y)\right) \\
&= C \int_{\partial Q} |f|^p \, d\mathscr{H}^{n-1},
\end{aligned}$$

with

$$C = C(n,p) := \frac{\tilde{C}^p}{2^{(n-1)(p-1)}} \int_0^{\frac{1}{2}} \frac{1}{\rho^{(n-1)(p-1)}}\, d\rho < +\infty.$$

Hence, 1. and 2. follow in once. We remark that the condition $p \in \big[1, n/(n-1)\big)$ is needed in order to guarantee the convergence of the integral in ρ.

We still need to prove 3. Pick any $\varphi \in C^\infty(\mathbb{R}^n)$. By the coarea formula we have

$$\int_Q V \cdot \nabla\varphi \, d\mathcal{L}^n = \frac{1}{2^{n-1}} \int_0^{\frac{1}{2}} \frac{1}{\rho^n} \int_{\partial Q_{2\rho}(0)} f\left(\frac{x}{2\rho}\right)\big(x \cdot \nabla\varphi(x)\big)\, d\mathscr{H}^{n-1}(x)\, d\rho$$

$$
\begin{aligned}
&= 2\int_0^{\frac{1}{2}} \int_{\partial Q} f(y)\big(y\cdot\nabla\varphi(2\rho y)\big)\, d\mathscr{H}^{n-1}(y)\, d\rho \\
&= \int_{\partial Q} f(y) \int_0^{\frac{1}{2}} \frac{d}{d\rho}\big(\varphi(2\rho y)\big)\, d\rho\, d\mathscr{H}^{n-1}(y) \\
&= \int_{\partial Q} f\varphi\, d\mathscr{H}^{n-1} - \Big(\int_{\partial Q} f\, d\mathscr{H}^{n-1}\Big)\varphi(0)
\end{aligned}
$$

and 3. follows. □

2.4. Proof of Theorem 1.1

As mentioned at the beginning of Section 2, Theorem 1.1 is equivalent to Theorem 2.1. We give here a proof of the latter. Our proof will unfold through the following steps (to skip the technical details, we outline here the proof for case $p \in (1, +\infty)$). Let $V \in L^p_{\mathbb{Z}}(Q^n_1(0))$.

1. For a fixed suitable $\varepsilon > 0$, we consider the cubic decomposition $\mathscr{C}_\varepsilon$ given by applying Lemma 2.1 to V. We smoothen the restriction of V to the $(n-1)$-skeleton S_ε of $\mathscr{C}_\varepsilon$ by applying Lemma 2.3 on every $(n-1)$-cube contained in S_ε. We denote the smoothing of V on S_ε by V_ε. For every good cube $Q \in \mathscr{C}_\varepsilon$ we approximate V on Q with $W_\varepsilon + (V)_Q$, where W_ε is the divergence-free extension of the mean-vanishing boundary datum $(V_\varepsilon - (V)_Q)\cdot\nu_{\partial Q}$ provided by Lemma 2.4. For every bad cube $Q \in \mathscr{C}_\varepsilon$ we approximate V on Q with the radial extension of the boundary datum $V_\varepsilon|_{\partial Q}\cdot\nu_{\partial Q}$ (see Lemma 2.5).
2. We show that the approximating vector field $\tilde{V}_\varepsilon$ which we have build in point 1 has distributional divergence given by a linear combination of delta distributions with integer coefficients, supported in the centers of the bad cubes.
3. We show that $\tilde{V}_\varepsilon$ converges strongly in L^p to V as $\varepsilon \to 0^+$.
4. By L^p-Hodge decomposition we can write $\tilde{V}_\varepsilon = d\varphi_\varepsilon + d^* A_\varepsilon$ with $\varphi_\varepsilon, A_\varepsilon \in W^{1,p}$. We approximate A_ε strongly in $W^{1,p}$ by a smooth 2-form A'_ε and we set $V_\varepsilon := d\varphi_\varepsilon + A'_\varepsilon$.

We can repeat this procedure for a sequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ converging to zero. The sequence $\{V_{\varepsilon_n}\}_{n\in\mathbb{N}}$ obtained this way lies in L^p_R (actually we might have to rescale the sequence slightly in order to deal with the region around the boundary) and tends to V in L^p. Finally we observe that if $p \in \big[n/(n-1), +\infty\big)$ the vector fields in L^p_R are divergence-free, therefore this property passes to the limit V.

Proof. Let $V \in L^p_{\mathbb{Z}}(Q_1(0))$ and let $\varepsilon \in E_V$ (constructed in Lemma 2.1). First, we notice that by using Lemma 2.3 separately on every face $F \in \mathscr{F}_\varepsilon$ we can build a vector field $V_\varepsilon \in C^\infty(S_\varepsilon)$ such that

$$
\int_{\partial Q} V_\varepsilon \cdot \nu_{\partial Q}\, d\mathscr{H}^{n-1} = \int_{\partial Q} V\cdot\nu_{\partial Q}\, d\mathscr{H}^{n-1} \qquad \forall\, Q \in \mathscr{C}_\varepsilon
$$

and

$$
\sum_{Q\in\mathscr{C}_\varepsilon} \int_{\partial Q} |V_\varepsilon - V|^p f(c_Q) d\mathscr{H}^{n-1} < \varepsilon.
$$

Let $\tilde{V}_\varepsilon$ be the vector field defined $\mathcal{L}^n$-a.e. on Ω_ε as follows:

1. if $Q \in \mathscr{C}_\varepsilon$ is a good cube, then we let $\tilde{V}_\varepsilon := W_\varepsilon + (V)_Q$ on Q, where W_ε is the extension of the datum $f := \big(V_\varepsilon - (V)_Q\big) \cdot \nu_{\partial Q}$ given by Lemma 2.4 (notice that for any good cube condition (2.9) is satisfied by our choice of f);
2. if $Q \in \mathscr{C}_\varepsilon$ is a bad cube, then we let

$$\tilde{V}_\varepsilon := \frac{1}{2^{n-1}} f\left(\frac{\varepsilon}{2}\frac{x - c_Q}{\|x - c_Q\|_\infty} + c_Q\right)\frac{x - c_Q}{\|x - c_Q\|_\infty^n}, \qquad \forall\, x \in Q,$$

with $f := V_\varepsilon\big|_{\partial Q} \cdot \nu_{\partial Q} \in L^\infty(\partial Q)$.

We recall that no bad cubes will appear in the cubic decomposition in case $p \in \big[n/(n-1), +\infty\big)$ (see Remark 2.1).

Claim 1. We claim that

$$\operatorname{div}(\tilde{V}_\varepsilon) = \sum_{Q \in \mathscr{C}_\varepsilon^b} d_Q \delta_{c_Q} \qquad \text{distributionally on } \Omega_\varepsilon,$$

where

$$d_Q := \int_{\partial Q} V_\varepsilon \cdot \nu_{\partial Q}\, d\mathscr{H}^{n-1} = \int_{\partial Q} V \cdot \nu_{\partial Q}\, d\mathscr{H}^{n-1} \in \mathbb{Z} \setminus \{0\}, \qquad \forall\, Q \in \mathscr{C}_\varepsilon^b.$$

Indeed, pick any $\varphi \in C_c^\infty(\Omega_\varepsilon)$. Let $Q \in \mathscr{C}_\varepsilon$ be a good cube. By the properties of the extension given by Lemma 2.4 and the divergence theorem we have

$$\begin{aligned}\int_Q \tilde{V}_\varepsilon \cdot \nabla\varphi\, d\mathcal{L}^n &= \int_{\partial Q} \big(V_\varepsilon - (V)_Q\big) \cdot \nu_{\partial Q}\varphi\, d\mathscr{H}^{n-1} + \int_{\partial Q} \big((V)_Q \cdot \nu_{\partial Q}\big)\varphi\, d\mathscr{H}^{n-1} \\ &= \int_{\partial Q} (V_\varepsilon \cdot \nu_{\partial Q})\varphi\, d\mathscr{H}^{n-1}.\end{aligned}$$

On the other hand, let $Q \in \mathscr{C}_\varepsilon$ be a bad cube. By (2.15), we have

$$\int_Q \tilde{V}_\varepsilon \cdot \nabla\varphi\, d\mathcal{L}^n = \int_{\partial Q} (V_\varepsilon \cdot \nu_{\partial Q})\varphi\, d\mathscr{H}^{n-1} - d_Q\langle \delta_{c_Q}, \varphi\rangle.$$

Hence we conclude that

$$\begin{aligned}\int_{\Omega_\varepsilon} \tilde{V}_\varepsilon \cdot \nabla\varphi\, d\mathcal{L}^n &= \sum_{Q \in \mathscr{C}_\varepsilon} \int_{\partial Q} (V_\varepsilon \cdot \nu_{\partial Q})\varphi\, d\mathscr{H}^{n-1} - \sum_{Q \in \mathscr{C}_\varepsilon^b} d_Q\langle \delta_{c_Q}, \varphi\rangle \\ &= -\sum_{Q \in \mathscr{C}_\varepsilon^b} d_Q\langle \delta_{c_Q}, \varphi\rangle.\end{aligned}$$

The claim follows.

Claim 2. We claim that $\|\tilde{V}_\varepsilon - V\|_{L^p(\Omega_\varepsilon,\mu)} \to 0$ as $\varepsilon \to 0^+$ in E_V.

Recall (2.7) and the estimate

$$\frac{|f - f(c_Q)|}{f(c_Q)} \leqslant C \qquad \text{in } Q, \text{ for any } Q \in \mathscr{C}_\varepsilon$$

which ha essentially the same proof. Notice that

$$\|\tilde{V}_\varepsilon - V\|_{L^p(\Omega_\varepsilon,\mu)}^p \leqslant (1 + C)(A_\varepsilon + B_\varepsilon),$$

with

$$A_\varepsilon := \sum_{Q\in\mathscr{C}_\varepsilon^g} \int_Q |\tilde{V}_\varepsilon - V|^p f(c_Q)\, d\mathcal{L}^n,$$

$$B_\varepsilon := \sum_{Q\in\mathscr{C}_\varepsilon^b} \int_Q |\tilde{V}_\varepsilon - V|^p f(c_Q)\, d\mathcal{L}^n.$$

By triangle inequality and by the estimate in Lemma 2.4, we have that

$$\begin{aligned} A_\varepsilon &\leqslant 2^{p-1}\left(\sum_{Q\in\mathscr{C}_\varepsilon^g} \int_Q |\tilde{V}_\varepsilon - (V)_Q|^p f(c_Q)\, d\mathcal{L}^n + \sum_{Q\in\mathscr{C}_\varepsilon^g} \int_Q |V - (V)_Q|^p f(c_Q)\, d\mathcal{L}^n \right) \\ &\leqslant 2^{p-1}\left(\sum_{Q\in\mathscr{C}_\varepsilon^g} \int_Q |W_\varepsilon|^p f(c_Q)\, d\mathcal{L}^n + \sum_{Q\in\mathscr{C}_\varepsilon^g} \int_Q |V - (V)_Q|^p f(c_Q)\, d\mathcal{L}^n \right) \\ &\leqslant 2^{p-1}\left(\varepsilon C_p \sum_{Q\in\mathscr{C}_\varepsilon^g} \int_{\partial Q} |V_\varepsilon - (V)_Q|^p\, f(c_Q) d\mathscr{H}^{n-1} + \sum_{Q\in\mathscr{C}_\varepsilon^g} \int_Q |V - (V)_Q|^p f(c_Q)\, d\mathcal{L}^n \right), \end{aligned}$$

where $C_p := C(p,Q)$ (see Remark 2.3). Again by triangle inequality and because of our choice of V_ε, we have

$$\begin{aligned} \varepsilon \sum_{Q\in\mathscr{C}_\varepsilon^g} \int_{\partial Q} |V_\varepsilon - (V)_Q|^p f(c_Q)\, d\mathscr{H}^{n-1} &\leqslant 2^{p-1}\left(\varepsilon \sum_{Q\in\mathscr{C}_\varepsilon^g} \int_{\partial Q} |V_\varepsilon - V|^p f(c_Q)\, d\mathscr{H}^{n-1} \right. \\ &\qquad \left. + \varepsilon \sum_{Q\in\mathscr{C}_\varepsilon^g} \int_{\partial Q} |V - (V)_Q|^p f(c_Q)\, d\mathscr{H}^{n-1} \right) \\ &\leqslant 2^{p-1}\left(2n\varepsilon^2 + \varepsilon \sum_{Q\in\mathscr{C}_\varepsilon^g} \int_{\partial Q} |V - (V)_Q|^p f(c_Q)\, d\mathscr{H}^{n-1} \right). \end{aligned}$$

Thus by Lemma 2.1 it follows that

$$\varepsilon \sum_{Q\in\mathscr{C}_\varepsilon^g} \int_{\partial Q} |V_\varepsilon - (V)_Q|^p f(c_Q)\, d\mathscr{H}^{n-1} \to 0 \qquad \text{as } \varepsilon \to 0^+ \text{ in } E_V.$$

Moreover, by (2.7) and recalling that $\mu = f\mathcal{L}^n$ we have

$$\begin{aligned} \sum_{Q\in\mathscr{C}_\varepsilon^g} \int_Q |V - (V)_Q|^p f(c_Q)\, d\mathcal{L}^n &\leqslant 2^p(1+C) \sum_{Q\in\mathscr{C}_\varepsilon^g} \fint_{Q_\varepsilon(0)} \int_Q |V(x+y) - V(y)|^p f(y)\, d\mathcal{L}^n(y) \\ &\leqslant 2^p(1+C) \fint_{Q_\varepsilon(0)} \|V(x+\cdot) - V\|^p_{L^p(\Omega_\varepsilon,\mu)} \to 0 \end{aligned}$$

as $\varepsilon \to 0^+$ in E_V. Hence, $A_\varepsilon \to 0$ as $\varepsilon \to 0^+$ in E_V.
On the other hand, by (2.14) we have

$$B_\varepsilon \leqslant 2^{p-1}\left(\sum_{Q\in\mathscr{C}_\varepsilon^b} \int_Q |\tilde{V}_\varepsilon|^p f(c_Q)\, d\mathcal{L}^n + \sum_{Q\in\mathscr{C}_\varepsilon^b} \int_Q |V|^p f(c_Q)\, d\mathcal{L}^n \right)$$

$$\leqslant 2^{p-1}\left(C\varepsilon \sum_{Q\in\mathscr{C}_\varepsilon^b}\int_{\partial Q}|V_\varepsilon|^p f(c_Q)\,d\mathscr{H}^{n-1}+\int_{\Omega_\varepsilon^b}|V|^p f(c_Q)\,d\mathcal{L}^n\right).$$

We notice that

$$\begin{aligned}\varepsilon \sum_{Q\in\mathscr{C}_\varepsilon^b}\int_{\partial Q}|V_\varepsilon|^p f(c_Q)\,d\mathscr{H}^{n-1} &\leqslant 2^{p-1}\left(\varepsilon \sum_{Q\in\mathscr{C}_\varepsilon^b}\int_{\partial Q}|V_\varepsilon - V|^p f(c_Q)\,d\mathscr{H}^{n-1}\right.\\ &\qquad\left. + \varepsilon \sum_{Q\in\mathscr{C}_\varepsilon^b}\int_{\partial Q}|V|^p f(c_Q)\,d\mathscr{H}^{n-1}\right)\\ &\leqslant 4^{p-1}\left(2n\varepsilon^2 + \varepsilon \sum_{Q\in\mathscr{C}_\varepsilon^b}\int_{\partial Q}|V-(V)_Q|^p f(c_Q)\,d\mathscr{H}^{n-1}\right.\\ &\qquad\left. + \varepsilon \sum_{Q\in\mathscr{C}_\varepsilon^b}\int_{\partial Q}|(V)_Q|^p f(c_Q)\,d\mathscr{H}^{n-1}\right).\end{aligned}$$

Moreover, by (2.7) we have

$$\begin{aligned}\varepsilon \sum_{Q\in\mathscr{C}_\varepsilon^b}\int_{\partial Q}|(V)_Q|^p f(c_Q)\,d\mathscr{H}^{n-1} &\leqslant \varepsilon \sum_{Q\in\mathscr{C}_\varepsilon^b}\int_{\partial Q}\left(⨍_Q |V|^p f(c_Q)\,d\mathcal{L}^n\right)d\mathscr{H}^{n-1}\\ &\leqslant \sum_{Q\in\mathscr{C}_\varepsilon^b}\varepsilon\left(⨍_Q |V|^p f(c_Q)\,d\mathcal{L}^n\right)\int_{\partial Q}d\mathscr{H}^{n-1}\\ &\leqslant 2n\sum_{Q\in\mathscr{C}_\varepsilon^b}\int_Q |V|^p f(c_Q)\,d\mathcal{L}^n = 2n\int_{\Omega_\varepsilon^b}|V|^p f(c_Q)\,d\mathcal{L}^n\\ &\leqslant (1+C)2n\int_{\Omega_\varepsilon^b}|V|^p f\,d\mathcal{L}^n.\end{aligned}$$

Thus, we have obtained

$$B_\varepsilon \leqslant C\left(\varepsilon^2 + \varepsilon \sum_{Q\in\mathscr{C}_\varepsilon^b}\int_{\partial Q}|V-(V)_Q|^p f(c_Q)\,d\mathscr{H}^{n-1} + \int_{\Omega_\varepsilon^b}|V|^p f\,d\mathcal{L}^n\right),$$

for some constant $C>0$ which does not depend on $\varepsilon \in E_V$. By Lemma 2.1 and Remark 2.1 we get that $B_\varepsilon \to 0$ as $\varepsilon \to 0^+$ in E_V. Hence, the claim follows.

Next we show that by rescaling V_ε we obtain a vector field with similar properties defined on the whole $Q_1(0)$. Let

$$\alpha_\varepsilon := \sup\{\alpha \in [1/2,1) \text{ s.t. } Q_1(0)\subset \alpha^{-1}\Omega_\varepsilon\}, \qquad \forall\,\varepsilon \in E_V.$$

Notice that $\alpha_\varepsilon \to 1^-$ as $\varepsilon \to 0^+$ in E_V. Define the vector field $\overline{V}_\varepsilon := \alpha_\varepsilon^{n-1}\tilde{V}_\varepsilon(\alpha_\varepsilon\,\cdot) : Q_1(0)\to \mathbb{R}^n$. It's straightforward that $\overline{V}_\varepsilon \in L^p(Q_1(0),\mu)$ in case $p>1$ and $\overline{V}_\varepsilon \in L^s(Q_1(0),\mu)$ for some $s>1$ in case $p=1$, for every given $\varepsilon\in E_V$. A direct computation also shows that the distributional divergence of $\overline{V}_\varepsilon$ on $Q_1(0)$ is given by

$$\operatorname{div}(\overline{V}_\varepsilon) = \sum_{Q\in\mathscr{C}_\varepsilon^b}\bar{d}_Q\delta_{\alpha_\varepsilon^{-1}c_Q},$$

with

$$\overline{d}_{Q'} = \begin{cases} d_{Q'} & \text{if } \alpha_\varepsilon^{-1} c_{Q'} \in Q_1(0), \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $\overline{V}_\varepsilon \to V$ in $L^p(Q_1(0),\mu)$. Indeed, we have

$$\begin{aligned}
\int_{Q_1(0)} |\overline{V}_\varepsilon - V|^p f \, d\mathcal{L}^n &= \int_{Q_1(0)} |\alpha_\varepsilon^{n-1}\tilde{V}_\varepsilon(\alpha_\varepsilon x) - V(x)|^p \, f(x) d\mathcal{L}^n(x) \\
&= \alpha_\varepsilon^{p(n-1)-n} \int_{\alpha_\varepsilon Q_1(0)} |\tilde{V}_\varepsilon(y) - \alpha_\varepsilon^{-(n-1)} V(\alpha_\varepsilon^{-1} y)|^p f(\alpha_\varepsilon^{-1} y) \, d\mathcal{L}^n(y) \\
&= \alpha_\varepsilon^{p(n-1)-n} \bigg(\int_{\Omega_\varepsilon} |\tilde{V}_\varepsilon(y) - \alpha_\varepsilon^{-(n-1)} V(\alpha_\varepsilon^{-1} y)|^p f(y) \, d\mathcal{L}^n(y) \\
&\quad + \int_{\Omega_\varepsilon} |\tilde{V}_\varepsilon(y) - \alpha_\varepsilon^{-(n-1)} V(\alpha_\varepsilon^{-1} y)|^p \frac{f(\alpha_\varepsilon^{-1} y) - f(y)}{f(y)} f(y) \, d\mathcal{L}^n(y) \bigg) \\
&\leqslant C_{n,p} \bigg(\int_{\Omega_\varepsilon} |\tilde{V}_\varepsilon - V|^p f \, d\mathcal{L}^n + \int_{\Omega_\varepsilon} |V - P_{\alpha_\varepsilon^{-1}} V|^p f \, d\mathcal{L}^n \bigg),
\end{aligned}$$

(see Lemma C.2 for the definition of $P_{\alpha_\varepsilon^{-1}} V$). By Lemma C.2 and since $\tilde{V}_\varepsilon \to V$ in $L^p(\Omega_\varepsilon,\mu)$, our claim follows.

Thus, we have built a vector field $\overline{V}_\varepsilon$ such that:

1. $\overline{V}_\varepsilon \in L^p(Q_1(0),\mu)$ and for $(p,q) \in [1,\frac{n}{n-1}) \times [0,1] \cup [\frac{n}{n-1},+\infty) \times (-\infty,0]$ we have $\overline{V}_\varepsilon \in L^s(Q_1(0),\mathcal{L}^n)$ for $s = p$ if $p > 1$ and $s > 1$ if $p = 1$[5];
2. the distributional divergence of $\overline{V}_\varepsilon$ on $Q_1(0)$ is given by a finite sum of delta distributions supported on a finite set $X_\varepsilon \subset Q_1(0)$ with integer weights $\{d_x \text{ s.t. } x \in X_\varepsilon\}$;
3. $\|\overline{V}_\varepsilon - V\|_{L^p(Q_1(0),\mu)} \to 0$ as $\varepsilon \to 0^+$ in E_V.

Now we are ready to reach the conclusions 1 and 2 of Theorem 2.1.

1. If $q \in [0,1]$ and $p \in \left[1,\frac{n}{n-1}\right)$ we possibly have $X_\varepsilon \neq \emptyset$, since bad cubes can appear in the cubic decompositions. Since $\overline{V}_\varepsilon$ always belongs to $L^s(Q_1(0))$ for some $s > 1$ (with $s = p$ if p itself is already greater than 1), we can Hodge-decompose $\overline{V}_\varepsilon^\flat$ as $\overline{V}_\varepsilon^\flat = d\varphi + d^*A$ for some $A \in \Omega^2_{W^{1,s}}(Q_1(0))$ and some $\varphi \in W^{1,s}(Q_1(0))$. Applying d^* to the previous decomposition we obtain

$$\Delta\varphi = d^*(\overline{V}_\varepsilon^\flat) = \operatorname{div}(\overline{V}_\varepsilon) = \sum_{x \in X_\varepsilon} d_x \delta_x.$$

By standard elliptic regularity, $\varphi \in C^\infty(Q_1(0) \smallsetminus X_\varepsilon)$. Choose $A_\varepsilon \in \Omega^2(Q_1(0))$ such that $\|A_\varepsilon - A\|_{\Omega^2_{W^{1,s}}(Q_1(0))} \leqslant \varepsilon$. Then $\|d^*A_\varepsilon - d^*A\|_{\Omega^1_{L^p(\mu)}(Q_1(0))} \leqslant \varepsilon$. Let $v_\varepsilon := d\varphi + d^*(A_\varepsilon)$ and let $U_\varepsilon := v_\varepsilon^\#$. Then $U_\varepsilon \in L^p_R(Q_1(0),\mu)$ for every $\varepsilon \in E_V$ and $U_\varepsilon \to V$ in $L^p(Q_1(0),\mu)$.

[5] In fact even when μ is different from $\mathcal{L}^n$, $\tilde{V}_\varepsilon$ is constructed through Lemmata 2.4 and 2.5 as extension of a smooth boundary datum, thus $\tilde{V}_\varepsilon$ lies in $L^r(\Omega_\varepsilon)$ for any $r \in [1,\frac{n}{n-1})$ if $1 \leqslant p < \frac{n}{n-1}$ and in $L^r(\Omega_\varepsilon)$ for any $r \in [1,\infty)$ if $p \geqslant \frac{n}{n-1}$. It follows that $\overline{V}_\varepsilon \in L^r(\Omega_\varepsilon)$ for any $r \in [1,\frac{n}{n-1})$ if $1 \leqslant p < \frac{n}{n-1}$ and $\overline{V}_\varepsilon \in L^r(\Omega_\varepsilon)$ for any $r \in [1,\infty)$ if $p \geqslant \frac{n}{n-1}$.

2. If $q \in (-\infty, 0]$ and $p \in \left[\frac{n}{n-1}, +\infty\right)$ no bad cubes are allowed in the cubic decomposition, thus $(\overline{V}_\varepsilon)_{\varepsilon \in E_V}$ is a sequence of divergence-free vector fields converging to V in $L^p(Q_1(0), \mu)$ as $\varepsilon \to 0^+$ in E_V. Hence V itself is divergence-free.

$\square$

Remark 2.4. Notice that if $p = 1$, $V_k \in L^s(Q_1^n(0))$ for any $k \in \mathbb{N}$ and for any $s \in \left[1, \frac{n}{n-1}\right)$.

Remark 2.5. Observe that this proof can be used to show that the analogous approximation result holds if we assume that V satisfies the first three conditions of Definition 1.2 and in addition we require that for every $\rho \in R_{F,x_0}$ we have that

$$\int_{\partial Q_\rho(x_0)} i^*_{\partial Q_\rho(x_0)} F \in S$$

for a set $S \subset \mathbb{R}$ such that $0 \in S$ and 0 is an isolated point in S. In this case the vector field V can be approximated in L^p by a sequence of vector fields $(V_n)_{n \in \mathbb{N}}$ smooth outside a finite set of points and such that for any $n \in \mathbb{N}$, $\operatorname{div}(V_n)$ is a finite sum of deltas with coefficients in S.

The following lemma corresponds to the 1-dimensional analogue of Theorem 2.1, for which we had assumed $n > 1$. In the following, given an interval $I \subset \mathbb{R}$, $p \in [1, \infty)$ and a subset $E \subset L^p(I)$, we denote by $\overline{E}^{L^p}$ the strong L^p-closure of E in $L^p(I)$.

Lemma 2.6. *Let $I \subset \mathbb{R}$ be an interval and $p \in [1, +\infty)$. Then*

$$\overline{L^p_R(I)}^{L^p} = L^p(I, \mathbb{Z}) + \{constants\} = L^p_{\mathbb{Z}}(I).$$

Proof. We start by showing the first equality. Notice that

$$L^p_R(I) = \left\{ V = c + \sum_{j \in J} a_j \chi_{I_j} \,:\, (I_j)_{j \in J} \text{ is a finite partition of } I, a_j \in \mathbb{Z}\, \forall\, j \in J,\, c \in [0,1) \right\}.$$

In other words, $L^p_R(I)$ consists of all integer-valued step functions and their translations by a constant.

First we show the inclusion " $\supset$ ": let $f = g + a$ with $g \in L^p(I, \mathbb{Z})$ and $a \in [0,1)$ and let $\varepsilon > 0$. For any $k \in \mathbb{N}$ set $g_k := \mathbb{1}_{|g| \leqslant k} g$. Then there exists $K \in \mathbb{N}$ such that $\|g_K - g\|_{L^p(I)} < \frac{\varepsilon}{2}$. Now for any $j \in \{-K, ..., K\}$ $g_K^{-1}(j) = g^{-1}(j)$ is a measurable set, therefore there exists a finite collection of disjoint intervals $(I_i^j)_{i \in J^j}$ such that $\mathcal{L}\left(\bigcup_{i \in J^j} I_i^j \triangle g^{-1}(j)\right) \leqslant \frac{\varepsilon}{2(2K+1)^2}$.
For any $j \in \{-K, ..., K\}$ set $A_j := \bigcup_{i \in J^j} I_i^j \setminus \left(\bigcup_{j' \neq j} \bigcup_{i \in J^{j'}} I_i^{j'}\right)$. Then A_j is a finite union of intervals and

$$\mathcal{L}(A_j \triangle g^{-1}(j)) \leqslant \mathcal{L}(A_j \setminus g^{-1}(j)) + \sum_{j' \neq j} \mathcal{L}\left(\bigcup_{i \in J^j} I_i^{j'} \cap g^{-1}(j)\right) \leqslant \frac{\varepsilon}{2(2K+1)}.$$

Set

$$\tilde{g}_K = \begin{cases} j & \text{if } x \in A_j \\ 0 & \text{otherwise} \end{cases}$$

Then by construction $\tilde{g}_K \in L^p_R(I)$ and $\|\tilde{g}_K - g\|_{L^p(I)} \leqslant \varepsilon$. We conclude that any $g \in L^p(I, \mathbb{Z})$ lies in the closure of $L^p_R(I)$. This shows " $\supset$ ". As $L^p(I,\mathbb{Z}) + \mathbb{R} = L^p(I,\mathbb{Z}) + [0,1)$ is closed in $L^p(I)$, the inclusion "$\subset$" holds as well.
Next we show the second equality. Let's start with "$\subset$". Let $g \in L^p(I,\mathbb{Z})$, $a \in \mathbb{R}$ and $f = g + a$. Let $x_0 \in I$, then for a.e. $r \in (0, \operatorname{dist}(x_0, \partial I))$ we have $f(x_0 + r) - f(x_0 - r) = g(x_0 + r) - g(x_0 - r) \in \mathbb{Z}$. Let $\tilde{R}_{F,x_0}$ denote the set of all such r. Set R_{R,x_0} to be the intersection of $\tilde{R}_{F,x_0}$ with the set of Lebesgue points of f. Then f satisfies Definition 1.2 and thus $f \in L^p_{\mathbb{Z}}(I)$.
To show "$\supset$" let $f \in L^p_{\mathbb{Z}}(I)$. Set $F : I \to \mathbb{S}^1$, $x \mapsto e^{i2\pi f(x)}$. Then F is a measurable bounded function. Notice that for any $x_0 \in I$, for a.e. $r \in (0, \operatorname{dist}(x_0, \partial I))$ there holds $F(x_0 - r) = F(x_0 + r)$. This implies that F is constant: this can be seen for instance approximating F by smooth functions with the same symmetry properties away from ∂I (convolving with a symmetric mollifier with small support), which then have to be constant. Choose $a \in \mathbb{R}$ such that $F \equiv e^{i2\pi a}$, then $f - a \in L^p(I,\mathbb{Z})$. This completes the proof. □

Remark 2.6. From Theorem 2.1 it follows immediately that

$$\overline{\Omega^{n-1}_{p,R}(Q_1(0))}^{L^p} = \Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0)).$$

To see this it is enough to check that $\Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$ is closed in $\Omega^{n-1}_p(Q_1(0))$, which can be shown by simple application of the coarea formula.

Notice that in case $p \in [n/(n-1), +\infty)$ we can approximate V strongly in L^p with smooth and divergence free vector fields. This is a straightforward consequence of Hodge decomposition.

2.5. A characterization of $\Omega^{n-1}_{p,\mathbb{Z}}$

First of all we apply Theorem 1.1 to obtain a useful characterization of the class $F \in \Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$.

Theorem 2.2. *Let $n \in \mathbb{N}$ such that $n > 0$, $p \in [1, +\infty)$. Let $F \in \Omega^{n-1}_p(Q^n_1(0))$. Then, the following are equivalent:*

1. *there exists $L \in \mathcal{R}_1(Q_1(0))$ such that $\partial L = *dF$ in $(W^{1,\infty}_0(Q_1(0)))^*$ and*

$$\mathbb{M}(L) = \sup_{\substack{\varphi \in \mathcal{D}(Q_1(0)),\\ ||d\varphi||_{L^\infty} \leqslant 1}} \int_{Q_1(0)} F \wedge d\varphi.$$

2. *for every Lipschitz function $f : \overline{Q_1(0)} \to [a,b] \subset \mathbb{R}$ such that $f|_{\partial Q_1(0)} \equiv b$, we have*

$$\int_{f^{-1}(t)} i^*_{f^{-1}(t)} F \in \mathbb{Z}, \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in [a,b];$$

3. *$F \in \Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$.*

Proof. We just need to show that $1 \Rightarrow 2$, $2 \Rightarrow 3$ and $3 \Rightarrow 1$. We prove these implications separately.

$1 \Rightarrow 2$. Assume 1. Let $L \in \mathcal{R}_1(Q_1(0))$ be given by

$$\langle L, \omega \rangle = \int_\Gamma \theta \langle \omega, \vec{L} \rangle \, d\mathscr{H}^1, \qquad \forall\, \omega \in \mathcal{D}^1(Q_1(0)),$$

where $\Gamma \subset Q_1(0)$ is a locally 1-rectifiable set, $\vec{L}$ is a Borel measurable unitary vector field on Γ and $\theta \in L^1(\Gamma, \mathscr{H}^1)$ is a $\mathbb{Z}$-valued function. Pick any Lipschitz function $f : \overline{Q_1(0)} \to \mathbb{R}$ such that $f = b$ on $\partial Q_1(0)$ and let $\varphi \in C_c^\infty((-\infty, b))$ be such that $\int_{\mathbb{R}} \varphi \, d\mathcal{L}^1 = 0$. By the coarea formula we have

$$\int_{Q_1(0)} F \wedge f^*(\varphi \operatorname{vol}_{\mathbb{R}}) = \int_{-\infty}^{+\infty} \varphi(t) \bigg(\int_{f^{-1}(t)} i^*_{f^{-1}(t)} F \bigg) \, dt.$$

At the same time, by the coarea formula for countably 1-rectifiable sets, we have

$$\langle L, f^*(\varphi \operatorname{vol}_{\mathbb{R}}) \rangle = \int_\Gamma \theta \langle f^*(\varphi \operatorname{vol}_{\mathbb{R}}), \vec{L} \rangle \, d\mathscr{H}^1 = \int_{-\infty}^{+\infty} \varphi(t) \bigg(\int_{\Gamma \cap f^{-1}(t)} \tilde{\theta} \bigg) \, dt,$$

where $\tilde{\theta} : \Gamma \to \mathbb{Z}$ is given by $\tilde{\theta} := \operatorname{sgn}(\langle f^* \operatorname{vol}_{\mathbb{R}}, \vec{L} \rangle)\theta$. Let $\Phi \in C_c^\infty((-\infty, b))$ satisfy $d\Phi = \varphi \operatorname{vol}_{\mathbb{R}}$. Notice that since $f = b$ on $\partial Q_1(0)$ we have $f^*\Phi \in W_0^{1,\infty}(Q_1(0))$. Then, by hypothesis, we have

$$\begin{aligned} \int_{Q_1(0)} F \wedge f^*(\varphi \operatorname{vol}_{\mathbb{R}}) &= \int_{Q_1(0)} F \wedge d f^* \Phi = \langle *dF, f^*\Phi \rangle = \langle \partial L, f^* \Phi \rangle \\ &= \langle L, d(f^*\Phi) \rangle = \langle L, f^*(d\Phi) \rangle = \langle L, f^*(\varphi \operatorname{vol}_{\mathbb{R}}) \rangle. \end{aligned}$$

Therefore

$$\int_{-\infty}^{\infty} \varphi(t) \bigg(\int_{f^{-1}(t)} i^*_{f^{-1}(t)} F - \int_{\Gamma \cap f^{-1}(t)} \tilde{\theta} \bigg) \, dt = 0, \qquad \forall\, \varphi \in C_c^\infty((-\infty, b)) \text{ s.t. } \int_{\mathbb{R}} \varphi = 0.$$

We conclude that

$$\int_{f^{-1}(t)} i^*_{f^{-1}(t)} F - \int_{\Gamma \cap f^{-1}(t)} \tilde{\theta} = c, \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in [a, b],$$

for some constant $c \in \mathbb{R}$. We claim that $c = 0$. In fact let $m \in \mathbb{N} \smallsetminus \{0\}$. Integrating both sides on $(-m, m)$ we get

$$\int_{\{|f| < m\}} F \wedge f^* \operatorname{vol}_{\mathbb{R}} - \int_{\Gamma \cap \{|f| < m\}} \theta \langle f^* \operatorname{vol}_{\mathbb{R}}, \vec{L} \rangle = 2mc. \tag{2.16}$$

Since $f^* \operatorname{vol}_{\mathbb{R}} = df$, we have

$$\int_{Q_1(0)} F \wedge f^* \operatorname{vol}_{\mathbb{R}} - \int_\Gamma \theta \langle f^* \operatorname{vol}_{\mathbb{R}}, \vec{L} \rangle = \int_{Q_1(0)} F \wedge df - \langle L, df \rangle = \langle *dF - \partial L, f \rangle = 0.$$

Thus, by letting $m \to +\infty$ in (2.16), we get that the left-hand-side converges to 0 whilst the right-hand-side diverges to $+\infty$, unless $c = 0$. Hence we conclude that $c = 0$, i.e.

$$\int_{f^{-1}(t)} i^*_{f^{-1}(t)} F = \int_{\Gamma \cap f^{-1}(t)} \tilde{\theta} \in \mathbb{Z}, \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in [a, b],$$

since $\Gamma \cap f^{-1}(t)$ consists of finitely many points for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$.
$2 \Rightarrow 3$. Assume 2. Given $x_0 \in Q_1(0)$, let $f_{x_0} := \min\left\{\|\cdot - x_0\|_\infty, \frac{r_{x_0}}{2}\right\}$. We claim that we can find R_{F,x_0} as in Definition 1.2. Indeed, let $L \subset Q_1(0)$ be the set of the Lebesgue points of F. Let $r_{x_0} := 2\,\mathrm{dist}_\infty(x_0, \partial Q_1(0))$. Then, by the coarea formula, we have

$$0 = \mathcal{L}^n\big(Q_{r_{x_0}}(x_0) \smallsetminus L\big) = \frac{1}{2^n}\int_0^{r_{x_0}} \mathscr{H}^{n-1}\big((Q_1(0) \smallsetminus L) \cap \partial Q_\rho(x_0)\big)\, d\rho,$$

which implies that there exists a set $E_{x_0} \subset (0, r_{x_0})$ such that

1. $\mathcal{L}^1\big((0, r_{x_0}) \smallsetminus E_{x_0}\big) = 0$;
2. for every $\rho \in E_{x_0}$, $\mathscr{H}^{n-1}$-a.e. $x \in \partial Q_\rho(x_0)$ is a Lebesgue point for F.

Hence, for every $\rho \in E_{x_0}$ it makes sense to consider the pointwise restriction of F to $\partial Q_\rho(x_0)$. Notice that, by the coarea formula, we have

$$\int_{E_{x_0}} \left(\int_{\partial Q_\rho(x_0)} |F|^p \, d\mathscr{H}^{n-1}\right) d\rho = 2^n \int_{Q_{r_{x_0}}(x_0)} |F|^p \, d\mathcal{L}^n < +\infty,$$

which implies

$$\int_{\partial Q_\rho(x_0)} |F|^p \, d\mathscr{H}^{n-1} < +\infty, \qquad \text{for } \mathcal{L}^1\text{-a.e. } \rho \in (0, E_{x_0}).$$

Moreover, by Statement 2., we have

$$\int_{f_{x_0}^{-1}(\rho)} i^*_{f_{x_0}^{-1}(\rho)} F = \int_{\partial Q_\rho(x_0)} i^*_{\partial Q_\rho(x_0)} F \in \mathbb{Z}, \qquad \text{for } \mathcal{L}^1\text{-a.e. } \rho \in (0, E_{x_0}).$$

Our claim follows immediately.
$3 \Rightarrow 1$. Assume 3. By Theorem 1.1, we can find a sequence $\{F_k\}_{k\in\mathbb{N}} \subset \Omega^{n-1}_{p,R}(Q_1(0))$ such that $F_k \to F$ strongly in L^p. Since the estimate

$$|\langle T_{F_k} - T_F, \omega\rangle| \leqslant C||F_k - F||_{L^p}$$

holds for every $\omega \in \mathcal{D}^1(Q_1(0))$ such that $||\omega||_{L^\infty} \leqslant 1$ and for every $k \in \mathbb{N}$, we conclude that

$$\sup_{\substack{\varphi \in W_0^{1,\infty}(Q_1(0)),\\ \|d\varphi\|_{L^\infty} \leqslant 1}} \langle *dF_k - *dF, \varphi\rangle \leqslant C||F_k - F||_{L^p} \to 0 \quad \text{as } k \to \infty. \tag{2.17}$$

Fix any $\varepsilon \in (0,1)$. By (2.17) we can find a subsequence $\{F_{k_j(\varepsilon)}\}_{j\in\mathbb{N}} \subset \Omega^{n-1}_{p,R}(Q_1(0))$ such that

$$\|*dF_{k_j(\varepsilon)} - *dF_{k_{j+1}(\varepsilon)}\|_{(W_0^{1,\infty}(Q_1(0)))^*} \leqslant \frac{\varepsilon}{2^j}, \qquad \text{for every } j \in \mathbb{N}.$$

For every $h \in \mathbb{N}$, let L_h^ε be a minimal connection for the singular set of $F_{k_h(\varepsilon)}$ in the sense of Definition 1.3 (the existence of such a minimal connection is proved in Proposition A.1). Analogously, for every $j \in \mathbb{N}$, let $L^\varepsilon_{j,j+1}$ be the minimal connection for the singular set of $F_{k_j(\varepsilon)} - F_{k_{j+1}(\varepsilon)}$. Define the following sequence of integer 1-currents on $Q_1(0)$:

$$\tilde{L}_h^\varepsilon := \begin{cases} L_0^\varepsilon & \text{if } h = 0,\\ L_0^\varepsilon - \displaystyle\sum_{j=0}^{h-1} L^\varepsilon_{j,j+1} & \text{if } h > 0, \end{cases} \qquad \text{for every } h \in \mathbb{N}.$$

Clearly

$$\partial \tilde{L}_h^\varepsilon = \partial L_0^\varepsilon - \sum_{j=0}^{h-1} \partial L_{j,j+1}^\varepsilon = \partial L_0^\varepsilon - \sum_{j=0}^{h-1} (\partial L_j^\varepsilon - \partial L_{j+1}^\varepsilon) = \partial L_h^\varepsilon = *dF_{k_h(\varepsilon)}.$$

Moreover, since $L_{j,j+1}^\varepsilon$ is a minimal connection, it holds that

$$\mathbb{M}(L_{j,j+1}^\varepsilon) = \| *dF_{k_j(\varepsilon)} - *dF_{k_{j+1}(\varepsilon)} \|_{(W_0^{1,\infty}(Q_1(0)))^*} \leqslant \frac{\varepsilon}{2^j}, \qquad \text{for every } j \in \mathbb{N}.$$

Thus,

$$\mathbb{M}(\tilde{L}_{h+1}^\varepsilon - \tilde{L}_h^\varepsilon) = \mathbb{M}(L_{h,h+1}^\varepsilon) \leqslant \frac{\varepsilon}{2^h}, \qquad \text{for every } h \in \mathbb{N},$$

which amounts to saying that the sequence $\{\tilde{L}_h^\varepsilon\}_{h\in\mathbb{N}}$ is a Cauchy sequence in mass. Hence, by the closure of integer currents under mass convergence (see Lemma C.1), there exists an integer 1-current $\tilde{L}^\varepsilon \in \mathcal{R}_1(Q_1(0))$ such that

$$\mathbb{M}(\tilde{L}_h^\varepsilon - \tilde{L}^\varepsilon) \to 0 \quad \text{ as } h \to \infty,$$

Notice that

$$\partial \tilde{L}^\varepsilon = \lim_{h\to\infty} \partial \tilde{L}_h^\varepsilon = \lim_{h\to\infty} *dF_{k_h(\varepsilon)} = *dF \quad \text{ in } (W_0^{1,\infty}(Q_1(0)))^*.$$

The family of integer 1-cycles $\{\tilde{L}^\varepsilon - \tilde{L}^{1/2}\}_{0<\varepsilon<1} \subset \mathcal{R}_1(Q_1(0))$ is uniformly bounded in mass. Indeed, first we notice that by (2.10) it holds that

$$\mathbb{M}(L_h^\varepsilon) = \| *dF_{k_h(\varepsilon)} \|_{(W_0^{1,\infty}(Q_1(0)))^*} \leqslant C, \quad \forall\, h \in \mathbb{N}, \forall\, \varepsilon \in (0,1),$$

where $C > 0$ is a constant independent on h and ε. Thus, we have

$$\mathbb{M}(\tilde{L}_h^\varepsilon) \leqslant \mathbb{M}(L_0^\varepsilon) + \sum_{j=0}^{h-1} \mathbb{M}(L_{j,j+1}^\varepsilon) \leqslant C + \sum_{j=0}^{+\infty} \frac{\varepsilon}{2^j} \leqslant C + \varepsilon \leqslant C + 1 \quad \forall\, h \in \mathbb{N}, \forall\, \varepsilon \in (0,1).$$

Since $\tilde{L}_h^\varepsilon \to \tilde{L}^\varepsilon$ in mass as $h \to +\infty$ for every $\varepsilon \in (0,1)$, we have

$$\mathbb{M}(\tilde{L}^\varepsilon - \tilde{L}^{1/2}) \leqslant \mathbb{M}(\tilde{L}^\varepsilon) + \mathbb{M}(\tilde{L}^{1/2}) \leqslant 2(C+1).$$

Hence, by standard compactness arguments for currents (see for instance [22], Theorem 7.5.2), we can find a sequence $\varepsilon_k \to 0$ and an integer 1-cycle $\tilde{L} \in \mathcal{R}_1(Q_1(0))$ with finite mass such that $\tilde{L}^{\varepsilon_k} - \tilde{L}^{1/2} \to \tilde{L}$ weakly in $\mathcal{D}_1(Q_1(0))$ as $k \to +\infty$. If we let $L := \tilde{L}^{1/2} + \tilde{L}$ then we get $\tilde{L}^{\varepsilon_k} \to L$ weakly in $\mathcal{D}_1(Q_1(0))$. By construction, L is again an integer 1-current with finite mass such that $\partial L = \partial \tilde{L}^{1/2} = *dF$ in $(W^{1,\infty}(Q_1(0)))^*$. We claim that

$$\mathbb{M}(L) = \inf_{\substack{T \in \mathcal{M}_1(Q_1(0)),\\ \partial T = *dF}} \mathbb{M}(T).$$

By contradiction, assume that we can find $T \in \mathcal{M}_1(Q_1(0))$ such that $\partial T = *dF$ and

$$\mathbb{M}(T) < \mathbb{M}(L) \leqslant \liminf_{k\to\infty} \mathbb{M}(\tilde{L}^{\varepsilon_k}),$$

where the last inequality follows by weak convergence and lower semicontinuity of the mass. Then, we can find some $h \in \mathbb{N}$ such that

$$\mathbb{M}(T) < \mathbb{M}(\tilde{L}^{\varepsilon_h}) - 2\varepsilon_h.$$

Moreover, since $\mathbb{M}(L_0^{\varepsilon} - \tilde{L}^{\varepsilon}) \leqslant 2\varepsilon$ for every $0 < \varepsilon < 1$, it holds that

$$\mathbb{M}(L_0^{\varepsilon_h} - \tilde{L}^{\varepsilon_h}) \leqslant 2\varepsilon_h.$$

We define $\tilde{T} := T + L_0^{\varepsilon_h} - \tilde{L}^{\varepsilon_h}$ and we notice that $\partial \tilde{T} = *dF_{k_0(\varepsilon_h)}$. Moreover, by the minimality of $L_0^{\varepsilon_h}$, we conclude that

$$\mathbb{M}(L_0^{\varepsilon_h}) \leqslant \mathbb{M}(\tilde{T}) \leqslant \mathbb{M}(T) + \mathbb{M}(L_0^{\varepsilon_h} - \tilde{L}^{\varepsilon_h}) < \mathbb{M}(L_0^{\varepsilon_h}),$$

which is a contradiction. Thus, our claim follows.
Since $L \in \mathcal{R}_1(Q)$, we get that

$$\mathbb{M}(L) = \inf_{\substack{T \in \mathcal{R}_1(Q_1(0)),\\ \partial T = *dF}} \mathbb{M}(T) = \inf_{\substack{T \in \mathcal{M}_1(Q_1(0)),\\ \partial T = *dF}} \mathbb{M}(T)$$

and, by Lemma A.2, we have

$$\inf_{\substack{T \in \mathcal{M}_1(Q_1(0)),\\ \partial T = *dF}} \mathbb{M}(T) = \sup_{\substack{\varphi \in \mathcal{D}(Q_1(0)),\\ ||d\varphi||_{L^\infty} \leqslant 1}} \int_M F \wedge d\varphi.$$

Hence, 1. follows. ☐

Remark 2.7. We notice that in the proof of Theorem 2.2 we have never used the minimality property of L while showing that $1 \Rightarrow 2$. Hence whenever $F \in \Omega_p^{n-1}(Q_1^n(0))$ admits a connection we have $F \in \Omega_{p,\mathbb{Z}}^{n-1}(Q_1^n(0))$ in the sense of Definition 1.2 and the conclusions of Theorem 1.1 hold for F.

By the previous remark we can deduce the following result from the proof of Theorem 2.2.

Corollary 2.1. *Let $F \in \Omega_p^{n-1}(Q_1^n(0))$ and assume that there exists an integer-multiplicity rectifiable 1-current of finite mass $I \in \mathcal{R}_1(Q_1(0))$ such that $\partial I = *dF$. Then there exists an integer-multiplicity rectifiable 1-current $L \in \mathcal{R}_1(Q_1(0))$ of finite mass such that $\partial L = *dF$ and*

$$\mathbb{M}(L) = \inf_{\substack{T \in \mathcal{D}_1(Q_1(0))\\ \partial T = *dF}} \mathbb{M}(T).$$

In other words, whenever there exists a connection for F, then there exists a minimal connection for F.

2.6. The case of $\partial Q_1^{n+1}(0)$

In order to extend the previous results to more general manifolds we introduce the following definition.

Definition 2.2. Let M be a Lipschitz m-manifold embedded in $\mathbb{R}^n$. Let $p \in [1, \infty)$. Set

$$\Omega^{m-1}_{p,R,\infty}(M) := \left\{ \alpha \in \Omega^{m-1}_p(M) \cap \Omega^{m-1}_{L^\infty_{loc}}(M \setminus S) \,:\, *d\alpha = \sum_{p \in S} d_p \delta_p \right\},$$

where $S \subset M$ is a finite set and $d_p \in \mathbb{Z}$ for every $p \in S$.
If M is smooth we also define

$$\Omega^{m-1}_{p,R}(M) := \left\{ \alpha \in \Omega^{m-1}_p(M) \cap \Omega^{m-1}(M \setminus S) \,:\, *d\alpha = \sum_{p \in S} d_p \delta_p \right\},$$

where $S \subset M$ is a finite set and $d_p \in \mathbb{Z}$ for every $p \in S$.

Notice that, in particular, a 1-form $\alpha \in \Omega^{m-1}_{p,R,\infty}(M)$ is L^p-integrable on M and locally bounded away from a finite set of points S. The previous definition is motivated by the following observation: let M be a Lipschitz m-manifold, N a smooth m-manifold, $\varphi : M \to N$ a bi-Lipschitz map. Let $F \in \Omega^{m-1}_{p,R}(N)$. Then $\varphi^* F \in \Omega^{m-1}_{p,R,\infty}(M)$ (see Lemma 2.7).

Corollary 2.2. *Let $n \in \mathbb{N}$ be such that $n \geqslant 2$. Let $F \in \Omega^{n-2}_{p,\mathbb{Z}}(\partial Q^n_1(0))$ in the sense of Definition 1.4. Then, the following facts hold:*

1. *if $p \in \big[1, (n-1)/(n-2)\big)$, then there exists a sequence $\{F_k\}_{k \in \mathbb{N}} \subset \Omega^{n-2}_{p,R,\infty}(\partial Q^n_1(0))$ such that $F_k \to F$ strongly in L^p;*
2. *if $p \in \big[(n-1)/(n-2), +\infty\big)$, then $*dF = 0$ distributionally on ∂Q^n.*

Proof. Let $N := (0, ..., 0, \frac{1}{2}) \in \partial Q^n_1(0) \subset \mathbb{R}^n$ and let

$$U := \big\{(x_1, ..., x_{n-1}, x_n) \in \partial Q^n_1(0) \text{ s.t. } x_n = \frac{1}{2}\big\}$$

be the upper face of $\partial Q^n_1(0)$. Let $q := (n-1) - (n-2)p$. For every $x = (x_1, ..., x_n) \in \mathbb{R}^n$, we let $x' := (x_1, ..., x_{n-1}) \in \mathbb{R}^{n-1}$. Define $\Phi : \partial Q^n_1(0) \setminus N \subset \mathbb{R}^n \to Q^{n-1}_1(0)$ by

$$\Phi(x) := \begin{cases} \left(\dfrac{1}{2} - \dfrac{\sqrt{2}}{4}\|x'\|_\infty^{\frac{1}{2}}\right)\dfrac{x'}{\|x'\|_\infty} & \text{on } U \setminus N, \\ g(x) & \text{on } \partial Q^n_1(0) \setminus U, \end{cases}$$

where the map $g : \overline{\partial Q^n_1(0) \setminus U} \to \overline{Q^{n-1}_{1/2}(0)}$ is any bi-Lipschitz homeomorphism such that $g \equiv \left(\frac{1}{2} - \frac{\sqrt{2}}{4}\|x'\|_\infty^{\frac{1}{2}}\right)\frac{x'}{\|x'\|_\infty}$ on ∂U. Notice that Φ is an homeomorphism from $\partial Q^n_1(0) \setminus N$ to $Q^{n-1}_1(0)$. We denote its inverse map by Ψ.
We have that Ψ is Lipschitz on $Q^{n-1}_1(0)$ and Φ is Lipschitz on every compact set $K \subset \partial Q^n_1(0) \setminus N$, since there exists $C > 0$ such that

$$|d\Phi(x)| \leqslant \frac{C}{\|x - N\|_\infty^{\frac{1}{2}}}, \qquad \forall\, x \in \partial Q^n_1(0) \setminus N,$$
$$|d\Psi(y)| \leqslant C\left(\frac{1}{2} - \|y\|_\infty\right), \qquad \forall\, y \in Q^{n-1}_1(0).$$

Define $\tilde{F} := \Psi^* F$ and fix any $\varepsilon \in (0, 1/4)$. Notice that

$$\int_{Q_1^{n-1}(0)} \left(\frac{1}{2} - \|\cdot\|_\infty\right)^q |\tilde{F}|^p \, d\mathscr{H}^{n-1} \leqslant C \int_{Q_1^{n-1}(0)} \left(\frac{1}{2} - \|\cdot\|_\infty\right)^{n-1} |F \circ \Psi|^p \, d\mathscr{H}^{n-1}$$
$$\leqslant C \int_{\partial Q_1^n(0)} |F|^p \, d\mathscr{H}^{n-1} < +\infty.$$

Moreover if $I \in \mathcal{R}_1(\partial Q_1^n(0))$ and $*dF = \partial I$, then $\Phi_* I \in \mathcal{R}_1(\partial Q_1^{n-1}(0))$ and $*d\tilde{F} = \partial \Phi_* I$ (see Lemma 2.7). This implies that $\tilde{F} \in \Omega^{n-2}_{p,\mathbb{Z}}(Q_1^{n-1}(0), \mu)$ with $\mu := (\frac{1}{2} - \|\cdot\|_\infty)^q \, \mathcal{L}^{n-1}$ in the sense of Definition 1.2.
Let's consider first the case $p \in \big[1, (n-1)/(n-2)\big)$. Notice that in this case $q > 0$.
From the construction used in the proof of Theorem 2.1 with $f := (\frac{1}{2} - \|\cdot\|_\infty)^q$ it follows that there exists a $(n-2)$-form $\tilde{F}_\varepsilon \in \Omega^{n-2}_{p,R}(\Omega_{\varepsilon, a_\varepsilon})$ such that $\|\tilde{F}_\varepsilon - \tilde{F}\|_{L^p(S_{\varepsilon, a_\varepsilon})} \leqslant \varepsilon$ and

$$\|\tilde{F}_\varepsilon - \tilde{F}\|^p_{L^p(\Omega_{\varepsilon, a_\varepsilon}, \mu)} \to 0 \tag{2.18}$$

as $\varepsilon \to 0^+$ in $E_{\tilde{F}}$.
Define $F_\varepsilon := \Phi^*_{a_\varepsilon} \tilde{F}_\varepsilon$ on $\partial Q_1^n(0) \smallsetminus U_\varepsilon$, with $\Phi_{a_\varepsilon} := \Phi + a_\varepsilon$ and $U_\varepsilon := \Phi^{-1}_{a_\varepsilon}(Q_1^{n-1}(0) \smallsetminus \Omega_{\varepsilon, a_\varepsilon})$.
Notice that

$$\begin{aligned}
\|F_\varepsilon - F\|^p_{L^p(\partial Q_1^n(0) \smallsetminus U_\varepsilon)} &\leqslant C \int_{\partial Q_1^n(0) \smallsetminus U_\varepsilon} \frac{1}{\|\cdot - N\|_\infty^{(n-2)p}} |\tilde{F}_\varepsilon \circ \Phi_{a_\varepsilon} - \tilde{F} \circ \Phi_{a_\varepsilon}|^p \, d\mathscr{H}^{n-1} \\
&\quad + C \int_{\partial Q_1^n(0) \smallsetminus U_\varepsilon} \frac{1}{\|\cdot - N\|_\infty^{(n-2)p}} |\tilde{F} \circ \Phi_{a_\varepsilon} - \tilde{F} \circ \Phi|^p \, d\mathscr{H}^{n-1} \\
&\leqslant C \bigg(\int_{\Omega_{\varepsilon, a_\varepsilon}} |\tilde{F}_\varepsilon - \tilde{F}|^p f \, d\mathscr{H}^{n-1} \\
&\quad + \int_{Q^{n-1}_{1-\varepsilon}(0)} |\tilde{F}(\cdot - a_\varepsilon) - \tilde{F}|^p f \, d\mathscr{H}^{n-1} \bigg).
\end{aligned}$$

The first term tends to zero as $\varepsilon \to 0^+$ in $E_{\tilde{F}}$ by (2.18), while the second tends to zero as $\varepsilon \to 0^+$ by (2.6). Therefore we have

$$\|F_\varepsilon - F\|^p_{L^p(\partial Q_1^n(0) \smallsetminus U_\varepsilon)} \to 0$$

as $\varepsilon \to 0^+$ in $E_{\tilde{F}}$.
Now we notice that

$$\begin{aligned}
\int_{\partial U_\varepsilon} i^*_{\partial U_\varepsilon} F_\varepsilon &= - \int_{\partial(\partial Q_1^n(0) \smallsetminus U_\varepsilon)} i^*_{\partial(\partial Q_1^n(0) \smallsetminus U_\varepsilon)} F_\varepsilon \\
&= \int_{\partial \Omega_{\varepsilon, a_\varepsilon}} i^*_{\partial \Omega_{\varepsilon, a_\varepsilon}} \tilde{F}_\varepsilon = \int_{\partial \Omega_{\varepsilon, a_\varepsilon}} i^*_{\partial \Omega_{\varepsilon, a_\varepsilon}} \tilde{F} =: b_\varepsilon \in \mathbb{Z}.
\end{aligned}$$

If $b_\varepsilon = 0$, then we use Lemma 2.4 to extend F_ε inside U_ε. If $b_\varepsilon \neq 0$, then we use Lemma 2.5 to extend F_ε inside U_ε (notice that U_ε is an $(n-1)$-cube of side-length $8\varepsilon^2$ contained in U and centered at N, for ε sufficiently small). In both cases, the following estimate holds:

$$\int_{U_\varepsilon} |F_\varepsilon|^p \, d\mathscr{H}^{n-1} \leqslant C \varepsilon^2 \int_{\partial U_\varepsilon} |F_\varepsilon|^p \, d\mathscr{H}^{n-2}$$

$$\leqslant C\varepsilon^2 \left(\int_{\partial U_\varepsilon} |F - F_\varepsilon|^p d\mathscr{H}^{n-2} + \int_{\partial U_\varepsilon} |F|^p d\mathscr{H}^{n-2} \right).$$

Notice that the first term on the right hand side tends to zero as $\varepsilon \to 0^+$ in $E_{\tilde{F}}$, since $\|\tilde{F}_\varepsilon - \tilde{F}\|_{L^p(S_{\varepsilon,a_\varepsilon})} \leqslant \varepsilon$ for any $\varepsilon \in E_{\tilde{F}}$. In order to control the second term, pick any $\delta > 0$ sufficiently small and notice that, by coarea formula, we have

$$\begin{aligned}
\fint_0^\delta \varepsilon^2 \int_{\partial U_\varepsilon} |F|^p \, d\mathscr{H}^{n-2} \, d\mathcal{L}^1(\varepsilon) &\leqslant \int_0^\delta \varepsilon \int_{Q^{n-1}_{8\varepsilon^2}(N)} |F|^p \, d\mathscr{H}^{n-2} \, d\mathcal{L}^1(\varepsilon) \\
&\leqslant \frac{1}{16} \int_0^{8\delta^2} \int_{\partial Q^{n-1}_\zeta(N)} |F|^p \, d\mathscr{H}^{n-2} \, d\mathcal{L}^1(\zeta) \\
&\leqslant \frac{2^{n-1}}{16} \int_{Q^{n-1}_{8\delta^2}(N)} |F|^p \, d\mathscr{H}^{n-1} \to 0^+
\end{aligned}$$

as $\delta \to 0^+$. This implies that we can pick a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}} \subset E_{\tilde{F}}$ such that $\varepsilon_j \to 0^+$ and

$$\varepsilon_j^2 \int_{\partial U_{\varepsilon_j}} |F|^p \, d\mathscr{H}^{m-1} \to 0^+$$

as $j \to +\infty$. Thus

$$\|F_{\varepsilon_j} - F\|^p_{L^p(U_{\varepsilon_j})} \leqslant 2^{p-1} \left(\|F_{\varepsilon_j}\|^p_{L^p(U_{\varepsilon_j})} + \|F\|^p_{L^p(U_{\varepsilon_j})} \right) \to 0$$

as $j \to +\infty$, since $\mathscr{H}^{n-1}(U_{\varepsilon_j}) \to 0^+$ as $\varepsilon_j \to 0^+$. Hence, we conclude that

$$\|F_{\varepsilon_j} - F\|^p_{L^p(\partial Q^n_1(0))} \to 0$$

as $j \to +\infty$. Moreover, by construction we have that $*dF_{\varepsilon_j}$ is a finite sum of Dirac-deltas with integer coefficients, for any $j \in \mathbb{N}$. Thus arguing as in the final step of the proof of Theorem 2.1 (i.e. by Hodge decomposition), for any $j \in \mathbb{N}$ we can find $\hat{F}_{\varepsilon_j} \in \Omega^1_{p,R,\infty}(\partial Q^n_1(0))$ such that $\|F_{\varepsilon_j} - \hat{F}_{\varepsilon_j}\|_{L^p(\partial Q^n_1(0))} < \varepsilon_j$. The sequence $\{\hat{F}_j\}_{j\in\mathbb{N}}$ then has the desired properties. This concludes the proof in the case $p \in \big[1, (n-1)/(n-2)\big)$.
If $p \in \big[(n-1)/(n-2), +\infty\big)$ notice that $q \leqslant 0$. Therefore by Remark 2.1 we may assume, up to passing to a subsequence, that no bad cube appears in the construction of $\tilde{F}_\varepsilon$. Hence repeating the first part of the proof as in the previous case, we get $b_{\varepsilon_j} = 0$ and $*dF_{\varepsilon_j} = 0$ on $\partial Q^n_1(0)$ for any $j \in \mathbb{N}$. Thus we obtain that F can be approximated in $\Omega^{n-1}_p(\partial Q^n_1(0))$ by a sequence $(F_{\varepsilon_j})_{j\in\mathbb{N}}$ such that for any $j \in \mathbb{N}$ there holds $*dF_{\varepsilon_j} = 0$, and this property passes to the limit. □

Remark 2.8. Notice that if $p = 1$, for any $k \in \mathbb{N}$ we have that $F_k \in \Omega^{n-1}_q(\partial Q^n_1(0))$ for some $q > 1$ which does not depend on k (compare with Remark 2.4).

In the following lemma, given a bi-Lipschitz homeomorphism $\varphi : M \to N$ we adopt the notation $\varphi_* F := (\varphi^{-1})^* F$ for every k-form F on M. Instead, given a k-current T on M, by $\varphi_* T$ we simply denote the usual push-forward for currents.

Lemma 2.7. *Let $M, N \subset \mathbb{R}^n$ be Lipschitz m-manifolds in $\mathbb{R}^n$. Let $\varphi : M \to N$ be a bi-Lipschitz homeomorphism.*

Let $F \in \Omega_p^{m-1}(M)$ and assume that there exists a 1-rectifiable current $I \in \mathcal{R}_1(M)$ of finite mass such that $*dF = \partial I$ in $(W_0^{1,\infty}(M))^*$. Then $\varphi_* F \in \Omega_p^{m-1}(N)$ and $*d(\varphi_* F) = \partial \varphi_* I$.
If $\{F_k\}_{k\in\mathbb{N}}$ is a sequence in $\Omega_{p,R,\infty}^{m-1}(M)$ and $F_k \to F$ in $\Omega_p^{m-1}(M)$ as $k \to \infty$, then $(\varphi_* F_k)_{k\in\mathbb{N}}$ is a sequence in $\Omega_{p,R,\infty}^{m-1}(N)$ and $\varphi_* F_k \to \varphi_* F$ in $\Omega_p^{m-1}(N)$ as $k \to \infty$.
If in addition we assume that M is smooth and closed or a bounded simply connected Lipschitz domain, and for any $k \in \mathbb{N}$ we have $F_k \in \Omega_q^{m-1}(M)$ for some $q > 1$ (possibly dependent on k), then F can be approximated in $\Omega_p^{m-1}(M)$ by $(m-1)$-forms in $\Omega_{p,R}^{m-1}(M)$.

Proof. Assume that $*dF = \partial I$ holds in $(W_0^{1,\infty}(M))^*$. Then for any $f \in W_0^{1,\infty}(N)$ we have that $\varphi^* f \in W_0^{1,\infty}(M)$ and thus

$$\langle *d(\varphi_* F), f\rangle = \int_N \varphi_* F \wedge df = \int_M F \wedge d\varphi^* f = \langle \partial I, \varphi^* f\rangle = \langle I, \varphi^* df\rangle = \langle \partial \varphi_* I, f\rangle,$$

therefore $*d(\varphi_* F) = \partial \varphi_* I$ in $(W_0^{1,\infty}(N))^*$.
Now assume that $(F_k)_{k\in\mathbb{N}}$ is a sequence in $\Omega_{p,R}^{m-1}(M)$ such that $F_k \to F$ in $\Omega_p^{m-1}(M)$ as $k \to \infty$. Then for any $k \in \mathbb{N}$ there exists $I_k \in \mathcal{R}_1(M)$ of finite mass and so that ∂I_k supported in a finite subset of M such that $*dF_k = \partial I_k$. As we saw above, $*d(\varphi_* F_k) = \partial \varphi_* I_k$, therefore $\varphi_* F_k \in \Omega_{p,R,\infty}^{m-1}(N)$. Moreover we have $\varphi_* F_k \to \varphi_* F$ in $\Omega_p^{m-1}(N)$ as $k \to \infty$.
Finally if M is smooth and closed or a bounded Lipschitz domain and for any $k \in \mathbb{N}$ we have that $F_k \in \Omega_q^{m-1}(M)$ for some $q > 1$, we can improve the approximating sequence $\{F_k\}_{k\in\mathbb{N}}$ as follows: for any $k \in \mathbb{N}$, let $\alpha_k \in \Omega_{W^{1,q}}^{m-2}(M)$, $\beta_k \in \Omega_{W^{1,q}}^{m}(M)$ and $h_k \in \Omega_h^{m-1}(M)$ (the space of harmonic $(m-1)$-forms on M) such that $F_k = d\alpha_k + d^*\beta_k + h_k$. Then $*\Delta\beta_k = *dF_k$. Since $F_k \in \Omega_{p,R}^{m-1}(M)$, dF_k is supported in a finite set of points, thus β is smooth in M outside of a finite number of points. Now let $\tilde{\alpha}_k \in \Omega^{m-2}(M)$ such that $\|\alpha_k - \tilde{\alpha}_k\|_{W^{1,p}} \leqslant \frac{1}{2^k}$ and set $\tilde{F}_k := d\tilde{\alpha}_k + d\beta_k + h_k$. Then by construction $\tilde{F}_k \in \Omega_{p,R}^{m-1}(M)$ and $\tilde{F}_k \to F$ in $\Omega_p^{m-1}(M)$ as $k \to \infty$. □

Theorem 1.1, Corollary 2.2, Lemma 2.7 and Remark 2.7 can be combined to obtain the following general statement.

Theorem 2.3. *Let $M \subset \mathbb{R}^n$ be any embedded m-dimensional Lipschitz submanifold of $\mathbb{R}^n$ which is bi-Lipschitz equivalent either to $Q_1^m(0)$ or $\partial Q_1^{m+1}(0)$. Then:*

1. *if $p \in [1, m/(m-1))$, $\overline{\Omega_{p,R,\infty}^{m-1}(M)}^{L^p} = \Omega_{p,\mathbb{Z}}^{m-1}(M)$. Moreover, if M is smooth we have $\overline{\Omega_{p,R}^{m-1}(M)}^{L^p} = \overline{\Omega_{p,R,\infty}^{m-1}(M)}^{L^p}$,*
2. *if $p \in [m/(m-1), +\infty)$, $\Omega_{p,\mathbb{Z}}^{m-1}(M) = \{F \in \Omega_p^{m-1}(M)$ s.t. $*dF = 0\}$.*

Proof. Assume first that there exists a bi-Lipschitz homeomorphism $\varphi : Q_1^m(0) \to M$ and $p \in [1, m/(m-1))$. Let $F \in \Omega_{p,\mathbb{Z}}^{m-1}(M)$. By Lemma 2.7, $\varphi^* F \in \Omega_{p,\mathbb{Z}}^{m-1}(Q_1^m(0))$. Therefore by Theorem 1.1 there exists a sequence $\{\tilde{F}_k\}_{k\in\mathbb{N}}$ in $\Omega_{p,R}^{m-1}(Q_1(0))$ such that $\tilde{F}_k \to \varphi^* F$ in $\Omega_p^{n-1}(Q_1(0))$. By Lemma 2.7 $\varphi_* \tilde{F}_k \in \Omega_{p,R,\infty}(Q_1(0))$ and $\varphi_* \tilde{F}_k \to F$ in $\Omega_p^{m-1}(M)$. On the other hand, $\Omega_{p,\mathbb{Z}}^{m-1}(M)$ is closed in $\Omega_p^{m-1}(M)$: let $\{F_k\}_{k\in\mathbb{N}}$ be a sequence in $\Omega_{p,\mathbb{Z}}^{m-1}(M)$ converging to some F in $\Omega_p^{m-1}(M)$. Then for any $k \in \mathbb{N}$ there is some $I_k \in \mathcal{R}_1(M)$ of finite mass such that $*dF_k = \partial I_k$, and ∂I_k can be written as in (2.19) (by Corollary 2.3, which is independent from this result). Since $F_k \to F$ in $\Omega_p^{m-1}(M)$, $\{\partial I_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $(W^{1,\infty}(M))^*$. By [28, Proposition A.1], there exists $I \in \mathcal{R}_1(M)$ of finite mass such that $\partial I_k \to \partial I$ in $(W^{1,\infty}(M))^*$. We deduce that $*dF = \partial I$, therefore $F \in \Omega^{m-1}_{p,\mathbb{Z}}(M)$. We conclude that $\overline{\Omega^{m-1}_{p,R,\infty}(M)}^{L^p} = \Omega^{m-1}_{p,\mathbb{Z}}(M)$.

Assume now that M is smooth and $F \in \overline{\Omega^{m-1}_{p,R,\infty}(M)}^{L^p}$. Let $\{F_k\}_{k\in\mathbb{N}}$ be a sequence in $\Omega^{m-1}_{p,R,\infty}(M)$ approximating F in $\Omega^{m-1}_p(M)$. By the last part of Lemma 2.7, there exists a sequence in $\Omega^{m-1}_{p,R}(M)$ approximating F in $\Omega^{m-1}_p(M)$ (if $p=1$, Remark 2.4 guarantees that the sequence can be chosen such that $F_k \in \Omega_q(M)$ for any $k \in \mathbb{N}$, for some $q > 1$), therefore $F \in \overline{\Omega^{m-1}_{p,R}(M)}^{L^p}$.

Next we consider the case $p \in [m/(m-1), +\infty)$. Let $F \in \Omega^{m-1}_{p,\mathbb{Z}}(M)$. By Lemma 2.7 $\varphi^* F \in \Omega^{m-1}_{p,\mathbb{Z}}(Q_1(0))$. Then by Theorem 1.1 there holds $*d\varphi^* F = 0$. Applying Lemma 2.7 we obtain $*dF = 0$.

Finally, if M is bi-Lipschitz equivalent to $\partial Q^{m+1}_1(0)$ we can repeat precisely the same argument, using Corollary 2.2 instead of Theorem 1.1, Corollary 2.4 instead of Corollary 2.3 and Remark 2.8 instead of Remark 2.4. □

2.7. Corollaries of Theorem 1.1

Finally we present a couple of Corollaries of Theorem 1.1.

First we show that the boundary of any $I \in \mathcal{R}_1(D)$ having finite mass can be approximated strongly in $(W^{1,\infty}_0(D))^*$ by finite sums of deltas with integer coefficients. This is a well-known result from the theory of currents. In fact, it follows by the $\mathbb{M}$-approximability of any integer-multiplicity rectifiable current with Lipschitz images of integer-multiplicity polyhedral chains ($\mathbb{M}$ here denotes the mass of currents). Nevertheless, by showing that for every $I \in \mathcal{R}_1(D)$ with finite mass there exists a vector field $V \in L^1(D)$ such that

$$\operatorname{div}(V) = \partial I \quad \text{in } (W^{1,\infty}_0(D))^*$$

we can get an independent proof of this fact as an application of Theorem 2.1.

Corollary 2.3. *Let $D \subset \mathbb{R}^n$ be any open and bounded domain in $\mathbb{R}^n$ which is bi-Lipschitz equivalent to $Q^n_1(0)$. Let $I \in \mathcal{R}_1(D)$ with finite mass. Then, there exists a vector field $V \in L^1(D)$ such that*

$$\operatorname{div}(V) = \partial I \quad \textit{in } (W^{1,\infty}_0(D))^*.$$

Thus ∂I can be approximated strongly in $(W^{1,\infty}_0(D))^*$ by finite sums of deltas with integer coefficients. More precisely there exist sequences of points $(p_i)_{i\in\mathbb{N}}$ and $(n_i)_{i\in\mathbb{N}}$ in D such that

$$\partial I = \sum_{i\in\mathbb{N}} (\delta_{p_i} - \delta_{n_i}) \ \textit{in } (W^{1,\infty}_0(D))^* \textit{ and } \sum_{i\in\mathbb{N}} |p_i - n_i| < \infty. \tag{2.19}$$

Proof. By Lemma 2.7, it is enough to consider the case $D = Q_1(0)$. Let I be as above. By [1, Theorem 5.6] there exists a map $u \in W^{1,n-1}(Q_1(0), \mathbb{S}^{n-1})$ such that

$$*d\left(\frac{1}{n}\sum_{i=1}^{n}(-1)^{i-1} u_i \bigwedge_{j\neq i} du_j\right) = \alpha_{n-1}\partial I,$$

where α_{n-1} denotes the volume of the $(n-1)$-dimensional ball.
Set

$$\omega := \frac{1}{n\alpha_{n-1}} \sum_{i=1}^{n} (-1)^{i-1} u_i \bigwedge_{j\neq i} du_j$$

Notice that $\omega \in \Omega_1^{n-1}(Q_1(0))$ and $*d\omega = \partial I$. Now let $V := (*\omega)^\sharp$. Then $V \in L^1(Q_1(0))$ and

$$\operatorname{div}(V) = \partial I.$$

By Theorem 2.1, there exists a sequence $(V_k)_{k\in\mathbb{N}}$ in $L^1_R(Q_1(0))$ such that $V_k \to V$ in L^1. Then

$$\operatorname{div}(V_k) \to \operatorname{div}(V) = \partial I \quad \text{in } (W_0^{1,\infty}(Q_1(0)))^*.$$

As for any $k \in \mathbb{N}$ we have that $\operatorname{div}(V_k)$ is a finite sum of deltas with integer coefficients, by [28, Proposition A.1][6] (2.19) holds. □

Corollary 2.4. *Let M be a complete Lipschitz m-manifold, with or without boundary, compactly contained in the open cube $Q_2(0)$. Let $I \in \mathcal{R}_1(Q_2(0))$ be a rectifiable current of finite mass supported on M. Then there exist two sequences of points $(p_i)_{i\in\mathbb{N}}$ and $(n_i)_{i\in\mathbb{N}}$ in M such that*

$$\partial I = \sum_{i\in\mathbb{N}} (\delta_{p_i} - \delta_{n_i}) \text{ in } (W^{1,\infty}(Q_2(0)))^* \text{ and } \sum_{i\in\mathbb{N}} |p_i - n_i| < \infty.$$

Proof. Let $I \in \mathcal{R}_1(M)$ be a rectifiable 1-current of finite mass supported in $M \subset Q_2(0)$. By Corollary 2.3 there exists a vector field $V \in L^1(Q_2(0))$ such that $\operatorname{div}(V) = \partial I$. Thus we can apply the arguments of the proof of Theorem 2.1 to V. For any $\varepsilon \in E_V$ this yields a vector field $\tilde{V}_\varepsilon \in L^1(\Omega_{a_\varepsilon,\varepsilon})$ with the following properties: all bad cubes $Q \in \mathscr{C}_{a_\varepsilon,\varepsilon}$ are such that $\overline{Q} \cap M \neq \emptyset$, therefore the topological singularities of $\tilde{V}_\varepsilon$ lie at a distance of at most $\sqrt{n}\varepsilon$ from M. Moreover notice that if ε is sufficiently small, $\int_{Q_2(0)} \operatorname{div}(\tilde{V}_\varepsilon)\, d\mathcal{L}^n = 0$ (one can see this by testing $\operatorname{div}(\tilde{V}_\varepsilon)$ against a function $\varphi \in C_c^\infty(Q_2(0))$ such that $\varphi \equiv 1$ in a neighbourhood of M). Thus $\operatorname{div}(\tilde{V}_\varepsilon)$ can represented by

$$\operatorname{div}(V_\varepsilon) = \sum_{i=1}^{Q^\varepsilon} (\delta_{p_i^\varepsilon} - \delta_{n_i^\varepsilon})$$

for some $Q^\varepsilon \in \mathbb{N}$ and points p_i^ε and n_i^ε (possibly repeated) in a $\sqrt{n}\varepsilon$-neighbourhood of M. By the argument of Lemma 2.2 (with $Q_2(0)$ in place of $Q_1(0)$) we have $\varepsilon Q^\varepsilon \to 0$ as $\varepsilon \to 0^+$ in E_V. Now for any $i \in \{1, ..., Q^\varepsilon\}$ let $\tilde{p}_i^\varepsilon$ and $\tilde{n}_i^\varepsilon$ in M such that $|p_i^\varepsilon - \tilde{p}_i^\varepsilon| < 2\sqrt{n}\varepsilon$ and $|n_i^\varepsilon - \tilde{n}_i^\varepsilon| < 2\sqrt{n}\varepsilon$. Let $I_{p_i^\varepsilon} \in \mathcal{R}_1(Q_2(0))$ be the rectifiable current given by integration on the segment joining p_i^ε and $\tilde{p}_i^\varepsilon$ oriented from p_i^ε to $\tilde{p}_i^\varepsilon$ and let $I_{n_i^\varepsilon} \in \mathcal{R}_1(Q_2(0))$ be the

[6] Here Proposition A.1 in [28] is applied to the following metric space: for any $x, y \in Q_1(0)$ let $\overline{d}(x,y) = \min\{d(x,y), \operatorname{dist}(x, \partial Q_1(0)) + \operatorname{dist}(y, \partial Q_1(0))\}$, where d denotes the Euclidean distance in $Q_1(0)$. Let $(\widetilde{Q}_1(0), \overline{d})$ denote the completion of $Q_1(0)$ with respect to the distance $\overline{d}$. Then Lipschitz functions on $(\widetilde{Q}_1(0), \overline{d})$ corresponds to functions in $W_0^{1,\infty}(Q_1(0))$ (with same Lipschitz constant) modulo additive constants.

rectifiable current given by integration on the segment joining n_i^ε and $\tilde{n}_i^\varepsilon$ oriented from $\tilde{n}_i^\varepsilon$ to n_i^ε. Let $I_\varepsilon \in \mathcal{R}_1(Q_2(0))$ be a rectifiable 1-current of finite mass such that $\operatorname{div}(\tilde{V}_\varepsilon) = \partial I_\varepsilon$. Set $\tilde{I}_\varepsilon = I_\varepsilon + \sum_{i=1}^{Q^\varepsilon}(I_{p_i^\varepsilon} + I_{n_i^\varepsilon})$. Then

$$\partial \tilde{I}_\varepsilon = \sum_{i=1}^{Q^\varepsilon}(\delta_{\tilde{p}_i^\varepsilon} - \delta_{\tilde{n}_i^\varepsilon})$$

is supported in M. Moreover we have

$$\|\partial I - \partial \tilde{I}_\varepsilon\|_{(W^{1,\infty}(M))^*} \leqslant \|\partial I - \partial I_\varepsilon\|_{(W^{1,\infty}(Q_2(0)))^*} + \|\partial I_\varepsilon - \partial \tilde{I}_\varepsilon\|_{(W^{1,\infty}(Q_2(0)))^*}$$

(here M is endowed with the euclidean distance in $Q_2(0)$; notice that we are making use of the fact that any Lipschitz function on M can be extended to a Lipschitz function on $Q_2(0)$ with same Lipschitz constant). Now since $\|\tilde{V}_\varepsilon - V\|_{L^p(\Omega_{a_\varepsilon,\varepsilon})} \to 0$ as $\varepsilon \to 0$ in E_V and ∂I, ∂I_ε are supported in a compact subset of $Q_2(0)$, the first term on the right hand side tends to zero as $\varepsilon \to 0^+$ in E_V. Moreover the second term is bounded by $4\sqrt{n}\varepsilon Q^\varepsilon$ (see for instance Lemma 2 in [8]) and thus tends to 0 as $\varepsilon \to 0^+$ in E_V. This shows that ∂I belongs to the (strong) $(W^{1,\infty}(M))^*$ closure of the class of 0-currents T on M such that

$$T = \sum_{j\in J}(\delta_{p_j} - \delta_{n_j}) \text{ in } (W^{1,\infty}(M))^* \text{ and } \sum_{j\in J}|p_j - n_j| < \infty \tag{2.20}$$

for a countable set J and points p_j, n_j in M. By [28, Proposition A.1] applied to the complete metric space (M,d) (where d denotes the Euclidean distance in $Q_2(0)$) this space is closed in $(W^{1,\infty}(M))^*$, therefore ∂I is also of this form. Since any Lipschitz function $\varphi \in W^{1,\infty}(Q_2(0))$ has a Lipschitz trace $\varphi\big|_M$ on M and $\langle \partial I, \varphi\rangle_{Q_2(0)} = \langle \partial I, \varphi\big|_M\rangle_M$, we conclude that ∂I can be represented as in 2.20 also as an element of $(W^{1,\infty}(Q_2(0))^*$. □

Theorem 2.1 could also be useful to obtain approximation results for Sobolev maps with values into manifolds. For instance it can be used to recover the following result in dimension 2. For $p \geqslant 2$ this is due to R. Schoen and K. Uhlenbeck (for $p = 2$, see [30], Section 4) and F. Bethuel and X. Zheng (for $1 < p < 2$, see [6, Theorem 4]), while for $p > 2$ is a direct consequence of Sobolev embedding.

Corollary 2.5. *Let $u \in W^{1,p}(Q_1^2(0), \mathbb{S}^1)$ for some $p \in (1,\infty)$.*
If $p < 2$, then

$$\frac{1}{2\pi}\operatorname{div}(u \wedge \nabla^\perp u) = \partial I, \tag{2.21}$$

where $I \in \mathcal{R}_1(Q_1^2(0))$ is a 1-rectifiable current of finite mass, and u can be approximated in $W^{1,p}$ by a sequence of functions in

$$\mathcal{R} := \left\{v \in W^{1,p}(Q_1^2(0), \mathbb{S}^1); v \in C^\infty(Q_1^2(0) \setminus A, \mathbb{S}^1),\ \textit{where } A \textit{ is some finite set}\right\}.$$

If $p \geqslant 2$, then[7]

$$\operatorname{div}(u \wedge \nabla^\perp u) = 0, \tag{2.22}$$

and u can be approximated in $W^{1,p}$ by a sequence of functions in $C^\infty(Q_1^2(0), \mathbb{S}^1)$.

[7] Here we make use of the following notation: for vectors $A = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ we write $A \wedge B = a_1 b_2 - a_2 b_1$. For a function v on $\mathbb{R}^2$ we write $\nabla^\perp v = \begin{pmatrix} \partial_2 v, \\ -\partial_1 v \end{pmatrix}$.

Proof. If $p > 2$, by Sobolev embedding u is uniformly continuous, therefore it can be approximated in L^p by smooth functions with values in $\mathbb{S}^1$. Let $\{u_n\}_{n\in\mathbb{N}}$ be such an approximating sequence. Notice that for any $n \in \mathbb{N}$ there holds

$$\operatorname{div}(u_n \wedge \nabla^\perp u_n) = 0.$$

Since $u_n \to u$ in L^p, $\operatorname{div}(u_n \wedge \nabla^\perp u_n) \to \operatorname{div}(u_n \wedge \nabla^\perp u_n)$ in $\mathcal{D}'(Q_1^2(0))$, therefore (2.22) holds true.

Next we consider the case $p \leqslant 2$. We claim that the vector field $u \wedge \nabla^\perp u$ belongs to $L^p_{\mathbb{Z}}(Q_1(0))$. In fact notice that for any $x_0 \in Q_1(0)$, for a.e. $\rho \in (0, 2\operatorname{dist}_\infty(x_0, \partial Q_1(0)))$ $\partial Q_\rho(x_0)$ consists $\mathscr{H}^{n-1}$-a.e. of Lebesgue points of $u \wedge \nabla^\perp u$. Moreover for almost any such ρ we have

$$\frac{1}{2\pi}\int_{\partial Q_\rho(x_0)} (u \wedge \nabla^\perp u) \cdot \nu_{\partial Q_\rho(x_0)} = \deg\left(u|_{\partial Q_\rho(x_0)}\right) \in \mathbb{Z}.$$

Hence the vector field $\frac{1}{2\pi} u \wedge \nabla^\perp u$ belongs to $L^p_{\mathbb{Z}}(Q_1(0))$.

If $p = 2$ by Theorem 2.1 $\operatorname{div}(u \wedge \nabla^\perp u) = 0$. If $p < 2$ by Theorem 2.1 there exists a sequence of vector fields $(V_n)_{n\in\mathbb{N}}$ in $L^p_R(Q_1(0))$ such that

$$V_n \to \frac{1}{2\pi} u \wedge \nabla^\perp u \quad \text{in } L^p(Q_1(0)) \text{ as } n \to \infty.$$

For any $n \in \mathbb{N}$ by Hodge decomposition there exist $a_n \in W^{1,p}(Q_1(0))$, $b_n \in W^{1,p}_0(Q_1(0))$ such that

$$2\pi V_n = \nabla^\perp a_n + \nabla b_n.$$

For any $n \in \mathbb{N}$ let $\tilde{a}_n \in C^\infty(Q_1(0))$ be such that $\|\tilde{a}_n - a_n\|_{L^p} \leqslant \frac{1}{n}$. Moreover notice that there exists $d_n \in W^{1,p}(Q_1(0), \mathbb{S}^1) \cap C^\infty(Q_1(0) \setminus A)$, where A is a finite set, such that

$$\nabla b_n = d_n \wedge \nabla^\perp d_n.$$

In fact

$$\Delta b_n = 2\pi \operatorname{div}(V_n) = 2\pi \sum_{i=1}^{Q^n} d_i^n \delta_{p_i^n}$$

for some $Q^n \in \mathbb{N}$, $p_i^n \in Q_1(0)$ and $d_i^n \in \mathbb{Z}$, thus $b_n = -\sum_{i=1}^{Q^n} \log|x - p_i^n|^{d_i^n} + h_n$ for an harmonic function h_n. Then d_n can be chosen to be

$$d_n(x) = e^{-i\tilde{h}_n} \prod_{i=1}^{Q^n} \left(\frac{x - p_i^n}{|x - p_i^n|}\right)^{d_n},$$

where $\tilde{h}_n$ is the harmonic conjugate of h_n (the product has to be understood as complex multiplication in $\mathbb{C} \simeq \mathbb{R}^2$).

For any $n \in \mathbb{N}$ set $u_n := e^{i\tilde{a}_n} d_n$. Then by construction $u_n \in \mathcal{R}$ and

$$u_n \wedge \nabla^\perp u_n \to u \wedge \nabla^\perp u \quad \text{in } L^p(Q_1(0)) \text{ as } n \to \infty.$$

Therefore there is $c \in [0, 2\pi)$ so that up to a subsequence

$$e^{ic} u_n \to u \quad \text{in } L^p(Q_1(0)) \text{ as } n \to \infty.$$

□

Remark 2.9. Equation (2.21) was obtained in [11, Theorem 3'] with the help of the approximation result of F. Bethuel and X. Zheng. For analogous results in higher dimensions, we address the reader to [10], Chapter 3, and the references therein.

3. The weak L^p-closure of $\Omega^{n-1}_{p,\mathbb{Z}}(Q^n_1(0))$

In the present section we follow the ideas presented in [27] to prove that the space $\Omega^{n-1}_{p,\mathbb{Z}}(Q^n_1(0))$ is weakly sequentially closed for every $n \geqslant 2$ and $p \in (1,+\infty)$. The main reason why such techniques couldn't be used before in this context for $n \neq 3$ was the lack of a strong approximation theorem like Theorem 1.1 for general dimension n. Such result is needed in order to define a suitable notion of distance between the cubical slices of a form $F \in \Omega^{n-1}_{p,\mathbb{Z}}(Q^n(0))$, given by $(x \mapsto x_0 + \rho x)^* i^*_{\partial Q_\rho(x_0)} F$ for $\mathcal{L}^1$-a.e. ρ (see subsections 3.1 and 3.2 for the precise definition). Once we have turned the space of the cubical slices of F into a metric space, we will show that the "slice function" associated to F, given by $\rho \mapsto (x \mapsto x_0 + \rho x)^* i^*_{\partial Q_\rho(x_0)} F$, is locally $\frac{1}{p'}-$Hölder continuous (see subsection 3.3). Moreover we will see that if $\{F_k\}_{k\in\mathbb{N}} \subset \Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$ converges weakly in L^p, then the sequence of the slice functions associated to each F_k is locally uniformly $\frac{1}{p'}-$Hölder continuous. Finally, we will use the previous facts together with some technical lemmata to conclude the proof of Theorem 1.3 for $D = Q^n_1(0)$. Notice that by Theorem 1.1 the result is clear if $p \in \big[n/(n-1), \infty\big)$, here we will focus on the case $p \in \big(1, n/(n-1)\big)$.

3.1. Slice distance on $\mathbb{S}^{n-1}$

Throughout the following section, we will assume that $p \in \big(1, n/(n-1)\big)$. Moreover, we will denote by "$*$" the Hodge star operator associated with the standard round metric on $\mathbb{S}^{n-1}$. We will denote by Z the linear subspace of $\Omega^{n-1}_p(\mathbb{S}^{n-1})$ given by

$$Z := \left\{ h \in \Omega^{n-1}_p(\mathbb{S}^{n-1}) \text{ s.t. } \int_{\mathbb{S}^{n-1}} h \in \mathbb{Z} \right\}.$$

Remark 3.1. It's clear that Z is weakly (and thus strongly) L^p-closed in $\Omega^{n-1}_p(\mathbb{S}^{n-1})$. Indeed, let $\{h_k\}_{k\in\mathbb{N}} \subset Z$ be any sequence such that $h_k \rightharpoonup h$ weakly in $\Omega^{n-1}_p(\mathbb{S}^{n-1})$, i.e.

$$\int_{\mathbb{S}^{n-1}} \varphi h_k \to \int_{\mathbb{S}^{n-1}} \varphi h, \qquad \forall\, \varphi \in L^{p'}(\mathbb{S}^{n-1}).$$

Then, the statement follows by picking $\varphi \equiv 1$ and noticing that a convergent sequence of integer numbers is eventually constant.

Fix any arbitrary point $q \in \mathbb{S}^{n-1}$. We define the functions $d, \tilde{d} : Z \times Z \to [0,+\infty]$ by

$$d(h_1,h_2) := \inf\left\{ ||\alpha||_{L^p} \text{ s.t. } *(h_1 - h_2) = d^*\alpha + \partial I + \left(\int_{\mathbb{S}^{n-1}} h_1 - h_2 \right)\delta_q \right\},$$

with $\alpha \in \Omega^1_p(\mathbb{S}^{n-1})$, $I \in \mathcal{R}_1(\mathbb{S}^{n-1})$, and

$$\tilde{d}(h_1,h_2) := \inf\left\{ ||\alpha||_{L^p} \text{ s.t. } *(h_1 - h_2) = d^*\alpha + \partial I + \left(\int_{\mathbb{S}^{n-1}} h_1 - h_2 \right)\delta_q \right\},$$

with $\alpha \in \Omega^1_p(\mathbb{S}^{n-1})$, $I \in \mathcal{R}_1(\mathbb{S}^{n-1}) \cap \mathcal{N}_1(\mathbb{S}^{n-1})$.

Remark 3.2 (d and $\tilde{d}$ are always finite on Z). We claim that $d, \tilde{d} < +\infty$. Since obviously $d(h_1, h_2) \leqslant \tilde{d}(h_1, h_2)$, it is enough to show that $\tilde{d}(h_1, h_2) < +\infty$, for every $h_1, h_2 \in Z$. This just amounts to saying that given any $h_1, h_2 \in Z$ we can always find $\alpha \in \Omega^1_p(\mathbb{S}^{n-1})$, $I \in \mathcal{R}_1(\mathbb{S}^{n-1}) \cap \mathcal{N}_1(\mathbb{S}^{n-1})$ satisfying

$$*(h_1 - h_2) = d^*\alpha + \partial I + \bigg(\int_{\mathbb{S}^{n-1}} h_1 - h_2 \bigg) \delta_q.$$

Indeed, let

$$a := \int_{\mathbb{S}^{n-1}} h_1 - h_2 \in \mathbb{Z}.$$

Consider the following first order differential system on $\mathbb{S}^{n-1}$:

$$\begin{cases} d^*\omega = *(h_1 - h_2) - a\delta_q =: F, \\ d\omega = 0. \end{cases}$$

Since $p \in \big(1, n/(n-1)\big)$, $F \in \mathcal{L}(W^{1,p'}(\mathbb{S}^{n-1}))$. Moreover, $\langle F, 1 \rangle = 0$. Hence, by Lemma B.2, we know that the previous differential system has a solution $\alpha \in \Omega^1_p(\mathbb{S}^{n-1})$ and the statement follows.

Remark 3.3. As observed above, it is clear that $d(h_1, h_2) \leqslant \tilde{d}(h_1, h_2)$, for every $h_1, h_2 \in Z$. We claim that actually $d(h_1, h_2) = \tilde{d}(h_1, h_2)$, for every $h_1, h_2 \in Z$. In order to prove the remaining inequality, fix any $h_1, h_2 \in Z$ and let $\{\alpha_k\}_{k \in \mathbb{N}} \subset \Omega^1_p(\mathbb{S}^{n-1})$, $\{I_k\}_{k \in \mathbb{N}} \subset \mathcal{R}_1(\mathbb{S}^{n-1})$ be such that

$$\begin{cases} *(h_1 - h_2) = d^*\alpha_k + \partial I_k + a\delta_q, \quad \forall\, k \in \mathbb{N}, \\ ||\alpha_k||_{L^p} \to d(h_1, h_2) \text{ as } k \to \infty, \end{cases}$$

with

$$a := \int_{\mathbb{S}^{n-1}} h_1 - h_2.$$

By Corollary B.1, the linear differential equation

$$\Delta u = *(h_1 - h_2) - a\delta_q$$

has a weak solution $\psi \in \dot{W}^{1,p}(\mathbb{S}^{n-1})$. Let $\omega_k := d\psi - \alpha_k$, for every $k \in \mathbb{N}$. Notice that

$$d^*\omega_k = \partial I_k, \qquad \forall\, k \in \mathbb{N}.$$

By Theorem 2.3, for every $k \in \mathbb{N}$ there exists a sequence $\{\omega^j_k\}_{j \in \mathbb{N}} \subset \Omega^1_{p,R}(\mathbb{S}^{n-1})$ such that $r^j_k := \omega_k - \omega^j_k \to 0$ strongly in L^p as $j \to \infty$. By construction, it follows that

$$*(h_1 - h_2) = d^*(\alpha_k + r^j_k) + d^*\omega^j_k + a\delta_q, \qquad \forall\, k, j \in \mathbb{N}.$$

We observe that by Proposition A.1 for every $k, j \in \mathbb{N}$ there exist $I^j_k \in \mathcal{R}_1(\mathbb{S}^{n-1}) \cap \mathcal{N}_1(\mathbb{S}^{n-1})$ such that $d^*\omega^j_k = \partial I^j_k$. This implies that

$$\tilde{d}(h_1, h_2) \leqslant ||\alpha_k + r^j_k||_{L^p} \leqslant ||\alpha_k||_{L^p} + ||r^j_k||_{L^p}, \qquad \forall\, k, j \in \mathbb{N}.$$

By letting first $j \to \infty$ and then $k \to \infty$ in the previous inequality, our claim follows.

Proposition 3.1. *(Z,d) is a metric space.*

Proof. We need to check symmetry, triangular inequality and non-degeneracy.
Symmetry. This is clear since both the L^p-norm and the space $\mathcal{R}_1(\mathbb{S}^{n-1})$ are invariant under sign change.
Triangular inequality. Let $h_1, h_2, h_3 \in Z$. By definition of infimum, for every $\varepsilon > 0$ we can write

$$\begin{cases} *(h_1 - h_2) = d^*\alpha_\varepsilon + \partial I_\varepsilon + \left(\displaystyle\int_{\mathbb{S}^{n-1}} h_1 - h_2\right)\delta_q, \\ *(h_2 - h_3) = d^*\alpha'_\varepsilon + \partial I'_\varepsilon + \left(\displaystyle\int_{\mathbb{S}^{n-1}} h_2 - h_3\right)\delta_q, \end{cases}$$

with $\alpha_\varepsilon, \alpha'_\varepsilon \in \Omega^1_p(\mathbb{S}^{n-1})$ and $I_\varepsilon, I'_\varepsilon \in \mathcal{R}_1(\mathbb{S}^{n-1})$ satisfying

$$\begin{cases} ||\alpha_\varepsilon||_{L^p} \leqslant d(h_1, h_2) + \varepsilon \\ ||\alpha'_\varepsilon||_{L^p} \leqslant d(h_2, h_3) + \varepsilon. \end{cases}$$

We notice that

$$*(h_1 - h_3) = d^*(\alpha_\varepsilon + \alpha'_\varepsilon) + \partial(I_\varepsilon + I'_\varepsilon) + \left(\int_{\mathbb{S}^{n-1}} h_1 - h_3\right)\delta_q, \qquad \forall\, \varepsilon > 0.$$

Then, by definition of d, we have

$$d(h_1, h_3) \leqslant ||\alpha_\varepsilon + \alpha'_\varepsilon||_{L^p} \leqslant ||\alpha_\varepsilon||_{L^p} + ||\alpha'_\varepsilon||_{L^p} \leqslant d(h_1, h_2) + d(h_2, h_3) + 2\varepsilon, \qquad \forall\, \varepsilon > 0.$$

By letting $\varepsilon \to 0^+$ in the previous inequality, we get our claim.
Non-degeneracy. Assume that $d(h_1, h_2) = 0$, for some $h_1, h_2 \in Z$. Let

$$a := \int_{\mathbb{S}^{n-1}} h_1 - h_2 \in \mathbb{Z}.$$

Then, since $d = \tilde{d}$ (see Remark 3.3) and by definition of $\tilde{d}$, there exist $\{\alpha_k\}_{k\in\mathbb{N}} \subset \Omega^1_p(\mathbb{S}^{n-1})$ and $\{I_k\}_{k\in\mathbb{N}} \subset \mathcal{R}_1(\mathbb{S}^{n-1}) \cap \mathcal{N}_1(\mathbb{S}^{n-1})$ such that

$$*(h_1 - h_2) = d^*\alpha_k + \partial I_k + a\delta_q \tag{3.1}$$

and $\alpha_k \to 0$ strongly in L^p as $k \to \infty$. Observe that

$$\partial I_k \to *(h_1 - h_2) - a\delta_q \quad \text{ in } (W^{1,\infty}(\mathbb{S}^{n-1}))^*.$$

Now for any $k \in \mathbb{N}$, ∂I_k can be represented as

$$\partial I_k = \sum_{j=1}^{J_k} (\delta_{p^k_j} - \delta_{n^k_j})$$

for some $J_k \in \mathbb{N}$ and points p^k_j, n^k_j in $\mathbb{S}^{n-1}$. But the space of distributions of the form

$$\sum_{j\in J} (\delta_{p_j} - \delta_{n_j}) \text{ such that } \sum_{j\in J} |p_j - n_j| < \infty \tag{3.2}$$

(for a countable set J and points p_j, n_j in $\mathbb{S}^{n-1}$) is closed with respect to the (strong) topology of $(W^{1,\infty}(\mathbb{S}^{n-1}))^*$ (see Proposition A.1 in [28]), thus there exists a distribution T as in (3.2) such that

$$*\,(h_1 - h_2) = T + a\delta_q. \tag{3.3}$$

But this implies that $*\,(h_1 - h_2) = 0$, since the left-hand-side in (3.3) is in L^p whilst the right-hand-side is in L^p if and only if it is equal to zero. □

Remark 3.4. Notice that the proof above relies on the fact that $d = \tilde{d}$, which was proved using Theorem 2.3. As the proof of Theorem 2.3 was quite involved, we remark here that there is a way to skip that passage: in the proof of the non-degeneracy of d it is not necessary to assume that $\{I_k\}_{k\in\mathbb{N}}$ lies in $\mathcal{N}_1(\mathbb{S}^{n-1})$. For any $k \in \mathbb{N}$, even if we only assume that I_k in (3.1) lies in $\mathcal{R}_1(\mathbb{S}^{n-1})$, Corollary 2.4 that ∂I_k is of the form (3.2), and thus the limit of $\{I_k\}_{k\in\mathbb{N}}$ in $(W^{1,\infty}(\mathbb{S}^{n-1}))^*$ will also be of that form.

Proposition 3.2. *Let $\{h_k\}_{k\in\mathbb{N}} \subset Z$ and $h \in Z$. Then the following are equivalent:*

1. *$\{h_k\}_{k\in\mathbb{N}} \subset Z$ is uniformly bounded in L^p and $d(h_k, h) \to 0$ as $k \to \infty$;*
2. *$h_k \rightharpoonup h$ weakly in L^p as $k \to \infty$.*

Proof. We prove separately the two implications.

$2 \Rightarrow 1$. Pick any subsequence of $\{h_k\}_{k\in\mathbb{N}}$ (not relabelled). For every $k \in \mathbb{N}$, let

$$a_k := \langle h_k - h, 1\rangle = \int_{\mathbb{S}^{n-1}} h_k - h.$$

Since $h_k \rightharpoonup h$ weakly in L^p as $k \to \infty$, it follows that $a_k \to 0$ as as $k \to \infty$. Since $\{a_k\}_{k\in\mathbb{N}} \subset \mathbb{Z}$, there exists $K \in \mathbb{N}$ such that $a_k = 0$ for every $k \geqslant K$. Fix any $k \geqslant K$. By Lemma B.2, the linear differential system

$$\begin{cases} d^*\omega = *(h_k - h) \\ d\omega = 0, \end{cases}$$

respectively (if $n = 2$)

$$\begin{cases} d^*\omega = *(h_k - h) \\ \displaystyle\int_{\mathbb{S}^1} \omega = 0, \end{cases}$$

has a unique weak solution $\alpha_k \in \Omega^1_p(\mathbb{S}^{n-1})$. By Remark B.3 we have

$$||\alpha_k||_{W^{1,p}} \leqslant C\big(||d\alpha_k||_{L^p} + ||d^*\alpha_k||_{L^p}\big) = C||h_k - h||_{L^p}.$$

Since $\{h_k\}_{k\in\mathbb{N}}$ is weakly convergent, we know that it is also uniformly bounded in L^p. Then $\{\alpha_k\}_{k\geqslant K}$ is uniformly bounded in $W^{1,p}$. Hence, by weak compactness in $W^{1,p}$, there exists a subsequence $\{\alpha_{k_l}\}_{l\in\mathbb{N}} \subset \{\alpha_k\}_{k\geqslant K}$ and a one-form $\alpha \in \Omega^1_{W^{1,p}}(\mathbb{S}^{n-1})$ such that $\alpha_{k_l} \rightharpoonup \alpha$ weakly in $W^{1,p}$. By Rellich-Kondrakov theorem, it follows that $\alpha_{k_l} \to \alpha$ strongly in L^p. We claim that $\alpha = 0$. Indeed (if $n \geqslant 3$)

$$(\alpha, \omega)_{L^p - L^{p'}} = \lim_{l\to\infty} (\alpha_{k_l}, d\varphi + d^*\beta)_{L^p - L^{p'}}$$

$$
\begin{aligned}
&= \lim_{l\to\infty} (d^*\alpha_{k_l}, \varphi)_{L^p - L^{p'}} \\
&= \lim_{l\to\infty} \int_{\mathbb{S}^{n-1}} (h_{k_l} - h) \wedge \varphi = 0, \qquad \forall\, \omega = d\varphi + d^*\beta \in \Omega^1(\mathbb{S}^{n-1}),
\end{aligned}
$$

where the second equality holds because α_{k_l} is distributionally closed. If $n = 2$ we have analogously

$$
\begin{aligned}
(\alpha, \omega)_{L^p - L^{p'}} &= \lim_{l\to\infty} (\alpha_{k_l}, d\varphi + \eta)_{L^p - L^{p'}} \\
&= \lim_{l\to\infty} (\alpha_{k_l}, d\varphi + d\varphi)_{L^p - L^{p'}} \\
&= \lim_{l\to\infty} (d^*\alpha_{k_l}, \varphi)_{L^p - L^{p'}} \\
&= \lim_{l\to\infty} \int_{\mathbb{S}^1} (h_{k_l} - h) \wedge \varphi = 0, \qquad \forall\, \omega = d\varphi + \eta \in \Omega^1(\mathbb{S}^1),
\end{aligned}
$$

where $\eta \in \Omega^1(\mathbb{S}^1)$ is an harmonic 1-form on $\mathbb{S}^1$ (hence a constant 1-form) and thus vanishes when paired with α_{k_l}.

Hence, we have shown that $\alpha_{k_l} \to 0$ strongly in L^p as $l \to \infty$. As $*(h_{k_l} - h) = d^*\alpha_{k_l}$ for every $l \in \mathbb{N}$, we have

$$
d(h_{k_l}, h) \leqslant ||\alpha_{k_l}||_{L^p} \to 0, \qquad \text{as } l \to \infty.
$$

We have just proved that any subsequence of $\{h_k\}_{k\in\mathbb{N}}$ has a further subsequence converging to h with respect to d, therefore 1. follows.

$1 \Rightarrow 2$. Pick any subsequence of $\{h_k\}_{k\in\mathbb{N}}$ (not relabelled). Since $\{h_k\}_{k\in\mathbb{N}} \subset Z$ is uniformly bounded in L^p, by weak L^p-compactness there exists a subsequence $\{h_{k_l}\}_{l\in\mathbb{N}}$ of $\{h_k\}_{k\in\mathbb{N}}$ and a $h_w \in Z$ such that $h_{k_l} \rightharpoonup h_w$ weakly in L^p. Since we have just shown that $2 \Rightarrow 1$, we know that $d(h_{k_l}, h_w) \to 0$ as $l \to \infty$. By uniqueness of the limit, we get $h_w = h$. We have just proved that any subsequence of $\{h_k\}_{k\in\mathbb{N}} \subset Z$ has a further subsequence converging to h weakly in L^p, hence 2. follows. □

3.2. Slice distance on $\partial Q_1^n(0)$

Let $Q_1(0) \subset \mathbb{R}^n$ be the unit cube in $\mathbb{R}^n$ centered at the origin and let $\Psi : \mathbb{S}^{n-1} \to \partial Q_1(0)$ be a bi-Lipschitz homeomorphism. We let Y be the linear subspace of $\Omega_p^{n-1}(\partial Q_1(0))$ given by

$$
Y := \left\{ h \in \Omega_p^{n-1}(\partial Q_1(0)) \text{ s.t. } \int_{\partial Q_1(0)} h \in \mathbb{Z} \right\}.
$$

Remark 3.5. Notice that $h \in Y$ if and only if $\Psi^* h \in Z$. Indeed, given any $h \in Z$ we have

$$
C_\Psi^{-1} \int_{\partial Q_1(0)} |h|^p \, d\mathscr{H}^{n-1} \leqslant \int_{\mathbb{S}^{n-1}} |\Psi^* h|^p \, d\mathscr{H}^{n-1} \leqslant C_\Psi \int_{\partial Q_1(0)} |h|^p \, d\mathscr{H}^{n-1},
$$

with $C_\Psi := \big(\max\{\|d\Psi\|_{L^\infty}, \|d\Psi^{-1}\|_{L^\infty}\}\big)^{(n-1)(p-1)}$, and

$$
\int_{\mathbb{S}^{n-1}} \Psi^* h = \int_{\partial Q_1(0)} h.
$$

Thus, the functions $d_\Psi, \tilde{d}_\Psi : Y \times Y \to [0, +\infty)$ given by

$$d_\Psi(h_1, h_2) := d(\Psi^* h_1, \Psi^* h_2) \qquad \forall\, h_1, h_2 \in Y, \tag{3.4}$$
$$\tilde{d}_\Psi(h_1, h_2) := \tilde{d}(\Psi^* h_1, \Psi^* h_2) \qquad \forall\, h_1, h_2 \in Y, \tag{3.5}$$

are well-defined and coincide on $Y \times Y$ by Remarks 3.2 and 3.3. Moreover, (Y, d_Ψ) is a metric space as a direct consequence of Proposition 3.1 and the following statement is a corollary of Proposition 3.2.

Corollary 3.1. *Let $\{h_k\}_{k\in\mathbb{N}} \subset Y$ and $h \in Y$. Then, the following are equivalent:*

1. *$\{h_k\}_{k\in\mathbb{N}} \subset Y$ is uniformly bounded in L^p and $d_\Psi(h_k, h) \to 0$ as $k \to \infty$;*
2. *$h_k \rightharpoonup h$ weakly in L^p as $k \to \infty$.*

Remark 3.6. Let $\Psi_1, \Psi_2 : \mathbb{S}^{n-1} \to \partial Q_1(0)$ be bi-Lipschitz homeomorphisms. We claim that the distances d_{Ψ_1} and d_{Ψ_2} induced on Y by Ψ_1 and Ψ_2 respectively are equivalent. Indeed notice that given any bi-Lipschitz map $\Lambda : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ we have

$$d(\Lambda^* h_1, \Lambda^* h_2) \leqslant \|d\Lambda\|_{L^\infty}^{n-2} \|d\Lambda^{-1}\|_{L^\infty}^{\frac{n-1}{p}} d(h_1, h_2), \qquad \forall\, h_1, h_2 \in Z. \tag{3.6}$$

To see this observe that if $\alpha \in \Omega^1_p(\mathbb{S}^{n-1})$ is a competitor in the definition of $d(h_1, h_2)$ then the form $(-1)^{n-2} * \Lambda^*(*\alpha) \in \Omega^1_p(\mathbb{S}^{n-1})$ is a competitor in the definition of $d(\Lambda^* h_1, \Lambda^* h_2)$. Hence

$$d(\Lambda^* h_1, \Lambda^* h_2) \leqslant \|*\Lambda^*(*\alpha)\|_{L^p} \leqslant \|d\Lambda\|_{L^\infty}^{n-2} \|d\Lambda^{-1}\|_{L^\infty}^{\frac{n-1}{p}} \|\alpha\|_{L^p},$$

for every competitor α in the definition of $d(h_1, h_2)$. By taking the infimum on all the competitors in the previous inequality, (3.6) follows. Applying (3.6) to $\Lambda = \Psi_1^{-1} \circ \Psi_2$ we obtain

$$\begin{aligned} d_{\Psi_2}(h_1, h_2) &= d(\Psi_2^* h_1, \Psi_2^* h_2) = d\big((\Psi_1^{-1} \circ \Psi_2)^* \Psi_1^* h_1, (\Psi_1^{-1} \circ \Psi_2)^* \Psi_1^* h_2\big) \\ &\leqslant C_{\Psi_1\Psi_2} d(\Psi_1^* h_1, \Psi_1^* h_2) = C_{\Psi_1\Psi_2} d_{\Psi_1}(h_1, h_2) \qquad \forall\, h_1, h_2 \in Y, \end{aligned}$$

and analogously

$$d_{\Psi_1}(h_1, h_2) \leqslant C_{\Psi_1\Psi_2} d_{\Psi_2}(h_1, h_2) \qquad \forall\, h_1, h_2 \in Y,$$

with $C_{\Psi_1\Psi_2} := \max\big\{\|d(\Psi_2^{-1} \circ \Psi_1)\|_{L^\infty}, \|d(\Psi_1^{-1} \circ \Psi_2)\|_{L^\infty}\big\}^{n-2+\frac{n-1}{p}}$.

3.3. Slice functions and their properties

Definition 3.1 (Slice functions)**.** Let $F \in \Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$. Given any arbitrary $x_0 \in Q_1(0)$, we let $\rho_0 := 2\,\mathrm{dist}_\infty(x_0, \partial Q_1(0))$.
We call the *slice function of F at x_0* the map $s : \mathrm{Dom}(s) \subset (0, \rho_0) \to Y$ given by

$$s(\rho) := (x \mapsto \rho x + x_0)^* i^*_{\partial Q_\rho(x_0)} F, \qquad \forall\, \rho \in \mathrm{Dom}(s),$$

where $\mathrm{Dom}(s)$ is the subset of $(0, \rho_0)$ defined as follows: $\rho \in \mathrm{Dom}(s)$ if and only if the following conditions hold:

1. $\mathscr{H}^{n-1}$-a.e. point in $\partial Q_\rho(x_0)$ is a Lebesgue point for F,
2. $|F| \in L^p(\partial Q_\rho(x_0), \mathscr{H}^{n-1})$,
3. ρ is a Lebesgue point for the L^p-function

$$(0, \rho_0) \ni \rho \mapsto \int_{\partial Q_\rho(x_0)} i^*_{\partial Q_\rho(x_0)} F,$$

4. $(x \mapsto \rho x + x_0)^* i^*_{\partial Q_\rho(x_0)} F \in Y$.

Remark 3.7. Notice that $\mathrm{Dom}(s)$ has $\mathcal{L}^1$ full measure in $(0, \rho_0)$. Moreover $s \in L^p\big((0, \rho_0); Y\big)$, in the following sense: let $j_\rho : x \mapsto \rho x + x_0$, then we have

$$\|s(\rho)\|^p_{L^p} = \int_{\partial Q_1(0)} |j^*_\rho F|^p \, d\mathscr{H}^{n-1} = \int_{\partial Q_\rho(x_0)} |F|^p \rho^{(n-1)(p-1)} \, d\mathscr{H}^{n-1}$$

and thus

$$\int_0^{\rho_0} ||s(\rho)||^p_{L^p} \, d\rho = \int_0^{\rho_0} \int_{\partial Q_\rho} |F|^p \rho^{(n-1)(p-1)} \, d\mathscr{H}^{n-1} d\rho \leqslant 2 \int_{Q_{\rho_0}(x_0)} |F|^p d\mathscr{H}^n$$

For the following let $\Psi : \mathbb{S}^{n-1} \to \partial Q_1(0)$ be the bi-Lipschitz map given by

$$\Psi(x) := \frac{x}{2\|x\|_\infty}, \qquad \forall\, x \in \mathbb{S}^{n-1}. \tag{3.7}$$

Proposition 3.3. *Let $x_0 \in Q_1(0)$ and set $\rho_0 := 2\,\mathrm{dist}(x_0, \partial Q_1(0))$. Fix any $F \in \Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$ and let $s \in L^p\big((0, \rho_0), Y\big)$ be the slice function of F at x_0. Let $K \subset (0, \rho_0)$ be compact. Then, there exists a subset $E \subset K$ such that $\mathcal{L}^1(K \smallsetminus E) = 0$ and a representative $\tilde{s}$ of s defined pointwise on E such that*

$$d_\Psi\big(\tilde{s}(\rho_1), \tilde{s}(\rho_2)\big) \leqslant C_{p,K} ||F||_{L^p} |\rho_1 - \rho_2|^{\frac{1}{p'}}, \qquad \forall\, \rho_1, \rho_2 \in E, \tag{3.8}$$

where d_Ψ denotes the distance introduced in (3.4) with respect to the map Ψ defined in (3.7) and

$$C_{p,K} := C_p \max_{\rho \in K} \rho^{\frac{1-n}{p}},$$

with C_p depending only on p.

Proof. Denote by $T_F \in \mathcal{D}_1(Q_1(0))$ the 1-current on $Q_1(0)$ given by

$$\langle T_F, \omega \rangle = \int_{Q_1(0)} F \wedge \omega, \qquad \forall\, \omega \in \mathcal{D}^1(Q_1(0)).$$

Since $F \in \Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$, by Theorem 2.2 there exists $I \in \mathcal{R}_1(Q_1(0))$ such that $\mathbb{M}(I) < +\infty$ and $*dF = \partial I$. By definition of integral 1-current, there exist a locally 1-rectifiable set $\Gamma \subset Q_1(0)$, a Borel measurable unitary vector field $\vec{I}$ on Γ and a positive $\mathbb{Z}$-valued $\mathscr{H}^1 \llcorner \Gamma$-integrable function $\theta \in L^1(\Gamma, \mathscr{H}^1)$ such that

$$\langle I, \omega \rangle = \int_\Gamma \theta \langle \omega, \vec{I} \rangle \, d\mathscr{H}^1, \qquad \forall\, \omega \in \mathcal{D}^1(Q_1(0)).$$

By the coarea formula, there exists $G \subset K$ such that $\mathcal{L}^1(K \setminus G) = 0$ and such that $\Gamma \cap \partial Q_\rho(x_0)$ is a finite set for every $\rho \in G$.
Consider the map $\Phi : \mathbb{S}^{n-1} \times [0, \rho_0] \to \mathrm{Im}(\Phi) = \overline{Q_{\rho_0}(x_0)} \subset \overline{Q_1(0)}$ given by

$$\Phi(y,t) := x_0 + t\Psi(y), \qquad \forall\, (y,t) \in \mathbb{S}^{n-1} \times [0, \rho_0].$$

Notice that $\Phi\big|_{\mathbb{S}^{n-1}\times[\rho_1,\rho_2]}$ is a bi-Lipschitz homeomorphism onto its image for every $\rho_1, \rho_2 \in (0,1)$.
We claim that estimate (3.8) holds on a full-measure subset of G. Indeed, fix any $\rho_1, \rho_2 \in G$. Without loss of generality, assume that $\rho_2 > \rho_1$. Let $\hat{\Phi} := \Phi\big|_{\mathbb{S}^{n-1}\times[\rho_1,\rho_2]}$. Define $\pi := \mathrm{pr}_1 \circ \Phi^{-1} : \overline{Q_{\rho_0}(x_0)} \to \mathbb{S}^{n-1}$, where $\mathrm{pr}_1 : \mathbb{S}^{n-1} \times [0, \rho_0] \to \mathbb{S}^{n-1}$ is the canonical projection on the first factor, and notice that π is a Lipschitz and proper map. Then $\pi_*\big(T_F \llcorner \mathrm{Im}(\hat{\Phi})\big) \in \mathcal{D}_1(\mathbb{S}^{n-1})$ can be expressed as follows: for any $\omega \in \Omega^1(\mathbb{S}^{n-1})$

$$\begin{aligned}
\langle \pi_*\big(T_F \llcorner \mathrm{Im}(\hat{\Phi})\big), \omega\rangle &= \langle T_F \llcorner \mathrm{Im}(\hat{\Phi}), \pi^*\omega\rangle = \int_{\mathrm{Im}(\hat{\Phi})} F \wedge \pi^*\omega \\
&= \int_{\mathrm{Im}(\hat{\Phi})} (\Phi^{-1})^*(\Phi^* F \wedge \Phi^*\pi^*\omega) \\
&= \int_{\mathbb{S}^{n-1}\times[\rho_1,\rho_2]} \Phi^* F \wedge \mathrm{pr}_1^*\omega \\
&= \int_{\mathbb{S}^{n-1}\times[\rho_1,\rho_2]} \mathrm{pr}_1^*\omega \wedge *(*\Phi^* F) \\
&= \int_{\mathbb{S}^{n-1}\times[\rho_1,\rho_2]} \langle \mathrm{pr}_1^*\omega, *\Phi^* F\rangle \ \mathrm{vol}_{\mathbb{S}^{n-1}\times[\rho_1,\rho_2]}(y,t) \\
&= \int_{\rho_1}^{\rho_2} \bigg(\int_{\mathbb{S}^{n-1}} \langle \mathrm{pr}_1^*\omega, *\Phi^* F\rangle \ \mathrm{vol}_{\mathbb{S}^{n-1}}(y) \bigg)\, dt \\
&= \int_{\rho_1}^{\rho_2} \bigg(\int_{\mathbb{S}^{n-1}} \langle \omega, i^*_{\mathbb{S}^{n-1}\times\{t\}} * \Phi^* F\rangle \ \mathrm{vol}_{\mathbb{S}^{n-1}}(y) \bigg)\, dt \\
&= \int_{\rho_1}^{\rho_2} \bigg(\int_{\mathbb{S}^{n-1}} \omega \wedge \big(* i^*_{\mathbb{S}^{n-1}\times\{t\}} * \Phi^* F\big) \bigg)\, dt \\
&= \int_{\mathbb{S}^{n-1}} \omega \wedge \bigg(\int_{\rho_1}^{\rho_2} \big(* i^*_{\mathbb{S}^{n-1}\times\{t\}} * \Phi^* F\big)\, dt \bigg) \\
&= (-1)^{n-1} \int_{\mathbb{S}^{n-1}} \omega \wedge \alpha,
\end{aligned}$$

where

$$\alpha := (-1)^{n-1} \int_{\rho_1}^{\rho_2} \big(* i^*_{\mathbb{S}^{n-1}\times\{t\}} * \Phi^* F\big)\, dt \in \Omega^{n-2}_p(\mathbb{S}^{n-1}). \tag{3.9}$$

In particular,

$$\begin{aligned}
\langle \partial\pi_*\big(T_F \llcorner \mathrm{Im}(\hat{\Phi})\big), \varphi\rangle &= \langle \pi_*\big(T_F \llcorner \mathrm{Im}(\hat{\Phi})\big), d\varphi\rangle \\
&= (-1)^{n-1} \int_{\mathbb{S}^{n-1}} d\varphi \wedge \alpha = \langle d^*(*\alpha), \varphi\rangle, \qquad \forall\, \varphi \in C^\infty(\mathbb{S}^{n-1}).
\end{aligned}$$

Recall that the restriction of an integral current to a measurable set is still an integral current. Moreover, the push-forward of an integral current through a Lipschitz and proper map remains an integral current (see [22, Chapter 7, §7.5]). Then, $\tilde{I} := -\pi_*\big(I \llcorner \mathrm{Im}(\hat{\Phi})\big) \in \mathcal{R}_1(\mathbb{S}^{n-1})$.
So far, we have shown that

$$\partial\pi_*\big((T_F - I)\llcorner \mathrm{Im}(\hat{\Phi})\big) = \partial\pi_*\big(T_F \llcorner \mathrm{Im}(\hat{\Phi})\big) - \partial\pi_*\big(I \llcorner \mathrm{Im}(\hat{\Phi})\big) = d^*(*\alpha) + \partial\tilde{I}.$$

Let $\zeta \in C_c^\infty((-1,1))$ such that $\displaystyle\int_{\mathbb{R}} \zeta = 1$. For any $\varepsilon \in (0, \min\{\rho_1, \rho_0 - \rho_2\})$ set $\zeta_\varepsilon = \dfrac{1}{\varepsilon}\zeta\left(\dfrac{\cdot}{\varepsilon}\right)$ and let χ_ε be the unique solution of

$$\begin{cases} \chi_\varepsilon'(x) = \zeta_\varepsilon(x-\rho_1) - \zeta_\varepsilon(x-\rho_2) \\ \chi_\varepsilon(0) = 0. \end{cases}$$

Let $\psi \in C^\infty(\mathbb{S}^{n-1}\times[0,\rho_0])$ and let $\mathrm{pr}_2 : \mathbb{S}^{n-1}\times[0,\rho_0] \to [0,\rho_0]$ be the projection on the second factor. We compute

$$\begin{aligned} \langle (\Phi^{-1})_* I, \psi\, d(\chi_\varepsilon \circ \mathrm{pr}_2)\rangle &= \int_{\Phi^{-1}(\Gamma)} \theta_{(\Phi^{-1})_* I}\psi(\chi_\varepsilon' \circ \mathrm{pr}_2)\langle d\mathrm{pr}_2, \vec{I}_{(\Phi^{-1})_* I}\rangle \, d\mathscr{H}^1 \\ &= \int_{\rho_1-\varepsilon}^{\rho_1+\varepsilon} \zeta_\varepsilon(t)\bigg(\int_{\Phi^{-1}(\Gamma)\cap(\mathbb{S}^{n-1}\times\{t\})} \psi\tilde{\theta}\, d\mathscr{H}^0\bigg)\, d\mathcal{L}^1(t) \\ &\quad - \int_{\rho_2-\varepsilon}^{\rho_2+\varepsilon} \zeta_\varepsilon(t)\bigg(\int_{\Phi^{-1}(\Gamma)\cap(\mathbb{S}^{n-1}\times\{t\})} \psi\tilde{\theta}\, d\mathscr{H}^0\bigg)\, d\mathcal{L}^1(t) \end{aligned}$$

with $\tilde{\theta} = \theta_{(\Phi^{-1})_* I}\mathrm{sgn}(\langle d\mathrm{pr}_2, \vec{I}_{(\Phi_{-1})_* I}\rangle) \in L^1(\Phi^{-1}(\Gamma), \mathbb{Z})$. Moreover

$$\begin{aligned} \langle (\Phi^{-1})_* T_F, \psi\, d(\chi_\varepsilon \circ \mathrm{pr}_2)\rangle &= \int_{\mathbb{S}^{n-1}\times[0,\rho_0]} \psi(\chi_\varepsilon' \circ \mathrm{pr}_2)\Phi^* F \wedge d\mathrm{pr}_2 \\ &= \int_{\rho_1-\varepsilon}^{\rho_1+\varepsilon} \zeta_\varepsilon(t)\left(\int_{\mathbb{S}^{n-1}\times\{t\}} \psi\big(\Phi\big|_{\mathbb{S}^{n-1}\times\{t\}}\big)^* F\right) d\mathcal{L}^1(t) \\ &\quad - \int_{\rho_2-\varepsilon}^{\rho_2+\varepsilon} \zeta_\varepsilon(t)\left(\int_{\mathbb{S}^{n-1}\times\{t\}} \psi\big(\Phi\big|_{\mathbb{S}^{n-1}\times\{t\}}\big)^* F\right) d\mathcal{L}^1(t). \end{aligned}$$

Now observe that

$$\langle (\Phi^{-1})_*(T_F - I), (\chi_\varepsilon \circ \mathrm{pr}_2)\, d\psi\rangle \to \langle ((\Phi^{-1})_*(T_F - I))\llcorner(\mathbb{S}^{n-1}\times[\rho_1,\rho_2]), d\psi\rangle$$

as $\varepsilon \to 0^+$, by dominated convergence. On the other hand, since $\partial(T_F - I) = 0$, we have

$$\langle (\Phi^{-1})_*(T_F - I), (\chi_\varepsilon \circ \mathrm{pr}_2)\, d\psi\rangle = \langle (\Phi^{-1})_* I, \psi\, d(\chi_\varepsilon \circ \mathrm{pr}_2)\rangle - \langle (\Phi^{-1})_* T_F, \psi\, d(\chi_\varepsilon \circ \mathrm{pr}_2)\rangle.$$

Therefore for almost every $\rho_1, \rho_2 \in (0,\rho_0)$ (depending on ψ) we have

$$\begin{aligned} \langle \partial((\Phi^{-1})_*(T_F - I)\llcorner(\mathbb{S}^{n-1}\times[\rho_1,\rho_2])), \psi\rangle &= \int_{\Phi^{-1}(\Gamma)\cap(\mathbb{S}^{n-1}\times\{\rho_1\})} \psi\tilde{\theta}\, d\mathscr{H}^0 \\ &\quad - \int_{\Phi^{-1}(\Gamma)\cap(\mathbb{S}^{n-1}\times\{\rho_2\})} \psi\tilde{\theta}\, d\mathscr{H}^0 \end{aligned} \tag{3.10}$$

$$
\begin{aligned}
&- \int_{\mathbb{S}^{n-1}\times\{\rho_1\}} \psi(\Phi|_{\mathbb{S}^{n-1}\times\{\rho_1\}})^* F \\
&+ \int_{\mathbb{S}^{n-1}\times\{\rho_2\}} \psi(\Phi|_{\mathbb{S}^{n-1}\times\{\rho_2\}})^* F.
\end{aligned}
$$

Now let $\{\psi_k\}_{k\in\mathbb{N}} \subset C^\infty(\mathbb{S}^{n-1}\times[0,\rho_0])$ be a countable sequence dense in $C^1(\mathbb{S}^{n-1}\times[0,\rho_0])$. For every $k\in\mathbb{N}$, let $E_k\subset G$ be the set such that (3.10) holds with $\psi=\psi_k$ (i.e. the set of the $\rho\in G$ which are "ζ_ε-Lebesgue points" of the integrands in (3.10), with $\psi=\psi_k$) and define

$$
E := \bigcap_{k\in\mathbb{N}} E_k.
$$

Then $\mathcal{L}^1(E)=\mathcal{L}^1(K)$ and for every $\rho_1,\rho_2\in E$ estimate (3.10) holds with $\psi=\psi_k$ for every $k\in\mathbb{N}$. By density of $\{\psi_k\}_{k\in\mathbb{N}}$ in $C^1(\mathbb{S}^{n-1}\times[0,\rho_0])$, we can pass to the limit in (3.10) and get that for any given couple of parameters $\rho_1,\rho_2\in E$ such identity holds for every $\psi\in C^\infty(\mathbb{S}^{n-1}\times[0,\rho_0])$. In particular, for every $\rho_1,\rho_2\in E$, $\varphi\in C^\infty(\mathbb{S}^{n-1})$ we have

$$
\begin{aligned}
\langle \partial\pi_*((T_F-I)\llcorner \mathrm{Im}(\hat{\Phi})),\varphi\rangle &= \langle \partial((\Phi^{-1})_*(T_F-I)\llcorner(\mathbb{S}^{n-1}\times[\rho_1,\rho_2])),\mathrm{pr}_1^*\varphi\rangle \\
&= \sum_{x\in\Gamma_{\rho_1}} \tilde{\theta}(x,\rho_1)\varphi(x) - \sum_{x\in\Gamma_{\rho_2}} \tilde{\theta}(x,\rho_2)\varphi(x) \\
&\quad - \int_{\mathbb{S}^{n-1}} \varphi\,\Psi^* s(\rho_1) + \int_{\mathbb{S}^{n-1}} \varphi\,\Psi^* s(\rho_2),
\end{aligned}
$$

where

$$
\begin{aligned}
\Gamma_{\rho_1} &:= \mathrm{pr}_1(\Phi^{-1}(\Gamma)\cap(\mathbb{S}^{n-1}\times\{\rho_1\}))\subset\mathbb{S}^{n-1}, \\
\Gamma_{\rho_2} &:= \mathrm{pr}_1(\Phi^{-1}(\Gamma)\cap(\mathbb{S}^{n-1}\times\{\rho_2\}))\subset\mathbb{S}^{n-1}
\end{aligned}
$$

are finite set for any ρ_1, $\rho_2\in G$.

Gathering together what we have proved so far, we have

$$
\big(\Psi^* s(\rho_2)-\Psi^* s(\rho_1)\big) = d^*(\alpha) + \partial I' + \left(\int_{\mathbb{S}^{n-1}} \Psi^* s(\rho_2)-\Psi^* s(\rho_1)\right)\delta_q,
$$

where α was defined in (3.9) and $I'\in\mathcal{R}_1(\mathbb{S}^{n-1})$ is any rectifiable one-current of finite mass such that

$$
\partial I' = \sum_{x\in\Gamma_{\rho_2}} \tilde{\theta}(x,\rho_2)\delta_x - \sum_{x\in\Gamma_{\rho_1}} \tilde{\theta}(x,\rho_1)\delta_x + \partial\tilde{I} + \left(\sum_{x\in\Gamma_{\rho_1}} \tilde{\theta}(x,\rho_1) - \sum_{x\in\Gamma_{\rho_2}} \tilde{\theta}(x,\rho_2)\right)\delta_q,
$$

i.e. $*\alpha$ is a competitor in the definition of $d\big(\Psi^* s(\rho_2),\Psi^* s(\rho_1)\big)$. Hence in order to estimate $d\big(\Psi^* s(\rho_2),\Psi^* s(\rho_1)\big)$ we just need to find an upper bound for $||*\alpha||_{L^p}$.

Notice that $|d\Phi|\leqslant t|d\Psi|+\frac{\sqrt{n}}{2}$. Moreover since

$$
\Phi^{-1}(x) = \left(\Psi^{-1}\left(\frac{x-x_0}{2\|x-x_0\|_\infty}\right), 2\|x-x_0\|_\infty\right),
$$

we have $|J\Phi^{-1}(x)| \leqslant 2^{-(n-2)} \left(\frac{\|d\Psi^{-1}\|_{L^\infty}}{\|x-x_0\|_\infty}\right)^{n-1}$. Therefore

$$\|*\alpha\|^p_{L^p(\mathbb{S}^{n-1})} \leqslant \int_{\mathbb{S}^{n-1}} \left|\int_{\rho_1}^{\rho_2} |\Phi^* F| dt\right|^p d\mathscr{H}^{n-1} \leqslant |\rho_1 - \rho_2|^{\frac{p}{p'}} \int_{\mathbb{S}^{n-1}\times[\rho_1,\rho_2]} |\Phi^* F|^p d\mathscr{H}^{n-1} dt$$
$$\leqslant \left(2^{-(n-2)} \left(\|d\Psi\|_{L^\infty} + \frac{\sqrt{n}}{2}\right)^{p(n-1)} \frac{\|d\Psi^{-1}\|^{n-1}_{L^\infty}}{\rho_1^{n-1}}\right) |\rho_1 - \rho_2|^{\frac{p}{p'}} \|F\|^p_{L^p(Q_1(0))}$$

and our claim follows.

□

3.4. Proof of Theorem 1.3 for $Q^n_1(0)$

For the proof of Theorem 1.3 we need two technical Lemmata.

Lemma 3.1. *Let $\{f_k\}_{k\in\mathbb{N}} \subset L^1(0,1)$ be such that $||f_k||_{L^1} \leqslant C$ for any $k \in \mathbb{N}$. Then there exist a sequence of compact subsets $\{W_h\}_{h\geqslant 2}$ of $(0,1)$ such that for every $h \in \mathbb{N}$ such that $n \geqslant 2$ the following properties hold:*

1. *$\mathcal{L}^1(W_h) = 1 - \dfrac{C+2}{h}$;*
2. *$W_h \subset (1/h, 1)$;*
3. *for almost every $\rho \in W_h$ and every $k \in \mathbb{N}$ there exists $k' > k$ such that $|f_{k'}(\rho)| \leqslant h$.*

Proof. Let $h \in \mathbb{N}$ be such that $n \geqslant 2$. For any $l \in \mathbb{N}$ let

$$A^h_l := \bigcap_{k=l}^{\infty} f_k^{-1}([-h,h]^c).$$

Notice that for any $l \in \mathbb{N}$ $A^h_l \subset A^h_{l+1}$ and set

$$I_h = \bigcup_{l=1}^{\infty} A^h_l.$$

Let $m \in \mathbb{N}$ and let $k \geqslant m$. Notice that

$$C \geqslant \int_0^1 |f_k(\rho)|\, d\rho \geqslant \int_{A^h_m} |f_k(\rho)|\, d\rho \geqslant h\mathcal{L}^1(A^h_m).$$

By letting $m \to \infty$ in the previous inequality, we obtain

$$\mathcal{L}^1(I_h) \leqslant \frac{C}{h}.$$

Set $E_h := I_h^c \cap (1/h, 1)$ (where the complement is taken in $(0,1)$), then we have

$$\mathcal{L}^1(E_h) = \mathcal{L}^1((I_h \cup (0,1/h])^c) = 1 - \mathcal{L}^1(I_h \cup (0,1/h]) \geqslant 1 - \frac{C+1}{h}.$$

Moreover, E_h and any of its subsets satisfy properties 2. and 3.. Finally, since E_h is measurable, we can find a compact set $W_h \subset E_h$ such that

$$\mathcal{L}^1(W_h) = 1 - \frac{C+2}{h}.$$

By construction, W_h satisfies 1, 2 and 3. □

Since we do not know if the space (Y,d) is complete, we will also need the following Lemma.

Lemma 3.2. *Let $K \subset [0,1]$ be compact and let $S \subset K$ be dense and countable. Let $\{f_k\}_{k\in\mathbb{N}} \subset C^0(K,Y)$ be such that*

1. *$\{f_k\}_{k\in\mathbb{N}}$ is uniformly Cauchy from K to (Y,d) (i.e. it is a Cauchy sequence with respect to uniform convergence);*
2. *for some $C>0$,*
$$\sup_{\rho\in S}\sup_{k\in\mathbb{N}} ||f_k(\rho)||_{L^p} \leqslant C;$$
3. *for some $A>0$ and some $\alpha\in(0,1]$, we have*
$$d\big(f_k(\rho), f_k(\rho')\big) \leqslant A|\rho-\rho'|^\alpha, \qquad \forall\, \rho,\rho' \in S.$$

Then, there exists $f \in C^0(K,Y)$ such that $f_k \to f$ uniformly.

Proof. Fix any $\rho\in S$. By assumption 2, $\{f_k(\rho)\}_{k\in\mathbb{N}}$ is bounded in L^p and therefore it has a subsequence converging weakly in L^p to a limit $f(\rho)\in Y$ (recall that Y is closed with respect to the weak L^p convergence). By Corollary 3.1, such a subsequence converges in (Y,d) to the same limit $f(\rho)$. Assumption 1 implies that $f_k(\rho) \xrightarrow{d} f(\rho)$ (for the whole original sequence). Fix any $\rho\in K$ and let $\{\rho_i\}_{i\in\mathbb{N}}\subset S$ be such that $\rho_i\to\rho$. We claim that there exists $f_\rho\in Y$ such that $f(\rho_i)\xrightarrow{d} f_\rho$ and f_ρ does not depend on the choice of the sequence $\{\rho_i\}_{i\in\mathbb{N}}$. Indeed since the L^p norm is weakly lower semi continuous we have

$$||f(\rho_i)||_{L^p} \leqslant \liminf_{k\to\infty} ||f_k(\rho_i)||_{L^p} \leqslant C,$$

for every $i\in\mathbb{N}$. Then there exists a subsequence $\{\rho_{i_j}\}_{j\in\mathbb{N}}$ and a $f_\rho\in L^p$ such that $f(\rho_{i_j}) \rightharpoonup f_\rho$ weakly in L^p. Since Y is L^p-weakly closed we have $f_\rho\in Y$. By Corollary 3.1 we also have $f(\rho_{i_j}) \xrightarrow{d} f_\rho$. Relabel this subsequence as $\{\rho_i\}_{i\in\mathbb{N}}$. To see that f_ρ doesn't depend on the subsequence assume that $\{\tilde\rho_i\}_{i\in\mathbb{N}}$ is another sequence in S with $\tilde\rho_i\to\rho$ and $f(\tilde\rho_i)\xrightarrow{d}\tilde f_\rho$. To see that $f_\rho=\tilde f_\rho$, first notice that by hypothesis 3. and triangle inequality we have

$$\begin{aligned} d\big(f(\rho_i), f(\tilde\rho_i)\big) &\leqslant d\big(f(\rho_i), f_k(\rho_i)\big) + d\big(f_k(\rho_i), f_k(\tilde\rho_i)\big) + d\big(f_k(\tilde\rho_i), f(\tilde\rho_i)\big) \\ &\leqslant d\big(f(\rho_i), f_k(\rho_i)\big) + A|\rho_i-\tilde\rho_i|^\alpha + d\big(f_k(\tilde\rho_i), f(\tilde\rho_i)\big). \end{aligned}$$

for every $i\in\mathbb{N}$. Hence, passing to the limit as $k\to\infty$ in the previous inequality, we get

$$d\big(f(\rho_i), f(\tilde\rho_i)\big) \leqslant A|\rho_i-\tilde\rho_i|^\alpha.$$

Thus we finally obtain

$$\begin{aligned} d(f_\rho,\tilde f_\rho) &\leqslant d\big(f_\rho, f(\rho_i)\big) + d\big(f(\rho_i), f(\tilde\rho_i)\big) + d\big(f(\tilde\rho_i), \tilde f_\rho\big) \\ &\leqslant d\big(f_\rho, f(\rho_i)\big) + A|\rho_i-\tilde\rho_i|^\alpha + d\big(f(\tilde\rho_i), \tilde f_\rho\big) \end{aligned}$$

and, passing to the limit as $i\to\infty$, we get $d(f_\rho,\tilde f_\rho)=0$, i.e. $f_\rho=\tilde f_\rho$. This concludes the proof of the claim.
For any $\rho\in K$ let

$$f(\rho) := \lim_{i\to\infty} f(\rho_i),$$

where $\{\rho_i\}_{i\in\mathbb{N}} \subset S$ is any sequence such that $\rho_i \to \rho$ and the limit is understood with respect to d. By the previous claim, f is a well-defined function on K.
Finally we claim that $f_k \xrightarrow{d} f$ uniformly, as $k \to \infty$. To see this, let $\varepsilon > 0$; by assumption 1 there exists $L \in \mathbb{N}$ such that for any $m, n \in \mathbb{N}$ with $m, n \geqslant L$ we have

$$\sup_{\rho\in S} d(f_m(\rho), f_n(\rho)) < \varepsilon.$$

Let $\rho \in K$ and let $\{\rho_i\}_{i\in\mathbb{N}}$ be a sequence in S such that $\rho_i \to \rho$. Then for any $k \in \mathbb{N}$ such that $h \geqslant L$ we have

$$d(f_k(\rho), f(\rho)) = \lim_{i\to\infty} d(f_k(\rho_i), f(\rho_i)) = \lim_{i\to\infty}\lim_{m\to\infty} d(f_k(\rho_i), f_m(\rho_i)) \leqslant \varepsilon.$$

This shows that f is the uniform limit of $\{f_k\}_{k\in\mathbb{N}}$ in K, with respect to d. As $f_k \in C^0(K, Y)$ for any $k \in \mathbb{N}$, we have that $f \in C^0(K, Y)$ (this also follows directly from the construction of f). □

We prove now Theorem 1.3 for the case $D = Q_1^n(0)$. In Remark 3.8 we will discuss how to deduce the general case from Theorem 3.1.

Theorem 3.1 (Weak closure for $Q_1^n(0)$). *Fix any $n \in \mathbb{N}$ such that $n \geqslant 2$ and assume that $p \in \big(1, n/(n-1)\big)$. Then $\Omega^{n-1}_{p,\mathbb{Z}}(Q_1^n(0))$ is weakly sequentially closed.*

Proof. Assume that $F \in \Omega^{n-1}_p(Q_1(0))$ belongs to the weak L^p-closure of $\Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$, i.e. there exists $\{F_k\}_{k\in\mathbb{N}} \subset \Omega^{n-1}_{p,R}(Q_1(0))$ such that $F_k \xrightarrow{L^p} F$. What we need to show is that $F \in \Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$, which amounts to saying that

$$\int_{\partial Q_\rho(x_0)} i^*_{\partial Q_\rho(x_0)} F \in \mathbb{Z}, \tag{3.11}$$

for every $x_0 \in Q_1(0)$ and for a.e. $\rho \in \big(0, 2\,\mathrm{dist}_\infty(x_0, \partial Q_1(0))\big)$. Without losing generality, we will just show (3.11) for $x_0 = 0$.
Step 1. For any $k \in \mathbb{N}$ let s_k be the slice function of F_k at 0. Fix any $h \in \mathbb{N}$ such that $h \geqslant 2$ and let $W_h \subset (1/h, 1)$ be the compact set given by applying Lemma 3.1 with $f_k = \|s_k\|_{L^p}$ and $C = 2^{\frac{1}{p}} \sup_{k\in\mathbb{N}}\|F_k\|_{L^p}$ (see Remark 3.7). Let $E_h^k \subset W_h$ denote the subset associated to W_h and s_k by Proposition 3.3, let $\tilde{E}_h = \bigcap_{k\in\mathbb{N}} E_h^k$ and let s_k denote its $\frac{1}{p'}$-Hölder representative on $\tilde{E}_h$, for any $k \in \mathbb{N}$. By property 3. in Lemma 3.1, for almost every $\rho \in \tilde{E}_h$ we can find a subsequence $\{s_{k_\rho}(\rho)\}_{k_\rho\in\mathbb{N}} \subset \{s_k(\rho)\}_{k\in\mathbb{N}}$ such that $s_{k_\rho}(\rho)$ is uniformly bounded in L^p by h. Denote by E_h the set of all such ρ and observe that $\mathcal{L}^1(E_h) = \mathcal{L}^1(W_h) = 1 - \frac{C+2}{h}$. Then for any $\rho \in E_h$, $\{s_{k_\rho}(\rho)\}_{k_\rho\in\mathbb{N}}$ has a subsequence that converges weakly in L^p or, equivalently, with respect to d_Ψ (see Corollary 3.1).
Let $S_h \subset E_h$ be a countable dense subset. By a diagonal extraction argument, we find a subsequence $\{s_{k_l}\}_{l\in\mathbb{N}}$ such that $\{s_{k_l}(\rho)\}_{l\in\mathbb{N}}$ is convergent with respect to d_Ψ and weakly in L^p, and is uniformly bounded in L^p by h, for every $\rho \in S_h$.
Step 2. Next we claim that for any $l \in \mathbb{N}$, s_{k_l} can be extended to a $\frac{1}{p'}$-Hölder continuous function on $\overline{E_h}$ with the same Hölder constant (which is bounded uniformly in l) and L^p

norm bounded by h.
In fact let $f \in \{s_{k_l}\}_{l\in\mathbb{N}}$, let $\rho \in \overline{E_h}$, let $\{\rho_i\}_{i\in\mathbb{N}}$ be a sequence in S_h such that $\rho_i \to \rho$ as $i \to \infty$ (observe that such a sequence exists, since S_h is dense in E_h, which in turn is dense in $\overline{E_h}$). Since $\{f(\rho_i)\}_{i\in\mathbb{N}} \subset Y$ is uniformly bounded in L^p, by weak L^p-compactness there exists a subsequence $\{f(\rho_{i_j})\}_{j\in\mathbb{N}}$ of $\{f(\rho_i)\}_{i\in\mathbb{N}}$ such that $f(\rho_{i_j}) \rightharpoonup f_\rho$ weakly in L^p for some $f_\rho \in Y$. By Corollary 3.1, we know that $d_\Psi\big(f(\rho_{i_j}), f_\rho\big) \to 0$. Since f is $\frac{1}{p'}$-Hölder continuous on $\overline{E_h}$, f_ρ does not depend on the sequence $\{\rho_i\}_{i\in\mathbb{N}}$ and thus $d_\Psi(f(\rho_i), f_\rho) \to 0$. Hence, the function

$$\tilde{f}(\rho) := \lim_{i\to\infty} f(\rho_i)$$

(where the limit has to be understood with respect to d_Ψ) is well-defined on $\overline{E_h}$ and satisfies (3.8) and $\|\tilde{f}(\rho)\|_{L^p(\overline{E_h})} \leqslant h$ on $\overline{E_h}$.
In the following, in order to simplify the notation, we will denote again by s_{k_l} the $\frac{1}{p'}$-Hölder extension of s_{k_l} to $\overline{E_h}$, for any $l \in \mathbb{N}$.
Step 3. We show that $\{s_{k_l}\}_{l\in\mathbb{N}}$ converges uniformly on $\overline{E_h}$ to some $s \in C^0(\overline{E_h}, Y)$.
Fix any $\varepsilon > 0$. By Step 2. we know that the sequence $\{s_{k_l}\}_{l\in\mathbb{N}}$ is equicontinuous from $\overline{E_h}$ to (Y, d_Ψ). Therefore we can choose $\delta > 0$ such that

$$d_\Psi\big(s_{k_l}(\rho), s_{k_l}(\rho')\big) < \varepsilon, \qquad \forall\, \rho, \rho' \in \overline{E_h} \text{ s.t. } |\rho - \rho'| < \delta \text{ and } \forall\, l \in \mathbb{N}.$$

Notice that $\{(\rho - \delta, \rho + \delta)\}_{\rho\in S_h}$ is an open cover of $\overline{E_h}$. Since $\overline{E_h}$ is compact, we can find a finite set $\{\rho_1, ..., \rho_m\} \subset S_h$ such that $\{(\rho_j - \delta, \rho_j + \delta)\}_{j=1,...,m}$ is a finite open cover of $\overline{E_h}$.
Now let $\rho \in \overline{E_h}$. Observe that there exists a point $\rho_j \in \{\rho_1, ..., \rho_m\}$ such that $\rho \in (\rho_j - \delta, \rho_j + \delta)$, i.e. $|\rho - \rho_j| < \delta$. By our choice of δ, this implies $d\big(s_{k_l}(\rho), s_{k_l}(\rho_j)\big) < \varepsilon$, for every $l \in \mathbb{N}$. By triangle inequality, we have

$$\begin{aligned} d_\Psi\big(s_{k_l}(\rho), s_{k_m}(\rho)\big) &\leqslant d_\Psi\big(s_{k_l}(\rho), s_{k_l}(\rho_j)\big) + d_\Psi\big(s_{k_l}(\rho_j), s_{k_m}(\rho_j)\big) + d_\Psi\big(s_{k_m}(\rho_j), s_{k_m}(\rho)\big) \\ &< 2\varepsilon + d_\Psi\big(s_{k_l}(\rho_j), s_{k_m}(\rho_j)\big). \end{aligned}$$

But since $\rho_j \in S_h$, we know that there exists $L_j > 0$ such that

$$d_\Psi\big(s_{k_l}(\rho_j), s_{k_m}(\rho_j)\big) < \varepsilon, \qquad \forall\, l, m \geqslant L_j.$$

Hence, by letting $L := \max_{j=1,\ldots,m} L_j$, we have that

$$d_\Psi\big(s_{k_l}(\rho), s_{k_m}(\rho)\big) < 3\varepsilon, \qquad \forall\, l, m \geqslant L, \forall\, \rho \in \overline{E_h}.$$

Here we have just proved that the sequence $\{s_{k_l}\}_{l\in\mathbb{N}}$ is uniformly Cauchy on $\overline{E_h}$ with respect to d_Ψ. Since $\{s_{k_l}\}_{l\in\mathbb{N}}$ satisfies all the hypotheses of Lemma 3.2, we get that there exists $s \in C^0(\overline{E_h}, Y)$ such that $s_{k_l} \to s$ uniformly on $\overline{E_h}$ with respect to d_Ψ.
Notice that since $\|s_{k_l}(\rho)\|_{L^p} \leqslant h$ for any $l \in \mathbb{N}$, for any $\rho \in E_h$, and since $s_{k_l}(\rho) \to s(\rho)$ with respect to d_Ψ for any $\rho \in \overline{E_h}$, by Corollary 3.1 there holds $s_{k_l}(\rho) \rightharpoonup s(\rho)$ in L^p for any $\rho \in E_h$.
Step 4. Let $s_0 : E_h \to Y$ be the restriction to E_h of the slice function of F at 0. We claim that $s = s_0$ a.e. in E_h. To show this we will prove that

$$\int_{E_h} \varphi(\rho) \int_{\partial Q_1(0)} \psi\big(s_{k_l}(\rho) - s_0(\rho)\big)\, d\rho \to 0, \qquad \text{as } l \to \infty,$$

for every $\varphi \in C_c^\infty((0,1))$ and for every $\psi \in \mathrm{Lip}(\partial Q_1(0))$. Indeed, an explicit computation gives

$$\begin{aligned}\int_{E_h} \varphi(\rho) \int_{\partial Q_1(0)} \psi\big(s_{k_l}(\rho) - s_0(\rho)\big)\, d\rho \\ &= \int_{E_h} \varphi(\rho)\bigg(\int_{\partial Q_\rho(0)} \psi\Big(\frac{\cdot}{\rho}\Big) i^*_{\partial Q_\rho(0)} F_{k_l} - \int_{\partial Q_\rho} \psi\Big(\frac{\cdot}{\rho}\Big) i^*_{\partial Q_\rho(0)} F \bigg)\, d\rho \\ &= 2^n \int_{Q_1(0)} \mathbb{1}_{E_h}(2\|\cdot\|_\infty)\varphi(2\|\cdot\|_\infty)\psi\bigg(\frac{\cdot}{\|\cdot\|_\infty}\bigg) d\|\cdot\|_\infty \wedge (F_{k_l} - F) \to 0 \text{ as } l \to \infty,\end{aligned}$$

since

$$\mathbb{1}_{E_h}(2\|\cdot\|_\infty)\varphi(2\|\cdot\|_\infty)\psi\bigg(\frac{\cdot}{\|\cdot\|_\infty}\bigg) d\|\cdot\|_\infty \in \Omega^1_{p'}(Q_1(0))$$

and $F_{k_l} \xrightarrow{L^p} F$ in $Q_1(0)$. On the other hand, since $s_{k_l}(\rho) \rightharpoonup s(\rho)$ in L^p for any $\rho \in E_h$,

$$\int_{E_h} \varphi(\rho) \int_{\partial Q_1(0)} \psi\big(s_{k_l}(\rho) - s(\rho)\big)\, d\rho \to 0 \qquad \text{as } l \to \infty$$

for every $\varphi \in C_c^\infty((0,1))$ and for every $\psi \in \mathrm{Lip}(\partial Q_1(0))$. Terefore we obtain

$$\int_{E_h} \varphi(\rho) \int_{\partial Q_1(0)} \psi\big(s(\rho) - s_0(\rho)\big)\, d\rho = 0, \qquad \forall\, \varphi \in C_c^\infty((0,1)), \forall\, \psi \in \mathrm{Lip}(\partial Q_1(0)).$$

This means that $s_0(\rho) = s(\rho) \in Y$ for a.e. $\rho \in E_h$.
Step 5. Finally we show that (3.11) holds for almost any $\rho \in (0,1)$.
In fact for any $\rho \in E_h$ such that $s_0(\rho) = s(\rho)$ we have

$$\int_{\partial Q_\rho(0)} i^*_{\partial Q_\rho} F = \int_{\partial Q_1(0)} s_0(\rho) = \int_{\partial Q_1(0)} s(\rho) = \lim_{k_l \to \infty} \int_{\partial Q_1(0)} s_{k_l}(\rho) \in \mathbb{Z},$$

since $s_{k_l}(\rho) \rightharpoonup s(\rho)$ in L^p. Thus (3.11) holds for $\mathcal{L}^1$-a.e. $\rho \in E_h$. Since the previous step can be repeated for any $h \in \mathbb{N}$ such that $h \geqslant 2$, and since

$$\lim_{h \to +\infty} \mathcal{L}^1(E_h) = \lim_{h \to +\infty} 1 - \frac{C+2}{h} = 1,$$

we conclude that (3.11) holds for $\mathcal{L}^1$-a.e. $\rho \in (0,1)$. □

Remark 3.8. Let $D \subset \mathbb{R}^n$ be any open and bounded domain which is bi-Lipschitz equivalent to $Q_1(0)$. From Theorem 3.1 follows that $\Omega^{n-1}_{p,\mathbb{Z}}(D)$ (see Definition 1.4) is a weakly sequentially closed subspace of $\Omega^{n-1}_p(D)$. Indeed, let $\varphi : Q_1(0) \to D$ be any bi-Lipschitz homeomorphism and let $\{F_k\}_{k\in\mathbb{N}} \subset \Omega^{n-1}_{p,\mathbb{Z}}(D)$ be such that $F_k \xrightarrow{L^p} F$ on D. Then, by Lemma 2.7 we have $\{\varphi^* F_k\}_{k\in\mathbb{N}} \subset \Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$ and as φ is bi-Lipschitz we have $\varphi^* F_k \xrightarrow{L^p} \varphi^* F$ on $Q_1(0)$. By the Weak Closure Theorem (Theorem 3.1), $\varphi^* F \in \Omega^{n-1}_{p,\mathbb{Z}}(Q_1(0))$. Thus $F \in \Omega^{n-1}_{p,\mathbb{Z}}(D)$ (again by Lemma 2.7). This shows that Theorem 1.3 holds true.

Observe that Theorem 1.3 does not hold if $n = 1$. In fact in this case the following holds.

Lemma 3.3. *Let I be a bounded connected interval in $\mathbb{R}$. Let $p \in [1, \infty)$. The sequential weak closure of $L^p_{\mathbb{Z}}(I)$ in $L^p(I)$ is $L^p(I)$. Moreover*

Proof. Since $C^0(\overline{I})$ is dense in $L^p(I)$, it is enough to show that any function $f \in C^0(\overline{I})$ can be approximated weakly in $L^p(I)$ by functions in $L^p_{\mathbb{Z}}(I)$. Without loss of generality we can assume that $I = [0, 1)$. For any $n \in \mathbb{N}$ let's define $f_n : I \to \mathbb{R}$ as follows:
for any $k \in \{1, ..., 2^n\}$ let $I^n_k := [\frac{k-1}{2^n}, \frac{k}{2^n})$, for any $k \in \{1, ..., 2^n\}$ let $c_k := ⨍_{I^n_k} f(x)dx$ and for any $x \in I^n_k$ set

$$f_n(x) := \begin{cases} \lceil c_k \rceil & \text{if } x - \frac{k-1}{2^n} \leqslant \frac{c_k}{2^n \lceil c_k \rceil} \\ 0 & \text{otherwise.} \end{cases}$$

Then $f_n \in L^p(I, \mathbb{Z})$ and $\int_{I^n_k} f_n(x)dx = \int_{I^n_k} f(x)dx$ for any $k \in \{1, ..., 2^n\}$, for any $n \in \mathbb{N}$. Moreover notice that since f is bounded, the sequence $(f_n)_{n\in\mathbb{N}}$ is bounded in $L^p(I)$. Therefore if $p > 1$ $(f_n)_{n\in\mathbb{N}}$ converges weakly in $L^p(I)$, up to a subsequence, to a function $\tilde{f} \in L^p(I)$. Testing against continuous functions on $\overline{I}$ it is easy to check that $f = \tilde{f}$.
If $p = 1$ we have to check that, up to a subsequence,

$$\lim_{n\to\infty} \int_I f_n g = \int_I f g \quad \forall\, g \in L^\infty(I). \tag{3.12}$$

Since $L^\infty(I) \subset L^q(I)$ for any $q > 1$, (3.12) follows from the case $p > 1$ (with $p = q'$). □

Remark 3.9. Let $n \geqslant 2$ and let $D \subset \mathbb{R}^n$ be any open, bounded and Lipschitz domain in $\mathbb{R}^n$. It is still unknown if the space $L^1_{\mathbb{Z}}(D)$ is weakly sequentially closed. Surely it is not weakly-$*$ closed, a proof of this fact can be achieved by generalising the arguments in [27, Section 8].

A. Minimal connections for forms with finitely many integer singularities

In the following appendices we gather some of the results that have been used in the present work. These results are not new and have been included here for the reader's convenience.

Throughout Appendix A, we will denote by $M^m \subset \mathbb{R}^n$ some arbitrary embedded Lipschitz and connected m-dimensional submanifold of $\mathbb{R}^n$. Let $p \in [1,\infty]$. We will denote by $\Omega^1_{p,R,\infty}(M)$ the space introduced in Definition 2.2.

Lemma A.1. *Let $F \in \Omega^{m-1}_{p,R,\infty}(M)$. Then, there exists a connection for F.*

Proof. Throughout the following proof, given any couple of points $x, y \in M$ we will denote by (x, y) an arbitrarily chosen oriented Lipschitz curve with finite length joining x and y. By assumption, it holds that

$$*dF = \sum_{j=1}^{N} d_j \delta_{x_j}, \qquad \text{for some } d_1, ..., d_N \in \mathbb{Z} \smallsetminus \{0\} \text{ and } x_1, ..., x_N \in M.$$

We define

$$\{i_1, ..., i_p\} := \big\{j \in \{1, ..., N\} \text{ s.t. } d_j > 0\big\},$$
$$\{j_1, ..., j_q\} := \big\{j \in \{1, ..., N\} \text{ s.t. } d_j < 0\big\},$$
$$d := \sum_{j=1}^{N} d_j \in \mathbb{Z}.$$

We build a family $\mathscr{F} = \{I_\alpha\}_{\alpha \in A}$ of oriented Lipschitz curves in M as follows. If there is no point x_j such that $d_j < 0$, then we set $\mathscr{F} = \emptyset$. Else, we start from x_{i_1} and we add to the family $\mathscr{F}$ the curves $\big(x_{j_1}, x_{i_1}\big), ..., \big(x_{j_{k_1}}, x_{i_1}\big)$, until we reach the condition $k_1 = q$ or the condition

$$r_1 := d_{i_1} + \sum_{l=1}^{k_1} d_{j_l} \leqslant 0.$$

If $k_1 = q$, then we stop. Else, we move to the point x_{i_2}.

If $r_1 = 0$, then we add to $\mathscr{F}$ the segments $\big(x_{j_{k_1+1}}, x_{i_2}\big), ..., \big(x_{j_{k_2}}, x_{i_2}\big)$, where $k_2 \in \{1, ..., q\}$ is the smallest value such that

$$r_2 := d_{i_2} + \sum_{l=k_1+1}^{k_2} d_{j_l} \leqslant 0.$$

If there is no $k \in \{1, ..., q\}$ such that

$$d_{i_2} + \sum_{l=k_1+1}^{k} d_{j_l} \leqslant 0,$$

then we add to $\mathscr{F}$ the segments $\big(x_{j_{k_1+1}}, x_{i_2}\big), ..., \big(x_{j_q}, x_{i_2}\big)$ and we set $k_2 = q$.

If $r_1 < 0$, then we add to $\mathscr{F}$ the segments $\left(x_{j_{k_1}}, x_{i_2}\right)$ and $\left(x_{j_{k_1+1}}, x_{i_2}\right), ..., \left(x_{j_{k_2}}, x_{i_2}\right)$, where $k_2 \in \{1, ..., q\}$ is the smallest value such that

$$r_2 := d_{i_2} + r_1 + \sum_{l=k_1+1}^{k_2} d_{j_l} \leqslant 0.$$

If there is no $k \in \{1, ..., q\}$ such that

$$d_{i_2} + r_1 + \sum_{l=k_1+1}^{k} d_{j_l} \leqslant 0,$$

then we add to $\mathscr{F}$ the segments $\left(x_{j_{k_1}}, x_{i_2}\right)$ and $\left(x_{j_{k_1+1}}, x_{i_2}\right), ..., \left(x_{j_q}, x_{i_2}\right)$ and we set $k_2 = q$. We proceed iteratively in this way, moving on to the subsequent points x_{i_s} until $k_s = q$ or $s = p$. Then, the construction of the family $\mathscr{F}$ is complete. We let x_{i_h} be the last node that is visited before the iteration stops and, for every $I_\alpha = (x_j, x_i) \in \mathscr{F}$, we define its multiplicity m_α as

$$m_\alpha := \begin{cases} |d_j| - |r_l| & \text{if } i = i_l \text{ and } j = i_{k_l}, \\ \min\{|d_i|, |r_{l-1}|\} & \text{if } i = i_l \text{ and } j = i_{k_{l-1}}, \\ \min\{|d_j|, |d_i|\} & \text{else.} \end{cases}$$

Finally, we divide three cases:

1. *Case* $d = 0$. Notice that this is always the case if M has no boundary. We define the integer 1-current $I \in \mathcal{R}_1(M)$ given by

$$\langle I, \omega \rangle := \sum_{\alpha \in A} m_\alpha \int_{I_\alpha} \omega, \qquad \text{for every } \omega \in \mathcal{D}^1(M).$$

2. *Case* $d > 0$. We fix a point $x_0 \in \partial M$ and we let $I_s^p := (x_0, x_{i_s})$, for every $s = h, ..., p$. We define the integer 1-current $I \in \mathcal{R}_1(M)$ given by

$$\langle I, \omega \rangle := \sum_{\alpha \in A} m_\alpha \int_{I_\alpha} \omega + r_h \int_{I_h^b} \omega + \sum_{s=h+1}^{p} d_{i_s} \int_{I_s^b} \omega, \qquad \text{for every } \omega \in \mathcal{D}^1(M).$$

3. *Case* $d < 0$. We fix a point $x_0 \in \partial M$ and we let $I_s^n := (x_{j_s}, x_0)$, for every $s = k_h, ..., q$ We define the integer 1-current $I \in \mathcal{R}_1(M)$ given by

$$\langle I, \omega \rangle := \sum_{\alpha \in A} m_\alpha \int_{I_\alpha} \omega + |r_h| \int_{I_h^b} \omega + \sum_{s=k_h+1}^{q} |d_{i_s}| \int_{I_s^b} \omega, \qquad \text{for every } \omega \in \mathcal{D}^1(M).$$

By direct computation, we verify that I has the desired properties and the statement follows. □

Lemma A.2. *Let $F \in \Omega^{m-1}_{p,\mathbb{Z}}(M)$ (see Definition 1.4). Then,*

$$\inf_{\substack{T \in \mathcal{D}_1(M), \\ \partial T = *dF}} \mathbb{M}(T) = \inf_{\substack{T \in \mathcal{M}_1(M), \\ \partial T = *dF}} \mathbb{M}(T) = \sup_{\substack{\varphi \in W_0^{1,\infty}(M), \\ ||d\varphi||_{L^\infty} \leqslant 1}} \int_M F \wedge d\varphi < +\infty, \tag{A.1}$$

where $\mathcal{M}_1(M)$ denotes the set of all the 1-currents with finite mass on M. Moreover, the infimum on the left-hand-side of the previous chain of equalities is achieved.

Proof. By definition, there exists an integer 1-current $I \in \mathcal{R}_1(M)$ with finite mass such that $\partial I = *dF$. Hence

$$\inf_{\substack{T\in\mathcal{D}_1(M),\\ \partial T=*dF}} \mathbb{M}(T) \leqslant \mathbb{M}(I) < +\infty.$$

The first equality in (A.1) is clear.
Notice that for every $T \in \mathcal{M}_1(M)$ such that $\partial T = *dF$ it holds that

$$\int_M F \wedge d\varphi = \langle *dF, \varphi\rangle = \langle \partial T, \varphi\rangle = \langle T, d\varphi\rangle \leqslant \mathbb{M}(T)||d\varphi||_{L^\infty(M)}, \qquad \forall\, \varphi \in W_0^{1,\infty}(M).$$

Hence,

$$\inf_{\substack{T\in\mathcal{M}_1(M),\\ \partial T=*dF}} \mathbb{M}(T) \geqslant \sup_{\substack{\varphi\in W_0^{1,\infty}(M),\\ ||d\varphi||_{L^\infty}\leqslant 1}} \int_M F \wedge d\varphi. \tag{A.2}$$

To prove that the previous inequality is actually an equality, it suffices to show that the supremum on its right-hand-side is greater than the mass of some 1-current with finite mass T on M such that $\partial T = *dF$. Define the vector subspace $X \subset \Omega^1_\infty(M)$ given by

$$X := \{\omega \in \Omega^1_\infty(M) \text{ s.t. } \omega = d\varphi, \text{ for some } \varphi \in W_0^{1,\infty}(M)\}.$$

Consider the linear functional $\phi : X \subset (\Omega^1(M), \|\cdot\|_{L^\infty}) \to \mathbb{R}$ given by

$$\langle \phi, \omega\rangle = \int_M F \wedge \omega, \qquad \forall\, \omega \in X.$$

By (A.2) we get that ϕ is continuous on X, i.e.

$$||\phi||_{\mathcal{L}(X)} = \sup_{\substack{\omega\in X,\\ ||\omega||_{L^\infty}\leqslant 1}} \int_M F\wedge\omega = \sup_{\substack{\varphi\in W_0^{1,\infty}(M),\\ ||d\varphi||_{L^\infty}\leqslant 1}} \int_M F\wedge d\varphi \leqslant \inf_{\substack{T\in\mathcal{M}_1(M),\\ \partial T=*dF}} \mathbb{M}(T) < +\infty.$$

By Hahn-Banach theorem, we can extend ϕ to a linear functional $T : \Omega^1(M) \to \mathbb{R}$ such that

$$||T||_{\mathcal{L}(\Omega^1_\infty(M))} = ||\phi||_{\mathcal{L}(X)} = \sup_{\substack{\varphi\in W_0^{1,\infty}(M),\\ ||d\varphi||_{L^\infty}\leqslant 1}} \int_M F\wedge d\varphi$$

But then, T is a 1-current on M having finite mass and such that

$$\mathbb{M}(T) \leqslant ||T||_{\mathcal{L}(\Omega^1_\infty(M))} = \sup_{\substack{\varphi\in W_0^{1,\infty}(M),\\ ||d\varphi||_{L^\infty}\leqslant 1}} \int_M F\wedge d\varphi.$$

Moreover,

$$\langle \partial T, \varphi\rangle = \langle T, d\varphi\rangle = \langle \phi, d\varphi\rangle = \int_M F\wedge d\varphi = \langle *dF, \varphi\rangle, \qquad \forall\, \varphi \in W_0^{1,\infty}(M).$$

Hence,

$$\mathbb{M}(T) = \inf_{\substack{\tilde{T}\in\mathcal{D}_1(M),\\ \partial\tilde{T}=*dF}} \mathbb{M}(\tilde{T}) = \inf_{\substack{\tilde{T}\in\mathcal{M}_1(M),\\ \partial\tilde{T}=*dF}} \mathbb{M}(\tilde{T}) = \sup_{\substack{\varphi\in W_0^{1,\infty}(M),\\ ||d\varphi||_{L^\infty}\leqslant 1}} \int_M F\wedge d\varphi < +\infty$$

and the statement follows. □

Proposition A.1. *Let $F \in \Omega^{m-1}_{p,R}(M)$. Then, there exists an integer 1-current $L \in \mathcal{R}_1(M)$ such that $\partial L = *dF$ and*

$$\mathbb{M}(L) = \inf_{\substack{T \in \mathcal{D}_1(M), \\ \partial T = *dF}} \mathbb{M}(T) = \sup_{\substack{\varphi \in W^{1,\infty}_0(M), \\ ||d\varphi||_{L^\infty} \leqslant 1}} \int_M F \wedge d\varphi.$$

In particular,

$$\mathbb{M}(L) \leqslant C||F||_{L^p}.$$

Proof. Notice that by [15, Chapter 1, Section 3.4, Theorem 8], we have

$$\inf_{\substack{T \in \mathcal{R}_1(M), \\ \partial T = *dF}} \mathbb{M}(T) = \inf_{\substack{T \in \mathcal{D}_1(M), \\ \partial T = *dF}} \mathbb{M}(T).$$

Since the mass $\mathbb{M}(\,\cdot\,)$ is lower semicontinuous with respect to the weak convergence in $\mathcal{D}_1(M)$ and since $\mathbb{M}$-bounded subsets of the competition class $\mathcal{R}_1(M) \cap \{T \in \mathcal{D}_1(M) \text{ s.t. } \partial T = *dF\}$ are weakly sequentially compact (for a reference, see e.g. [22, Equation (7.5), Theorem 7.5.2]), by the direct method of calculus of variations we conclude that there exists an integer 1-current $L \in \mathcal{R}_1(M)$ such that $\partial L = *dF$ and

$$\mathbb{M}(L) = \inf_{\substack{T \in \mathcal{R}_1(M), \\ \partial T = *dF}} \mathbb{M}(T) = \inf_{\substack{T \in \mathcal{D}_1(M), \\ \partial T = *dF}} \mathbb{M}(T) = \sup_{\substack{\varphi \in W^{1,\infty}_0(M), \\ ||d\varphi||_{L^\infty} \leqslant 1}} \int_M F \wedge d\varphi,$$

where the last equality follows from Lemma A.2. The statement follows. □

B. Laplace equation on spheres

Let $n \in \mathbb{N}$ be such that $n \geqslant 2$ and fix any $p \in (1, +\infty)$. We let

$$\dot{W}^{1,p}(\mathbb{S}^{n-1}) := \left\{ u \in W^{1,p}(\mathbb{S}^{n-1}) \text{ s.t. } \bar{u} := \int_{\mathbb{S}^{n-1}} u \operatorname{vol}_{\mathbb{S}^{n-1}} = 0 \right\}.$$

We can endow the space $\dot{W}^{1,p}(\mathbb{S}^{n-1})$ with the usual $W^{1,p}$-norm induced by $W^{1,p}(\mathbb{S}^{n-1})$, given by

$$||u||_{W^{1,p}} := ||u||_{L^p} + ||du||_{L^p}, \qquad \forall\, u \in \dot{W}^{1,p}(\mathbb{S}^{n-1}).$$

Lemma B.1 (Poincaré inequality on $\dot{W}^{1,p}$)**.** *There exists a constant $C > 0$ such that*

$$\int_{\mathbb{S}^{n-1}} |u|^p \operatorname{vol}_{\mathbb{S}^{n-1}} \leqslant C \int_{\mathbb{S}^{n-1}} |du|^p \operatorname{vol}_{\mathbb{S}^{n-1}}, \qquad \forall\, u \in \dot{W}^{1,p}(\mathbb{S}^{n-1}).$$

Proof. By contradiction, assume that for every $k > 0$ there exists $u_k \in \dot{W}^{1,p}(\mathbb{S}^{n-1})$ such that $||u_k||_{L^p} = 1$ and

$$1 > k \int_{\mathbb{S}^{n-1}} |du_k|^p \operatorname{vol}_{\mathbb{S}^{n-1}}.$$

This implies immediately that $||du_k||_{L^p} \to 0$ as $k \to \infty$. In particular, the sequence $\{u_k\}_{k\in\mathbb{N}}$ is bounded with respect to the $W^{1,p}$-norm. Hence, by weak compactness of $W^{1,p}(\mathbb{S}^{n-1})$, there exists a subsequence $\{u_{k_j}\}_{j\in\mathbb{N}}$ of $\{u_k\}_{k\in\mathbb{N}}$ such that $u_{k_j} \rightharpoonup u$ in $W^{1,p}(\mathbb{S}^{n-1})$. Moreover, by Rellich-Kondrakov theorem, we have $u_{k_j} \to u$ strongly in $L^p(\mathbb{S}^{n-1})$. Since $du_{k_j} \to 0$ strongly in L^p we get $du = 0$. Then, u is constant on $\mathbb{S}^{n-1}$. Since $u_{k_j} \rightharpoonup u$ in $L^p(\mathbb{S}^{n-1})$, it follows that

$$0 = \lim_{j\to\infty} \int_{\mathbb{S}^{n-1}} u_{k_j}\, vol_{\mathbb{S}^{n-1}} = \int_{\mathbb{S}^{n-1}} u\, vol_{\mathbb{S}^{n-1}}$$

and this leads to $u = 0$. But this is absurd, since by strong L^p-convergence of $\{u_{k_j}\}_{j\in\mathbb{N}}$ to u we obtain $||u||_{L^p} = 1$. □

Remark B.1. By Lemma B.1, we conclude that we can endow $\dot{W}^{1,p}$ with the following much more convenient norm:

$$||u||_{\dot{W}^{1,p}} := ||du||_{L^p}, \qquad \forall\, u \in \dot{W}^{1,p}(\mathbb{S}^{n-1}).$$

Moreover, such a norm is equivalent to $W^{1,p}$-norm.

Remark B.2. Notice that a linear functional on $W^{1,p}(\mathbb{S}^{n-1})$ restricts to an element of $(\dot{W}^{1,p'}(\mathbb{S}^{n-1}))^*$ if and only if it is $W^{1,p}$-continuous and $\langle F, 1\rangle = 0$.

Lemma B.2. *Let $F \in (W^{1,p'}(\mathbb{S}^{n-1}))^*$ be such that $\langle F, 1\rangle = 0$. Then, the following facts hold.*

1. *If $n \geqslant 3$ the linear differential system*

$$\begin{cases} d^*\omega = F \\ d\omega = 0 \end{cases}$$

has a unique weak solution $\alpha \in \Omega^1_p(\mathbb{S}^{n-1})$.

2. *If $n = 2$ the linear differential system*

$$\begin{cases} d^*\omega = F \\ \displaystyle\int_{\mathbb{S}^1} \omega = 0 \end{cases}$$

has a unique weak solution $\alpha \in \Omega^1_p(\mathbb{S}^1)$.

In both cases, α satisfies the following estimate:

$$||\alpha||_{L^p} \leqslant C||F||_{\mathcal{L}(\dot{W}^{1,p'}(\mathbb{S}^{n-1}))},$$

for some constant $C > 0$ depending only n.

Proof. Notice the since $\langle F, 1\rangle = 0$, by Remark B.2 F restricts to an element of $(\dot{W}^{1,p'}(\mathbb{S}^{n-1}))^*$. Consider the linear functional $\phi : \Omega^1(\mathbb{S}^{n-1}) \to \mathbb{R}$ given by

$$\langle \phi, \omega\rangle = \langle F, u\rangle, \qquad \forall\, \omega = du + d^*\beta + \eta \in \Omega^1(\mathbb{S}^{n-1}),$$

where η is a harmonic 1-form on $\mathbb{S}^{n-1}$ (in case $n = 2$). Since ϕ is continuous and linear on $\Omega^1(\mathbb{S}^{n-1})$ with respect to the $L^{p'}$-norm, by Hahn-Banach theorem there exists a unique (recall that L^p-spaces are strictly convex) extension $\Phi \in (\Omega^1_{p'}(\mathbb{S}^{n-1}))^*$ of ϕ such that

$$||\Phi||_{(\Omega^1_{p'}(\mathbb{S}^{n-1}))^*} = ||\phi||_{(\Omega^1(\mathbb{S}^{n-1}))^*} \leqslant C||F||_{\mathcal{L}(\dot{W}^{1,p'}(\mathbb{S}^{n-1}))}.$$

By Riesz representation theorem, there exists a unique $\alpha \in \Omega^1_p(\mathbb{S}^{n-1})$ such that

$$\langle \alpha, \omega \rangle_{L^p - L^{p'}} := \int_{\mathbb{S}^{n-1}} \alpha \wedge *\omega = \langle \Phi, \omega \rangle, \qquad \forall\, \omega \in \Omega^1_{p'}(\mathbb{S}^{n-1}) \tag{B.1}$$

and

$$||\alpha||_{L^p} = ||\Phi||_{(\Omega^1_{p'}(\mathbb{S}^{n-1}))^*} \leqslant C||F||_{(\dot{W}^{1,p'}(\mathbb{S}^{n-1}))^*}.$$

Finally applying equation (B.1) we get

$$\langle \alpha, du \rangle_{L^p - L^{p'}} = \langle \Phi, du \rangle = \langle \phi, du \rangle = \langle F, u \rangle, \qquad \forall\, u \in C^\infty(\mathbb{S}^{n-1}),$$

and

$$\langle \alpha, d^*\beta \rangle_{L^p - L^{p'}} = \langle \Phi, d^*\beta \rangle = \langle \phi, d^*\beta \rangle = 0, \qquad \forall\, \beta \in \Omega^2(\mathbb{S}^{n-1}).$$

The two previous equations are exactly the weak forms of the equations $d^*\alpha = F$ and $d\alpha = 0$ respectively. Moreover, in case $n = 2$, we have

$$\int_{\mathbb{S}^1} \alpha = \int_{\mathbb{S}^1} \alpha \wedge *1 = \langle \alpha, *1 \rangle_{L^p - L^{p'}} = \langle \Phi, *1 \rangle = \langle F, 1 \rangle = 0.$$

This concludes about the existence of a solution to the differential systems given in points 1 and 2. For what concerns uniqueness, assume that α and α' are two solutions of the differential system given in point 1 (resp. 2) and define $\beta = \alpha - \alpha'$. Then, we distinguish the two cases:

Case $n \geqslant 3$. In this case, β satisfies

$$\begin{cases} d^*\beta = 0 \\ d\beta = 0. \end{cases}$$

Hence, β is a harmonic 1-form on $\mathbb{S}^{n-1}$ for $n \geqslant 1$, which implies $\beta = 0$.

Case $n = 2$. In this case, β satisfies

$$\begin{cases} d^*\beta = 0 \\ \displaystyle\int_{\mathbb{S}^1} \beta = 0. \end{cases}$$

Hence, β is a harmonic 1-forms on $\mathbb{S}^1$, which implies $\beta = c\,\mathrm{vol}_{\mathbb{S}^1}$ for some $c \in \mathbb{R}$. But since β has vanishing integral on $\mathbb{S}^1$, we get $\beta = 0$. □

Definition B.1 (Sobolev spaces of differential forms)**.** Fix any $k \in \mathbb{N} \setminus \{0\}$. We define the *Sobolev space of $W^{1,p}$-regular differential k-forms on $\mathbb{S}^{n-1}$* by

$$\Omega^k_{W^{1,p}}(\mathbb{S}^{n-1}) := \big\{\omega \in \Omega^k_p(\mathbb{S}^{n-1}) \text{ s.t. } d\omega, d^*\omega \in L^p\big\}.$$

We endow such space with the norm

$$||\omega||_{W^{1,p}} := ||\omega||_{L^p} + ||d\omega||_{L^p} + ||d^*\omega||_{L^p}, \qquad \forall\, \omega \in \Omega^k_{W^{1,p}}(\mathbb{S}^{n-1}).$$

Remark B.3. It can be shown (see [31, §3 and §4]) that such Sobolev spaces are completely equivalent to the usual ones, namely the space of k-forms having local coefficients in $W^{1,p}$. Moreover, in case $n \geqslant 3$ there exists $C > 0$ such that

$$||\omega||_{W^{1,p}} \leqslant C\big(||d\omega||_{L^p} + ||d^*\omega||_{L^p}\big), \qquad \forall\, \omega \in \Omega^1_{W^{1,p}}(\mathbb{S}^{n-1}). \tag{B.2}$$

Indeed, let

$$\begin{aligned} X &:= \{d\alpha \text{ s.t. } \alpha \in \Omega^1_{W^{1,p}}(\mathbb{S}^{n-1})\}, \\ Y &:= \{d^*\beta \text{ s.t. } \beta \in \Omega^1_{W^{1,p}}(\mathbb{S}^{n-1})\}. \end{aligned}$$

By [31, Proposition 7.1], both X and Y are closed linear subspaces respectively of $\Omega^2_p(\mathbb{S}^{n-1})$ and $\dot{W}^{1,p}(\mathbb{S}^{n-1})$. Then, $X \oplus Y$ is a Banach space with respect to the standard norm on the direct sum of two Banach spaces. We claim that the liner operator $T : \Omega^1_{W^{1,p}}(\mathbb{S}^{n-1}) \to X \oplus Y$ given by $T\omega = (d\omega, d^*\omega)$, for every $\omega \in \Omega^1_{W^{1,p}}(\mathbb{S}^{n-1})$ is a continuous linear bijection between Banach spaces. Indeed, the fact that T is injective follows form the fact that there no non-zero harmonic forms on $\mathbb{S}^{n-1}$ for $n \geqslant 3$. Hence, we just need to show that T is surjective. Pick any $(d\alpha, d^*\beta) \in X \oplus Y$. By Lemma B.2, the linear differential system

$$\begin{cases} d^*\omega = d^*(\beta - \alpha) \\ d\omega = 0 \end{cases}$$

has a unique weak solution $\tilde{\omega} \in \Omega^1_p(\mathbb{S}^{n-1})$. Since by construction we have $d\tilde{\omega}, d^*\tilde{\omega} \in L^p$, we conclude that $\tilde{\omega} \in \Omega^1_{W^{1,p}}(\mathbb{S}^{n-1})$. Then, by letting $\omega := \tilde{\omega} + \alpha \in \Omega^1_{W^{1,p}}(\mathbb{S}^{n-1})$ we see that

$$T\omega = (d\omega, d^*\omega) = (d\tilde{\omega} + d\alpha, d^*\tilde{\omega} + d^*\alpha) = (d\alpha, d^*\beta)$$

and we have proved our claim. This proves that T has a continuous inverse and the statement follows with $C = ||T^{-1}||_{\mathcal{L}(X \oplus Y, \Omega^1_{W^{1,p}}(\mathbb{S}^{n-1}))}$.

In case $n = 2$, the estimate (B.2) still holds for every $\omega \in \Omega^1_{W^{1,p}}(\mathbb{S}^1)$ such that

$$\int_{\mathbb{S}^1} \omega = 0.$$

The proof is completely analogous.

Remark B.4 (L^p-Hodge decomposition). Let

$$\begin{aligned} X &:= \{d\varphi \text{ s.t. } \varphi \in \dot{W}^{1,p}(\mathbb{S}^{n-1})\}, \\ Y &:= \{d^*\beta \text{ s.t. } \beta \in \Omega^2_{W^{1,p}}(\mathbb{S}^{n-1})\}, \end{aligned}$$

$$Z := \{\eta \in \Omega^1(\mathbb{S}^{n-1}) \text{ s.t. } \Delta\eta = (dd^* + d^*d)\eta = 0\}.$$

Then, as a particular consequence of the L^p-Hodge decomposition theorem (see e.g. [31, Proposition 6.5]), the operator $T : X \oplus Y \oplus Z \to \Omega^1_p(\mathbb{S}^{n-1})$ given by

$$T(d\varphi, d^*\beta, \eta) := d\varphi + d^*\beta + \eta$$

is a continuous and linear isomorphism between Banach spaces. Hence, T has a continuous inverse. We let

$$C_H := ||T^{-1}||_{\mathcal{L}(\Omega^1_p(\mathbb{S}^{n-1}), X\oplus Y)}.$$

We conclude that for every $\omega \in \Omega^1_p(\mathbb{S}^{n-1})$ there exist $\varphi \in \dot{W}^{1,p}(\mathbb{S}^{n-1})$, $\beta \in \Omega^2_{W^{1,p}}(\mathbb{S}^{n-1})$ and $\eta \in Z$ such that $\omega = d\varphi + d^*\beta + \eta$ and

$$||d\varphi||_{L^p} + ||d^*\beta||_{L^p} + ||\eta||_{L^p} \leqslant C_H||\omega||_{L^p}. \tag{B.3}$$

Lemma B.3 (A weak version of Poincarè lemma)**.** *Let $n \geqslant 3$ and let $\alpha \in \Omega^1_p(\mathbb{S}^{n-1})$ be such that $d\alpha = 0$ weakly on $\mathbb{S}^{n-1}$. Then, there exists a Sobolev function $\varphi \in \dot{W}^{1,p}(\mathbb{S}^{n-1})$ such that $d\varphi = \alpha$ weakly on $\mathbb{S}^{n-1}$.*

Proof. We follow the notation of Remark B.4 and we notice that, since $n \geqslant 3$ we have $Z = \{0\}$. Hence, we write $\alpha = d\varphi + d^*\beta$, for $\varphi \in \dot{W}^{1,p}(\mathbb{S}^{n-1})$ and $\beta \in \Omega^2_{W^{1,p}}(\mathbb{S}^{n-1})$. We observe that

$$\begin{aligned}
(d^*\beta, \omega)_{L^p - L^{p'}} &= (d^*\beta, d\psi + d^*\gamma)_{L^p-L^{p'}} \\
&= (d^*\beta, d^*\gamma)_{L^p-L^{p'}} \\
&= (\alpha - d\varphi, d^*\gamma)_{L^p-L^{p'}} \\
&= (\alpha, d^*\gamma)_{L^p-L^{p'}} = 0, \qquad \forall\, \omega = d\psi + d^*\gamma \in \Omega^1(\mathbb{S}^{n-1}).
\end{aligned}$$

This implies $d^*\beta = 0$ and the statement follows. □

Corollary B.1 (Laplace equation on spheres)**.** *Let $F \in \mathcal{L}(W^{1,p'}(\mathbb{S}^{n-1}))$ such that $\langle F, 1\rangle = 0$. Then, the linear differential equation*

$$\Delta u = F$$

has a unique weak solution $\varphi \in \dot{W}^{1,p}(\mathbb{S}^{n-1})$ satisfying

$$||\varphi||_{W^{1,p}} \leqslant C||F||_{(\dot{W}^{1,p'}(\mathbb{S}^{n-1}))^*}.$$

Proof. First, we face the case $n \geqslant 3$. By Lemma B.2 we can find $\alpha \in \Omega^1_p(\mathbb{S}^{n-1})$ satisfying

$$\begin{cases} d^*\alpha = F \\ d\alpha = 0 \end{cases}$$

and

$$||\alpha||_{L^p} \leqslant C||F||_{(\dot{W}^{1,p'}(\mathbb{S}^{n-1}))^*}.$$

Since $d\alpha = 0$, by Lemma B.3 there exists $\varphi \in \dot{W}^{1,p}(\mathbb{S}^{n-1})$ such that $\alpha = d\varphi$. Hence, we get

$$\Delta\varphi = d^*d\varphi = d^*\alpha = F.$$

Moreover, by Lemma B.1, we have

$$||\varphi||_{W^{1,p}} \leqslant C||d\varphi||_{L^p} = C||\alpha||_{L^p} \leqslant C||F||_{(\dot{W}^{1,p'}(\mathbb{S}^{n-1}))^*}.$$

This concludes the proof in case $n \geqslant 3$.
If $n = 2$, then by Lemma B.2 we can find $\alpha \in \Omega^1_p(\mathbb{S}^{n-1})$ satisfying

$$\begin{cases} d^*\alpha = F \\ \displaystyle\int_{\mathbb{S}^1} \alpha = 0 \end{cases}$$

and

$$||\alpha||_{L^p} \leqslant C||F||_{(\dot{W}^{1,p'}(\mathbb{S}^{n-1}))^*}.$$

By setting $\varphi := *\alpha$, the statement follows. □

C. Some technical lemmata

In this section we will make use of the following notation: let T be an m-rectifiable current in $\mathbb{R}^n$, then T can be represented as follows:

$$\langle T, \omega\rangle = \int_{\mathbb{R}^n} \theta\langle\omega, \xi\rangle d\mathscr{H}^m\big|_{\Sigma} \quad \forall\omega \in \Omega^m(\mathbb{R}^n),$$

where Σ is a locally m-rectifiable set,

$$\theta : \Sigma \to \mathbb{Z}$$

is a locally $\mathscr{H}^m$-integrable, non-negative function and

$$\xi : \Sigma \to \Lambda_m\mathbb{R}^n$$

is an $\mathscr{H}^m$-measurable function such that for $\mathscr{H}^m$-almost every point $x \in \Sigma$, $\xi(x)$ is a simple unit m-vector in $T_x\Sigma$.
In this case we write

$$T = \tau(\Sigma, \theta, \xi).$$

Lemma C.1. *For any $k \in \mathbb{N}$ let*

$$T_k = \tau(\Sigma_k, \theta_k, \xi_k)$$

be an m-rectifiable current on $\mathbb{R}^n$ of finite mass. Assume that $(T_k)_{k\in\mathbb{N}}$ is a Cauchy sequence with respect to the convergence in mass.
Then there exists an m-rectifiable current

$$T = \tau(\Sigma, \theta, \xi)$$

such that

$$T_k \to T \quad (k \to \infty) \quad \textit{in mass}.$$

Proof. Replacing the original sequence by a subsequence if necessary, we may assume that for any $k \in \mathbb{N}$

$$\mathbb{M}(T_k - T_{k+1}) \leqslant 2^{-k}.$$

Now for any $k \in \mathbb{N}$ let

$$\tilde{T}_k := \begin{cases} T_1 & \text{if } k = 1 \\ T_k - T_{k-1} & \text{if } k > 1. \end{cases}$$

Then for any $k \in \mathbb{N}$ we have

$$T_k = \sum_{i=1}^{k} \tilde{T}_i.$$

For any $k \in \mathbb{N}$ write

$$\tilde{T}_k = \langle \tilde{\Sigma}_k, \tilde{\theta}_k, \tilde{\xi}_k \rangle.$$

Notice that

$$\sum_{i=1}^{k} \tilde{\theta}_i \tilde{\xi}_i = \theta_k \xi_k \qquad \mathscr{H}^m\text{- a.e. on } \Sigma,$$

for every $k \in \mathbb{N}$. Set

$$\Sigma := \bigcup_{k \in \mathbb{N}} \big(\tilde{\Sigma}_k \smallsetminus \tilde{\theta}_k^{-1}(0)\big).$$

Then Σ is m-rectifiable as countable union of m-rectifiable sets. Moreover $\mathscr{H}^m(\Sigma) < \infty$. In fact

$$\mathscr{H}^m(\Sigma) \leqslant \sum_{k \in \mathbb{N}} \mathscr{H}^m\big(\tilde{\Sigma}_k \smallsetminus \tilde{\theta}_k^{-1}(0)\big) \leqslant \sum_{k \in \mathbb{N}} \int_{\mathbb{R}^n} |\tilde{\theta}_k| \, d\mathscr{H}^m \llcorner \tilde{\Sigma}_k = \sum_{k \in \mathbb{N}} \mathbb{M}(\tilde{T}_k) < \infty.$$

Next let

$$\theta = \sum_{k \in \mathbb{N}} \tilde{\theta}_k,$$

where $\tilde{\theta}_k$ is extended by zero on $\Sigma \smallsetminus \tilde{\Sigma}_k$ for any $k \in \mathbb{N}$.
By Beppo-Levi Theorem

$$\begin{aligned} \int_{\mathbb{R}^n} |\theta| \, d\mathscr{H}^m \llcorner \Sigma &= \sum_{k \in \mathbb{N}} \int_{\mathbb{R}^n} |\tilde{\theta}_k| \, d\mathscr{H}^m \llcorner \Sigma \\ &= \sum_{k \in \mathbb{N}} \int_{\mathbb{R}^n} |\tilde{\theta}_k| \, d\mathscr{H}^m \llcorner \tilde{\Sigma}_k = \sum_{k \in \mathbb{N}} \mathbb{M}(\tilde{T}_k) < \infty. \end{aligned} \tag{C.1}$$

Therefore θ is finite $\mathscr{H}^m$-a.e. in Σ, i.e. for $\mathscr{H}^m$-a.e. $x \in \Sigma$ there are only finitely many $k \in \mathbb{N}$ so that $\theta_k(x) \neq 0$. In particular the sum

$$\sum_{k \in \mathbb{N}} \tilde{\theta}_k(x) \xi_k(x)$$

is well defined and finite for $\mathscr{H}^m$-a.e. $x \in \Sigma$ (again ξ_k is extended by zero on $\Sigma \setminus \tilde{\Sigma}_k$ for any $k \in \mathbb{N}$) and we can write

$$\theta(x)\xi(x) = \sum_{k\in\mathbb{N}} \tilde{\theta}_k(x)\xi_k(x)$$

for some $\theta(x) \in \mathbb{Z}_{\geqslant 0}$ and for some simple unit m-vector $\xi(x)$ in $T_x\Sigma$, for $\mathscr{H}^m$-a.e. $x \in \Sigma$. Observe that θ is an $\mathscr{H}^m$-measurable function on Σ as the absolute value of the a.e.-limit of $\mathscr{H}^m$-measurable functions. Analogously, ξ is an $\mathscr{H}^m$-measurable map on Σ as the a.e.-limit of $\mathscr{H}^m$-measurable maps. We set

$$T := \tau(\Sigma, \theta, \xi)$$

and we claim that

$$T_k \to T \quad (k \to \infty) \quad \text{in mass.}$$

In fact we know that since the space of m-currents is complete under the convergence in mass (as a dual space), there exists an m-current T' such that

$$T_k \to T' \quad (k \to \infty) \quad \text{in mass.}$$

To see that $T = T'$ observe that for any $\omega \in \mathcal{D}^m(\mathbb{R}^n)$

$$\theta_k\langle \omega, \xi_k\rangle = \sum_{i=1}^{k} \tilde{\theta}_i\langle \omega, \tilde{\xi}_i\rangle \to \theta\langle \omega, \xi\rangle \quad \mathscr{H}^m\text{-a.e. in } \Sigma,$$

thus by (C.1) and Dominated Convergence Theorem we conclude that

$$\langle T_k, \omega\rangle \to \langle T, \omega\rangle \quad (k \to \infty).$$

In particular $T = T'$. □

Lemma C.2. *Let $\alpha \in (1, +\infty)$, $q \in (-\infty, 1]$, $\varepsilon \in (0, 1)$ and let $\Omega \subset Q^n_{1-\varepsilon}(0)$ be open, Lipschitz and bounded. For any $p \in [1, +\infty)$ and $\mu := f\mathcal{L}^n$ with $f = (\frac{1}{2} - \|\cdot\|_\infty)^q$, consider the continuous linear operator $P_\alpha : L^p(\Omega, \mu; \mathbb{R}^n) \to L^p(\Omega, \mu; \mathbb{R}^n)$ given by*

$$(P_\alpha V)(x) := \begin{cases} \alpha^{n-1}V(\alpha x) & \text{if } x \in \alpha^{-1}\Omega, \\ 0 & \text{on } \Omega \setminus \alpha^{-1}\Omega. \end{cases}$$

Then:

1. *For every $\alpha \in (1, +\infty)$ such that $|1 - \alpha^{-1}| \leqslant \varepsilon$ holds that*

$$\|P_\alpha V\|_{L^p(\mu)} \leqslant C\alpha^{n-1-\frac{n}{p}}\|V\|_{L^p(\mu)},$$

   for some constant $C > 0$ depending only on q and p.
2. *For every $V \in L^p(\Omega, \mu; \mathbb{R}^n)$ we have that $P_\alpha V \to V$ in $L^p(\Omega, \mu; \mathbb{R}^n)$ as $\alpha \to 1^+$.*

Proof. Fist we prove 1. Fix any $V \in L^p(\Omega, \mu; \mathbb{R}^n)$ and compute

$$\begin{aligned}\int_\Omega |P_\alpha V|^p \, d\mu &= \alpha^{p(n-1)} \int_{\alpha^{-1}\Omega} |V(\alpha x)|^p \, d\mu(x) \\ &\leqslant \alpha^{p(n-1)-n} \bigg(\int_\Omega |V(y)|^p \, d\mu(y) + \int_\Omega |V(y)|^p \frac{f(\alpha^{-1}y) - f(y)}{f(y)} \, d\mu(y) \bigg).\end{aligned}$$

As in (2.5) we can estimate

$$\left| \frac{f(\alpha^{-1}y) - f(y)}{f(y)} \right| \leqslant q \left(\frac{1}{2} - \|y\|_\infty \right)^{-1} \|y\|_\infty (1 - \alpha^{-1}) \leqslant C$$

for any $y \in Q_{1-\varepsilon}(0)$ and any $\alpha \geqslant 1$ such that $|1 - \alpha^{-1}| \leqslant \varepsilon$, for some constant C depending only on q. Therefore

$$\int_\Omega |P_\alpha V|^p \, d\mu \leqslant (C+1) \alpha^{p(n-1)-n} \int_\Omega |V(y)|^p \, d\mu(y).$$

Hence 1. follows. We are left to prove 2.. Fix any $\delta > 0$ and let $V_\delta \in C^0_c(\Omega; \mathbb{R}^n)$ be such that

$$\|V_\delta - V\|_{L^p(\mu)} \leqslant \delta.$$

By 1, we have

$$\begin{aligned}\|P_\alpha V - V\|_{L^p(\mu)} &\leqslant \|P_\alpha (V - V_\delta)\|_{L^p(\mu)} + \|P_\alpha V_\delta - V_\delta\|_{L^p(\mu)} + \|V_\delta - V\|_{L^p(\mu)} \\ &\leqslant (C\alpha^{n-1-\frac{n}{p}} + 1)\delta + \|P_\alpha V_\delta - V_\delta\|_{L^p(\mu)},\end{aligned}$$

for every $\alpha \in (1, +\infty)$ such that $|1 - \alpha^{-1}| \leqslant \varepsilon$. Since V_δ is continuous and compactly supported, it follows from dominated convergence that $\|P_\alpha V_\delta - V_\delta\|_{L^p(\mu)} \to 0$ as $\alpha \to 1^+$. Hence, by letting $\alpha \to 1^+$ in the previous inequality we get

$$\limsup_{\alpha \to 1^+} \|P_\alpha V - V\|_{L^p(\mu)} \leqslant (C+1)\delta.$$

As $\delta > 0$ was arbitrary, 2 follows. □

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