Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XIX (2019), 191-215

On the boundedness of slc surfaces of general type

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Abstract. The purpose of this note is to give a new proof of Alexeev's boundedness result for stable surfaces which is independent of the base field and to highlight some important consequences of this result.

Mathematics Subject Classification (2010): 74K20 (primary); 74B20 (secondary).

Let k be an algebraically closed field, an slc *model* (X, B) is a projective semi-log canonical pair such that $K_X + B$ is ample (see Definition 1.3 below). The main result of this paper is the following.

Theorem 1 (Alexeev). *Fix a constant* $\upsilon > 0$ *and a DCC set* $\mathscr{C} \subset [0, 1] \cap \mathbb{Q}$. *Then there exists an integer* $r > 0$ *such that for any algebraically closed field* k *and any two dimensional* slc *model* (X, B) *defined over* k *with* $\operatorname{coeff}(B) \subseteq \mathscr{C}$ *and* $(K_X + B)^2 = \upsilon$, $r(K_X + B)$ *is very ample.*

Remark. This result was originally proved by Alexeev in a series of papers, [2–4] and [5]. The results there are stated for surfaces defined over a fixed algebraically closed field (of any characteristc) however, as pointed out to us by Alexeev, they actually hold independently of the field. We believe that this was known to some experts, however there are some subtleties in Alexeev's arguments that make the proof of the results over an arbitrary field not entirely routine. In this paper we propose an alternative proof which we believe simplifies and makes Alexeev's original approach more transparent. The main differences are: A substantial simplification of the arguments needed from [3]; the use of recent effective Matsusaka results of [17] and [7] (instead of the original papers of Matsusaka and Kollár); and the use of ultraproducts (*cf.* [16] and [6]) to simplify some of the arguments of [4]. Of course the entire paper is heavily influenced by [4].

CDH was supported by NSF research grants no: DMS-1300750, DMS-1265285 and by a grant from the Simons Foundation; Award Number: 256202.
SJK was supported in part by NSF Grants DMS-1301888 and DMS-1565352 and the Craig McKibben and Sarah Merner Endowed Professorship in Mathematics.

Received October 3, 2016; accepted in revised form July 3, 2017.
Published online February 2019.

The following result, which is of independent interest, is a key step in the proof of Theorem 1. This was also proved by Alexeev, but we provide an independent proof.

Theorem 2 (Alexeev). *Fix a DCC set $\mathscr{C}$. Let $\mathscr{V} = \{(K_X + B)^2\}$ where (X, B) is a two dimensional* slc *model defined over k, an algebraically closed field, with* coeff$(B) \subseteq \mathscr{C}$*. Then $\mathscr{V}$ is also a DCC set. In particular, there exists a number $\delta > 0$, depending only on $\mathscr{C}$, such that if $0 < v \in \mathscr{V}$, then $v \geq \delta$.*

Corollary 3. *Fix constants $\varepsilon, \upsilon > 0$ and a DCC set $\mathscr{C} \subset [0, 1] \cap \mathbb{Q}$. Then the set of all two dimensional ε-log canonical pairs (X, B) defined over k with* coeff$(B) \subseteq \mathscr{C}$*, $K_X + B$ nef and big and $(K_X + B)^2 \leq \upsilon$ is degree bounded,* i.e.*, there exists a constant $d > 0$ such that for any pair (X, B) as above there is a very ample divisor H on X such that $H^2 \leq d$ and $B_{\text{red}} \cdot H \leq d$.*

We have the following interesting applications which should allow the construction of moduli spaces of (semi-log-canonical) canonically polarized surfaces for $p \gg 0$.

Theorem 4. *Fix a constant $\upsilon \in \mathbb{Q}$ and a DCC set $\mathscr{C} \subset [0, 1] \cap \mathbb{Q}$. Then there exists a number $p_0 > 0$ such that if L is an algebraically closed field of characteristic $p > p_0$, (X, B) a pair defined over L such that* $\dim X = 3$*, $f : X \to S = \operatorname{Spec} L[[t]]$ a projective morphism with connected fibers such that, with $\eta \in S$ denoting the generic point of S,* coeff$(B_\eta) \subseteq \mathscr{C}$*, (X_η, B_η) is semi-log canonical, and $K_{X_\eta} + B_\eta$ is ample with $(K_{X_\eta} + B_\eta)^2 = \upsilon$, then there exist a separable finite morphism $S' \to S$, a projective morphism $f' : X' \to S'$, and a pair (X', B') such that (X'_s, B'_s) is semi-log canonical and $K_{X'_s} + B'_s$ is ample for all $s \in S'$, and $(X'_{\eta'}, B'_{\eta'})$ is isomorphic to $(X_\eta, B_\eta) \times_\eta \eta'$ where $\eta' \in S'$ denotes the generic point.*

Theorem 4 will follow as Corollary 2.12 to the somewhat more technical Theorem 2.11 which we only state later. It also implies Corollary 2.13, a variant of the above statement.

Finally, using Theorem 4 we will we prove another variant:

Theorem 5. *Fix a constant $\upsilon \in \mathbb{Q}$ and a DCC set $\mathscr{C} \subset [0, 1] \cap \mathbb{Q}$. For each $m > 0$ let L_m be an algebraically closed field of characteristic $p_m > 0$ such that* $\lim p_m = \infty$ *and let $k = [L_m]$. Further let (X_m, B_m) be a pair defined over L_m such that* $\dim X_m = 3$*, and let $f_m : X_m \to S_m$ be a projective morphism with connected fibers to a smooth curve. Assume that for each $m \in \mathbb{Z}$ with $\eta \in S_m$ denoting the generic point of S_m,* coeff$(B_{m,\eta}) \subseteq \mathscr{C}$*, $(X_{m,\eta}, B_{m,\eta})$ is semi-log canonical, and $K_{X_{m,\eta}} + B_{m,\eta}$ is ample with $(K_{X_{m,\eta}} + B_{m,\eta})^2 = \upsilon$.*

Then for all but finitely many m's there exist a separable finite morphism $\sigma_m : S'_m \to S_m$, a projective morphism $X'_m \to S'_m$, and a pair (X'_m, B'_m) such that $(X'_{m,s}, B'_{m,s})$ is semi-log canonical and $K_{X'_{m,s}} + B'_{m,s}$ is ample for all $s \in S'_m$, and $(X'_{m,s}, B'_{m,s})$ is isomorphic to $(X_{m,\sigma_m(s)}, B_{m,\sigma_m(s)})$ for general $s \in S'_m$.

As an ancillary result we establish an important connection between bounding the degree of a projective variety and bounding the degree of its defining polynomials.

This is, of course, related to the fascinating Eisenbud-Goto conjecture [8], but we only need a much weaker statement. It turns out that the proof is not complicated at all, but still the statement might be of some independent interest.

Theorem 6 (Theorem 1.8). *There exists a function $\beta : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, such that if $d, q, n \in \mathbb{N}$, L is a field and $X \subseteq \mathbb{P}^n$ is a projective variety over L such that* $\dim X = q$ *and* $\deg X \leq d$, *then $I(X) \subseteq L[x_0, \dots, x_n]$ can be generated by homogenous polynomials of degree at most $\beta(d, q)$.*

ACKNOWLEDGEMENTS. The authors are grateful to Valery Alexeev for useful comments and especially for explaining why the results of [4] hold independently of the field and to Zsolt Patakfalvi and the anonymous referee for helpful remarks.

1. Preliminaries

1.1. Definitions

We follow the definitions of [13] (in particular for discrepancies, terminal, klt and lc pairs). A *pair* (X, B) consists of a demi-normal variety X (see Definition 1.3) and an effective $\mathbb{Q}$-divisor B on X such that none of the irreducible components of B is contained in $\operatorname{Sing} X$. The set of coefficients appearing in the irreducible decomposition $B = \sum_{i=1}^{r} b_i B_i$ is denoted by $\operatorname{coeff}(B) = \{b_i | i = 1, \dots, r\}$ and we let $B_{\text{red}} = \sum_{i=1}^{r} B_i$. Recall that a pair (X, B) is ε*-klt* (respectively ε*-lc*) if X is normal and $a(X, B) > \varepsilon - 1$ (respectively $a(X, B) \geq \varepsilon - 1$) where $a(X, B)$ is the total discrepancy of (X, B), in particular $b_i < 1 - \varepsilon$ (respectively $b_i \leq 1 - \varepsilon$).

We say that $\mathscr{C} \subset \mathbb{R}$ is a *DCC set* if given any non-increasing sequence $(a_i)_{i \in \mathbb{N}}$ of elements of $\mathscr{C}$ then $(a_i)_{i \in \mathbb{N}}$ is constant for all $i \gg 0$. The typical example is $I = \{1 - \frac{1}{m} | m \in \mathbb{N}\}$. We let $I_+ = \{0\} \cup \{i = \sum_{p=1}^{l} i_p | i_1, \dots, i_l \in I\}$ and $D(I) = \{a = \frac{m-1+f}{m},\ m \in \mathbb{N},\ f \in I_+ \cap [0, 1]\}$. It is well known that I is a DCC set if and only if and only if $D(I)$ is a DCC set. Recall the following (see, *e.g.*, [4, 2.7]).

Lemma 1.1 (Shokurov's log adjunction formula). *Let $(X, S+B)$ be a log canonical surface pair where $B = \sum b_i B_i$ and S is a prime divisor with normalization $\nu : S^\nu \to S$, then*

$$(K_X + S + B)|_{S^\nu} = K_{S^\nu} + \operatorname{Diff}_{S^\nu}(B) = K_{S^\nu} + \operatorname{Diff}(0)_{S^\nu} + B|_{S^\nu}$$

where the coefficients of $\operatorname{Diff}_{S^\nu}(B)$ *are* 1 *or of the form* $(n - 1 + \sum a_i b_i)/n \in [0, 1]$ *for some $n, a_i \in \mathbb{N}$. In particular,* $\operatorname{coeff}(\operatorname{Diff}_{S^\nu}(B)) \subseteq D(I)$ *if* $\operatorname{coeff}(B) \subseteq I$.

For later use we recall the following elementary observation.

Lemma 1.2. *Let $(X, S + B)$ be a log canonical surface pair where S is a prime divisor with normalization $\nu : S^\nu \to S$. Then for any $1 \geq \lambda \geq 0$ we have*

$$(K_X + S + \lambda B)|_{S^\nu} \geq K_{S^\nu} + \lambda \mathrm{Diff}_{S^\nu}(B).$$

Proof. Let $B = \sum b_i B_i$. It suffices to show that $(n - 1 + \lambda \sum a_i b_i)/n \geq \lambda(n - 1 + \sum a_i b_i)/n$. □

For $a, b \in \mathbb{R}$ set $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$. Similarly, for $A = \sum a_i A_i$ and $A' = \sum a'_i A_i$ $\mathbb{R}$-divisors set $A \vee A' = \sum(a_i \vee a'_i)A_i$ and $A \wedge A' = \sum(a_i \wedge a'_i)A_i$. A pair (X, B) is a *simple normal crossings pair* or an *snc pair* if X is smooth and the support of B consists of smooth divisors meeting transversely.

Definition 1.3. A scheme X is called *demi-normal* if it is seminormal, S_2 and G_1 or equivalently if it is S_2 and its codimension 1 points are either regular points or nodes (*cf.* [13, 5.1, 10.14]). Let X be a demi-normal scheme with normalization $\pi : \widetilde{X} \to X$ and conductors $D \subset X$ and $\widetilde{D} \subset \widetilde{X}$. Let $B \subset X$ be an effective $\mathbb{Q}$-divisor whose support does not contain any irreducible component of D and $\widetilde{B} \subset \widetilde{X}$ the divisorial part of $\pi^{-1}(B)$.

The pair (X, B) is called *semi-log canonical* or *slc* if X is demi-normal, K_X+B is $\mathbb{Q}$-Cartier and $(\widetilde{X}, \widetilde{B} + \widetilde{D})$ is log canonical. An slc *model* (or *semi log canonical model*) is a projective pair (X, B) which is slc and such that $K_X + B$ is ample.

Let $\pi : X \to U$ be a projective morphism of normal varieties, then by definition $\pi_*\mathscr{O}_X(D) = \pi_*\mathscr{O}_X(\lfloor D \rfloor)$. Given an $\mathbb{R}$-divisor D on a normal projective variety X, the *volume* of D is defined as

$$\mathrm{vol}(D) = \lim_{m\to\infty} \frac{h^0(\mathscr{O}_X(\lfloor mD \rfloor))}{m^n/n!}.$$

If D is nef, then $\mathrm{vol}(D) = D^{\dim X}$. Note that $\mathrm{vol}(\lambda D) = \lambda^{\dim X}\mathrm{vol}(D)$ for any $\lambda > 0$. It is easy to see that if $f : X \to Y$ is a morphism of normal projective varieties, then $\mathrm{vol}(D) \leq \mathrm{vol}(f_*D)$ and if E is an $\mathbb{R}$-Cartier divisor on Y such that $D - f^*E \geq 0$ is f-excepional, then $\mathrm{vol}(E) = \mathrm{vol}(D)$.

Let X be a quasi-projective variety then a *b-divisor* $\mathbf{B}$ over X is given by a collection of divisors $\mathbf{B}_{X'}$ on X' for any birational morphism $X' \to X$ with the property that if $X'' \to X$ is another birational morphism and ν is a valuation corresponding to a divisor on X' and X'', then $\mathrm{mult}_\nu(\mathbf{B}_{X'}) = \mathrm{mult}_\nu(\mathbf{B}_{X''})$. In other words a b-divisor over X is defined by its multiplicity along any divisor over X. Similarly one defines $\mathbb{R}$-b-divisors etc.

Let (X, B) be a pair. Then typical examples of $\mathbb{R}$-b-divisors are as follows.

(1) The *discrepancy b-divisor*, $\mathbf{A} = \mathbf{A}_B$ is defined by the equation

$$K_{X'} = \nu^*(K_X + B) + \mathbf{A}_{B,X'}$$

for any birational morphism $\nu : X' \to X$;

(2) The b-divisor $\mathbf{L} = \mathbf{L}_B$ is defined by the equation

$$K_{X'} + \mathbf{L}_{B,X'} = \nu^*(K_X + B) + E_{B,X'}$$

where $\mathbf{L}_{B,X'}$ and $E_{B,X'}$ are effective with no common components, for any birational morphism $\nu : X' \to X$;

(3) The b-divisor $\mathbf{M} = \mathbf{M}_B$ is defined by $\mathrm{mult}_E(\mathbf{M}) = \mathrm{mult}_E(B)$ if E is a divisor on X and $\mathrm{mult}_E(\mathbf{M}) = 1$ otherwise.

We have (*cf.* [9, 5.3]):

Proposition 1.4. *Let (X, B) be a projective snc pair , $f : Y \to X$ a log resolution of (X, B), and $g : X \to Z$ a birational projective morphism such that (Z, g_*B) is also an snc pair . Then*

(1) $\mathrm{vol}(K_X + B) = \mathrm{vol}(K_Y + \Gamma)$ *for any $\mathbb{R}$-divisor Γ such that $\Gamma - \mathbf{L}_{B,Y} \geq 0$ is f-exceptional;*

(2) $\mathrm{vol}(K_X + B) = \mathrm{vol}(K_X + \Theta)$ *where $\Theta = B \wedge \mathbf{L}_{g_*B,X}$.*

Definition 1.5. We say that a set of varieties $\mathscr{X}$ is *degree bounded* if there exists a constant $m > 0$ such that for each $X \in \mathscr{X}$ there is a very ample divisor H on X with $H^{\dim(X)} < m$. A set of pairs $\mathscr{P}$ is *degree bounded* if there exists an integer $m > 0$ such that for each $(X, B) \in \mathscr{P}$ there is a very ample divisor H on X with $H^{\dim(X)} < m$ and $H^{\dim(X)-1} \cdot B_{\mathrm{red}} < m$. A set of pairs $\mathscr{B}$ is *log birationally degree bounded* if there exists a degree bounded set of pairs $\mathscr{P}$ such that for any $(X, B) \in \mathscr{B}$ there exists a pair $(Z, D) \in \mathscr{X}$ and a birational map $f : Z \dashrightarrow X$ such that D_{red} contains the strict transform of B_{red} and all f-exceptional divisors.

1.2. Ultraproducts

We briefly recall a few results about ultrafilters and ultraproducts that will be needed in what follows. The interested reader may consult [16] and [6] for more background.

We fix $\mathscr{U}$ a *non-principal ultrafilter* on $\mathbb{N}$ for the sequel. So $\mathscr{U}$ is a non-empty collection of infinite subsets of $\mathbb{N}$ such that

(1) If $A \subset B \subset \mathbb{N}$ and $A \in \mathscr{U}$, then $B \in \mathscr{U}$;
(2) If $A, B \in \mathscr{U}$, then $A \cap B \in \mathscr{U}$;
(3) For any $A \subset \mathbb{N}$, either A or $\mathbb{N} \setminus A$ are in $\mathscr{U}$.

We say that a property $P(m)$ holds for *almost all* $m \in \mathbb{N}$ if $\{m \in \mathbb{N} | P(m) \text{ holds}\} \in \mathscr{U}$. Let $(A_m)_{m\in\mathbb{N}}$ be a sequence of rings then the *ultraproduct* $[A_m]$ is defined by $[A_m] := \left(\prod_{m\in\mathbb{N}} A_m\right) / \sim$ where $\sim$ is the equivalence relation defined by $(a_m) \sim (b_m)$ iff $a_m = b_m$ for almost all $m \in \mathbb{N}$. $[a_m] \in [A_m]$ denotes the equivalence class corresponding to the sequence $(a_m)_{m\in\mathbb{N}}$. Note that a_m only needs to be defined for almost all $m \in \mathbb{N}$. If, for almost all $m \in \mathbb{N}$, the A_m are reduced (respectively fields), then so is $[A_m]$. Given a sequence of homomorphisms of rings $f_m : A_m \to B_m$ then $[f_m] : [A_m] \to [B_m]$ is a homomorphism of rings defined by $[f_m]([a_m]) = [f_m(a_m)]$.

Suppose now that L_m is a sequence of fields and $k = [L_m]$. For any fixed integer $n > 0$, we define the ring of *internal polynomials*

$$k[x_1, \dots, x_n]_{\text{int}} = [L_m[x_1, \dots, x_n]] .$$

Note that the name is misleading as the elements of $k[x_1, \dots, x_n]_{\text{int}}$ are not necessarily polynomials. There exists a natural embedding

$$k[x_1, \dots, x_n] \hookrightarrow k[x_1, \dots, x_n]_{\text{int}}$$

whose image is the set of elements $g = [g_m] \in k[x_1, \dots, x_n]_{\text{int}}$ of bounded degree (*i.e.*, such that there exists an integer d with $\deg(g_m) \le d$ for almost all $m \in \mathbb{N}$). For an ideal $\mathfrak{a} \subseteq k[x_1, \dots, x_n]$ we put $\mathfrak{a}_{\text{int}} := \mathfrak{a} \cdot k[x_1, \dots, x_n]_{\text{int}}$.

We have the following:

Theorem 1.6 ([18, Theorem 1.1]). *The extension $k[x_1, \dots, x_n] \hookrightarrow k[x_1, \dots, x_n]_{\text{int}}$ is faithfully flat. In particular, for an ideal $\mathfrak{a} \subseteq k[x_1, \dots, x_n]$, $\mathfrak{a}_{\text{int}} \cap k[x_1, \dots, x_n] = \mathfrak{a}$.*

This theorem implies that the ideals of $k[x_1, \dots, x_n]_{\text{int}}$ generated in bounded degree are in a one-to-one correspondence with the ideals of $k[x_1, \dots, x_n]$, and hence they are all of the form $[\mathfrak{a}_m]$ for a sequence of ideals $\mathfrak{a}_m \subseteq L_m[x_1, \dots, x_n]$, which are all generated in bounded degree.

Given the fields L_m for $m \in \mathbb{N}$ and assuming the above constructions, the symbol $[X_m]$ stands for equivalence classes of sequences of schemes X_m of finite type over L_m with respect to the equivalence relation: $[X_m] \sim [Y_m]$ iff $X_m = Y_m$ for almost all $m \in \mathbb{N}$. This $[X_m]$ is called an *internal scheme* over k. An *internal morphism* $[f_m]$ is defined by (the equivalence class of) a sequence of morphisms $f_m : X_m \to Y_m$ where as usual $[f_m] \sim [g_m]$ iff $f_m = g_m$ for almost all $m \in \mathbb{N}$. Similarly, an *internal coherent sheaf* on an internal scheme $[X_m]$, denoted by the symbol $[\mathscr{F}_m]$, is defined as an equivalence class of sequences of coherent sheaves $\mathscr{F}_m$ on X_m by the usual equivalence relation, $[\mathscr{F}_m] \sim [\mathscr{G}_m]$ iff $\mathscr{F}_m \simeq \mathscr{G}_m$ for almost all $m \in \mathbb{N}$, where $\mathscr{F}_m$ and $\mathscr{G}_m$ are coherent sheaves on X_m for almost all $m \in \mathbb{N}$.

Claim 1.7. There exists a functor $\mathsf{int} : X \mapsto X_{\text{int}}$ from separated schemes of finite type over k to internal schemes.

Construction. First assume that X is affine and define int as follows: let $X \hookrightarrow \mathbb{A}_k^N$ be a closed embedding defined by the ideal $\mathfrak{a} \subset k[x_1, \ldots, x_N]$. As observed above $\mathfrak{a}_{\text{int}} = [\mathfrak{a}_m]$ for an appropriate sequence of ideals, and then we define X_m in $\mathbb{A}_{L_m}^N$ by the ideal $\mathfrak{a}_m$ (it is enough to do this for almost all $m \in \mathbb{N}$). Now we set $X_{\text{int}} := [X_m]$. A similar construction applies to morphisms (for details see [6, pages 1468-1469]) which implies that the above defined X_{int} is independent of the embedding we chose at the beginning and hence the construction is functorial. By patching on an open cover of X we obtain X_{int} in the general case and similarly the same for morphisms. □

Similarly to the functor $X \mapsto X_{\text{int}}$ one may also define a functor $\mathscr{F} \mapsto \mathscr{F}_{\text{int}}$ from coherent sheaves on X to internal coherent sheaves on X_{int}. The construction is relatively straightforward; for details and basic properties see [6, pages 1471-1472].

Notice that since the construction of the functor $X \mapsto X_{\text{int}}$ is based on the defining ideal sheaf of X and hence for a divisor $D \subset X$, the internal subscheme $D_{\text{int}} \subset X_{\text{int}}$ is a divisor with corresponding divisorial sheaf $\mathscr{O}_{X_{\text{int}}}(D_{\text{int}}) \simeq [\mathscr{O}_{X_m}(D_m)]$. In other words, for a divisor, the corresponding *internal divisor* may be obtained either as an internal scheme or an internal coherent sheaf. By [6, 3.5(i)] Cartier divisors correspond to Cartier divisors. An *internal pair* (X, D) consists of an internal scheme X and an internal $\mathbb{Q}$-divisor $D \subset X$. For a pair (X, D) over k, we will use the notation $(X, D)_{\text{int}} := (X_{\text{int}}, D_{\text{int}})$.

Next, we establish the important connection between bounding the degree of a projective variety and bounding the degree of its defining polynomials promised in the introduction.

Theorem 1.8. *There exists a function $\beta : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, such that if $d, q, n \in \mathbb{N}$, L is a field and $X \subseteq \mathbb{P}^n$ is a projective variety over L such that* $\dim X = q$ *and* $\deg X \leq d$*, then $I(X) \subseteq L[x_0, \ldots, x_n]$ can be generated by homogenous polynomials of degree at most $\beta(d, q)$.*

Proof. Note that if $\deg X \leq d$, then there exists a linear subspace $\mathbb{P}^{q+d-1} \simeq P \subseteq \mathbb{P}^n$ such that $X \subseteq P$ and hence we may assume that $n \leq q+d-1$. We utilize an idea of Mumford: For any linear subspace $T \subseteq \mathbb{P}^n$ of dimension $n-q-2$ let H_T denote the join of X and T, *i.e.*, the union of lines determined by pairs of points given by $X \times T$. Alternatively, H_T may be defined as follows: consider the projection map $\pi_T : \mathbb{P}^n \dashrightarrow \mathbb{P}^{q+1}$ and let $H_T := \overline{\pi_T^{-1}\pi_T(X)}$. Then $\deg H_T \leq \deg X \leq d$ and it is easy to see (*cf.* proof of [15, Theorem 1, page 32]) that

$$X \underset{\text{set-theoretically}}{=} \bigcap_{T \cap X = \emptyset} H_T.$$

It follows that there exists an ideal $J \subseteq L[x_0, \ldots, x_n]$ that can be generated by homogenous polynomials of degree at most d and such that $\sqrt{J} = I(X)$. Then the statement follows by [18, Theorem 2.10(ii)]. □

Proposition 1.9. *Fix $d > 0$ and let L_m be a sequence of fields. Let X_m be a sequence of projective varieties defined over L_m of bounded degree and dimension for almost all $m \in \mathbb{N}$. Then there exists a projective variety X defined over $k = [L_m]$ such that $X_{\mathrm{int}} = [X_m]$. Furthermore, this X admits an embedding to a projective space over k such that its ideal sheaf is locally generated by polynomials of bounded degree.*

Proof. Since the X_m have bounded degree and dimension (for almost all $m \in \mathbb{N}$), we may assume that there are fixed integers $d, N > 0$ such that for almost all $m \in \mathbb{N}$, X_m is embedded in $\mathbb{P}^N_{L_m}$ with degree $\leq d$. Let $\mathbb{A}^N_k =: U \subset \mathbb{P}^N_k$ be a standard open affine subset and let $U_{\mathrm{int}} = [U_m]$ be the corresponding internal scheme. Consider $X_m \cap U_m \subseteq U_m \simeq \mathbb{A}^N_{L_m}$ with defining ideal $\mathfrak{a}_m$. It follows from Proposition 1.8 that the $\mathfrak{a}_m$ are generated by polynomials of bounded degree and then so is $\mathfrak{a}_{\mathrm{int}} = [\mathfrak{a}_m] \subseteq k[x_1, \dots, x_N]_{\mathrm{int}}$. Let $\mathfrak{a} := \mathfrak{a}_{\mathrm{int}} \cap k[x_1, \dots, x_N]$ and $X^U = Z(\mathfrak{a}) \subseteq \mathbb{A}^N_k = U$. By the construction we have that $[X_m \cap U_m] = (X^U)_{\mathrm{int}}$. Glueing the various X^U as before we obtain the required closed subscheme $X \subseteq \mathbb{P}^N_k$. □

Lemma 1.10. *Let $X \subset \mathbb{P}^N_k$ where $k = [L_m]$ and $\mathscr{L}$ a semiample line bundle on X. If $X_{\mathrm{int}} = [X_m]$ and $\mathscr{L}_{\mathrm{int}} = [\mathscr{L}_m]$, then $\mathscr{L}_m$ is semiample for almost all $m \in \mathbb{N}$.*

Proof. Since $\mathscr{L}$ is semiample, there is an integer $r > 0$ such that $\mathscr{L}^{\otimes r}$ defines a morphism $\phi : X \to \mathbb{P}^M_k$ with $\phi^*\mathscr{O}_{\mathbb{P}^M_k}(1) \simeq \mathscr{L}^{\otimes r}$. Since $(\mathbb{P}^M_k)_{\mathrm{int}} = [\mathbb{P}^M_{L_m}]$ and $(\mathscr{O}_{\mathbb{P}^M_k}(1))_{\mathrm{int}} = [\mathscr{O}_{\mathbb{P}^M_{L_m}}(1)]$, it follows that $\mathscr{L}^{\otimes r}_m = \phi^*_m(\mathscr{O}_{\mathbb{P}^M_{L_m}}(1))$ for almost all $m \in \mathbb{N}$ (where $(\phi)_{\mathrm{int}} = [\phi_m]$) by [6, 3.5(iii)]. Thus $\mathscr{L}_m$ is semiample for almost all $m \in \mathbb{N}$. □

Lemma 1.11. *Let (X, B) be a log canonical pair projective over $k = \bar{k} = [L_m]$ and let $\nu : X \dashrightarrow Y$ be a good minimal model (respectively log canonical model), then (X_m, B_m) is a log canonical pair projective over L_m and $\nu_m : X_m \dashrightarrow Y_m$ is a good minimal model (respectively log canonical model) for (X_m, B_m) for almost all $m \in \mathbb{N}$, where $\nu_{\mathrm{int}} = [\nu_m]$, $Y_{\mathrm{int}} = [Y_m]$ and $B_{\mathrm{int}} = [B_m]$.*

Proof. Since $r(K_X + B)$ is Cartier, so is $r(K_{X_m} + B_m)$ for almost all $m \in \mathbb{N}$ by [6, 3.5(i)]. Let $\mu : X' \to X$ be a log resolution of (X, B), then following [6, Proof of 4.1], $\mu_m : X'_m \to X_m$ is a log resolution for almost all $m \in \mathbb{N}$ and $(K_{X'/X})_{\mathrm{int}} = [K_{X'_m/X_m}]$. Since the coefficients of $\mathbf{A}_{X'} = K_{X'} - \mu^*(K_X + B)$ are ≥ -1, the same holds for the coefficients of $\mathbf{A}_{X'_m} = K_{X'_m} - \mu^*_m(K_{X_m} + B_m)$ for almost all $m \in \mathbb{N}$ (*cf.* [6, Section 3]). We only discuss the case of good minimal models (the other case is very similar). By assumption $K_Y + B_Y$ is semiample where $B_Y = \nu_* B$. By Lemma 1.10, $K_{Y_m} + B_{Y_m}$ is semiample for almost all $m \in \mathbb{N}$. Let $p : W \to X$ and $q : W \to Y$ resolve ν, then $p^*(K_X + B) \geq q^*(K_Y + B_Y)$ where the inequality is strict along ν-exceptional divisors on Y. But then, if $p_m : W_m \to X_m$ and $q_m : W_m \to Y_m$ are the corresponding morphisms, we have $p^*_m(K_{X_m} + B_m) \geq q^*_m(K_{Y_m} + B_{Y_m})$ where the inequality is strict along ν_m-exceptional divisors on Y_m. □

1.3. Effective Matsusaka and birational boundedness

We begin by recalling the following effective version of Matsusaka's theorem and a vanishing theorem due to di Cerbo, Fanelli and Terakawa.

Theorem 1.12. *Let X be a smooth surface over a field of characteristic $p > 0$ and D a big and nef Cartier divisor on X. Let $q_0 := (2\mathrm{vol}(K_X) + 9)/(p-1)$, then $H^i(\mathcal{O}_X(K_X + qD)) = 0$ for all $i > 0$ and $q > q_0$.*

Proof. See [17] and [7, 5.9]. □

Theorem 1.13. *Let X be a smooth surface and D be a big and nef Cartier divisor. Let $l = D^2 - 5$ (respectively $l = D^2 - 9$) with $l \geq 0$ then if $|K_X + D|$ has a base point at $x \in X$ (respectively $|K_X + D|$ does not separate points $x, y \in X$) then*

1. *If X is not of general type nor quasi-elliptic of Kodaira dimension 1, then there exists a curve $C \subset X$ containing x (respectively containing one of the points x, y) such that $D \cdot C < 2$ (respectively $D \cdot C < 4$);*
2. *If X is of general type with $D^2 > \mathrm{vol}(K_X)+6$ (respectively $D^2 > \mathrm{vol}(K_X)+9$) or is quasi-elliptic of Kodaira dimension 1 then there exists a curve $C \subset X$ containing x (respectively containing one of the points x, y) such that $D \cdot C \leq 7$ (respectively $D \cdot C \leq 17$).*

Proof. See the main theorem of [17] and [7, 4.9, 4.11]. □

Corollary 1.14. *Let X be a normal surface and D a nef and big Cartier divisor such that $D^2 \geq \mathrm{vol}(K_X)$, then $|K_X + qD|$ defines a birational morphism for any $q \geq 18$.*

Proof. Let $\mu : X' \to X$ be a resolution and $D' = \mu^* D$. Pick a general point $x \in X'$ (respectively general points $x, y \in X'$). If C is a curve on X' containing x (respectively containing x or y), then $qD' \cdot C \geq q \geq 18$ and $(qD')^2 = (qD)^2 \geq \mathrm{vol}(K_X) + 10 \geq \mathrm{vol}(K_{X'}) + 10$. By Theorem 1.13, x is not a base point of $|K_{X'} + qD'|$ and $|K_{X'} + qD'|$ separates x and y. Thus $|K_X + qD|$ defines a birational morphism. □

We will also need the following result which is analogous to [9, 3.1].

Theorem 1.15. *Fix $A \in \mathbb{N}$, $\delta > 0$. Let (X, B) be a log canonical surface such that the coefficients of B are $\geq \delta$, $\mathrm{vol}(q(K_X + B)) \leq A$ and $|K_X + q(K_X + B)|$ is birational for some $q > 0$, then (X, B) is log birationally degree bounded.*

Proof. The proof follows [9, 3.1]. For the convenience of the reader we include a sketch which highlights the main changes necessary to avoid the use of Kawamata-Viehweg vanishing.

By a standard reduction (*cf.* [9, 3.1]) we may assume that X is smooth and $|K_X + q(K_X + B)|$ induces a morphism $\phi : X \to Z$, *i.e.*, $|K_X + q(K_X + B)| = |M| + E$ where $|M|$ is basepoint-free. Let H be a very ample Cartier divisor on Z

so that $M = \phi^* H$. It suffices to show that H^2 and $\phi_* B_{\mathrm{red}} \cdot H$ are bounded from above.

Clearly

$$H^2 = M^2 = \mathrm{vol}(M) \le \mathrm{vol}(K_X + q(K_X + B)) \le (q+1)^2 \mathrm{vol}(K_X + B) \le 2^2 A.$$

Let D_0 be the sum of the components of B that are not ϕ-exceptional. Note that if $G \in |M|$, then there is an effective $\mathbb{Q}$-divisor $C = E + B - \delta D_0 \ge 0$ such that

$$\delta D_0 + G + C \sim_{\mathbb{Q}} (q+1)(K_X + B). \tag{1.1}$$

Let $\alpha = 2A + 10$. Since $B \ge 0$ and $q > 0$,

$$\mathrm{vol}(q(K_X + B)) \ge \mathrm{vol}(K_X + B) \ge \mathrm{vol}(K_X)$$

and so $\alpha \ge 2\mathrm{vol}(K_X) + 10$. Then

$$\begin{aligned}\phi_* D_0 \cdot H &= D_0 \cdot G \le D_0 \cdot 2\alpha G \le 4\mathrm{vol}(K_X + D_0 + 2\alpha G)\\ &\le 4\mathrm{vol}\left(\left(1 + \left(\frac{1}{\delta} + 2\alpha\right)(q+1)\right)(K_X + B)\right) \le 16\left(1 + \frac{1}{\delta} + 2\alpha\right)^2 A.\end{aligned} \tag{1.2}$$

Here the first (in)equality follows as $G \sim M = \phi^* H$, the second is trivial, the third by Lemma 1.16 below, the fourth by (1.1), and the fifth since $\frac{1+(1/\delta+2\alpha)(q+1)}{q} \le 2(1 + 1/\delta + 2\alpha)$. □

Lemma 1.16. *Let X be a normal surface, M a Cartier divisor such that $|M|$ is base point free and the induced map $\phi = \phi_{|M|} : X \to Z$ is birational. Let $L = 2\alpha M$ for some integer $\alpha \ge \mathrm{vol}(K_X) + 10$ and D be a sum of distinct prime divisors, then*

$$D \cdot L \le 4\mathrm{vol}(K_X + D + L).$$

Proof. By standard reductions, we may assume that (X, D) is an snc pair and the components of D are disjoint and not ϕ-exceptional (*cf.* [9, 3.2]). Now consider the short exact sequence

$$0 \to \mathscr{O}_X(K_X + mL) \to \mathscr{O}_X(K_X + mL + D) \to \mathscr{O}_D(K_D + mL|_D) \to 0. \tag{1.3}$$

Since $R^1\phi_*\mathscr{O}_X(K_X + mL) = 0$ by the Grauert-Riemenschneider vanishing theorem (see [13, 10.4] for a version that applies here) using the projection formula and Serre vanishing, it follows that

$$H^1(\mathscr{O}_X(K_X + mL)) = H^1(\phi_*\mathscr{O}_X(K_X + mL)) = H^1(\phi_*\mathscr{O}_X(K_X) \otimes \mathscr{O}_Z(mH)) = 0$$

for all $m \gg 0$ and hence

$$H^0(\mathscr{O}_X(K_X + mL + D)) \to H^0(\mathscr{O}_D(K_D + mL|_D)) \tag{1.4}$$

is surjective.

Claim 1.17. It follows that $h^0(\mathcal{O}_X(K_X+L))>0$ and no component of D is contained in the base locus of $|K_X+\alpha M+D|$.

Proof. Since $\alpha \geq 2\mathrm{vol}(K_X)+10$, by Theorem 1.12, $H^1(\mathcal{O}_X(K_X+\alpha M))=0$ and hence

$$H^0(\mathcal{O}_X(K_X+\alpha M+D)) \to H^0(\mathcal{O}_D(K_D+\alpha M|_D)) = \oplus H^0(\mathcal{O}_{D_i}(K_{D_i}+\alpha M|_{D_i}))$$

is surjective where $D=\sum D_i$ and each D_i is a prime divisor. Since the components of D are not ϕ exceptional, $M\cdot D_i>0$, $H^0(\mathcal{O}_{D_i}(K_{D_i}+\alpha M|_{D_i}))\neq 0$ for all i and so a general element of $H^0(\mathcal{O}_X(K_X+\alpha M+D))$ does not vanish along any component of D.

The proof that $h^0(\mathcal{O}_X(K_X+L))>0$ is similar (and easier). □

Now consider the following commutative diagram

$$\begin{array}{ccc} \mathcal{O}_X(K_X+mL+D) & \longrightarrow & \mathcal{O}_D(K_D+mL|_D) \\ \downarrow & & \downarrow \\ \mathcal{O}_X((2m-1)(K_X+L+D)) & \longrightarrow & \mathcal{O}_D((2m-1)(K_D+L|_D)) \end{array}$$

where the vertical maps are induced by a general divisor in $|(m-1)(2K_X+L+2D)|=|2(m-1)(K_X+\alpha M+D)|$. Since no component of D is contained in the support of this divisor (Claim 1.17), it follows that

$$\begin{aligned} & h^0(\mathcal{O}_X((2m-1)(K_X+L+D))) \\ & \quad - h^0(\mathcal{O}_X((2m-2)(K_X+L+D)+K_X+L)) \\ &= \dim \mathrm{Im}\Big[H^0(\mathcal{O}_X((2m-1)(K_X+L+D))) \\ & \qquad \to H^0(\mathcal{O}_D((2m-1)(K_D+L|_D)))\Big] \\ &\geq h^0(\mathcal{O}_D(K_D+mL|_D)). \end{aligned} \tag{1.5}$$

Let $P(m)=h^0(\mathcal{O}_X(m(K_X+L+D)))$, then $P(m)=m^2\mathrm{vol}(K_X+L+D)+o(m^2)$. Since $h^0(\mathcal{O}_X(K_X+L))>0$ (see Claim 1.17), we have $h^0(\mathcal{O}_X((2m-2)(K_X+L+D)+K_X+L))\geq P(2m-2)$ and hence by (1.5), we have

$$\begin{aligned} P(2m-1)-P(2m-2) \geq Q(m) &:= h^0(\mathcal{O}_D(K_D+mL|_D)) \\ &= mL\cdot D+o(m). \end{aligned} \tag{1.6}$$

Comparing leading terms of $P(m)$ and $Q(m)$, it follows that

$$4\mathrm{vol}(K_X+L+D)\geq L\cdot D.$$ □

2. The proofs of the main results

2.1. Preliminary results

Lemma 2.1. *Fix $\mathscr{C} \subset [0, 1]$ a DCC set, then there exists a constant $V > 0$ such that if (X, B) is a klt surface such that $\rho(X) = 1$, $\mathrm{coeff}(B) \subset \mathscr{C}$ and $K_X + B \sim_{\mathbb{Q}} 0$, then $(-K_X)^2 \leq V$.*

Proof. Suppose that $(-K_X)^2 > V$, then for any smooth point $x \in X$ there exists a $\mathbb{Q}$-divisor $G \sim_{\mathbb{Q}} -K_X$ such that $\mathrm{mult}_x(G) > V^{1/2}$ (*cf.* [14, 1.1.31]). Since $\rho(X) = 1$, we may assume that all components of G contain a general point $x \in X$ and in particular are not contained in the support of B. Let $\Phi = (1-\delta)B + \delta G$ such that (X, Φ) is log canonical but not klt. Notice that $0 < \delta < 2/V^{1/2}$ (*cf.* [14, 9.3.2]). Perturbing G we may in fact assume that there is a unique non-klt center Z for (X, Φ).

If Z is a divisor, then (since $\rho(X) = 1$) we may assume that $\delta G = Z$. Restricting to Z we have

$$0 \equiv (K_X + (1-\delta)B + \delta G)|_Z = K_Z + \mathrm{Diff}_Z((1-\delta)B).$$

Since $\deg \mathrm{Diff}_Z((1-\delta)B) \geq 0$ then $\deg(K_Z) \in \{0, -2\}$. If $B \neq 0$, then

$$2 = \deg \mathrm{Diff}_Z((1-\delta)B)$$

(see Lemma 1.1) easily implies that δ is bounded from below (*cf.* [10, 5.2]) and hence $(-K_X)^2$ is bounded from above. If $B = 0$ then $K_X \equiv 0$ and the claim is trivial.

Therefore we may assume that $\dim Z = 0$. Let $\nu : X' \to X$ be the extraction of the corresponding curve E of discrepancy -1 so that $K_{X'} + E + \Phi' = \nu^*(K_X + \Phi) \sim_{\mathbb{Q}} 0$ where $\Phi' = \nu_*^{-1}\Phi$. Write $K_{X'} + B' + aE = \nu^*(K_X + B) \sim_{\mathbb{Q}} 0$ where $a < 1$. We run the first step of the $K_{X'} + \Phi' \equiv -E$ minimal model program. If the induced rational map is a Mori fiber space $X' \to W$, then restricting to a general fiber F we let $\Phi'' = \Phi'|_F$, $E'' = E|_F$ and $B'' = B'|_F$, and if the induced rational map is a divisorial contraction $\pi : X' \to F$, then we let Φ'', E'' and B'' be the pushforwards of Φ', E and B'. We have that

$$K_F + \Phi'' + E'' \equiv 0, \qquad K_F + B'' + aE'' \equiv 0,$$

$K_F + B'' + E'' \equiv (1-a)E''$ is ample and since $\Phi'' \geq (1-\delta)B''$, then

$$\delta B'' \equiv (1-a)E'' + \Phi'' - (1-\delta)B'' \geq 0.$$

It follows that $B'' \neq 0$ and $K_F + (1-\eta)B'' + E'' \equiv 0$ for some $0 < \eta < \delta$.

If $\dim F = 1$, then since the coefficients of B'' are in the DCC set $\mathscr{C}$, there exists a constant $\beta > 0$ such that $\deg(\eta B'') = \deg(K_F + B'' + E'') \geq \beta$. But then, since $K_F + B'' + aE'' \equiv 0$, we have

$$2 \geq \deg(B'') \geq \beta/\eta > \beta/\delta \geq \beta V^{1/2}/2$$

and so V and hence $(-K_X)^2$ are bounded from above.

If $\dim F = 2$, let $(K_F + B'' + E'')|_{E''} = K_{E''} + \mathrm{Diff}_{E''}(B'')$. Since (F, E'') is purely log terminal, then E'' is smooth and by adjunction the coefficients of $\mathrm{Diff}_{E''}(B'')$ are in the DCC set $D(\mathscr{C})$ and so

$$-2 + \deg\big(\mathrm{Diff}_{E''}(B'')\big) = \big(K_F + B'' + E''\big) \cdot E'' \geq \beta - 2 > 0$$

where $\beta = \min\{\sum b'_i | b'_i \in D(\mathscr{C}), \sum b'_i > 2\}$. Fix λ such that $K_{E''} + \lambda \mathrm{Diff}_{E''}(B'') \equiv 0$, then as E'' is rational,

$$\lambda = \frac{2}{\deg\big(\mathrm{Diff}_{E''}(B'')\big)} \leq \frac{2}{\beta} < 1.$$

However, by Lemma 1.2, we have

$$\big(K_F + \lambda B'' + E''\big)|_{E''} \geq K_{E''} + \lambda \mathrm{Diff}_{E''}\big(B''\big) \equiv 0 \equiv \big(K_F + (1 - \eta)B'' + E''\big)|_{E''}$$

and so $1 - \delta < 1 - \eta \leq \lambda \leq 2/\beta < 1$. But then

$$0 < 1 - \frac{2}{\beta} < \delta < \frac{2}{\sqrt{V}} < \frac{2}{\sqrt{(-K_X)^2}},$$

which implies that $(-K_X)^2$ is bounded from above. □

Remark 2.2. The following two auxilliary results, (2.3) and (2.4), were also proved by Alexeev and Alexeev-Mori. In fact, an effective version of (2.4) is proven in [5, 4.6]. Here we provide independent proofs.

Lemma 2.3. *Fix $\mathscr{C} \subset [0, 1]$ a DCC set, then there exists an $\varepsilon > 0$ such that if (X, B) is a projective klt surface such that $\rho(X) = 1$,* coeff$(B) \subset \mathscr{C}$ *and $K_X + B \sim_{\mathbb{Q}} 0$, then (X, B) is ε Kawamata log terminal.*

Proof. Suppose that the claim is false. Then there is a sequence of pairs (X_n, B_n) as above with total discrepancy $a(X_n, B_n) = \varepsilon_n - 1$ such that ε_n is a decreasing sequence with limit 0. Let $\mathscr{C}' = \mathscr{C} \cup \{1 - \varepsilon_n\}_{n \in \mathbb{N}}$ then $\mathscr{C}'$ is a DCC set. Suppose that (X_n, B_n) does not contain a component of coefficient $1 - \varepsilon_n$. Let $\nu : X' \to X = X_n$ be a projective birational morphism extracting the corresponding divisor E so that $\rho(X'/X) = 1$ and the exceptional divisor is E. We may write $K_{X'} + B' + eE = \nu^*(K_X + B)$ where $e = 1 - \varepsilon_n$. Since $\rho(X') = 2$ there is a second extremal ray R_2 (here $R_1 = [E]$). Since $(K_{X'} + B') \cdot R_2 = -eE \cdot R_2 < 0$, it follows that R_2 is $K_{X'} + B'$ negative and hence it can be contracted. Let $\mu : X' \to X''$ be the corresponding contraction. If $\dim X'' = 1$, then let $F \simeq \mathbb{P}^1$ be a general fiber. We have

$$0 = \deg\big(K_{X'} + B' + eE\big)|_F = -2 + \big(B' + eE\big) \cdot F = -2 + \sum b_i + e.$$

Since $b_i \in \mathscr{C}$, it is easy to see that $e = 1 - \varepsilon_n$ is constant for $n \gg 0$ which is impossible. Therefore, we may assume that $X' \to X''$ is a birational contraction.

Then $K_{X''} + B'' + eS = \mu_*(K_{X'} + B' + eE) \sim_{\mathbb{Q}} 0$. Replacing X by X'' and B by $B'' + eS$, we may assume that B contains a component S of coefficient $e = 1 - \varepsilon_n$.

Write $B = B' + eS$. We then have

$$\varepsilon S^2 = (1-e)S^2 = \big(K_X + B' + S\big) \cdot S = \deg\big(K_S + \mathrm{Diff}_S(B')\big) \geq \beta - 2,$$

where $\beta = \min\{\sum b_i' | b_i' \in D(\mathscr{C}'),\ \sum b_i' > 2\} > 2$. But then

$$(-K_X)^2 = B^2 \geq (1-\varepsilon)^2 S^2 \geq \frac{(1-\varepsilon)^2}{\varepsilon}(\beta - 2),$$

where $\lim_{n\to\infty}(1-\varepsilon_n)^2(\beta-2)/\varepsilon_n = +\infty$ contradicting Lemma 2.1. □

Lemma 2.4. *Fix $\mathscr{C} \subset [0,1]$ a DCC set, then there exists a constant $\delta > 0$ such that if $(X, B = \sum_{i=1}^r b_i B_i)$ is a klt surface such that $K_X + B$ is big, and $b_i \in \mathscr{C}$, then $K_X + (1-\delta)B$ is big.*

Proof. If this were not the case, then there is a sequence of klt surfaces (X_n, B_n) and a decreasing sequence of numbers $\delta_n > 0$ such that $\lim \delta_n = 0$ and $\kappa(K_{X_n} + (1-\delta_n)B_n) \in \{0, 1\}$. After running a $(K_{X_n} + (1-\delta_n)B_n)$-minimal model program, we may assume that $K_{X_n} + (1-\delta_n)B_n$ is nef. Now we run a K_{X_n}-minimal model program. After finitely many divisorial contractions, we may assume that we have a Mori fiber space $f : X_n' \to Z_n$. Since each divisorial contraction is automatically $K_{X_n} + (1-\delta_n)B_n$-trivial (see [10, 5.1, 5.2]), we may assume that $K_{X_n'} + (1-\delta_n)B_n'$ is nef and f is $K_{X_n'} + (1-\delta_n)B_n'$-trivial.

If $\dim Z_n = 1$, let $F_n \simeq \mathbb{P}^1$ be a general fiber. We have

$$0 = \big(K_{X_n'} + (1-\delta_n)B_n'\big) \cdot F_n = -2 + (1-\delta_n)\sum n_i b_i,$$

where $b_i \in \mathscr{C}$ and $n_i \in \mathbb{N}$. Note that $(1-\delta_n)B_n' \cdot F_n \neq 0$. Therefore $2/(1-\delta_n)$ is a decreasing sequence contained in the DCC set $\{\sum n_i b_i | n_i \in \mathbb{N},\ b_i \in \mathscr{C}\}$. Thus δ_n is evenytually constant as required.

If $\dim Z_n = 0$, then $\rho(X_n') = 1$ and $-K_{X_n'}$ is ample. Since $K_{X_n'} + (1-\delta_n)B_n' \equiv 0$ and the coefficients of $(1-\delta_n)B_n'$ belong to a DCC set, say $\mathscr{C}'$, by Lemma 2.3 there exists an $\varepsilon > 0$ such that each $(X_n', (1-\delta_n)B_n')$ is ε-klt and so by Lemmas 2.6 and 2.5, there is an integer $N > 0$ such that $NK_{X_n'}$ is Cartier. Now consider

$$N\big(-K_{X_n'}\big)^2 = -(1-\delta_n)B_n' \cdot NK_{X_n'}.$$

Since $NK_{X_n'}$ is Cartier (and $K_{X_n'}$ is a Weil divisor), by Lemma 2.1 $N(-K_{X_n'})^2 \in \{1, \ldots, NV\}$ a finite set of positive integers. Therefore, after passing to a subsequence, we may assume that $N(-K_{X_n'})^2$ is constant. But then $N(-K_{X_n'})^2/(1-\delta_n)$ cannot be an integer for $n \gg 0$ and this is a contradiction since $B_n' \cdot (-NK_{X_n'}) \in \mathbb{Z}$. □

Lemma 2.5. *Fix $\varepsilon > 0$ then there exists a constant $\varrho = \varrho(\varepsilon)$ such that if (X, B) is a projective ε-log canonical surface and $-(K_X + B)$ is nef, then* rk Pic$(X) \leq \varrho$. *In particular the number of exceptional divisors of negative discrepancy $a_E(X, B) < 0$ is at most ϱ.*

Proof. Let $f : X' \to X$ be a projective birational morphism such that $K_{X'} + B' = f^*(K_X + B)$ where $B' \geq 0$ and $a_E(X', B') \geq 0$ for any divisor E exceptional over X' (in other words f extracts precisely the divisors of negative discrepancy $a_E(X, B) < 0$). Clearly $-(K_{X'} + B')$ is nef, X' is smooth and coeff$(B') \in (0, 1 - \varepsilon]$. By [4, Theorem 6.3] (see also [5, Theorem 1.8]) there exists a constant $\varrho = \varrho(\varepsilon)$ such that $\rho(X) \leq \rho(X') \leq \varrho$. Finally the number of exceptional divisors of negative discrepancy is just $\rho(X') - \rho(X) \leq \rho(X') - 1$ and the lemma follows. □

Lemma 2.6. *Fix $k \in \mathbb{N}$ and $\varepsilon > 0$. There exists an integer $N = N(k, \varepsilon)$ such that if (X, B) is an ε-klt surface singularity such that the number of exceptional divisors of discrepancy $a_E(X, B) < 0$ is $\leq k$ then NK_X is Cartier and NG is Cartier for any integral Weil divisor G contained in the support of B.*

Proof. (See also [4] and [5]) Let $\nu : X' \to X$ be a partial resolution extracting all divisors of discrepancy $a_E(X, B) < 0$, in particular X' has at most du Val singularities which are not contained in the support of B' where $K_{X'} + B' = \nu^*(K_X + B)$. By the classification of klt singularities [1], the weights of each curve in the corresponding graph are bounded by $2/\varepsilon$ (*cf.* [4, Proof of 7.5]) and so there are only finitely many possibilities for the corresponding graph. Let $G = K_X$ or G be a component of the support of B and G' its strict transform. Then we may write $\nu^*G = G' + \sum e_i E_i$ where the E_i are exceptional divisors and the denominators of the e_i divide $t = |\det(E_k \cdot E_{k'})|$. But then $t(G' + \sum e_i E_i)$ is integral. Since X' has only du Val singularities, $t\nu^*G$ is Cartier. By the Basepoint-free theorem tG is Cartier (*cf.* the proof of [5, 4.7]). □

2.2. Proof of Theorem 2

We follow some ideas from [5] and [4] applying techniques from [6]. We may assume that $\mathscr{C} \supset \{1 - \frac{1}{a} | a \in \mathbb{N}\} \cup \{1\}$. Note that it suffices to prove the theorem for log canonical pairs. To see this, consider an slc model (X, B) and its normalization $\nu : \cup X_i \to X$. Writing $K_{X_i} + B_i = (\nu|_{X_i})^*(K_X + B)$, we have log canonical models (X_i, B_i) such that coeff$(B_i) \in \mathscr{C}$ and $(K_X + B)^2 = \sum (K_{X_i} + B_i)^2$. The claim now follows easily since if $\mathscr{D}$ is a DCC set, then so is $\mathscr{D}' = \{\sum d_i | d_i \in \mathscr{D}\}$.

Suppose now, by way of contradiction, that (X_m, B_m) is a sequence of slc surfaces defined over the algebraically closed field L_m of characteristic $p_m > 0$, such that coeff$(B_m) \subseteq \mathscr{C}$ and

$$\mathrm{vol}(K_{X_m} + B_m) > \mathrm{vol}(K_{X_{m+1}} + B_{m+1}). \tag{2.1}$$

In particular we may fix a constant $V > 0$ such that vol$(K_{X_m} + B_m) \leq V$ for all $m \in \mathbb{N}$.

Passing to a log resolution, we may assume that (X_m, B_m) is an snc pair. In fact, given a birational morphism $X'_m \to X_m$ let B'_m be the strict transform of B_m plus the exceptional divisor so that $\mathrm{vol}(K_{X'_m} + B'_m) = \mathrm{vol}(K_{X_m} + B_m)$ (*cf.* Proposition 1.4) and $\mathrm{coeff}(B'_m) \subseteq \mathscr{C}$. Then we replace (X_m, B_m) by (X'_m, B'_m). Since (X_m, B_m) is a snc pair and bigness is an open condition, replacing the coefficients that equal 1 by $1 - \frac{1}{r}$ for some $r \gg 0$, we may assume that (X_m, B_m) is klt.

Claim 2.7. [4, 7.6] We may assume that the pairs (X_m, B_m) are log birationally bounded, *i.e.*, there exists a constant $d > 0$ and birational maps $f_m : X_m \dashrightarrow Z_m$ and very ample divisors H_m on Z_m such that $H_m^2 \leq d$ and $H_m \cdot B_{Z_m} \leq d$ where B_{Z_m} is the sum of the strict transform of B_m and the $Z_m \dashrightarrow X_m$ exceptional divisors.

By Lemma 2.4 it follows easily that there is a finite set of rational numbers $\mathscr{C}'$ depending only on $\mathscr{C}$ and divisors $0 \leq D_m \leq B_m$ such that $K_{X_m} + D_m$ is big and $\mathrm{coeff}(D_m) \subseteq \mathscr{C}'$. By [4, 7.3] we may also assume that the number of components of B_m is bounded by a constant (depending only on $\mathscr{C}$). Let $\mu_m : X_m \to X'_m$ be a minimal model for $K_{X_m} + D_m$ and $D'_m = (\mu_m)_* D_m \leq B'_m = (\mu_m)_* B_m$. Note that $K_{X'_m} + D'_m$ is klt and big. Let $f'_m : X'_m \to Z_m$ be the be the corresponding log canonical model for $K_{X_m} + D_m$ and $f_m : X_m \to Z_m$ the induced morphism. Since the number of components of D_m is bounded, it follows easily that the number of divisors E over Z_m of discrepancy $a_E(Z_m, (f_m)_* D_m) < 0$ is bounded from above. By Lemma 2.6, there exists an integer $N > 0$ depending only on $\mathscr{C}$ and V such that $G_m = N(K_{Z_m} + (f_m)_* D_m)$ is ample and Cartier. By Corollary 1.14, $|K_{X_m} + q f_m^* G_m|$ is birational for all $q \geq 18$ and hence so is $|K_{X_m} + 18N(K_{X_m} + D_m)|$. Since $B_m \geq D_m$, it follows that $|K_{X_m} + 18N(K_{X_m} + B_m)|$ is birational. Since $\mathrm{vol}(18N(K_{X_m} + B_m)) \leq (18N)^2 V$, by Theorem 1.15 (with $q = 18N$) it follows that the pairs (X_m, B_m) are log birationally degree bounded.

Claim 2.8. We may assume that $f_m : X_m \to Z_m$ is a morphism given by a finite sequence of blow ups along smooth strata of $(Z_m, \widehat{B}_{Z_m})$ where $\widehat{B}_{Z_m} = (B_{Z_m})_{\mathrm{red}}$ and that $(Z_m, \widehat{B}_{Z_m})$ is an snc pair and is degree bounded.

Proof. Let $([Z_m], [\widehat{B}_{Z_m}])$ be the internal pair associated to the sequence of pairs $(Z_m, \widehat{B}_{Z_m})$ where $\widehat{B}_{Z_m} = (B_{Z_m})_{\mathrm{red}}$. Since Z_m and $\widehat{B}_{Z_m}$ are degree bounded, it follows by Proposition 1.9 that there exists a pair $(Z, \widehat{B})$ defined over $k = [L_m]$ such that $(Z, \widehat{B})_{\mathrm{int}} = ([Z_m], [\widehat{B}_{Z_m}])$. Let $\nu : Z' \to Z$ be a log resolution of $(Z, \widehat{B})$ and $\widehat{B}' = \nu_*^{-1} B + \mathrm{Ex}(\nu)$. If $(Z', \widehat{B}')_{\mathrm{int}} = ([Z'_m], [\widehat{B}'_{Z'_m}])$, then it is easy to see that Z'_m and $\widehat{B}'_{Z'_m}$ are degree bounded (for almost all $m \in \mathbb{N}$). Replacing (X_m, B_m) by an appropriate birational model (X'_m, B'_m), we may assume that $f'_m : X'_m \to Z'_m$ is a morphism with $(f'_m)_*(B'_m) \leq \widehat{B}'_{Z'_m}$. Replacing (X_m, B_m) by (X'_m, B'_m) and $X_m \to Z_m$ by $X'_m \to Z'_m$ we may assume that f_m is a morphism, (Z_m, B_{Z_m}) is an snc pair and is degree bounded.

Let $(Z, \widehat{B})$ be the projective pair (over k) defined above so that $(Z, \widehat{B})_{\mathrm{int}} = ([Z_m], [\widehat{B}_{Z_m}])$ where as above $\widehat{B}$ and $\widehat{B}_{Z_m}$ denote the reduced divisors. Let $X''_m \to Z_m$ be a finite sequence of strata such that every divisor E on X_m of discrepancy

$a_E(Z_m, B_{Z_m}) < 0$ is a divisor on X''_m. Let B''_m be the strict transform of B_m plus the sum of all $X''_m \to Z_m$ exceptional divisors which are not also $X''_m \dashrightarrow X_m$ exceptional, taken with coefficient $1 - 1/r$ for some $r \gg 0$. Then one can see easily that $\mathrm{vol}(K_{X_m} + B_m) = \mathrm{vol}(K_{X''_m} + B''_m)$ (*cf.* Proposition 1.4). Replacing (X_m, B_m) by the pair (X''_m, B''_m), we may therefore assume that each X_m is obtained from Z_m via a finite sequence of blow ups along smooth strata of (Z_m, B_{Z_m}). □

For almost all $m \in \mathbb{N}$, the strata of $(Z_m, \widehat{B}_{Z_m})$ are in one-to-one correspondence with the strata of $(Z, \widehat{B})$ (*cf.* [6, 3.8]). Therefore, we define (X^m, B^m) by blowing up the corresponding strata on $(Z, \widehat{B})$ and choosing the coefficients of B^m to match those of B_m. Let ν be any divisorial valuation over Z. Since the coefficients belong to a DCC set, after passing to a subsequence, we may assume that the sequence $\mathbf{M}_{B^m}(\nu)$ is non decreasing and hence that $\lim \mathbf{M}_{B^m}(\nu)$ exists. Notice that if $\mathbf{M}_{B^m}(\nu) \neq 0$, then ν corresponds to either a component of $\widehat{B}$ or to a divisor exceptional over Z. If moreover $\mathbf{M}_{B^m}(\nu) \notin \{0, 1\}$ then the corresponding divisor is obtained by blowing up Z along some strata of $\widehat{B}$. Therefore, there are only countably many divisorial valuations ν for which $\mathbf{M}_{B^m}(\nu) \neq \mathbf{M}_{B^k}(\nu)$. By a standard diagonalization argument, we may assume that after passing to a subsequence, there is a well defined b-divisor over Z defined by $\mathbf{B}(\nu) = \lim \mathbf{M}_{B^m}(\nu)$ for any valuation ν over Z. Let $\Phi := \mathbf{B}_Z$. Let B^m_Z be the pushforward of B^m to Z, then $\mathbf{B}_Z = \lim B^m_Z$ Since $\mathscr{C}$ satisfies the DCC, it follows that

$$B^m_Z \leq \Phi \qquad \text{for almost all } m \in \mathbb{N}. \tag{2.2}$$

Claim 2.9. We may assume that

$$\mathbf{L}_\Phi \leq \mathbf{B}, \tag{2.3}$$

where $\Phi = \mathbf{B}_Z$.

Proof. We follow the proof of [9, 5.7] checking that our choices do not affect the volume of $K_{X_m} + B_m$. Let $(Z', \mathbf{B}')$ be the reduction of $(Z, \mathbf{B})$ defined in [9, 5.7], so that if $\Phi' = \mathbf{B}'_{Z'}$, then we have the inequality of b-divisors

$$\mathbf{L}_{\Phi'} \leq \mathbf{B}'.$$

Recall that the reduction $(Z', \mathbf{B}')$ is given by a finite sequence of cuts where a cut is defined as follows: given a birational morphism of smooth projective varieties $\mu : Z' \to Z$ and a subset Σ of the μ exceptional divisors, for every valuation $\sigma \in \Sigma$, let $\Gamma_\sigma = (\mathbf{L}_\Phi \wedge \mathbf{B})_{Y_\sigma}$, where $Y_\sigma \to Z$ is the divisorial contraction of the divisor over Z corresponding to σ which defined in [9, 5.4] and $\Phi = \mathbf{B}_Z$. Let $\Theta = \wedge_{\sigma \in \Sigma} (\mathbf{L}_{\Gamma_\sigma})_{Z'}$, the minimum of the divisors $(\mathbf{L}_{\Gamma_\sigma})_{Z'}$. The cut of $(Z, \mathbf{B})$, associated to $Z' \to Z$ and Σ, is the pair $(Z', \mathbf{B}')$, where $\mathbf{B}' = \mathbf{B} \wedge \mathbf{M}_\Theta$, so that $\mathbf{B}'_{Z'} = \Theta \wedge \mathbf{B}_{Z'}$ and $\mathbf{B}'(\nu) = \mathbf{B}(\nu)$ for any valuation ν corresponding to an exceptional divisor over Z'. We may assume that $Z' \to Z$ is given by a finite sequence of blow ups along strata of $\widehat{B}$ and so we let $Z'_m \to Z_m$ be obtained by the corresponding sequence of blow ups along strata of B_{Z_m} for almost all $m \in \mathbb{N}$. After possibly blowing up X^m and replacing B^m by its strict transform plus the exceptional divisor, we may assume that $X^m \to Z$ factors via a morphism $X^m \to Z'$ and similarly we have morphisms $X_m \to Z'_m$ for almost all $m \in \mathbb{N}$.

Now consider the divisors B'^m on X^m defined by $B'^m = B^m \wedge (\mathbf{M}_{\Theta^m})_{X^m}$ where $\Theta^m = \wedge_{\sigma\in\Sigma}(\mathbf{L}_{\Gamma^m_\sigma})_{Z'}$, $\Gamma^m_\sigma = (\mathbf{L}_{B^m_Z})_{Y_\sigma} \wedge B_{Y_\sigma}$ where B^m_Z and $B^m_{Y_\sigma}$ are the pushforwards of B^m to Z and Y_σ. Then, as in the proof of [9, 5.7], we may assume that $\mathbf{B}' = \lim \mathbf{M}_{B'^m}$. Let B'_m be the divisors on X_m corresponding to B'^m. We will show that $\operatorname{vol}(K_{X_m} + B_m) = \operatorname{vol}(K_{X_m} + B'_m)$. Assuming this, we may replace B_m by B'_m and the claim follows.

We define $\Phi_m = B_{Z_m}$ and

$$\Gamma_{m,\sigma} = (\mathbf{L}_{\Phi_m})_{Y_{m,\sigma}} \wedge B_{Y_{m,\sigma}},$$

where $B_{Y_{m,\sigma}}$ is the pushforward of B_m to $Y_{m,\sigma}$. Let $\Theta_m := \wedge_{\sigma\in\Sigma}(\mathbf{L}_{\Gamma_{m,\sigma}})_{Z'_m}$. It is easy to see that the divisors B_{Z_m}, $B_{Y_{m,\sigma}}$, $\Gamma_{m,\sigma}$ and Θ_m correspond to the divisors B^m_Z, $B^m_{Y_\sigma}$, Γ^m_σ and Θ^m so that we have

$$B'_m = B_m \wedge (\mathbf{M}_{\Theta_m})_{X_m}.$$

It then follows that

$$B_m \wedge (\mathbf{L}_{\Theta_m})_{X_m} \leq B'_m = B_m \wedge (\mathbf{M}_{\Theta_m})_{X_m} \leq B_m. \tag{2.4}$$

Thus, by (2.4) and Proposition 1.4

$$\operatorname{vol}(K_{X_m} + B_m) = \operatorname{vol}\big(K_{X_m} + B'_m\big).$$

Replacing B_m by B'_m the claim follows. □

Claim 2.10. For almost all $m \in \mathbb{N}$ we have $\operatorname{vol}(K_{Z_m} + t\Phi_{Z_m}) = \operatorname{vol}(K_Z + t\Phi)$ for all $t \in [0, 1] \cap \mathbb{Q}$.

Proof. There are finitely many birational morphisms $\{\psi^i : Z \to W^i\}_{i\in I}$ such that for any $t \in [0, 1]$, there exists an $i \in I$ such that ψ^i is a minimal model for $K_Z+t\Phi$. Let $[\psi^i_m] : [Z_m] \to [W^i_m]$ be the corresponding morphism of internal schemes. It is easy to see that for almost all $m \in \mathbb{N}$ this is a minimal model for $K_{Z_m} + t\Phi_{Z_m}$ and $(K_{W^i_m} + t\psi^i_{m,*}\Phi_{Z_m})^2 = (K_{W^i} + t\psi_*\Phi_Z)^2$ (*cf.* Lemma 1.11). Therefore, the claim follows. □

Now we observe that

$$\operatorname{vol}(K_{X_m} + B_m) \leq \operatorname{vol}(K_{Z_m} + B_{Z_m}) \leq \operatorname{vol}(K_{Z_m} + \Phi_{Z_m}),$$

where the first inequality follows as B_{Z_m} is the pushforward of B_m, and the second as $B_{Z_m} \leq \Phi_{Z_m}$ *cf.* (2.2).

On the other hand, for any $\varepsilon > 0$, the pair $(Z, (1-\varepsilon)\Phi)$ is klt with simple normal crossings and hence there is a terminalization $h : Y \to Z$ (given by a finite sequence of blow ups along strata of $(Z, (1-\varepsilon)\Phi)$) so that $(Y, \Psi := \mathbf{L}_{(1-\varepsilon)\Phi,Y})$ is terminal. We have that for some $\lambda > 0$,

$$\Psi \le (1-\lambda)\mathbf{L}_{\Phi,Y} \le \mathbf{L}_{\Phi,Y} \le \mathbf{B}_Y, \tag{2.5}$$

where the last inequality follows from (2.3). For almost all $m \in \mathbb{N}$ we may consider $h_m : Y_m \to Z_m$ given by the same sequence of blow ups along strata of (Z_m, Φ_{Z_m}). Then, denoting by $\mathbf{B}_{Y_m}$ and Ψ_m the divisors on Y_m corresponding to $\mathbf{B}_Y$ and Ψ, since $\mathbf{B}_Y = \lim \mathbf{M}_{B^m,Y}$, comparing coefficients of divisors on Y, by (2.5), for infinitely many $m \in \mathbb{N}$ we have $\Psi \le \mathbf{M}_{B^m,Y}$ and hence also $\Psi_m \le \mathbf{M}_{B_m,Y_m}$. It follows that then

$$\begin{aligned} \operatorname{vol}(K_Z + (1-\varepsilon)\Phi) &= \operatorname{vol}\big(K_{Z_m} + (1-\varepsilon)\Phi_{Z_m}\big) = \operatorname{vol}\big(K_{Y_m} + \Psi_m\big) \\ &\le \operatorname{vol}\big(K_{Y_m} + \mathbf{M}_{B_m,Y_m}\big) = \operatorname{vol}\big(K_{X_m} + B_m\big), \end{aligned} \tag{2.6}$$

where the first (in)equality follows from Claim 2.10, the second since (Y_m, Ψ_m) is a terminalization of $(Z_m, (1-\varepsilon)\Phi_{Z_m})$ (observe that $\Psi_m = (\mathbf{L}_{(1-\varepsilon)\Phi_{Z_m}})_{Z_m}$ and apply Proposition 1.4), the third since $\Psi_m \le \mathbf{M}_{B_m,Y_m}$, and the fourth by Proposition 1.4. Taking the limit as $\varepsilon \to 0$, by (2.1) we obtain

$$\operatorname{vol}(K_Z + \Phi) \le \lim \operatorname{vol}\big(K_{X_m} + B_m\big) < \operatorname{vol}\big(K_{X_m} + B_m\big).$$

Combining this with the above equations and Claim 2.10, we have that

$$\operatorname{vol}(K_Z + \Phi) < \operatorname{vol}\big(K_{X_m} + B_m\big) \le \operatorname{vol}\big(K_{Z_m} + \Phi_{Z_m}\big) = \operatorname{vol}(K_Z + \Phi) \tag{2.7}$$

for infinitely many m. This is the required contradiction and it completes the proof of Theorem 2. □

2.3. Proof of Theorem 1

It suffices to show that for any sequence of projective log canonical surfaces (X_m, B_m) with fixed volume $(K_{X_m} + B_m)^2 = v$ and $\operatorname{coeff}(B_m) \subseteq \mathscr{C}$, there exists an integer $r > 0$ such that $r(K_{\bar{X}_m} + \bar{B}_m)$ is very ample where $\phi_m : X_m \to \bar{X}_m$ is the log canonical model of (X_m, B_m) and $\bar{B}_m = \phi_{m,*}B_m$. Arguing by contradiction (and passing to a subsequence), assume that $m!(K_{\bar{X}_m} + \bar{B}_m)$ is not very ample for all $m > 0$. Following the notation in the above proof, let $Z \to W$ be the log canonical model of $(Z, \Phi = \mathbf{B}_Z)$. Let $[Z_m] \to [W_m]$ be the corresponding morphism of internal schemes, so that for almost all $m \in \mathbb{N}$ we have morphisms $h_m : Z_m \to W_m$ which are log canonical models for (Z_m, Φ_{Z_m}). We have

$$\operatorname{vol}\big(K_{X_m} + B_m\big) = \operatorname{vol}\big(K_{Z_m} + \Phi_{Z_m}\big) \ge \operatorname{vol}\big(K_{Z_m} + B_{Z_m}\big) \ge \operatorname{vol}\big(K_{X_m} + B_m\big),$$

where the first (in)equality follows since all inequalities in (2.7) are actually equalities, the second since $\Phi_{Z_m} \geq B_{Z_m}$ and the last as $K_{Z_m} + B_{Z_m}$ is the pushforward of $K_{X_m} + B_m$.

Since $K_{Z_m} + B_{Z_m}$ is big and has a log canonical model, and $\Phi_{Z_m} \geq B_{Z_m}$, it follows by [11, 2.2.2] that $Z_m \to W_m$ is a log canonical model for (Z_m, B_{Z_m}). In particular, $(h_m)_* B_{Z_m} = (h_m)_* \Phi_{Z_m}$ is rational and hence so is $h_* \Phi$. But then by the result over the fixed field k (see [4, 9.2]), we know that there is an integer r depending only on $\upsilon = \operatorname{vol}(K_{X_m} + B_m)$ and $\mathscr{C}$ such that $r(K_W + h_* \Phi)$ is Cartier and very ample. But then $r(K_{W_m} + h_{m,*}\Phi)$ is Cartier and very ample for infinitely many $m > 0$. This is the required contradiction and the assertion of Theorem 1 follows.

2.4. Proof of Corollary 3

We may assume that $1 - \varepsilon \in \mathscr{C}$. It suffices to show that any sequence of ε-log canonical projective pairs (X_m, B_m) with $\dim X_m = 2$, $\operatorname{coeff}(B_m) \in \mathscr{C}$, $K_{X_m} + B_m$ nef and big and $\operatorname{vol}(K_{X_m} + B_m) \leq v$ is degree bounded.

Following the proof of Theorem 2, we may assume that $(Z, \mathbf{B}_Z)$ is an snc pair with coefficients $\leq 1 - \varepsilon$. Replacing Z by an appropriate birational model, we may in fact assume that $(Z, \mathbf{B}_Z)$ is terminal and hence so are (Z_m, B_{Z_m}). But then $\operatorname{vol}(K_{X_m} + B_m) = \operatorname{vol}(K_{Z_m} + B_{Z_m})$ for almost all $m \in \mathbb{N}$ by Proposition 1.4 and so we may assume that $(X_m, B_m) = (Z_m, B_{Z_m})$. Notice that we have replaced X_m by an appropriate birational model and B_m by its strict transform plus the exceptional divisors with coefficient $(1 - \varepsilon)$, hence $K_{X_m} + B_m$ may no longer be nef. Let B^m be the divisors on Z corresponding to B_m on X_m. Since the support of B^m has finitely many components and $\mathscr{C}$ is a DCC set, after passing to a subsequence, we may assume that $B^m \leq B^{m+1} \leq B^{m+2} \leq \ldots \lim B^i = B^\infty$. Let B^∞_m be the corresponding divisors on X_m, so that $B^\infty_m \geq B_m$. We claim that $K_Z + B^\infty$ is big. If this were not the case, then Z would be covered by curves C with $(K_Z + B^\infty) \cdot C \leq 0$. But then, the same would be true for (X_m, B_m) as for almost all $m \in \mathbb{N}$ we have

$$0 \geq \big(K_Z + B^\infty\big) \cdot C = \big(K_{X_m} + B^\infty_m\big) \cdot C_m \geq \big(K_{X_m} + B_m\big) \cdot C_m,$$

where $(C)_{\text{int}} = [C_m]$. This contradicts the fact that $K_{X_m} + B_m$ is big. Since being big is an open condition, it follows that $K_Z + (1 - \delta)B^\infty$ is big for all $0 < \delta \ll 1$ and we may assume that $(1 - \delta)B^\infty \leq B^m \leq B^\infty$. The set of all minimal/canonical models $Z \to W$ for pairs (Z, G) with $(1 - \delta)B^\infty \leq G \leq B^\infty$ is bounded. Arguing as in the proof of Claim 2.10 the corresponding rational maps $Z_m \to W_m$ give minimal/canonical models for (X_m, B_m) for almost all $m \in \mathbb{N}$. Corollary 3 follows easily.

2.5. Proof of Theorems 4 and 5

In the sequel we will use the following notation: If $f_m : X_m \to S_m$ is a morphism of schemes and $s \in S_m$ a point, then $X_{m,s}$ denotes the fiber $(X_m)_s = f_m^{-1}(s)$. More generally, if $S'_m \to S_m$ is a morphism and $s \in S'_m$ a point, then $X_{m,s}$ denotes the fiber product $X_m \times_{S_m} \{s\}$.

Theorem 2.11. *Fix a constant $\upsilon \in \mathbb{Q}$, a DCC set $\mathscr{C} \subset [0, 1] \cap \mathbb{Q}$. For each $m \in \mathbb{Z}$, let L_m be an algebraically closed field of $\operatorname{char} L_m = p_m > 0$ such that $\lim p_m = \infty$. Let $k = [[L_m]]$, S a smooth 1-dimensional scheme defined over k, and $S_{\text{int}} = [S_m]$ the corresponding internal scheme. Further let (X_m, B_m) be a pair defined over L_m such that $\dim X_m = 3$, and let $f_m : X_m \to S_m$ be a projective morphism with connected fibers. Assume that for each $m \in \mathbb{Z}$, $\operatorname{coeff}(B_{m,\eta}) \subseteq \mathscr{C}$, $(X_{m,\eta}, B_{m,\eta})$ is semi-log canonical, and $K_{X_{m,\eta}} + B_{m,\eta}$ is ample with $(K_{X_{m,\eta}} + B_{m,\eta})^2 = \upsilon$ where η denotes the generic point of S_m.*

Then there exist a finite separable morphism $S' \to S$, and a projective semistable family of semi-log canonical models $(X', B') \to S'$ such that considering the corresponding internal objects, for an infinite subset $V \subseteq \mathbb{Z}$ and for each $m \in V$ there exist an induced separable finite morphism $\sigma_m : S'_m \to S_m$, a projective morphism $X'_m \to S'_m$, and a pair (X'_m, B'_m) such that $K_{X'_{m,s}} + B'_{m,s}$ is ample, $(X'_{m,s}, B'_{m,s})$ is semi-log canonical (in particular reduced) for all $s \in S'_m$, and $(X'_{m,\eta'}, B'_{m,\eta'})$ is isomorphic to $(X_{\eta'}, B_{\eta'})$ for where $\eta \in S_m$ and $\eta' \in S'_m$ are the generic points.

Proof. Let $F_m = K(S_m)$ be the field of rational functions of S_m, $\overline{F}_m$ its algebraic closure, and $(\overline{X}_{m,\eta}, \overline{B}_{m,\eta})$ the geometric general fiber of f_m (obtained by the base change $\operatorname{Spec} \overline{F}_m \to S_m$). Since $\operatorname{coeff}(\overline{B}_{m,\eta}) \subseteq \mathscr{C}$ and $\operatorname{vol}(K_{\overline{X}_{m,\eta}} + \overline{B}_{m,\eta}) = \operatorname{vol}(K_{X_{m,\eta}} + B_{m,\eta}) = \upsilon$, by Theorem 1 there is a fixed integer $r > 0$ (independent of m) such that $r(K_{\overline{X}_{m,\eta}} + \overline{B}_{m,\eta})$ is very ample.

As $X_{m,\eta}$ and the components of $B_{m,\eta}$ have bounded degree (*cf.* Theorem 1.15), there exists, by Proposition 1.9, a pair (X°, B°) defined over $F = [F_m]$ such that $(X^\circ, B^\circ)_{\text{int}} = ([X_{m,\eta}], [(B_{m,\eta})_{\text{red}}])$. Since $r(K_{\overline{X}_{m,\eta}} + \overline{B}_{m,\eta})$ is very ample and hence in particular Cartier, it follows that $\operatorname{coeff}(B_m) \subseteq \left\{ \frac{a}{r} \middle| 0 \le a \le r \right\}$ and so, after passing to an infinite subset of $\mathbb{Z}$, we may assume that actually $B^\circ_{\text{int}} = [B_{m,\eta}]$.

Let $k = [L_m]$ and note that it is algebraically closed of $\operatorname{char} k = 0$ by [16, 2.4.1, 2.4.2]. By definition $S_{\text{int}} = [S_m]$, so $K(S) = F = [F_m]$ by Theorem 1.6 and the construction of the functor $Z \mapsto Z_{\text{int}}$. Since $\operatorname{char} k = 0$, after a possible base change, resolving and taking the relative semi-log canonical model, one obtains a semistable family of semi-log canonical models $(X', B') \to S'$ with a finite separable morphism $S' \to S$. Considering the corresponding internal objects $X'_{\text{int}} = [X'_m]$, $B'_{\text{int}} = [B'_m]$, and $S'_{\text{int}} = [S'_m]$ proves Theorem 2.11. □

Corollary 2.12 (Theorem 4). *Fix a constant $\upsilon \in \mathbb{Q}$ and a DCC set $\mathscr{C} \subset [0, 1] \cap \mathbb{Q}$. Then there exists a number $p_0 > 0$ such that if L is an algebraically closed field of characteristic $p > p_0$, (X, B) a pair defined over L such that $\dim X = 3$, $f : X \to S = \operatorname{Spec} L[[t]]$ a projective morphism with connected fibers such that, $\operatorname{coeff}(B_{m,\eta}) \subseteq \mathscr{C}$, $(X_{m,\eta}, B_{m,\eta})$ is semi-log canonical, and $K_{X_{m,\eta}} + B_{m,\eta}$ is ample with $(K_{X_{m,\eta}} + B_{m,\eta})^2 = \upsilon$, then there exist a separable finite morphism $S' \to S$, a projective morphism $f' : X' \to S'$, and a pair (X', B') such that considering the* *corresponding internal objects for an infinite subset* $V \subseteq \mathbb{Z}$ *and for each* $m \in V$, $(X'_{m,s}, B'_{m,s})$ *is semi-log canonical and* $K_{X'_{m,s}} + B'_{m,s}$ *is ample for all* $s \in S'$, *and* $(X'_{m,\eta'}, B'_{m,\eta'})$ *is isomorphic to* $(X_{\eta'}, B_{\eta'})$ *where* $\eta \in S_m$ *and* $\eta' \in S'_m$ *are the generic points.*

Corollary 2.13. *Fix constants* $\upsilon \in \mathbb{Q}$, $g \in \mathbb{N}$ *and a DCC set* $\mathscr{C} \subset [0,1] \cap \mathbb{Q}$. *Then there exists a number* $p_0 > 0$ *such that if L is an algebraically closed field of characteristic* $p > p_0$, (X, B) *a pair defined over L such that* $\dim X = 3$, $f : X \to S$ *a projective morphism with connected fibers, where S is a smooth curve over L whose geometric genus is at most g, such that,* $\operatorname{coeff}(B_{m,\eta}) \subseteq \mathscr{C}$, $(X_{m,\eta}, B_{m,\eta})$ *is log canonical, and* $K_{X_{m,\eta}} + B_{m,\eta}$ *is ample with* $(K_{X_{m,\eta}} + B_{m,\eta})^2 = \upsilon$, *then there exist a separable finite morphism* $S' \to S$, *a projective morphism* $f' : X' \to S'$, *and a pair* (X', B') *such that considering the corresponding internal objects, for an infinite subset* $V \subseteq \mathbb{Z}$ *and for each* $m \in V$, $(X'_{m,s}, B'_{m,s})$ *is semi-log canonical and* $K_{X'_{m,s}} + B'_{m,s}$ *is ample for all* $s \in S'$, *and* $(X'_{m,\eta'}, B'_{m,\eta'})$ *is isomorphic to* $(X_{\eta'}, B_{\eta'})$ *where* $\eta \in S_m$ *and* $\eta' \in S'_m$ *are the generic points.*

Remark 2.14. Note that the situation of Corollary 2.13 arises for instance if S is the "reduction mod p" of a fixed curve defined in characteristic zero.

Proof of Corollaries 2.12 *and* 2.13. Let L_m be a sequence of algebraically closed fields of characteristic $p_m > 0$ such that $\lim p_m = \infty$, (X_m, B_m) a sequence of pairs and $X_m \to S_m$ a sequence of morphisms defined over L_m, where either $S_m = L_m[[t]]$ for each m or S_m is a smooth curve over L_m whose geometric genus is at most g for each m. Suppose that the conclusion of the appropriate corollary fails for each m. Let $k := [L_m]$ and either let $S := \operatorname{Spec} k[[t]]$ or let S be the smooth curve provided by Proposition 1.9 (*cf.* [6, 3.7]). With these definitions the assumptions of Theorem 2.11 are satisfied and hence we obtain a contradiction. □

Theorem 2.15 (Theorem 5). *Fix a constant* $\upsilon \in \mathbb{Q}$ *and a DCC set* $\mathscr{C} \subset [0,1] \cap \mathbb{Q}$. *For each* $m > 0$ *let* L_m *be an algebraically closed field of characteristic* $p_m > 0$ *such that* $\lim p_m = \infty$ *and let* $k = [L_m]$. *Further let* (X_m, B_m) *be a pair defined over* L_m *such that* $\dim X_m = 3$, *and let* $f_m : X_m \to S_m$ *be a projective morphism with connected fibers to a smooth curve. Assume that for each* $m \in \mathbb{Z}$, $\operatorname{coeff}(B_{m,\eta}) \subseteq \mathscr{C}$, $(X_{m,\eta}, B_{m,\eta})$ *is semi-log canonical, and* $K_{X_{m,\eta}} + B_{m,\eta}$ *is ample with* $(K_{X_{m,\eta}} + B_{m,\eta})^2 = \upsilon$ *where* η *is the generic point of* S_m.

Then for all but finitely many m's there exist a separable finite morphism $\sigma_m : S'_m \to S_m$, *a projective morphism* $X'_m \to S'_m$, *and a pair* (X'_m, B'_m) *such that* $(X'_{m,s}, B'_{m,s})$ *is semi-log canonical and* $K_{X'_{m,s}} + B'_{m,s}$ *is ample for all* $s \in S'_m$, *and* $(X'_{m,s}, B'_{m,s})$ *is isomorphic to* $(X_{m,\sigma_m(s)}, B_{m,\sigma_m(s)})$ *for general* $s \in S'_m$.

Proof. Suppose that the conclusion of the theorem fails, *i.e.*, that it fails for infinitely many primes p_m. Passing to a subsequence, we may assume that the conclusion fails for every prime p_m and we aim to find a contradiction. Since the statement is local over the base, we may assume that $S_m = \operatorname{Spec}(R_m)$ where R_m is a DVR with closed point s_m. Let $\widehat{S}_m$ be the formal neighborhood of $s_m \in S_m$ and $\widehat{X}_m$ be the formal neighborhood of $f_m^{-1}(s_m) \subset X_m$ with induced morphism $\widehat{f}_m : \widehat{X}_m \to \widehat{S}_m$. We have $\widehat{S}_m = L_m[[t]]$ and arguing as in Corollary 2.12 there is a finite cover $\widehat{S}' \to \widehat{S} = k[[t]]$ and a family of semi-log canonical models $(\widehat{X}', \widehat{B}') \to \widehat{S}'$ which over the generic fiber is induced by $([\widehat{X}_{m,\eta}], [(\widehat{B}_{m,\eta})_{\mathrm{red}}])$. Since $\widehat{S}'$ is a normal complete 1-dimensional DVR, we may assume that $\widehat{S}' \simeq k[[s]]$ and $\sigma : \widehat{S}' \to \widehat{S}$ is induced by the inclusion $\sigma^* : k[[t]] \to k[[s]]$. Let $\sigma^*(t) = s^r g(s)$ where $g(s) \in k[[s]]$ is a unit. Considering the corresponding internal objects $\widehat{X}'_{\mathrm{int}} = [\widehat{X}'_m]$, $\widehat{B}'_{\mathrm{int}} = [\widehat{B}'_m]$ and $\widehat{S}'_{\mathrm{int}} = [\widehat{S}_m]$, then $\widehat{S}'_m = L_m[[s]] \to \widehat{S}_m = L_m[[t]]$ where $t = s^r g(s)$.

Claim 2.16. Let $S'_m \to S_m$ be a finite cover ramified at s_m to order r and s'_m be the corresponding closed point on S'_m. Then $\widehat{S}'_m$ is isomorphic to the completion of S'_m along s'_m.

Proof. Let $\gamma : \widetilde{S}'_m \to S_m$ be the morphism induced by the above finite cover where $\widetilde{S}'_m$ is the completion of S'_m along s'_m, then $\widetilde{S}'_m = \mathrm{Spec}(L_m[[s]])$ and γ is determined by $\gamma^*(t) = s^r h_m(s)$ where we view $t \in R_m$ a local parameter of S_m at s_m. Let $g_m(s) \in L_m[[s]]$ be the elements corresponding to $g(s) \in k[[s]]$ and $\alpha_m(s), \beta_m(s) \in k[[s]]$ such that $(\alpha_m(s))^r = h_m(s)$ and $(\beta_m(s))^r = g_m(s)$, then $\alpha_m(s), \beta_m(s) \in k[[s]]$ are units. Let $\tau_m : L_m[[s]] \to L_m[[s]]$ be a isomorphism such that $\tau_m(\alpha_m(s)) = \beta_m(s)$, then τ_m induces the required isomorphism $\widehat{S}'_m \to \widetilde{S}'_m$. □

Consider now $\widetilde{f}_m : \widetilde{X}_m \to S'_m$ a log resolution of $X_m \times_{S_m} S'_m$ such that if $\widetilde{B}_m$ is the strict transform of B_m plus the reduced exceptional divisor and the reduced fiber $(\widetilde{f}_m^{-1}(s'_m))_{\mathrm{red}}$, then $\widetilde{B}_m$ has simple normal crossings support. Let $\widehat{\widetilde{X}}_m$ be the completion of $\widetilde{X}_m$ along $\widetilde{f}_m^{-1}(s'_m)$. Consider a common resolution $\pi_m : W_m \to \widehat{\widetilde{X}}_m$ and $\varrho_m : W_m \to \widehat{X}'_m$. We write

$$\pi_m^*\Big(K_{\widehat{\widetilde{X}}_m} + \widehat{\widetilde{B}}_m\Big) = \varrho_m^*\Big(K_{\widehat{X}'_m} + \widehat{B}'_m + \widehat{X}'_{m,s'_m}\Big) + G_m.$$

It is easy to see that $\pi_{m,*}(G_m) \geq 0$ and since $\varrho_m^*(K_{\widehat{X}'_m} + \widehat{B}'_m) - \pi_m^*(K_{\widehat{\widetilde{X}}_m} + \widehat{\widetilde{B}}_m)$ is π_m-nef, then by the negativity lemma, $G_m \geq 0$ (notice that restricting to the central fiber and applying the usual negativity lemma, we obtain that $G_m|_{(W_m)_{s'_m}} \geq 0$, and hence $G_m \geq 0$ as we are working over a formal neighborhood of s'_m).

Claim 2.17. The ring $\bigoplus_{q \geq 0} \widetilde{f}_{m,*}\mathscr{O}_{\widetilde{X}_m}(q(K_{\widetilde{X}_m} + \widetilde{B}_m))$ is a finitely generated $\mathscr{O}_{S'_m}$-algebra.

Proof. Since the statement is triviallly true over the open subset $S'_m \setminus \{s'_m\}$, we may localize S'_m at s'_m. Since the natural functor from coherent sheaves over S'_m to coherent sheaves over $\widehat{S}'_m$ is exact, it suffices to check the analogous statement for the $\mathscr{O}_{\widehat{S}'_m}$-algebra

$$\bigoplus_{q\geq 0} \widehat{\widetilde{f}}_{m,*}\mathscr{O}_{\widehat{\widetilde{X}}_m}\left(q\left(K_{\widehat{\widetilde{X}}_m}+\widehat{\widetilde{B}}_m\right)\right).$$

By what we have observed above, this algebra is isomorphic to

$$\bigoplus_{q\geq 0} \widehat{f}'_{m,*}\mathscr{O}_{\widehat{X}'_m}\left(q\left(K_{\widehat{X}'_m}+\widehat{B}'_m+\widehat{X}_{m,s'_m}\right)\right) \simeq \bigoplus_{q\geq 0} \widehat{f}'_{m,*}\mathscr{O}_{\widehat{X}'_m}\left(q\left(K_{\widehat{X}'_m}+\widehat{B}'_m\right)\right)$$

which is finitely generated (since $K_{\widehat{X}'_m}+\widehat{B}'_m$ is ample over $\widehat{S}'_m$. □

We now consider $X'_m = \mathrm{Proj}(\bigoplus_{q\geq 0} \widetilde{f}_{m,*}\mathscr{O}_{\widetilde{X}_m}(q(K_{\widetilde{X}_m}+\widetilde{B}_m)))$. Note that the special fiber X'_{m,s'_m} is isomorphic to the special fiber of $\widehat{X}'_m \to \widehat{S}'_m$ and in particular it is reduced. By construction $X'_m \to S'_m$ is a family of log canonical models and these log canonical models determine semi-log canonical models over $S'_m \setminus \{s'_m\}$ and over a formal neigborhood of $s'_m \in S'_m$. Since these semi-log canonical models agree over the generic point of the formal neigborhood of $s'_m \in S'_m$, we obtain a semi-log canonical model over the whole of S'_m (which is automatically projective over S'_m since the relative log canonical divisor is relatively ample). This is the required contradiction and the proof is complete. □

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