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A land of monotone plenty

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Abstract. A fundamental concept in optimal transport is c-cyclical monotonicity: it allows to link the optimality of transport plans to the geometry of their support sets. Recently, related concepts have been successfully applied in the multi-marginal version of the transport problem as well as in the martingale transport problem which arises from model-independent finance.

We establish a unifying concept of *c-monotonicity/finitistic optimality* which describes the geometric structure of optimizers of a generalized moment problem. This allows us to strengthen known results in optimal martingale transport and for a transport problem with a continuum of marginals.

If the optimization problem can be formulated as a multi-marginal transport problem, potentially with additional linear constraints, our contribution is parallel to a recent result of Zaev.

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1. Introduction

1.1. Motivation from optimal transport

Given probabilities μ and ν on Polish spaces X and Y, and a cost function $c : X \times Y \to \mathbb{R}_+$, the Monge-Kantorovich problem is to find a cost minimizing transport plan. More precisely, writing $\Pi(\mu, \nu)$ for the set of all measures on $X \times Y$ with X-marginal μ and Y-marginal ν, the problem is to find

$$\inf \left\{ \int c \, d\gamma : \gamma \in \Pi(\mu, \nu) \right\} \tag{OT}$$

and to identify an optimal transport plan $\gamma^* \in \Pi(\mu, \nu)$.

The concept of c-cyclical monotonicity leads to a geometric characterization of optimal couplings. Its relevance for (OT) has been fully recognized by Gangbo and McCann [15], based on earlier work of Knott and Smith [23] and Rüschendorf [33] among others.

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A set $\Gamma \subseteq X \times Y$ is *c-cyclically monotone* if any measure α, that is finite and supported on finitely many points in Γ, is a cost-minimizing transport between its marginals. *I.e.*, if α' has the same marginals as α, then

$$\int c\, d\alpha \leq \int c\, d\alpha'. \tag{1.1}$$

A transport plan γ is called *c-cyclically monotone* if it is concentrated on such a set Γ, *i.e.*, if there is such a Γ with $\gamma(\Gamma) = 1$.[1]

Connecting optimality and c-cyclical monotonicity is technically intricate. A series of contributions ([2,5,9,32,34] among others) led to the following clear-cut characterization:

Theorem 1.1 (Monotonicity principle). *Let $c : X \times Y \to [0, \infty)$ be Borel measurable and assume that $\gamma \in \Pi(\mu, \nu)$ is a transport plan with finite costs $\int c\, d\gamma \in \mathbb{R}_+$. Then γ is optimal if and only if γ is c-cyclically monotone.*

The importance of this result stems from the observation that it is often an elementary and feasible task to see whether a transport behaves optimally on a finite number of points. But this would be a priori of no help for a problem where single points do not carry positive mass. Theorem 1.1 provides the required remedy to this obstacle as it establishes the connection to optimality on a "pointwise" level.

1.2. Recent developments and aims of this article

Recently several variants of (OT) have been discussed in the literature: the multi-marginal transport problem (see [11,20,22,28,29]), the martingale transport problem ([4, 6–8, 12–14, 16, 17, 26] among others), and problems where a continuum of marginals is prescribed, [30, 31]. Having cyclical monotonicity in mind, the problem in [30] seems of particular interest: here, Pass presents a solution that is of Monge-type and appears very natural, yet the proof of its optimality and uniqueness appears rather technical and relies on assumptions that might be difficult to verify in practice. But it is apparent that this solution is the only transport that fulfills an infinite-dimensional analogue of cyclical monotonicity. We were thus drawn to the question whether a suitable notion of c-cyclical monotonicity could prove useful for such extended problems by reducing the technical level and leading to stronger results.

[1] The more familiar way of stating c-cyclical monotonicity for a set Γ is to assert that for any $(x_1, y_1), \ldots, (x_n, y_n) \in \Gamma$, with the convention $y_{n+1} = y_1$,

$$\sum_{i=1}^{n} c\big(x_i, y_i\big) \leq \sum_{i=1}^{n} c\big(x_i, y_{i+1}\big).$$

We have used the equivalent formulation above as it is not inherently two-dimensional and serves our exposition more directly. For the equivalence see [35, Exercise 2.21].

The main goal of this article is therefore to establish a monotonicity principle as an analogue of the "necessary"-part of Theorem 1.1 in a rather wide generality. More precisely, we use the framework of a generalized moment problem (GMP), define a general notion of c-monotonicity and then establish that optimizers are c-monotone. To this end, we build on ideas from [7], where, mimicking the idea of c-cyclical monotonicity, a notion of "finitistic optimality" was already introduced for the martingale transport problem and optimizers were shown to fulfill that criterion.[2]

Our Theorem 1.4 allows to obtain improved versions of the results from [7] and [22, Proposition 2.3], and it includes one implicaton of the classical result stated in Theorem 1.1. Finally, we use the result to prove a strengthened version of Pass' Monge-type result. In contrast to Pass' original derivation we do not require additional assumptions on the payoff functional or the prescribed marginals which might be difficult to verify in his intended applications.

We note that, although (GMP) constitutes a classical problem in probability, and it is well known that (OT) and its variants fit into this framework (see, *e.g.*, [21] and [25]), the general optimality criterion of c-monotonicity we state in Theorem 1.4 is new to the best of our knowledge.

We point out a particular novelty of the approach in this article: in all the instances where the monotonicity principle was previously known, the minimization problem (GMP) admits a well understood dual problem and it is known that there is no duality gap. In the literature on the Monge-Kantorovich problem, it is well known that the absence of a duality gap can be used to show that optimal transport plans are cyclically monotone, see, *e.g.*, [35, Exercise 2.38]. In fact, assuming certain regularity assumptions, this argument could be used to establish Theorem 1.4 whenever there is no duality gap. The advantage of the approach presented below is twofold. On the one hand it allows to derive the desired implication virtually without regularity assumptions. More importantly, it is applicable also in situations where duality is either unknown or known to fail (see [3, Section 3.4] for such cases).

We conclude this section with a precise statement of the problem, the definition of c-monotonicity and the optimality criterion of Theorem 1.4. For sake of readability we postpone its proof to the last section, Section 4. Section 2 shows how some problems can be written in our framework. We also give a counterexample on the "sufficiency"-part of Theorem 1.1 in the general situation. Section 3 deals with the problem from [30] in light of Theorem 1.4.

1.3. The basic optimization problem

Let E be a Polish space and $c : E \to \mathbb{R}$ a Borel measurable cost function.

We fix a set $\mathcal{F}$ of Borel-measurable functions on E and write $\Pi_{\mathcal{F}}$ for the set of probability measures γ on E for which $\int f \, d\gamma = 0$ for all $f \in \mathcal{F}$.[3] The

[2] In fact, in many instances of the martingale transport problem, finitistic optimality is also sufficient for optimality.

[3] By asserting that $\int f \, d\gamma = 0$ we implicitly state that this integral exists.

generalized moment problem is then to minimize the total cost choosing from $\Pi_{\mathcal{F}}$, *i.e.*,

$$\min_{\gamma \in \Pi_{\mathcal{F}}} \int c \, d\gamma. \tag{GMP}$$

1.4. A general concept of *c*-monotonity and main result

Our general definition of c-monotonicity applicable to (GMP) is the following:

Definition 1.2. For a measure α on the Polish space E and a set $\mathcal{F}$ of measurable functions $E \to \mathbb{R}$, a competitor of α is a measure α' on E such that $\alpha(E) = \alpha'(E)$, and for all $f \in \mathcal{F}$ one has

$$\int f \, d\alpha = \int f \, d\alpha'. \tag{1.2}$$

If, in addition, α is finitely supported, *i.e.*, concentrated on finitely many points, we require this property also for a competitor.

A set $\Gamma \subseteq E$ is called *finitely minimal/c-monotone* if each measure α, which is finite and concentrated on finitely many points in Γ, is cost minimizing amongst its competitors. A measure γ is called *finitely minimal/c-monotone* if it is concentrated on a finitely minimal/c-monotone set.

Establishing that optimizers of problem (GMP) are finitely minimal will need an assumption on the family $\mathcal{F}$:

Assumption 1.3.

(1) There exists a function $g : E \to [0, \infty)$ such that each element of $\mathcal{F}$ is bounded by some multiple of g. *I.e.*, for each $f \in \mathcal{F}$ there is a constant $a_f \in \mathbb{R}_+$ such that $|f| \leq a_f g$;
(2) All functions in $\mathcal{F}$ are continuous, or $\mathcal{F}$ is at most countable.

These properties are satisfied in all examples encountered in this article.

Theorem 1.4. *Let E be a Polish space and $c : E \to \mathbb{R}$ a Borel measurable function. Let $\mathcal{F}$ be a family of Borel-measurable functions on E satisfying Assumption* 1.3 *and assume that γ^* is such that*

$$\min_{\gamma \in \Pi_{\mathcal{F}}} \int c \, d\gamma = \int c \, d\gamma^* \in \mathbb{R}.$$

Then γ^* is finitely minimal/c-monotone.

1.5. Connection with [37]

In an independent work, Zaev [37] obtains (among a number of further developments) a result which is related to Theorem 1.4. His article is concerned with the multi-marginal transport problem described in Subsection 2.1, allowing for additional linear constraints. In our notation this corresponds to problem (GMP) on a set E which is a product $X_1 \times \ldots \times X_n$ of Polish probability spaces and where $\mathcal{F}$ is a superset of the set $\mathcal{F}_2$ defined in (2.4); several important extensions of the transport problem can be phrased in this form. Under continuity and (weak) integrability assumptions Zaev establishes the existence of an optimizer, an extension of the classical Monge-Kantorovich duality as well as a necessary geometric condition for optimizers. The latter statement is equivalent to the assertion of Theorem 1.4 (applied to the setup of [37]). The proof given in [37] is based on his duality result and different from the approach pursued here.

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2. Examples

2.1. Optimal transport and its multi-marginal version

The Monge-Kantorovich problem (OT) fits the framework of (GMP): a measure γ on $E = X \times Y$ is a transport plan in $\Pi(\mu, \nu)$ if and only if

$$\int \varphi(x)\, d\gamma(x, y) = \int \varphi(x)\, d\mu(x),$$

and

$$\int \psi(y)\, d\gamma(x, y) = \int \psi(y)\, d\nu(y)$$

for all continuous bounded functions $\varphi : X \to \mathbb{R}$, $\psi : Y \to \mathbb{R}$. Therefore (OT) is equivalent to (GMP) with

$$\mathcal{F}_1 = \left\{\varphi \circ p_X - \int \varphi\, d\mu : \varphi \in C_b(X)\right\} \cup \left\{\psi \circ p_Y - \int \psi\, d\nu : \psi \in C_b(Y)\right\}.$$

Regarding its statement, the multi-marginal problem is mainly an extension in notation: $\mu_1, \ldots, \mu_n$ are probability measures on Polish spaces $X_1, \ldots, X_n$, the set $\Pi(\mu_1, \ldots, \mu_n)$ consists of the probability measures γ on $E = X_1 \times \ldots \times X_n$ with $p_i(\gamma) = \mu_i$ for $i = 1, \ldots, n$, and the problem is to find

$$\inf \left\{\int c\, d\gamma : \gamma \in \Pi(\mu_1, \ldots, \mu_n)\right\}, \tag{2.1}$$

which is equivalent to (GMP) with

$$\mathcal{F}_2 = \left\{ \varphi \circ p_i - \int \varphi \, d\mu_i : \varphi \in C_b(X_i), 1 \leq i \leq n \right\}. \tag{2.2}$$

Note that c-monotonicity for these problems is just cyclical monotonicity as it is stated in the introduction.

2.2. Model-independent finance – Martingale transport

Starting with the Monge-Kantorovich problem, but looking for an optimizer only among martingale measures, yields the problem of optimal martingale transport (in its most basic formulation). This is closely related to model independent finance, a field that is concerned with determining the possible price range of financial assets under the martingale-paradigm of mathematical finance, see for instance [1,10,17,18]. Roughly speaking, the payoff of a financial asset is represented by a cost function depending on the evolution of the price of an underlying stock. Due to the martingale-pricing paradigm, an arbitrage-free price of the asset is computed as its expected payoff under a martingale measure that is calibrated to market information. The task is hence to find minimum and maximum prices with the help of suitable martingale measures.

Here we have $E = \mathbb{R}^n_+$ or $\mathbb{R}^n$, and $c : E \to \mathbb{R}$. A probability measure γ on E is a martingale measure if and only if for each $l < n$ one has equality and real values in

$$\int x_{l+1} \, \varphi(x_1, \ldots, x_l) \, d\gamma = \int x_l \, \varphi(x_1, \ldots, x_l) \, d\gamma$$

for each continuous bounded function $\varphi : \mathbb{R}^l \to \mathbb{R}$. If we can observe the current value $\xi \in \mathbb{R}$ of the stock price, we only have to consider martingales where all marginals have expectation ξ.

We therefore consider

$$\mathcal{F}^{(\mathrm{mart})} = \{p_1 - \xi\} \cup \left\{ (p_{l+1} - p_l) \, \big(\varphi \circ p_{\{1,\ldots,l\}}\big) : \varphi \in C_b\big(\mathbb{R}^l\big), 1 \leq l < n \right\}. \tag{2.3}$$

The martingale condition (with expectation ξ) then corresponds to $\int f \, d\gamma = 0$ for all $f \in \mathcal{F}^{(\mathrm{mart})}$.

Further market information can be encoded through additional functions. For instance, it is often a reasonable idealization to assume that the marginal distributions of the stock price at particular time instances can be derived from market data. The case of a given marginal distribution at the terminal time t_n has been particularly intriguing.[4] In the present context this corresponds to $p_n(\gamma) = \mu$ for some

[4] This case is naturally connected to the Skorokhod embedding problem, we refer to the survey of Obłój [27].

probability μ, *i.e.*, specifying

$$\mathcal{H} = \left\{ \varphi \circ p_n - \int \varphi \, d\mu : \varphi \in C_b(\mathbb{R}) \right\}. \tag{2.4}$$

More recently the case with all intermediate marginals given has been considered, too. This corresponds to $\mathcal{H} = \mathcal{F}_2$ (where $X_1 = \ldots = X_n = \mathbb{R}$). The main problem of model independent finance can hence be seen as (GMP) with $\mathcal{F}_4 = \mathcal{F}^{(mart)} \cup \mathcal{H}$.

The article [7] discussed the case with $\mathcal{F} = \mathcal{F}^{(mart)} \cup \mathcal{F}_2$ and $n = 2$. The notion of finite optimality introduced there can easily be seen to be equivalent with c-monotonicity for this problem. In fact, that notion and the variational lemma in [7] characterizing optimality via finite optimality have served as a basis for Definition 1.2 and Theorem 1.4.

2.3. A counterexample to sufficiency

It is natural to ask whether the converse of Theorem 1.4 holds true as well, *i.e.*, if finite optimality is also sufficient for optimality overall, at least under additional regularity assumptions on the function c and the underlying spaces. This is not the case as shown by the following counterexample in the context of transport plans which are invariant under group actions (see, *e.g.*, [24]).

Example 2.1. Let $X = Y = (0, 1)$, and $\mu = \nu = \lambda$. For some irrational number $\xi > 0$, let $T : (0, 1) \to (0, 1)$ denote the operator $x \mapsto x \oplus \xi$ (addition of ξ modulo 1). We want to minimize the cost $c(x, y) = (y - x)^2$ among the transport plans π that are $T \otimes T$-invariant, *i.e.*, the transport plans π for which $\pi = \big(T \otimes T\big)(\pi)$. These transport plans are characterized as those for which

$$\int h\big(T \otimes T\big) \, d\pi = \int h \, d\pi \quad \text{for all } h \in C_b(X \times Y).$$

The unique minimizer here is the uniform distribution on the diagonal, but each other transport plan is also concentrated on a finitely minimal set, as each subset of $X \times Y$ is finitely minimal: every finite and finitely supported α is its only competitor. For a competitor α', the signed measure $\alpha - \alpha'$ is $T \otimes T$-invariant, and hence a continuous measure. The only finitely supported such measure is zero, hence $\alpha = \alpha'$.

3. A continuum marginal transport problem revisited

In this section we discuss in some detail the problem introduced by Pass in [30]. For an interval $I = [0, T]$ we consider a family $(\mu_t)_{t \in I}$ of probability measures on $\mathbb{R}$ such that $t \mapsto \mu_t$ is weakly continuous. We consider the space $\mathcal{R}[0, T]$ of Riemann-integrable functions $[0, T] \to \mathbb{R}$ and write $\Pi_{\mathcal{R}}(\mu_t)$ for the set of probability measures with marginals $(\mu_t)_{t \in I}$ on the space $\mathcal{R}[0, T]$. ($\Pi_{\mathcal{R}}(\mu_t)$ is nonempty, see Lemma 3.1 below).

Given a concave function $h : \mathbb{R} \mapsto \mathbb{R}$, the goal is to determine

$$\inf_{\gamma \in \Pi_{\mathcal{R}}(\mu_t)} \int h\left(\int_0^T f(t)\, dt\right) d\gamma(f), \tag{B}$$

the space $\mathcal{R}[0, T]$ is not a Polish space, and so this problem is not exactly an instance of (GMP). We will nevertheless be able to use Theorem 1.4 for deriving an optimality result.

We denote by $q_t : (0, 1) \to \mathbb{R}$ the quantile function of μ_t, *i.e.*, q_t is the generalized inverse of μ_t's distribution function: $q_t(x) = \inf\{y : \mu_t\big((-\infty, y]\big) \geq x\}$. The map $q : (0, 1) \to \mathbb{R}^{[0,T]}$ defined by $x \mapsto q_.(x)$ pushes forward Lebesgue measure λ from $(0, 1)$ to a measure π^* on $\mathbb{R}^{[0,T]}$ that can be described as a uniform distribution on the quantile paths of (μ_t).

Notably $t \mapsto q_t(x)$ is in general not continuous[5], consider, *e.g.*, $T = 1$ and $\mu_t = t\delta_{\{0\}} + (1 - t)\delta_{\{1\}}$ (see Example 3.10 for a counterexample with a family of absolutely continuous measures). In fact, there might even be quantile paths that are not Riemann-integrable: we present the (somewhat lengthy) argument in Example 3.9 at the end of this section.

Nevertheless, the measure π^* can be regarded as a measure on $\mathcal{R}[0, T]$ due to the following lemma.

Lemma 3.1. *If $(\mu_t)_{t\in[0,T]}$ is weakly continuous, then for the measure π^* described above, $\pi^*\big(\mathcal{R}[0, T]\big) = 1$.*

Proof. First note that each quantile path is bounded on the compact interval $[0, T]$. This is an easy consequence of weak continuity.[6] We hence need to show that for λ-a.e. $x \in (0, 1)$ the path $t \mapsto q_t(x)$ is continuous in λ-a.e. $t \in [0, 1]$. Set

$$U = \{(x, t) \in (0, 1) \times [0, 1] : q_.(x) \text{ not continuous in } t\}$$

and

$$U' = \{(x, t) \in (0, 1) \times [0, 1] : q_t(.) \text{ not continuous in } x\}.$$

We have $U \subseteq U'$ due to weak continuity. The set U' is a Borel set: note that $q_t(.)$ is continuous in x if and only if it is right-continuous in x, and the function $(x, t) \mapsto q_t(x)$ is measurable. So U' is the complement of the Borel set

$$\bigcap_n \bigcup_m \left\{(x, t) : q_t\left(x + \frac{1}{m}\right) - q_t(x) < \frac{1}{n}\right\}.$$

[5] Continuity of $t \mapsto q_t(x)$ is claimed to follow from continuity of $t \mapsto \mu_t$ in [30], but this is a glitch.

[6] Otherwise there would be a convergent sequence $t_n \to t$ such that either $q_{t_n}(x) \to \infty$ or $q_{t_n}(x) \to -\infty$. We only consider the first case: pick a $y > x$ and set $G = q_t(y)$. There is an N such that for all $n \geq N$ we have $q_{t_n}(x) > 2G$, and hence also $q_{t_n}(x') > 2G$ for all $x' > x$. We can find an $x' \in (x, y)$ such that $q_t(.)$ is continuous in x'. Therefore, $q_{t_n}(x')$ should converge to $q_t(x')$ but this is impossible as $q_t(x') \leq G < 2G < q_{t_n}(x')$.

Moreover, U' is a null set by Fubini's theorem. Therefore, U is a Lebesgue null set, and we must have $\lambda(U_x)$ for λ-a.e. x. $\square$

Pass shows that π^* is the unique minimizer of (B). Among other conditions, he is building on the assumption that the quantile functions satisfy a property of uniform Riemann-integrability which may be difficult to verify in practice. We will show the following strengthened result.

Theorem 3.2. *Let $h : \mathbb{R} \to \mathbb{R}$ be concave and $(\mu_t)_{t\in I}$ a family of probability measures on $\mathbb{R}$, weakly continuous in t and such that*

$$\int_0^T \int |x|\, d\mu_t(x)\, dt < \infty, \quad \textit{and} \int |h|\, d\mu_t < \infty \ \textit{for all } t \in [0, T].$$

Then π^* is a minimizer of (B). *If the infimum in* (B) *is finite and h is strictly concave, then π^* is the unique minimizer.*

Without loss of generality from now on we will work with $I = [0, T] = [0, 1]$.

For completeness and to fix ideas, we discuss a result that can be seen as a finite-dimensional predecessor to [30] and has been well-known for at least several decades. We mention the note by [19] for a simple geometric proof and further references, and for a more general result [11]. We denote by π_n^* the n-dimensional analogue of the measure π^* introduced above. *I.e.*, given n probability measures $\mu_1, \dots, \mu_n$ on $\mathbb{R}$, then π_n^* is the push forward of Lebesgue measure λ on $(0, 1)$ to $\mathbb{R}^n$ via $x \mapsto \big(q_1(x), \dots, q_n(x)\big)$, where, as before, q_i is the quantile function of μ_i.

Theorem 3.3. *Let $h : \mathbb{R} \to \mathbb{R}$ be strictly concave and $\mu_1, \dots, \mu_n$ be probability measures on $\mathbb{R}$ such that*

$$\int |x|\, d\mu_i < \infty \quad \textit{and} \quad \int |h|\, d\mu_i < \infty, \ \textit{for } 1 \le i \le n.$$

Then π_n^* is the unique minimizer of

$$\inf_{\gamma\in\Pi_n(\mu_1,\dots,\mu_n)} \int h(x_1 + \cdots + x_n)\, d\gamma(x). \tag{3.1}$$

It is intuitive to see why the monotonicity principle is useful for Theorem 3.3. Finite optimality of a set A (in $\mathbb{R}^n$) implies that A must be a monotone set, *i.e.*, $\le$ must be a total order on A: if f and g are both in A, then either $f \le g$ or $g \le f$. Else, set $f' = \max\{f, g\}$ and $g' = \min\{f, g\}$, and let α be the measure $\frac{1}{2}\delta_f + \frac{1}{2}\delta_g$ and α' the measure $\frac{1}{2}\delta_{f'} + \frac{1}{2}\delta_{g'}$. Then α' is a measure with the same marginals as α (on $\mathbb{R}^n$, or $\mathcal{R}[0, 1]$, respectively). But due to strict concavity of h, it is easy to see that α' leads to lower costs than α in both cases (3.1) and (B), contradicting the definition of finite optimality. The argument of optimality of π_n^* (or π^*, respectively) is then completed by another well-known fact, a proof of which we include for reader's convenience.

Lemma 3.4. *Let γ be a probability measure on $\mathbb{R}^n$ with marginals $\mu_1, \dots, \mu_n$. If there is a monotone Borel set M with $\gamma(M) = 1$, then $\gamma = \pi_n^*$.*

Let γ be a probability measure on $\mathbb{R}^{[0,1]}$ with marginals $\big(\mu_t\big)_{t \in I}$. If there is a monotone Borel set M with $\gamma(M) = 1$, then $\gamma = \pi^*$.

Proof. The second part is a simple consequence of the first one since the distribution of a continuous time process is determined by its finite dimensional marginal distributions. Hence, let γ be as in the first statement. For arbitrary points $a_1, \dots, a_n \in \mathbb{R}$, we show that for $I = (-\infty, a_1] \times \cdots \times (-\infty, a_n]$ we have $\gamma(I) = \pi_n^*(I)$. Set $z = \sup\{x : q_i(x) \le a_i \quad \text{for } i = 1, \dots, n\}$. Then we have $\pi_n^*(I) = z$, and for at least one i_0 we have $\mu_{i_0}\big((-\infty, a_{i_0}]\big) = z$. We can hence conclude that $\gamma(I) \le z$. In fact, equality must hold. Observe that from the definition of z we have $\mu_i\big((-\infty, a_i]\big) \ge z$ for all $i = 1, \dots, n$. Hence $\gamma(I) < z$ would imply that for each i there is an element $(b_1^{(i)}, \dots, b_n^{(i)}) \in \Gamma$ such that $b_i^{(i)} \le a_i$, and $b_{j_i}^{(i)} > a_{j_i}$ for some $j_i \ne i$. This contradicts the monotonicity of M. □

Proof of Theorem 3.3. The set $\Pi(\mu_1 \dots, \mu_n)$ is weakly compact. Due to the assumptions on first moments and h-moments of the marginal measures μ_i, the operator to be minimized is lower semi-continuous and bounded. Hence there is a finite minimizer. Strict concavity of h and the above outlined application of the monotonicity principle yield that each finite minimizer must be concentrated on a finitely minimal, hence monotone set. By the preceding lemma, each minimizer must be equal to π_n^*. □

Now we turn to proving Theorem 3.2: we will solve a problem for a countable index set as an intermediate step, where we also add monotonicity and boundedness (from above) to the assumptions on h. We then use the intermediate result in the proof of Theorem 3.2 at the end of this section. Writing $Q = [0, 1] \cap \mathbb{Q}$, we define $\Pi_Q(\mu_q)$ as the set of probability measures on $\mathbb{R}^Q$ with marginals $(\mu_q)_{q \in Q}$. Furthermore, we fix a sequence of finite partitions $(\mathcal{P}_n)$ of $[0, 1]$ with $\mathcal{P}_n \subseteq \mathcal{P}_{n+1} \subseteq Q$ and $\bigcup_n \mathcal{P}_n = Q$. We then replace the original problem (B) by

$$\inf_{\gamma \in \Pi_Q(\mu_q)} \int h\left(\limsup_{n \to \infty} \sum_{t_i \in \mathcal{P}_n} f_{t_i}(t_i - t_{i-1})\right) d\gamma(f). \tag{B'}$$

Writing π_Q^* for the Q-analogue of π^*, we claim che following.

Proposition 3.5. *Let $h : \mathbb{R} \to \mathbb{R}_{\le 0}$ be non-positive, concave and increasing. Provided that $\int |x| \, d\mu_q(x) < \infty$ and $\int |h| \, d\mu_q < \infty$ for all $q \in Q$, the measure π_Q^* is a minimizer of Problem* (B').

The proof is preceded by Lemmas 3.6, 3.7, and 3.8. The assumptions here on h and the marginals are as in Proposition 3.5.

Lemma 3.6. $\Pi_Q\big(\mu_q\big)$ *is weakly compact.*

Proof. By Prokhorov's theorem: let $\varepsilon > 0$ be arbitrary. Then, with $Q = \{q_1, q_2, \dots\}$, for each q_k there exists a compact set $K_k \subseteq \mathbb{R}$ with $\mu_{q_k}(K_k) > 1 - \frac{\varepsilon}{2^k}$. The set $K = \Pi_{k=1}^{\infty} K_k$ is a compact subset of $\mathbb{R}^Q$. For a measure $\gamma \in \Pi_Q(\mu_q)$ we have

$$\gamma(K) = \lim_{n\to\infty} \gamma\big(p^{-1}_{q_1,q_2,\dots,q_n}(K_1 \times K_2 \times \cdots \times K_n)\big).$$

As for each n

$$\gamma\big(p^{-1}_{q_1,q_2,\dots,q_n}(K_1 \times K_2 \times \cdots \times K_n)\big) > 1 - \sum_{k=1}^{n} \frac{\varepsilon}{2^k} \geq 1 - \varepsilon$$

we have $\gamma(K) \geq 1 - \varepsilon$, and Prokhorov's theorem can be applied. □

We introduce some notation:

$$\begin{aligned}
s_n &: \mathbb{R}^Q \to \mathbb{R}, && f \mapsto \sum_{t_i \in \mathcal{P}_n} f_{t_i}(t_i - t_{i-1}),\\
s_n^{(h)} &: \mathbb{R}^Q \to \mathbb{R}, && f \mapsto \sum_{t_i \in \mathcal{P}_n} h(f_{t_i})(t_i - t_{i-1}),\\
\varphi_n &: \mathbb{R}^Q \to \mathbb{R} \cup \{\infty\}, && f \mapsto \sup_{k\geq n} s_k(f),\\
\varphi &: \mathbb{R}^Q \to \mathbb{R} \cup \{-\infty, \infty\}, && f \mapsto \inf_n \varphi_n(f) = \limsup_n s_n(f).
\end{aligned}$$

We continue with the following.

Lemma 3.7. *For each n, the operators defined on $\Pi_Q(\mu_q)$,*

$$S_n : \gamma \mapsto \int h \circ s_n \, d\gamma$$

and

$$\Phi_n : \gamma \mapsto \int h \circ \varphi_n \, d\gamma$$

are lower-semi-continuous (with respect to weak convergence topology) and have minimizers. The values of the minima are finite.

Proof. The existence of minimizers will follow from lower-semi-continuity of the operators and compactness of $\Pi_Q(\mu_q)$. Hence, let $(\gamma_l)_{l\in\mathbb{N}}$ be a sequence in $\Pi_Q(\mu_q)$ converging weakly to some γ_0.

We have

$$\varphi_n \geq s_n$$

and hence, by monotonicity and concavity of h that

$$h \circ \varphi_n \geq h \circ s_n \geq s_n^{(h)}.$$

For each $\gamma \in \Pi_Q(\mu_q)$,

$$\int s_n^{(h)}\,d\gamma = \sum_{t_i \in \mathcal{P}_n} (t_i - t_{i-1}) \int h(f_{t_i})\,d\gamma(f) = \sum_{t_i \in \mathcal{P}_n} (t_i - t_{i-1}) \int h\,d\mu_{t_i}.$$

Hence in particular

$$\lim_{l\to\infty} \int s_n^{(h)}\,d\gamma_l = \int s_n^{(h)}\,d\gamma_0.$$

As $s_n^{(h)}$ is continuous, the prerequisites in [36, Lemma 4.3] are met for both S_n and Φ_n, and applying that result we get

$$\liminf_{l\to\infty} S_n(\gamma_l) \geq S_n(\gamma_0)$$

and

$$\liminf_{l\to\infty} \Phi_n(\gamma_l) \geq \Phi_n(\gamma_0).$$

Finally, the finiteness of the minimal values follows from h being bounded from above, the assumption on finite h-moments of the marginals, and $h \circ \varphi_n \geq h \circ s_n \geq s_n^{(h)}$. □

Lemma 3.8. *For each $n \in \mathbb{N}$, the measure π_Q^* minimizes Φ_n on $\Pi_Q(\mu_q)$.*

Proof. We first show that, when h is strictly concave, the following stronger assertion is true: π_Q^* is the unique measure in $\Pi_Q(\mu_q)$ doing the following:

(0) it minimizes Φ_n,
(1) among the minimizers of Φ_n it minimizes S_1,
(2) among the measures fulfilling (0) and (1), it minimizes S_2,
⋮
(k) among the measures fulfilling (0), (1), …, ($k-1$), it minimizes S_k
⋮

We show existence of a measure fulfilling all the conditions (0), (1), …: write K_0 for the set of minimizers of Φ_n. By the previous lemma, $K_0 \neq \emptyset$. Also, K_0 is compact: for it is a closed subset of the compact set $\Pi_Q(\mu_q)$, where closedness is due to the semi-continuity of Φ_n. Hence, among the minimizers of Φ_n, there is a minimizer of the lower-semi-continuous operator S_1. Writing K_1 for the set of these minimizers, by the same argument as above, K_1 is nonempty and compact. Hence, the set K_2 of minimizers of S_2 on K_1 is nonempty and again compact. By induction we obtain a decreasing sequence of compact nonempty sets K_k. Hence the set $K = \bigcap_k K_k$ is nonempty and each of its elements fulfills properties (0), (1), … Pick such an element and denote it by π_0. We now apply the monotonicity principle to show that π_0 must indeed be equal to π_Q^*: the element π_0 is concentrated on a set Γ that is finitely optimal for each of the problems (k). Observe first that finite optimality of Γ for problem (0) alone does not need to imply that Γ is monotone.[7] However, finite optimality of Γ for problem (1) (*i.e.*, the optimization of S_1 on the set K_0) does imply that Γ must be monotone on $\mathcal{P}_1$, that is, if $f, g \in \Gamma$, then either $f|_{\mathcal{P}_1} \leq g|_{\mathcal{P}_1}$ or $f|_{\mathcal{P}_1} \geq g|_{\mathcal{P}_1}$. For if there were f, g not ordered on $\mathcal{P}_1$, then write $f' = \mathbf{1}_{\mathcal{P}_1} \max(f, g) + \mathbf{1}_{\mathcal{P}_1^c} f$ and $g' = \mathbf{1}_{\mathcal{P}_1} \min(f, g) + \mathbf{1}_{\mathcal{P}_1^c} g$. Set $\alpha = \frac{1}{2}\delta_f + \frac{1}{2}\delta_g$ and $\alpha' = \frac{1}{2}\delta_{f'} + \frac{1}{2}\delta_{g'}$, where δ_f denotes the Dirac-measure on f, etc. Then apparently $S_1(\alpha') < S_1(\alpha)$, but α' is also a competitor of α: it clearly has the same marginals, and we have $\varphi_n(f') = \varphi_n(f)$ and $\varphi_n(g') = \varphi_n(g)$, as manipulating a function $f \in \mathbb{R}^Q$ on finitely many points does not change the value of φ_n. Hence, also $\Phi_n(\alpha') = \int h \circ \varphi_n \, d\alpha' = \int h \circ \varphi_n \, d\alpha = \Phi_n(\alpha)$. The existence of an S_1-better competitor is a contradiction to finite optimality, so Γ must indeed be monotone on $\mathcal{P}_1$. Now for problem (2), we also find that Γ must be monotone on $\mathcal{P}_2$: let $f, g \in \Gamma$, and assume, due to monotonicity of Γ on $\mathcal{P}_1$, that $f|_{\mathcal{P}_1} \geq g|_{\mathcal{P}_1}$. If f and g were not ordered on $\mathcal{P}_2$, then the same construction of f', g', α and α' as above (with $\mathcal{P}_2$ in place of $\mathcal{P}_1$) will give a contradiction to finite optimality: note that $s_1(f') = s_1(f)$ and $s_1(g') = s_1(g)$, as $f' = f$ and $g' = g$ on $\mathcal{P}_1$. Hence, $\Phi_n(\alpha') = \int h \circ \varphi_n \, d\alpha' = \int h \circ \varphi_n \, d\alpha = \Phi_n(\alpha)$ and $S_1(\alpha') = \int h \circ s_1 \, d\alpha' = \int h \circ s_1 \, d\alpha = S_1(\alpha)$, and α' is really a competitor of α.

Iterating this argument one finds that Γ must indeed be monotone on each $\mathcal{P}_k$, and henceforth monotone. But then π_0 must be π_Q^*, because π_Q^* is the only measure in $\Pi_Q(\mu_q)$ concentrated on a monotone set. This last statement follows easily from Lemma 3.4.

Finally, we discuss the case where h is concave, but not necessarily strictly concave. Then, due to the finiteness of $\int |x| \, d\mu_q$ for all $q \in Q$, there is, for each $k \in \mathbb{N}$, a strictly concave function h_k such that $\int |h_k| \, d\mu_q < \infty$ for all $q \in \mathcal{P}_k$. Then by adapting the above argument, it is easy to see that π_Q^* is the only measure in $\Pi_Q\big(\mu_q\big)$ that

(0) minimizes Φ_n

(1′) among the minimizers of Φ_n, it minimizes $\int h_1(s_1) \, d\gamma$,

$\vdots$

(k') among the measures fulfilling (0), …, (k-1'), it minimizes $\int h_k(s_k) \, d\gamma$,

$\vdots$

□

Proof of Proposition 3.5. Let γ be a measure in $\Pi_Q(\mu_q)$. Then for each n, according to the previous lemma it holds

$$\int h \circ \varphi_n \, d\gamma \geq \int h \circ \varphi_n \, d\pi_Q^*.$$

[7] What finite optimality does imply is the following: if f, g are in Γ, and $\varphi_n(f) > \varphi_n(g)$, then one must have $\varphi_n\big((f-g)^+\big) = 0$. This is a weaker condition than $\leq$ being an order on Γ, and explains why one works with the sequence of problems (k) rather than just with problem (0).

As h is increasing and non-positive, and φ_n decreases to $\varphi = \limsup_n s_n$, an application of monotone convergence finishes the proof. □

Finally we can prove Theorem 3.2.

Proof of Theorem 3.2. Note first that due to the regularity assumption that $\int_0^1 \int |x| d\mu_t \, dt < \infty$, without loss of generality we can assume that h only takes values in $\mathbb{R}_{\leq 0}$: otherwise, we would replace it by $h - ax - b$ for suitable values $a, b \in \mathbb{R}$, and take care of the fact that the integral $\int\int_0^1 f(t) dt \, d\gamma(f)$ is finite and invariant among all $\gamma \in \Pi_{\mathcal{R}}(\mu_t)$. This is a consequence of the said regularity assumption and Fubini's theorem applied to the function $(t, f) \mapsto f(t)$, which, on the product $[0, 1] \times \mathcal{R}$ is $\lambda \otimes \gamma$-a.e. equal to a measurable function.

If we further assume for the time being that h is increasing, we can apply Proposition 3.5 to see the optimality of π^* as follows: let p_Q be the projection $\mathbb{R}^I \to \mathbb{R}^Q$, and write p for its restriction on $\mathcal{R}[0, 1]$. For an arbitrary $\gamma \in \Pi_{\mathcal{R}}(\mu_t)$, the measure $p(\gamma)$ is in $\Pi_Q(\mu_q)$ and clearly

$$\int h\left(\int_0^1 f \, dt\right) d\gamma = \int h\left(\limsup_{n\to\infty} \sum_{t_i \in \mathcal{P}_n} f_{t_i}(t_i - t_{i-1})\right) d\, p(\gamma).$$

Moreover, for the right-hand-side one also has, due to Theorem 3.5,

$$\int h\left(\limsup_{n} \sum_{t_i \in \mathcal{P}_n} f_{t_i}(t_i - t_{i-1})\right) d\, p(\gamma) \geq \int h\left(\limsup_{n} \sum_{t_i \in \mathcal{P}_n} f_{t_i}(t_i - t_{i-1})\right) d\, \pi^*_Q.$$

As the right-hand-side of this equals $\int h\big(\int_0^1 f \, dt\big) \, d\pi^*$ we have

$$\int h\left(\int_0^1 f \, dt\right) d\gamma \geq \int h\left(\int_0^1 f \, dt\right) d\pi^*.$$

If h is not increasing, then assume first it is decreasing. If in problem (B') we replace lim sup by lim inf one can show, with the statement and proof of Proposition 3.5 and the above argument suitably adapted, that π^* must be again optimal. Finally, if h is neither increasing nor decreasing, then it can still be written as a sum $h_1 + h_2$, where h_1 is concave, increasing and non-positive, and h_2 is concave, decreasing and non-positive, and again π^* is an optimizer (h_1 and h_2 will satisfy the regularity assumptions as long as h does).

If the minimum is finite and h is strictly concave, each other minimizer must be concentrated on a finitely minimal, hence monotone set and thus be equal to π^*. □

We close this section with two examples which complement Lemma 3.1 and may help to clarify the role of Riemann-integrability.

Example 3.9. Weak continuity of $t \mapsto \mu_t$ does not imply that all quantile paths $x \mapsto q_t(x)$ are Riemann-integrable.

First we construct a set C' that bears some resemblance to the Cantor-set, but is considerably bigger: starting with the unit interval $[0, 1]$, we cut out the middle section $A_1 = \big(a_1^{(1)}, b_1^{(1)}\big)$ with length $1/10$. From the remaining two intervals, we cut out the middle sections $\big(a_1^{(2)}, b_1^{(2)}\big)$, $\big(a_2^{(2)}, b_2^{(2)}\big)$ such that their combined length is $1/100$. In the n-th step, we cut out the 2^{n-1} middle sections $\big(a_1^{(n)}, b_1^{(n)}\big), \dots,$ $\big(a_{2^{n-1}}^{(n)}, b_{2^{n-1}}^{(n)}\big)$ such that their combined length is $1/10^n$. We denote the whole set cut out in the n-th step by A_n and set $C' = [0, 1] - A_1 - A_2 - \dots$. We have $\lambda(C') = 8/9$, and its indicator function is continuous in x if and only if $x \notin C'$. Therefore, $\mathbb{1}_{C'}$ is not Riemann-integrable.

Next, we construct a function f on $[0, 1]$ as the supremum of functions f_n: let f_1 be the function on $[0, 1]$ that on $\big[a_1^{(1)}, b_1^{(1)}\big]$ linearly interpolates between $(a_1^{(1)}, 0)$, $((a_1^{(1)} + b_1^{(1)})/2, 1/10)$ and $(b_1^{(1)}, 0)$, and is equal to zero elsewhere. Let f_n be the function that, for each $i \in \{1, \dots, 2^{n-1}\}$, on $\big[a_i^{(n)}, b_i^{(n)}\big]$ linearly interpolates between $(a_i^{(n)}, 0)$, $((a_i^{(n)} + b_i^{(n)})/2, 1/10^n)$ and $(b_i^{(n)}, 0)$ and is equal to zero elsewhere. Then set $f = \sup_n f_n$. Note that f is continuous on $[0, 1]$.

We consider the family of probability distributions given by

$$\mu_t = \big(1/2 - f(t)\big)\delta_0 + \big(1/2 + f(t)\big)\delta_1.$$

Due to the continuity of f, the family $\big(\mu_t\big)_{t\in[0,1]}$ is weakly continuous in t. For each t, the quantile function $q_t(.)$ jumps from 0 to 1 after $1/2 - f(t)$. We hence find that

$$q_t(1/2) = 1 - \mathbb{1}_{C'}(t).$$

So the path $t \mapsto q_t(1/2)$ is not Riemann-integrable.

Example 3.10. In Lemma 3.1 it is not possible to replace the set of Riemann-integrable functions with the set of continuous functions, even if the family (μ_t) is assumed to consist of absolutely continuous measures:

To see this, let f be a function $[0, 1] \to [0, 1]$ with the following properties: f is strictly increasing and absolutely continuous, $f(0) = 0$ and $f(1) = 1$ and $f(x) > x$ elsewhere.

Then define, for each $t \in [0, 1]$, distribution functions on $[0, 1]$ by

$$F_t(x) = \begin{cases} f(x) & \text{for } x \in [0, t] \\ f(t) & \text{for } x \in [t, f(t)] \\ x & \text{for } x \geq f(t). \end{cases}$$

It is obvious that the family $(\mu_t)_{t\in[0,1]}$ corresponding to the family $(F_t)_{t\in[0,1]}$ is weakly continuous in t, and all μ_t are absolutely continuous with respect to λ. But all quantile paths have a discontinuity: let $x \in (0, 1)$, and t such that $f(t) = x$. Then the quantile path of x, *i.e.*, $q_.(x) = q_.(f(t))$, is discontinuous in t:

$$q_t(x) = q_t(f(t)) = t,$$

but for all $t' < t$ we will have

$$q_{t'}(x) = q_{t'}(f(t)) = f(t).$$

However $f(t) > t$, so $q_.(x)$ is not continuous in t.

4. Proof of Theorem 1.4

In the proof of Theorem 1.4 we will make use of the following result from [5], which is a consequence of a duality result by Kellerer [20].

Lemma 4.1 ([5, Proposition 2.1]). *Let (E, m) be a Polish probability space, and M an analytic*[8] *subset of E^l, then one of the following holds true:*

(i) *There exist m-null sets $M_1, \ldots, M_l \subseteq E$ such that $M \subseteq \bigcup_{i=1}^{l} p_i^{-1}(M_i)$;*
(ii) *There is a measure η on E^l such that $\eta(M) > 0$ and $p_i(\eta) \le m$ for $i = 1, \ldots, l$.*

Proof of Theorem 1.4. Without loss of generality we assume that $|c| \le g$. We want to find a finitely minimal set Γ with $\gamma^*(\Gamma) = 1$. To obtain this, it is sufficient to show that for each $l \in \mathbb{N}$ there is a set Γ_l with $\gamma^*(\Gamma_l) = 1$ such that the following holds: for any finite measure α concentrated on at most l points in Γ_l and satisfying $\alpha(E) \le 1$ as well as $\int g\, d\alpha \le l$, there is no c-better competitor α' on at most l points and satisfying $\int g\, d\alpha' \le l$. For then $\Gamma := \bigcap_{l \in \mathbb{N}} \Gamma_l$ is finitely minimal.

Hence, fix l and define a subset of E^l,

$$\begin{aligned} M = \{&(z_1, \ldots, z_l) \in E^l : \\ &\text{there exists a measure } \alpha \text{ on } E, \alpha(E) \le 1, \textstyle\int g\, d\alpha \le l, \operatorname{supp} \alpha \subseteq \{z_1, \ldots, z_l\}, \\ &\text{so that there is a } c\text{-better competitor } \alpha', \textstyle\int g\, d\alpha' \le l, |\operatorname{supp} \alpha'| \le l\}. \end{aligned}$$

Note that M is the projection of the set

$$\begin{aligned} \hat{M} = \Big\{ &(z_1, \ldots, z_l, \alpha_1, \ldots, \alpha_l, z'_1, \ldots, z'_l, \alpha'_1, \ldots, \alpha'_l,) \in E^l \times \mathbb{R}^l_+ \times E^l \times \mathbb{R}^l_+ : \\ &\sum \alpha_i \le 1, \sum \alpha_i g(z_i) \le l, \sum \alpha_i = \sum \alpha'_i, \sum \alpha'_i g(z'_i) \le l, \\ &\sum \alpha_i f(z_i) = \sum \alpha'_i f(z'_i) \text{ for all } f \in \mathcal{F} \text{ and } \sum \alpha_i c(z_i) > \sum \alpha'_i c(z'_i) \Big\}, \end{aligned}$$

[8] The result in [5, Proposition 2.1] is stated only for Borel sets, but the same proof applies in the case where M is analytic.

onto the first l coordinates. By our Assumption 1.3, the set $\hat{M}$ is Borel, hence M is analytic.

We apply Lemma 4.1 to the space (E, γ^*) and the set M: if (i) holds, then define $N := \bigcup_{i=1}^l M_i$. Then $\Gamma_l := E \setminus N$ has full measure, $\gamma^*(\Gamma_l) = 1$. From the definitions of M and N it can be directly seen that Γ_l is as needed.

If (i) does not hold, (ii) has to. Hence, let us derive a contradiction from it.

Write p_i for the projection of an element of E^l onto its i-th component. We may assume that the measure η in (ii) is concentrated on M, and also fulfills $p_i(\eta) \leq \frac{1}{l}\gamma^*$ for $i = 1, \dots, l$.

We now apply the Jankow – von Neumann selection theorem to the set $\hat{M}$ to define a mapping

$$z \mapsto \big(\alpha_1(z), \dots, \alpha_l(z), z'_1(z), \dots, z'_l(z), \alpha'_1(z), \dots, \alpha'_l(z)\big)$$

such that

$$\big(z, \alpha_1(z), \dots, \alpha_l(z), z'_1(z), \dots, z'_l(z), \alpha'_1(z), \dots, \alpha'_l(z)\big) \in \hat{M}$$

for $z \in M$, and the mapping is measurable with respect to the σ-field generated by the analytic subsets of E^l. Setting

$$\alpha_z := \sum_i \alpha_i(z)\delta_{z_i}, \alpha'_z := \sum_i \alpha'_i(z)\delta_{z'_i(z)}$$

we thus obtain kernels $z \mapsto \alpha_z$ and $z \mapsto \alpha'_z$ from E^l with the σ-field generated by its analytic subsets to E with its Borel-sets. We use these kernels to define measures ω, ω' on the Borel-sets on E through

$$\omega(B) = \int \alpha_z(B)\, d\eta(z) \quad \text{and} \quad \omega'(B) = \int \alpha'_z(B)\, d\eta(z).$$

By construction $\omega \leq \gamma^*$. Moreover ω' is a c-better competitor of ω: for each $f \in \mathcal{F}$ we have

$$\int f\, d\omega' = \iint f\, d\alpha'_z d\eta(z) = \iint f\, d\alpha_z d\eta(z) = \int f\, d\omega.$$

Note that the first and last equality are justified since $\int g\, d\alpha_z, \int g\, d\alpha'_z \leq l$ for all z. Similarly, since $|c| \leq g$, we obtain

$$\int c\, d\omega' = \iint c\, d\alpha'_z d\eta(z) < \iint c\, d\alpha_z d\eta(z) = \int c\, d\omega.$$

Summing up, we obtain a probability measure $\gamma' := \gamma^* - \omega + \omega'$ with $\int c\, d\gamma' < \int c\, d\gamma^*$ and $\gamma' \in \Pi_{\mathcal{F}}$. This contradicts the optimality of γ^*. $\square$

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