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On a singular Liouville-type equation and the Alexandrov isoperimetric inequality

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Abstract. We obtain a generalized version of an inequality, first derived by C. Bandle in the analytic setting, for weak subsolutions of a singular Liouville-type equation. As an application we obtain a new proof of the Alexandrov isoperimetric inequality on singular abstract surfaces. Interestingly enough, motivated by this geometric problem, we obtain a seemingly new characterization of local metrics on Alexandrov's surfaces of bounded curvature. At least to our knowledge, the characterization of the equality case in the isoperimetric inequality in such a weak framework is new as well.

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1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be an open, smooth and bounded domain, K a measurable function on Ω, and ω be a signed measure of bounded total variation in Ω and $\omega = \omega_+ - \omega_-$ be its Jordan decomposition, that is, for a Borel set $E \subseteq \Omega$, it holds $\omega_\pm(E) = \sup_{U \subset E}(\pm\omega(U))$. Then $\omega_\pm$ are non negative and mutually orthogonal measures of bounded total variation on Ω and we define $f = f_+ - f_-$, where f_+ and f_- are two superharmonic functions constructed as follows,

$$f_\pm(x) = h_\pm(x) + \int_\Omega G(x, y) d\omega_\pm(y), \tag{1.1}$$

where $h_\pm$ are harmonic in Ω. Here $G(x, y)$ denotes the Green's function of $-\Delta$ in Ω. We are concerned with some quantitative estimates for subsolutions of the

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Liouville-type equation,

$$-\Delta u = 2Ke^f\, e^u \quad \text{in} \quad \Omega. \tag{1.2}$$

By assuming Ω simply connected, $\partial\Omega$ analytic, u an analytic and $C^0(\overline{\Omega})$ subsolution of (1.2) with $K(x) \equiv K_0$ in Ω for some $K_0 \geq 0$, in a pioneering paper [4], C. Bandle proved that

$$L^2(\partial\Omega) \geq (4\pi - \omega_+(\Omega) - K_0 M(\Omega))\, M(\Omega), \tag{1.3}$$

where

$$L(\partial\Omega) = \int_{\partial\Omega} e^{\frac{f+u}{2}} d\ell, \quad \text{and} \quad M(\Omega) = \int_{\Omega} e^{f+u} dx.$$

Here and in the rest of this paper $d\ell$ and dx are used to denote the integration with respect to the 1-dimensional and 2-dimensional Hausdorff measures $\mathcal{H}^1$ and $\mathcal{H}^2$ respectively.

The inequality (1.3) is sharp and in [4] the case where the equality holds is characterized as well, see also [5]. Actually (1.3) admits a beautiful geometric interpretation in terms of the Alexandrov isoperimetric inequality [2], as discussed in [4] and more extensively in [5]. As we will see later on, the original geometric setting of the problem in terms of singular isothermal coordinates [22], suggests that (1.3) should hold in a more general form. This is our motivation and indeed our main aim is to obtain a generalized version of (1.3) in a weak framework. To state our result, we need some definitions first.

Definition 1.1. We say that $E \subset \mathbb{R}^2$ is a *simple* domain, if it is an open and bounded domain whose boundary ∂E is the support of a rectifiable Jordan curve. We will also say that $E \subset \mathbb{R}^2$ is a *regular* domain if it is a connected, open and bounded domain whose boundary ∂E is the union of finitely many rectifiable Jordan curves.

Definition 1.2. Let $S \subset \Omega$ be a finite set. We say that

$$f \in L^{p,\mathrm{loc}}_{\mathrm{loc}}(\Omega \setminus S) \text{ or either } u \in W^{2,p,\mathrm{loc}}_{\mathrm{loc}}(\Omega \setminus S)\,, \quad \text{for some } p > 2,$$

if for each open and relatively compact set $U \Subset \Omega \setminus S$ there exists $p = p_U > 2$ such that,

$$f \in L^{p_U}(U) \text{ or either } u \in W^{2,p_U}(U)\,.$$

Also, by setting $B_r(S) = \bigcup_{p\in S} B_r(p)$, we say that,

$$f \in L^{p,\mathrm{loc}}(\Omega \setminus S) \text{ or either } u \in W^{2,p,\mathrm{loc}}(\Omega \setminus S)\,, \quad \text{for some } p > 2,$$

if for each $r > 0$, there exists $p_r > 2$ such that,

$$f \in L^{p_r}(\Omega \setminus B_r(S)) \text{ or either } u \in W^{2,p_r}(\Omega \setminus B_r(S))\,.$$

Definition 1.3. Let $f_\pm$ be two superharmonic functions in Ω taking the form (1.1), $u \in L^1_{\mathrm{loc}}(\Omega)$ and $Ke^{f+u} \in L^1_{\mathrm{loc}}(\Omega)$. For any fixed and relatively compact Borel set $E \Subset \Omega$, we define,

$$\mathcal{K}_+(E; K_0) = \sup_{U \subseteq E} \left\{ \frac{1}{2}\omega(U) + \int_U [K - K_0] e^{f+u} dx \right\}, \tag{1.4}$$

where the supremum is taken over all Borel sets $U \subseteq E$.

Finally we will need the following result about the local exponential integrability of e^f. Although similar exponential estimates for logarithmic potentials are well known, see [21] or more recently [11] and [31], it seems that the statement which is really needed here has been introduced only very recently in [1].

Proposition 1.4. *Let $f_\pm$ be two superharmonic functions satisfying* (1.1) *in Ω and*

$$S_{2\pi} = \{x \in \Omega \,:\, \omega_+(x) \geq 2\pi\}. \tag{1.5}$$

Then $S_{2\pi}$ is finite, $d_1 = \frac{1}{4}\mathrm{dist}(S_{2\pi}, \partial\Omega) > 0$ and we have:

(i) *It holds $e^{-(f_- - h_-)} \in L^\infty(\Omega)$ and $e^{(f_+ - h_+)} \in L^{p_0,\mathrm{loc}}(\Omega \setminus S_{2\pi})$ for some $p_0 > 2$;*
(ii) *If*

$$\textit{forall } x \in \Omega, \textit{ it holds } \omega_+(x) < 4\pi. \tag{1.6}$$

Then $e^{(f_+ - h_+)} \in L^{q_0}(\Omega)$ for some $q_0 > 1$.

Let $K_0 \geq 0$, $f_\pm$ be two superharmonic functions taking the form (1.1) and satisfying (1.6), $q_0 > 1$ and $p_0 > 2$ be defined as in Proposition 1.4 and $K \in L^{n,\mathrm{loc}}_{\mathrm{loc}}(\Omega \setminus S_{2\pi}) \cap L^s_{\mathrm{loc}}(\Omega)$, for some $n > \frac{2p_0}{p_0 - 2}$ and some $s > \frac{q_0}{q_0 - 1}$. Our main results consist in finding weak but still sufficient conditions to ensure that the following inequality

$$L^2(\partial E) \geq (4\pi - 2\mathcal{K}_+(E; K_0) - K_0 M(E))\, M(E), \tag{1.7}$$

holds, with a full characterization of the equality sign, where E is any relatively compact subdomain $E \Subset \Omega$.

Theorem 1.5. *Assume that*

$$\begin{aligned} &Ke^{f+u} \in L^1(\Omega), \textit{ where } u \in L^1(\Omega) \\ &\textit{is a solution of } (1.2) \textit{ in the sense of distributions.} \end{aligned} \tag{1.8}$$

Then $u \in W^{2,p,\mathrm{loc}}_{\mathrm{loc}}(\Omega \setminus S_{2\pi}) \cap W^{2,q}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$, for some $p > 2$ and some $q \in (1, 2)$, and in particular u is a strong solution of (1.2)*, that is,*

$$-\Delta u = 2Ke^f e^u \quad \textit{for a.a. } x \in \Omega. \tag{1.9}$$

Moreover, for any fixed simple and relatively compact subdomain $E \Subset \Omega$, we have $M(E) < +\infty$ and (1.7) *holds.*

Theorem 1.6. *Assume that* $u \in W^{2,p,\mathrm{loc}}_{\mathrm{loc}}(\Omega \setminus S_{2\pi}) \cap W^{2,q}_{\mathrm{loc}}(\Omega)$, *for some* $p > 2$ *and some* $q > 1$, *is a strong subsolution of* (1.2), *that is,*

$$-\Delta u \leq 2K e^f e^u \quad \textit{for a.a. } x \in \Omega. \tag{1.10}$$

Then, for any fixed simple and relatively compact subdomain $E \Subset \Omega$, *we have* $M(E) < +\infty$ *and* (1.7) *holds.*

Theorem 1.7. *Let the assumptions of either Theorem* 1.5 *or Theorem* 1.6 *be satisfied. Then we have:*

(i) *The equality sign in* (1.7) *is attained if and only if* u *is a strong solution of* (1.2) *in* E *and,*

$$e^{f(z)+u(z)} = \frac{\tau^2 \left|\Phi_0'(z)(\Phi_0(z))^{-\alpha}\right|^2}{\left(1 + \frac{K_0\tau^2}{4(1-\alpha)^2}|\Phi_0(z)|^{2(1-\alpha)}\right)^2}, \quad \textit{for } z \in E, \tag{1.11}$$

for some $\tau \neq 0$, *where* $\alpha = \frac{1}{4\pi}\omega_+(E)$ *and* Φ_0 *is a conformal map of* E *onto the disk of unit radius,* $|\Phi_0(z)| < 1$ *with* $\Phi_0(z_0) = 0$, *for some* $z_0 \in E$;

(ii) *If* $\omega \perp e^{f+u}\mathcal{H}^2$, *then the equality holds if and only if, in addition to the above conditions, one has* $K \equiv K_0$ *for a.a.* $z \in E$ *and* $\omega = 4\pi\alpha\delta_{z=z_0}$, *that is,* $f(z) = h(z) + 4\pi\alpha G(z, z_0) = h(z) - 2\alpha \log|\Phi_0(z)|$, *for some harmonic function* h *in* E.

Remark 1.8. By using the fact that $\omega_+ \perp \omega_-$, it is easy to check that if $\omega_- \perp e^{f+u}\mathcal{H}^2$, then we have

$$2\mathcal{K}_+(E; K_0) = \omega_+(E) + 2\int_E [K - K_0]^+ e^{f+u} dx,$$

where $K^+ = \max\{K, 0\}$, while in general the equality sign should be replaced by the inequality sign.

As far as one is just concerned with the inequality and not with the characterization of the equality sign, then, if the assumptions of Theorem 1.5 are satisfied, then (1.7) holds under much weaker conditions. The proof of this fact is based on Theorem 1.5 and on some results and arguments in [11] about the regularity properties of Liouville-type equations.

Corollary 1.9. *Let* $K_0 \geq 0$, *and* $f_\pm$ *be two superharmonic functions taking the form* (1.1) *and satisfying* (1.6), *and* $K \in L^1(\Omega; e^{f+u}\mathcal{H}^2)$ *where* $u \in L^1(\Omega)$ *is a solution of* (1.2) *in the sense of distributions. Then:*

(i) *It holds* $u \in W^{1,r}_{\mathrm{loc}}(\Omega)$ *for any* $r \in (1, 2)$ *and* $e^{t|u|} \in L^1_{\mathrm{loc}}(\Omega)$ *for any* $t \geq 1$;
(ii) *The inequality* (1.7) *holds.*

Motivated by the study of a cosmic string equation, in a recent paper [8] we derived (1.7) in the easier situation where $\omega_- \equiv 0$ while ω_+ is proportional to a Dirac delta. The problem here is more subtle, and the crux of the proof is to attach to each strong subsolution of (1.2) an auxiliary function (which we will denote by η) which satisfies a Liouville type equation with Dirichlet boundary condition on E, and which admits a suitable locally absolutely continuous weighted rearrangement (which we will denote by η^*). The difficulty arises since, in view of the generality suggested by the geometric application, no assumption is made about ω, with the unique exception of the "no-cusp" hypothesis (1.6). As a consequence, the term e^{f_+}, which is part of the weight factor in the weighted rearrangement, can come with almost any kind of singularity. In particular, the standard argument [25] yielding the absolute continuity of η^*, does not work in this case, neither in the slightly improved form used to handle conical singularities, see [8]. We succeed in solving this problem by a careful decomposition of the singular set of ω_+, see the definition of $S_{2\pi}$ in (1.5). The point is that $S_{2\pi}$ is finite in Ω, while, locally in its complement, we come up with enough summability for e^f to guarantee that η^* is absolutely continuous. This approach, recently pursued in [1] to prove a regularity result for a class of singular surfaces introduced by Alexandrov [3], motivates the peculiar notations introduced in Definition 1.2. In particular, the assumptions about K and u, are essentially the minimal requirements to match the regularity of η as allowed by the properties of f derived in this way. The characterization of the equality case in this weak contest is new as well.

To avoid repetitions we provide a unified proof of Theorems 1.5, 1.6 and 1.7, which is divided in four steps. In the first and second step we construct η, its weighted rearrangement η^* and prove that η^* is locally absolutely continuous. Step four contains the discussion about the equality case. Step three is the adaptation in our setting of the part of Bandle's argument which is concerned with the derivation of a differential inequality and its consequences.

In the second part of this paper, and in the same spirit of [4], we will apply (1.7) to derive a new proof of the Alexandrov isoperimetric inequality for $K_0 \geq 0$ on abstract surfaces of bounded curvature, see (4.9) in Theorem 4.7. We refer the reader to [5,6,13,17,19] and the references therein for a detailed exposition of the proof and of the interesting history of Alexandrov's inequality and to [26,27] for other more recent proofs. See also [9,23]. While in the above references one can find various proofs of the inequality (4.9), we were not able to find a proof of the characterization of the equality case in the weak context pursued here, which seems therefore to be new even in the geometric setting.

Besides, to apply our estimates to this problem, we need to prove a seemingly new characterization of the structure of the metrics in local isothermal coordinates for certain classes of singular surfaces, see Theorem 4.4. This intermediate result can also be seen as another result in the description of the regularity properties of isothermal coordinates systems on Alexandrov's surfaces of bounded curvature recently pursued in [1]. Finally, some explicit examples are discussed to illustrate these results, including the isoperimetric inequality (4.9) on various singular surfaces homeomorphic to the 2-sphere.

We conclude this introduction with a remark about the case where E is not simple but just regular, that is, the possibility that E could be connected but not simply connected.

Remark 1.10. If Ω is simply connected and the assumptions of either Theorem 1.5 or of Theorem 1.6 are satisfied, but if the set $E \Subset \Omega$ is just assumed to be regular, then it is straightforward to check that our proof yields the following inequality,

$$L^2(\partial E) > (4\pi - 2\mathcal{K}_+(E_s; K_0) - K_0 M(E))\, M(E), \tag{1.12}$$

where E_s is the interior of the closure of the union of E with the bounded components of $\mathbb{R}^2 \setminus E$ (the "holes" of E) , which we denote by $(E)_B$, that is,

$$E_s = \overset{\circ}{\overline{E \cup (E)_B}}.$$

In other words we still have an inequality of the form (1.7), but we have a worse isoperimetric ratio, which is essentially obtained by subtracting the terms of the total curvature relative to the "holes" of E. This is not a technical point, and in fact it is possible to construct counterexamples to the inequality where these terms are omitted, see for example [9, page 14]. The proof of this inequality is really the same as that of Theorems 1.5, 1.6, but for the fact that in (3.22) and in (3.23) below we use the Huber inequality (2.2) for the non contractible domain E. In particular this is also why we obtain the strict inequality in this case. It is straightforward to check that if the (weaker) assumptions of Corollary 1.9 are satisfied, then (1.12) holds with the strict inequality replaced by the $\geq$ sign.

This paper is organized as follows. In section 2 we prove Proposition 1.4 and discuss the Huber's inequality. In Section 3 we prove Theorems 1.5, 1.6, 1.7 and Corollary 1.9. Sections 4 and 5 are devoted to the discussion of the Alexandrov isoperimetric inequality and the related examples.

2. Preliminary estimates: exponential summability of subharmonic functions and Huber's inequality

The local exponential integrability of e^{f_+} as claimed in Proposition 1.4 is not new, see [1]. We provide the proof of Proposition 1.4 for the sake of completeness.

Proof of Proposition 1.4. We will denote by d_Ω the diameter of Ω. Clearly $S_{2\pi}$ is finite since ω_+ is finite, whence obviously $\mathrm{dist}(S_{2\pi}, \partial\Omega) > 0$. Let $d_1 = \frac{1}{4}\mathrm{dist}(S_{2\pi}, \partial\Omega)$ and let us set $\Omega_d = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < d\}$. Then $\omega_+(\Omega_d) \searrow 0^+$ as $d \searrow 0^+$, whence there exists $d_0 > 0$ such that $\omega^+(\Omega_d) < \frac{\pi}{2}$, for each $d < 4d_0$. We choose d_0 possibly smaller to satisfy $4d_0 < d_1$. It is not difficult to see that there exists $C_0 > 0$ such that,

$$(f_+(x) - h_+(x)) - C_0 \leq w_0(x) := \frac{1}{2\pi} \int_{\Omega_{2d_0}} \log\left(\frac{d_\Omega}{|x-y|}\right) d\omega_+, \text{ for all } x \in \Omega_{d_0}.$$

By Jensen's inequality and Fubini-Tonelli theorem we can estimate,

$$\int_{\Omega_{d_0}} \exp\left(\frac{3\pi w_0}{\omega_+(\Omega_{2d_0})}\right) dx$$

$$\leq \int_{\Omega_{d_0}} dx \int_{\Omega_{2d_0}} \left(\frac{d_\Omega}{|x-y|}\right)^{\frac{3}{2}} \frac{d\omega_+(y)}{\omega_+(B_{2d_0})} = \int_{\Omega_{2d_0}} \frac{d\omega_+(y)}{\omega_+(\Omega_{2d_0})} \int_{\Omega_{d_0}} \left(\frac{d_\Omega}{|x-y|}\right)^{\frac{3}{2}} dx$$

$$\leq \int_{\Omega_{2d_0}} \frac{d\omega_+(y)}{\omega_+(\Omega_{2d_0})} \int_{B_{d_\Omega}(y)} \left(\frac{d_\Omega}{|x-y|}\right)^{\frac{3}{2}} dx = \pi(2d_\Omega)^2,$$

where we used the fact that $\Omega_{d_0} \subset B_{d_\Omega}(y)$. This inequality shows that $e^{(f_+-h_+)} \in L^6(\Omega_{d_0})$.

(i) Since $-(f_- - h_-)$ is negative, then $e^{-(f_- - h_-)} \in L^\infty(\Omega)$. Let $\Omega_0 = \{\Omega \setminus \Omega_{\frac{d_0}{2}}\} \setminus B_r(S_{2\pi})$, with $0 < r < d_1$, and let us fix $x_0 \in \overline{\Omega_0}$. Since $\omega_+(x_0) < 2\pi$, then we can find $\varepsilon > 0$ such that there exists $R > 0$ depending on x_0 and ε, such that the ball centred at x_0, say $B_{2R} := B_{2R}(x_0)$, satisfies $B_{2R} \Subset \{\Omega \setminus \Omega_{d_0}\} \setminus S_{2\pi}$ and $\omega_+(B_{2R}) \leq 2\pi - 2\varepsilon$. As above there exists $C > 0$ such that,

$$(f_+(x) - h_+(x)) - C \leq w(x) := \frac{1}{2\pi}\int_{B_{2R}} \log\left(\frac{4R}{|x-y|}\right) d\omega_+,$$
$$\text{for all } x \in D_{x_0} \equiv D_0 := B_R(x_0),$$

and for any $\delta < 4\pi$ we can estimate,

$$\int_{D_0} \exp\left(\frac{(4\pi-\delta)w}{\omega_+(B_{2R})}\right) dx$$

$$\leq \int_{D_0} dx \int_{B_{2R}} \left(\frac{d_\Omega}{|x-y|}\right)^{2-\frac{\delta}{2\pi}} \frac{d\omega_+(y)}{\omega_+(B_{2R})} = \int_{B_{2R}} \frac{d\omega_+(y)}{\omega_+(B_{2R})} \int_{D_0} \left(\frac{d_\Omega}{|x-y|}\right)^{2-\frac{\delta}{2\pi}} dx$$

$$\leq \int_{B_{2R}} \frac{d\omega_+(y)}{\omega_+(B_{2R})} \int_{B_{d_\Omega}(y)} \left(\frac{d_\Omega}{|x-y|}\right)^{2-\frac{\delta}{2\pi}} dx = \frac{(2\pi d_\Omega)^2}{\delta}.$$

Therefore, in particular by choosing $\delta < \varepsilon$, we see that $p_{D_0} := \frac{(4\pi-\delta)}{\omega_+(D_0)} > \frac{(4\pi-\varepsilon)}{\omega_+(B_{2R})} > 2$ so that $e^{(f_+-h_+)} \in L^{p_{D_0}}(D_0)$, for some $p_{D_0} > 2$ depending on x_0 and R.

At this point we define $\mathcal{B} = \bigcup_{x\in\overline{\Omega_0}} D_x$, where each D_x, constructed as above, comes with its own $p_{D_x} > 2$. Clearly $\mathcal{B}$ is an open cover of $\overline{\Omega_0}$, and since $\overline{\Omega_0}$ is compact, then we can extract a finite cover D_{x_j}, with $j = 1, \ldots, N$, and set $p_U := \min\{6, \min_{j=1,\ldots,N} p_{D_{x_j}}\}$. Therefore $e^{(f_+-h_+)} \in L^{p_U}(\Omega \setminus B_r(S_{2\pi}))$, for some $p_U > 2$, which proves (i).

(ii) Let us define $\Omega_1 = \Omega \setminus \Omega_{\frac{d_0}{2}}$. We use (1.6), as in the proof of (i) to conclude that $e^{(f_+-h_+)} \in L^q(\Omega_1)$ for some $q > 1$. Therefore we find $e^{(f_+-h_+)} \in L^{q_0}(\Omega)$ where $q_0 = \min\{6, q\} > 1$, as claimed. $\square$

Next we present the well known Huber inequality [18] as well as a generalization, suitable to be applied to regular (whence in particular non simply connected) domains.

Theorem 2.1 (Huber inequality, [18]). *Let $\Omega \subset \mathbb{R}^2$ be open and bounded and $E \Subset \Omega$ be a simple and relatively compact subset. Let f be the difference of two superharmonic functions in Ω taking the form* (1.1). *Then it holds*

$$\left(\int_{\partial E} e^{\frac{f}{2}} d\ell\right)^2 \geq (4\pi - \omega_+(E)) \int_E e^f dx. \tag{2.1}$$

The equality holds in (2.1) *if and only if, in complex notations,*

$$f(z) = c + 2\log\left|\Phi'(z)(\Phi(z)^{-\alpha_E})\right|$$

where $\alpha_E = \frac{1}{4\pi}\omega_+(E)$ and Φ is a conformal map of E onto the disk of unitary radius $|w| = |\Phi(z)| < 1$ with $\Phi(z_0) = 0$ for some $z_0 \in E$.

We will need the following generalization of the Huber result.

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^2$ be open and bounded and $E \Subset \Omega$ be a simple and relatively compact subset. Let f be the difference of two superharmonic functions in Ω taking the form* (1.1). *If $U \subseteq E$ is a regular domain, then it holds*

$$\left(\int_{\partial U} e^{\frac{f}{2}} d\ell\right)^2 \geq (4\pi - \omega_+(E)) \int_U e^f dx. \tag{2.2}$$

In particular, if U is not simply connected, then the inequality is strict.

Proof. In view of Theorem 2.1 we are left to discuss the cases where U is not simply connected and prove in particular that in all those cases the inequality is strict. Obviously the strict inequality is trivially satisfied if $\omega_+(E) \geq 4\pi$, whence we assume without loss of generality that $\omega_+(E) < 4\pi$. Let us assume for the moment that $U = U_1 \setminus \overline{U_0}$ for a pair of simple domains such that $U_0 \Subset U_1$ and $\partial U = \partial U_1 \cup \partial U_0$. So $U_1 = U \cup \overline{U_0}$ and in this case, by assumption we have $E = U_1$ and in particular $\omega_+(U_0) < \omega_+(U_1) < 4\pi$. For any domain $U \subset \mathbb{R}^2$, let us set

$$\ell(\partial U) = \int_{\partial U} e^{\frac{f}{2}} d\ell, \quad \text{and} \quad M(U) = \int_U e^f dx.$$

Thus we may use (2.1) to obtain

$$\begin{aligned}
\ell^2(\partial U) = \ell^2(\partial U_1 \cup \partial U_0) &> \ell^2(\partial U_1) + \ell^2(\partial U_0) \\
&\geq (4\pi - \omega_+(U_1))\, M(U_1) + (4\pi - \omega_+(U_0))\, M(U_0) \\
&> (4\pi - \omega_+(U_1))\, M(U_1) > (4\pi - \omega_+(U_1))\, M(U),
\end{aligned}$$

which is (2.2) in this particular case. The case where $\mathbb{R}^2 \setminus U$ has finitely many bounded components readily follows by an induction argument on the number of "holes" of U. Obviously the inequality is always strict whenever U is not simply connected. □

3. The proof of Theorems 1.5, 1.6, 1.7 and of Corollary 1.9

This section is devoted to the proof of Theorems 1.5, 1.6, 1.7 and of Corollary 1.9.

Proof of Theorems 1.5, 1.6, 1.7. Once the result has been established for $K_0 \neq 0$, then the case $K_0 = 0$ is worked out by an elementary limiting argument, which is why we will just discuss the case $K_0 > 0$. □

We recall that by assumption $\frac{np_0}{n+p_0} > 2$ and $\frac{sq_0}{s+q_0} > 1$. First of all, we have the following,

Lemma 3.1. *The following facts hold:*

(a) *If* (1.8) *holds and if* $K \in L^s_{\mathrm{loc}}(\Omega)$ *for some* $s > \frac{q_0}{q_0-1}$, *then,*

$$Ke^{f+u} \in L^r_{\mathrm{loc}}(\Omega),$$

and $u \in L^\infty_{\mathrm{loc}}(\Omega) \cap W^{2,r}_{\mathrm{loc}}(\Omega)$, *for any* $1 \le r \le \frac{sq_0}{s+q_0}$. *In particular* u *is a strong solution of* (1.2)*;*

(b) *If* (1.8) *holds and if* $K \in L^{n,\mathrm{loc}}_{\mathrm{loc}}(\Omega \setminus S_{2\pi}) \cap L^s_{\mathrm{loc}}(\Omega)$ *for some* $n > \frac{2p_0}{p_0-2}$ *and some* $s > \frac{q_0}{q_0-1}$, *then,*

$$Ke^{f+u} \in L^{k,\mathrm{loc}}_{\mathrm{loc}}(\Omega \setminus S_{2\pi}) \cap L^r_{\mathrm{loc}}(\Omega), \tag{3.1}$$

and $u \in W^{2,k,\mathrm{loc}}_{\mathrm{loc}}(\Omega \setminus S_{2\pi}) \cap W^{2,r}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$, *for any* $2 < k \le \frac{np_0}{n+p_0}$ *and* $1 \le r \le \frac{sq_0}{s+q_0}$. *In particular* u *is a strong solution of* (1.2).

Proof. (a) By assumption we have $\frac{sq_0}{s+q_0} > 1$ and then, in view of Proposition 1.4, we also have $Ke^f \in L^q_{\mathrm{loc}}(\Omega)$, for all $1 < q \le \frac{sq_0}{s+q_0}$. On the other hand, since $Ke^{f+u} \in L^1(\Omega)$, then, by [11, Remark 2], we have $e^{|u|} \in L^k_{\mathrm{loc}}(\Omega)$ for any $k > 0$, and therefore in particular $e^u \in L^{q'}(\Omega)$, where $q' = \frac{q}{q-1} < +\infty$. Thus we can apply another result in [11] (see [11, Remark 5]), which yields $u \in L^\infty_{\mathrm{loc}}(\Omega)$. So, by standard elliptic estimates, we conclude also that $u \in W^{2,r}_{\mathrm{loc}}(\Omega)$ and in particular that u is a strong solution of (1.2).

(b) Next, let us fix a compact set $U \subset \Omega \setminus S_{2\pi}$ and observe that, by assumption, $K \in L^n(U)$ for some $n > \frac{2p_U}{p_U-2}$. Therefore $\frac{np_U}{n+p_U} > 2$ and then, in view of Proposition 1.4, we also have $Ke^f \in L^p(U)$, $\forall\, 2 < p \le \frac{np_U}{n+p_U}$. Since U is arbitrary, then we conclude that $Ke^{f+u} \in L^{k,\mathrm{loc}}_{\mathrm{loc}}(\Omega \setminus S_{2\pi}) \cap L^r_{\mathrm{loc}}(\Omega)$, for any $2 < k \le \frac{np_U}{n+p_U}$ and $1 < r \le \frac{sq_0}{s+q_0}$. As above, by standard elliptic estimates, we conclude also that $u \in W^{2,k,\mathrm{loc}}_{\mathrm{loc}}(\Omega \setminus S_{2\pi}) \cap W^{2,r}_{\mathrm{loc}}(\Omega)$ and in particular that u is a strong solution of (1.2). □

Lemma 3.1 shows that if (1.8) holds, then $u \in W^{2,p,\mathrm{loc}}_{\mathrm{loc}}(\Omega \setminus S_{2\pi}) \cap W^{2,q}_{\mathrm{loc}}(\Omega) \cap L^{\infty}_{\mathrm{loc}}(\Omega)$, for some $p > 2$ and $q > 1$, and moreover that u is a strong solution of (1.2). Whence we are reduced to the analysis of the case where $u \in W^{2,p,\mathrm{loc}}_{\mathrm{loc}}(\Omega \setminus S_{2\pi}) \cap W^{2,q}_{\mathrm{loc}}(\Omega) \cap L^{\infty}_{\mathrm{loc}}(\Omega)$, for some $p > 2$ and $q > 1$, satisfies (1.10). In particular, in the rest of the proof, we will use the fact that, by the Sobolev embedding Theorem, $u \in C^0_{\mathrm{loc}}(\Omega)$. Clearly, in view of (3.1), $M(E)$ is finite. We divide the proof into four steps.

Step 1. Since $E \Subset \Omega$ is relatively compact and simple, then we can find an open, simply connected, relatively compact and smooth domain Ω_0 such that,

$$E \Subset \Omega_0 \Subset \Omega.$$

Since $S_{2\pi}$ is finite and since $\omega_{\pm}(\Omega_0) < +\infty$, then we can choose Ω_0 such that, for some $N \in \mathbb{N}$,

$$S^0_{2\pi} := S_{2\pi} \cap \Omega_0 = \{q_1, \dots, q_N\} \subset \Omega_0 \text{ and } \partial\Omega_0 \cap S_{2\pi} = \emptyset. \tag{3.2}$$

Clearly, in view of (1.10), we have,

$$-\Delta u \le 2Ke^f e^u = 2[K - K_0]e^f e^u + 2K_0 e^f e^u \quad \text{for a.a. } x \in \Omega_0. \tag{3.3}$$

Next, let us define,

$$\phi(x) := -\Delta u - 2[K - K_0]e^f e^u - 2K_0 e^f e^u, \quad \text{for } x \in \Omega_0. \tag{3.4}$$

Since $u \in W^{2,p,\mathrm{loc}}_{\mathrm{loc}}(\Omega \setminus S_{2\pi}) \cap W^{2,q}(\Omega) \cap C^0_{\mathrm{loc}}(\Omega)$, for some $p > 2$ and some $q > 1$, and in view of (3.1) and of Proposition 1.4, we see from (3.3) that,

$$\phi(x) \le 0, \quad \text{for a.a. } x \in \Omega_0 \quad \text{and} \quad \phi \in L^{p,\mathrm{loc}}_{\mathrm{loc}}(\Omega_0 \setminus S^0_{2\pi}) \cap L^q(\Omega_0),$$

for some $p > 2$ and some $q > 1$. Therefore, in view of [16, Theorem 9.15, Corollary 9.18 and Lemma 9.17] we see that the linear problem,

$$\Delta w = \phi \ \text{ in } \ \Omega_0, \qquad w = 0 \ \text{ on } \ \partial\Omega_0, \tag{3.5}$$

admits a unique strong solution $w \in W^{2,p,\mathrm{loc}}_{\mathrm{loc}}(\Omega_0 \setminus S^0_{2\pi}) \cap W^{2,q}(\Omega_0) \cap C^0(\overline{\Omega_0})$, for some $p > 2$ and some $q > 1$. Obviously w is superharmonic (see [16, Section 2.8 and Example 2.7, 2.8]).

Next let f_1 be the Perron's (see [16, Section 2.8]) solution of $\Delta f_1 = 0$ in E, $f_1 = -u$ on ∂E. Since $u \in C^0(\overline{E})$, then f_1 is well defined and continuous up to the boundary (see [16, Section 2.8]). Let us also define f_2 to be the unique $W^{2,p,\mathrm{loc}}_{\mathrm{loc}}(\Omega_0 \setminus S^0_{2\pi}) \cap W^{2,q}(\Omega_0) \cap C^0(\overline{\Omega_0})$ (for some $p > 2$ and some $q > 1$) solution of the linear problem,

$$-\Delta f_2 = 2[K - K_0]e^f e^u \ \text{ in } \ \Omega_0, \qquad f_2 = 0 \ \text{ on } \ \partial\Omega_0.$$

With these definitions, we may finally set $\eta = u + w + f_1 - f_2$. Then, we see that $\eta \in W^{2,p,\mathrm{loc}}_{\mathrm{loc}}(E \setminus S^0_{2\pi}) \cap W^{2,q}(E) \cap C^0(\overline{E})$ for some $p > 2$ and some $q > 1$ and satisfies,

$$-\Delta\eta = 2K_0\, e^{\psi}\, e^{\eta} \quad \text{for a.a. } x \in E, \quad \eta = 0 \ \text{ on } \ \partial E, \tag{3.6}$$

where

$$\psi = f_+ + f_2 - f_- - w - f_1. \tag{3.7}$$

By Sobolev embedding theorem we conclude that,

$$\eta_+ \in C^1_{\mathrm{loc}}\big(E \setminus S^0_{2\pi}\big). \tag{3.8}$$

Since $\eta \in W^{2,q}(E)$, for some $q > 1$, then by using Sobolev embedding once more we see that $\eta \in W^{1,2}(E) \cap C^0(E)$. Then by the maximum principle for weak solutions (see for example [16, Theorem 8.1]) we deduce that $\eta \geq 0$. In particular, by the strong maximum principle for weak supersolutions (see for example [16, Theorem 8.18]) we also check that η is strictly positive in E. In particular, we conclude that,

$$\eta(x) > 0 \quad \text{for all} \quad x \in E \quad \text{and} \quad \eta(x) = 0 \iff x \in \partial E. \tag{3.9}$$

Step 2. Let us set $t_+ = \max_{\overline{E}} \eta$,

$$d\tau = e^{\psi} dx, \qquad \sigma = e^{\frac{\psi}{2}} d\ell,$$

and let us define,

$$\begin{aligned} &\Omega(t) = \{x \in E \mid \eta(x) > t\}, \quad t \in [0, t_+), \\ \text{and} \quad &\Gamma(t) = \{x \in E \mid \eta(x) = t\}, \quad t \in [0, t_+], \end{aligned}$$

and

$$\mu(t) = \int_{\Omega(t)} d\tau.$$

Since η satisfies (3.6), then $\Gamma(t)$ has null measure, whence we conclude that μ is continuous. Moreover, in view of (3.9), we find that,

$$\Omega(0) = E, \qquad \Gamma(0) = \partial E, \qquad \mu(0) = \int_E d\tau. \tag{3.10}$$

Clearly we can extend μ on $[0, t_+]$ by setting $\mu(t_+) = \lim_{t \nearrow t_+} \mu(t) = 0^+$, whence $\mu \in C^0([0, t_+])$. Next, by using (3.6) once more, it is not difficult to see that the 2-dimensional measure of the set $\{x \in E \ : \ \nabla\eta(x) = 0\}$ vanishes. Therefore, by a well known consequence of the co-area formula (see for example [12, page 158]) and of Sard's Lemma for Sobolev functions [14] (here we use also (3.2)), we see that,

$$\frac{d\mu(t)}{dt} = -\int_{\Gamma(t)} \frac{e^{\psi}}{|\nabla\eta|}\, d\ell, \tag{3.11}$$

for a.a. $t \in [0, t_+]$.

At this point, for any $s \in [0, \mu(0)) \equiv [\mu(t_+), \mu(0))$, we introduce a weighted rearrangement of η,

$$\eta^*(s) = |\{t \in [0, t_+] : \mu(t) > s\}|, \tag{3.12}$$

where $|U|$ denotes the Lebesgue measure of a Borel set $U \subset \mathbb{R}$. By setting $\eta^*(\mu(0)) = 0$, then $\eta^* \in C^0([0, \mu(0)])$ is the inverse of μ on $[0, t_+]$ and coincides with the distribution function of μ. Actually η^* is strictly decreasing, whence differentiable almost everywhere. A crucial point at this stage is to prove that η^* is not just continuous and differentiable almost everywhere, but also locally absolutely continuous. It turns out that in fact it is locally Lipschitz in $(0, \mu(0))$ as shown in the following Lemma.

Lemma 3.2. *For any* $0 < \overline{a} \le a < b \le \overline{b} < \mu(0)$*, there exist* $\overline{C} = \overline{C}(\overline{a}, \overline{b}, S_{2\pi}, \mathcal{K}_+(E; K_0)) > 0$ *such that,*

$$\eta^*(a) - \eta^*(b) \le \overline{C}(b - a). \tag{3.13}$$

Proof. In view of (3.2) and (3.8), we see that $|\nabla\eta| \le C_U$ on any $U \Subset E \setminus S^0_{2\pi}$. Let us then set $t_i = \eta(x_i)$ and $x_i \in S_{2\pi}$ for $i = 1, \dots, m$, with $m \le N$, and $t_0 = \eta^*(a)$ and $t_{m+1} = \eta^*(b)$. For any

$$\varepsilon < \min\left\{\frac{|\eta^*(a) - \eta^*(b)|}{4(m + 1)}, \ \frac{1}{4}\min_{i=0,\dots,m}\{t_{i+1} - t_i\}\right\},$$

we can find $\delta = \delta_\varepsilon$ such that $\eta^{-1}[t_i + \varepsilon, t_{i+1} - \varepsilon] \cap B_\delta(S_{2\pi}) = \emptyset$ for any $i = 0, \dots, m$, where $B_\delta(S_{2\pi})$ is a δ-neighbourhood of the set $S_{2\pi}$. Therefore, in particular, we can find $C_\varepsilon > 0$ such that $|\nabla\eta(x)| \le C_\varepsilon$, for all $x \in \eta^{-1}[t_i + \varepsilon, t_{i+1} - \varepsilon]$. At this point, since $K_0 > 0$, then we can assume without loss of generality that $2\gamma_E(K_0) := 4\pi - 2\mathcal{K}_+(E; K_0) > 0$ (otherwise $4\pi - 2\mathcal{K}_+(E; K_0) - K_0 M(E) < 0$ and (1.7) would be trivially satisfied). Therefore we can use the coarea formula (see [12, page 158]) and Huber's isoperimetric inequality (2.2), to conclude that,

$$\begin{aligned}
b - a &= \mu(\eta^*(b)) - \mu(\eta^*(a)) = \int_{\eta > \eta^*(b)} d\tau - \int_{\eta > \eta^*(a)} d\tau = \int_{\eta^*(b) < \eta \le \eta^*(a)} d\tau \\
&\ge \int_{\eta^*(b) < \eta < \eta^*(a)} d\tau = \int_{\eta^*(b)}^{\eta^*(a)} \left(\int_{\Gamma(t)} \frac{d\sigma}{|\nabla\eta|}\right) dt = \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} \left(\int_{\Gamma(t)} \frac{d\sigma}{|\nabla\eta|}\right) dt \\
&\ge \sum_{i=0}^{m} \int_{t_i+\varepsilon}^{t_{i+1}-\varepsilon} \left(\int_{\Gamma(t)} \frac{d\sigma}{|\nabla\eta|}\right) dt \ge \frac{1}{C_\varepsilon} \sum_{i=0}^{m} \int_{t_i+\varepsilon}^{t_{i+1}-\varepsilon} \left(\int_{\Gamma(t)} d\sigma\right) dt \\
&\ge \frac{\sqrt{2\gamma_E(K_0)}}{C_\varepsilon} \sum_{i=0}^{m} \int_{t_i+\varepsilon}^{t_{i+1}-\varepsilon} \sqrt{\left(\int_{\Omega(t)} d\tau\right)} \\
&\ge \frac{\sqrt{2\gamma_E(K_0)}}{C_\varepsilon} \sqrt{\left(\int_{\Omega(\eta^*(\overline{b}))} d\tau\right)} \sum_{i=0}^{m} \int_{t_i+\varepsilon}^{t_{i+1}-\varepsilon} dt \\
&= C(\overline{a}, \overline{b}, S_{2\pi}, \mathcal{K}_+(E; K_0))|\eta^*(a) - \eta^*(b) - 2(m+1)\varepsilon| \ge \frac{1}{4} C|\eta^*(a) - \eta^*(b)|,
\end{aligned}$$

for a strictly positive constant C depending on $\overline{a}, \overline{b}, S_{2\pi}, \mathcal{K}_+(E; K_0)$, as claimed. □

Step 3. In view of (3.11) we obtain,

$$\frac{d\eta^*(s)}{ds} = -\left(\int_{\Gamma(\eta^*(s))} \frac{e^{\psi}}{|\nabla\eta|}\,d\ell\right)^{-1}, \tag{3.14}$$

for any $s \in I^*$, where $[0, \mu(0)] \setminus I^*$ is a set of null measure and, by setting $I := \eta^*(I^*)$, then $\mu(I) = I^*$. Next, let us define,

$$F(s) = 2K_0 \int_{\Omega(\eta^*(s))} e^{\eta}d\tau, \quad \text{for } s \in [0, \mu(0)],$$

where,

$$F(\mu(0)) = 2K_0 \int_E e^{\eta}d\tau = 2K_0 M(E), \tag{3.15}$$

and we have set,

$$F(0) = \lim_{s \searrow 0^+} F(s) = 0^+. \tag{3.16}$$

Clearly $F(s)$ is strictly increasing and continuous on $[0, \mu(0)]$ and in particular locally Lipschitz in $(0, \mu(0))$, since in fact it satisfies,

$$|F(s) - F(s_0)| \le C|\mu(\eta^*(s)) - \mu(\eta^*(s_0))| = C|s - s_0|,$$
$$\text{for all } 0 = \mu(t_+) < s_0 < s < \mu(0),$$

for a suitable constant $C > 0$. In particular it holds,

$$\int_{\Omega(\eta^*(s))} e^{u}d\tau = \int_0^s e^{\eta^*(\lambda)}d\lambda, \quad \text{for all } s \in [0, \mu(0)],$$

so that,

$$\begin{aligned}\frac{dF(s)}{ds} = 2K_0 e^{\eta^*(s)}, \quad \text{and} \quad \frac{d^2F(s)}{ds^2} &= 2K_0\frac{d\eta^*(s)}{ds}\,e^{\eta^*(s)} \\ &= \frac{d\eta^*(s)}{ds}\frac{dF(s)}{ds}, \quad \text{for all } s \in I^*.\end{aligned} \tag{3.17}$$

We remark that since $\eta^*(s)$ is differentiable almost everywhere, then the formula for the first derivative of $F(s)$ shows that in fact $\frac{dF(s)}{ds}$ is differentiable almost everywhere as well.

For any $s \in I^*$ the Cauchy-Schwartz inequality yields,

$$\begin{aligned}\left(\int_{\Gamma(\eta^*(s))} d\sigma\right)^2 &\le \left(\int_{\Gamma(\eta^*(s))} \frac{e^{\psi}}{|\nabla\eta|}\,d\ell\right)\left(\int_{\Gamma(\eta^*(s))} |\nabla\eta|d\ell\right) \\ &= \left(-\frac{d\eta^*(s)}{ds}\right)^{-1}\left(\int_{\Gamma(\eta^*(s))}\left(-\frac{\partial\eta}{\partial\nu_+}\right)d\ell\right),\end{aligned} \tag{3.18}$$

where $\nu_+ = \frac{\nabla\eta}{|\nabla\eta|}$ is the exterior unit normal to $\Omega(\eta^*(s))$ and we have used (3.14). Obviously, we can assume without loss of generality that $\eta^{-1}(S_{2\pi} \cap E) \notin I$, so that, since η satisfies (3.8), then (3.6) readily implies that

$$\int_{\Gamma(\eta^*(s))} \left(-\frac{\partial\eta}{\partial\nu_+}\right) d\ell = \int_{\Omega(\eta^*(s))} 2K_0\, e^{\eta} d\tau,$$

for any $s \in I^*$. Therefore, in particular we deduce that

$$\int_{\Gamma(\eta^*(s)))} \left(-\frac{\partial\eta}{\partial\nu_+}\right) d\ell = \int_{\Omega(\eta^*(s))} 2K_0\, e^{\eta} d\tau = F(s),$$

for any $s \in I^*$. Plugging this identity in (3.18) we find

$$\left(\int_{\Gamma(\eta^*(s))} d\sigma\right)^2 \leq \left(-\frac{d\eta^*(s)}{ds}\right)^{-1} F(s), \tag{3.19}$$

for any $s \in I^*$. Clearly, in view of (3.7), we have

$$-\Delta\psi = \omega_+ - \omega_- + \phi - \Delta f_2 \leq \omega + 2[K - K_0]e^f\, e^u, \tag{3.20}$$

whence

$$\sup_{U \subset E} \left\{ \int_U (-\Delta\psi) \right\} \leq 2\mathcal{K}_+(E; K_0), \tag{3.21}$$

and we can apply generalized Huber's inequality (2.2) to conclude that

$$\left(\int_{\Gamma(\eta^*(s))} d\sigma\right)^2 \geq [4\pi - 2\mathcal{K}_+(E; K_0)]\mu(\eta^*(s)) \equiv [4\pi - 2\mathcal{K}_+(E; K_0)]s, \tag{3.22}$$

for any $s \in I^* \cap (0, \mu(0))$.

Remark 3.3. If $4\pi - 2\mathcal{K}(E; K_0^+) < 0$, then (3.22) trivially satisfied.

To simplify the exposition let us set,

$$2\gamma_E(K_0) = 4\pi - 2\mathcal{K}_+(E; K_0).$$

Hence, substituting (3.22) in (3.19), we obtain,

$$2\gamma_E(K_0)s \leq \left(-\frac{d\eta^*(s)}{ds}\right)^{-1} F(s), \text{ for any } s \in I^* \cap (0, \mu(0)).$$

So, multiplying by $\frac{dF(s)}{ds}\left(-\frac{d\eta^*(s)}{ds}\right)$, we come up with the inequality,

$$2\frac{dF(s)}{ds}\left(\frac{d\eta^*(s)}{ds}\right)\gamma_E(K_0)s + \frac{dF(s)}{ds}F(s) \geq 0, \text{ for any } s \in I^* \cap (0, \mu(0)),$$

and conclude that,

$$\frac{d}{ds}\left[2\gamma_E(K_0)s\frac{dF(s)}{ds} - 2\gamma_E(K_0)F(s) + \frac{1}{2}(F(s))^2\right] \geq 0,$$

for any $s \in I^* \cap (0, \mu(0))$. Let $P_+(s)$ denote the functions in the square brackets. Since F and η^* are both continuous and locally Lipschitz continuous in $[0, \mu(0)]$ and since, in view of (3.17), $\frac{dF(s)}{ds}$ is continuous and locally Lipschitz continuous in $[0, \mu(0)]$ as well, then we come up with the inequality,

$$P_+(\mu(0)) - P_+(0) \geq 0.$$

Therefore we can use (3.15), (3.16) and (3.17) to obtain,

$$\left[2\gamma_E(K_0)\mu(0)2K_0e^{\eta^*(\mu(0))} - 2\gamma_E(K_0)(2K_0M(E)) + 2(K_0)^2M^2(E)\right] \geq 0.$$

Since $\eta^*(\mu(0)) = 0$, this is equivalent to the following inequality,

$$2\gamma_E(K_0)\mu(0) - 2\gamma_E(K_0)M(E) + K_0M^2(E) \geq 0.$$

So, by using the inequality (3.22) once more and (3.10) we find,

$$\begin{aligned} L^2(\partial E) = \left(\int_{\partial E} e^{\frac{u}{2}} ds\right)^2 \equiv \left(\int_{\Gamma(0)} d\sigma\right)^2 &\geq 2\gamma_E(K_0)\mu(0) \\ &\geq 2\gamma_E(K_0)M(E) - K_0M^2(E) \\ &= (4\pi - 2\mathcal{K}_+(E; K_0) - K_0M(E))M(E), \end{aligned} \tag{3.23}$$

which is (1.7) as claimed.

Step 4. We will discuss here the case where the equality holds in (1.7).

First of all, there is no chance to have the equality in (1.7) if the strict inequality holds in (3.21). Therefore, because of (3.20), we see that we must have $\phi = 0$ for a.a. $x \in E$, that is, in view of (3.3) and (3.4), we also conclude that u must be a solution of (1.2) in E, and not just a subsolution as in (1.10).

Next we must have the equality sign in the Huber inequality used in (3.22) for a.a. $s \in I^* \cap (0, \mu(0))$ and in (3.23) for $s = \mu(0)$. Therefore, in view of (2.1) and (2.2), we conclude that for each $t \in I \cup \{0\}$, we have,

(a) $\Omega(t)$ is simply connected and $\psi(z) = c_t + 2\log\left|\Phi_t'(z)(\Phi_t(z))^{-\alpha_{\Omega(t)}}\right|$,

with $z \in \overline{\Omega(t)}$,

where $\alpha_{\Omega(t)} = \frac{1}{2\pi}\mathcal{K}_+(\Omega(t); K_0)$, $c_t \in \mathbb{R}$ and Φ_t is a conformal map of $\Omega(t)$ onto the disk of unit radius $|\mathrm{w}| = |\Phi_t(z)| < 1$ with $\Phi_t(z_t) = 0$, for some $z_t \in \Omega(t)$. Here ψ is the function defined in (3.7). Since ϕ vanishes, then we have the equality sign in (3.20) and (3.21) which therefore do not provide other conditions. However, in view of Sard's lemma for Sobolev functions, we can assume without loss of generality that $\Omega(t)$ is simple for each $t \in \overline{I \cup \{0\}}$, so that each Φ_t can be extended to a univalent and continuous map from $\overline{\Omega(t)}$ to a closed unit disk, see for example [20, Theorem 2.6]. At this point, by setting $\mathrm{w} = \Phi_0(z)$, and in view of (a), we conclude that

$$v(\mathrm{w}) := \eta\big(\Phi_0^{-1}(\mathrm{w})\big),$$

is a strong solution of

$$-\Delta v = 2K_0 e^{c_0} |\mathrm{w}|^{-2\alpha}\, e^{v} \;\text{ in }\; \{|\mathrm{w}| < 1\}, \qquad v = 0 \;\text{ on }\; |w| = 1,$$

where $\alpha = \alpha_E \equiv \alpha_{\Omega(0)}$. In particular we have that the level lines of v are concentric circles centred at the origin, that is, v is radial. Actually, by using the Brezis-Merle estimates for Liouville type equations (see [11, Remark 5]) and standard elliptic theory, we see that v is analytic far away from the origin and of class $W^{2,q}(B_1)$, for a suitable $q > 1$ depending on α.

Thus, by a straightforward evaluation we find that,

$$v(\mathrm{w}) = \log \frac{\tau_0^2}{\left(1 + \frac{K_0 e^{c_0} \tau_0^2}{4(1-\alpha)^2} |w|^{2(1-\alpha)}\right)^2}, \;\text{ for } |\mathrm{w}| < 1,$$

for a suitable constant $\tau_0 \neq 0$, to be fixed in order to satisfy the Dirichlet boundary condition.

As a consequence we find that,

$$\eta(z) = \log \frac{\tau^2 e^{-c_0}}{\left(1 + \frac{K_0 \tau^2}{4(1-\alpha)^2} |\Phi_0(z)|^{2(1-\alpha)}\right)^2}, \;\text{ with } z \in E,$$

for some $\tau \neq 0$ and then, since in particular $e^{\psi(z)} = e^{c_0} \left|\Phi_0'(z)(\Phi_0(z))^{-\alpha}\right|^2$, we see that,

$$\eta(z) = \log \frac{\tau^2 e^{-\psi(z)} \left|\Phi_0'(z)(\Phi_0(z))^{-\alpha}\right|^2}{\left(1 + \frac{K_0 \tau^2}{4(1-\alpha)^2} |\Phi_0(z)|^{2(1-\alpha)}\right)^2}, \;\text{ with } z \in E.$$

Since $\eta + \psi = f + u$, then we finally conclude that

$$e^{f(z) + u(z)} = \frac{\tau^2 \left|\Phi_0'(z)(\Phi_0(z))^{-\alpha}\right|^2}{\left(1 + \frac{K_0 \tau^2}{4(1-\alpha)^2} |\Phi_0(z)|^{2(1-\alpha)}\right)^2}, \;\text{ for } z \in E,$$

as claimed in (i) of Theorem 1.7. Finally, by using the well known fact that the logarithm of the modulus of a non vanishing holomorphic function is harmonic, we find that,

$$\begin{aligned}
2Ke^{f+u} &= -\Delta u \\
&= \Delta f - \Delta \log\left(\left|\Phi_0^{'}(z)(\Phi_0(z))^{-\alpha}\right|^2\right) \\
&\quad + 2\Delta \log\left(1 + \frac{K_0\tau^2}{4(1-\alpha)^2}|\Phi_0(z)|^{2(1-\alpha)}\right) \\
&= \Delta f + 4\pi\alpha\delta_{z=z_0} + 2K_0e^{f+u} = -\omega + 4\pi\alpha\delta_{z=0} + 2K_0e^{f+u},
\end{aligned}$$

in the sense of distributions in E and classically in $E \setminus \{0\}$. Therefore, if $\omega \perp e^{f+u}\mathcal{H}^2$, then this identity can be satisfied if and only if,

$$2Ke^{f+u} \equiv 2K_0e^{f+u} \text{ for a.a. } z \in E, \tag{3.24}$$

and $\omega = 4\pi\alpha\delta_{z=z_0}$. In other words

$$f(z) = h(z) + 2\alpha G(z, z_0) = h(z) - 2\alpha \log|\Phi_0(z)|, \tag{3.25}$$

for some h harmonic in E. At this point (3.24) and (3.25) readily imply that $K \equiv K_0$ for a.a. $z \in E$, which proves (ii) of Theorem 1.7.

Proof of Corollary 1.9. (i) In this situation we just know that $u \in L^1_{\rm loc}(\Omega)$ and $Ke^{f+u} \in L^1_{\rm loc}(\Omega)$. So we also have $\Delta u \in L^1_{\rm loc}(\Omega)$ and then in particular, by the Green representation formula, $|\nabla u| \in L^1_{\rm loc}(\Omega)$. By [11, Remark 2] we find $e^{t|u|} \in L^1_{\rm loc}(\Omega)$ for any $t \geq 1$ and letting $\Omega_0 \Subset \Omega$ be any open, smooth and relatively compact subset, we have $u \in L^1(\partial\Omega_0)$ by standard trace embeddings. Let $u = u_1 + u_2$, where u_1 is the unique weak solution (in the sense of Stampacchia [24]) of the Dirichlet problem,

$$\begin{cases} -\Delta u_1 = 2Ke^{f+u} & \text{in } \Omega_0 \\ u_1 = 0 & \text{on } \Omega_0, \end{cases}$$

and u_2 satisfies,

$$\begin{cases} -\Delta u_2 = 0 & \text{in } \Omega_0 \\ u_2 = u & \text{on } \Omega_0. \end{cases}$$

Then $u_2(x) = -\int_{\partial\Omega_0} u(y)\frac{\partial G_0}{\partial\nu}(x-y)d\ell_y$, where G_0 is the Green function of $-\Delta$ relative to Ω_0, and since $u \in L^1(\partial\Omega_0)$, then $u_2 \in L^\infty_{\rm loc}(\Omega_0)$. Moreover, $u_1 \in W^{1,r}_0(\Omega_0)$ for any $r \in (1, 2)$ by the results in [24] and then we find $u \in W^{1,r}_{\rm loc}(\Omega)$.

(ii) Let E be any relatively compact and simple subset, we can find an open, smooth, simple and relatively compact subset Ω_1 such that $E \Subset \Omega_1 \Subset \Omega$. Let $K_n \in C^0(\overline{\Omega_1})$ be any sequence satisfying,

$$K_n \leq K \text{ a.e. in } \Omega_1 \text{ and } K_n e^{f+u} \to K e^{f+u}, \text{ as } n \to +\infty, \text{ in } L^1(\Omega_1). \quad (3.26)$$

Next, let $v_n = v_{n,1} + u_2$, where $v_{n,1}$ is the unique weak solution (in the sense of Stampacchia [24]) of the Dirichlet problem,

$$\begin{cases} -\Delta v_{n,1} = 2K_n e^{f+u} & \text{in } \Omega_1 \\ v_{n,1} = 0 & \text{on } \Omega_1, \end{cases}$$

and u_2 satisfies,

$$\begin{cases} -\Delta u_2 = 0 & \text{in } \Omega_1 \\ u_2 = u & \text{on } \Omega_1. \end{cases}$$

Obviously, as in (i) we find $u_2 \in L^{\infty}_{\rm loc}(\Omega_1)$. In particular, by the Green representation formula, it is not difficult to see that,

$$v_n \leq u \text{ a.e. in } \Omega_1. \quad (3.27)$$

Let us observe that, by Theorem 4.4, $e^{f+u} = e^{\rho} \in L^{p_0,\rm loc}_{\rm loc}(\Omega \setminus S_{2\pi}) \cap L^{q_0}_{\rm loc}(\Omega)$ for some $p_0 > 2$ and $q_0 > 1$, whence by standard elliptic estimates and the Sobolev embedding we find $v_n \in W^{2,q_0}(\Omega_1) \cap C^0(\overline{\Omega_1})$. By using (3.26) with well known results in [24], we conclude that $v_n \to u$ in $W^{1,r}_{\rm loc}(\Omega_1)$, for any $r \in (1,2)$. At this point we observe that v_n is a solution of,

$$-\Delta v_n = 2\widehat{K_n} e^f e^{v_n} \text{ in } \Omega_1,$$

where,

$$\widehat{K_n} = K_n e^{u - v_n} \text{ satisfies } \sup_{\overline{\Omega_1}} |\widehat{K_n}| \leq C_n e^u.$$

By (i) we have $\widehat{K_n} \in L^t(\Omega_1)$ for any $t \geq 1$. On the other side, by Proposition 1.4, we also find that $e^f \in L^{s,\rm loc}_{\rm loc}(\Omega \setminus S_{2\pi}) \cap L^m_{\rm loc}(\Omega)$ for some $s > 2$ and $m > 1$. Therefore we can apply Theorem 1.5 on Ω_1 with $K = \widehat{K_n}$ and $u = v_n$, to conclude that,

$$\left(\int_{\partial E} e^{\frac{f+v_n}{2}} d\ell\right)^2 \geq \left(4\pi - 2\mathcal{K}_{+,n}(E; K_0) - K_0 \int_E e^{f+v_n}\right) \int_E e^{f+v_n}, \quad (3.28)$$

where,

$$\mathcal{K}_{+,n}(E; K_0) = k_{s,+}(E) + \int_E [\widehat{K_n} - K_0]^+ e^{f+v_n} dx.$$

Since $v_n \to u$ in $W^{1,r}_{\mathrm{loc}}(\Omega_1)$ and in view of (3.26), along a subsubsequence (which we will not relabel) we have $v_n \to u$ a.e. in Ω_1 and $K_n e^{f+u} \to K e^{f+u}$, as $n \to +\infty$, a.e. in Ω_1. Then, by (3.27) and the dominated convergence theorem we conclude that,

$$\int_E e^{f+v_n} \to \int_E e^{f+u}, \text{ as } n \to +\infty,$$
$$\mathcal{K}_{+,n}(E; K_0) \to \mathcal{K}_+(E; K_0), \text{ as } n \to +\infty,$$

and,

$$\int_{\partial E} e^{\frac{f+v_n}{2}} d\ell \to \int_{\partial E} e^{\frac{f+u}{2}} d\ell,$$

where for the second limit we observe that,

$$[\widehat{K_n} - K_0]^+ e^{f+v_n} = [K_n e^{u-v_n} - K_0]^+ e^{f+v_n} \leq [K e^{u-v_n}]^+ e^{f+v_n} = [K]^+ e^{f+u}.$$

It is understood that the last limit holds true whenever $\int_{\partial E} e^{\frac{f+u}{2}} d\ell$ is finite, otherwise (1.7) is trivially satisfied since $M(E) < +\infty$. Therefore, in the limit $n \to +\infty$, along the given subsequence we recover (1.7), as claimed. □

4. Application to the Alexandrov isoperimetric inequality

The notion of surface of bounded curvature (*SBC* for short) was introduced by A.D. Alexandrov [2], as a model to describe surfaces with a wide variety of singularities. A detailed discussion of this subtle subject is behind the scope of our work, and we refer the reader to [3,22] for a complete account about the subject, and to [30] for a shorter exposition of some of the main results. Here we will just use an equivalent local description of these objects.

Indeed, according to a series of results due to Huber and Reshetnyak, see [22], an *SBC* without boundary can be equivalently defined as a Riemann surface $\mathcal{M}$ equipped with a metric $\mathfrak{g}$, which admits an atlas of local charts $\mathcal{U} = \{U_j, \Phi_j\}_{j\in J}$, such that each Φ_j is an isometry of U_j on $\Omega_j = \Phi_j(U_j)$, with $\Omega_j \subset \mathbb{R}^2(\simeq \mathbb{C})$, a smooth, open and bounded set, such that $\mathfrak{g}$ in local coordinates takes the form of a quadratic differential, $\Phi_j^{\#}(\mathfrak{g}) = e^{\rho_j(z)}|dz|^2$, with $z = x + iy \in \mathbb{C}$. Here # denotes the standard pull-back, $|dz|^2$ is the Euclidean metric and $\rho \equiv \rho_j = \rho_+ - \rho_-$, where $\rho_\pm$ are two superharmonic functions defined by

$$\rho_\pm(z) = h^0_\pm(z) + \int_{\Omega_j} \Gamma(z,y) d\omega^0_\pm(y), \quad \text{and } \Gamma(z,y) = \frac{1}{2\pi}\log\left(\frac{1}{|z-y|}\right), \tag{4.1}$$

with $h^0_\pm$ harmonic in Ω_j. Here $\omega^0_\pm$ are the mutually orthogonal non negative measures defined by the Jordan decomposition of a measure of bounded total variation on Ω_j, $\omega^0 = \omega^0_+ - \omega^0_-$. Any such system of coordinates is said to be isothermal and any metric taking the form $e^{\rho(z)}|dz|^2$ with ρ as in (4.1) is said to be subharmonic. Among other things, the definition is completed by the transitions rules between charts of functions and holomorphic forms, thus including the metric, see [22] for further details.

This is why we will focus our attention on the local model of an *SBC*.

Definition 4.1. An *abstract surface of bounded curvature* (*ASBC* for short) is a pair $\mathcal{S} = \big(\Omega, e^{\rho(z)}|dz|^2\big)$, where $\Omega \subset \mathbb{R}^2$ is open, smooth and bounded and $\rho = \rho_+ - \rho_-$, with $\rho_\pm$ as defined in (4.1).

Hence, if $\mathcal{S} = \big(\Omega, e^{\rho(z)}|dz|^2\big)$ is an *ASBC*, according to Reshetnyak (see [22, Theorem 8.1.7]), the *total curvature* $\mathcal{K}$, is the measure of finite total variation defined as follows,

Definition 4.2. Let $\mathcal{S} = \big(\Omega, e^{\rho(z)}|dz|^2\big)$ be an *ASBC*. The total curvature $\mathcal{K}(E)$ of a Borel set $E \subseteq \Omega$ is defined by:

$$2\mathcal{K}(E) := \omega^0(E) = \omega^0_+(E) - \omega^0_-(E).$$

Remark 4.3. We remark that, with this definition, the total curvature is well defined and finite for any Borel set $E \subseteq \Omega$. Nevertheless, if for some $z_0 \in \Omega$ it holds $\omega^0_+(z_0) \geq 4\pi$, then the lengths and areas of sets containing z_0, as defined via the metric $g = e^{\rho(z)}|dz|^2$ (see (4.6), (4.7) below) are not well defined in general. Any point $z_0 \in \Omega$ which satisfies $\omega^0_+(z_0) \geq 4\pi$ is said to be a *cusp*.

From now on we will assume that $\mathcal{S} = \big(\Omega, e^{\rho(z)}|dz|^2\big)$ is an *ASBC with no cusps*, that is, we assume that,

$$\text{for all } z \in \Omega, \ \text{ it holds } \omega^0_+(z) < 4\pi. \tag{4.2}$$

Let $S_{2\pi} = \{x \in \Omega \ : \ \omega^0_+(z) \geq 2\pi\}$. We have the following seemingly new result about the structure of subharmonic metrics with no cusps. Interestingly enough it is sharp, see Example 5.1 below for further details. The proof is based on various results and arguments in [11] about the regularity properties of Liouville-type equations. Here $\mathcal{H}^2$, denotes the 2-dimensional Hausdorff measure.

Theorem 4.4. *Let $\mathcal{S} = \big\{\Omega, e^{\rho(z)}|dz|^2\big\}$ be an ASBC with no cusps. Then $e^\rho \in L^{p_0,\mathrm{loc}}_{\mathrm{loc}}(\Omega \setminus S_{2\pi}) \cap L^{q_0}_{\mathrm{loc}}(\Omega)$ for some $p_0 > 2$ and some $q_0 > 1$. Moreover, there exists $K \in L^1_{\mathrm{loc}}\big(\Omega; e^\rho\mathcal{H}^2\big)$ and a Radon measure k_s on Ω, satisfying $k_s \perp e^\rho\mathcal{H}^2$, such that, letting $k_s = k_{s,+} - k_{s,-}$ be the Jordan decomposition of k_s, then ρ can be decomposed as $\rho = u + f$, where $f = f_+ - f_-$, with $f_\pm$ satisfying* (1.1) *with $\omega_\pm = 2k_{s,\pm}$ and $h_\pm$ suitable harmonic functions and where $u \in L^1_{\mathrm{loc}}(\Omega)$ is a solution of,*

$$-\Delta u = 2Ke^{f+u} \quad in \quad \Omega, \tag{4.3}$$

in the sense of distributions. In particular, one of the following holds:

(i) *We have $K \in L^s_{\mathrm{loc}}(\Omega)$, for some $s > \frac{q_0}{q_0-1}$ and then u is a strong solution of* (4.3) *which satisfies $u \in L^\infty_{\mathrm{loc}}(\Omega) \cap W^{2,r}_{\mathrm{loc}}(\Omega)$, for all $1 \leq r \leq \frac{sq_0}{s+q_0}$;*

(ii) *We have* $u \in W^{1,r}_{\rm loc}(\Omega)$ *for any* $r \in (1,2)$ *and* $e^{t|u|} \in L^1_{\rm loc}(\Omega)$ *for any* $t \geq 1$.

In both cases,

$$e^{\rho(z)}|dz|^2 \equiv e^{u(z)+f(z)}|dz|^2, \text{ with } z \in \Omega, \quad \text{and } Ke^{f+u} \in L^1_{\rm loc}(\Omega),$$

and

$$\mathcal{K}(E) = \int_E Ke^{f+u} + k_s(E), \tag{4.4}$$

for any relatively compact Borel set $E \Subset \Omega$. *Moreover, if* $\rho = u + f$ *for a pair* $\{u, f\}$ *as above, then, for any fixed* h *harmonic in* Ω, *the pair* $\{u_h, f_h\} := \{u - h, f + h\}$ *satisfies the same properties with* $\rho = u_h + f_h$.

Proof. Let $H(z, y) = G(z.y) - \Gamma(z, y)$ be the regular part of the Green function on Ω. Then $m_\pm(z) = \int_\Omega H(z, y)d\omega^0_\pm(y)$ are harmonic in Ω, and $\rho + m_+ - m_-$ takes the form $\rho_+ - \rho_-$ for a suitable pair $\rho_\pm$ satisfying (1.1). Therefore, by Proposition 1.4, we find $e^\rho \in L^{p_0,\rm loc}_{\rm loc}(\Omega \setminus S_{2\pi}) \cap L^{q_0}_{\rm loc}(\Omega)$ for some $p_0 > 2$ and some $q_0 > 1$. Then $e^\rho\mathcal{H}^2$ is a Radon measure on Ω, and so it is well defined the Lebesgue decomposition of $\mathcal{K}$ with respect to $e^\rho\mathcal{H}^2$,

$$\mathcal{K} = Ke^\rho\mathcal{H}^2 + k_s \text{ with } K \in L^1_{\rm loc}\left(\Omega; e^\rho\mathcal{H}^2\right), \quad \text{and } k_s \perp e^\rho\mathcal{H}^2, \tag{4.5}$$

where k_s is a Radon measure on Ω. We first observe that, since $\rho \in L^1_{\rm loc}(\Omega)$, then $-\Delta\rho = \omega^0_+ - \omega^0_-$ holds in the sense of distributions in Ω, whence, by (4.5) and the definition of $\mathcal{K}$, we see that the following equality

$$-\Delta\rho = 2Ke^\rho + 2k_s,$$

holds as well, in the sense of distributions in Ω. Let $f = f_+ - f_-$ be defined by (1.1) with $\omega_\pm = 2k_{s,\pm}$, $h_\pm = 0$, and let us set,

$$u := \rho - f.$$

Clearly $u \in L^1_{\rm loc}(\Omega)$, and since $-\Delta f = 2k_s$ in the sense of distributions, then we deduce that

$$-\Delta u = 2Ke^{f+u} + 2k_s + \Delta f = 2Ke^{f+u},$$

that is, u satisfies (4.3) in the sense of distributions in Ω.

At this point, the fact that $\mathcal{K}(E)$ takes the form (4.4) is a straightforward consequence of the fact that $k_s \perp e^\rho\mathcal{H}^2$. Moreover we observe that, if K satisfies the assumption in (i), then all the assumptions of Lemma 3.1 (a) are satisfied and then the conclusion readily follows.

So we are left with the case where K does not satisfy the assumption in (i), that is, we just know that $K \in L^1(\Omega; e^{f+u}\mathcal{H}^2)$ where $u \in L^1(\Omega)$ is a solution of (4.3) in the sense of distributions. Therefore all the assumptions of Corollary 1.9(i) are satisfied and then the desired conclusion follows.

Finally it is obvious that the representation $\rho = u + f$ with all the properties established above still holds for $\{u_h, f_h\}$ where h is an arbitrary harmonic function in Ω. $\square$

Let $E \Subset \Omega$ be any regular and relatively compact subset and suppose that (4.2) holds. Then we define the *length* of ∂E,

$$L(\partial E) = \int_{\partial E} e^{\frac{f+u}{2}} d\ell, \tag{4.6}$$

and the *area* of E,

$$M(E) = \int_{E} e^{f+u} dx. \tag{4.7}$$

Definition 4.5. For $K_0 \in \mathbb{R}$ and for any and relatively compact Borel set $E \Subset \Omega$, we define the *positive variation of the total curvature* of E with respect to K_0,

$$\mathcal{K}_+(E; K_0) = \sup_{U \subseteq E} \left\{ \mathcal{K}(U) - K_0 \int_U e^{f+u} dx \right\}, \tag{4.8}$$

where the supremum is taken over all Borel sets $U \subseteq E$.
Because of (4.4), and since $k_{s,+} \perp e^{f+u}\mathcal{H}^2$, then $\mathcal{K}_+(E; K_0)$ takes the form,

$$\mathcal{K}_+(E; K_0) = k_{s,+}(E) + \int_E [K - K_0]^+ e^{f+u} dx.$$

Definition 4.6. For fixed $\alpha > -1$ and $K_0 > 0$, a *spherical* $\{K_0, \alpha\}$*-cone* is the *ASBC* defined by $\left\{B_1, |\mathrm{w}|^{-2\alpha} e^{v(\mathrm{w})} |d\mathrm{w}|^2\right\}$ where $B_1 = \{\mathrm{w} \in \mathbb{C} \,:\, |\mathrm{w}| < 1\}$ and,

$$e^{v(\mathrm{w})} = \frac{\tau_0^2}{\left(1 + \frac{K_0 \tau_0^2}{4(1-\alpha)^2} |\mathrm{w}|^{2(1-\alpha)}\right)^2}, \text{ with } |\mathrm{w}| < 1,$$

for some $\tau_0 \neq 0$.
It is worth to remark that the function v in Definition 4.6 is of class $L^\infty(B_1) \cap W^{2,p}_{\mathrm{loc}}(B_1 \setminus \{0\}) \cap W^{2,q}(B_1)$ for any $p > 2$ and for any $q < \frac{1}{|\alpha|}$ and it is a strong solution of $-\Delta v = 2K_0 |\mathrm{w}|^{-2\alpha} e^v$ in B_1.

In view of Theorems 1.5, 1.7, Corollary 1.9 and Theorem 4.4, and in the same spirit of [4], for $K_0 \geq 0$ we obtain a new proof of the Alexandrov [2] isoperimetric inequality on an ASBC. At least to our knowledge the characterization of the equality sign in this weak framework is new.

Theorem 4.7. *Let* $\mathcal{S} = \left\{\Omega, e^\rho |dz|^2\right\}$ *be an ASBC with no cusps and fix* $K_0 \geq 0$. *Then the curvature takes the form* (4.4) *for some* u, f, K, k_s *as in Theorem 4.4 and for any simple and relatively compact subset* $E \Subset \Omega$*, it holds*

$$L^2(\partial E) \geq (4\pi - 2\mathcal{K}_+(E; K_0) - K_0 M(E))\, M(E). \tag{4.9}$$

In particular, if K satisfies the assumption of Theorem 4.4(i) *and also* $K \in L^{n,\mathrm{loc}}_{\mathrm{loc}}(\Omega \setminus S_{2\pi})$, *for some* $n > \frac{2p_0}{p_0-2}$, *then the equality in* (4.9) *holds if and only if:*

- *The ASBC* $(E, e^{\rho}|dz|^2)$ *is isometric to a spherical* $\{K_0, \alpha\}$*-cone with* $\alpha = \frac{1}{2\pi}k_{s,+}(E)$;
- *It holds* $\rho = u + f$ *and* e^{f+u} *takes the form* (1.11), *where u is a solution of* (4.3) *with* $K \equiv K_0$ *for a.a.* $z \in E$ *and* $k_s = 2\pi\alpha\delta_{z=z_0}$, *for some* $z_0 \in E$, *that is,* $f(z) = h(z) + 2\alpha G(z, z_0) = h(z) - 2\alpha \log|\Phi_0(z)|$, *for some function h harmonic in E.*

Proof. Since $\mathcal{S}$ is an *ASBC* with no cusps, then, by Theorem 4.4, the curvature takes the form (4.4) where $u \in L^1_{\mathrm{loc}}(\Omega)$ is a solution of (4.3) in the sense of distributions, f takes the form (1.1) with $h_\pm$ harmonic and $\omega_\pm = 2k_{s,\pm}$ and $Ke^{u+f} \in L^1(\Omega)$. If K satisfies the assumption in Theorem 4.4(i) and also $K \in L^{n,\mathrm{loc}}_{\mathrm{loc}}(\Omega \setminus S_{2\pi})$, for some $n > \frac{2p_0}{p_0-2}$, then all the hypothesis of Theorems 1.5, 1.7 are satisfied as well. As a consequence, the inequality (4.9) holds and the equality sign is attained if and only if (1.11) holds, that is,

$$e^{f(z)+u(z)}|dz|^2 = |\sigma\Phi_0(z)|^{-2\alpha} e^{v(\sigma\Phi_0(z))}|d(\sigma\Phi_0(z))|^2 = |\mathrm{w}|^{-2\alpha}e^{v(\mathrm{w})}|d\mathrm{w}|^2,$$
$$\text{with } \sigma = \sqrt[1-\alpha]{\tau},$$

for any $B_1 \ni \mathrm{w} = \Phi_0(z)$, with $z \in E$, as claimed. In particular, since $\omega = 2k_s \perp e^{u+f}\mathcal{H}^2$ by construction, then Theorem 1.7(ii) can be applied as well. This observation completes the discussion of the equality case.

Clearly, to conclude the proof, it is enough to show that (4.9) holds in case (ii) of Theorem 4.4 is satisfied. However this is just the content of Corollary 1.9(ii) which immediately yields the desired conclusion. □

5. Examples

We recall that a point P on an *SBC* is said to be a conical singularity of order $\alpha > -1$ if in an isothermal chart $\{\Omega, z\}$ such that $z(P) = 0$, the metric takes the form $e^{\rho(z)}|dz|^2 = |z|^{2\alpha}e^{u(z)}|dz|^2$, where $u \in C^0(\Omega) \cap C^2(\Omega \setminus \{0\})$.

In this section δ_p denotes the Dirac delta with pole at $p \in \mathbb{R}^2$.

Example 5.1. We use [11, Example 1] to construct an *ASBC* of the form $\left\{B_1, e^{\rho}|dz|^2\right\}$ such that $\{u, f, K, k_s\}$ as obtained in Theorem 4.4 have the following properties:

- Either $e^{\rho} \in L^{\infty}(B_1)$ or $e^{\rho} \in L^q(B_1)$, for any $q \geq 1$;
- We have $K \in L^1(e^{\rho}\mathcal{H}^2, B_1) \cap L^1(B_1)$ but there is no $s > 1$ such that $K \in L^s(B_1)$;
- Function u is not locally bounded;
- Function u has all the properties claimed in Theorem 4.4(ii).

Let $0 \neq a < 1$, and for $z \in B_1 \setminus \{0\}$ let us set $u(z) = -a \log\left(\log\left(\frac{e}{|z|}\right)\right)$ and

$$K(z) = -\frac{a}{2}|z|^{-2}\left(\log\left(\frac{e}{|z|}\right)\right)^{-(2-a)}.$$

The superharmonic function $\rho(z) = \int_{B_1} G(z, y) d\omega^0(y)$, where

$$\omega^0(y) = 2K(y)e^{u(y)}d\mathcal{H}^2,$$

takes the form $\rho = \rho_+ - \rho_-$ as in (4.1) with $h^0_\pm = 0$, and $\omega^0_- = 0$ and $\omega^0_+(y) = 2K(y)e^{u(y)}d\mathcal{H}^2$ if $a < 0$, while $\omega^0_-(y) = 2K(y)e^{u(y)}d\mathcal{H}^2$ and $\omega^0_+ = 0$ if $a \in (0, 1)$. Since $Ke^u \in L^1(B_1)$, then $\omega^0 \ll e^u\mathcal{H}^2$ and so we find $\{u, f, K, k_s\}$ as claimed in Theorem 4.4 by setting $f = 0$, $k_s = 0$, $K \equiv K$ and $u \equiv u$. In fact we see that u is a solution of,

$$\begin{cases} -\Delta u = 2Ke^u & \text{in } B_1 \\ u = 0 & \text{on } B_1, \end{cases}$$

that is, in particular $u \equiv \rho$, and so we find,

$$e^{\rho(z)} = \left(\log\left(\frac{e}{|z|}\right)\right)^{-a}, \text{ for } z \in B_1.$$

If $a \in (0, 1)$, then $e^\rho \in L^\infty(B_1)$ and $K \in L^1(e^\rho\mathcal{H}^2, B_1) \cap L^1(B_1)$ but $u(z) \to -\infty$ as $z \to 0$. If $a < 0$, then $e^\rho \in L^q(B_1)$ for any $q \geq 1$. It holds $K \in L^1(e^\rho\mathcal{H}^2, B_1) \cap L^1(B_1)$ but $u(z) \to +\infty$ as $z \to 0$. In both cases, there is no $s > 1$ such that $K \in L^s(B_1)$, and so there is no chance that K satisfies the assumption of Theorem 4.4(i). On the other side, in both cases it is easy to check that u has all the properties claimed in Theorem 4.4(ii). Clearly Theorem 4.7 applies and then (4.9) holds on $\{B_1, e^\rho|dz|^2\}$.

Example 5.2. Let $\mathbb{S}^2_{\alpha_1,\alpha_2}$ be the *SBC* defined by the isothermal charts $\{\Omega_i, \varphi_i\}_{i=1,2}$ and the local metrics $\{g_i\}_{i=1,2}$ constructed as follows. For $r_0 \geq 4$ and $-1 < \alpha_1 \leq \alpha_2 \leq 0$, we define,

$$\Omega_2 = \{z \in \mathbb{C} : |z| < r_0\}, \quad \text{with} \quad \varphi_2 = z \quad \text{and} \quad g_2 = e^\rho|dz|^2,$$
$$\Omega_1 = \left\{z \in \mathbb{C} \cup \{\infty\} : |z| > \frac{1}{r_0}\right\}, \quad \text{with} \quad \varphi_1 = \frac{1}{z} \quad \text{and} \quad g_1 = \varphi_1^\#(g_2),$$

where,

$$\rho(z) = \begin{cases} \log\left(\dfrac{4(1+\alpha_2)^2|z|^{2\alpha_2}}{\left(1+|z|^{2(1+\alpha_2)}\right)^2}\right) & \text{if } |z| < 1 \\ \log\left(\dfrac{4(1+\alpha_2)^2|z|^{2\alpha_1}}{\left(1+|z|^{2(1+\alpha_1)}\right)^2}\right) & \text{if } |z| \in [1, +\infty). \end{cases} \tag{5.1}$$

This is a compact surface without boundary, homeomorphic to the two sphere, with two conical singularities, $z = \infty$ of order α_1 and $z = 0$ of order α_2. For $\alpha_1 = \alpha_2 < 0$ we are reduced to the classical "american football" [28], with constant Gaussian curvature $K \equiv 1$. Instead, if $\alpha_1 < \alpha_2 \leq 0$, we have the glueing of two caps of american footballs with gaussian curvatures 1 and $\frac{(1+\alpha_1)^2}{(1+\alpha_2)^2}$ respectively, with different conical singularities, see [7, 15] for more details about this singular surface.

We consider a decomposition in the $\{\Omega_2, \varphi_2\}$ chart, as claimed in Theorem 4.4, of the form $\rho(z) = f(z) + u(z)$, where,

$$u(z) = \begin{cases} \log\left(\dfrac{4(1+\alpha_2)^2}{\left(1+|z|^{2(1+\alpha_2)}\right)^2}\right) & \text{if } |z| < 1 \\ \log\left(\dfrac{4(1+\alpha_2)^2|z|^{2(\alpha_1-\alpha_2)}}{\left(1+|z|^{2(1+\alpha_1)}\right)^2}\right) & \text{if } |z| \in [1,+\infty), \end{cases}$$

and

$$f(z) = f(z;\alpha_2) = 2\alpha_2 \log|z|, \;\; |z| \in (0,+\infty).$$

Clearly we have $u \in W^{2,k}_{\text{loc}}(\mathbb{R}^2 \setminus \{0\}) \cap W^{2,r}_{\text{loc}}(\mathbb{R}^2)$, for any $k > 2$ and $1 < r < \frac{2}{|\alpha_1|}$, which is also a strong solution of $-\Delta u = 2K|z|^{2\alpha_2}e^u$ in $\mathbb{R}^2$, with,

$$K(z) = \begin{cases} 1 & \text{if } |z| \in [0,1) \\ \dfrac{(1+\alpha_1)^2}{(1+\alpha_2)^2} & \text{if } |z| \in (1,+\infty). \end{cases}$$

So $K \in L^\infty(\mathbb{R}^2)$ and putting

$$k_{s,2} = 2\pi|\alpha_2|\delta_{z=0},$$

we find,

$$\mathcal{K}(E) = \int_E K e^{f+u} d\mathcal{H}^2 + k_{s,2}(E), \quad \text{for } E \Subset \{|z| < r_0\},$$

which is the total curvature of a relatively compact Borel set E in the $\{\Omega_2, \varphi_2\}$ chart. For a generic Borel set $E_0 \subseteq \mathbb{C} \cup \{\infty\}$, we can consider the analogue decomposition for g_1 which takes the form $g_1 = e^{\rho_1}|d\mathrm{w}|^2$, with $\rho_1 = f_1 + u_1$, where

$$u_1(\mathrm{w}) = \begin{cases} \log\left(\dfrac{4(1+\alpha_2)^2}{\left(1+|\mathrm{w}|^{2(1+\alpha_1)}\right)^2}\right) & \text{if } |\mathrm{w}| < 1 \\ \log\left(\dfrac{4(1+\alpha_2)^2|\mathrm{w}|^{2(\alpha_2-\alpha_1)}}{\left(1+|\mathrm{w}|^{2(1+\alpha_2)}\right)^2}\right) & \text{if } |\mathrm{w}| \in [1,+\infty), \end{cases}$$

with $f_1(\cdot) = f(\cdot, \alpha_1)$, and eventually find the total curvature of any Borel set $E_0 \subseteq \mathbb{C} \cup \{\infty\}$,

$$\mathcal{K}(E_0) = \int_{E_{0,2}} K e^{f+u} d\mathcal{H}^2 + k_{s,2}(E_{0,2}) + \int_{\varphi_1(E_{0,1})} K_1 e^{f_1+u_1} d\mathcal{H}^2 + k_{s,1}(\varphi_1(E_{0,1})), \quad (5.2)$$

where $E_{0,2} = E_0 \cap \{|z| < r_0\}$ and $E_{0,1} = E_0 \cap \{|z| \geq r_0\}$ while $K_1 = K \circ \varphi_1$, and,

$$k_{s,1} = 2\pi |\alpha_1| \delta_{\mathrm{w}=0}.$$

Next, to simplify the notations let us set

$$\sigma_{1,2} = \frac{(1+\alpha_1)^2}{(1+\alpha_2)^2} \leq 1.$$

It is easy to check that the area of $\mathbb{S}^2_{\alpha_1,\alpha_2}$ is $2\pi(1+\alpha_2) + \frac{1}{\sigma_{1,2}} 2\pi(1+\alpha_1)$ while, by using (5.2), we see that the total curvature of $\mathbb{S}^2_{\alpha_1,\alpha_2}$ is 4π, in agreement with the fact that, as well known [3], the Gauss-Bonnet formula holds even in this singular context. Please observe that this is just an equivalent formulation of the singular Gauss-Bonnet formula, see [29], which asserts that the global integral of the absolutely continuous part of the Gaussian curvature equals the singular Euler characteristic, yielding in this particular case the well-known identity,

$$\int_{B_1} K e^{f+u} d\mathcal{H}^2 + \int_{\varphi_1((B_1)^c)} K_1 e^{f_1+u_1} d\mathcal{H}^2 = 2\pi(2+\alpha_1+\alpha_2).$$

If E is a simple set surrounding the origin, then we can always take r_0 large enough to guarantee that $E \Subset \{|z| < r_0\}$ so that the inequality (4.9) takes the form,

$$\begin{aligned} L^2(\partial E) \geq \big(&4\pi(1+\alpha_2) - 2[1-K_0]^+ M(E \cap B_1) \\ &- 2[\sigma_{1,2} - K_0]^+ M(E \cap (B_1)^c) - K_0 M(E)\big) M(E). \end{aligned}$$

In particular, if K is not constant in E, then the inequality is always strict and if we choose $K_0 = 1$, then it reduces to the well known Bol's [10] inequality,

$$L^2(\partial E) \geq (4\pi(1+\alpha_2) - M(E))\, M(E).$$

If $E = B_R$ with $R \leq 1$, then $K \equiv 1$ in E and since,

$$L^2(\partial B_R) = \left(\int_0^{2\pi} \frac{2(1+\alpha_2) R^{\alpha_2}}{1+R^{2(1+\alpha_2)}} d\ell \right)^2 = \frac{16\pi^2(1+\alpha_2)^2 R^{2\alpha_2}}{(1+R^{2(1+\alpha_2)})^2},$$

and,

$$M(B_R) = \int_{B_R} \frac{4(1+\alpha_2)^2 |x|^{2\alpha_2}}{\left(1+|x|^{2(1+\alpha_2)}\right)^2} dx = \frac{4\pi(1+\alpha_2) R^{2\alpha_2}}{1+R^{2(1+\alpha_2)}},$$

then we find the equality in (4.9) with $K_0 = 1$,

$$\begin{aligned} L^2(\partial B_R) &= \frac{16\pi^2(1+\alpha_2)^2 R^{2\alpha_2}}{\left(1+R^{2(1+\alpha_2)}\right)^2} \\ &= \left(4\pi(1+\alpha_2) - \frac{4\pi(1+\alpha_2)R^{2\alpha_2}}{1+R^{2(1+\alpha_2)}}\right)\frac{4\pi(1+\alpha_2)R^{2\alpha_2}}{1+R^{2(1+\alpha_2)}} \\ &= (4\pi(1+\alpha_2) - M(B_R))\, M(B_R). \end{aligned}$$

Example 5.3. This example illustrates the failure of Theorem 4.4 on a surface homeomorphic to the two-sphere with a cusp and in the same time the kind of singularity which yields a curvature function K which is unbounded but in $L^r(E)$ for some $r > 1$.

Let us consider the same charts $\{\Omega_i, \varphi_i\}_{i=1,2}$ as in Example 5.2, where this time the metric $g_2(z) = e^{\rho(z)}|dz|^2$ is defined as follows,

$$\rho(z) = \begin{cases} \log\left(\dfrac{2}{\left(2-|z|^{\frac{1}{2}}\right)^2}\right) & \text{if } |z| < 1 \\ \log\left(\dfrac{8|z|^{\frac{3}{2}}}{\left(1+|z|^{\frac{1}{2}}\right)^2}\right) & \text{if } |z| \in [1, +\infty). \end{cases} \tag{5.3}$$

We consider a decomposition as claimed in Theorem 4.4 in the $\{\Omega_2, \varphi_2\}$ chart, $\rho(z) = f(z) + u(z)$, where we choose $f = 0$ so that $u = \rho$, which satisfies $u \in W^{2,k,\mathrm{loc}}_{\mathrm{loc}}(\mathbb{R}^2 \setminus \{0\}) \cap W^{2,r}_{\mathrm{loc}}(\mathbb{R}^2)$, for any $k > 2$ and $1 < r < \frac{4}{3}$, and is a strong solution of $-\Delta u = 2Ke^{f+u}$ in $\mathbb{R}^2$, where,

$$K(z) = \begin{cases} -\dfrac{1}{4}\dfrac{1}{|z|^{\frac{3}{2}}} & \text{if } |z| \in [0, 1) \\ \dfrac{1}{32}\dfrac{1}{|z|^3} & \text{if } |z| \in (1, +\infty). \end{cases}$$

The total curvature of a relatively compact Borel set E in the $\{\Omega_2, \varphi_2\}$ chart takes the form,

$$\mathcal{K}(E) = \int_E Ke^u dx, \quad \text{for } E \Subset \{|z| < r_0\},$$

with $K \in L^r(E) \cap L^\infty_{\mathrm{loc}}(E \setminus \{0\})$ for any $1 < r < \frac{4}{3}$.

On the other hand, let us check whether or not the assumption (4.2) is satisfied on a generic relatively compact Borel set in the $\{\Omega_1, \varphi_1\}$ chart. The metric takes the form,

$$g_1(\mathrm{w}) = \varphi_1^{\#}(g_2) = e^{\rho_1(\mathrm{w})}|d\mathrm{w}|^2,$$

where,

$$\rho_1(\mathrm{w}) = \begin{cases} \log\left(\dfrac{8|\mathrm{w}|^{-\frac{9}{2}}}{\left(1+|\mathrm{w}|^{\frac{1}{2}}\right)^2}\right) & \text{if } |\mathrm{w}| \in [0,1] \\[2ex] \log\left(\dfrac{2|\mathrm{w}|^{-3}}{\left(2|\mathrm{w}|^{\frac{1}{2}}-1\right)^2}\right) & \text{if } |\mathrm{w}| \in (1,+\infty). \end{cases} \tag{5.4}$$

Therefore, it is readily seen that ρ_1 takes the form (4.1) with $\omega_+^0(0) = \frac{9\pi}{2} > 4\pi$ which violates (4.2). This singular surface is still homeomorphic to the two sphere, but it has a cusp at $z = \infty$. As a consequence, while the curvature is always well defined in the sense of measures, the area of a compact Borel set in the $\{\Omega_1, \varphi_1\}$ chart is not, since e^{ρ_1} is not an $L^1_{\mathrm{loc}}(\mathbb{R}^2)$ function. In particular, there is no chance to use the argument in the proof of Theorem 4.4, which should be based on the Lebesgue decomposition of $\mathcal{K} = \omega^0$ with respect to $e^{\rho_1}\mathcal{H}^2$, since the latter is not even a Radon measure in this case. It is worth to mention that, nevertheless, the product $(K \circ \varphi_1)e^{\rho_1}$ is an $L^1_{\mathrm{loc}}(\mathbb{R}^2)$ function which could be used in principle as the density of the total curvature. On the other hand, the right hand side of the Alexandrov's isoperimetric inequality (4.9) is not well defined in general.

However Theorem 4.4 and Theorem 4.7 can be applied in the $\{\Omega_2, \varphi_2\}$ chart, so that (4.9) holds therein. In particular, if E is any open and relatively compact Borel set in Ω_2, then the equality is always strict, since K is never constant in E.

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