

A new type of non-topological bubbling solutions to a competitive Chern-Simons model

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Abstract. We study a non-Abelian Chern-Simons system in \mathbb{R}^2 , including the simple Lie algebras A_2 and B_2 . In a previous work, we proved the existence of radial non-topological solutions with prescribed asymptotic behaviors via the degree theory. We also constructed a sequence of bubbling solutions with only one component blowing up partially at infinity. In this paper, we construct a sequence of radial non-topological bubbling solutions of another type via the shooting argument. One component of these bubbling solutions locally converge to a non-topological solution of the Chern-Simons-Higgs scalar equation, but both components blow up partially in different regions at infinity at the same time. This generalizes a recent work by Choe, Kim and the second author, where the $SU(3)$ case (*i.e.*, A_2) was studied. Our result is new even for the $SU(3)$ case and also confirms the difference between the $SU(3)$ case and the B_2 case.

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1. Introduction

In this paper, we study a non-Abelian Chern-Simons system of rank 2:

$$\begin{pmatrix} \Delta u_1 \\ \Delta u_2 \end{pmatrix} + K \begin{pmatrix} e^{u_1} \\ e^{u_2} \end{pmatrix} - K \begin{pmatrix} e^{u_1} & 0 \\ 0 & e^{u_2} \end{pmatrix} K \begin{pmatrix} e^{u_1} \\ e^{u_2} \end{pmatrix} = \begin{pmatrix} 4\pi N_1 \delta_0 \\ 4\pi N_2 \delta_0 \end{pmatrix} \text{ in } \mathbb{R}^2, \quad (1.1)$$

where N_1, N_2 are non-negative integers, δ_0 denotes the Dirac measure at 0, and $K = (a_{ij})$ is a 2×2 matrix satisfying

$$a_{11}, a_{22} > 0, \quad a_{12}, a_{21} < 0 \quad \text{and} \quad a_{11}a_{22} - a_{12}a_{21} > 0. \quad (1.2)$$

Clearly (1.1) can be considered as a perturbation of the following Liouville system with a singular source:

$$\begin{pmatrix} \Delta u_1 \\ \Delta u_2 \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} e^{u_1} \\ e^{u_2} \end{pmatrix} = \begin{pmatrix} 4\pi N_1 \delta_0 \\ 4\pi N_2 \delta_0 \end{pmatrix} \text{ in } \mathbb{R}^2. \quad (1.3)$$

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See [1–3]. Under the assumption (1.2), then in literature, (1.3) is also said to be *competitive*, as compared to the *cooperative* case where $a_{12}, a_{21} > 0$.

In the last few decades, various Chern-Simons field theories [12] have been widely studied, largely motivated by their applications to the physics of high critical temperature superconductivity. Another interesting feature of Chern-Simons field theories is that they provide a gauge invariant mechanism of mass generation [11]. These Chern-Simons theories can be reduced to systems of nonlinear partial differential equations, which have posed many mathematically challenging problems to analysts. Our first motivation to analyse (1.1) comes from the relativistic non-Abelian self-dual Chern-Simons model, which was proposed by Kao and Lee [23] and Dunne [13, 14]. Following [13, 14], the relativistic non-Abelian self-dual Chern-Simons model is defined in the $(2 + 1)$ Minkowski space $\mathbb{R}^{1,2}$, and the gauge group is a compact Lie group with a semi-simple Lie algebra \mathcal{G} . The Chern-Simons Lagrangian action density \mathcal{L} in $2 + 1$ dimensional spacetime involves the Higgs field ϕ and the \mathcal{G} -valued gauge field $A = (A_0, A_1, A_2)$. In general, the Euler-Lagrangian equation of \mathcal{L} is too difficult to deal with. Therefore, we only consider the energy minimizers of the Lagrangian functional, which turn out to be the solutions of the following self-dual Chern-Simons equations:

$$D_- \phi = 0 \quad \text{and} \quad F_{+-} = \frac{1}{\kappa^2} [\phi - [\phi, \phi^\dagger], \phi], \phi^\dagger, \quad (1.4)$$

where $D_- = D_1 - iD_2$ while $\kappa > 0$ and $F_{+-} = \partial_+ A_- - \partial_- A_+ + [A_+, A_-]$ with $A_\pm = A_1 \pm iA_2$, $\partial_\pm = \partial_1 \pm i\partial_2$ and $[\cdot, \cdot]$ is the Lie bracket over \mathcal{G} . In [14], Dunne considered a simplified form of the self-dual system (1.4), in which the fields ϕ and A are algebraically restricted:

$$\phi = \sum_{a=1}^r \phi^a E_a,$$

where r is the rank of the gauge Lie algebra, E_a is the simple root step operator, and ϕ^a are complex-valued functions. Let

$$u_a = \log |\phi^a| \quad \text{for} \quad a = 1, \dots, r.$$

Then system (1.4) can be reduced to the following system of nonlinear partial differential equations

$$\Delta u_a + \frac{1}{\kappa^2} \left(\sum_{b=1}^r K_{ab} e^{u_b} - \sum_{b=1}^r \sum_{c=1}^r e^{u_b} K_{bc} e^{u_c} K_{ac} \right) = 4\pi \sum_{j=1}^{N_a} \delta_{p_j^a}, \quad \text{for } 1 \leq a \leq r, \quad (1.5)$$

where $K = (K_{ab})$ is the Cartan matrix of a semi-simple Lie algebra, $\{p_j^a\}_{j=1}^{N_a}$ are zeros of ϕ^a ($a = 1, \dots, r$), and δ_p denotes the Dirac measure concentrated at p in \mathbb{R}^2 . See [29] for the derivation of (1.5) from (1.4). For example, there are

three types of Cartan matrices of rank 2, which correspond to the semi-simple Lie algebras A_2 , B_2 and G_2 respectively:

$$\mathcal{A}_2(i.e., SU(3)) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \mathcal{B}_2 = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}, \mathcal{G}_2 = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}. \quad (1.6)$$

Let $((K^{-1})_{ab})$ denote the inverse of the matrix K . Assume that

$$\sum_{b=1}^r (K^{-1})_{ab} > 0 \quad \text{for } a = 1, 2, \dots, r. \quad (1.7)$$

A solution $\mathbf{u} = (u_1, \dots, u_r)$ of (1.5) is called a *topological solution* if

$$u_a(x) \rightarrow \ln \sum_{b=1}^r (K^{-1})_{ab} \quad \text{as } |x| \rightarrow +\infty \quad \text{for } a = 1, \dots, r;$$

a solution \mathbf{u} is called a *non-topological solution* if

$$u_a(x) \rightarrow -\infty \quad \text{as } |x| \rightarrow +\infty \quad \text{for } a = 1, \dots, r.$$

For any configuration p_j^a in \mathbb{R}^2 , the existence of topological solutions to (1.5) was proved by Yang [29] in 1997. However, the existence question of non-topological solutions (and mixed-type solutions, see below) is much more difficult than the one for topological solutions, and has remained open for a long time. Only recently, with the help of the classification result in [24], the first existence result of non-topological solutions to (1.5) with $K = \mathcal{A}_2, \mathcal{B}_2$ and \mathcal{G}_2 have been obtained by Ao, Wei and the second author [1,2] by the finite-dimensional reduction method through a perturbation from the Liouville system (1.3). However, the understanding of the structure of non-topological solutions is still far from complete.

1.1. Main result

In this paper, we focus on the radially symmetric solutions of (1.5) when all the vortices coincide at the origin. We only consider the rank 2 and competitive case, namely K is a 2×2 matrix satisfying (1.2). Moreover, we may assume, without loss of generality, that $\kappa = 1$. Then system (1.5) coincides with (1.1). In particular, when $K = \mathcal{A}_2$, then (1.1) becomes the following $SU(3)$ Chern-Simons system

$$\begin{cases} \Delta u_1 + 2(e^{u_1} - 2e^{2u_1} + e^{u_1+u_2}) - (e^{u_2} - 2e^{2u_2} + e^{u_1+u_2}) = 4\pi N_1 \delta_0 \\ \Delta u_2 + 2(e^{u_2} - 2e^{2u_2} + e^{u_1+u_2}) - (e^{u_1} - 2e^{2u_1} + e^{u_1+u_2}) = 4\pi N_2 \delta_0 \end{cases} \quad \text{in } \mathbb{R}^2. \quad (1.8)$$

When $K = \mathcal{B}_2$, then system (1.1) becomes the following \mathcal{B}_2 Chern-Simons system

$$\begin{cases} \Delta u_1 + 2e^{u_1} - e^{u_2} - 4e^{2u_1} + 2e^{2u_2} = 4\pi N_1 \delta_0 \\ \Delta u_2 + 2e^{u_2} - 2e^{u_1} - 4e^{2u_2} + 2e^{u_1+u_2} + 4e^{2u_1} = 4\pi N_2 \delta_0 \end{cases} \quad \text{in } \mathbb{R}^2. \quad (1.9)$$

As in [21], in order to simplify the expression of system (1.1), we consider the transformation

$$(u_1, u_2) \rightarrow \left(u_1 + \ln \frac{a_{22} - a_{12}}{a_{11}a_{22} - a_{12}a_{21}}, u_2 + \ln \frac{a_{11} - a_{21}}{a_{11}a_{22} - a_{12}a_{21}} \right)$$

and let

$$(a_1, a_2) = \left(\frac{-a_{12}(a_{11} - a_{21})}{a_{11}a_{22} - a_{12}a_{21}}, \frac{-a_{21}(a_{22} - a_{12})}{a_{11}a_{22} - a_{12}a_{21}} \right).$$

Clearly the assumption (1.2) implies $a_1 > 0$ and $a_2 > 0$. Then system (1.1) becomes

$$\begin{cases} \Delta u_1 + (1 + a_1)(e^{u_1} - (1 + a_1)e^{2u_1} + a_1e^{u_1+u_2}) \\ \quad - a_1(e^{u_2} - (1 + a_2)e^{2u_2} + a_2e^{u_1+u_2}) = 4\pi N_1\delta_0 \\ \Delta u_2 + (1 + a_2)(e^{u_2} - (1 + a_2)e^{2u_2} + a_2e^{u_1+u_2}) \\ \quad - a_2(e^{u_1} - (1 + a_1)e^{2u_1} + a_1e^{u_1+u_2}) = 4\pi N_2\delta_0 \end{cases} \quad \text{in } \mathbb{R}^2. \quad (1.10)$$

For the three types of Cartan matrices (1.6), we find that

$$(a_1, a_2) = \begin{cases} (1, 1), & \text{if } K = \mathcal{A}_2 \\ (2, 3), & \text{if } K = \mathcal{B}_2 \\ (5, 9), & \text{if } K = \mathcal{G}_2. \end{cases} \quad (1.11)$$

In particular, system (1.8) is invariant under the above transformation.

Clearly, to study system (1.1), we only need to consider system (1.10). It is more interesting to us that, when (a_1, a_2) take some other special values but not (1.11), system (1.10) also arises in some other physical models, such as the Lozano-Marqués-Moreno-Schaposnik model [25] and the Gudnason model [15, 16]. Lozano *et al.* [25] considered the bosonic sector of $\mathcal{N} = 2$ supersymmetric Chern-Simons-Higgs theory when the gauge group is $U(1) \times SU(N)$ and has N_f flavors of fundamental matter fields. They investigated so-called local Z_N string-type solutions when $N_f = N$ and obtained a system of nonlinear differential equations (see [25, (19)-(22)]) which, under a suitable change of variables and unknowns, can be transformed into (1.10) with $(a_1, a_2) = (\frac{k-1}{N}, \frac{(N-1)(k-1)}{N})$ and $k > 0$. If $k > 1$, then $a_1, a_2 > 0$ and $a_1 + a_2 = 1$. Gudnason [15, 16] considered a $\mathcal{N} = 2$ supersymmetric Yang-Mills-Chern-Simons-Higgs theory with the general gauge group $G = U(1) \times G'$, where G' is a non-Abelian simple Lie group represented by matrices. When the gauge group are $U(1) \times SO(2M)$ and $U(1) \times US_p(2M)$, the so-called master equations are a system of nonlinear differential equations (see [15, (3.64)-(3.65)] or [17, (2.1)-(2.2)]). Letting $M = 1$ and using a suitable transformation, this system coincides with (1.10) with $a_1 = a_2 = \frac{\beta^* - \alpha^*}{2\alpha^*}$ and $\alpha^*, \beta^* > 0$. If $\beta^* > \alpha^*$, then $a_1 = a_2 > 0$. See [21] for these two transformations.

Therefore, it is worth for us to study system (1.10) with generic $a_1, a_2 > 0$ rather than only (1.11). As in [21], we easily see that a solution (u_1, u_2) of (1.10) is a *topological solution* if $(u_1, u_2) \rightarrow (0, 0)$ as $|x| \rightarrow +\infty$; a *non-topological solution* if $(u_1, u_2) \rightarrow (-\infty, -\infty)$ as $|x| \rightarrow +\infty$; a *mixed-type solution* if either $(u_1, u_2) \rightarrow (\ln \frac{1}{1+a_1}, -\infty)$ or $(u_1, u_2) \rightarrow (-\infty, \ln \frac{1}{1+a_2})$ as $|x| \rightarrow +\infty$.

We postpone other mathematical details about (1.10) and first state our main result. To simplify the notations, in the sequel we denote

$$A = (1 + a_1)(1 + a_2) \quad \text{and} \quad B = a_1 a_2. \tag{1.12}$$

Then $A - B = 1 + a_1 + a_2 > 1$. Denote $B(0, R) := \{x \in \mathbb{R}^2 \mid |x| < R\}$. The main result of this paper is to prove the existence of a new type of non-topological bubbling solutions.

Theorem 1.1 (Theorem 2.3). *Assume that $a_1, a_2 > 0$ satisfy*

$$3(1 + a_1)(1 + a_2) - 4a_1 a_2 > 0 \tag{1.13}$$

and N_1, N_2 are non-negative integers satisfying

$$(A - 4B)(N_1 + 1) < 2a_1(1 + a_1)(N_2 + 1) \quad \text{if} \quad A - 4B > 0. \tag{1.14}$$

Let (α_1, α_2) satisfy $\alpha_1 \geq 1, \alpha_2 > 1$ and

$$\begin{aligned} & (3A - 4B)\alpha_1 + \frac{1 + a_1}{a_2}(A - 2B)\alpha_2 \\ &= AN_1 + \frac{1 + a_1}{a_2}AN_2 + \left(4 + 2\frac{1 + a_1}{a_2}\right)(A - B), \end{aligned} \tag{1.15}$$

$$\frac{4B - A}{A}(\alpha_1 - 1) + \frac{2a_1}{1 + a_2}(\alpha_2 - 1) - (N_1 + 1) > 0. \tag{1.16}$$

Then system (1.10) admits a sequence of radial non-topological bubbling solutions $(u_{1,n}, u_{2,n})$ such that $\sup_{\mathbb{R}^2} u_{2,n} \rightarrow -\infty$ as $n \rightarrow \infty$. Furthermore, there exist two intersection points $R_{3,n} \gg R_{1,n} \gg 1$ of $u_{1,n}$ and $u_{2,n}$ such that:

- (i) *It holds $u_{1,n} \rightarrow U$ in $C^2_{\text{loc}}(B(0, R_{1,n}))$ as $n \rightarrow \infty$, where U is the unique radial solution of*

$$\begin{cases} \Delta U + (1 + a_1)e^U - (1 + a_1)^2 e^{2U} = 4\pi N_1 \delta_0 & \text{in } \mathbb{R}^2, \\ U(x) = -2\gamma \ln |x| + O(1) & \text{as } |x| \rightarrow \infty \end{cases} \tag{1.17}$$

with

$$\gamma = \frac{4B - A}{A}(\alpha_1 - 1) + \frac{2a_1}{1 + a_2}(\alpha_2 - 1) + 1; \quad (1.18)$$

(ii) It holds $\int_{R_{1,n}}^{R_{3,n}} r e^{u_{1,n}} dr \rightarrow 0$, $\int_0^{R_{1,n}} r e^{u_{2,n}} dr \rightarrow 0$, $\int_{R_{3,n}}^{\infty} r e^{u_{2,n}} dr \rightarrow 0$ and

$$\begin{aligned} \int_{R_{1,n}}^{R_{3,n}} r e^{u_{2,n}} dr &\rightarrow \frac{4}{1 + a_2} \left(\frac{a_2}{1 + a_1}(\gamma + N_1) + N_2 + 1 \right), \\ \int_{R_{3,n}}^{\infty} r e^{u_{1,n}} dr &\rightarrow \frac{4}{1 + a_1}(\alpha_1 - 1) \end{aligned} \quad (1.19)$$

as $n \rightarrow \infty$;

(iii) There exist constants $\alpha_{1,n} > 1$ and $\alpha_{2,n} > 1$ such that

$$u_{k,n}(r) = -2\alpha_{k,n} \ln r + O(1) \text{ as } r \rightarrow \infty, \text{ for } k = 1, 2,$$

and $(\alpha_{1,n}, \alpha_{2,n}) \rightarrow (\alpha_1, \alpha_2)$ as $n \rightarrow \infty$.

1.2. Motivations and remarks

In order to get a full understanding of Theorem 1.1 (such as, why we assume (1.15)), we will shortly discuss some known results. Define a quadratic form $J : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$J(x, y) = \frac{a_2(1 + a_2)}{2}x^2 + a_1a_2xy + \frac{a_1(1 + a_1)}{2}y^2. \quad (1.20)$$

Recently, in [20, 21], Huang and the second author classified all radially symmetric solutions of (1.1). Among other things, they proved the following interesting result.

Theorem A ([21]). *Let $a_1, a_2 > 0$ and $(u_1, u_2) \neq (0, 0)$ be a radially symmetric solution of system (1.10). Then both $u_1 < 0$ and $u_2 < 0$ in \mathbb{R}^2 , and one of the following conclusions holds.*

- (i) *The pair (u_1, u_2) is a topological solution;*
- (ii) *The pair (u_1, u_2) is a mixed-type solution;*
- (iii) *The pair (u_1, u_2) is a non-topological solution and there exist constants $\alpha_1, \alpha_2 > 1$ such that*

$$u_k(x) = -2\alpha_k \ln |x| + O(1) \text{ as } |x| \rightarrow +\infty, \text{ for } k = 1, 2. \quad (1.21)$$

Consequently, $e^{u_1}, e^{u_2} \in L^1(\mathbb{R}^2)$. Moreover, (α_1, α_2) satisfies

$$J(\alpha_1 - 1, \alpha_2 - 1) > J(N_1 + 1, N_2 + 1). \quad (1.22)$$

The inequality (1.22) comes from the following Pohozaev identity (see [21]):

$$\begin{aligned} & J(\alpha_1 - 1, \alpha_2 - 1) - J(N_1 + 1, N_2 + 1) \\ &= \frac{A - B}{4} \int_0^\infty r \left[a_2(1 + a_1)e^{2u_1} + a_1(1 + a_2)e^{2u_2} - 2a_1a_2e^{u_1+u_2} \right] dr. \end{aligned} \quad (1.23)$$

Therefore, (1.22) is a necessary condition for the existence of radially symmetric non-topological solutions satisfying the asymptotic condition (1.21). In view of Theorem A, it is natural to consider the following question.

Question. Fix $\alpha_1, \alpha_2 > 1$ satisfying (1.22). Is there a radially symmetric non-topological solution of system (1.10) subject to the prescribed asymptotic condition (1.21)?

If we let $N_1 = N_2 = N$ with $a_1 = a_2$ and $u_1 = u_2 = u$ in (1.10), then system (1.10) turns to be the following Chern-Simons-Higgs scalar equation

$$\Delta u + e^u(1 - e^u) = 4\pi N\delta_0 \quad \text{in } \mathbb{R}^2. \quad (1.24)$$

Equation (1.24) is known as the $SU(2)$ Chern-Simons model for the Abelian case; see [19, 22]. For recent developments, we refer the reader to [3, 4, 6–8, 27, 28] and references therein. Remark that the Pohozaev identity plays a key role in studying non-topological solutions of (1.24). Let u be a radial non-topological solution of (1.24) satisfying $u(x) = -2\alpha \ln |x| + O(1)$ near ∞ . Then the Pohozaev identity implies

$$(\alpha - 1)^2 - (N + 1)^2 = \frac{1}{2} \int_0^\infty r e^{2u} dr > 0,$$

which implies $\alpha > N + 2$. In 2002, Chan, Fu and the second author [4] proved that the inequality $\alpha > N + 2$ is also a sufficient condition for the existence of radial non-topological solutions satisfying $u(x) = -2\alpha \ln |x| + O(1)$ near ∞ . However, as pointed out in [5, 9], this might not hold for system (1.10). The reason is following: there might be a sequence of solutions $(u_{1,n}, u_{2,n})$ such that only one component blows up, but the other one does not, *i.e.*, the so-called phenomena of *partial blowup*; see Theorem C for instance. As a result, only one of the L^1 norms of $e^{2u_{1,n}}$ and $e^{2u_{2,n}}$ tends to 0 as $n \rightarrow \infty$, which implies that the quantity $J(\alpha_1 - 1, \alpha_2 - 1) - J(N_1 + 1, N_2 + 1)$ might not converge to 0, namely it has a gap. Therefore, the inequality (1.22) might not be a sufficient condition for the existence of radial non-topological solutions satisfying (1.21).

In a previous work [5], we found a sufficient condition for the above question. As in [5], we define

$$\Omega := \{(\alpha_1, \alpha_2) \mid \alpha_1, \alpha_2 > 1, J(\alpha_1 - 1, \alpha_2 - 1) > J(N_1 + 1, N_2 + 1)\}, \quad (1.25)$$

and

$$S := \{(\alpha_1, \alpha_2) \mid \alpha_1, \alpha_2 > 0 \text{ and } (\alpha_1, \alpha_2) \text{ satisfies (1.27)-(1.30)}\}, \quad (1.26)$$

where

$$(A - 2B)\alpha_2 - a_2(1 + a_2)\alpha_1 < a_2(1 + a_2)N_1 + AN_2 + 2(A - B), \quad (1.27)$$

$$(A - 2B)\alpha_1 - a_1(1 + a_1)\alpha_2 < a_1(1 + a_1)N_2 + AN_1 + 2(A - B), \quad (1.28)$$

$$(3A - 4B)\alpha_1 + \frac{1 + a_1}{a_2}(A - 2B)\alpha_2 > AN_1 + \frac{1 + a_1}{a_2}AN_2 \\ + \left(4 + 2\frac{1 + a_1}{a_2}\right)(A - B), \quad (1.29)$$

$$(3A - 4B)\alpha_2 + \frac{1 + a_2}{a_1}(A - 2B)\alpha_1 > AN_2 + \frac{1 + a_2}{a_1}AN_1 \\ + \left(4 + 2\frac{1 + a_2}{a_1}\right)(A - B). \quad (1.30)$$

Theorem B ([5]). *Assume that N_1, N_2 are non-negative integers and $a_1, a_2 > 0$ satisfy*

$$(1 + a_1)(1 + a_2) > (6 - 2\sqrt{5})a_1a_2. \quad (1.31)$$

Let Ω and S be defined in (1.25)-(1.26). Then $S \cap \Omega \neq \emptyset$, and for any fixed $(\alpha_1, \alpha_2) \in S \cap \Omega$, system (1.10) admits a radially symmetric non-topological solution (u_1, u_2) satisfying the prescribed asymptotic condition (1.21).

Remark that $S \cap \Omega \neq \emptyset$ if and only if (a_1, a_2) satisfies (1.31), i.e., (1.31) is a necessary condition for Theorem B. For example, Theorem B can be applied to the $SU(3)$ system (1.8) and the \mathcal{B}_2 system (1.9). The counterpart of Theorem B for the $SU(3)$ system (1.8) was firstly obtained by Choe, Kim and the second author [9], and Theorem B is a generalization of their result to the generic system (1.10). Applying Theorem B to the \mathcal{B}_2 case, we conclude that if

$$(\alpha_1, \alpha_2) \in S \cap \Omega = \left\{ (\alpha_1, \alpha_2) \left| \begin{array}{l} \alpha_1 > N_1 + N_2 + 3 \\ \alpha_2 > 2N_1 + N_2 + 4 \end{array} \right. \right\}, \quad (1.32)$$

then (1.9) has a radial non-topological solution satisfying (1.21).

We proved Theorem B via the Leray-Schauder degree theory. To do this, we proved a uniform boundedness result for radial solutions satisfying (1.21) whenever $(\alpha_1, \alpha_2) \in S \cap \Omega$. Then a natural question is whether the set $S \cap \Omega$ is the optimal range of (α_1, α_2) for the existence of radial solutions satisfying (1.21). This question has not been settled yet (Theorem 1.1 gives a negative answer for the \mathcal{B}_2 case; see Remark 1.3 below). However, in the same paper [5], we also proved the existence of partially bubbling solutions along some part of $\partial(S \cap \Omega)$.

Theorem C ([5]). *Assume that N_1, N_2 are non-negative integers and $a_1, a_2 > 0$ satisfy*

$$(1 + a_1)(1 + a_2) > 2a_1a_2. \quad (1.33)$$

Let $(\alpha_1, \alpha_2) \in \Omega$ satisfy $\alpha_1 \neq \alpha_2$ and

$$(A - 2B)\alpha_2 - a_2(1 + a_2)\alpha_1 = a_2(1 + a_2)N_1 + AN_2 + 2(A - B). \quad (1.34)$$

Then system (1.10) admits a sequence of radial non-topological bubbling solutions $(u_{1,n}, u_{2,n})$ such that $\sup_{\mathbb{R}^2} u_{2,n} \rightarrow -\infty$ as $n \rightarrow \infty$. Furthermore, there exists a intersection point $R_{1,n} \gg 1$ of $u_{1,n}$ and $u_{2,n}$ such that:

- (i) It holds $u_{1,n} \rightarrow U$ in $C_{\text{loc}}^2(B(0, R_{1,n}))$, where U is the unique radial solution of (1.17) with $\gamma = \alpha_1 + \frac{2\alpha_1}{1+\alpha_2}(\alpha_2 - 1)$. Besides,

$$\lim_{n \rightarrow \infty} \int_{R_{1,n}}^{\infty} r e^{u_{1,n}} dr = 0; \tag{1.35}$$

- (ii) There exists $(\alpha_{1,n}, \alpha_{2,n}) \in \Omega$ such that

$$u_{k,n}(r) = -2\alpha_{k,n} \ln r + O(1) \text{ as } r \rightarrow \infty, \text{ for } k = 1, 2,$$

and $(\alpha_{1,n}, \alpha_{2,n}) \rightarrow (\alpha_1, \alpha_2)$ as $n \rightarrow \infty$.

Theorem C proves the existence of bubbling solutions along the boundary of (1.27). For these bubbling solutions, only the second component blows up. We call this type of bubbling solutions of *type I*. Inspired by Theorem B, there might exist another type of bubbling solutions along the boundary of (1.29) (or equivalently (1.30)), which we call of *type II*. But for type II, the estimate (1.35) no longer holds, which means that both components of bubbling solutions blow up at infinity, namely the asymptotics of type II are more complicated.

Therefore, Theorem 1.1 gives precisely the existence of bubbling solutions of type II. We conclude this section by some further comments about Theorem 1.1.

Remark 1.2. For the $SU(3)$ case, we have $(a_1, a_2) = (1, 1)$. Then it is easy to check that the range of (α_1, α_2) given in Theorem 1.1 is exactly

$$2\alpha_1 + \alpha_2 = N_1 + 2N_2 + 6 \quad \text{and} \quad 1 \leq \alpha_1 < N_2 + 2. \tag{1.36}$$

We remark that the counterpart of Theorem 1.1 for the $SU(3)$ system (1.8) was firstly proved by Choe, Kim and the second author [10] under the following assumption

$$2\alpha_1 + \alpha_2 = N_1 + 2N_2 + 6 \quad \text{and} \quad 1 < \alpha_1 < N_2 + 2, \tag{1.37}$$

where the assumption $\alpha_1 > 1$ plays a crucial role in their proof. Theorem 1.1 improves their result on two aspects. First, for the $SU(3)$ case, Theorem 1.1 covers the special case $\alpha_1 = 1$ (note that $(\alpha_1, \alpha_2) = (1, N_1 + 2N_2 + 4) \in \partial(S \cap \Omega)$) which was not considered in [10]. Remark that the case $\alpha_1 = 1$ is different from the case $\alpha_1 > 1$. Indeed, we can see from (1.19) that, if $\alpha_1 > 1$, then (1.35) no longer holds, namely the bubbling solutions are of type II. However, the case $\alpha_1 = 1$ satisfies (1.35), namely the bubbling solutions are of type I just as in Theorem C. This phenomenon is reasonable, because the intersection point of line (1.34) with line (1.15), which exists provided $A > 2B$, is exactly

$$(\alpha_1, \alpha_2) = \left(1, \frac{a_2(1 + a_2)}{A - 2B}(N_1 + 1) + \frac{A}{A - 2B}(N_2 + 1) + 1 \right).$$

Thus the case $\alpha_1 = 1$ can be seen as a critical case that connects bubbling solutions of type I with bubbling solutions of type II. Observe that $\alpha_1 > 1$ was assumed in Theorem C, so our study of the case $\alpha_1 = 1$ is also a complement of Theorem C. Second, Theorem 1.1 generalizes their result to the generic system (1.10). Theorem 1.1 indicates that, there exist bubbling solutions of type II along the boundary of (1.29). This fact, together with Theorem C, shows that the set $S \cap \Omega$ is an optimal range¹ in view of the degree theory.

Remark 1.3. For the $SU(3)$ system (1.8), we still do not know whether the set $S \cap \Omega$ is the optimal range for the existence of non-topological solutions, but we strongly believe so in view of Theorems B, C and 1.1; see [9, 10]. However, the generic system (1.10) is more involved than the $SU(3)$ system (1.8). One example is the \mathcal{G}_2 case where $(a_1, a_2) = (5, 9)$ and so $3A - 4B = 0$. Therefore, none of Theorems B, C and 1.1 can be applied to the \mathcal{G}_2 case, and understanding the non-topological solution structure for the \mathcal{G}_2 case remains open. Another example is the \mathcal{B}_2 case, where $(a_1, a_2) = (2, 3)$ and so $A - 2B = 0$. Then it is easy to check that the range of (α_1, α_2) given in Theorem 1.1 is exactly

$$\alpha_1 = N_1 + N_2 + 3 \quad \text{and} \quad \alpha_2 > 1. \quad (1.38)$$

From here, we conclude that the set $S \cap \Omega$ (see (1.32)) given in Theorem B, which is optimal in view of the degree theory, is not the optimal range for the existence of non-topological solutions to the \mathcal{B}_2 system (1.9).

Remark 1.4. Theorem B can not be applied to the case $A \leq (6 - 2\sqrt{5})B$. Therefore, Theorem 1.1 also gives the first existence result of radial non-topological solutions for the case $\frac{4}{3}B < A \leq (6 - 2\sqrt{5})B$.

Remark 1.5. Clearly, by Theorem A-(iii), assumptions $\alpha_1 \geq 1, \alpha_2 > 1$ are necessary for Theorem 1.1. As mentioned before, $\gamma > N_1 + 2$ is a necessary and sufficient condition for the existence of radial solutions for (1.17). Therefore, (1.18) indicates that (1.16) is a necessary condition for Theorem 1.1. In fact, (1.16) is also needed to guarantee that (α_1, α_2) satisfies inequality (1.22) (see Lemma 2.2 below), which is obviously necessary by the Pohozaev identity. On the other hand, assumptions (1.13)-(1.14) are also necessary conditions for Theorem 1.1, because they are the necessary and sufficient condition to guarantee $\{(\alpha_1, \alpha_2) \mid \alpha_1 \geq 1, \alpha_2 > 1 \text{ and satisfy (1.15)-(1.16)}\} \neq \emptyset$; see Lemma 2.1 below.

Theorem 1.1 will be proved via the shooting method in Section 2.

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¹ Here we mean that the a priori estimates can not hold for any open connected set containing $S \cap \Omega$ as a proper subset. In other words, if there is another connected range \tilde{S} for the existence of solutions by the degree theory, then $\tilde{S} \cap S \cap \Omega = \emptyset$.

2. Construction via the shooting method

In this section, we will prove Theorem 1.1 by constructing bubbling solutions via the shooting method. In the sequel, we assume that $a_1, a_2 > 0$ satisfy

$$3(1 + a_1)(1 + a_2) - 4a_1a_2 > 0. \quad (2.1)$$

Recall the notations A, B in (1.12). Assume that N_1, N_2 are non-negative integers satisfying

$$(A - 4B)(N_1 + 1) < 2a_1(1 + a_1)(N_2 + 1) \text{ if } A - 4B > 0. \quad (2.2)$$

Define

$$\Sigma := \{(\alpha_1, \alpha_2) \mid \alpha_1 \geq 1, \alpha_2 > 1, g(\alpha_1, \alpha_2) = 0, h(\alpha_1, \alpha_2) > 0\}, \quad (2.3)$$

where

$$\begin{aligned} g(\alpha_1, \alpha_2) := & (3A - 4B)\alpha_1 + \frac{1 + a_1}{a_2}(A - 2B)\alpha_2 - AN_1 \\ & - \frac{1 + a_1}{a_2}AN_2 - \left(4 + 2\frac{1 + a_1}{a_2}\right)(A - B), \end{aligned} \quad (2.4)$$

$$h(\alpha_1, \alpha_2) := \frac{4B - A}{A}(\alpha_1 - 1) + \frac{2a_1}{1 + a_2}(\alpha_2 - 1) - (N_1 + 1). \quad (2.5)$$

Lemma 2.1. $\Sigma \neq \emptyset$ if and only if (2.1)-(2.2) hold.

Proof. Denote $\tilde{N}_k = N_k + 1$ for $k = 1, 2$. Clearly $\Sigma \neq \emptyset$ is equivalent to

$$\tilde{\Sigma} := \left\{ (x, y) \mid x \geq 0, y > 0, \tilde{g}(x, y) = 0, \tilde{h}(x, y) > 0 \right\} \neq \emptyset,$$

where

$$\begin{aligned} \tilde{g}(x, y) := & (3A - 4B)x + \frac{1 + a_1}{a_2}(A - 2B)y - A\tilde{N}_1 - \frac{1 + a_1}{a_2}A\tilde{N}_2, \\ \tilde{h}(x, y) := & \frac{4B - A}{A}x + \frac{2a_1}{1 + a_2}y - \tilde{N}_1. \end{aligned}$$

If $3A - 4B \leq 0$, then $\{(x, y) \mid x \geq 0, y > 0, \tilde{g}(x, y) = 0\} = \emptyset$. Therefore, (2.1) is a necessary condition to guarantee $\tilde{\Sigma} \neq \emptyset$. In the following, we always assume that (2.1) holds. Then it is trivial to see that $\tilde{\Sigma} \neq \emptyset$ in the case $A - 2B \leq 0$. Consider the remaining case $A - 2B > 0$. Observe that the intersection point of $\tilde{g}(x, y) = 0$ with the y -axis is $(0, \frac{a_2(1+a_2)\tilde{N}_1 + A\tilde{N}_2}{A-2B})$. If $A - 4B \leq 0$, a direct computation shows that

$$\tilde{h}\left(0, \frac{a_2(1+a_2)\tilde{N}_1 + A\tilde{N}_2}{A-2B}\right) > 0 \quad (2.6)$$

holds automatically, which implies $\tilde{\Sigma} \neq \emptyset$. If $A - 4B > 0$, it is easy to see that $\tilde{\Sigma} \neq \emptyset$ if and only if (2.6) holds, which is just equivalent to (2.2). This completes the proof. \square

In the sequel, we fix any $(\alpha_1, \alpha_2) \in \Sigma$. We will prove the existence of bubbling solutions near (α_1, α_2) just as stated in Theorem 1.1.

Inspired by the blowup analysis in our previous work [5], we define

$$\gamma := \frac{4B - A}{A}(\alpha_1 - 1) + \frac{2a_1}{1 + a_2}(\alpha_2 - 1) + 1. \quad (2.7)$$

Then $h(\alpha_1, \alpha_2) > 0$ gives

$$\gamma > N_1 + 2. \quad (2.8)$$

Clearly, $g(\alpha_1, \alpha_2) = 0$ and (2.7) yield

$$\alpha_1 = -\frac{A - 2B}{A}(\gamma - 1) + \frac{2B}{A}(N_1 + 1) + \frac{2a_1}{1 + a_2}(N_2 + 1) + 1, \quad (2.9)$$

$$\begin{aligned} \alpha_2 &= \frac{a_2}{1 + a_1} \frac{3A - 4B}{A}(\gamma - 1) + \frac{a_2}{1 + a_1} \frac{A - 4B}{A}(N_1 + 1) \\ &\quad + \frac{A - 4B}{A}(N_2 + 1) + 1. \end{aligned} \quad (2.10)$$

By $\alpha_1 \geq 1$ we obtain

$$\gamma \leq 1 + \frac{2B}{A - 2B}(N_1 + 1) + \frac{2a_1(1 + a_1)}{A - 2B}(N_2 + 1) \text{ if } A > 2B. \quad (2.11)$$

As in Lemma 2.1, for convenience, we always denote

$$\tilde{\gamma} = \gamma - 1, \quad \tilde{\alpha}_k = \alpha_k - 1 \text{ and } \tilde{N}_k = N_k + 1, \text{ for } k = 1, 2. \quad (2.12)$$

Lemma 2.2. *The inequality $h(\alpha_1, \alpha_2) > 0$ implies $J(\alpha_1 - 1, \alpha_2 - 1) > J(N_1 + 1, N_2 + 1)$.*

Proof. By the definition (1.20) of J , a direct computation shows

$$\begin{aligned} J(x, y) &= J(-x, -y) \\ &= J\left(x, -\frac{2a_2}{1 + a_1}x - y\right) = J\left(-x, \frac{2a_2}{1 + a_1}x + y\right) \\ &= J\left(-x - \frac{2a_1}{1 + a_2}y, y\right) = J\left(x + \frac{2a_1}{1 + a_2}y, -y\right). \end{aligned} \quad (2.13)$$

Since (2.9)-(2.10) give

$$\begin{aligned} \tilde{\alpha}_1 &= \frac{2B - A}{A}\tilde{\gamma} + \frac{2B}{A}\tilde{N}_1 + \frac{2a_1}{1 + a_2}\tilde{N}_2, \\ \tilde{\alpha}_2 &= \frac{a_2}{1 + a_1}(\tilde{\gamma} + \tilde{N}_1) + \tilde{N}_2 - \frac{2a_2}{1 + a_1}\tilde{\alpha}_1, \end{aligned}$$

we can derive

$$\begin{aligned}
 J(\tilde{\alpha}_1, \tilde{\alpha}_2) &= J\left(\tilde{\alpha}_1, \frac{a_2}{1+a_1}(\tilde{\gamma} + \tilde{N}_1) + \tilde{N}_2 - \frac{2a_2}{1+a_1}\tilde{\alpha}_1\right) \\
 &= J\left(\tilde{\alpha}_1, -\frac{a_2}{1+a_1}(\tilde{\gamma} + \tilde{N}_1) - \tilde{N}_2\right) \\
 &= J\left(\frac{2a_1}{1+a_2}\left(\frac{a_2}{1+a_1}(\tilde{\gamma} + \tilde{N}_1) + \tilde{N}_2\right) - \tilde{\gamma}, -\frac{a_2}{1+a_1}(\tilde{\gamma} + \tilde{N}_1) - \tilde{N}_2\right) \\
 &= J\left(-\tilde{\gamma}, \frac{a_2}{1+a_1}(\tilde{\gamma} + \tilde{N}_1) + \tilde{N}_2\right) \\
 &= J(\tilde{N}_1, \tilde{N}_2) + \frac{a_2(1+a_1+a_2)}{2(1+a_1)}(\tilde{\gamma}^2 - \tilde{N}_1^2).
 \end{aligned}$$

By $\gamma > N_1 + 2$, we conclude $J(\alpha_1 - 1, \alpha_2 - 1) > J(N_1 + 1, N_2 + 1)$. \square

Since $\gamma > N_1 + 2$, by [4, Theorem 2.1], there is a unique radially symmetric solution U of the Chern-Simons-Higgs equation

$$\begin{cases} \Delta U + (1+a_1)e^U - (1+a_1)^2e^{2U} = 4\pi N_1\delta_0 & \text{in } \mathbb{R}^2 \\ U(x) = -2\gamma \ln|x| + O(1) & \text{as } |x| \rightarrow \infty. \end{cases} \quad (2.14)$$

Moreover, $U < -\ln(1+a_1)$ in \mathbb{R}^2 and

$$\begin{aligned}
 \int_0^\infty r \left[(1+a_1)e^U - (1+a_1)^2e^{2U} \right] dr &= 2(\gamma + N_1), \\
 \lim_{r \rightarrow \infty} \left[r^2e^{U(r)} + |rU'(r) + 2\gamma| \right] &= 0.
 \end{aligned} \quad (2.15)$$

Let $V(|x|) = V(x) := U(x) - 2N_1 \ln|x|$, then $V(0) := \lim_{r \rightarrow 0} V(r)$ is well defined; see [4].

To use the shooting method, we consider an initial value problem of system (1.10) in a radial variable. Denote

$$F_k(r) := (1+a_k)e^{2u_k(r)} - e^{u_k(r)} - a_k e^{u_1(r)+u_2(r)}, \quad \text{for } k = 1, 2,$$

for convenience. Clearly

$$|F_k| \leq (1+a_k)e^{u_k} \leq 1+a_k \quad \text{whenever } u_1, u_2 \leq 0, \quad (2.16)$$

and it is easy to check that

$$\begin{cases} F_k < -\frac{1}{2}e^{u_k} < 0 & \text{if } u_k < -\ln 2(1+a_1+a_2), \\ F_{3-k} < F_k < 0 & \text{if } u_k < u_{3-k} < -\ln 2(1+a_1+a_2), \end{cases} \quad k = 1, 2. \quad (2.17)$$

We will study the following initial value problem

$$\begin{cases} u_1''(r) + \frac{1}{r}u_1'(r) = (1 + a_1)F_1(r) - a_1F_2(r) & r > 0 \\ u_2''(r) + \frac{1}{r}u_2'(r) = (1 + a_2)F_2(r) - a_2F_1(r) & r > 0 \\ u_1(r) = 2N_1 \ln r + V(0) + o(1) & r \rightarrow 0 \\ u_2(r) = 2N_2 \ln r + \ln \varepsilon + o(1) & r \rightarrow 0, \end{cases} \quad (2.18)$$

where $\varepsilon \in (0, 1)$. Clearly, the solution of (2.18) depends on ε and we denote it by $(u_{1,\varepsilon}, u_{2,\varepsilon})$. Consequently, $F_k(r) = F_{k,\varepsilon}(r)$ also depends on ε . For the sake of convenience, when no confusion can arise, we will omit the subscript ε . The main result of this section is following, and Theorem 1.1 is a direct corollary.

Theorem 2.3. *Assume (2.1)-(2.2) and fix any $(\alpha_1, \alpha_2) \in \Sigma$. Then there exists sufficiently small $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$, system (2.18) has an entire solution $(u_{1,\varepsilon}, u_{2,\varepsilon})$. Furthermore, there exist two intersection points $R_{3,\varepsilon} \gg R_{1,\varepsilon} \gg 1$ of $u_{1,\varepsilon}$ and $u_{2,\varepsilon}$ such that:*

- (i) *It holds $u_{1,\varepsilon} \rightarrow U$ in $C_{\text{loc}}^2(B(0, R_{1,\varepsilon}))$ and $\sup_{\mathbb{R}^2} u_{2,\varepsilon} \rightarrow -\infty$ as $\varepsilon \rightarrow 0$;*
(ii) *It holds $\int_{R_{1,\varepsilon}}^{R_{3,\varepsilon}} r e^{u_{1,\varepsilon}} dr \rightarrow 0$, $\int_0^{R_{1,\varepsilon}} r e^{u_{2,\varepsilon}} dr \rightarrow 0$, $\int_{R_{3,\varepsilon}}^\infty r e^{u_{2,\varepsilon}} dr \rightarrow 0$ and*

$$\begin{aligned} \int_{R_{1,\varepsilon}}^{R_{3,\varepsilon}} r e^{u_{2,\varepsilon}} dr &\rightarrow \frac{2}{1 + a_2} \left(\frac{2a_2}{1 + a_1} (\tilde{\gamma} + \tilde{N}_1) + 2\tilde{N}_2 \right), \\ \int_{R_{3,\varepsilon}}^\infty r e^{u_{1,\varepsilon}} dr &\rightarrow \frac{4}{1 + a_1} (\alpha_1 - 1) \end{aligned}$$

as $\varepsilon \rightarrow 0$;

- (iii) *There exists $(\alpha_{1,\varepsilon}, \alpha_{2,\varepsilon}) \in \Omega$ such that*

$$u_{k,\varepsilon}(r) = -2\alpha_{k,\varepsilon} \ln r + O(1) \text{ as } r \rightarrow \infty, \text{ for } k = 1, 2,$$

and $(\alpha_{1,\varepsilon}, \alpha_{2,\varepsilon}) \rightarrow (\alpha_1, \alpha_2)$ as $\varepsilon \rightarrow 0$.

In the rest of this section, we prove Theorem 2.3, which is quite long and delicate, and we divide it into several lemmas. The basic strategy is similar to that in [10] where the $SU(3)$ case was studied. Since the solution $(u_{1,\varepsilon}, u_{2,\varepsilon})$ of the initial value problem (2.18) exists locally, the key point is to prove that $(u_{1,\varepsilon}, u_{2,\varepsilon})$ exists globally for $r \in (0, +\infty)$ (i.e., does not blow up at finite r) provided that $\varepsilon > 0$ is sufficiently small. This is the most difficult part of the proof. To overcome this difficulty, we need to carry on a delicate analysis of the asymptotic behavior of $(u_{1,\varepsilon}, u_{2,\varepsilon})$ as $r \rightarrow +\infty$. For example, we need to understand what happens if $u_{1,\varepsilon}$ and $u_{2,\varepsilon}$ intersect and how many times they intersect. The main tool is the well-known Pohozaev identity together with the blow up analysis, which makes our argument rather technical and involved.

Moreover, as pointed out in [5, 18, 21], the generic situation poses new analytical difficulties compared to the $SU(3)$ case. For example, in [10] they used many helpful inequalities, which hold in the $SU(3)$ case because of $(a_1, a_2) = (1, 1)$ but can not hold for general $a_1, a_2 > 0$. This requires us to develop generic ideas to avoid using this kind of inequalities. It is interesting that our general idea turns out to be somewhat simpler. More importantly, our general idea also works for the critical case $\alpha_1 = 1$. This is reasonable in view of mathematics. Roughly speaking, people usually exploit a more special method when the problem is more special. When the problem is more general, people need to develop a more generic method, the idea of which might be more natural and hence simpler.

In the sequel, we denote positive constants independent of ε (possibly different in different places) by C, C_0, C_1, \dots . Let δ be a constant such that

$$0 < \delta < \frac{\min\{1, a_1, a_2, B\} \cdot \min\{1, \hat{\alpha}_1, \tilde{\alpha}_2, \gamma - (N_1 + 2)\}}{100(1 + a_1 + a_2)^4}, \tag{2.19}$$

where $\hat{\alpha}_1 := \tilde{\alpha}_1 = \alpha_1 - 1$ if $\alpha_1 > 1$ and $\hat{\alpha}_1 := 1$ if $\alpha_1 = 1$. Then by (2.14)-(2.15), we can fix a constant $R_0 > 1$ large enough such that

$$\begin{aligned} R_0^2 e^{U(R_0)} + |R_0 U'(R_0) + 2\gamma| < \delta^3, \quad \text{and} \quad \int_{R_0}^{\infty} r e^{U(r)} dr < \delta^3, \\ \left| \int_0^{R_0} r \left[e^U - (1 + a_1) e^{2U} \right] dr - \frac{2(\gamma + N_1)}{1 + a_1} \right| < \delta^3. \end{aligned} \tag{2.20}$$

Repeating the argument of [5, Lemma 5.2], we can prove the existence of small $\varepsilon_1 > 0$ such that for each $\varepsilon \in (0, \varepsilon_1)$, problem (2.18) admits a solution $(u_{1,\varepsilon}, u_{2,\varepsilon})$ on $[0, R_0]$ which satisfies:

- (1) Both $u_{1,\varepsilon} < 0$ and $u_{2,\varepsilon} < 0$ on $[0, R_0]$, $u_{2,\varepsilon}(R_0) < u_{1,\varepsilon}(R_0) < 2 \ln \delta$ and $R_0^2 e^{u_{1,\varepsilon}(R_0)} < \delta$;
- (2) Inequalities $|R_0 u'_{1,\varepsilon}(R_0) + 2\gamma| < \delta^2$ and $|R_0 u'_{2,\varepsilon}(R_0) - \frac{2a_2}{1+a_1}(\gamma + N_1) - 2N_2| < \delta^2$;
- (3) Convergence $|u_{1,\varepsilon} - U| \rightarrow 0$ and $u_{2,\varepsilon} \rightarrow -\infty$ uniformly on $[0, R_0]$ as $\varepsilon \rightarrow 0$.

Furthermore, by following the argument of [5, Lemma 5.3], for each $\varepsilon \in (0, \varepsilon_1)$, there exists $R_1 = R_{1,\varepsilon} > R_0$ such that

- (4) $u_{1,\varepsilon}(R_{1,\varepsilon}) = u_{2,\varepsilon}(R_{1,\varepsilon})$, $u_{2,\varepsilon} < u_{1,\varepsilon} < 2 \ln \delta$ on $[R_0, R_{1,\varepsilon}]$ and

$$\begin{cases} |ru'_{1,\varepsilon}(r) + 2\gamma| < \delta, \\ \left| ru'_{2,\varepsilon}(r) - \frac{2a_2}{1+a_1}(\gamma + N_1) - 2N_2 \right| < \delta, \end{cases} \quad \text{for all } r \in [R_0, R_{1,\varepsilon}]. \tag{2.21}$$

In particular, together with (2.19), we have

$$ru'_{1,\varepsilon}(r) < -2\gamma + \delta < -4 \quad \text{for any } r \in [R_0, R_{1,\varepsilon}]. \tag{2.22}$$

Lemma 2.4. *Let $\varepsilon \rightarrow 0$, then $R_{1,\varepsilon} \rightarrow \infty$ and $R_{1,\varepsilon}^2 e^{u_k(R_{1,\varepsilon})} \rightarrow 0$ for $k = 1, 2$ and $\int_0^{R_{1,\varepsilon}} r e^{u_{2,\varepsilon}} dr \rightarrow 0$. Furthermore,*

$$\begin{aligned} R_{1,\varepsilon} u'_{1,\varepsilon}(R_{1,\varepsilon}) &= -2\gamma + o(1), \\ R_{1,\varepsilon} u'_{2,\varepsilon}(R_{1,\varepsilon}) &= \frac{2a_2}{1+a_1}(\gamma + N_1) + 2N_2 + o(1). \end{aligned}$$

Proof. By property (3) above, $u_1(R_0) - u_2(R_0) \rightarrow \infty$ as $\varepsilon \rightarrow 0$. Since (2.21) gives $|ru'_k(r)| \leq C$ on $[R_0, R_1]$, we easily obtain $R_1 \rightarrow \infty$ as $\varepsilon \rightarrow 0$. Then it follows from (2.22) and property (1) that

$$R_1^2 e^{u_2(R_1)} = R_1^2 e^{u_1(R_1)} < \frac{1}{R_1^2} R_0^4 e^{u_1(R_0)} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

By (2.21) again, we know that $u'_2(r) > 0$ for $r \in [R_0, R_1]$. Then by property (3), we have

$$\int_0^{R_1} r e^{u_2(r)} dr = o(1) + \int_{R_0}^{R_1} r e^{u_2(r)} dr \leq o(1) + R_1^2 e^{u_2(R_1)} = o(1)$$

as $\varepsilon \rightarrow 0$. Moreover, $e^{u_2} \rightarrow 0$ uniformly on $[0, R_1]$, which implies

$$\left| (1+a_1)e^{u_1(r)+u_2(r)} + F_2(r) \right| \leq 2(A-B)e^{u_2(r)} \rightarrow 0 \text{ uniformly on } [0, R_1].$$

Recall that

$$u''_1(r) + \frac{1}{r}u'_1(r) = \left[(1+a_1)^2 e^{2u_1} - (1+a_1)e^{u_1} \right] - a_1 \left[(1+a_1)e^{u_1+u_2} + F_2 \right].$$

By the standard continuous dependence on data in the ODE theory and $R_1 \rightarrow \infty$ as $\varepsilon \rightarrow 0$, we conclude that $|u_1 - U| \rightarrow 0$ uniformly on any compact subset $K \Subset [0, \infty)$. This, together with $e^{u_1(r)} \leq r^{-4} R_0^4 e^{u_1(R_0)}$ for $r \in [R_0, R_1]$, easily yields

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_0^{R_1} r \left[e^{u_1} - (1+a_1)e^{2u_1} \right] dr &= \int_0^\infty r \left[e^U - (1+a_1)e^{2U} \right] dr \\ &= \frac{2(\gamma + N_1)}{1+a_1}. \end{aligned}$$

Moreover,

$$\lim_{\varepsilon \rightarrow 0} \int_0^{R_1} r (e^{2u_2} + e^{u_1+u_2}) dr = 0.$$

Consequently, by integrating (2.18) over $(0, R_1)$, we obtain

$$\begin{aligned} R_1 u'_1(R_1) &= 2N_1 - (1 + a_1) \int_0^{R_1} r \left[e^{u_1} - (1 + a_1)e^{2u_1} \right] dr + o(1) \\ &= -2\gamma + o(1), \\ R_1 u'_2(R_1) &= 2N_2 + a_2 \int_0^{R_1} r \left[e^{u_1} - (1 + a_1)e^{2u_1} \right] dr + o(1) \\ &= \frac{2a_2}{1 + a_1}(\gamma + N_1) + 2N_2 + o(1) \end{aligned}$$

as $\varepsilon \rightarrow 0$. This completes the proof. \square

For each $\varepsilon \in (0, \varepsilon_1)$, we define

$$R^* = R_\varepsilon^* := \sup\{r > R_{1,\varepsilon} \mid u_{1,\varepsilon} < 0, u_{2,\varepsilon} < 0 \text{ on } [0, r]\}.$$

Our final goal is to prove $R_\varepsilon^* = \infty$ provided $\varepsilon > 0$ sufficiently small. To this goal, we define

$$R_2 = R_{2,\varepsilon} := \sup\{r \in (R_{1,\varepsilon}, R_\varepsilon^*) \mid u'_{2,\varepsilon} > 0 \text{ on } [R_{1,\varepsilon}, r]\}.$$

First, we recall the following Pohozaev identity (cf. [21, Lemma 7.2] or [5, Lemma 2.2])

$$\begin{aligned} &\frac{d}{dr} \left\{ J(ru'_1(r) + 2, ru'_2(r) + 2) + (1 + a_1 + a_2)r^2 \left[a_2 e^{u_1} + a_1 e^{u_2} \right. \right. \\ &\quad \left. \left. - \frac{a_2(1 + a_1)}{2} e^{2u_1} - \frac{a_1(1 + a_2)}{2} e^{2u_2} + a_1 a_2 e^{u_1 + u_2} \right] \right\} \\ &= (1 + a_1 + a_2)r \left[a_2(1 + a_1)e^{2u_1} + a_1(1 + a_2)e^{2u_2} - 2a_1 a_2 e^{u_1 + u_2} \right], \end{aligned} \quad (2.23)$$

where J is defined in (1.20).

Lemma 2.5. *There exists a small $\varepsilon_2 \in (0, \varepsilon_1)$ such that for each $\varepsilon \in (0, \varepsilon_2)$, $u_{1,\varepsilon} < u_{2,\varepsilon}$ on $(R_{1,\varepsilon}, R_{2,\varepsilon})$.*

Proof. Assume by contradiction that there exist a sequence $\varepsilon_n \downarrow 0$ and $r_n \in (R_{1,\varepsilon_n}, R_{2,\varepsilon_n})$ such that $u_{1,\varepsilon_n}(r_n) = u_{2,\varepsilon_n}(r_n)$ and $u_{1,\varepsilon_n} < u_{2,\varepsilon_n}$ on $(R_{1,\varepsilon_n}, r_n)$. Clearly $u'_{1,\varepsilon_n}(r_n) \geq u'_{2,\varepsilon_n}(r_n) > 0$ for all n . We will omit the subscript ε_n in the following argument for convenience. We divide the proof into seven steps.

Step 1. We claim that $u_2(r_n) \rightarrow -\infty$ as $\varepsilon_n \downarrow 0$.

If not, we may assume, up to a subsequence, that $u_2(r_n) \geq c_0$ for some constant $c_0 < \ln \delta$. Recall $u_2(R_1) \rightarrow -\infty$. For n large, there exist $b_n, d_n \in (R_1, r_n)$ such that $b_n < d_n$, $u_2(d_n) = c_0 - 1$, $u_2(b_n) = c_0 - 2$, $u_2 \leq c_0 - 1$ on $[R_1, d_n]$ and

$u_2 \geq c_0 - 2$ on $[b_n, d_n]$. Consequently, $u_1 < u_2 \leq c_0 - 1 < \ln \delta$ on $(R_1, d_n]$ and so (2.17) gives $F_2 < F_1 < 0$ on $(R_1, d_n]$, which implies

$$(ru'_2(r))' = r[(1 + a_2)F_2 - a_2F_1] < rF_2 < -\frac{1}{2}re^{u_2(r)} \quad \text{on } (R_1, d_n]. \quad (2.24)$$

This, together with Lemma 2.4, yields $0 \leq ru'_2(r) \leq R_1u'_2(R_1) \leq C$ uniformly on $[R_1, d_n]$. Then by the mean value theorem, there exists $e_n \in (b_n, d_n)$ such that

$$\frac{1}{d_n - b_n} = \frac{u_2(d_n) - u_2(b_n)}{d_n - b_n} = u'_2(e_n) \leq \frac{C}{R_1},$$

which implies $d_n - b_n \rightarrow \infty$. Consequently,

$$\begin{aligned} 0 \leq d_nu'_2(d_n) &= b_nu'_2(b_n) + \int_{b_n}^{d_n} (ru'_2)' dr \leq R_1u'_2(R_1) - \frac{1}{2} \int_{b_n}^{d_n} re^{u_2(r)} dr \\ &\leq C - \frac{1}{2}e^{c_0-2} \int_{b_n}^{d_n} r dr \rightarrow -\infty \quad \text{as } n \rightarrow \infty, \end{aligned}$$

a contradiction.

Step 2. We claim the existence of constant $C > 0$ independent of ε_n such that

$$u_k(r) + 2 \ln r \leq C \quad \text{uniformly for } r \in [R_1, r_n] \quad \text{for } k = 1, 2. \quad (2.25)$$

By Step 1, we may assume $u_1 < u_2 \leq u_2(r_n) < \ln \delta$ on (R_1, r_n) for all n . Then $[(1 + a_2)ru'_1 + a_1ru'_2]' = (A - B)rF_1 \leq -\frac{A-B}{2}re^{u_1}$ and $[a_2ru'_1 + (1 + a_1)ru'_2]' = (A - B)rF_2 \leq -\frac{A-B}{2}re^{u_2}$ on $[R_1, r_n]$, which imply

$$\begin{aligned} 0 &\leq (1 + a_2)r_nu'_1(r_n) + a_1r_nu'_2(r_n) \\ &\leq (1 + a_2)R_1u'_1(R_1) + a_1R_1u'_2(R_1) - \frac{A - B}{2} \int_{R_1}^{r_n} re^{u_1} dr, \\ 0 &\leq a_2r_nu'_1(r_n) + (1 + a_1)r_nu'_2(r_n) \\ &\leq a_2R_1u'_1(R_1) + (1 + a_1)R_1u'_2(R_1) - \frac{A - B}{2} \int_{R_1}^{r_n} re^{u_2} dr. \end{aligned}$$

By this and Lemma 2.4, we obtain

$$\int_{R_1}^{r_n} r(e^{u_1} + e^{u_2}) dr \leq C \quad \text{for all } n. \quad (2.26)$$

Since $e^{u_1} \leq e^{u_2} \leq e^{u_2(r_n)} \rightarrow 0$ on $[R_1, r_n]$, it follows that

$$\lim_{n \rightarrow \infty} \int_{R_1}^{r_n} r(e^{2u_1} + e^{2u_2} + e^{u_1+u_2}) dr = 0. \quad (2.27)$$

Combining (2.27) with Lemma 2.4, we integrate the Pohozaev identity (2.23) over (R_1, r) for any $r \in (R_1, r_n]$, which yields that

$$\begin{aligned} \frac{A-B}{2}r^2(a_2e^{u_1} + a_1e^{u_2}) &\leq J(ru'_1(r) + 2, ru'_2(r) + 2) + (A-B)r^2 \\ &\times \left[a_2e^{u_1} + a_1e^{u_2} - \frac{a_2(1+a_1)}{2}e^{2u_1} - \frac{a_1(1+a_2)}{2}e^{2u_2} + a_1a_2e^{u_1+u_2} \right] \\ &= J(R_1u'_1(R_1) + 2, R_1u'_2(R_1) + 2) + o(1) \leq C \end{aligned}$$

holds for all $r \in (R_1, r_n]$. This proves (2.25).

Step 3. Denote $(x_1, y_1) := (2-2\gamma, 2 + \frac{2a_2}{1+a_1}(\gamma+N_1)+2N_2)$. Clearly the following system

$$\begin{cases} J(x, y) = J(x_1, y_1) \\ (1+a_2)x + a_1y = (1+a_2)x_1 + a_1y_1, \end{cases} \quad (2.28)$$

has at most two distinct solutions, one is just (x_1, y_1) , and we denote the other one, if exists, by (x_2, y_2) .

Fix any $\theta \in [2.5, 3] \setminus \{2-x_1, 2-x_2\}$. Since $r_nu'_1(r_n) > 0$ and (2.22) gives $R_1u'_1(R_1) < -4$, there exists $t_n \in (R_1, r_n)$ such that $t_nu'_1(t_n) = -\theta$ and $ru'_1(r) < -\theta$ for all $r \in [R_1, t_n)$. We claim the existence of constant C independent of ε_n such that

$$u_2(t_n) + 2 \ln t_n \geq C \text{ for } n \text{ sufficiently large.} \quad (2.29)$$

Observe that $r^\theta e^{u_1(r)}$ is decreasing for $r \in [R_1, t_n]$, which implies

$$\int_{R_1}^{t_n} r e^{u_1} dr \leq R_1^\theta e^{u_1(R_1)} \int_{R_1}^{t_n} r^{1-\theta} dr \leq \frac{R_1^2 e^{u_1(R_1)}}{\theta-2} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Since $[(1+a_2)ru'_1 + a_1ru'_2]' = (A-B)rF_1$ and $|F_1| \leq (1+a_1)e^{u_1}$, we can obtain

$$(1+a_2)t_nu'_1(t_n) + a_1t_nu'_2(t_n) = (1+a_2)R_1u'_1(R_1) + a_1R_1u'_2(R_1) + o(1) \quad (2.30)$$

as $n \rightarrow \infty$. This, together with Lemma 2.4, shows that

$$(1+a_2)(2-\theta) + a_1 \left(2 + \lim_{n \rightarrow \infty} t_nu'_2(t_n) \right) = (1+a_2)x_1 + a_1y_1.$$

Since $2-\theta \notin \{x_1, x_2\}$, so $J(2-\theta, 2 + \lim_{n \rightarrow \infty} t_nu'_2(t_n)) - J(x_1, y_1) \neq 0$. Recall from (2.25) that $t_n^2 e^{(u_i+u_j)(t_n)} \leq Ct_n^{-2} \rightarrow 0$ as $n \rightarrow \infty$ for $1 \leq i, j \leq 2$. Then by (2.27) and Lemma 2.4, we easily deduce via integrating the Pohozaev identity (2.23) over (R_1, t_n) that

$$\begin{aligned} &(A-B)t_n^2 \left(a_2e^{u_1(t_n)} + a_1e^{u_2(t_n)} \right) \\ &= J(2+R_1u'_1(R_1), 2+R_1u'_2(R_1)) - J(2+t_nu'_1(t_n), 2+t_nu'_2(t_n)) + o(1) \quad (2.31) \\ &\rightarrow J(x_1, y_1) - J \left(2-\theta, 2 + \lim_{n \rightarrow \infty} t_nu'_2(t_n) \right) \neq 0 \end{aligned}$$

as $n \rightarrow \infty$. This proves (2.29) since $u_1(t_n) < u_2(t_n)$.

Step 4. For $k = 1, 2$, we consider the scaled functions

$$\hat{u}_{k,n}(r) := u_{k,\varepsilon_n}(t_n r) + 2 \ln t_n, \quad \frac{R_{1,\varepsilon_n}}{t_n} \leq r \leq \frac{r_n}{t_n}.$$

Then $(\hat{u}_{1,n}, \hat{u}_{2,n})$ satisfies

$$\begin{cases} \hat{u}_{1,n}'' + \frac{1}{r} \hat{u}_{1,n}' = (1 + a_1) \left((1 + a_1) t_n^{-2} e^{2\hat{u}_{1,n}} - e^{\hat{u}_{1,n}} - a_1 t_n^{-2} e^{\hat{u}_{1,n} + \hat{u}_{2,n}} \right) \\ \quad - a_1 \left((1 + a_2) t_n^{-2} e^{2\hat{u}_{2,n}} - e^{\hat{u}_{2,n}} - a_2 t_n^{-2} e^{\hat{u}_{1,n} + \hat{u}_{2,n}} \right) \\ \hat{u}_{2,n}'' + \frac{1}{r} \hat{u}_{2,n}' = (1 + a_2) \left((1 + a_2) t_n^{-2} e^{2\hat{u}_{2,n}} - e^{\hat{u}_{2,n}} - a_2 t_n^{-2} e^{\hat{u}_{1,n} + \hat{u}_{2,n}} \right) \\ \quad - a_2 \left((1 + a_1) t_n^{-2} e^{2\hat{u}_{1,n}} - e^{\hat{u}_{1,n}} - a_1 t_n^{-2} e^{\hat{u}_{1,n} + \hat{u}_{2,n}} \right). \end{cases} \quad (2.32)$$

By (2.25) and (2.29), we see that

$$|\hat{u}_{2,n}(1)| \leq C \quad \text{for sufficiently large } n. \quad (2.33)$$

Step 5. We claim that $\frac{R_{1,\varepsilon_n}}{t_n} \rightarrow 0$ and $u_{1,\varepsilon_n}(t_n) - u_{1,\varepsilon_n}(R_{1,\varepsilon_n}) \rightarrow -\infty$ as $n \rightarrow \infty$.

Assume by contradiction that up to a subsequence, $t_n/R_1 \leq C$ for all n . Similarly as (2.24), we can prove that $0 \leq r u_2'(r) \leq R_1 u_2'(R_1) \leq C$ uniformly on $[R_1, r_n]$. Consequently,

$$u_2(r) - u_2(R_1) \leq C \ln \frac{r}{R_1} \leq C \quad \text{uniformly for } r \in [R_1, t_n],$$

which implies thanks to Lemma 2.4 that

$$\int_{R_1}^{t_n} r e^{u_1} dr \leq \int_{R_1}^{t_n} r e^{u_2} dr \leq R_1^2 e^{u_2(R_1) + C} \left[\left(\frac{t_n}{R_1} \right)^2 - 1 \right] \rightarrow 0$$

as $n \rightarrow \infty$. Recalling the first equation in (2.18) and $|F_k| \leq (1 + a_k) e^{u_k}$, we obtain

$$\begin{aligned} -\theta &= t_n u_1'(t_n) \leq R_1 u_1'(R_1) + \int_{R_1}^{t_n} r [(1 + a_1)|F_1| + a_1|F_2|] dr \\ &= -2\gamma + o(1) \quad \text{as } n \rightarrow \infty, \end{aligned}$$

a contradiction with $\theta \leq 3$ and $\gamma > N_1 + 2$. This proves $R_1/t_n \rightarrow 0$ as $n \rightarrow \infty$. Consequently, we deduce from $r u_1'(r) \leq -\theta$ on $[R_1, t_n]$ that $u_1(t_n) - u_1(R_1) \leq -\theta \ln \frac{t_n}{R_1} \rightarrow -\infty$ as $n \rightarrow \infty$.

Step 6. We claim that $\frac{r_n}{t_n} \rightarrow \infty$ as $n \rightarrow \infty$.

By (2.25), (2.27) and the Pohozaev identity (2.23), it follows that

$$J(r u_1'(r) + 2, r u_2'(r) + 2) = J(R_1 u_1'(R_1) + 2, R_1 u_2'(R_1) + 2) + O(1) \leq C$$

uniformly for $r \in [R_1, r_n]$. Hence

$$|ru'_1(r)| + |ru'_2(r)| \leq C \text{ uniformly for } r \in [R_1, r_n]. \quad (2.34)$$

Then by $u_2(r_n) - u_2(R_1) > 0$ and Step 5, we have

$$\begin{aligned} C \ln \frac{r_n}{t_n} &\geq \int_{t_n}^{r_n} u'_1(r) dr = u_1(r_n) - u_1(t_n) \\ &= u_2(r_n) - u_2(R_1) + u_1(R_1) - u_1(t_n) \rightarrow \infty \end{aligned}$$

as $n \rightarrow \infty$. This proves the claim.

Step 7. We conclude the proof by obtaining a contradiction.

By $u_2(t_n) > u_2(R_1) = u_1(R_1)$, (2.33) and Step 5, we have

$$\hat{u}_{1,n}(1) = u_1(t_n) - u_2(t_n) + \hat{u}_{2,n}(1) \leq u_1(t_n) - u_1(R_1) + C \rightarrow -\infty$$

as $n \rightarrow \infty$. Combining this with (2.33) and (2.34), we conclude that $\hat{u}_{2,n}$ is uniformly bounded in $C_{\text{loc}}((0, \infty))$, while $\hat{u}_{1,n} \rightarrow -\infty$ uniformly on any compact subset $K \subset\subset (0, \infty)$ as $n \rightarrow \infty$. Up to a subsequence, we may assume that $\hat{u}_{2,n} \rightarrow \hat{u}$ in $C_{\text{loc}}^2((0, \infty))$, where \hat{u} satisfies

$$\begin{cases} \hat{u}'' + \frac{1}{r}\hat{u}' = -(1 + a_2)e^{\hat{u}} & \text{for } r \in (0, \infty), \\ \int_0^\infty e^{\hat{u}} r dr \leq \liminf_{n \rightarrow \infty} \int_{R_1}^{r_n} e^{u_2} r dr < +\infty. \end{cases}$$

Reminding that $u'_2(r) > 0$ on (R_1, r_n) , we easily conclude that $\hat{u}'(r) \geq 0$ for any $r > 0$, namely \hat{u} is increasing on $(0, \infty)$, which contradicts to $\int_0^\infty r e^{\hat{u}} dr < \infty$. This completes the proof. \square

Lemma 2.6. *There exists a small $\varepsilon_3 \in (0, \varepsilon_2)$ such that for each $\varepsilon \in (0, \varepsilon_3)$, it holds*

$$\begin{aligned} R_{2,\varepsilon} &< \infty, \\ u'_{2,\varepsilon}(R_{2,\varepsilon}) &= 0, \\ u_{1,\varepsilon}(R_{2,\varepsilon}) &< u_{2,\varepsilon}(R_{2,\varepsilon}) < \ln \delta, \\ |u_{2,\varepsilon}(R_{2,\varepsilon}) + 2 \ln R_{2,\varepsilon}| &\leq C \\ u_{k,\varepsilon}(r) + 2 \ln r &\leq C \text{ uniformly for } r \in [R_{1,\varepsilon}, R_{2,\varepsilon}], \quad k = 1, 2. \end{aligned} \quad (2.35)$$

Furthermore, $\frac{R_{2,\varepsilon}}{R_{1,\varepsilon}} \rightarrow \infty$ as $\varepsilon \rightarrow 0$.

Proof. We divide the proof into four steps.

Step 1. We claim that $R_{2,\varepsilon} < \infty$, $u'_{2,\varepsilon}(R_{2,\varepsilon}) = 0$ and $u_{1,\varepsilon}(R_{2,\varepsilon}) < u_{2,\varepsilon}(R_{2,\varepsilon}) < \ln \delta$ for $\varepsilon > 0$ sufficiently small.

Lemma 2.5 shows $u_1 < u_2$ on (R_1, R_2) . By repeating Step 1 of Lemma 2.5, we can prove

$$\sup_{R_1 \leq r < R_2} u_2(r) \rightarrow -\infty \text{ as } \varepsilon \rightarrow 0. \quad (2.36)$$

So $u_1 < u_2 < \ln \delta - 1$ on (R_1, R_2) for $\varepsilon > 0$ small enough. Then (2.24) holds for any $r \in (R_1, R_2)$. Recalling $u'_2 > 0$ on $[R_1, R_2]$, we have for any $r \in (R_1, R_2)$ that

$$-C \leq ru'_2(r) - R_1u'_2(R_1) \leq -\frac{1}{2} \int_{R_1}^r te^{u_2(t)} dt \leq -\frac{1}{4} e^{u_2(R_1)} (r^2 - R_1^2). \quad (2.37)$$

Letting $r \uparrow R_2$, it follows that $R_2 < \infty$ and so $u'_2(R_2) = 0$. The proof of Lemma 2.5 also yields $u_1(R_2) < u_2(R_2)$ for $\varepsilon > 0$ sufficiently small.

Step 2. We claim that (2.35) holds provided $\varepsilon > 0$ is sufficiently small.

In fact, since $u_1 < u_2 < \ln \delta$ on $(R_1, R_2]$, (2.37) also implies $\int_{R_1}^{R_2} r(e^{u_1} + e^{u_2}) dr \leq C$. This fact, together with (2.36), implies

$$\lim_{\varepsilon \rightarrow 0} \int_{R_1}^{R_2} r(e^{2u_1} + e^{2u_2} + e^{u_1+u_2}) dr = 0. \quad (2.38)$$

The rest argument is the same as Step 2 of Lemma 2.5.

Step 3. We claim that $\frac{R_{2,\varepsilon}}{R_{1,\varepsilon}} \rightarrow \infty$ as $\varepsilon \rightarrow 0$.

Remembering $ru'_2(r) \leq R_1u'_2(R_1) \leq C$ for $r \in [R_1, R_2]$, we have $u_2(r) \leq u_2(R_1) + C \ln \frac{r}{R_1}$ for all $r \in [R_1, R_2]$. Consequently,

$$\begin{aligned} -R_1u'_2(R_1) &= \int_{R_1}^{R_2} (ru'_2(r))' dr = \int_{R_1}^{R_2} r[(1+a_2)F_2 - a_2F_1] dr \\ &\geq (1+a_2) \int_{R_1}^{R_2} rF_2 dr \geq -(1+a_2)^2 \int_{R_1}^{R_2} re^{u_2(r)} dr \\ &\geq -(1+a_2)^2 R_1^2 e^{u_2(R_1)} \left[\left(\frac{R_2}{R_1} \right)^{C+2} - 1 \right]. \end{aligned}$$

This proves the claim because Lemma 2.4 gives $R_1^2 e^{u_2(R_1)} \rightarrow 0$ and $R_1u'_2(R_1) \rightarrow C > 0$ as $\varepsilon \rightarrow 0$.

Step 4. We prove the existence of constant C independent of ε such that $u_{2,\varepsilon}(R_{2,\varepsilon}) + 2 \ln R_{2,\varepsilon} \geq C$ provided $\varepsilon > 0$ is sufficiently small.

Assume by contradiction that there exist a sequence $\varepsilon_n \downarrow 0$ such that $u_{2,\varepsilon_n}(R_{2,\varepsilon_n}) + 2 \ln R_{2,\varepsilon_n} \rightarrow -\infty$ as $n \rightarrow \infty$. We will omit the subscript ε_n for convenience. Since u_2 is increasing on $[R_1, R_2]$, we have

$$r^2 e^{u_1(r)} \leq r^2 e^{u_2(r)} \leq R_2^2 e^{u_2(R_2)} \rightarrow 0 \text{ for any } r \in [R_1, R_2]. \quad (2.39)$$

We consider two cases separately.

Case 1. Up to a subsequence, $\sup_{[R_1, R_2]} r u_1'(r) \leq -2.5$ for all n .

Then $r^{2.5} e^{u_1(r)}$ is decreasing on $[R_1, R_2]$, which implies

$$\int_{R_1}^{R_2} r e^{u_1} dr \leq 2R_1^2 e^{u_1(R_1)} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (2.40)$$

Let (x_1, y_1) and (x_2, y_2) be in Step 3 of Lemma 2.5. Clearly $y_1 > 2$. Fix any $\theta \in (0, \frac{y_1-2}{2}) \setminus \{y_2 - 2\}$. Since $u_2'(R_2) = 0$ and

$$u_2'(R_1) = \frac{2a_2}{1+a_1}(\gamma + N_1) + 2N_2 + o(1) = y_1 - 2 + o(1) > \frac{y_1 - 2}{2} \text{ for } n \text{ large,}$$

there exists $t_n \in (R_1, R_2)$ such that $t_n u_2'(t_n) = \theta$. By (2.40) we see that (2.30) holds as $n \rightarrow \infty$, which implies

$$(1+a_2) \left(2 + \lim_{n \rightarrow \infty} t_n u_1'(t_n) \right) + a_1(2+\theta) = (1+a_2)x_1 + a_1 y_1.$$

On the other hand, by (2.38) and (2.39), we can prove via the Pohozaev identity (2.23) that (compare with (2.31))

$$J \left(2 + \lim_{n \rightarrow \infty} t_n u_1'(t_n), 2 + \theta \right) - J(x_1, y_1) = 0.$$

That is, $(2 + \lim_{n \rightarrow \infty} t_n u_1'(t_n), 2 + \theta)$ is also a solution of (2.28), which yields a contradiction with $2 + \theta \notin \{y_1, y_2\}$. So Case 1 is impossible.

Case 2. Up to a subsequence, $\sup_{[R_1, R_2]} r u_1'(r) > -2.5$ for all n .

In this case, since $R_1 u_1'(R_1) < -4$ by (2.22), we can repeat the argument of Step 3 in Lemma 2.5 to obtain the existence of $t_n \in (R_1, R_2)$ such that (2.29) holds. Since u_2 is increasing on $[R_1, R_2]$, so

$$C \leq u_2(t_n) + 2 \ln t_n \leq u_2(R_2) + 2 \ln R_2 \rightarrow -\infty$$

as $n \rightarrow \infty$, also a contradiction. So Case 2 is also impossible. This completes the proof. \square

Lemma 2.6 implies $R_{2,\varepsilon} < R_\varepsilon^*$ for each $\varepsilon \in (0, \varepsilon_3)$. Consider the following scaled functions:

$$\bar{u}_k(r) = \bar{u}_{k,\varepsilon}(r) := u_{k,\varepsilon}(R_{2,\varepsilon} r) + 2 \ln R_{2,\varepsilon} \text{ for } k=1, 2 \text{ and } \varepsilon \in (0, \varepsilon_3), \quad (2.41)$$

where $\frac{R_{1,\varepsilon}}{R_{2,\varepsilon}} \leq r \leq 1$. Then (\bar{u}_1, \bar{u}_2) satisfies

$$\begin{cases} \bar{u}_1'' + \frac{1}{r} \bar{u}_1' = (1+a_1) \left((1+a_1) R_2^{-2} e^{2\bar{u}_1} - e^{\bar{u}_1} - a_1 R_2^{-2} e^{\bar{u}_1 + \bar{u}_2} \right) \\ \quad - a_1 \left((1+a_2) R_2^{-2} e^{2\bar{u}_2} - e^{\bar{u}_2} - a_2 R_2^{-2} e^{\bar{u}_1 + \bar{u}_2} \right), \\ \bar{u}_2'' + \frac{1}{r} \bar{u}_2' = (1+a_2) \left((1+a_2) R_2^{-2} e^{2\bar{u}_2} - e^{\bar{u}_2} - a_2 R_2^{-2} e^{\bar{u}_1 + \bar{u}_2} \right) \\ \quad - a_2 \left((1+a_1) R_2^{-2} e^{2\bar{u}_1} - e^{\bar{u}_1} - a_1 R_2^{-2} e^{\bar{u}_1 + \bar{u}_2} \right). \end{cases} \quad (2.42)$$

Moreover, $|\bar{u}_2(1)| \leq C$ uniformly for ε by Lemma 2.6. Now we claim that

$$\lim_{\varepsilon \rightarrow 0} \bar{u}_1(1) = -\infty. \quad (2.43)$$

In fact, since $\bar{u}_1 \leq \bar{u}_2 \leq \bar{u}_2(1) \leq C$ on $[R_1/R_2, 1]$, it follows from (2.42) that $|(r\bar{u}'_k)'| \leq C$ uniformly for $r \in [R_1/R_2, 1]$ and $k = 1, 2$. Consequently,

$$r\bar{u}'_1(r) \leq \frac{R_1}{R_2}\bar{u}'_1\left(\frac{R_1}{R_2}\right) + Cr = R_1u'_1(R_1) + Cr \leq -4 + Cr,$$

and so $\bar{u}_1(r) \geq -4 \ln r + \bar{u}_1(1) - C$ for any $r \in [R_1/R_2, 1]$. Recalling $\int_{R_1}^{R_2} r e^{u_1} dr \leq C$ uniformly for all $\varepsilon \in (0, \varepsilon_3)$ by Step 2 of Lemma 2.6, we have

$$C \geq \int_{R_1}^{R_2} r e^{u_1} dr = \int_{\frac{R_1}{R_2}}^1 r e^{\bar{u}_1} dr \geq \frac{1}{2} \left(\left(\frac{R_2}{R_1} \right)^2 - 1 \right) e^{\bar{u}_1(1) - C}.$$

This proves (2.43) since $R_2/R_1 \rightarrow \infty$ as $\varepsilon \rightarrow 0$.

Again by $|(r\bar{u}'_k)'| \leq C$ on $[R_1/R_2, 1]$ for $k = 1, 2$, it follows that $\bar{u}_1 \rightarrow -\infty$ uniformly on any compact subset $K \Subset (0, 1]$ as $\varepsilon \rightarrow 0$ and \bar{u}_2 is uniformly bounded in $C_{\text{loc}}((0, 1])$.

Lemma 2.7. *It holds $\lim_{\varepsilon \rightarrow 0} \int_{R_{1,\varepsilon}}^{R_{2,\varepsilon}} r e^{u_{1,\varepsilon}} dr = 0$. Consequently,*

$$\lim_{\varepsilon \rightarrow 0} R_{2,\varepsilon} u'_{1,\varepsilon}(R_{2,\varepsilon}) = -2 \frac{A-B}{A} \gamma + \frac{2B}{A} N_1 + \frac{2a_1}{1+a_2} N_2. \quad (2.44)$$

Proof. Assume by contradiction that there exists a sequence $\varepsilon_n \downarrow 0$ such that

$$\lim_{n \rightarrow \infty} \int_{R_{1,\varepsilon_n}}^{R_{2,\varepsilon_n}} r e^{u_{1,\varepsilon_n}} dr > 0. \quad (2.45)$$

Again we will omit the subscript ε_n for convenience. We consider two cases separately.

Case 1. Up to a subsequence, $\sup_{[R_1, R_2]} r u'_1(r) \leq -2.5$ for all n .

Then (2.40) holds, a contradiction with (2.45). So Case 1 is impossible.

Case 2. Up to a subsequence, $\sup_{[R_1, R_2]} r u'_1(r) > -2.5$ for all n .

In this case, since $R_1 u'_1(R_1) < -4$ by (2.22), we can repeat the argument of Step 3 in Lemma 2.5. In particular, there exist a constant $\theta \in (2.5, 3)$ and a sequence $t_n \in (R_1, R_2)$ such that $t_n u'_1(t_n) = -\theta$, $r u'_1(r) < -\theta$ for $r \in [R_1, t_n]$ and $u_2(t_n) + 2 \ln t_n \geq C$ for n large. Then by the same argument used in (2.40), we have

$$\int_{R_1}^{t_n} r e^{u_1} dr \leq \frac{1}{\theta - 2} R_1^2 e^{u_1(R_1)} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (2.46)$$

On the other hand, since $\bar{u}_2 \leq \bar{u}_2(1) \leq C$ on $[R_1/R_2, 1]$, we have

$$C \leq t_n^2 e^{u_2(t_n)} = \left(\frac{t_n}{R_2}\right)^2 e^{\bar{u}_2(\frac{t_n}{R_2})} \leq C \left(\frac{t_n}{R_2}\right)^2,$$

which implies $t_n/R_2 \geq C > 0$ for all n . Reminding that $\bar{u}_1 \rightarrow -\infty$ uniformly on $[C, 1]$, we conclude that

$$\int_{t_n}^{R_2} r e^{u_1} dr = \int_{\frac{t_n}{R_2}}^1 r e^{\bar{u}_1} dr \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Combining this with (2.46), we obtain a contradiction with (2.45) again.

Therefore, $\lim_{\varepsilon \rightarrow 0} \int_{R_1}^{R_2} r e^{u_1} dr = 0$. Consequently, by the same argument as (2.30), we have

$$(1 + a_2)R_2 u_1'(R_2) + a_1 R_2 u_2'(R_2) = (1 + a_2)R_1 u_1'(R_1) + a_1 R_1 u_2'(R_1) + o(1).$$

Then (2.44) follows directly from Lemma 2.4 and $u_2'(R_2) = 0$. \square

For each fixed $\varepsilon \in (0, \varepsilon_3)$, we define

$$R_3 = R_{3,\varepsilon} := \sup \{r \in [R_{2,\varepsilon}, R_\varepsilon^*) \mid u_1 < u_2 \text{ on } [R_{2,\varepsilon}, r]\}.$$

Then $R_2 < R_3 \leq R^*$. If there exists $t \in (R_2, R_3)$ such that $u_2'(t) = 0$ and $u_2'(r) < 0$ for $r \in (R_2, t)$, then Lemma 2.6 yields $u_1 < u_2 < \ln \delta$ on $[R_2, t]$. Consequently, $F_2 < F_1 < 0$ on $[R_2, t]$ and so

$$0 = t u_2'(t) - R_2 u_2'(R_2) = \int_{R_2}^t r [(1 + a_2)F_2 - a_2 F_1] dr < 0,$$

a contradiction. Therefore,

$$u_2'(r) < 0 \text{ for any } r \in (R_2, R_3). \tag{2.47}$$

Consider the scaled functions \bar{u}_k defined in (2.41) for $r \in \left(\frac{R_1}{R_2}, \frac{R_3}{R_2}\right)$. By (2.47) and the definition of R_3 , we have $\bar{u}_1(r) < \bar{u}_2(r) \leq \bar{u}_2(1) \leq C$ for all $r \in \left(\frac{R_1}{R_2}, \frac{R_3}{R_2}\right)$. This, together with (2.42), gives $|(r\bar{u}_k)'(r)| \leq Cr$ for all $r \in \left(\frac{R_1}{R_2}, \frac{R_3}{R_2}\right)$. Consequently, $\bar{u}_1 \rightarrow -\infty$ uniformly on any compact subset $K \Subset \left(\frac{R_1}{R_2}, \frac{R_3}{R_2}\right)$ as $\varepsilon \rightarrow 0$ and \bar{u}_2 is uniformly bounded in $C_{\text{loc}}\left(\left(\frac{R_1}{R_2}, \frac{R_3}{R_2}\right)\right)$. Since $\bar{u}_2(1) - \bar{u}_1(1) \rightarrow \infty$ as $\varepsilon \rightarrow 0$, we conclude from the definition of R_3 that

$$\lim_{\varepsilon \rightarrow 0} \frac{R_3}{R_2} = \infty. \tag{2.48}$$

Then, for any constant $b > 1$, it holds

$$\lim_{\varepsilon \rightarrow 0} \int_{R_1}^{bR_2} r e^{u_1} dr = \lim_{\varepsilon \rightarrow 0} \int_{R_1}^{R_2} r e^{u_1} dr + \lim_{\varepsilon \rightarrow 0} \int_1^b r e^{\bar{u}_1} dr = 0. \tag{2.49}$$

Lemma 2.8. *It holds $\bar{u}_{2,\varepsilon} \rightarrow \omega_2$ in $C_{\text{loc}}^2((0, \infty))$ as $\varepsilon \rightarrow 0$, where*

$$\omega_2(r) = \ln \frac{2D^2(D^2 - 4)r^{D-2}}{(1 + a_2)(D + 2 + (D - 2)r^D)^2} \text{ for } r \in (0, \infty). \quad (2.50)$$

Here $D := \frac{2a_2}{1+a_1}(\gamma + N_1) + 2N_2 + 2$. Consequently,

$$\int_0^\infty r e^{\omega_2} dr = \frac{2}{1 + a_2} D. \quad (2.51)$$

Proof. Consider (2.38) and Lemma 2.4. By integrating the Pohozaev identity over (R_1, R_2) , we obtain

$$\begin{aligned} & J(R_2 u_1'(R_2) + 2, R_2 u_2'(R_2) + 2) + (A - B) R_2^2 \left[a_2 e^{u_1(R_2)} + a_1 e^{u_2(R_2)} \right. \\ & \left. - \frac{a_2(1 + a_1)}{2} e^{2u_1(R_2)} - \frac{a_1(1 + a_2)}{2} e^{2u_2(R_2)} + a_1 a_2 e^{u_1(R_2) + u_2(R_2)} \right] \\ & = J(R_1 u_1'(R_1) + 2, R_1 u_2'(R_1) + 2) + o(1), \text{ as } \varepsilon \rightarrow 0. \end{aligned}$$

Recall that $R_2^2 e^{u_1(R_2)} = e^{\bar{u}_1(1)} \rightarrow 0$ and $R_2^2 e^{u_i(R_2) + u_j(R_2)} = R_2^{-2} e^{\bar{u}_i(1) + \bar{u}_j(1)} \rightarrow 0$ for $1 \leq i, j \leq 2$ as $\varepsilon \rightarrow 0$. Combining these with Lemma 2.4 and (2.44), we have

$$\begin{aligned} a_1(A - B) R_2^2 e^{u_2(R_2)} & = J(R_1 u_1'(R_1) + 2, R_1 u_2'(R_1) + 2) \\ & \quad - J(R_2 u_1'(R_2) + 2, R_2 u_2'(R_2) + 2) + o(1) \\ & = J\left(2 - 2\gamma, \underbrace{\frac{2a_2}{1 + a_1}(\gamma + N_1) + 2N_2 + 2}_{=: D}\right) \\ & \quad - J\left(\underbrace{2 - 2\frac{A - B}{A}\gamma + \frac{2B}{A}N_1 + \frac{2a_1}{1 + a_2}N_2}_{=: \frac{a_1}{1+a_2}D - (2\gamma - 2) - \frac{2a_1}{1+a_2}}\right) + o(1) \\ & = \frac{a_1(A - B)}{2(1 + a_2)}(D^2 - 4) + o(1), \text{ as } \varepsilon \rightarrow 0. \end{aligned} \quad (2.52)$$

Hence $e^{\bar{u}_2(1)} = R_2^2 e^{u_2(R_2)} \rightarrow \frac{1}{2(1+a_2)}(D^2 - 4)$ as $\varepsilon \rightarrow 0$.

Recalling that \bar{u}_2 is uniformly bounded in $C_{\text{loc}}((\frac{R_1}{R_2}, \frac{R_3}{R_2}))$, up to a subsequence, we may assume that $\bar{u}_2 \rightarrow \omega_2$ in $C_{\text{loc}}^2((0, \infty))$. Clearly, ω_2 satisfies

$$\begin{cases} \omega_2'' + \frac{1}{r}\omega_2' = -(1 + a_2)e^{\omega_2} \\ \omega_2(r) \leq \omega_2(1) = \ln \frac{D^2 - 4}{2(1 + a_2)} \end{cases} \text{ for } r \in (0, \infty). \quad (2.53)$$

Consequently, a simple contrary argument shows $\int_0^\infty r e^{\omega_2} dr < \infty$. Recalling $0 \leq r \bar{u}'_2(r) = R_2 r u'_2(R_2 r) \leq R_1 u'_2(R_1) = D - 2 + o(1)$ for all $r \in [\frac{R_1}{R_2}, 1]$, we obtain (note $\omega'_2(1) = 0$)

$$2\gamma_2 := \lim_{r \rightarrow 0} r \omega'_2(r) \in (0, D - 2].$$

In conclusion, ω_2 is a radial solution of the Liouville equation with singular sources

$$\Delta v + (1 + a_2)e^v = 4\pi \gamma_2 \delta_0 \text{ in } \mathbb{R}^2 \quad \text{and} \quad \int_0^\infty r e^v dr < \infty.$$

By a well-known classification result due to Prajapat and Tarantello [26], it holds

$$\omega_2(r) + \ln(1 + a_2) = \ln \frac{8\lambda(1 + \gamma_2)^2 r^{2\gamma_2}}{(1 + \lambda r^{2\gamma_2+2})^2}$$

for some constant $\lambda > 0$. By $\omega'_2(1) = 0$ and $\omega_2(1) = \ln \frac{D^2-4}{2(1+a_2)}$, a direct computation gives $\gamma_2 = \frac{D-2}{2}$ and $\lambda = \frac{D-2}{D+2}$. Consequently, we see that (2.50)-(2.51) hold. The above argument actually shows that $\bar{u}_2 \rightarrow \omega_2$ in $C_{\text{loc}}^2((0, \infty))$ as $\varepsilon \rightarrow 0$ (i.e., not only along a subsequence). This completes the proof. \square

Lemma 2.9. *There exists a small $\varepsilon_4 \in (0, \varepsilon_3)$ such that for each $\varepsilon \in (0, \varepsilon_4)$, there holds $R_{3,\varepsilon} < R_\varepsilon^*$. Consequently, $u_{1,\varepsilon}(R_{3,\varepsilon}) = u_{2,\varepsilon}(R_{3,\varepsilon})$.*

Proof. Assume by contradiction that there exists a sequence $\varepsilon_n \downarrow 0$ such that $R_{3,\varepsilon_n} = R_{\varepsilon_n}^*$. Since $u_{1,\varepsilon_n} < u_{2,\varepsilon_n} \leq u_{2,\varepsilon_n}(R_{2,\varepsilon_n}) < \ln \delta$ on $(R_{1,\varepsilon_n}, R_{3,\varepsilon_n})$, we see from the definition of $R_{\varepsilon_n}^*$ that $R_{3,\varepsilon_n} = \infty$, namely $(u_{1,\varepsilon_n}, u_{2,\varepsilon_n})$ is an entire solution and $u_{1,\varepsilon_n} < u_{2,\varepsilon_n} < \ln \delta$ on $(R_{1,\varepsilon_n}, \infty)$. By Theorem A, we see that $(u_{1,\varepsilon_n}, u_{2,\varepsilon_n})$ is a non-topological solution and there exist constants $\beta_{k,\varepsilon_n} > 1$ such that $r u'_{k,\varepsilon_n}(r) \rightarrow -2\beta_{k,\varepsilon_n}$ as $r \rightarrow \infty$ for $k = 1, 2$. Clearly, $\beta_{2,\varepsilon_n} \leq \beta_{1,\varepsilon_n}$ for all n . Again we will omit the subscript ε_n for convenience.

By Lemma 2.8, we can fix a large constant $b > 1$ such that $b^2 e^{\omega_2(b)} < \delta$ and $\int_b^\infty r e^{\omega_2} dr < \delta/2$. By the dominated convergence theorem,

$$\int_{R_1}^{bR_2} r e^{u_2} dr = \int_{\frac{R_1}{R_2}}^b r e^{\bar{u}_2} dr \rightarrow \int_0^b r e^{\omega_2} dr \text{ as } n \rightarrow \infty.$$

Recall (2.49), (2.51) and $u_1 < u_2 \leq u_2(R_2) \rightarrow -\infty$ on (R_1, ∞) . Then for n sufficiently large, we have

$$\begin{aligned} |\bar{u}_2(b) - \omega_2(b)| < \delta, \quad \text{and} \quad \left| \int_{R_1}^{bR_2} r e^{u_2} dr - \frac{2}{1+a_2} D \right| < \delta, \\ (A - B)^2 \int_{R_1}^{bR_2} r (e^{u_1} + e^{2u_1} + e^{2u_2} + e^{u_1+u_2}) dr < \delta. \end{aligned}$$

Recalling (2.18) and (2.21), we have

$$\begin{aligned}
bR_2u'_1(bR_2) &\geq R_1u'_1(R_1) + a_1 \int_{R_1}^{bR_2} r e^{u_2} dr \\
&\quad - \int_{R_1}^{bR_2} r \left[(1+a_1)e^{u_1} + a_1(1+a_1)e^{u_1+u_2} + a_1(1+a_2)e^{2u_2} \right] dr \\
&\geq -2\gamma - \delta + a_1 \left(\frac{2}{1+a_2} D - \delta \right) - \delta \\
&= -2 \frac{A-2B}{A} \gamma + \frac{4B}{A} N_1 + \frac{4a_1}{1+a_2} (N_2+1) - (2+a_1)\delta,
\end{aligned}$$

and

$$\begin{aligned}
bR_2u'_2(bR_2) &\leq R_1u'_2(R_1) - (1+a_2) \int_{R_1}^{bR_2} r e^{u_2} dr \\
&\quad + \int_{R_1}^{bR_2} r \left[(1+a_2)^2 e^{2u_2} + a_2 e^{u_1} + a_1 a_2 e^{u_1+u_2} \right] dr \\
&\leq \frac{2a_2}{1+a_1} (\gamma + N_1) + 2N_2 + \delta - (1+a_2) \left(\frac{2}{1+a_2} D - \delta \right) + \delta \quad (2.54) \\
&= -\frac{2a_2}{1+a_1} (\gamma + N_1) - 2N_2 - 4 + (3+a_2)\delta \\
&< -4 - \frac{4a_2}{1+a_1} - 2(2+a_1)\delta,
\end{aligned}$$

where we have used $\frac{2a_2}{1+a_1}(\gamma - 2) > 7(1+a_1+a_2)\delta$ (by (2.19)) to obtain the last inequality. Recalling $u_1 < u_2 \leq u_2(R_2) < \ln \delta$ on $[R_2, \infty)$, we have $F_2 < F_1 < 0$ and so $(ru'_2)'(r) = r[(1+a_2)F_2 - a_2F_1] < 0$ on $[R_2, \infty)$. Consequently, $ru'_2(r) \leq bR_2u'_2(bR_2)$ for any $r \geq bR_2$, which implies

$$-2\beta_2 \leq -4 - \frac{4a_2}{1+a_1} - 2(2+a_1)\delta. \quad (2.55)$$

On the other hand, by $ru'_2(r) \leq bR_2u'_2(bR_2) < -4$ for $r \geq bR_2$, we also have

$$\begin{aligned}
\int_{bR_2}^{\infty} r e^{u_1} dr &\leq \int_{bR_2}^{\infty} r e^{u_2} dr \leq (bR_2)^4 e^{u_2(bR_2)} \int_{bR_2}^{\infty} r^{-3} dr \\
&= \frac{1}{2} b^2 e^{\bar{u}_2(b)} \leq \frac{1}{2} b^2 e^{\omega_2(b)+\delta} \leq \frac{1}{2} \delta e^\delta < \delta.
\end{aligned} \quad (2.56)$$

Consequently,

$$\begin{aligned}
 -2\beta_1 &= bR_2u'_1(bR_2) + \int_{bR_2}^{\infty} (ru'_1(r))' dr \\
 &\geq bR_2u'_1(bR_2) \\
 &\quad - \int_{bR_2}^{\infty} r \left[(1+a_1)e^{u_1} + a_1(1+a_1)e^{u_1+u_2} + a_1(1+a_2)e^{2u_2} \right] dr \\
 &\geq bR_2u'_1(bR_2) - (2+a_1)\delta \\
 &\geq -2\frac{A-2B}{A}\gamma + \frac{4B}{A}N_1 + \frac{4a_1}{1+a_2}(N_2+1) - 2(2+a_1)\delta.
 \end{aligned}$$

This, together with (2.55) and $\beta_2 \leq \beta_1$, gives

$$\frac{A-2B}{A}(\gamma-1) \geq \frac{2B}{A}(N_1+1) + \frac{2a_1}{1+a_2}(N_2+1) + 1 + \frac{2a_2}{1+a_1}.$$

However, $\alpha_1 \geq 1$ and (2.9) give

$$\frac{A-2B}{A}(\gamma-1) \leq \frac{2B}{A}(N_1+1) + \frac{2a_1}{1+a_2}(N_2+1),$$

which yields a contradiction. This completes the proof. \square

Lemma 2.10. *There hold $\lim_{\varepsilon \rightarrow 0} \int_{R_{1,\varepsilon}}^{R_{3,\varepsilon}} r e^{u_{1,\varepsilon}} dr = 0$ and $\lim_{\varepsilon \rightarrow 0} \int_{R_{1,\varepsilon}}^{R_{3,\varepsilon}} r e^{u_{2,\varepsilon}} dr = \frac{2}{1+a_2}D$ and $\lim_{\varepsilon \rightarrow 0} R_{3,\varepsilon}^2 e^{u_{k,\varepsilon}(R_{3,\varepsilon})} = 0$ for $k = 1, 2$. Consequently,*

$$\lim_{\varepsilon \rightarrow 0} R_{3,\varepsilon}u'_{1,\varepsilon}(R_{3,\varepsilon}) = -2\frac{A-2B}{A}\gamma + \frac{4B}{A}N_1 + \frac{4a_1}{1+a_2}(N_2+1) \geq -2, \quad (2.57)$$

$$\lim_{\varepsilon \rightarrow 0} R_{3,\varepsilon}u'_{2,\varepsilon}(R_{3,\varepsilon}) = -\frac{2a_2}{1+a_1}(\gamma+N_1) - 2N_2 - 4 < -4. \quad (2.58)$$

Furthermore, $\lim_{\varepsilon \rightarrow 0} R_{3,\varepsilon}u'_{1,\varepsilon}(R_{3,\varepsilon}) = -2$ if and only if $A-2B > 0$ and $\alpha_1 = 1$.

Proof. Given any $\mu \in (0, \delta)$, there exists a large constant $b_\mu > 1$ such that $b_\mu^2 e^{\omega_2(b_\mu)} < \mu$ and $\int_{b_\mu}^{\infty} r e^{\omega_2} dr < \mu/2$. Then by a similar argument used in Lemma 2.9, we have for $\varepsilon > 0$ sufficiently small that

$$|\bar{u}_2(b_\mu) - \omega_2(b_\mu)| < \mu \quad \text{and} \quad \left| \int_{R_1}^{b_\mu R_2} r e^{u_2} dr - \frac{2}{1+a_2}D \right| < \mu,$$

$$(A-B)^2 \int_{R_1}^{b_\mu R_2} r (e^{u_1} + e^{2u_1} + e^{2u_2} + e^{u_1+u_2}) dr < \mu,$$

$$\left| R_1 u'_2(R_1) - \frac{2a_2}{1+a_1}(\gamma+N_1) - 2N_2 \right| < \mu.$$

Consequently, by repeating the argument of (2.54)-(2.56), we can prove

$$\int_{b_\mu R_2}^{R_3} r e^{u_1} dr \leq \int_{b_\mu R_2}^{R_3} r e^{u_2} dr \leq \frac{1}{2} b_\mu^2 e^{\omega_2(b_\mu)+\mu} < \mu,$$

and

$$R_3^2 e^{u_1(R_3)} = R_3^2 e^{u_2(R_3)} < (b_\mu R_2)^2 e^{u_2(b_\mu R_2)} = b_\mu^2 e^{\bar{u}_2(b_\mu)} \leq b_\mu^2 e^{\omega_2(b_\mu)+\mu} \leq 2\mu,$$

namely $R_3^2 e^{u_k(R_3)} < 2\mu$ and $\int_{R_1}^{R_3} r e^{u_1} dr < 2\mu$ and $|\int_{R_1}^{R_3} r e^{u_2} dr - \frac{2}{1+a_2} D| < 2\mu$ for $\varepsilon > 0$ sufficiently small. This proves

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} R_3^2 e^{u_k(R_3)} &= 0, \\ \lim_{\varepsilon \rightarrow 0} \int_{R_1}^{R_3} r e^{u_1} dr &= 0, \\ \lim_{\varepsilon \rightarrow 0} \int_{R_1}^{R_3} r e^{u_2} dr &= \frac{2}{1+a_2} D. \end{aligned} \tag{2.59}$$

Recalling $u_1 < u_2 \leq u_2(R_2) \rightarrow -\infty$ on (R_1, R_3) as $\varepsilon \rightarrow 0$, we have

$$\lim_{\varepsilon \rightarrow 0} \int_{R_1}^{R_3} r (e^{2u_1} + e^{2u_2} + e^{u_1+u_2}) dr = 0.$$

Consequently, we integrate (2.18) over (R_1, R_3) to derive

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} R_3 u_1'(R_3) &= \lim_{\varepsilon \rightarrow 0} \left[R_1 u_1'(R_1) + a_1 \int_{R_1}^{R_3} r e^{u_2} dr \right] = -2\gamma + \frac{2a_1}{1+a_2} D \\ &= -2 \frac{A-2B}{A} \gamma + \frac{4B}{A} N_1 + \frac{4a_1}{1+a_2} (N_2+1) \geq -2, \quad (\text{by (2.11)}) \end{aligned}$$

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} R_3 u_2'(R_3) &= \lim_{\varepsilon \rightarrow 0} \left[R_1 u_2'(R_1) - (1+a_2) \int_{R_1}^{R_3} r e^{u_2} dr \right] \\ &= \frac{2a_2}{1+a_1} (\gamma + N_1) + 2N_2 - 2D = -\frac{2a_2}{1+a_1} (\gamma + N_1) - 2N_2 - 4. \end{aligned}$$

This completes the proof. \square

Fix any constant $\vartheta \in [0, \delta)$ such that

$$\vartheta = 0 \text{ if } \alpha_1 > 1 \quad \text{and} \quad \vartheta > 0 \text{ if } \alpha_1 = 1. \tag{2.60}$$

Then by Lemma 2.10, there exists $\varepsilon_5 \in (0, \varepsilon_4)$ such that $R_{3,\varepsilon}^2 e^{u_{1,\varepsilon}(R_{3,\varepsilon})} < \delta$ and $R_{3,\varepsilon} u_1'(R_{3,\varepsilon}) > -2 - \vartheta > R_{3,\varepsilon} u_2'(R_{3,\varepsilon}) + 1$ for any $\varepsilon \in (0, \varepsilon_5)$. For each $\varepsilon \in (0, \varepsilon_5)$, we define

$$\begin{aligned} R_4 &= R_{4,\varepsilon} := \sup \{ r \in [R_{3,\varepsilon}, R_\varepsilon^*) \mid r u_{1,\varepsilon}'(r) > -2 - \vartheta \}, \\ R_5 &= R_{5,\varepsilon} := \sup \{ r \in (R_{3,\varepsilon}, R_\varepsilon^*) \mid u_{1,\varepsilon} > u_{2,\varepsilon} \text{ on } (R_{3,\varepsilon}, r) \}. \end{aligned}$$

Clearly, $R_{4,\varepsilon}, R_{5,\varepsilon} > R_{3,\varepsilon}$ for all $\varepsilon \in (0, \varepsilon_5)$.

Lemma 2.11. *There exists a small $\varepsilon_6 \in (0, \varepsilon_5)$ such that for each $\varepsilon \in (0, \varepsilon_6)$, $u_{2,\varepsilon} < u_{1,\varepsilon} < \ln \delta$ on $(R_{3,\varepsilon}, R_{5,\varepsilon})$, $ru'_{1,\varepsilon}(r)$ is strictly decreasing on $(R_{3,\varepsilon}, R_{5,\varepsilon})$ and $R_{4,\varepsilon} \leq R_{5,\varepsilon}$. In particular, if $\alpha_1 > 1$, then $R_{4,\varepsilon} < R_{5,\varepsilon}$, namely $R_{4,\varepsilon}u'_{1,\varepsilon}(R_{4,\varepsilon}) = -2$, $ru'_{1,\varepsilon}(r) > -2$ on $[R_{3,\varepsilon}, R_{4,\varepsilon})$ and $ru'_{1,\varepsilon}(r) < -2$ on $(R_{4,\varepsilon}, R_{5,\varepsilon})$.*

Proof. We divide the proof into four steps.

Step 1. We claim that

$$\sup_{[R_{3,\varepsilon}, R_{5,\varepsilon})} u_{1,\varepsilon} \rightarrow -\infty \text{ as } \varepsilon \rightarrow 0. \quad (2.61)$$

Suppose by contradiction that there exist a sequence $\varepsilon_n \downarrow 0$ and a constant $c_0 < \ln \delta$ such that $\sup_{[R_{3,\varepsilon_n}, R_{5,\varepsilon_n})} u_{1,\varepsilon_n} \geq c_0$ for all n . We will omit the subscript ε_n for convenience. The following proof is similar to Step 1 of Lemma 2.5. For n large, there exist $b_n, d_n \in (R_3, R_5)$ such that $b_n < d_n$ while $u_1(d_n) = c_0 - 1$ and $u_1(b_n) = c_0 - 2$ and $u_1 \leq c_0 - 1$ on $[R_3, d_n]$ and $u_1 \geq c_0 - 2$ on $[b_n, d_n]$. Clearly, $u'_1(d_n) \geq 0$ and $u_2 < u_1 < \ln \delta$ and so $F_1 < F_2 < 0$ on $(R_3, d_n]$, which implies

$$(ru'_1(r))' = r[(1 + a_1)F_1 - a_1F_2] < rF_1 < -\frac{1}{2}re^{u_1(r)} \text{ on } (R_3, d_n]. \quad (2.62)$$

Then $0 \leq ru'_1(r) \leq R_3u'_1(R_3) \leq C$ for any $r \in [R_3, d_n]$, which yields $d_n - b_n \rightarrow \infty$. Consequently,

$$\begin{aligned} 0 &\leq d_nu'_1(d_n) = b_nu'_1(b_n) + \int_{b_n}^{d_n} (ru'_1)' dr \leq R_3u'_1(R_3) - \frac{1}{2} \int_{b_n}^{d_n} re^{u_1} dr \\ &\leq C - \frac{1}{2}e^{c_0-2} \int_{b_n}^{d_n} r dr \rightarrow -\infty \text{ as } n \rightarrow \infty, \end{aligned}$$

a contradiction.

Step 2. By Step 1, for $\varepsilon > 0$ sufficiently small, we have $u_2 < u_1 < \ln \delta$ on (R_3, R_5) , which implies that (2.62) holds on (R_3, R_5) and so $ru'_1(r)$ is strictly decreasing on $[R_3, R_5)$.

Step 3. We prove that $R_{4,\varepsilon} \leq R_{5,\varepsilon}$ for $\varepsilon > 0$ sufficiently small.

Assume by contradiction that there exist a sequence $\varepsilon_n \downarrow 0$ such that $R_{5,\varepsilon_n} < R_{4,\varepsilon_n}$. Again we omit the subscript ε_n for convenience. Consequently, $R_5 < \infty$ and $ru'_1(r) > -2 - \vartheta$ for $r \in [R_3, R_5]$. Since $[a_2ru'_1 + (1 + a_1)ru'_2]' = (A - B)rF_2 < 0$ on $[R_3, R_5]$, we have

$$ru'_2(r) \leq \frac{a_2}{1 + a_1} [R_3u'_1(R_3) + 2 + \vartheta] + R_3u'_2(R_3) =: l_n$$

uniformly for $r \in [R_3, R_5]$. Recalling (2.12) and (2.57)-(2.58), it is easy to see that

$$\begin{aligned} l &:= \lim_{n \rightarrow \infty} l_n \\ &= \frac{4a_2(B - A)}{(1 + a_1)A} \tilde{\gamma} - \frac{2a_2(A - 2B)}{(1 + a_1)A} \tilde{N}_1 - 2 \frac{A - 2B}{A} \tilde{N}_2 - 2 + \frac{a_2\vartheta}{1 + a_1}. \end{aligned}$$

We claim $l < -2 - \delta$. Recall $3A - 4B > 0$ with $\tilde{\gamma} > 1$ and $\vartheta < \delta$ and (2.19). Clearly $l < -2 - \frac{a_2}{1+a_1} \frac{A-B}{A} \tilde{\gamma} < -2 - \delta$ if $A \geq 2B$. Let us consider the remaining case $\frac{4}{3}B < A < 2B$. By $\alpha_2 > 1$ and (2.10), we easily obtain

$$\begin{aligned} \frac{a_2}{1+a_1} \tilde{\gamma} &> \frac{a_2}{1+a_1} \frac{4B-A}{3A-4B} \tilde{N}_1 + \frac{4B-A}{3A-4B} \tilde{N}_2 \\ &> \frac{a_2}{1+a_1} \frac{2B-A}{A-B} \tilde{N}_1 + \frac{2B-A}{A-B} \tilde{N}_2, \end{aligned} \quad (2.63)$$

where we have used $\frac{4B-A}{3A-4B} > \frac{2B-A}{A-B}$. So we also get $l < -2 - \frac{a_2}{1+a_1} \frac{A-B}{A} \tilde{\gamma} < -2 - \delta$.

Hence, for n sufficiently large, $ru'_2(r) \leq l_n < -2 - \vartheta < ru'_1(r)$ for all $r \in [R_3, R_5]$, namely $u_2 - u_1$ is strictly decreasing on $[R_3, R_5]$, which contradicts to $u_2(R_5) - u_1(R_5) = 0$.

Step 4. Let $\alpha_1 > 1$, then $\vartheta = 0$. We prove that $R_4 < R_5$ for $\varepsilon > 0$ sufficiently small.

Assume by contradiction that $R_4 = \infty$ for some $\varepsilon > 0$ sufficiently small. Then $R_5 = \infty$ and so $ru'_1(r) > -2$ for all $r \geq R_3$, which implies $\int_{R_3}^{\infty} re^{u_1} dr = \infty$. On the other hand, Step 2 shows that (2.62) holds on (R_3, ∞) , so

$$\infty = \frac{1}{2} \int_{R_3}^{\infty} re^{u_1} dr \leq \limsup_{r \rightarrow \infty} [-tu'_1(t)] \Big|_{R_3}^r \leq R_3 u'_1(R_3) + 2 \leq C,$$

a contradiction. So $R_4 < \infty$ for $\varepsilon > 0$ sufficiently small. Then by repeating the argument of Step 3, we finally conclude that $R_4 < R_5$ for $\varepsilon > 0$ sufficiently small. \square

Lemma 2.12. For each $\varepsilon \in (0, \varepsilon_6)$ it holds $u_{k,\varepsilon}(r) + 2 \ln r \leq C$ uniformly for $r \in [R_{3,\varepsilon}, R_{5,\varepsilon})$ and $k = 1, 2$. Furthermore, $\lim_{\varepsilon \rightarrow 0} \int_{R_{3,\varepsilon}}^{R_{5,\varepsilon}} re^{u_{2,\varepsilon}} dr = 0$ and

$$\begin{aligned} &\lim_{\varepsilon \rightarrow 0} [a_2 ru'_{1,\varepsilon}(r) + (1+a_1)ru'_{2,\varepsilon}(r)] \\ &= -4a_2 \frac{A-B}{A} \gamma - 2a_2 \frac{A-2B}{A} N_1 - 2 \frac{A-2B}{1+a_2} N_2 - 4 \frac{A-B}{1+a_2} \end{aligned} \quad (2.64)$$

uniformly for all $r \in [R_{3,\varepsilon}, R_{5,\varepsilon})$.

Proof. We separate the proof into three steps.

Step 1. For each $\varepsilon \in (0, \varepsilon_6)$, we claim that

$$u_k(r) + 2 \ln r \leq C \text{ uniformly for } r \in [R_3, R_5]. \quad (2.65)$$

Lemma 2.11 shows that $u_2 < u_1 < \ln \delta$ and $(ru'_1)' \leq -\frac{1}{2}re^{u_1}$ on $[R_3, R_5]$. Hence

$$-2 - \vartheta < ru'_1(r) \leq R_3 u'_2(R_3) - \frac{1}{2} \int_{R_3}^r re^{u_1} dr, \quad \text{for all } r \in (R_3, R_4),$$

which implies

$$\int_{R_3}^{R_4} r (e^{u_1} + e^{u_2}) dr \leq 2 \int_{R_3}^{R_4} r e^{u_1} dr \leq C.$$

This, together with (2.61), gives

$$\lim_{\varepsilon \rightarrow 0} \int_{R_3}^{R_4} r \left[e^{2u_1} + e^{2u_2} + e^{u_1+u_2} \right] dr = 0.$$

Then by repeating the argument of Step 2 in Lemma 2.5, we have $u_k(r) + 2 \ln r \leq C$ for all $r \in [R_3, R_4]$ and $k = 1, 2$. If $R_4 = R_5$, we are done. If $R_4 < R_5$, by $ru'_1(r) < -2 - \vartheta$ for $r \in (R_4, R_5)$, we conclude that

$$u_2(r) + 2 \ln r < u_1(r) + 2 \ln r \leq u_1(R_4) + 2 \ln R_4 \leq C$$

for all $r \in [R_4, R_5]$. This proves (2.65).

Step 2. Recalling (2.10) and $\tilde{\alpha}_2 = \alpha_2 - 1 > 0$, we claim that for $\varepsilon > 0$ sufficiently small,

$$ru'_2(r) \leq -2 - \frac{3}{2}\tilde{\alpha}_2 \text{ uniformly for } r \in [R_3, R_5]. \quad (2.66)$$

By (2.65) we have

$$\lim_{\varepsilon \rightarrow 0} \int_{R_3}^{R_5} r \left[e^{2u_1} + e^{2u_2} + e^{u_1+u_2} \right] dr = 0. \quad (2.67)$$

Consequently, integrating the Pohozaev identity (2.23) over $[R_3, r]$ gives

$$\begin{aligned} J(ru'_1(r) + 2, ru'_2(r) + 2) &\leq J(ru'_1(r) + 2, ru'_2(r) + 2) \\ &\quad + (A - B)r^2 \left[a_2 e^{u_1} + a_1 e^{u_2} - \frac{a_2(1+a_1)}{2} e^{2u_1} \right. \\ &\quad \left. - \frac{a_1(1+a_2)}{2} e^{2u_2} + a_1 a_2 e^{u_1+u_2} \right] \\ &= J(R_3 u'_1(R_3) + 2, R_3 u'_2(R_3) + 2) + o(1) \end{aligned} \quad (2.68)$$

uniformly for all $r \in [R_3, R_5]$ as $\varepsilon \rightarrow 0$. On the other hand, since $[a_2 ru'_1 + (1 + a_1) ru'_2]' = (A - B)r F_2 < 0$ on $[R_3, R_5]$, we have for any $r \in [R_3, R_5]$ that

$$\begin{aligned} a_2(ru'_1(r) + 2) + (1 + a_1)(ru'_2(r) + 2) \\ \leq a_2(R_3 u'_1(R_3) + 2) + (1 + a_1)(R_3 u'_2(R_3) + 2) =: \eta_\varepsilon. \end{aligned} \quad (2.69)$$

Recalling (2.12) and (2.57)-(2.58), we have

$$\eta := \lim_{\varepsilon \rightarrow 0} \eta_\varepsilon = -4a_2 \frac{A - B}{A} \tilde{\gamma} - 2a_2 \frac{A - 2B}{A} \tilde{N}_1 - 2 \frac{A - 2B}{1 + a_2} \tilde{N}_2.$$

Using (2.63) if $A < 2B$, we easily see $\eta < 0$. Hence $\eta_\varepsilon < 0$ for $\varepsilon > 0$ sufficiently small. Note from (1.20) that

$$J(x, y) = \frac{a_2(A - B)}{2(1 + a_1)}x^2 + \frac{a_1}{2(1 + a_1)}[a_2x + (1 + a_1)y]^2.$$

This, together with (2.68)-(2.69) and $\eta_\varepsilon < 0$, easily yields

$$[ru'_1(r) + 2]^2 \leq [R_3u'_1(R_3) + 2]^2 + o(1)$$

and so $ru'_1(r) \geq -R_3u'_1(R_3) - 4 + o(1)$ uniformly for any $r \in [R_3, R_5]$ as $\varepsilon \rightarrow 0$. Substituting this inequality into (2.69) and recalling (2.57)-(2.58), we finally obtain

$$\begin{aligned} ru'_2(r) &\leq \frac{2a_2}{1 + a_1}R_3u'_1(R_3) + R_3u'_2(R_3) + \frac{4a_2}{1 + a_1} + o(1) \\ &= -\frac{2a_2(3A - 4B)}{(1 + a_1)A}\tilde{\gamma} - \frac{2a_2(A - 4B)}{(1 + a_1)A}\tilde{N}_1 - 2\frac{A - 4B}{A}\tilde{N}_2 - 2 + o(1) \\ &= -2\tilde{\alpha}_2 - 2 + o(1) \quad (\text{by (2.10)}) \end{aligned}$$

uniformly for any $r \in [R_3, R_5]$ as $\varepsilon \rightarrow 0$. This proves (2.66).

Step 3. We prove $\lim_{\varepsilon \rightarrow 0} \int_{R_3, \varepsilon}^{R_5, \varepsilon} r e^{u_2, \varepsilon} dr = 0$ and (2.64).

By (2.66), $r^{2+\tilde{\alpha}_2}e^{u_2(r)}$ is strictly decreasing for $r \in [R_3, R_5]$, so Lemma 2.10 gives

$$\int_{R_3}^{R_5} r e^{u_2} dr \leq \frac{1}{\tilde{\alpha}_2} R_3^2 e^{u_2(R_3)} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

Then by integrating $[a_2ru'_1 + (1 + a_1)ru'_2]' = (A - B)rF_2$ over $[R_3, r]$ for any $r \in [R_3, R_5]$ and recalling $|F_2| \leq (1 + a_2)e^{u_2}$, we easily obtain (2.64). This completes the proof. \square

Now we consider the cases $\alpha_1 = 1$ and $\alpha_1 > 1$ separately.

2.1. The critical case $\alpha_1 = 1$

In this subsection, we consider the critical case $\alpha_1 = 1$. Consequently, we see from (2.8), (2.9) and (2.12) that

$$A - 2B > 0 \quad \text{and} \quad \tilde{\gamma} = \frac{2B}{A - 2B}\tilde{N}_1 + \frac{2a_1(1 + a_1)}{A - 2B}\tilde{N}_2. \quad (2.70)$$

The following lemma provides an evidence that this critical case is different from the generic case $\alpha_1 > 1$.

Lemma 2.13. *There exists a small $\varepsilon_7 \in (0, \varepsilon_6)$ such that for each $\varepsilon \in (0, \varepsilon_7)$, $R_{4, \varepsilon} = R_{5, \varepsilon} = R_\varepsilon^* = +\infty$.*

Proof. Assume by contradiction that $R_{4,\varepsilon_n} < +\infty$ for a sequence $\varepsilon_n \downarrow 0$. We will omit the subscript ε_n for convenience. Then $R_4 u'_1(R_4) = -2 - \vartheta$. This, together with (2.64) and (2.70), gives

$$\begin{aligned} & \lim_{n \rightarrow \infty} R_4 u'_2(R_4) + 2 \\ &= \frac{4a_2(B-A)}{(1+a_1)A} \tilde{\gamma} - \frac{2a_2(A-2B)}{(1+a_1)A} \tilde{N}_1 - 2 \frac{A-2B}{A} \tilde{N}_2 + \frac{a_2 \vartheta}{1+a_1} \\ &= -\frac{1+a_2}{a_1} \tilde{\gamma} + \frac{a_2}{1+a_1} \vartheta. \end{aligned} \quad (2.71)$$

Similarly, by Lemma 2.10 we have $\lim_{n \rightarrow \infty} R_3 u'_1(R_3) + 2 = 0$ and

$$\lim_{n \rightarrow \infty} R_3 u'_2(R_3) + 2 = -\frac{2a_2}{1+a_1} (\tilde{\gamma} + \tilde{N}_1) - 2\tilde{N}_2 = -\frac{1+a_2}{a_1} \tilde{\gamma}.$$

Since we have assumed $R_4 < \infty$, it follows from (2.65) and (2.66) that $R_4^2 e^{u_i(R_4)+u_j(R_4)} \leq C R_4^{-2} \rightarrow 0$ for $1 \leq i, j \leq 2$ and $R_4^2 e^{u_2(R_4)} < R_3^2 e^{u_2(R_3)} \rightarrow 0$ as $n \rightarrow \infty$. Combining these with (2.67), we can repeat the proof of Lemma 2.8 to obtain (similar to (2.52))

$$\begin{aligned} & a_2(A-B) R_4^2 e^{u_1(R_4)} \\ &= J(R_3 u'_1(R_3) + 2, R_3 u'_2(R_3) + 2) - J(R_4 u'_1(R_4) + 2, R_4 u'_2(R_4) + 2) + o(1) \\ &= J\left(0, -\frac{1+a_2}{a_1} \tilde{\gamma}\right) - J\left(-\vartheta, -\frac{1+a_2}{a_1} \tilde{\gamma} + \frac{a_2}{1+a_1} \vartheta\right) + o(1) \\ &= -\frac{a_2(A-B)}{2(1+a_1)} \vartheta^2 + o(1) \text{ as } n \rightarrow \infty, \end{aligned}$$

which yields a contradiction with $\vartheta > 0$. \square

Now we can finish the proof of Theorem 2.3 for $\alpha_1 = 1$.

Completion of the proof of Theorem 2.3 for $\alpha_1 = 1$. Let $\varepsilon \in (0, \varepsilon_7)$, then $R_{4,\varepsilon} = R_{5,\varepsilon} = R_\varepsilon^* = +\infty$. Since Lemma 2.11 shows that $u_{2,\varepsilon}(r) < u_{1,\varepsilon}(r) < \ln \delta$ for any $r \in (R_{3,\varepsilon}, +\infty)$, we conclude that $(u_{1,\varepsilon}, u_{2,\varepsilon})$ is an entire solution. By Theorem A, there exists $(\alpha_{1,\varepsilon}, \alpha_{2,\varepsilon}) \in \Omega$ such that

$$u_{k,\varepsilon}(r) = -2\alpha_{k,\varepsilon} \ln r + O(1) \text{ as } r \rightarrow \infty \text{ for } k = 1, 2.$$

Consequently, $r u'_{k,\varepsilon}(r) \rightarrow -2\alpha_{k,\varepsilon}$ as $r \rightarrow \infty$. Then Lemma 2.11 and the definition of $R_{4,\varepsilon}$ yield $-2 - \vartheta \leq -2\alpha_{1,\varepsilon} < R_{3,\varepsilon} u'_{1,\varepsilon}(R_{3,\varepsilon})$, namely

$$2 + \vartheta \geq \limsup_{\varepsilon \rightarrow 0} 2\alpha_{1,\varepsilon} \geq \liminf_{\varepsilon \rightarrow 0} 2\alpha_{1,\varepsilon} \geq -\lim_{\varepsilon \rightarrow 0} R_{3,\varepsilon} u'_{1,\varepsilon}(R_{3,\varepsilon}) = 2.$$

Since $\vartheta \in (0, \delta)$ can be taken *a priori* arbitrary small, we conclude that

$$\lim_{\varepsilon \rightarrow 0} \alpha_{1,\varepsilon} = \alpha_1 = 1.$$

This, together with (2.64), easily implies

$$\begin{aligned} & a_2 + (1 + a_1) \lim_{\varepsilon \rightarrow 0} \alpha_{2,\varepsilon} \\ &= 2a_2 \frac{A - B}{A} \gamma + a_2 \frac{A - 2B}{A} N_1 + \frac{A - 2B}{1 + a_2} N_2 + 2 \frac{A - B}{1 + a_2}, \end{aligned}$$

so

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \alpha_{2,\varepsilon} &= \frac{2a_2}{1 + a_1} \frac{A - B}{A} \tilde{\gamma} + \frac{a_2}{1 + a_1} \frac{A - 2B}{A} \tilde{N}_1 + \frac{A - 2B}{A} \tilde{N}_2 + 1 \\ &= \frac{1 + a_2}{2a_1} \tilde{\gamma} + 1 \quad (\text{by (2.71)}) \\ &= \alpha_2. \quad (\text{by (2.7)}) \end{aligned}$$

By Step 3 in the proof of Lemma 2.12, we have $\int_{R_{3,\varepsilon}}^{\infty} r F_2 dr \rightarrow 0$ as $\varepsilon \rightarrow 0$. Consequently, by integrating $(ru'_{1,\varepsilon})' = (1 + a_1)rF_1 - a_1rF_2$ over $(R_{3,\varepsilon}, +\infty)$, we deduce from $-2\alpha_{1,\varepsilon} - R_{3,\varepsilon}u'_{1,\varepsilon}(R_{3,\varepsilon}) \rightarrow 0$ that $\int_{R_{3,\varepsilon}}^{\infty} r F_1 dr \rightarrow 0$ as $\varepsilon \rightarrow 0$. Then by (2.17) we conclude that

$$\lim_{\varepsilon \rightarrow 0} \int_{R_{3,\varepsilon}}^{\infty} r e^{u_{1,\varepsilon}} dr = 0. \quad (2.72)$$

Observe from (2.47) and (2.66) that $u'_{2,\varepsilon}(r) < 0$ for all $r > R_{2,\varepsilon}$. Besides, Lemma 2.4 shows that $\sup_{[0, R_{1,\varepsilon}]} u_{2,\varepsilon} \rightarrow -\infty$. Combining these with (2.36), we conclude $\sup_{\mathbb{R}^2} u_{2,\varepsilon} \rightarrow -\infty$ as $\varepsilon \rightarrow 0$. This completes the proof. \square

Remark 2.14. In Theorem C where bubbling solutions of type I are constructed, we assumed $\alpha_1 > 1$, which plays an essential role in the proof of Theorem C (see [5]). In particular, the conclusion $(\alpha_{1,\varepsilon}, \alpha_{2,\varepsilon}) \rightarrow (\alpha_1, \alpha_2)$ is a corollary of (2.72) in the proof of Theorem C. However, for the critical case $\alpha_1 = 1$ studied here, the idea used in Theorem C can not be applied and we have to argue the other way around: the conclusion (2.72) is a consequence of $(\alpha_{1,\varepsilon}, \alpha_{2,\varepsilon}) \rightarrow (\alpha_1, \alpha_2)$.

2.2. The generic case $\alpha_1 > 1$

In this subsection, we consider the generic case $\alpha_1 > 1$. Then $\vartheta = 0$ and $R_{4,\varepsilon} < R_{5,\varepsilon}$. The following lemma also provides an evidence that this case is different from the critical case $\alpha_1 = 1$.

Lemma 2.15. *Recalling $\alpha_1 > 1$ in (2.9), it holds (compare to (2.72))*

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{R_{3,\varepsilon}}^{R_{5,\varepsilon}} r e^{u_1} dr &= \frac{4}{1+a_1} \left(\frac{2B-A}{A} \tilde{\gamma} + \frac{2B}{A} \tilde{N}_1 + \frac{2a_1}{1+a_2} \tilde{N}_2 \right) \\ &= \frac{4(\alpha_1 - 1)}{1+a_1}. \end{aligned} \quad (2.73)$$

Proof. We separate the proof into five steps.

Step 1. We claim that

$$\lim_{\varepsilon \rightarrow 0} R_{4,\varepsilon}^2 e^{u_{1,\varepsilon}(R_{4,\varepsilon})} = \frac{E^2}{2(1+a_1)}, \quad (2.74)$$

where $E > 0$ is defined by

$$\begin{aligned} E &:= \lim_{\varepsilon \rightarrow 0} R_{3,\varepsilon} u'_{1,\varepsilon}(R_{3,\varepsilon}) + 2 \\ &= 2 \frac{2B-A}{A} \tilde{\gamma} + \frac{4B}{A} \tilde{N}_1 + \frac{4a_1}{1+a_2} \tilde{N}_2 = 2(\alpha_1 - 1). \end{aligned} \quad (2.75)$$

Recalling (2.64) and $R_4 u'_1(R_4) = -2$, it holds

$$F := \lim_{\varepsilon \rightarrow 0} R_4 u'_2(R_4) + 2 = -\frac{4a_2(A-B)}{(1+a_1)A} \tilde{\gamma} - \frac{2a_2(A-2B)}{(1+a_1)A} \tilde{N}_1 - 2 \frac{A-2B}{A} \tilde{N}_2.$$

Again, (2.65) and (2.66) imply $R_4^2 e^{u_i(R_4)+u_j(R_4)} \leq C R_4^{-2} \rightarrow 0$ for $1 \leq i, j \leq 2$ and

$$R_4^2 e^{u_2(R_4)} < R_3^2 e^{u_2(R_3)} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \quad (2.76)$$

Combining these with (2.67), we can repeat the proof of Lemma 2.8 to obtain (similarly to (2.52))

$$\begin{aligned} &a_2(A-B) R_4^2 e^{u_1(R_4)} \\ &= J(R_3 u'_1(R_3) + 2, R_3 u'_2(R_3) + 2) - J(R_4 u'_1(R_4) + 2, R_4 u'_2(R_4) + 2) + o(1) \\ &= J\left(E, \underbrace{-\frac{2a_2}{1+a_1}(\tilde{\gamma} + \tilde{N}_1) - 2\tilde{N}_2}_{=:G}\right) - J(0, F) + o(1) \\ &= J\left(-2\tilde{\gamma} - \frac{2a_1}{1+a_2}G, G\right) - J\left(0, \frac{A-2B}{A}G - \frac{2a_2}{1+a_1}\tilde{\gamma}\right) + o(1) \\ &= J(2\tilde{\gamma}, G) - J\left(0, \frac{A-2B}{A}G - \frac{2a_2}{1+a_1}\tilde{\gamma}\right) + o(1) \text{ (by (2.13))} \\ &= \frac{a_2(A-B)}{2(1+a_1)} \left(\frac{2a_1}{1+a_2}G + 2\tilde{\gamma}\right)^2 + o(1) \\ &= \frac{a_2(A-B)}{2(1+a_1)} E^2 + o(1) \text{ as } \varepsilon \rightarrow 0. \end{aligned}$$

Hence (2.74) holds.

Step 2. For $\varepsilon > 0$ sufficiently small, we consider the scaled functions

$$\tilde{u}_k(r) = \tilde{u}_{k,\varepsilon}(r) := u_{k,\varepsilon}(R_{4,\varepsilon}r) + 2 \ln R_{4,\varepsilon} \text{ for } k = 1, 2,$$

where $\frac{R_{3,\varepsilon}}{R_{4,\varepsilon}} \leq r < \frac{R_{5,\varepsilon}}{R_{4,\varepsilon}}$. Then $(\tilde{u}_1, \tilde{u}_2)$ satisfies

$$\begin{cases} \tilde{u}_1'' + \frac{1}{r}\tilde{u}_1' = (1+a_1) \left((1+a_1)R_4^{-2}e^{2\tilde{u}_1} - e^{\tilde{u}_1} - a_1R_4^{-2}e^{\tilde{u}_1+\tilde{u}_2} \right) \\ \quad - a_1 \left((1+a_2)R_4^{-2}e^{2\tilde{u}_2} - e^{\tilde{u}_2} - a_2R_4^{-2}e^{\tilde{u}_1+\tilde{u}_2} \right), \\ \tilde{u}_2'' + \frac{1}{r}\tilde{u}_2' = (1+a_2) \left((1+a_2)R_4^{-2}e^{2\tilde{u}_2} - e^{\tilde{u}_2} - a_2R_4^{-2}e^{\tilde{u}_1+\tilde{u}_2} \right) \\ \quad - a_2 \left((1+a_1)R_4^{-2}e^{2\tilde{u}_1} - e^{\tilde{u}_1} - a_1R_4^{-2}e^{\tilde{u}_1+\tilde{u}_2} \right). \end{cases} \quad (2.77)$$

By Lemma 2.11, it is easy to see that $u_1(r) + 2 \ln r \leq u_1(R_4) + 2 \ln R_4$ for any $r \in [R_3, R_5]$. So for any $\frac{R_3}{R_4} < r < \frac{R_5}{R_4}$ we have

$$\tilde{u}_2(r) + 2 \ln r < \tilde{u}_1(r) + 2 \ln r \leq \tilde{u}_1(1) = \ln \frac{E^2}{2(1+a_1)} + o(1). \quad (2.78)$$

Moreover, (2.76) gives $\tilde{u}_2(1) \rightarrow -\infty$ as $\varepsilon \rightarrow 0$.

Step 3. We claim that

$$\frac{R_3}{R_4} \rightarrow 0 \text{ and } \frac{R_5}{R_4} \rightarrow \infty \text{ as } \varepsilon \rightarrow 0.$$

Recall from Lemma 2.11 that $-2 \leq r\tilde{u}_1'(r) = R_4r u_1'(R_4r) \leq R_3u_1'(R_3) \leq C$ for $r \in [\frac{R_3}{R_4}, 1]$. By the mean value theorem, we have

$$\tilde{u}_1(1) - \ln \left(R_3^2 e^{u_1(R_3)} \right) - 2 \ln \frac{R_4}{R_3} = \tilde{u}_1(1) - \tilde{u}_1 \left(\frac{R_3}{R_4} \right) \leq C \left(\frac{R_4}{R_3} - 1 \right).$$

Recalling $R_3^2 e^{u_1(R_3)} \rightarrow 0$ and (2.78), we conclude that $\frac{R_3}{R_4} \rightarrow 0$ as $\varepsilon \rightarrow 0$.

By (2.78) and $R_4^{-2}e^{\tilde{u}_k(r)} = e^{u_k(R_4r)} < 1$ for any $r \in [\frac{R_3}{R_4}, \frac{R_5}{R_4}]$, it is easy to deduce from (2.77) that $|r\tilde{u}_k'(r)| \leq C/r$ for all $r \in [\frac{R_3}{R_4}, \frac{R_5}{R_4}]$. Consequently, \tilde{u}_1 is uniformly bounded in $C_{\text{loc}}((\frac{R_3}{R_4}, \frac{R_5}{R_4}))$ and $\tilde{u}_2 \rightarrow -\infty$ uniformly on any compact subset $K \Subset (\frac{R_3}{R_4}, \frac{R_5}{R_4})$. This, together with the definition of R_5 , yields $\frac{R_5}{R_4} \rightarrow \infty$ as $\varepsilon \rightarrow 0$.

Step 4. We claim that $\tilde{u}_1 \rightarrow \omega_1$ in $C_{\text{loc}}^2((0, \infty))$ as $\varepsilon \rightarrow 0$, where

$$\omega_1(r) = \ln \frac{2E^2r^{E-2}}{(1+a_1)(1+r^E)^2} \text{ for } r \in (0, \infty), \quad (2.79)$$

and E is seen in (2.75). Consequently,

$$\int_0^\infty r e^{\omega_1} dr = \frac{2}{1+a_1} E. \tag{2.80}$$

By Step 3, up to a subsequence, we may assume that $\tilde{u}_1 \rightarrow \omega_1$ in $C_{loc}^2((0, \infty))$, where ω_1 satisfies

$$\begin{cases} \omega_1'' + \frac{1}{r}\omega_1' = -(1+a_1)e^{\omega_1}, \\ \omega_1(r) + 2 \ln r \leq \omega_1(1) = \ln \frac{E^2}{2(1+a_1)}, \end{cases} \quad \text{for } r \in (0, \infty). \tag{2.81}$$

Since $\omega_1'(1) = -2$ and $r\omega_1'(r)$ is strictly decreasing on $(0, \infty)$, it is easy to prove that $\int_0^\infty r e^{\omega_1} dr < \infty$. Recalling $-2 \leq r\tilde{u}_1'(r) = R_4 r u_1'(R_4 r) \leq R_3 u_1'(R_3) = E - 2 + o(1)$ for all $r \in [R_3/R_4, 1]$, we easily obtain

$$2\gamma_1 := \lim_{r \rightarrow 0} r\omega_1'(r) \in (-2, E - 2].$$

In conclusion, ω_1 is a radial solution of the Liouville equation with singular sources

$$\Delta v + (1+a_1)e^v = 4\pi\gamma_1\delta_0 \text{ in } \mathbb{R}^2 \quad \text{and} \quad \int_0^\infty r e^v dr < \infty.$$

Since $\gamma_1 > -1$, again by the classification result due to Prajapat and Tarantello [26], there holds

$$\omega_1(r) + \ln(1+a_1) = \ln \frac{8\lambda(1+\gamma_1)^2 r^{2\gamma_1}}{(1+\lambda r^{2\gamma_1+2})^2}$$

for some constant $\lambda > 0$. By $\omega_1'(1) = -2$ and $\omega_1(1) = \ln \frac{E^2}{2(1+a_1)}$, a direct computation gives $\gamma_1 = \frac{E-2}{2}$ and $\lambda = 1$. This proves (2.79) and (2.80). Clearly, the above argument also shows that $\tilde{u}_1 \rightarrow \omega_1$ in $C_{loc}^2((0, \infty))$ as $\varepsilon \rightarrow 0$ (i.e., not only along a subsequence).

Step 5. We prove (2.73).

Given any $\mu \in (0, \delta)$. By (2.79)-(2.80), there exist small constant $b_\mu \in (0, 1)$ and large constant $d_\mu > 1$ such that

$$\begin{aligned} b_\mu^2 e^{\omega_1(b_\mu)} + d_\mu^2 e^{\omega_1(d_\mu)} + \left| \int_{b_\mu}^{d_\mu} r e^{\omega_1} dr - \frac{2E}{1+a_1} \right| &< \frac{\mu}{2}, \\ b_\mu \omega_1'(b_\mu) + 2 \geq \frac{2}{3} E \quad \text{and} \quad d_\mu \omega_1'(d_\mu) + 2 &\leq -\frac{2}{3} E. \end{aligned}$$

Consequently, since $\tilde{u}_1 \rightarrow \omega_1$ in $C^2([b_\mu, d_\mu])$, there exists sufficiently small $\varepsilon_\mu > 0$ such that for each $\varepsilon \in (0, \varepsilon_\mu)$, we have

$$b_\mu^2 e^{\tilde{u}_1(b_\mu)} + d_\mu^2 e^{\tilde{u}_1(d_\mu)} + \left| \int_{b_\mu}^{d_\mu} r e^{\tilde{u}_1} dr - \frac{2E}{1+a_1} \right| < \mu,$$

$$b_\mu \tilde{u}'_1(b_\mu) + 2 \geq \frac{1}{2}E \quad \text{and} \quad d_\mu \tilde{u}'_1(d_\mu) + 2 \leq -\frac{1}{2}E.$$

Recalling that $r\tilde{u}'_1(r) = R_4 r u'_1(R_4 r)$ is strictly decreasing on $(\frac{R_3}{R_4}, \frac{R_5}{R_4})$, we obtain $r\tilde{u}'(r) + 2 \leq -\frac{1}{2}E$ for all $r \in [d_\mu, \frac{R_5}{R_4})$, which implies that $r^{2+\frac{1}{2}E} e^{\tilde{u}_1(r)}$ decreases on $[d_\mu, \frac{R_5}{R_4})$ and so

$$(R_4 r)^2 e^{u_1(R_4 r)} = r^2 e^{\tilde{u}_1(r)} \leq d_\mu^2 e^{\tilde{u}_1(d_\mu)} < \mu \quad \text{for any } r \in \left[d_\mu, \frac{R_5}{R_4} \right).$$

Furthermore,

$$\int_{d_\mu R_4}^{R_5} r e^{u_1} dr = \int_{d_\mu}^{\frac{R_5}{R_4}} r e^{\tilde{u}_1} dr \leq \frac{2}{E} d_\mu^2 e^{\tilde{u}_1(d_\mu)} < \frac{2}{E} \mu.$$

Similarly, by $r\tilde{u}'(r) + 2 \geq \frac{1}{2}E$ for all $r \in [\frac{R_3}{R_4}, b_\mu]$, we can prove

$$\int_{R_3}^{b_\mu R_4} r e^{u_1} dr = \int_{\frac{R_3}{R_4}}^{b_\mu} r e^{\tilde{u}_1} dr \leq \frac{2}{E} b_\mu^2 e^{\tilde{u}_1(b_\mu)} < \frac{2}{E} \mu.$$

Since $\int_{b_\mu R_4}^{d_\mu R_4} r e^{u_1} dr = \int_{b_\mu}^{d_\mu} r e^{\tilde{u}_1} dr$, (2.73) follows immediately. This completes the proof. \square

Lemma 2.16. *Let $t_\varepsilon \in (R_{4,\varepsilon}, R_{5,\varepsilon}]$ be such that $t_\varepsilon < \infty$ and $\frac{t_\varepsilon}{R_{4,\varepsilon}} \rightarrow \infty$ as $\varepsilon \rightarrow 0$. Then $\lim_{\varepsilon \rightarrow 0} t_\varepsilon^2 e^{u_{k,\varepsilon}(t_\varepsilon)} = 0$ for $k = 1, 2$ and*

$$\lim_{\varepsilon \rightarrow 0} t_\varepsilon u'_{1,\varepsilon}(t_\varepsilon) = -2\alpha_1 \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} t_\varepsilon u'_{2,\varepsilon}(t_\varepsilon) = -2\alpha_2,$$

where α_1, α_2 are seen in (2.9)-(2.10).

Proof. Since $\frac{t}{R_4} \rightarrow \infty$, by repeating the argument of Step 5 in Lemma 2.15, it is easy to prove that $t^2 e^{u_2(t)} \leq t^2 e^{u_1(t)} \rightarrow 0$ and

$$\int_{R_3}^t r e^{u_1} dr \rightarrow \frac{2E}{1+a_1} \quad \text{as } \varepsilon \rightarrow 0.$$

Recalling $\int_{R_3}^{R_5} r e^{u_2} dr \rightarrow 0$ (see Lemma 2.12), (2.67), (2.57)-(2.58) and (2.75), we conclude

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} t u'_1(t) &= \lim_{\varepsilon \rightarrow 0} \left[R_3 u'_1(R_3) - (1 + a_1) \int_{R_3}^t r e^{u_1} dr \right] = E - 2 - 2E \\ &= -2\alpha_1, \\ \lim_{\varepsilon \rightarrow 0} t u'_2(t) &= \lim_{\varepsilon \rightarrow 0} \left[R_3 u'_2(R_3) + a_2 \int_{R_3}^t r e^{u_1} dr \right] \\ &= -\frac{2a_2}{1 + a_1} (\tilde{\gamma} + \tilde{N}_1) - 2\tilde{N}_2 - 2 + \frac{2a_2}{1 + a_1} E \\ &= -\frac{2a_2}{1 + a_1} \frac{3A - 4B}{A} \tilde{\gamma} - \frac{2a_2}{1 + a_1} \frac{A - 4B}{A} \tilde{N}_1 - 2 \frac{A - 4B}{A} \tilde{N}_2 - 2 \\ &= -2\alpha_2. \end{aligned}$$

This completes the proof. □

Lemma 2.17. *There exists a small $\varepsilon_7 \in (0, \varepsilon_6)$ such that for each $\varepsilon \in (0, \varepsilon_7)$, there holds $ru'_{2,\varepsilon}(r) < -2 - \tilde{\alpha}_2$ for all $r \in [R_{3,\varepsilon}, R_\varepsilon^*]$. Consequently,*

$$\lim_{\varepsilon \rightarrow 0} \int_{R_{3,\varepsilon}}^{R_\varepsilon^*} r e^{u_{2,\varepsilon}} dr \leq \frac{1}{\tilde{\alpha}_2} \lim_{\varepsilon \rightarrow 0} R_{3,\varepsilon}^2 e^{u_{2,\varepsilon}(R_{3,\varepsilon})} = 0. \tag{2.82}$$

Proof. Recall that $\tilde{\alpha}_k = \alpha_k - 1 > 0$ for $k = 1, 2$ and (2.66) gives that $ru'_{2,\varepsilon}(r) \leq -2 - \frac{3}{2}\tilde{\alpha}_2$ for $r \in [R_{3,\varepsilon}, R_{5,\varepsilon})$. Assume by contradiction that there exist a sequence $\varepsilon_n \downarrow 0$ and $t_n \in [R_{5,\varepsilon_n}, R_{\varepsilon_n}^*)$ such that $t_n u'_{2,\varepsilon_n}(t_n) = -2 - \tilde{\alpha}_2$ and $ru'_{2,\varepsilon}(r) < -2 - \tilde{\alpha}_2$ for $r \in [R_{3,\varepsilon_n}, t_n)$. We will omit the subscript ε_n for convenience. Clearly $R_5 < t_n < R^*$. Since $\frac{R_5}{R_4} \rightarrow \infty$, Lemma 2.16 yields $\lim_{n \rightarrow \infty} R_5^2 e^{u_k(R_5)} = 0$ for $k = 1, 2$ and

$$\lim_{n \rightarrow \infty} R_5 u'_1(R_5) = -2 - 2\tilde{\alpha}_1 \quad \text{and} \quad \lim_{n \rightarrow \infty} R_5 u'_2(R_5) = -2 - 2\tilde{\alpha}_2. \tag{2.83}$$

Since $ru'_2(r) < -2 - \tilde{\alpha}_2$ for $r \in [R_3, t_n)$, we have

$$\int_{R_3}^{t_n} r e^{u_2} dr \leq \frac{1}{\tilde{\alpha}_2} R_3^2 e^{u_2(R_3)} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{2.84}$$

Recall from (2.61) that $\sup_{[R_3, R_5]} u_1 \rightarrow -\infty$ as $n \rightarrow \infty$. Since $u_2 \leq u_1$ on $[R_3, R_5]$, we have $\sup_{[R_3, t_n]} u_2 \rightarrow -\infty$ as $n \rightarrow \infty$.

Step 1. We claim that $u'_1 < 0$ on $[R_5, t_n]$ for n sufficiently large. Consequently, $\sup_{[R_3, t_n]} u_1 \rightarrow -\infty$ as $n \rightarrow \infty$.

Suppose that, up to a subsequence, there exists $r_n \in [R_5, t_n]$ such that $u'_1(r_n) = 0$ and $u'_1(r) < 0$ for $r \in [R_5, r_n)$. Then for n sufficiently large, $u_1 < \ln \delta$ and so

$F_1 < 0, -F_2 < 2e^{u_2}$ on $[R_5, r_n]$, which imply that $(ru'_1)' = r[(1+a_1)F_1 - a_1F_2] \leq 2a_1re^{u_2}$ on $[R_5, r_n]$. Consequently,

$$0 - R_5u'_1(R_5) = \int_{R_5}^{r_n} (ru'_1)' dr \leq 2a_1 \int_{R_5}^{r_n} re^{u_2} dr \leq 2a_1 \int_{R_3}^{t_n} re^{u_2} dr \rightarrow 0$$

as $n \rightarrow \infty$, which yields a contradiction with (2.83).

Step 2. We claim that $ru'_1(r) < -2 - \tilde{\alpha}_1$ on $[R_5, t_n]$ for n large enough. Consequently,

$$\int_{R_5}^{t_n} re^{u_1} dr \leq \frac{1}{\tilde{\alpha}_1} R_5^2 e^{u_1(R_5)} \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{2.85}$$

By Step 1 and (2.83)-(2.84), we may take n large enough such that $(ru'_1)' \leq 2a_1re^{u_2}$ on $[R_5, t_n]$, $R_5u'_1(R_5) < -2 - \frac{3}{2}\tilde{\alpha}_1$ and $2a_1 \int_{R_5}^{t_n} re^{u_2} dr < \frac{1}{2}\tilde{\alpha}_1$. Then for any $r \in [R_5, t_n]$, we have

$$ru'_1(r) \leq R_5u'_1(R_5) + 2a_1 \int_{R_5}^r te^{u_2} dt < -2 - \tilde{\alpha}_1.$$

Step 3. We complete the proof.

Similarly as Steps 1-2, we may take n large enough such that $(ru'_2)' = r[(1+a_2)F_2 - a_2F_1] \leq 2a_2re^{u_1}$ on $[R_5, t_n]$, so we conclude from (2.83) and (2.85) that

$$-2 - \tilde{\alpha}_2 = t_nu'_2(t_n) \leq R_5u'_2(R_5) + 2a_1 \int_{R_5}^{t_n} re^{u_1} dr \rightarrow -2 - 2\tilde{\alpha}_2$$

as $n \rightarrow \infty$, a contradiction with $\tilde{\alpha}_2 > 0$. This completes the proof. □

We are now in a position to complete the proof of Theorem 2.3 for $\alpha_1 > 1$.

Completion of the proof of Theorem 2.3 for $\alpha_1 > 1$. For each $\varepsilon \in (0, \varepsilon_7)$, we take a number $t_\varepsilon \in (R_{4,\varepsilon}, R_{5,\varepsilon})$ such that $\frac{t_\varepsilon}{R_{4,\varepsilon}} \rightarrow \infty$ as $\varepsilon \rightarrow 0$. Then Lemma 2.11 gives $u_{2,\varepsilon}(t_\varepsilon) < u_{1,\varepsilon}(t_\varepsilon) < \ln \delta$. Moreover, by Lemma 2.16 we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} t_\varepsilon^2 e^{u_{k,\varepsilon}(t_\varepsilon)} &= 0, \\ \lim_{\varepsilon \rightarrow 0} t_\varepsilon u'_{1,\varepsilon}(t_\varepsilon) &= -2\alpha_1, \\ \lim_{\varepsilon \rightarrow 0} t_\varepsilon u'_{2,\varepsilon}(t_\varepsilon) &= -2\alpha_2. \end{aligned}$$

Combining these with (2.82), we can repeat the argument of Steps 1-2 in Lemma 2.17 to conclude the existence of $\varepsilon_8 \in (0, \varepsilon_7)$ such that $ru'_{1,\varepsilon}(r) < -2 - \tilde{\alpha}_1$ on $[t_\varepsilon, R_\varepsilon^*]$ for any $\varepsilon \in (0, \varepsilon_8)$. Consequently,

$$\int_{t_\varepsilon}^{R_\varepsilon^*} re^{u_{1,\varepsilon}} dr \leq \frac{1}{\tilde{\alpha}_1} t_\varepsilon^2 e^{u_{1,\varepsilon}(t_\varepsilon)} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \tag{2.86}$$

Now we consider $\varepsilon \in (0, \varepsilon_8)$. Recall from Lemma 2.17 that $ru'_{2,\varepsilon}(r) < -2 - \tilde{\alpha}_2$ on $[R_{3,\varepsilon}, R_\varepsilon^*)$. It turns out that both $u_{1,\varepsilon}$ and $u_{2,\varepsilon}$ decrease on $[t_\varepsilon, R_\varepsilon^*)$, which implies $u_{k,\varepsilon}(r) < u_{k,\varepsilon}(t_\varepsilon) < \ln \delta$ for any $r \in (t_\varepsilon, R_\varepsilon^*)$ and $k = 1, 2$. By the definition of R_ε^* , we conclude that $R_\varepsilon^* = \infty$, namely $(u_{1,\varepsilon}, u_{2,\varepsilon})$ is an entire solution for $\varepsilon \in (0, \varepsilon_8)$. By Theorem A, there exists $(\alpha_{1,\varepsilon}, \alpha_{2,\varepsilon}) \in \Omega$ such that

$$u_{k,\varepsilon}(r) = -2\alpha_{k,\varepsilon} \ln r + O(1) \text{ as } r \rightarrow \infty \text{ for } k = 1, 2.$$

Recall that $|(ru'_{k,\varepsilon})'| = r|(1 + a_k)F_k - a_k F_{3-k}| \leq Cr(e^{u_{1,\varepsilon}} + e^{u_{2,\varepsilon}})$ for $k = 1, 2$. By (2.82), (2.86) and $R_\varepsilon^* = \infty$, we obtain

$$|-2\alpha_{k,\varepsilon} - t_\varepsilon u'_{k,\varepsilon}(t_\varepsilon)| \leq C \int_{t_\varepsilon}^\infty r(e^{u_{1,\varepsilon}} + e^{u_{2,\varepsilon}})dr \rightarrow 0 \text{ as } \varepsilon \rightarrow 0$$

for $k = 1, 2$. Therefore, $\lim_{\varepsilon \rightarrow 0} (\alpha_{1,\varepsilon}, \alpha_{2,\varepsilon}) = (\alpha_1, \alpha_2)$.

Observe from (2.47) and Lemma 2.17 that $u'_{2,\varepsilon}(r) < 0$ for all $r > R_{2,\varepsilon}$. Besides, Lemma 2.4 shows that $\sup_{[0, R_{1,\varepsilon}]} u_{2,\varepsilon} \rightarrow -\infty$. Combining these with (2.36), we conclude $\sup_{\mathbb{R}^2} u_{2,\varepsilon} \rightarrow -\infty$ as $\varepsilon \rightarrow 0$.

This completes the proof. □

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