

Non-secant defectivity via osculating projections

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Abstract. We introduce a method to produce bounds for the non secant defectivity of an arbitrary irreducible projective variety, once we know how its osculating spaces behave in families and when the linear projections from them are generically finite.

Then we analyze the relative dimension of osculating projections of Grassmannians, and as an application of our techniques we prove that asymptotically the Grassmannian $\mathbb{G}(r, n)$, parametrizing r -planes in \mathbb{P}^n , is not h -defective for $h \leq \left(\frac{n+1}{r+1}\right)^{\lfloor \log_2(r) \rfloor}$. This bound improves the previous one $h \leq \frac{n-r}{3} + 1$, due to H. Abo, G. Ottaviani and C. Peterson, for any $r \geq 4$.

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1. Introduction

The h -secant variety $\text{Sec}_h(X)$, of an irreducible, non-degenerate n -dimensional variety $X \subset \mathbb{P}^N$, is the Zariski closure of the union of the linear spaces spanned by collections of h points on X . Secant varieties are central objects in both classical algebraic geometry [20, 63], and applied mathematics [41–43, 46].

The *expected dimension* of $\text{Sec}_h(X)$ is

$$\text{expdim}(\text{Sec}_h(X)) := \min\{nh + h - 1, N\}.$$

However, the actual dimension of $\text{Sec}_h(X)$ might be smaller than the expected one. Indeed, if the expected dimension is $nh + h - 1$ this happens when through a general point of \mathbb{P}^N there are infinitely many $(h - 1)$ -planes h -secant to X . According to a definition by F. Zak [63] we will say that X is *h -defective* if

$$\dim(\text{Sec}_h(X)) < \text{expdim}(\text{Sec}_h(X)).$$

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In this paper we introduce a method to produce bounds for the non secant defectivity of an arbitrary irreducible projective variety, based on the behavior of its osculating spaces and of the corresponding osculating projections. Then, with these techniques, we study the dimension of secant varieties of the Grassmannian $\mathbb{G}(r, n)$ parametrizing r -planes in \mathbb{P}^n . Grassmannians together with Veronese and Segre varieties form the triad of varieties parametrizing rank one tensors. Hence, a general point of their h -secant variety corresponds to a tensor of a given rank depending on h . For this reason, secant varieties of Grassmannians, Veroneses and Segres are particularly interesting in problems of tensor decomposition [25, 26, 34, 41, 47, 48].

Furthermore, secant varieties have been widely used to construct and study moduli spaces for all possible additive decompositions of a general tensor into a given number of rank one tensors [13, 31, 32, 44, 45, 52, 58].

The problem of determining the actual dimension of secant varieties, and its relation with the dimension of certain linear systems of hypersurfaces with double points, have a very long history in algebraic geometry, and can be traced back to the Italian school [55, 57, 60].

Since then the geometry of secant varieties has been studied and used by many authors in various contexts [20, 24, 39, 54], and the problem of secant defectivity has been widely studied for Veroneses, Segres and Grassmannians [1–4, 8, 12, 15, 17–19, 40].

Despite the long history of this subject, only in 1995 J. Alexander and A. Hirshowitz [4] classified secant defective Veronese varieties. Indeed, they proved that, except for the double Veronese embedding which is almost always defective, the degree d Veronese embedding of \mathbb{P}^n is not h -defective, with the following exceptions:

$$(d, n, h) \in \{(4, 2, 5), (4, 3, 9), (3, 4, 7), (4, 4, 14)\}.$$

Later on, K. Baur, J. Draisma, W. A. de Graaf proposed a conjecture on secant defectivity of Grassmannians in the spirit of Alexander-Hirshowitz result [11].

It is well-known that the secant variety $\text{Sec}_h(\mathbb{G}(1, n))$, that is the locus of skew-symmetric matrices of rank at most $2h$, is almost always defective. Therefore, throughout the paper we assume $r \geq 2$. Only four defective cases are known then, and we have the following conjecture.

Conjecture ([11, Conjecture 4.1]). If $r \geq 2$ then $\mathbb{G}(r, n)$ is not h -defective with the following exceptions:

$$(r, n, h) \in \{(2, 7, 3), (3, 8, 3), (3, 8, 4), (2, 9, 4)\}.$$

In [18] M. V. Catalisano, A. V. Geramita, and A. Gimigliano gave explicit bounds on (r, n, h) for $\mathbb{G}(r, n)$ not to be h -defective. Later, in [3] H. Abo, G. Ottaviani, and C. Peterson, improved these bounds, and showed that the conjecture is true for $h \leq 6$. Finally, in [15] A. Boralevi further improved this result by proving the conjecture for $h \leq 12$.

To the best of our knowledge, the best asymptotic bound for $\text{Sec}_h(\mathbb{G}(r, n))$ to have expected dimension was obtained by H. Abo, G. Ottaviani, and C. Peterson using monomial techniques.

Theorem ([3, Theorem 3.3]). *If $r \geq 2$ and*

$$h \leq \frac{n-r}{3} + 1$$

then $\text{Sec}_h(\mathbb{G}(r, n))$ has the expected dimension.

Our starting point, in order to study the dimension of $\text{Sec}_h(\mathbb{G}(r, n))$, is a result due to L. Chiantini and C. Ciliberto [20, Proposition 3.5] relating secant defectivity with the dimension of the general fiber of a general tangential projection. Given $x_1, \dots, x_h \in X \subset \mathbb{P}^N$ general points, we may consider a *general h -tangential projection* of X

$$\tau_{X,h} : X \subset \mathbb{P}^N \dashrightarrow \mathbb{P}^{N_h}$$

that is the linear projection with center $\langle T_{x_1} X, \dots, T_{x_h} X \rangle$. Then, by [20, Proposition 3.5] if $\tau_{X,h}$ is generically finite then X is not $(h+1)$ -defective.

Our approach consists in considering linear projections from higher order osculating spaces. If $p \in X \subset \mathbb{P}^N$ is a smooth point, the *m -osculating space* $T_p^m X$ of X at p is essentially the linear subspace of \mathbb{P}^N generated by the partial derivatives of order less or equal than m of a local parametrization of X at p , see Definition 3.1.

Given $p_1, \dots, p_l \in X$ general points, we denote by

$$\Pi_{T_{p_1}^{k_1}, \dots, T_{p_l}^{k_l}} : X \subset \mathbb{P}^N \dashrightarrow \mathbb{P}^{N_{k_1, \dots, k_l}}$$

the corresponding $(k_1 + \dots + k_l)$ -*osculating projection*, that is the linear projection with center $\langle T_{p_1}^{k_1} X, \dots, T_{p_l}^{k_l} X \rangle$.

When $X = \mathbb{G}(r, n)$ we manage to control the dimension of the general fiber of osculating projections from the span of a certain number of general osculating spaces. Indeed, in Corollary 4.5 we prove that, under suitable numerical hypothesis, such an osculating projection is birational.

Then, in Section 5 we construct flat degenerations of general tangential projections to linear projections which factor through suitable osculating projections. Since, by Proposition 5.7, the dimension of the general fiber can only increase under specialization, the birationality of a certain osculating projection yields that the general tangential projection degenerating to it is generically finite.

However, we are not able to check if the degree of the map is preserved, that is if the general tangential projection is birational as well. In order to do this, one needs to achieve a good control on the indeterminacy locus of the relevant tangential projection.

On the other hand, by [20, Proposition 3.5] knowing that a general tangential projection is generically finite is enough to conclude that, under the numerical hypothesis ensuring the birationality of the corresponding general osculating projection, $\mathbb{G}(r, n)$ is not defective. As a direct consequence of our main results in Theorem 6.3 we get the following.

Theorem. *Assume that $r \geq 2$, set*

$$\alpha := \left\lfloor \frac{n+1}{r+1} \right\rfloor$$

and write $r = 2^{\lambda_1} + \dots + 2^{\lambda_s} + \varepsilon$, with $\lambda_1 > \lambda_2 > \dots > \lambda_s \geq 1$, $\varepsilon \in \{0, 1\}$. If either

- $h \leq (\alpha - 1)(\alpha^{\lambda_1 - 1} + \dots + \alpha^{\lambda_s - 1}) + 1$, or
- $n \geq r^2 + 3r + 1$ and $h \leq \alpha^{\lambda_1} + \dots + \alpha^{\lambda_s} + 1$

then $\mathbb{G}(r, n)$ is not h -defective.

Note that the bounds in our main result give that asymptotically the Grassmannian $\mathbb{G}(r, n)$ is not $\left(\frac{n+1}{r+1}\right)^{\lfloor \log_2(r) \rfloor}$ -defective, while [3, Theorem 3.3] yields that $\mathbb{G}(r, n)$ is not $\frac{n}{3}$ -defective. In Section 6.1 we show that Theorem 6.3 improves [3, Theorem 3.3] for any $r \geq 4$. However, H. Abo, G. Ottaviani, and C. Peterson in [3] gave a much better bound, going with n^2 , in the case $r = 2$.

We would like to mention that, as remarked by C. Ciliberto and F. Russo in [24], the idea that the behavior of osculating projections reflects the geometry of the variety itself was already present in the work of G. Castelnuovo [16, pages 186-188].

Finally, we would like to stress that the machinery introduced in this paper could be used to produce bounds, for the non secant defectivity of an arbitrary irreducible projective variety, once we know how its osculating spaces behave in families and when the projections from them are generically finite. Indeed, in [6] we apply these techniques to Segre-Veronese varieties.

The paper is organized as follows. In Section 2 we recall some notions on secant varieties and tangential projections. In Section 3 we compute explicitly the osculating spaces of Grassmannians, and in Section 4 we study the relative dimension of general osculating projections. In Section 5, in order to extend our results on osculating projections to tangential projections, we investigate how rational maps degenerate in a 1-dimensional family. Finally, in Section 6 we take advantage of these techniques to prove our main result on the dimension of secant varieties of Grassmannians.

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2. Secant varieties

Throughout the paper we work over the field of complex numbers. In this section we recall the notions of secant varieties, secant defectivity and secant defect. We refer to [53] for a nice and comprehensive survey on the subject.

Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate variety of dimension n and let

$$\Gamma_h(X) \subset X \times \cdots \times X \times \mathbb{G}(h-1, N)$$

where $h \leq N$, be the closure of the graph of the rational map

$$\alpha : X \times \cdots \times X \dashrightarrow \mathbb{G}(h-1, N),$$

mapping h general points to their linear span $\langle x_1, \dots, x_h \rangle$. Observe that $\Gamma_h(X)$ is irreducible and reduced of dimension hn . Let $\pi_2 : \Gamma_h(X) \rightarrow \mathbb{G}(h-1, N)$ be the natural projection. We denote

$$\mathcal{S}_h(X) := \pi_2(\Gamma_h(X)) \subset \mathbb{G}(h-1, N).$$

Again $\mathcal{S}_h(X)$ is irreducible and reduced of dimension hn . Finally, let

$$\mathcal{I}_h = \{(x, \Lambda) \mid x \in \Lambda\} \subset \mathbb{P}^N \times \mathbb{G}(h-1, N)$$

with natural projections π_h and ψ_h onto the factors. Furthermore, observe that $\psi_h : \mathcal{I}_h \rightarrow \mathbb{G}(h-1, N)$ is a \mathbb{P}^{h-1} -bundle on $\mathbb{G}(h-1, N)$.

Definition 2.1. Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate variety. The *abstract h -secant variety* is the irreducible variety

$$\text{Sec}_h(X) := (\psi_h)^{-1}(\mathcal{S}_h(X)) \subset \mathcal{I}_h.$$

The *h -secant variety* is

$$\text{Sec}_h(X) := \pi_h(\text{Sec}_h(X)) \subset \mathbb{P}^N.$$

It immediately follows that $\text{Sec}_h(X)$ is a $(hn + h - 1)$ -dimensional variety with a \mathbb{P}^{h-1} -bundle structure over $\mathcal{S}_h(X)$. We say that X is *h -defective* if

$$\dim \text{Sec}_h(X) < \min\{\dim \text{Sec}_h(X), N\}.$$

The number

$$\delta_h(X) = \min\{\dim \text{Sec}_h(X), N\} - \dim \text{Sec}_h(X)$$

is called the *h -defect* of X . We say that X is *h -defective* if $\delta_h(X) > 0$.

Now, let $x_1, \dots, x_h \in X \subset \mathbb{P}^N$ be general points, and let $T_{x_i} X$ be the tangent space of X at x_i . We will call the linear projection

$$\tau_{X,h} : X \subseteq \mathbb{P}^N \dashrightarrow \mathbb{P}^{N_h},$$

where $N_h = N - \dim(\langle T_{x_1} X, \dots, T_{x_h} X \rangle) - 1$, with center $\langle T_{x_1} X, \dots, T_{x_h} X \rangle$ a *general h -tangential projection* of X . Finally, let $X_h = \tau_{X,h}(X)$. We will need the first part of the following result due to L. Chiantini and C. Ciliberto.

Proposition 2.2 ([20, Proposition 3.5]). *Let $X \subset \mathbb{P}^N$ be a irreducible, non-degenerate, projective variety of dimension n , then the following facts hold:*

- *If $\dim(X_h) = \dim(X)$, that is $\tau_{X,h} : X \dashrightarrow X_h$ is generically finite, then X is not $(h+1)$ -defective;*
- *If $N - \dim(\langle T_{x_1} X, \dots, T_{x_h} X \rangle) - 1 \geq n$ and $\dim(X_h) < \dim(X)$, that is $\tau_{X,h} : X \dashrightarrow X_h$ has positive dimensional general fibers, then X is $(h+1)$ -defective.*

3. Osculating spaces of Grassmannians

Let $X \subset \mathbb{P}^N$ be an integral projective variety of dimension n , let $p \in X$ be a smooth point, and

$$\begin{aligned} \phi : \mathcal{U} \subseteq \mathbb{C}^n &\longrightarrow \mathbb{C}^N \\ (t_1, \dots, t_n) &\longmapsto \phi(t_1, \dots, t_n) \end{aligned}$$

with $\phi(0) = p$, be a local parametrization of X in a neighborhood of $p \in X$.

For any $m \geq 0$ let $O_p^m X$ be the affine subspace of \mathbb{C}^N passing through $p \in X$, and whose direction is given by the subspace generated by the vectors $\phi_I(0)$, where $I = (i_1, \dots, i_n)$ is a multi-index such that $|I| \leq m$ and

$$\phi_I = \frac{\partial^{|I|} \phi}{\partial t_1^{i_1} \dots \partial t_n^{i_n}}. \quad (3.1)$$

Definition 3.1. The *m -osculating space* $T_p^m X$ of X at p is the projective closure in \mathbb{P}^N of the affine subspace $O_p^m X \subseteq \mathbb{C}^N$.

For instance, $T_p^0 X = \{p\}$, and $T_p^1 X$ is the usual tangent space of X at p . When no confusion arises we will write T_p^m instead of $T_p^m X$.

Osculating spaces can be defined intrinsically. Let \mathcal{L} be an invertible sheaf on X and $V = H^0(X, \mathcal{L})$, and $\Delta \subset X \times X$ the diagonal. The rank $\binom{n+m}{m}$ locally free sheaf

$$J_m(\mathcal{L}) = \pi_{1*} \left(\pi_2^*(\mathcal{L}) \otimes \mathcal{O}_{X \times X} / \mathcal{I}_\Delta^{m+1} \right)$$

is called the *m -jet bundle* of \mathcal{L} . Note that the fiber of $J_m(\mathcal{L})$ at $p \in X$ is

$$J_m(\mathcal{L})_p \cong H^0 \left(X, \mathcal{L} \otimes \mathcal{O}_X / \mathfrak{m}_p^{m+1} \right)$$

and the quotient map

$$j_{m,p} : V \rightarrow H^0\left(X, \mathcal{L} \otimes \mathcal{O}_X/\mathfrak{m}_p^{m+1}\right)$$

is nothing but the evaluation of the global sections and their derivatives of order at most m at the point $p \in X$. Let

$$j_m : V \otimes \mathcal{O}_X \rightarrow J_m(\mathcal{L})$$

be the corresponding vector bundle map. Then, there exists an open subset $U_m \subseteq X$ where j_m is of maximal rank $r_m \leq \binom{n+m}{m}$.

The linear space $\mathbb{P}(j_{m,p}(V)) = T_p^m X \subseteq \mathbb{P}(V)$ is the m -osculating space of X at $p \in X$. The integer r_m is called the *general m -osculating dimension* of \mathcal{L} on X .

Note that while the dimension of the tangent space at a smooth point is always equal to the dimension of the variety, higher order osculating spaces can be strictly smaller than expected even at a general point. In general, we have

$$\dim(T_p^m X) = \min\left\{\binom{n+m}{n} - 1 - \delta_{m,p}, N\right\} \quad (3.2)$$

where $\delta_{m,p}$ is the number of independent differential equations of order less or equal than m satisfied by X at p .

Projective varieties having general m -osculating dimension smaller than expected were introduced and studied in [14, 56, 59, 61, 62], and more recently in [9, 10, 27–30, 33, 37, 38, 49, 51].

Examples of algebraic surfaces with defective higher order osculating spaces are rational normal scrolls and developable surfaces, that is cones or tangent developables of curves. As an example, which will be useful later on in the paper, we consider tangent developables of rational normal curves.

Proposition 3.2. *Let $C_n \subseteq \mathbb{P}^n$ be a rational normal curve of degree n in \mathbb{P}^n , and let $Y_n \subseteq \mathbb{P}^n$ be its tangent developable. Then*

$$\dim(T_p^m Y_n) = \min\{m+1, n\}$$

for $p \in Y_n$ general, and $m \geq 1$.

Proof. We may work on an affine chart. Then Y_n is the surface parametrized by

$$\begin{aligned} \phi : \mathbb{A}^2 &\longrightarrow \mathbb{A}^n \\ (t, u) &\mapsto (t+u, t^2+2tu, \dots, t^n+nt^{n-1}u). \end{aligned}$$

Note that

$$\frac{\partial^m \phi}{\partial t^{m-k} \partial u^k} = 0$$

for any $k \geq 2$. Furthermore, we have

$$\frac{\partial^m \phi}{\partial t^m} - \frac{\partial^m \phi}{\partial t^{m-1} \partial u} = u \frac{\partial^{m+1} \phi}{\partial t^m \partial u}$$

for any $m \geq 1$.

Therefore, for any $m \geq 1$ we get just two non-zero partial derivatives of order m , and one partial derivative is given in terms of smaller order partial derivatives. Furthermore, in the notation of (3.2) we have $\delta_{m,p} = \frac{m(m+1)}{2} - 1$ for any $1 \leq m \leq n-1$, where $p \in Y_n$ is a general point. \square

From now on we will assume that $n \geq 2r+1$. We will denote by $e_0, \dots, e_n \in \mathbb{C}^{n+1}$ both the vectors of the canonical basis of \mathbb{C}^{n+1} and the corresponding points in $\mathbb{P}^n = \mathbb{P}(\mathbb{C}^{n+1})$.

Throughout the paper we will always view $\mathbb{G}(r, n)$ as a projective variety in its Plücker embedding, that is the morphism induced by the determinant of the universal quotient bundle $\mathcal{Q}_{\mathbb{G}(r,n)}$ on $\mathbb{G}(r, n)$:

$$\begin{aligned} \varphi_{r,n} : \quad \mathbb{G}(r, n) &\longrightarrow \mathbb{P}^N := \mathbb{P}(\bigwedge^{r+1} \mathbb{C}^{n+1}) \\ \langle v_0, \dots, v_r \rangle &\longmapsto [v_0 \wedge \dots \wedge v_r] \end{aligned}$$

where $N = \binom{n+1}{r+1} - 1$. Now, let

$$\Lambda := \{I \subset \{0, \dots, n\}, |I| = r+1\}.$$

For each $I = \{i_0, \dots, i_r\} \in \Lambda$, with $i_0 < i_1 < \dots < i_r$, let $e_I \in \mathbb{G}(r, n)$ be the point corresponding to $e_{i_0} \wedge \dots \wedge e_{i_r} \in \bigwedge^{r+1} \mathbb{C}^{n+1}$. We will denote by p_I the Plücker on on \mathbb{P}^N .

Furthermore, we define a distance on Λ as

$$d(I, J) = |I| - |I \cap J| = |J| - |I \cap J|$$

for each $I, J \in \Lambda$. Note that, with respect to this distance, the diameter of Λ is $r+1$.

In the following we give an explicit description of osculating spaces of Grassmannians at fundamental points.

Proposition 3.3. *For any $s \geq 0$ we have*

$$T_{e_I}^s(\mathbb{G}(r, n)) = \langle e_J \mid d(I, J) \leq s \rangle = \{p_J = 0 \mid d(I, J) > s\} \subseteq \mathbb{P}^N.$$

In particular, $T_{e_I}^s(\mathbb{G}(r, n)) = \mathbb{P}^N$ for any $s \geq r+1$.

Proof. We may assume that $I = \{0, \dots, r\}$ and consider the usual parametrization of $\mathbb{G}(r, n)$:

$$\phi : \mathbb{C}^{(r+1)(n-r)} \rightarrow \mathbb{G}(r, n)$$

given by

$$A = (a_{ij}) = \begin{pmatrix} 1 & \dots & 0 & a_{0,r+1} & \dots & a_{0n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & a_{r,r+1} & \dots & a_{rn} \end{pmatrix} \mapsto (\det(A_J))_{J \in \Lambda}$$

where A_J is the $(r+1) \times (r+1)$ matrix obtained from A considering just the columns indexed by J .

Note that each variable appears in degree at most one in the coordinates of ϕ . Therefore, deriving two times with respect to the same variable always gives zero.

Thus, in order to describe the osculating spaces we may take into account just partial derivatives with respect to different variables. Moreover, since the degree of $\det(A_J)$ with respect to $a_{i,j}$ is at most $r+1$ all partial derivatives of order greater or equal than $r+2$ are zero. Hence, it is enough to prove the proposition for $s \leq r+1$.

Given $J = \{j_0, \dots, j_r\} \subset \{0, \dots, n\}$, $k \in \{0, \dots, r\}$, and $k' \in \{r+1, \dots, n\}$ we have

$$\frac{\partial \det(A_J)}{\partial a_{k,k'}} = \begin{cases} 0 & \text{if } k' \notin J \\ (-1)^{l+1+k'} \det(A_{J,k,k'}) & \text{if } k' = j_l \end{cases}$$

where $A_{J,k,k'}$ denotes the submatrix of A_J obtained deleting the line indexed by k and the column indexed by k' . More generally, for any $m \geq 1$ and for any

$$\begin{aligned} J &= \{j_0, \dots, j_r\} \subset \{0, \dots, n\}, \\ K' &= \{k'_1, \dots, k'_m\} \subset \{r+1, \dots, n\}, \\ K &= \{k_1, \dots, k_m\} \subset \{0, \dots, r\} \end{aligned}$$

we have

$$\begin{aligned} & \frac{\partial^m \det(A_J)}{\partial a_{k_1, k'_1} \dots \partial a_{k_m, k'_m}} \\ &= \begin{cases} (\pm 1) \det(A_{J, (k_1, k'_1), \dots, (k_m, k'_m)}) & \text{if } K' \subset J \text{ and } |K| = |K'| = m \leq d \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

where $d = d(J, \{0, \dots, r\}) = \deg(\det(A_J))$. Therefore

$$\frac{\partial^m \det(A_J)}{\partial a_{k_1, k'_1} \dots \partial a_{k_m, k'_m}}(0) = \begin{cases} \pm 1 & \text{if } J = K' \cup (\{0, \dots, r\} \setminus K) \\ 0 & \text{otherwise} \end{cases}$$

and

$$\frac{\partial^m \phi}{\partial a_{k_1, k'_1} \dots \partial a_{k_m, k'_m}}(0) = \pm e_{K' \cup (\{0, \dots, r\} \setminus K)}.$$

Note that $d(K' \cup (\{0, \dots, r\} \setminus K), \{0, \dots, r\}) = m$, and that any J with $d(J, \{0, \dots, r\}) = m$ may be written in the form $K' \cup (\{0, \dots, r\} \setminus K)$.

Finally, we get that

$$\left\langle \frac{\partial^{|I|} \phi}{\partial^I a_{i,j}}(0) \mid |I| = m \right\rangle = \langle e_J \mid d(J, \{0, \dots, r\}) = m \rangle$$

which proves the statement. \square

Now, it is easy to compute the dimension of the osculating spaces of $\mathbb{G}(r, n)$.

Corollary 3.4. *For any point $p \in \mathbb{G}(r, n)$ we have*

$$\dim T_p^s \mathbb{G}(r, n) = \sum_{l=1}^s \binom{r+1}{l} \binom{n-r}{l}$$

for any $0 \leq s \leq r$, while $T_p^s \mathbb{G}(r, n) = \mathbb{P}^N$ for any $s \geq r+1$.

Proof. Since $\mathbb{G}(r, n) \subset \mathbb{P}^N$ is homogeneous under the action the algebraic subgroup of $PGL(N+1)$ the statement follows, by standard combinatorial computations, from Proposition 3.3. \square

4. Osculating projections

In this section we study linear projections of Grassmannians from their osculating spaces. In order to help the reader get acquainted with the ideas of the proofs, we start by studying in detail projections from a single osculating space.

Let $0 \leq s \leq r$ be an integer, and $I \in \Lambda$. By Proposition 3.3 the projection of $\mathbb{G}(r, n)$ from $T_{e_I}^s$ is given by

$$\begin{aligned} \Pi_{T_{e_I}^s} : \mathbb{G}(r, n) &\dashrightarrow \mathbb{P}^{N_s} \\ (p_I)_{I \in \Lambda} &\mapsto (p_J)_{J \in \Lambda \mid d(I, J) > s}. \end{aligned}$$

Moreover, given $I' = \{i'_0, \dots, i'_s\} \subset I$ with $|I'| = s+1$ we can consider the linear projection

$$\begin{aligned} \pi_{I'} : \mathbb{P}^n &\dashrightarrow \mathbb{P}^{n-s-1} \\ (x_i) &\mapsto (x_i)_{i \in \{0, \dots, n\} \setminus I'} \end{aligned}$$

which in turn induces the linear projection

$$\begin{aligned} \Pi_{I'} : \mathbb{G}(r, n) &\dashrightarrow \mathbb{G}(r, n-s-1) \\ [V] &\mapsto [\pi_{I'}(V)] \\ (p_I)_{I \in \Lambda} &\mapsto (p_J)_{J \in \Lambda \mid J \cap I' = \emptyset}. \end{aligned}$$

Note that the fibers of $\Pi_{I'}$ are isomorphic to $\mathbb{G}(r, r + s + 1)$. More precisely, let $y \in \mathbb{G}(r, n - s - 1)$ be a point, and consider a general point $x \in \overline{\Pi_{I'}^{-1}(y)} \subset \mathbb{G}(r, n)$ corresponding to an r -plane $V_x \subset \mathbb{P}^n$. Then we have

$$\overline{\Pi_{I'}^{-1}(y)} = \mathbb{G}\left(r, \left\langle V_x, e_{i'_0}, \dots, e_{i'_s} \right\rangle\right) \subset \mathbb{G}(r, n).$$

On the other hand, *a priori* it is not at all clear what the fibers of $\Pi_{T_{e_I}^s}$ are. In general the image of $\Pi_{T_{e_I}^s}$ is very singular, and its fibers may not be connected. In what follows we study the general fiber of the map $\Pi_{T_{e_I}^s}$ by factoring it through several projections of type $\Pi_{I'}$.

Lemma 4.1. *If $s = 0, \dots, r$ and $I' \subset I$ with $|I'| = s + 1$, then the rational map $\Pi_{I'}$ factors through $\Pi_{T_{e_I}^s}$. Moreover, $\Pi_{T_{e_I}^r} = \Pi_I$.*

Proof. Since $J \cap I' = \emptyset \Rightarrow d(I, J) > s$ then the center of $\Pi_{T_{e_I}^s}$ is contained in the center of $\Pi_{I'}$. Furthermore, if $s = r$ then $J \cap I = \emptyset \Leftrightarrow d(I, J) > r$. \square

Now, we are ready to describe the fibers of $\Pi_{T_{e_I}^s}$ for $0 \leq s \leq r$.

Proposition 4.2. *The rational map $\Pi_{T_{e_I}^s}$ is birational for every $0 \leq s \leq r - 1$, and*

$$\Pi_{T_{e_I}^r} : \mathbb{G}(r, n) \dashrightarrow \mathbb{G}(r, n - r - 1)$$

is a fibration with fibers isomorphic to $\mathbb{G}(r, 2r + 1)$.

Proof. For the second part of the statement it is enough to observe that $\Pi_{T_{e_I}^r} = \Pi_I$. Now, let us consider the first claim. Since $\Pi_{T_{e_I}^s}$ factors through $\Pi_{T_{e_I}^{s-1}}$ it is enough to prove that $\Pi_{T_{e_I}^{r-1}}$ is birational. By Lemma 4.1 for any $I_j = I \setminus \{i_j\}$, there exists a rational map τ_j such that the following diagram is commutative

$$\begin{array}{ccc} \mathbb{G}(r, n) & \overset{\Pi_{T_{e_I}^{r-1}}}{\dashrightarrow} & W \subseteq \mathbb{P}^{N_s} \\ & \searrow \Pi_{I_j} & \downarrow \tau_j \\ & & \mathbb{G}(r, n - r) \end{array}$$

where $W = \overline{\Pi_{T_{e_I}^{r-1}}(\mathbb{G}(r, n))}$. Now, let $x \in W$ be a general point, and $F \subset \mathbb{G}(r, n)$ be the fiber of $\Pi_{T_{e_I}^{r-1}}$ over x . Set $x_j = \tau_j(x) \in \mathbb{G}(r, n - r)$, and denote by $F_j \subset \mathbb{G}(r, n)$ the fiber of Π_{I_j} over x_j . Therefore

$$F \subseteq \bigcap_{j=0}^r F_j. \quad (4.1)$$

Now, note that if $y \in F$ is a general point corresponding to an r -plane $V_y \subset \mathbb{P}^n$ we have

$$F_j = \mathbb{G}(r, \langle V_y, e_{i_0}, \dots, \widehat{e_{i_j}}, \dots, e_{i_r} \rangle)$$

and hence

$$\bigcap_{j=0}^r F_j = \bigcap_{j=0}^r \mathbb{G}(r, \langle V_y, e_{i_0}, \dots, \widehat{e_{i_j}}, \dots, e_{i_r} \rangle) = \mathbb{G}(r, V_y) = \{y\}.$$

The last equality and (4.1) force $F = \{y\}$, and since we are working in characteristic zero $\Pi_{T_{e_l}^{r-1}}$ is birational. \square

Our next aim is to study linear projections from the span of several osculating spaces. In particular, we want to understand when such a projection is birational as we did in Proposition 4.2 for the projection from a single osculating space.

Clearly, there are some natural numerical constraints regarding how many coordinate points of $\mathbb{G}(r, n)$ we may take into account, and the order of the osculating spaces we want to project from.

First of all, by Proposition 4.2 the order of the osculating spaces cannot exceed $r - 1$. Furthermore, since in order to carry out the computations, we need to consider just coordinate points of $\mathbb{G}(r, n)$ corresponding to linearly independent linear subspaces of dimension $r + 1$ in \mathbb{C}^{n+1} we can use at most

$$\alpha := \left\lfloor \frac{n+1}{r+1} \right\rfloor$$

of them.

Now, let us consider the points $e_{I_1}, \dots, e_{I_\alpha} \in \mathbb{G}(r, n)$ where

$$I_1 = \{0, \dots, r\}, \dots, I_\alpha = \{(r+1)(\alpha-1), \dots, (r+1)\alpha-1\} \in \Lambda. \quad (4.2)$$

Again by Proposition 3.3 the projection from the span of the osculating spaces of $\mathbb{G}(r, n)$ of orders s_1, \dots, s_l at the points e_{I_1}, \dots, e_{I_l} is given by

$$\begin{aligned} \Pi_{T_{e_{I_1}}^{s_1}, \dots, e_{I_l}}^{s_l} : \mathbb{G}(r, n) &\dashrightarrow \mathbb{P}^{N_{s_1, \dots, s_l}} \\ (pI)_{I \in \Lambda} &\mapsto (pJ)_{J \in \Lambda \mid d(I_1, J) > s_1, \dots, d(I_l, J) > s_l} \end{aligned}$$

whenever $\{J \in \Lambda \mid d(I_1, J) \leq s_1 \text{ or } \dots \text{ or } d(I_l, J) \leq s_l\} \neq \Lambda$, and $l \leq \alpha$.

Furthermore, for any $I'_1 = \{i_0^1, \dots, i_{s_1}^1\} \subset I_1, \dots, I'_l = \{i_0^l, \dots, i_{s_l}^l\} \subset I_l$ we consider the projection

$$\begin{aligned} \pi_{I'_1, \dots, I'_l} : \mathbb{P}^n &\dashrightarrow \mathbb{P}^{n-l-\sum_1^{s_l} s_i} \\ (x_i)_{i=0, \dots, n} &\mapsto (x_i)_{i \in \{0, \dots, n\} \setminus (I'_1 \cup \dots \cup I'_l)} \end{aligned}$$

where $l \leq \alpha$ and $n - l - \sum_1^l s_i \geq r + 1$. The map $\pi_{I'_1, \dots, I'_l}$ in turn induces the projection

$$\begin{aligned} \Pi_{I'_1, \dots, I'_l} : \mathbb{G}(r, n) &\dashrightarrow \mathbb{G}\left(r, n - l - \sum_1^l s_i\right) \\ [V] &\mapsto [\pi_{I'_1, \dots, I'_l}(V)] \\ (pI)_{I \in \Lambda} &\mapsto (pJ)_{J \in \Lambda \mid J \cap (I'_1 \cup \dots \cup I'_l) = \emptyset}. \end{aligned}$$

Lemma 4.3. *Let I_1, \dots, I_α be as in (4.2), l, s_1, \dots, s_l be integers such that $0 \leq s_j \leq r - 1$, and $0 < l \leq \min\{\alpha, n - r - 1 - \sum_i s_i\}$. Then for any $I_1^l = \{i_0^1, \dots, i_{s_1}^1\} \subset I_1, \dots, I_l^l = \{i_0^l, \dots, i_{s_l}^l\} \subset I_l$ with $|I_j^l| = s_j + 1$ the rational maps $\Pi_{T_{e_{I_1}^{s_1}, \dots, e_{I_l}^{s_l}}}$ and $\Pi_{I'_1, \dots, I'_l}$ are well-defined and the latter factors through the former.*

Proof. Note that $J \cap (I'_1 \cup \dots \cup I'_l) = \emptyset$ yields $d(I_1, J) > s_1, \dots, d(I_l, J) > s_l$. Note also that the I_j 's are disjoint since the I_j^l 's are. Furthermore, since $\sum (s_i + 1) = l + \sum s_i \leq n - r - 1$ and $n \geq 2r + 1$, there are at least $r + 2$ elements in $\{0, \dots, n\} \setminus (I'_1 \cup \dots \cup I'_l)$. If k_1, \dots, k_{r+2} are such elements, then

$$K_j := \{k_1, \dots, \widehat{k_j}, \dots, k_{r+2}\} \in \{J \in \Lambda \mid d(I_j, J) > s_j, j = 1, \dots, l\}$$

for any $j = 1, \dots, r+2$ forces $\{J \in \Lambda \mid d(I_1, J) \leq s_1 \text{ or } \dots \text{ or } d(I_l, J) \leq s_l\} \neq \Lambda$. \square

Now, we are ready to prove the main result of this section.

Proposition 4.4. *Let I_1, \dots, I_α be as in (4.2), l, s_1, \dots, s_l be integers such that $0 \leq s_j \leq r - 1$, and $0 < l \leq \min\{\alpha, n - r - 1 - \sum_i s_i\}$. Then the projection $\Pi_{T_{e_{I_1}^{s_1}, \dots, e_{I_l}^{s_l}}}$ is birational.*

Proof. For any collection of subsets $I_i^l \subset I_i$ with $|I_i^l| = s_i + 1$ set $I' = \bigcup_i I_i^l$. By Lemma 4.3 there exists a rational map $\tau_{I'_1, \dots, I'_l}$ fitting in the following commutative diagram

$$\begin{array}{ccc} \mathbb{G}(r, n) & \overset{\Pi_{T_{e_{I_1}^{s_1}, \dots, e_{I_l}^{s_l}}}}{\dashrightarrow} & W \subseteq \mathbb{P}^{N_{s_1, \dots, s_l}} \\ & \searrow \Pi_{I'_1, \dots, I'_l} & \downarrow \tau_{I'_1, \dots, I'_l} \\ & & \mathbb{G}(r, n - l - \sum_1^l s_i) \end{array}$$

where $W = \overline{\Pi_{T_{e_{I_1}^{s_1}, \dots, e_{I_l}^{s_l}}}(\mathbb{G}(r, n))}$. Now, let $x \in W$ be a general point, and $F \subset \mathbb{G}(r, n)$ be the fiber of $\Pi_{T_{e_{I_1}^{s_1}, \dots, e_{I_l}^{s_l}}}$ over x . Set $x' = \tau_{I'_1, \dots, I'_l}(x) \in \mathbb{G}(r, n - l - \sum_1^l s_i)$, and denote by

$$F_{I'_1, \dots, I'_l} \subset \mathbb{G}(r, n)$$

the fiber of $\Pi_{I'_1, \dots, I'_l}$ over x' . Therefore

$$F \subseteq \bigcap_{I'_1, \dots, I'_l} F_{I'_1, \dots, I'_l} \quad (4.3)$$

where the intersection runs over all the collections of subsets $I'_i \subset I_i$ with $|I'_i| = s_i + 1$. Now, if $y \in F$ is a general point corresponding to an r -plane $V_y \subset \mathbb{P}^n$ we have

$$F_{I'_1, \dots, I'_l} = \mathbb{G}(r, \langle V_y, e_j \mid j \in I' \rangle)$$

and hence

$$\bigcap_{I'_1, \dots, I'_l} F_{I'_1, \dots, I'_l} = \bigcap_{I'_1, \dots, I'_l} \mathbb{G}(r, \langle V_y, e_j \mid j \in I' \rangle) = \mathbb{G}(r, V_y) = \{y\} \quad (4.4)$$

where again the first intersection is taken over all the subsets $I'_i \subset I_i$ with $|I'_i| = s_i + 1$.

Finally, to conclude it is enough to observe that (4.3) and (4.4) yield $F = \{y\}$, and since we are working in characteristic zero $\Pi_{T_{e_{I'_1}, \dots, e_{I'_l}}^{s_1, \dots, s_l}}$ is birational. \square

In what follows we just make Proposition 4.4 more explicit.

Corollary 4.5. *Set $\alpha := \lfloor \frac{n+1}{r+1} \rfloor$ and let I_1, \dots, I_α be as in (4.2). Then $\Pi_{T_{e_{I_1}, \dots, e_{I_{\alpha-1}}}^{r-1, \dots, r-1}}$ is birational. Furthermore, if $n \geq r^2 + 3r + 1$ then $\Pi_{T_{e_{I_1}, \dots, e_{I_\alpha}}^{r-1, \dots, r-1}}$ is birational.*

Now, set $r' := n - 2 - \alpha r$ and $r'' := \min\{n - 3 - \alpha(r - 1), r - 2\}$. If $2r + 1 < n < r^2 + 3r + 1$ then:

- It holds $r - 1 \geq r' \geq 0$ and $\Pi_{T_{e_{I_1}, \dots, e_{I_{\alpha-1}}, e_{I_\alpha}}^{r-1, \dots, r-1, r'}}$ is birational;
- It holds $r'' \geq 0$ and $\Pi_{T_{e_{I_1}, \dots, e_{I_{\alpha-1}}, e_{I_\alpha}}^{r-2, \dots, r-2, r''}}$ is birational.

Proof. First we apply Proposition 4.4 with $l = \alpha - 1$ and $s_1 = \dots = s_{\alpha-1} = r - 1$. In this case the constraint is $\alpha - 1 \leq n - r - 1 - (\alpha - 1)(r - 1)$, that is $\alpha \leq \frac{n-r-1}{r} + 1$. Note that this is always the case since

$$\alpha \leq \frac{n+1}{r+1} \leq \frac{n-1}{r} = \frac{n-r-1}{r} + 1.$$

If $l = \alpha$ and $s_1 = \dots = s_\alpha = r - 1$ the constraint in Proposition 4.4 is $\alpha \leq n - r - 1 - \alpha(r - 1)$, which is equivalent to $\alpha \leq \frac{n-r-1}{r}$. Now, it is enough to observe that

$$\frac{n+1}{r+1} \leq \frac{n-r-1}{r} \iff n \geq r^2 + 3r + 1.$$

If $n \geq r^2 + 3r + 1$, then the claim follows from the inequalities $\alpha \leq \frac{n+1}{r+1} \leq \frac{n-r-1}{r}$.

Now assume that $n < r^2 + 3r + 1$. First we check that $r' = n - 2 - \alpha r \leq r - 1$, that is $\alpha \geq \frac{n-1-r}{r}$. That follows from

$$\alpha \geq \frac{n+1}{r+1} - 1 = \frac{n-r}{r+1} \geq \frac{n-r-1}{r}$$

whenever $n \geq 2r + 1$. Next we check that $r', r'' \geq 0$. If $2r + 1 < n < 3r + 2$ then $\alpha = 2$, and $r' = n - 2 - 2r \geq 0$. If $n \geq 3r + 2$ we have

$$\alpha = \left\lfloor \frac{n+1}{r+1} \right\rfloor \leq \frac{n+1}{r+1} \leq \frac{n-2}{r}$$

and then $r' = n - 2 - \alpha r \geq 0$. Furthermore, note that

$$\alpha = \left\lfloor \frac{n+1}{r+1} \right\rfloor \leq \frac{n+1}{r+1} \leq \frac{n-3}{r-1}$$

and then $r'' = n - 3 - \alpha(r-1) \geq 0$.

Now, we apply Proposition 4.4 with $l = \alpha, s_1 = \dots = s_{\alpha-1} = r - 1$ and $s_\alpha = r'$. In this case the constraint in Proposition 4.4 is $\alpha \leq n - r - 1 - (\alpha - 1)(r - 1) - r'$ that is $r' \leq n - 2 - \alpha r$.

Finally, if $l = \alpha, s_1 = \dots = s_{\alpha-1} = r - 2$ and $s_\alpha = r''$, then the constraint in Proposition 4.4 is $\alpha \leq n - r - 1 - (\alpha - 1)(r - 2) - r''$, that is $r'' \leq n - 3 - \alpha(r - 1)$. \square

5. Degenerating tangential projections to osculating projections

In this section we construct explicit degenerations of tangential projections to osculating projections. We begin by studying how the span of two osculating spaces degenerates in a flat family of linear spaces parametrized by \mathbb{P}^1 .

We recall that the Grassmannian $\mathbb{G}(r, n)$ is rationally connected by rational normal curves of degree $r + 1$. Indeed, if $p, q \in \mathbb{G}(r, n)$ are general points, corresponding to the r -planes $V_p, V_q \subseteq \mathbb{P}^n$, we may consider a rational normal scroll $X \subseteq \mathbb{P}^n$ of dimension $r + 1$ containing V_p and V_q . Then the r -planes of X correspond to the points of a degree $r + 1$ rational normal curve in $\mathbb{G}(r, n)$ joining p and q .

The first step consists in studying how the span of two osculating spaces at two general points $p, q \in \mathbb{G}(r, n)$ behaves when q approaches p along a degree $r + 1$ rational normal curve connecting p and q .

Proposition 5.1. *Let $p, q \in \mathbb{G}(r, n) \subseteq \mathbb{P}^N$ be general points, $k_1, k_2 \geq 0$ integers such that $k_1 + k_2 \leq r - 1$, and $\gamma : \mathbb{P}^1 \rightarrow \mathbb{G}(r, n)$ a degree $r + 1$ rational normal curve with $\gamma(0) = p$ and $\gamma(\infty) = q$. Let us consider the family of linear spaces*

$$T_t = \left\langle T_p^{k_1}, T_{\gamma(t)}^{k_2} \right\rangle \quad \text{with } t \in \mathbb{P}^1 \setminus \{0\}$$

parametrized by $\mathbb{P}^1 \setminus \{0\}$, and let T_0 be the flat limit of $\{T_t\}_{t \in \mathbb{P}^1 \setminus \{0\}}$ in $\mathbb{G}(\dim(T_t), N)$. Then $T_0 \subset T_p^{k_1+k_2+1}$.

Proof. We may assume that $p = e_{I_1}$ and $q = e_{I_2}$, see (4.2), and that $\gamma : \mathbb{P}^1 \rightarrow \mathbb{G}(r, n)$ is the rational normal curve given by

$$\gamma([t : s]) = (se_0 + te_{r+1}) \wedge \cdots \wedge (se_r + te_{2r+1}).$$

We can work on the affine chart $s = 1$ and set $t = (t : 1)$. Consider the points

$$e_0, \dots, e_n, e_0^t = e_0 + te_{r+1}, \dots, e_r^t = e_r + te_{2r+1}, e_{r+1}^t = e_{r+1}, \dots, e_n^t = e_n \in \mathbb{P}^n$$

and the corresponding points of \mathbb{P}^N

$$e_I = e_{i_0} \wedge \cdots \wedge e_{i_r}, e_I^t = e_{i_0}^t \wedge \cdots \wedge e_{i_r}^t \quad \text{for } I \in \Lambda.$$

By Proposition 3.3 we have

$$T_t = \langle e_I \mid d(I, I_1) \leq k_1; e_I^t \mid d(I, I_1) \leq k_2 \rangle \quad \text{for } t \neq 0$$

and

$$T_p^{k_1+k_2+1} = \langle e_I \mid d(I, I_1) \leq k_1 + k_2 + 1 \rangle = \{p_I = 0 \mid d(I, I_1) > k_1 + k_2 + 1\}.$$

Therefore, in order to prove that $T_0 \subset T_p^{k_1+k_2+1}$ it is enough to exhibit, for any index $I \in \Lambda$ with $d(I, I_1) > k_1 + k_2 + 1$, a hyperplane $H_I \subset \mathbb{P}^N$ of type

$$p_I + t \left(\sum_{J \in \Lambda, J \neq I} f(t)_{I,J} p_J \right) = 0$$

such that $T_t \subset H_I$ for $t \neq 0$, where $f(t)_{I,J} \in \mathbb{C}[t]$ are polynomials. Clearly, taking the limit for $t \mapsto 0$, this will imply that $T_0 \subseteq \{p_I = 0\}$.

In order to construct such a hyperplane we need to introduce some other definitions. We define

$$\Delta(I, l) := \{(I \setminus J) \cup (J + r + 1) \mid J \subset I \cap I_1, |J| = l\} \subset \Lambda$$

for any $I \in \Lambda$, $l \geq 0$, where $L + \lambda := \{i + \lambda; i \in L\}$ is the translation of the set L by the integer λ . Note that $\Delta(I, 0) = \{I\}$ and $\Delta(I, l) = \emptyset$ for l big enough. For any $l > 0$ set

$$\begin{aligned} \Delta(I, -l) &:= \{J \mid I \in \Delta(J, l)\} \subset \Lambda; \\ s_I^+ &:= \max_{l \geq 0} \{l \mid \Delta(I, l) \neq \emptyset\} \in \{0, \dots, r+1\}; \\ s_I^- &:= \max_{l \geq 0} \{l \mid \Delta(I, -l) \neq \emptyset\} \in \{0, \dots, r+1\}; \\ \Delta(I)^+ &:= \bigcup_{0 \leq l} \Delta(I, l) = \bigcup_{0 \leq l \leq s_I^+} \Delta(I, l); \\ \Delta(I)^- &:= \bigcup_{0 \leq l} \Delta(I, -l) = \bigcup_{0 \leq l \leq s_I^-} \Delta(I, -l). \end{aligned}$$

Note that $0 \leq s_I^- \leq d(I, I_1)$ and $0 \leq s_I^+ \leq r + 1 - d(I, I_1)$, and for any l we have $J \in \Delta(I, l) \Rightarrow d(J, I) = |l|$ and $d(J, I_1) = d(I, I_1) + l$ and $d(J, I_2) = d(I, I_2) - l$.

Now, we write the e_I^t 's with $d(I, I_1) < k_2$, in the basis e_J , $J \in \Lambda$. For any $I \in \Lambda$ we have

$$\begin{aligned} e_I^t &= e_I + t \sum_{J \in \Delta(I, 1)} (\text{sign}(J)e_J) + \dots + t^l \sum_{J \in \Delta(I, l)} (\text{sign}(J)e_J) + \dots + t^{s_I^+} \sum_{J \in \Delta(I, s_I^+)} (\text{sign}(J)e_J) \\ &= \sum_{l=0}^{s_I^+} \left(t^l \sum_{J \in \Delta(I, l)} \text{sign}(J)e_J \right) = \sum_{J \in \Delta(I)^+} \left(t^{d(I, J)} \text{sign}(J)e_J \right) \end{aligned}$$

where $\text{sign}(J) = \pm 1$. Note that $\text{sign}(J)$ depends on J but not on I , hence we may replace e_J by $\text{sign}(J)e_J$, and write

$$e_I^t = \sum_{J \in \Delta(I)^+} t^{d(I, J)} e_J.$$

Therefore, we have

$$T_t = \left\langle e_I \mid d(I, I_1) \leq k_1; \sum_{J \in \Delta(I)^+} \left(t^{d(I, J)} e_J \right) \mid d(I, I_1) \leq k_2 \right\rangle.$$

Next, we define

$$\Delta := \{I \mid d(I, I_1) \leq k_1\} \cup \left(\bigcup_{d(I, I_1) \leq k_2} \Delta(I)^+ \right) \subset \Lambda.$$

Let $I \in \Lambda$ be an index with $d(I, I_1) > k_1 + k_2 + 1$. If $I \notin \Delta$ then $T_t \subset \{p_I = 0\}$ for any $t \neq 0$ and we are done.

Now, assume that $I \in \Delta$. For any e_K^t with non-zero Plücker coordinate p_I we have $I \in \Delta(K)^+$, that is $K \in \Delta(I)^-$. Now, we want to find a hyperplane H_I of type

$$F_I = \sum_{J \in \Delta(I)^-} t^{d(I, J)} c_J p_J = 0 \quad (5.1)$$

where $c_J \in \mathbb{C}$ with $c_I \neq 0$, and such that $T_t \subset H_I$ for $t \neq 0$. Note that then we can divide the equation by c_I , and get a hyperplane H_I of the required type:

$$p_I + \frac{t}{c_I} \left(\sum_{J \in \Delta(I)^-, J \neq I} t^{d(J, I) - 1} c_J p_J \right) = 0.$$

In the following we will write $s_I^- = s$ for short. Since

$$|\Delta(I)^-| = \sum_{l=0}^s |\Delta(I, -l)| = 1 + s + \binom{s}{2} + \cdots + \binom{s}{s-1} + 1 = 2^s$$

in equation (5.1) there are 2^s variables c_J . Now, we want to understand what conditions we get by requiring $T_t \subseteq \{F_I = 0\}$ for $t \neq 0$.

Given $K \in \Delta(I)^-$ we have $s_K^+ \geq d(I, K)$ and

$$\begin{aligned} F_I(e_K^t) &= F \left(\sum_{L \in \Delta(K)^+} (t^{d(K,L)} e_L) \right) = F \left(\sum_{l=0}^{s_K^+} \left(t^l \sum_{L \in \Delta(K,l)} e_L \right) \right) \\ &= F \left(\sum_{l=0}^{d(I,K)} \left(t^l \sum_{L \in \Delta(K,l)} e_L \right) \right) \\ &\stackrel{(5.1)}{=} \sum_{J \in \Delta(I)^- \cap \Delta(K)^+} t^{d(I,K)-d(J,K)} c_J \left(t^{d(J,K)} \right) = t^{d(I,K)} \left[\sum_{J \in \Delta(I)^- \cap \Delta(K)^+} c_J \right] \end{aligned}$$

that is

$$F_I(e_K^t) = 0 \quad \text{for all } t \neq 0 \Leftrightarrow \sum_{J \in \Delta(I)^- \cap \Delta(K)^+} c_J = 0.$$

Note that this is a linear condition on the coefficients c_J , with $J \in \Delta(I)^-$. Therefore,

$$\begin{aligned} T_t \subset \{F_I = 0\} \text{ for } t \neq 0 \\ \Leftrightarrow \begin{cases} F_I(e_L) = 0 & \text{for all } L \in \Delta(I)^- \cap B[I_1, k_1] \\ F_I(e_K^t) = 0 \forall t \neq 0 & \text{for all } K \in \Delta(I)^- \cap B[I_1, k_2] \end{cases} \quad (5.2) \\ \Leftrightarrow \begin{cases} c_L = 0 & \text{for all } L \in \Delta(I)^- \cap B[I_1, k_1] \\ \sum_{J \in \Delta(I)^- \cap \Delta(K)^+} c_J = 0 & \text{for all } K \in \Delta(I)^- \cap B[I_1, k_2] \end{cases} \end{aligned}$$

where $B[J, u] := \{K \in \Lambda \mid d(J, K) \leq u\}$. The number of conditions on the c_J 's, $J \in \Delta(I)^-$ is then

$$c := |\Delta(I)^- \cap B[I_1, k_1]| + |\Delta(I)^- \cap B[I_1, k_2]|.$$

The problem is now reduced to find a solution of the linear system given by the c equations (5.2) in the 2^s variables c_J 's say $J \in \Delta(I)^-$ such that $c_I \neq 0$. Therefore,

it is enough to find $s + 1$ complex numbers $c_I = c_0 \neq 0, c_1, \dots, c_s$ satisfying the following conditions

$$\begin{cases} c_j = 0 & \text{for all } j = s, \dots, d - k_1 \\ \sum_{l=0}^{d(I,K)} |\Delta(I)^- \cap \Delta(K, l)| c_{d(I,K)-l} = 0 & \text{for all } K \in \Delta(I)^- \cap B[I_1, k_2] \end{cases} \quad (5.3)$$

where $d = d(I, I_1) > k_1 + k_2 + 1$. Note that (5.3) can be written as

$$\begin{cases} c_j = 0 & \text{for all } j = s, \dots, d - k_1 \\ \sum_{k=0}^j \binom{j}{j-k} c_k = 0 & \text{for all } j = s, \dots, d - k_2 \end{cases}$$

that is

$$\begin{cases} c_s = 0 \\ \vdots \\ c_{d-k_1} = 0 \\ \binom{s}{0} c_s + \binom{s}{1} c_{s-1} + \dots + \binom{s}{s-1} c_1 + \binom{s}{s} c_0 = 0 \\ \vdots \\ \binom{d-k_2}{0} c_{d-k_2} + \binom{d-k_2}{1} c_{d-k_2-1} + \dots + \binom{d-k_2}{d-k_2-1} c_1 + \binom{d-k_2}{d-k_2} c_0 = 0. \end{cases} \quad (5.4)$$

Now, it is enough to show that the linear system (5.4) admits a solution with $c_0 \neq 0$. If $s < d - k_2$, the system (5.4) reduces to $c_s = \dots = c_{d-k_1} = 0$. In this case we may take $c_0 = 1, c_1 = \dots, c_s = 0$. Note that $d - k_1 > k_2 + 1 \geq 1$ and we can use the hyperplane $p_I = 0$.

From now on assume that $s \geq d - k_2$. Since $c_s = \dots = c_{d-k_1} = 0$ we may consider the second set of conditions in (5.4) and translate the problem into checking that the system (5.5) admits a solution involving the variables $c_0, c_1, \dots, c_{d-k_1+1}$ with $c_0 \neq 0$. Note that (5.4) takes the following form:

$$\begin{cases} \binom{s}{s-(d-k_1+1)} c_{d-k_1+1} + \binom{s}{s-(d-k_1)} c_{d-k_1} + \dots + \binom{s}{s-1} c_1 + \binom{s}{s} c_0 = 0 \\ \vdots \\ \binom{d-k_2}{k_1-1-k_2} c_{d-k_1+1} + \binom{d-k_2}{k_1-k_2} c_{d-k_1} + \dots + \binom{d-k_2}{d-k_2-1} c_1 + \binom{d-k_2}{d-k_2} c_0 = 0. \end{cases} \quad (5.5)$$

Therefore, it is enough to check that the $(s - d + k_2 + 1) \times (d - k_1 + 1)$ matrix

$$M = \begin{pmatrix} \binom{s}{s-(d-k_1+1)} & \binom{s}{s-(d-k_1)} & \dots & \binom{s}{s-1} \\ \vdots & \vdots & & \vdots \\ \binom{d-k_2}{k_1-1-k_2} & \binom{d-k_2}{k_1-k_2} & \dots & \binom{d-k_2}{d-k_2-1} \end{pmatrix} \quad (5.6)$$

has maximal rank. Note that $s \leq d$ and $d > k_1 + k_2 + 1$ yield $s - d + k_2 + 1 < d - k_1 + 1$. Then it is enough to show that the $(s - d + k_2 + 1) \times (s - d + k_2 + 1)$ submatrix

$$\begin{aligned} M' &= \begin{pmatrix} \binom{s}{s-(s-d+k_2+1)} & \binom{s}{s-(s-d+k_2)} & \cdots & \binom{s}{s-1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{d-k_2}{d-k_2-(s-d+k_2+1)} & \binom{d-k_2}{d-k_2-(s-d+k_2)} & \cdots & \binom{d-k_2}{d-k_2-1} \end{pmatrix} \\ &= \begin{pmatrix} \binom{s}{s+1-d+k_2} & \cdots & \binom{s}{1} \\ \vdots & \ddots & \vdots \\ \binom{d-k_2}{s+1-d+k_2} & \cdots & \binom{d-k_2}{1} \end{pmatrix} \end{aligned}$$

has non-zero determinant. Since the determinant of M' is equal to the determinant of the matrix of binomial coefficients

$$M'' := \left(\binom{i}{j} \right)_{\substack{d-k_2 \leq i \leq s \\ 1 \leq j \leq s+1-d+k_2}},$$

it is enough to observe that since $d - k_2 > k_1 + 1 \geq 1$ by [35, Corollary 2] we have $\det(M') = \det(M'') \neq 0$. \square

Essentially, Proposition 5.1 says that two general osculating spaces of $\mathbb{G}(r, n)$ behave well under degenerations. We formalize this concept as follows.

Assumption 5.2. Let $X \subset \mathbb{P}^N$ be an irreducible projective variety, $p, q \in X$ be general points, and $k_1, k_2 \geq 0$ integers. We will assume that there exists a smooth curve $\gamma : C \rightarrow X$, with $\gamma(t_0) = p$ and $\gamma(t_\infty) = q$ such that the flat limit T_{t_0} in $\mathbb{G}(\dim(T_t), N)$ of the family of linear spaces

$$T_t = \langle T_p^{k_1}, T_{\gamma(t)}^{k_2} \rangle, \quad t \in C \setminus \{t_0\}$$

parametrized by $C \setminus \{t_0\}$, is contained in $T_p^{k_1+k_2+1}$.

For our applications to Grassmannians we will always choose $C \cong \mathbb{P}^1$. Moreover, we would like to stress that there exist varieties, with small higher order osculating spaces, not satisfying Assumption 5.2.

Example 5.3. Let us consider the tangent developable $Y_n \subseteq \mathbb{P}^n$ of a degree n rational normal curve $C_n \subseteq \mathbb{P}^n$ as in Proposition 3.2.

Note that two general points $p = \phi(t_1, u_1)$ and $q = \phi(t_2, u_2)$ in Y_n can be joined by a smooth rational curve. Indeed, we may consider the curve

$$\begin{aligned} \xi(t) &= (t_1 + t(t_2 - t_1) + u_1 + t(u_2 - u_1), \dots, (t_1 + t(t_2 - t_1))^n \\ &\quad + n(t_1 + t(t_2 - t_1))^{n-1}(u_1 + t(u_2 - u_1))). \end{aligned}$$

Now, let $\gamma : C \rightarrow Y_n$ be a smooth curve with $\gamma(t_0) = p$ and $\gamma(t_\infty) = q$, and let T_{t_0} be the flat limit of the family of linear spaces

$$T_t = \langle T_p, T_{\gamma(t)} \rangle \quad \text{with } t \in C \setminus \{t_0\}.$$

Now, one can prove that if $n \geq 5$ then $T_p Y_n \cap T_q Y_n = \emptyset$ by a straightforward computation, or alternatively by noticing that by [7] Y_n is not 2-secant defective, and then by Terracini's lemma [53, Theorem 1.3.1] $T_p Y_n \cap T_q Y_n = \emptyset$. Now, $T_p Y_n \cap T_q Y_n = \emptyset$ implies that $\dim(T_t) = 5$ for any $t \in C$. On the other hand, by Proposition 3.2 we have $\dim(T_p^3 Y_n) = 4$. Hence, $T_{t_0} \not\subset T_p^3 Y_n$ as soon as $n \geq 5$.

Now we are ready to prove a stronger version of Proposition 5.1.

Proposition 5.4. *Let $p_1, \dots, p_\alpha \in \mathbb{G}(r, n) \subseteq \mathbb{P}^N$ be general points with $\alpha = \lfloor \frac{n+1}{r+1} \rfloor$ and $k \leq (r-1)/2$ a non-negative integer, and $\gamma_j : \mathbb{P}^1 \rightarrow \mathbb{G}(r, n)$ a degree $r+1$ rational normal curve with $\gamma_j(0) = p_1$ and $\gamma_j(\infty) = p_j$, for every $j = 2, \dots, \alpha$. Let us consider the family of linear spaces*

$$T_t = \langle T_{p_1}^k, T_{\gamma_2(t)}^k, \dots, T_{\gamma_\alpha(t)}^k \rangle \quad \text{for } t \in \mathbb{P}^1 \setminus \{0\}$$

parametrized by $\mathbb{P}^1 \setminus \{0\}$, and let T_0 be the flat limit of $\{T_t\}_{t \in \mathbb{P}^1 \setminus \{0\}}$ in $\mathbb{G}(\dim(T_t), N)$. Then $T_0 \subset T_p^{2k+1}$.

Proof. If $\alpha = 2$ it follows from the Proposition 5.1. Therefore, we may assume that $\alpha \geq 3$, $p_j = e_{I_j}$ (4.2) and that $\gamma_j : \mathbb{P}^1 \rightarrow \mathbb{P}^N$ is the rational curve given by

$$\gamma_j([t : s]) = (s e_0 + t e_{(r+1)(j-1)}) \wedge \cdots \wedge (s e_r + t e_{(r+1)j-1}).$$

We can work on the affine chart $s = 1$ and set $t = (t : 1)$. Consider the points

$$\begin{aligned} e_0, \dots, e_n, e_0^{j,t} &= e_0 + t e_{(r+1)(j-1)}, \dots, e_r^{j,t} = e_r + t e_{(r+1)j-1}, e_{r+1}^{j,t} \\ &= e_{r+1}, \dots, e_n^{j,t} = e_n \in \mathbb{P}^n \end{aligned}$$

and the corresponding points in \mathbb{P}^N

$$e_I = e_{i_0} \wedge \cdots \wedge e_{i_r}, e_I^{j,t} = e_{i_0}^{j,t} \wedge \cdots \wedge e_{i_r}^{j,t}, I = \{i_0, \dots, i_r\} \in \Lambda,$$

for $j = 2, \dots, \alpha$. By Proposition 3.3 we have

$$T_t = \left\langle e_I \mid d(I, I_1) \leq k; e_I^{j,t} \mid d(I, I_1) \leq k, j = 2, \dots, \alpha \right\rangle \quad \text{for } t \neq 0$$

and

$$T_{p_0}^{2k+1} = \langle e_I \mid d(I, I_1) \leq 2k+1 \rangle = \{p_I = 0 \mid d(I, I_1) > 2k+1\}.$$

Therefore, as in Proposition 5.1, in order to prove that $T_0 \subset T_p^{2k+1}$ it is enough to exhibit, for any index $I \in \Lambda$ with $d(I, I_1) > 2k + 1$, a hyperplane $H_I \subset \mathbb{P}^N$ of type

$$p_I + t \left(\sum_{J \in \Lambda, J \neq I} f(t)_{I,J} p_J \right) = 0$$

such that $T_t \subset H_I$ for $t \neq 0$, where $f(t)_{I,J} \in \mathbb{C}[t]$ are polynomials. The first part of the proof goes as in the proof of Proposition 5.1. Given $I \in \Lambda$ we define

$$\Delta(I, l)_j := \{(I \setminus J) \cup (J + (j-1)(r+1)) \mid J \subset I \cap I_1, |J| = l\} \subset \Lambda$$

for any $I \in \Lambda, l \geq 0, j = 2, \dots, \alpha$, where $L + \lambda := \{i + \lambda; i \in L\}$ is the translation of the set L by the integer λ . Note that $\Delta(I, 0)_j = \{I\}$ and $\Delta(I, l)_j = \emptyset$ for l big enough. For any $l > 0$ set

$$\begin{aligned} \Delta(I, -l)_j &:= \{J \mid I \in \Delta(J, l)_j\} \subset \Lambda; \\ s(I)_j^+ &:= \max_{l \geq 0} \{\Delta(I, l)_j \neq \emptyset\} \in \{0, \dots, r+1\}; \\ s(I)_j^- &:= \max_{l \geq 0} \{\Delta(I, -l)_j \neq \emptyset\} \in \{0, \dots, r+1\}; \\ \Delta(I)_j^+ &:= \bigcup_{0 \leq l} \Delta(I, l)_j = \bigcup_{0 \leq l \leq s(I)_j^+} \Delta(I, l)_j; \\ \Delta(I)_j^- &:= \bigcup_{0 \leq l} \Delta(I, -l)_j = \bigcup_{0 \leq l \leq s(I)_j^-} \Delta(I, -l)_j. \end{aligned}$$

Note that $0 \leq s(I)_j^- \leq d(I, I_1), 0 \leq s(I)_j^+ \leq r+1 - d(I, I_1)$, and for any l we have

$$J \in \Delta(I, l)_j \Rightarrow d(J, I) = |l| \text{ and } d(J, I_1) = d(I, I_1) + l \text{ and } d(J, I_j) = d(I, I_j) - l.$$

Now, we write $e_I^{j,t}$ and $d(I, I_1) < k$, in the basis e_J with $J \in \Lambda$. For any $I \in \Lambda$ we have

$$e_I^{j,t} = \sum_{J \in \Delta(I)_j^+} \left(t^{d(I,J)} \text{sign}(J) e_J \right)$$

where $\text{sign}(J) = \pm 1$. Since $\text{sign}(J)$ does depend on J but not on I we can replace e_J by $\text{sign}(J)e_J$. Then, we may write

$$e_I^t = \sum_{J \in \Delta(I)_j^+} \left(t^{d(I,J)} e_J \right),$$

and

$$T_t = \left\langle e_I \mid d(I, I_1) \leq k; \sum_{J \in \Delta(I)_j^+} \left(t^{d(I, J)} e_J \right) \mid d(I, I_1) \leq k, 2 \leq j \leq \alpha \right\rangle.$$

Next, we define

$$\Delta := \{I \mid d(I, I_1) \leq k\} \cup \left(\bigcup_{2 \leq j \leq \alpha} \bigcup_{d(I, I_1) \leq k} \Delta(I)_j^+ \right) \subset \Lambda.$$

Let $I \in \Lambda$ be an index with $d(I, I_1) =: D > 2k + 1$. If $I \notin \Delta$ then $T_t \subset \{p_I = 0\}$ for any $t \neq 0$ and we are done. Now, assume that $I \in \Delta$, and $I \in \Delta(K_1, l_1)_2 \cap \Delta(K_2, l_2)_3$ with

$$d(K_1, I_1), d(K_2, I_1) \leq k.$$

Consider the following sets

$$\begin{aligned} I^0 &:= I \cap I_1 \\ I^1 &:= I \cap (K_1 + (r + 1)) \subset I_2 \\ I^2 &:= I \cap (K_2 + 2(r + 1)) \subset I_3 \\ I^3 &:= I \setminus (I^0 \cup I^1 \cup I^2). \end{aligned}$$

Then $|I^1| = l_1$ and $|I^2| = l_2$. Set $u := |I^3|$, then

$$d(I, I_1) = l_1 + l_2 + u \leq l_1 + l_2 + 2u = d(K_1, I_1) + d(K_2, I_1) \leq 2k$$

contradicting $d(I, I_1) > 2k + 1$. Therefore, there is a unique j such that

$$I \in \bigcup_{d(J, I_1) \leq k} \Delta(J)_j^+.$$

Note that $\Delta(I, -s(I)_j^-)$ has only one element, say I' . Then

$$k + 1 - D + s(I)_j^- = k + 1 - d(I, I_1) + d(I, I') = k + 1 - d(I_1, I') > 0.$$

Now, consider the set of indexes

$$\Gamma := \{I\} \cup \Delta(I, -1)_j \cup \dots \cup \Delta\left(I, -\left(k+1-D+s(I)_j^-\right)\right)_j = \bigcup_{l=0}^{k+1-D+s(I)_j^-} \Delta(I, -l)_j \subset \Lambda.$$

Our aim now is to find a hyperplane of the form

$$H_I = \left\{ \sum_{J \in \Gamma} t^{d(I, J)} c_J p_J = 0 \right\} \quad (5.7)$$

such that $T_l \subset H_l$ and $c_l \neq 0$. First, we claim that

$$J \in \Gamma \Rightarrow J \notin \bigcup_{\substack{2 \leq i \leq \alpha \\ i \neq j}} \bigcup_{d(I, I_1) \leq k} \Delta(I)_i^+. \quad (5.8)$$

Indeed, assume that $J \in \Delta(I, -l)_j \cap \Delta(K, m)_i$ for some $K \in \Lambda$ with

$$d(K, I_1) \leq k \text{ and } i \neq j \text{ with } 0 \leq l \leq k + 1 - D + s(I)_j^-, m \geq 0.$$

Since $J \in \Delta(I, -l)_j$ then

$$|J \cap I_j| = |I \cap I_j| - l \geq s(I)_j^- - l \geq D - (k + 1) > k.$$

On other hand, since $J \in \Delta(K, m)_i$ with $j \neq i$ we have

$$|J \cap I_j| = |K \cap I_j| \leq d(K, I_1) \leq k.$$

A contradiction. Now, (5.8) yields that the hyperplane H_l given by (5.7) is such that

$$\left\langle e_l |d(I, I_1) \leq k; \sum_{J \in \Delta(I)_i^+} t^{d(I, J)} c_{(I, J)} e_J |d(I, I_1) \leq k, i = 2, \dots, \alpha, i \neq j \right\rangle \subset H_l, t \neq 0.$$

Therefore

$$T_l \subset H_l \text{ for } t \neq 0 \iff \left\langle \sum_{J \in \Delta(I)_j^+} t^{d(I, J)} e_J |d(I, I_1) \leq k \right\rangle \subset H_l \text{ for } t \neq 0.$$

Now, arguing as in the proof of Proposition 5.1 we obtain

$$T_l \subset H_l \text{ for } t \neq 0 \iff \sum_{J \in \Delta(K)_j^+ \cap \Gamma} c_J = 0 \text{ for all } K \in \Delta(I)_j^- \cap B[I_1, k] \quad (5.9)$$

and the problem is now reduced to find a solution of the linear system given by the $|\Delta(I)_j^- \cap B[I_1, k]|$ equations (5.9) in the $|\Delta(K)_j^+ \cap \Gamma|$ variables $c_J, J \in \Delta(K)_j^+ \cap \Gamma$, such that $c_l \neq 0$. We set $c_J = c_{d(I, J)}$ and, as in the proof of Proposition 5.1, we consider the linear system

$$\sum_{l=0}^{k+1-D+s(I)_j^-} \binom{D-i}{D-l-i} c_l = 0 \text{ for all } i = D - s(I)_j^-, \dots, k \quad (5.10)$$

with $k + 2 - D + s(I)_j^-$ variables $c_0, \dots, c_{k+1-D+s(I)_j^-}$ and $k + 1 - D + s(I)_j^-$ equations, where $D = d(I, I_1)$. Finally, arguing exactly as in the last part of the proof of Proposition 5.1 we have that (5.10) admits a solution with $c_0 \neq 0$. \square

We conclude this section with the definition of m -osculating regularity which essentially will be a measure of how many general osculating spaces of order k we can degenerate to an osculating space of order $2k + 1$.

Definition 5.5. Let $X \subset \mathbb{P}^N$ be an irreducible projective variety. We say that X has m -osculating regularity if given $p_1, \dots, p_m \in X$ general points, and an integer $k \geq 0$, there exist smooth curves $\gamma_j : C \rightarrow X$ with $\gamma_j(t_0) = p_1$ and $\gamma_j(t_\infty) = p_j$ for $j = 2, \dots, m$ such that the family of linear spaces

$$T_t = \left\langle T_{p_1}^k, T_{\gamma_2(t)}^k, \dots, T_{\gamma_m(t)}^k \right\rangle \quad \text{for } t \in C \setminus \{t_0\}$$

parametrized by $C \setminus \{t_0\}$ has flat limit T_{t_0} contained in T_p^{2k+1} .

Note that by Proposition 5.4 the Grassmannian $\mathbb{G}(r, n)$ has α -osculating regularity, where $\alpha = \lfloor \frac{n+1}{r+1} \rfloor$.

5.1. Limit linear systems

Let $X \subset \mathbb{P}^N$ be an irreducible rational variety of dimension n , $p_1, \dots, p_m \in X$ general points. We reinterpret the notion of m -osculating regularity in Definition 5.5 in terms of limit linear systems and collisions of fat points.

Let $\mathcal{H} \subseteq |\mathcal{O}_{\mathbb{P}^n}(d)|$ be the sublinear system of $|\mathcal{O}_{\mathbb{P}^n}(d)|$ inducing the birational map $i_{\mathcal{H}} : \mathbb{P}^n \dashrightarrow X \subset \mathbb{P}^N$, and $q_i = i_{\mathcal{H}}^{-1}(p_i)$.

Then X has m -osculating regularity if and only if there exists smooth curves $\gamma_i : C \rightarrow \mathbb{P}^n$, $i = 2, \dots, m$, with $\gamma_i(t_0) = q_1$ and $\gamma_i(t_\infty) = q_i$ for $i = 1, \dots, m$, such that the limit linear system \mathcal{H}_{t_0} of the family of linear systems \mathcal{H}_t given by the hypersurfaces in \mathcal{H} having at least multiplicity $s + 1$ at $q_1, \gamma_2(t), \dots, \gamma_m(t)$ contains the linear system $\mathcal{H}_{q_1}^{2s+2}$ of degree d hypersurfaces with multiplicity at least $2s + 2$ at q_1 .

Indeed, if $p_i = i_{\mathcal{H}}(q_i)$ for $i = 1, \dots, m$ then the linear system of hyperplanes in \mathbb{P}^N containing

$$T_t = \left\langle T_{p_1}^s, T_{i_{\mathcal{H}}(\gamma_2(t))}^s, \dots, T_{i_{\mathcal{H}}(\gamma_m(t))}^s \right\rangle$$

corresponds to the linear system \mathcal{H}_t . Similarly, the linear system of hyperplanes in \mathbb{P}^N containing $T_{p_1}^{2s+1}$ corresponds to the linear system $\mathcal{H}_{q_1}^{2s+2}$.

Therefore, the problem of computing the m -osculating regularity of a rational variety can be translated in terms of limit linear systems in \mathbb{P}^n given by colliding a number of fat points. This is a very hard and widely studied subject [21–23, 50].

5.2. Degenerating rational maps

In order to study the fibers of general tangential projections via osculating projections we need to understand how the fibers of rational maps behave under specialization. We refer to [36] for the general theory of rational maps relative to a base scheme.

Proposition 5.6. *Let C be a smooth and irreducible curve, $X \rightarrow C$ an integral scheme flat over C , and $\phi : X \dashrightarrow \mathbb{P}^n_C$ be a rational map of schemes over C . Let $d_0 = \dim(\overline{\phi|_{X_{t_0}}(X_{t_0})})$ with $t_0 \in C$. Then for $t \in C$ general we have $\dim(\overline{\phi|_{X_t}(X_t)}) \geq d_0$.*

In particular, if there exists $t_0 \in C$ such that $\phi|_{X_{t_0}} : X_{t_0} \dashrightarrow \mathbb{P}^n$ is generically finite, then for a general $t \in C$ the rational map $\phi|_{X_t} : X_t \dashrightarrow \mathbb{P}^n$ is generically finite as well.

Proof. Let us consider the closure $Y = \overline{\phi(X)} \subseteq \mathbb{P}^n_C$ of the image of X through ϕ . By taking the restriction $\pi|_Y : Y \rightarrow C$ of the projection $\pi : \mathbb{P}^n_C \rightarrow C$ we see that Y is a scheme over C .

Note that since Y is an irreducible and reduced scheme over the curve C we have that Y is flat over C . In particular, the dimension of the fibers $\pi|_Y^{-1}(t) = Y_t$ is a constant $d = \dim(Y_t)$ for any $t \in C$.

For $t \in C$ general the fiber $\pi|_Y^{-1}(t) = Y_t$ contains $\phi|_{X_t}(X_t)$ as a dense subset. Therefore, we have $d = \dim(\phi|_{X_t}(X_t)) \leq \dim(X_t)$ for $t \in C$ general.

Then, since $\phi|_{X_{t_0}}(X_{t_0}) \subseteq Y_{t_0}$ we have $\dim(\overline{\phi|_{X_{t_0}}(X_{t_0})}) \leq d = \dim(\overline{\phi|_{X_t}(X_t)})$ for $t \in C$ general.

Now, assume that $\dim(X_{t_0}) = \dim(\phi|_{X_{t_0}}(X_{t_0})) \leq d$. Therefore, we get

$$\dim(X_{t_0}) \leq d \leq \dim(X_t) = \dim(X_{t_0})$$

that yields $d = \dim(X_{t_0}) = \dim(X_t)$ for any $t \in C$. Hence, for a general $t \in C$ we have

$$\dim(X_t) = \dim(\overline{\phi|_{X_t}(X_t)})$$

that is $\phi|_{X_t} : X_t \dashrightarrow \overline{\phi|_{X_t}(X_t)} \subseteq \mathbb{P}^n$ is generically finite. \square

Now, let C be a smooth and irreducible curve, $X \subset \mathbb{P}^N$ an irreducible and reduced projective variety, and $f : \Lambda \rightarrow C$ a family of k -dimensional linear subspaces of \mathbb{P}^N parametrized by C .

Let us consider the invertible sheaf $\mathcal{O}_{\mathbb{P}^n \times C}(1)$, and the sublinear system $|\mathcal{H}_\Lambda| \subseteq |\mathcal{O}_{\mathbb{P}^n \times C}(1)|$ given by the sections of $\mathcal{O}_{\mathbb{P}^n \times C}(1)$ vanishing on $\Lambda \subset \mathbb{P}^n \times C$. We denote by $\pi_{\Lambda|X \times C}$ the restriction of the rational map $\pi_\Lambda : \mathbb{P}^n \times C \dashrightarrow \mathbb{P}^{n-k-1} \times C$ of schemes over C induced by $|\mathcal{H}_\Lambda|$.

Furthermore, for any $t \in C$ we denote by $\Lambda_t \cong \mathbb{P}^k$ the fiber $f^{-1}(t)$, and by $\pi_{\Lambda_t|X}$ the restriction to X of the linear projection $\pi_{\Lambda_t} : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-k-1}$ with center Λ_t .

Proposition 5.7. *Let $d_0 = \dim(\overline{\pi_{\Lambda_{t_0}|X}(X)})$ for $t_0 \in C$. Then*

$$\dim(\overline{\pi_{\Lambda_t|X}(X)}) \geq d_0$$

for $t \in C$ general.

Furthermore, if there exists $t_0 \in C$ such that $\pi_{\Lambda_{t_0}|X} : X \dashrightarrow \mathbb{P}^{n-k-1}$ is generically finite then $\pi_{\Lambda_t|X} : X \dashrightarrow \mathbb{P}^{n-k-1}$ is generically finite for $t \in C$ general.

Proof. The rational map $\pi_{\Lambda|X \times C} : X \times C \dashrightarrow \mathbb{P}^{n-k-1} \times C$ of schemes over C is just the restriction of the relative linear projection $\pi_{\Lambda} : \mathbb{P}^n \times C \dashrightarrow \mathbb{P}^{n-k-1} \times C$ with center Λ .

Therefore, the restriction of $\pi_{\Lambda|X \times C}$ to the fiber $X_t \cong X$ of $X \times C$ over $t \in C$ induces the linear projection from the linear subspace Λ_t , that is

$$\pi_{\Lambda|X_t} = \pi_{\Lambda_t|X}$$

for any $t \in C$. Now, to conclude it is enough to apply Proposition 5.6 with $\phi = \pi_{\Lambda|X \times C}$. \square

Essentially, Propositions 5.6 and 5.7 say that the dimension of the general fiber of the special map is greater or equal than the dimension of the general fiber of the general map. Therefore, when the special map is generically finite the general one is generically finite as well. We would like to stress that in this case, under suitable assumptions, [5, Lemma 5.4] says that the degree of the map can only decrease under specialization.

6. Non secant defectivity via osculating projections

In this section we use the techniques developed in Section 5 to study the dimension of secant varieties of Grassmannians. Our first step consists in reinterpreting Proposition 2.2 in terms of osculating projections. In order to do this, we need to describe how many tangent spaces we can take in such a way that the flat limit of the span of them is contained in a higher order osculating space.

First, given an irreducible projective variety satisfying Assumption 5.2 and having m -osculating regularity, we introduce a function $h_m : \mathbb{N}_{\geq 0} \rightarrow \mathbb{N}_{\geq 0}$ counting how many tangent spaces we can degenerate to a higher order osculating space.

Definition 6.1. Given an integer $m \geq 2$ we define a function

$$h_m : \mathbb{N}_{\geq 0} \rightarrow \mathbb{N}_{\geq 0}$$

as follows: $h_m(0) = 0$. For any $k \geq 1$ write

$$k + 1 = 2^{\lambda_1} + 2^{\lambda_2} + \dots + 2^{\lambda_l} + \varepsilon$$

where $\lambda_1 > \lambda_2 > \dots > \lambda_l \geq 1$ and $\varepsilon \in \{0, 1\}$, and define

$$h_m(k) := m^{\lambda_1-1} + m^{\lambda_2-1} + \dots + m^{\lambda_l-1}.$$

In particular $h_m(2k) = h_m(2k - 1)$ and $h_2(k) = \left\lfloor \frac{k+1}{2} \right\rfloor$.

Theorem 6.2. *Let $X \subset \mathbb{P}^N$ be an irreducible projective variety satisfying Assumption 5.2 and having m -osculating regularity. Assume that there exist integers $l, k_1, \dots, k_l \geq 1$ and general points $p_1, \dots, p_l \in X$ such that $\Pi_{T_{p_1, \dots, p_l}^{k_1, \dots, k_l}}$ is generically finite, and set*

$$h := \sum_{j=1}^l h_m(k_j).$$

Then X is not $(h + 1)$ -defective.

Proof. Let us consider a general tangential projection Π_T where

$$T = \left\langle T_{p_1^1}^1, \dots, T_{p_1^{h_m(k_1)}}^1, \dots, T_{p_l^1}^1, \dots, T_{p_l^{h_m(k_l)}}^1 \right\rangle$$

and $p_1^1 = p_1, \dots, p_l^1 = p_l$. Our argument consists in specializing the projection Π_T several times in order to reach a generically finite projection. For seek of notational simplicity along the proof we will assume $l = 1$. For the general case it is enough to apply the same argument l times.

Let us begin with the case $k_1 + 1 = 2^\lambda$. Then $h_m(k_1) = m^{\lambda-1}$. Since X has m -osculating regularity we can degenerate Π_T , in a family parametrized by a smooth curve, to a projection Π_{U_1} whose center U_1 is contained in

$$V_1 = \left\langle T_{p_1^1}^3, T_{p_1^{m+1}}^3, \dots, T_{p_1^{m^{\lambda-1}-m+1}}^3 \right\rangle.$$

Again, since X has m -osculating regularity we may specialize, in a family parametrized by a smooth curve, the projection Π_{V_1} to a projection Π_{U_2} whose center U_2 is contained in

$$V_2 = \left\langle T_{p_1^1}^7, T_{p_1^{m^2+1}}^7, \dots, T_{p_1^{m^{\lambda-1}-m^2+1}}^7 \right\rangle.$$

Proceeding recursively in this way in last step we get a projection $\Pi_{U_{\lambda-1}}$ whose center $U_{\lambda-1}$ is contained in

$$V_{\lambda-1} = T_{p_1^1}^{2^\lambda-1}.$$

When $k_1 + 1 = 2^\lambda$ our hypothesis means that $\Pi_{T_{p_1^1}^{k_1}}$ is generically finite. Therefore, $\Pi_{U_{\lambda-1}}$ is generically finite, and applying Proposition 5.7 recursively to the specializations in between Π_T and $\Pi_{U_{\lambda-1}}$ we conclude that Π_T is generically finite as well.

Now, more generally, let us assume that

$$k_1 + 1 = 2^{\lambda_1} + \dots + 2^{\lambda_s} + \varepsilon$$

with $\varepsilon \in \{0, 1\}$, and $\lambda_1 > \lambda_2 > \dots > \lambda_s \geq 1$. Then

$$h_m(k_1) = m^{\lambda_1-1} + \dots + m^{\lambda_s-1}.$$

By applying s times the argument for $k_1 + 1 = 2^{\lambda_1}$ in the first part of the proof we may specialize Π_T to a projection Π_U whose center U is contained in

$$V = \left\langle T_{p_1^1}^{2^{\lambda_1}-1}, T_{p_1^{m^{\lambda_1}-1+1}}^{2^{\lambda_2}-1}, \dots, T_{p_1^{m^{\lambda_1}-1+\dots+m^{\lambda_s-1}-1+1}}^{2^{\lambda_s}-1} \right\rangle.$$

Finally, we use Assumption 5.2 $s - 1$ times to specialize Π_V to a projection $\Pi_{U'}$ whose center U' is contained in

$$V' = T_{p_1^1}^{2^{\lambda_1}+\dots+2^{\lambda_s}-1}.$$

Note that $T_{p_1^1}^{2^{\lambda_1}+\dots+2^{\lambda_s}-1} = T_{p_1^1}^{k_1}$ if $\varepsilon = 0$, and $T_{p_1^1}^{2^{\lambda_1}+\dots+2^{\lambda_s}-1} = T_{p_1^1}^{k_1-1} \subset T_{p_1^1}^{k_1}$ if $\varepsilon = 1$. In any case, since by hypothesis $\Pi_{T_{p_1^1}^{k_1}}$ is generically finite, again by applying Proposition 5.7 recursively to the specializations in between Π_T and $\Pi_{U'}$ we conclude that Π_T is generically finite. Therefore, by Proposition 2.2 we get that X is not $(\sum_{j=1}^l h_m(k_j) + 1)$ -defective. \square

Now, we are ready to prove our main result on non-defectivity of Grassmannians.

Theorem 6.3. *Assume that $r \geq 2$, set*

$$\alpha := \left\lfloor \frac{n+1}{r+1} \right\rfloor$$

and let h_α be as in Definition 6.1. If either of the following conditions holds

- $n \geq r^2 + 3r + 1$ and $h \leq \alpha h_\alpha(r-1)$, or
- $n < r^2 + 3r + 1$ and r is even, and $h \leq (\alpha-1)h_\alpha(r-1) + h_\alpha(n-2-\alpha r)$, or
- $n < r^2 + 3r + 1$ and r is odd, and $h \leq (\alpha-1)h_\alpha(r-2) + h_\alpha(\min\{n-3-\alpha(r-1), r-2\})$

then $\mathbb{G}(r, n)$ is not $(h+1)$ -defective.

Proof. Since by Propositions 5.1 and 5.4 the Grassmannian $\mathbb{G}(r, n)$ satisfies Assumption 5.2 and has α -osculating regularity, it is enough to apply Corollary 4.5 together with Theorem 6.2. \square

Note that if we write

$$r = 2^{\lambda_1} + 2^{\lambda_2} + \dots + 2^{\lambda_s} + \varepsilon \tag{6.1}$$

with $\lambda_1 > \lambda_2 > \dots > \lambda_s \geq 1$ and $\varepsilon \in \{0, 1\}$, then

$$h_\alpha(r-1) = \alpha^{\lambda_1-1} + \dots + \alpha^{\lambda_s-1}.$$

Therefore, the first bound in Theorem 6.3 gives

$$h \leq \alpha^{\lambda_1} + \dots + \alpha^{\lambda_s}.$$

Furthermore, just considering the first summand in the second and third bound in Theorem 6.3 we get that $\mathbb{G}(r, n)$ is not $(h + 1)$ -defective for

$$h \leq (\alpha - 1) \left(\alpha^{\lambda_1 - 1} + \dots + \alpha^{\lambda_s - 1} \right).$$

Finally, note that (6.1) yields $\lambda_1 = \lfloor \log_2(r) \rfloor$. Hence, asymptotically we have $h_\alpha(r - 1) \sim \alpha^{\lfloor \log_2(r) \rfloor - 1}$, and by Theorem 6.3 $\mathbb{G}(r, n)$ is not $(h + 1)$ -defective for

$$h \leq \alpha^{\lfloor \log_2(r) \rfloor} = \left(\frac{n+1}{r+1} \right)^{\lfloor \log_2(r) \rfloor}.$$

Example 6.4. In order to help the reader in getting a concrete idea of the order of growth of the bound in Theorem 6.3 for $n \geq r^2 + 3r + 1$ we work out some cases in the following table:

r	$r^2 + 3r + 1$	h
4	29	$\left(\frac{n+1}{5}\right)^2 + 1$
6	55	$\left(\frac{n+1}{7}\right)^2 + \left(\frac{n+1}{7}\right) + 1$
8	89	$\left(\frac{n+1}{9}\right)^3 + 1$
10	131	$\left(\frac{n+1}{11}\right)^3 + \left(\frac{n+1}{11}\right) + 1$
12	181	$\left(\frac{n+1}{13}\right)^3 + \left(\frac{n+1}{13}\right)^2 + 1$
14	239	$\left(\frac{n+1}{15}\right)^3 + \left(\frac{n+1}{15}\right)^2 + \left(\frac{n+1}{15}\right) + 1$
16	305	$\left(\frac{n+1}{17}\right)^4 + 1$

Thanks to Theorem 6.3 it is straightforward to get a linear bound going with $\frac{n}{2}$.

Corollary 6.5. Assume that $r \geq 2$, and set

$$\alpha := \left\lfloor \frac{n+1}{r+1} \right\rfloor.$$

If either of the following holds

- $n \geq r^2 + 3r + 1$ and $h \leq \lfloor \frac{r}{2} \rfloor \alpha + 1$, or
- $n < r^2 + 3r + 1$ and r is even, and $h \leq \lfloor \frac{n+1}{2} \rfloor - \frac{r}{2}$, or
- $n < r^2 + 3r + 1$ and r is odd, and $h \leq \min \left\{ \frac{r-1}{2} \alpha + 1, \lfloor \frac{n}{2} \rfloor - \frac{r-1}{2} \right\}$

then $\mathbb{G}(r, n)$ is not h -defective.

Proof. Since $\alpha \geq 2$ we have $h_\alpha(k) \geq h_2(k) = \lfloor \frac{k+1}{2} \rfloor$. In particular, $h_\alpha(r-1) \geq \lfloor \frac{r}{2} \rfloor$ and $h_\alpha(r-2) \geq \lfloor \frac{r-1}{2} \rfloor$.

Now, it is enough to observe that

$$\frac{r}{2}(\alpha-1) + \left\lfloor \frac{n-2-\alpha r+1}{2} \right\rfloor + 1 = \left\lfloor \frac{n+1}{2} \right\rfloor - \frac{r}{2}$$

for r even, and

$$\frac{r-1}{2}(\alpha-1) + \left\lfloor \frac{n-3-\alpha(r-1)+1}{2} \right\rfloor + 1 = \left\lfloor \frac{n}{2} \right\rfloor - \frac{r-1}{2}$$

for r odd, and to apply Theorem 6.3. \square

6.1. Comparison with Abo-Ottaviani-Peterson bound

Finally, we show that Corollary 6.5 strictly improves [3, Theorem 3.3] for $r \geq 4$, whenever $(r, n) \notin \{(4, 10), (5, 11)\}$.

For $r \geq 4, n \geq 2r+1$ we define the following functions of r and n :

$$a := \left\lfloor \frac{r}{2} \right\rfloor \left\lfloor \frac{n+1}{r+1} \right\rfloor, \quad a' := \left\lfloor \frac{n-1}{2} \right\rfloor - \frac{r}{2}, \quad a'' := \left\lfloor \frac{n}{2} \right\rfloor - \frac{r+1}{2}, \quad b := \left\lfloor \frac{n-r}{3} \right\rfloor.$$

First we show that $a > b$. Indeed, if $r > 2$ is even then

$$a = \frac{r}{2} \left\lfloor \frac{n+1}{r+1} \right\rfloor > \frac{r}{2} \cdot \frac{n-r}{r+1} > \frac{n-r}{3} \geq \left\lfloor \frac{n-r}{3} \right\rfloor = b$$

and if $r > 5$ is odd then

$$a = \frac{r-1}{2} \left\lfloor \frac{n+1}{r+1} \right\rfloor > \frac{r-1}{2} \cdot \frac{n-r}{r+1} > \frac{n-r}{3} \geq \left\lfloor \frac{n-r}{3} \right\rfloor = b.$$

Furthermore, if $r = 5$ we write $n = 6\lambda + \varepsilon$ with $\varepsilon \in \{-1, 0, 1, 2, 3, 4\}$. Then we have

$$a = 2 \left\lfloor \frac{6\lambda + \varepsilon + 1}{6} \right\rfloor = 2\lambda > 2\lambda + \left\lfloor \frac{\varepsilon - 5}{3} \right\rfloor = \left\lfloor \frac{6\lambda + \varepsilon - 5}{3} \right\rfloor = b.$$

Now, we assume that $n < r^2 + 3r + 1$ and we show that $a' > b$ if r is even and $(r, n) \neq (4, 10)$, and that $a'' > b$ if r is odd and $(n, r) \neq (5, 11)$. Note that $a'(4, 10) = a''(5, 11) = b(4, 10) = b(5, 11) = 2$. If r is even

$$a' = \left\lfloor \frac{n-1}{2} \right\rfloor - \frac{r}{2} > \frac{n-1}{2} - 1 - \frac{r}{2} = \frac{n-r-3}{2} > \frac{n-r}{3} = b$$

whenever $n > r + 9$. Similarly, if r is odd and $n > r + 9$ we have

$$a'' = \left\lfloor \frac{n}{2} \right\rfloor - \frac{r+1}{2} > \frac{n}{2} - 1 - \frac{r+1}{2} = \frac{n-r-3}{2} > \frac{n-r}{3} = b.$$

Now, if $r > 8$ then $n \geq 2r + 1 \Rightarrow n > r + 9$. A finite number of cases are left, namely

$$(r, n) \in \left\{ (r, n); r = 4, 5, 6, 7, 8 \text{ and } r^2 + 3r + 1 > r + 9 \geq n \geq 2r + 1 \right\}.$$

These cases can be easily checked one by one.

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