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The rectified *n*-harmonic map flow with applications to homotopy classes

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Abstract. We introduce a rectified n-harmonic map flow from an n-dimensional closed Riemannian manifold to another closed Riemannian manifold. We prove existence of a global solution, which is regular except for a finite number of points, of the rectified n-harmonic map flow and establish an energy identity for the flow at each singular time. Finally, we present two applications of the rectified n-harmonic map flow to minimizing the n-energy functional and the Dirichlet energy functional in a homotopy class.

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1. Introduction

Let (M, g) be an n-dimensional compact Riemannian manifold without boundary, and let (N, h) be another m-dimensional compact Riemannian manifold without boundary (isometrically embedded into $\mathbb{R}^L$). The n-energy functional $E_n(u; M)$ of a map $u : (M, g) \to (N, h)$ is defined by

$$E_n(u; M) = \frac{1}{n} \int_M |\nabla u|^n \, dv.$$

A map u from M to N is said to be an n-harmonic map if u is a critical point of the n-energy functional; *i.e.*, it satisfies

$$\operatorname{div}\left[|\nabla u|^{n-2}\nabla u\right] + |\nabla u|^{n-2} A(u)(\nabla u, \nabla u) = 0 \quad \text{in } M, \tag{1.1}$$

where A is the second fundamental form of N.

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When $n = 2$, an n-harmonic map is a harmonic map. The fundamental question on harmonic maps, asked by Eells and Sampson [9] (see also [10]), is whether a given smooth map u_0 can be deformed to a harmonic map in its homotopy class $[u_0]$. Eells and Sampson [9] answered the question for the case that the sectional curvature of N is non-positive by introducing the heat flow for harmonic maps. In order to solve the Eells-Sampson question, it is very important to establish global existence of the harmonic map flow. When $n = 2$, Struwe [27] proved the existence of a unique global weak solution to the harmonic map flow for an arbitrary closed target manifold, which is smooth except for a finite set of singularities, where the flow blows up through a finite number of harmonic maps on S^2 (called *bubbles*). Chang, Ding and Ye [1] constructed an example that the harmonic map flow blows up at finite time. Ding and Tian [8] established the energy identity of the harmonic map flow at each blow-up time. Qing and Tian [23] proved that as $t \to \infty$, there is no neck between a limit map u_∞ and bubbles. Therefore, a given map u_0 can be deformed into a splitting sum of finite harmonic maps.

When $n > 2$, Chen and Struwe [4] showed global existence of a weak solution of the harmonic map flow, in which the weak solution is partially regular and has a complicated singular set. In general, it is difficult to apply the harmonic map flow to investigate the Eells-Sampson question. Motivated by the Eells-Simpson question, it is interesting to ask whether a given map $u_0 \in C^\infty(M, N)$ can be deformed to an n-harmonic map in the homotopy class $[u_0]$. Related to this question, Hungerbuhler [19] investigated the n-harmonic map flow in the following equation:

$$\frac{\partial u}{\partial t} = \operatorname{div}\left[|\nabla u|^{n-2}\nabla u\right] + |\nabla u|^{n-2} A(u)(\nabla u, \nabla u), \tag{1.2}$$

with initial value u_0, and generalized the result of Struwe [27] to prove that there exists a global weak solution $u : M \times [0, +\infty) \to N$ of the n-harmonic map flow (1.2) such that $u \in \mathbf{C}^{1,\alpha}(M \times (0, +\infty)\backslash\{\Sigma_k \times T_k\}_{k=1}^L)$ for a finite number of singular times $\{T_k\}_{k=1}^L$ and a finite number of singular closed sets $\Sigma_k \subset M$ for $k = 1, \ldots, L$ with an integer L, depending only M and u_0. Chen, Cheung, Choi, Law [2] constructed an example to show that the n-harmonic map flow (1.2) blows up at finite time for $n = 3$. However, it has been an open question whether the singular set Σ_k of the n-harmonic map flow at each singular time T_k is finite. Without the finiteness of the singular set Σ_k, it is difficult to control the loss of the energy at the singular time T_k. In order to overcome this difficulty, we introduce a rectified n-harmonic map flow in the following equation:

$$\left(1 - a + a|\nabla u|^{n-2}\right)\frac{\partial u}{\partial t} = \operatorname{div}\left[|\nabla u|^{n-2}\nabla u\right] + |\nabla u|^{n-2} A(u)(\nabla u, \nabla u), \tag{1.3}$$

with initial value $u(0) = u_0$ with a constant $a \in [0, 1]$. In particular, when $a = 0$, the flow (1.3) is the standard n-harmonic map flow. When $a = 1$, the flow (1.3) is an evolution equation involving the normalized n-Laplacian (*e.g.*, [5]).

In this paper, we assume that $n > 2$. We firstly prove:

Theorem 1.1. *For each $a \in (0, 1]$, there exists a global weak solution $u : M \times [0, +\infty) \to N$ of* (1.3) *with initial value $u_0 \in W^{1,n}(M)$ in which there are finite times $\{T_k\}_{k=1}^L$ and finitely many singular points $\{x^{j,k}\}_{j=1}^{l_k}$ such that u is regular in $M \times (0, +\infty)\backslash\{\{x^{j,k}\}_{j=1}^{l_k} \times T_k\}_{k=1}^L$ in the following sense:*

$$u \in C^{0,\alpha}_{\mathrm{loc}}\left(M \times (0, +\infty)\Big\backslash \left\{\left\{x^{1,k} \cdots , x^{l_k,k}\right\} \times T_k\right\}_{k=1}^{L}\right),$$
$$\nabla u \in L^{\infty}_{\mathrm{loc}}\left(M \times (0, +\infty)\Big\backslash \left\{\left\{x^{1,k} \cdots , x^{l_k,k}\right\} \times T_k\right\}_{k=1}^{L}\right).$$

As $t \to T_k$, $u(x, t)$ strongly converges to $u(x, T_k)$ in $W^{1,n+1}_{\mathrm{loc}}(M\backslash\{x^{1,k} \cdots , x^{l_k,k}\})$.

Theorem 1.1 generalized the result of Struwe [27]. For the proof of Theorem 1.1, one of key ideas is to obtain an ε-regularity estimate by improving the delicate proof of Hungerbuhler in [19] for the case of $a = 0$ based on a variant of Moser's iteration. Since the term $|\nabla u|^{n-2}\partial_t u$ in the flow (1.3) causes an extra difficulty, we have to carry out much more complicated analysis to obtain the boundedness of $|\nabla u|$ (see Lemma 2.4).

Remark 1.2. It will be very interesting if some can prove that the solution of the flow (1.3) in Theorem 1.1 is $C^{1,\alpha}$ away from singularities.

Secondly, we generalize the result of Ding-Tian [8] from two-dimensional case to n-dimensional cases and prove:

Theorem 1.3. *For each $a \in (0, 1]$, let $u : M \times [0, +\infty) \to N$ be a solution of* (1.3) *with initial value u_0 in Theorem* 1.1. *Let T_k be the above singular time. Then, there are a finite number of n-harmonic maps $\{\omega_{i,k}\}_{i=1}^{m_k}$ (also called bubbles) on S^n such that*

$$\lim_{t \nearrow T_k} E_n(u(t); M) = E_n(u(\cdot, T_k); M) + \sum_{i=1}^{m_k} E_n(\omega_{i,k}, S^n).$$

For the proof of the energy identity, Wang and Wei [30] proved an energy identity for a sequence of approximate n-harmonic maps by reducing multiple bubbles to a single bubble. In order to make proofs more clear, we give a detailed procedure of bubble-neck decomposition based on the method of Ding-Tian [8] and then prove the energy identity.

Next, we will present some applications of the rectified n-flow to the problem of minimizing the n-energy functional in a homotopy class $[u_0]$. When $n = 2$, Lemaire [20] and Schoen-Yau [26] established existence results of harmonic maps by minimizing the Dirichet energy in a homotopy class under the topological condition $\pi_2(N) = 0$. In [24], Sacks and Uhlenbeck established many existence results of minimizing harmonic maps in their homotopy classes by introducing the

'Sacks-Uhlenbeck functional'. Recently, the author and Yin [18] introduced the Sacks-Uhlenbeck flow on Riemannian surfaces to provide a new proof of the energy identity of a minimizing sequence in a homotopy class $[u_0]$. A similar approach on the Yang-Mills α-flow on 4-manifolds has been obtained by the author, Tian and Yin [16]. Expanding the idea in [18] with applications of a rectified n-flow, we prove:

Theorem 1.4. *For a homotopy class* $[u_0]$, *let* $\{u_k\}_{k=1}^{\infty}$ *be a minimizing sequence of* E_n *in the homotopy class* $[u_0]$ *and* u *the weak limit in* $W^{1,n}(M, N)$. *Then, there is a finite set* Σ *of singular points in* M *so that as* $k \to \infty$, u_k *converges strongly to* u *in* $W^{1,n}_{\mathrm{loc}}(M\backslash\Sigma, N)$ *and there are a finite number of* n*-harmonic maps* $\{\omega_i\}_{i=1}^{l}$ *on* S^{n-1} *such that*

$$\lim_{k\to\infty} E_n(u_k; M) = E_n(u_\infty; M) + \sum_{i=1}^{l} E_n\big(\omega_i, S^{n-1}\big).$$

If $\pi_n(N) = 0$, *the singular set* Σ *is empty and there is a minimizing map of the* n*-energy functional in the homotopy class* $[u_0]$.

We would like to point out that Duzaar and Kuwert [7] studied the decomposition of a minimizing sequences of the n-energy functional in a homotopy class $[u_0]$ with $N = S^l$, which could be used to prove an energy identity for the minimizing sequence of E_n. Our proof is completely different from one in [7]. By a modification of the rectified n-harmonic flow, we follow the idea of the α-flow [18] to obtain a new minimizing sequence $\{\tilde{u}_k\}_{k=1}^{\infty}$, having the same weak limit u of the minimizing sequence $\{u_k\}_{k=1}^{\infty}$ in the same homotopy class.

Furthermore, in order to prove the existence of a harmonic map in a given homotopy class $[u_0]$, it is a nature way to minimize the Dirichlet functional $E(u; M) = \frac{1}{2}\int_M |\nabla u|^2\, dv$ in the homotopy class. Indeed, there were successful results for $n = 2$, which were mentioned above [20,26] and [24]. In higher dimensions, it is very challenging to minimize the Dirichlet functional in a homotopy class. White [31] showed that if d is the greatest integer strictly less than p, a homotopy equivalence is well defined for neighboring maps after restriction to the d-skeleton of M and there exists a minimizer of the p-energy $E_p(u; M) = \frac{1}{p}\int_M |\nabla u|^p dv$ with prescribed d-homotopy type. White [31] raised an open problem about the partial regularity of the minimum solution of the p-energy with prescribed d-homotopy type. In particular, even for $p = 2$, the partial regularity theory of Schoen-Uhlenbeck [24] (also Giaquinta-Giusti [12]) on an energy minimizing map u in $W^{1,2}(M, N)$ cannot be applied since the Sobolev space $W^{1,2}(M, N)$ cannot be approximated by smooth maps and a minimizing map of the Dirichlet in $W^{1,2}(M, N)$ is not in the homotopy class.

Let $\{u_k\}_{k=1}^{\infty}$ be a minimizing sequence of the p-energy E_p in the homotopy class $[u_0]$ for $2 \le p \le n$ and let $u \in W^{1,p}(M, N)$ be the weak limit of $\{u_k\}_{k=1}^{\infty}$. Related to the above White problem, it is a very interesting problem whether the limit map u is a weakly p-harmonic map and partially regular. Motivated by recent results of [13] and [17], we partially answer the question by applying a modified n-flow and prove:

Theorem 1.5. *Let p be a number with $2 \le p \le n$. Assume that N is a homogenous Riemannian manifold without boundary. For a given homotopy class $[u_0]$, let $\{u_i\}_{i=1}^{\infty}$ be a minimizing sequence of the p-energy $E_p(u; M)$ in the homotopy class $[u_0]$. Then, there is a subsequence of $\{u_i\}_{i=1}^{\infty}$ such that u_i weakly converges to a weak p-harmonic map u. Moreover, u belongs to $C^{1,\alpha}(M \backslash \Sigma, N)$ for a closed singular set $\Sigma \subset M$ and $\mathcal{H}^{n-p}(\Sigma) < \infty$, where $\mathcal{H}^{n-p}$ denotes the Hausdorff measure.*

To prove Theorem 1.5, we employ a perturbation of the p-energy functional and its gradient flow in a homotopy class. This kind of perturbation of the Dirichlet functional was used by Uhlenbeck in [29] to reprove the Eells-Sampson result, by Giaquinta, the author and Yin [13] for proving partial regularity of minimizers of the relaxed functional of harmonic maps and also by the author and Yin [18] for proving partial regularity of minimizers of the relaxed functional of bi-harmonic maps.

The paper is organised as follows. In Section 2, we establish some basic estimates and global existence of weak solutions to the rectified n-flow. In Section 3, we prove the energy identity at a singular time and finish a proof of Theorem 1.3. In Section 4, we prove Theorem 1.4. In Section 5, we finish a proof of Theorem 1.5.

2. Some estimates and global existence

In local coordinates, the Riemannian metric g on M can be represented by

$$g = g_{ij} dx^i \otimes dx^j,$$

with a positive definitive symmetric $n \times n$ matrix (g_{ij}). The volume element dv of $(M; g)$ is defined by

$$dv = \sqrt{|g|}dx \quad \text{with } |g| = \det(g_{ij}).$$

Note that (N, h) is a m-dimensional compact Riemannian manifold without boundary, isometrically embedded into $\mathbb{R}^L$. For a map $u : M \to N$, the gradient norm $|\nabla u|$ is given by

$$|\nabla u(x)|^2 = \sum_{i,j,\alpha} g^{ij}(x) \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\alpha}{\partial x_j},$$

where $(g^{ij}) = (g_{ij})^{-1}$ is the inverse matrix of (g_{ij}). A $C^{1,\alpha}$-map u from M to N is called an n-harmonic map if it satisfies

$$\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_i} \left[|\nabla u|^{n-2} g^{ij} \sqrt{|g|} \frac{\partial}{\partial x_j} u \right] + |\nabla u|^{n-2} A(u)(\nabla u, \nabla u) = 0 \quad \text{in } M, \quad (2.1)$$

where A is the second fundamental form of N.

In order to show existence of the rectified n-flow (1.3), we consider an approximate n-functional

$$E_{n,\varepsilon}(u) = \int_M \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{n} |\nabla u|^n \right) dv \tag{2.2}$$

for a constant $\varepsilon > 0$. The Euler-Lagrange equation for the functional (2.2) is

$$\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_i} \left[\left(\varepsilon + |\nabla u|^{n-2} \right) g^{ij} \sqrt{|g|} \frac{\partial}{\partial x_j} u \right] + \left(\varepsilon + |\nabla u|^{n-2} \right) A(u)(\nabla u, \nabla u) = 0. \tag{2.3}$$

For each $\varepsilon > 0$ and $a \in [0, 1]$, the rectified gradient flow for the functional (2.2) is

$$\begin{aligned} &\left(1 - a + \varepsilon + a|\nabla u|^{n-2}\right) \frac{\partial u}{\partial t} \\ &= \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_i} \left[\left(\varepsilon + |\nabla u|^{n-2} \right) g^{ij} \sqrt{|g|} \frac{\partial}{\partial x_j} u \right] + \left(\varepsilon + |\nabla u|^{n-2} \right) A(u)(\nabla u, \nabla u) \end{aligned} \tag{2.4}$$

with initial value $u(0) = u_0$ in M. Multiplying $\partial_t u$ to both sides of (2.4), we have the following energy identity:

Lemma 2.1. *Assume that $\varepsilon > 0$ and $a \in [0, 1]$. Let $u(t)$ be a smooth solution to the flow (2.4) in $M \times [0, T)$ with initial value $u(0) = u_0$. Then for each s with $0 < s < T$, we have*

$$\begin{aligned} &\int_M \frac{\varepsilon}{2} |\nabla u(s)|^2 + \frac{1}{n} |\nabla u(s)|^n \, dv + \int_0^s \int_M \left(1 - a + \varepsilon + a|\nabla u|^{n-2}\right) \left| \frac{\partial u}{\partial t} \right|^2 dv \, dt \\ &= \int_M \frac{\varepsilon}{2} |\nabla u_0|^2 + \frac{1}{n} |\nabla u_0|^n \, dv. \end{aligned} \tag{2.5}$$

Moreover, we have the following local energy's inequality for $a > 0$:

Lemma 2.2 (Local energy inequality). *Assume that $\varepsilon > 0$ and $a \in (0, 1]$. Let $u(t)$ be a smooth solution to the flow (2.4) in $M \times [0, T]$ with initial value $u(0) = u_0$ and set $e_\varepsilon(u) = \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{n} |\nabla u|^n$. For any x_0 with $B_{2R}(x_0) \subset M$ and for any two $s, \tau \in [0, T)$ with $s < \tau$, we have*

$$\begin{aligned} &\int_{B_R(x_0)} e_\varepsilon(u)(\cdot, \tau) \, dv + \int_s^\tau \int_M (1 - a + \varepsilon + a|\nabla u|^{n-2}) |\partial_t u|^2 \, dv \, dt \\ &\leq \int_{B_{2R}(x_0)} e_\varepsilon(u)(\cdot, s) \, dv + C(a) \frac{(\tau - s)}{R^2} \int_M e_\varepsilon(u_0) \, dv\,, \end{aligned} \tag{2.6}$$

and

$$\begin{aligned}&\int_{B_R(x_0)} e_\varepsilon(u)(\cdot,s)\,dv-\int_{B_{2R}(x_0)} e_\varepsilon(u)(\cdot,\tau)\,dv\\ &\le C(a)\int_s^\tau\int_M(1-a+\varepsilon+a|\nabla u|^{n-2})|\partial_t u|^2\,dv\,dt \qquad (2.7)\\ &+C(a)\left(\frac{(\tau-s)}{R^2}\int_M e_\varepsilon(u_0)\,dv\int_s^\tau\int_M(1-a+\varepsilon+|\nabla u|^{n-2})|\partial_t u|^2\,dv\,dt\right)^{1/2},\end{aligned}$$

where $C(a)$ is a constant depending on $a>0$.

Proof. Let φ be a cut-off function with support in $B_{2R}(x_0)$ and $\varphi\equiv 1$ on $B_R(x_0)$ with $|\nabla\varphi|\le C/R$. Then

$$\begin{aligned}\frac{d}{dt}\int_M\varphi^2 e_\varepsilon(u)dv&=\int_M\varphi^2\left\langle\left(\varepsilon+|\nabla u|^{n-2}\right)\nabla u,\nabla\frac{\partial u}{\partial t}\right\rangle dv\\ &=-\int_M\varphi^2\left(1-a+\varepsilon+a|\nabla u|^{n-2}\right)\left|\frac{\partial u}{\partial t}\right|^2 dv\\ &\quad+\int_M\varphi\left(\varepsilon+|\nabla u|^{n-2}\right)\nabla u\#\nabla\varphi\#\frac{\partial u}{\partial t}\,dv.\end{aligned}$$

(2.6) follows from integrating in t over $[s,\tau]$ and using Young's inequality. Similarly, we have (2.7). □

We would like to point out that Lemma 2.2 does not hold for $a=0$.

Lemma 2.3. *Assume that $\varepsilon>0$ and $a\in[0,1]$. Let $u:M\to N$ be a smooth solution to the flow equation* (2.4) *in $M\times[0,T]$. There is a small constant $\varepsilon_0>0$ such that if the inequality*

$$\sup_{0\le t\le T}\int_{B_{2R_0}(x_0)}|\nabla u|^n\,dv\,dt<\varepsilon_0$$

holds for some positive R_0, then we have

$$\begin{aligned}&\int_0^T\int_{B_{R_0}(x_0)}|\nabla u|^{n+2}+|\nabla^2u|^2\left(\varepsilon+|\nabla u|^{n-2}\right)dv\,dt \qquad (2.8)\\ &\le C\left(1+T+TR_0^{-2}\right)\left[E_\varepsilon(u_0)+R_0^n\right],\end{aligned}$$

where the constant C does not depend on ε, a and u.

Proof. In a neighborhood of each point $x_0\in M$, we can choose an orthonormal frame $\{e_i\}_i^n$. We denote by ∇_i the first covariant derivative with respect to e_i and by $\nabla^2_{ji}u$ the second covariant derivatives of u and so on.

Let ϕ be a cut-off function with support in $B_{2R_0}(x_0)$ such that $\phi = 1$ in $B_{R_0}(x_0)$, $|\nabla\phi| \le CR_0^{-1}$ and $|\phi| \le 1$ in $B_{2R_0}(x_0)$. Multiplying (2.4) by $\phi^n \Delta u$, we have

$$\begin{aligned} I_1 =: &\int_{B_{2R_0}(x_0)} \left(1 - a + \varepsilon + a|\nabla u|^{n-2}\right) \langle \partial_t u, \Delta u\rangle \, \phi^n \, dv \\ = &\int_{B_{2R_0}(x_0)} \left\langle \nabla_k \left(\left(\varepsilon + |\nabla u|^{n-2}\right) \nabla_k u\right), \Delta u \right\rangle \phi^n \, dv \\ &+ \int_{B_{2R_0}(x_0)} \left\langle \left(\varepsilon + |\nabla u|^{n-2}\right) A(u)(\nabla u, \nabla u), \Delta u \right\rangle \phi^n \, dv \\ =: &\; I_2 + I_3. \end{aligned} \tag{2.9}$$

In order to prove (2.8), I_1, I_2 are main terms and I_3 can be estimated easily by

$$\begin{aligned} |I_3| \le &\frac{1}{4} \int_{B_{2R_0}(x_0)} \left(\varepsilon + |\nabla u|^{n-2}\right) \left|\nabla^2 u\right|^2 \phi^n \, dv \\ &+ C \int_{B_{2R_0}(x_0)} \left(\varepsilon + |\nabla u|^{n-2}\right) |\nabla u|^4 \phi^n \, dv \end{aligned} \tag{2.10}$$

due to the fact that $|A(u)(\nabla u, \nabla u)| \le C|\nabla u|^2$.

To estimate the term I_1 of (2.9), it follows from integrating by parts and using Young's inequality that

$$\begin{aligned} I_1 = &\int_{B_{2R_0}(x_0)} \left\langle \left(1 - a + \varepsilon + a|\nabla u|^{n-2}\right) \partial_t u, \Delta u \right\rangle \phi^n \, dv \\ = &-\frac{d}{dt} \int_{B_{2R_0}(x_0)} \left(\frac{1 - a + \varepsilon}{2} |\nabla u|^2 + \frac{a}{n} |\nabla u|^n\right) \phi^n \, dv \\ &- a \int_{B_{2R_0}(x_0)} \nabla_k \left(|\nabla u|^{n-2}\right) \partial_t u \cdot \nabla_k u \phi^n \, dv \\ &- n \int_{B_{2R_0}(x_0)} \left(1 - a + \varepsilon + a|\nabla u|^{n-2}\right) \partial_t u \cdot \nabla_k u \phi^{n-1} \nabla_k \phi \, dv \\ \le &-\frac{d}{dt} \int_{B_{2R_0}(x_0)} \left(\frac{1 - a + \varepsilon}{2} |\nabla u|^2 + \frac{a}{n} |\nabla u|^n\right) \phi^n \, dv \\ &+ C \int_{B_{2R_0}(x_0)} \left(1 - a + \varepsilon + a|\nabla u|^{n-2}\right) |\partial_t u|^2 \phi^n \, dv \\ &+ \frac{n-2}{4} \int_{B_{2R_0}(x_0)} |\nabla u|^{n-2} |\nabla(|\nabla u|)|^2 \phi^n \, dv \\ &+ C \int_{B_{2R_0}(x_0)} \left(1 - a + \varepsilon + a|\nabla u|^{n-2}\right) |\nabla u|^2 |\nabla\phi|^2 \phi^{n-2} \, dv. \end{aligned} \tag{2.11}$$

To estimate the first term I_2 of the right-hand side of (2.9), it follows from the well-known Ricci identity that

$$\nabla_k\nabla_l\left((\varepsilon+|\nabla u|^{n-2})\nabla u\right)=\nabla_l\nabla_k\left((\varepsilon+|\nabla u|^{n-2})\nabla u\right)+R_M\#\left(\left(\varepsilon+|\nabla u|^{n-2}\right)\nabla u\right)$$

with the Riemannian curvature R_M. Then, integrations by parts twice yield that

$$\begin{aligned}
I_2&=\int_{B_{2R_0}(x_0)}\left\langle\nabla_k\left(\left(\varepsilon+|\nabla u|^{n-2}\right)\nabla_k u\right),\Delta u\right\rangle\phi^n\,dv\\
&=\int_{B_{2R_0}(x_0)}\left\langle\nabla_l\left(\left(\varepsilon+|\nabla u|^{n-2}\right)\nabla_k u\right),\nabla_k\nabla_l u\right\rangle\phi^n\,dv\\
&\quad-\int_{B_{2R_0}(x_0)}\left\langle\nabla_k\left(\left(\varepsilon+|\nabla u|^{n-2}\right)\nabla_k u\right),\nabla_l u\right\rangle\nabla_l\phi^n\,dv\\
&\quad+\int_{B_{2R_0}(x_0)}\left\langle\nabla_l\left(\left(\varepsilon+|\nabla u|^{n-2}\right)\nabla_k u\right),\nabla_l u\right\rangle\nabla_k\phi^n\,dv\\
&\quad+\int_{B_{2R_0}(x_0)}\left\langle R_M\#\left(\left(\varepsilon+|\nabla u|^{n-2}\right)\nabla_k u\right),\nabla_l u\right\rangle\phi^n\,dv\\
&\geq\frac{3}{4}\int_{B_{2R_0}(x_0)}\left(\varepsilon+|\nabla u|^{n-2}\right)\left|\nabla^2 u\right|^2\phi^n\,dv\\
&\quad+\frac{n-2}{2}\int_{B_{2R_0}(x_0)}\left(\varepsilon+|\nabla u|^{n-2}\right)|\nabla|\nabla u||^2\phi^n\,dv\\
&\quad-C\int_{B_{2R_0}(x_0)}\left(\varepsilon+|\nabla u|^{n-2}\right)|\nabla u|^2\phi^{n-2}\left(\phi^2+|\nabla\phi|^2\right)dv.
\end{aligned}\tag{2.12}$$

Substituting (2.10)-(2.12) into (2.9) and using Young's inequality, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{B_{2R_0}(x_0)}\left(\frac{1-a+\varepsilon}{2}|\nabla u|^2+\frac{a}{n}|\nabla u|^n\right)\phi^n\,dv\\
&\quad+\frac{1}{2}\int_{B_{2R_0}(x_0)}\left(\varepsilon+|\nabla u|^{n-2}\right)\left|\nabla^2 u\right|^2\phi^n\,dv\\
&\leq C\int_{B_{2R_0}(x_0)}|\nabla u|^2\left(\varepsilon|\nabla u|^2+|\nabla u|^n\right)\phi^n\,dv\\
&\quad+C\int_{B_{2R_0}(x_0)}(1+|\nabla u|^n)\phi^{n-2}\left(\phi^2+|\nabla\phi|^2\right)dv\\
&\quad+C\int_{B_{2R_0}(x_0)}\left(1-a+\varepsilon+a|\nabla u|^{n-2}\right)|\partial_t u|^2\phi^n\,dv.
\end{aligned}\tag{2.13}$$

The main term in the right hand of (2.13) is the term involving $|\nabla u|^{n+2}\phi^n$ and other lower order terms can be treated by the Young inequality. By applying the Hölder and Sobolev inequalities, we have

$$
\begin{aligned}
&\int_0^T \int_{B_{2R_0}(x_0)} |\nabla u|^{n+2}\phi^n \, dv\, dt \\
&\le \left(\sup_{0\le t\le T} \int_{B_{2R_0}(x_0)} |\nabla u|^n \, dv\right)^{\frac{2}{n}} \int_0^T \left(\int_{B_{2R_0}(x_0)} |\nabla u|^{\frac{n^2}{n-2}} \phi^{\frac{n^2}{n-2}} \, dv\right)^{\frac{n-2}{n}} dt \\
&\le C\varepsilon_0^{\frac{2}{n}} \int_0^T \int_{B_{2R_0}(x_0)} \left|\nabla\left(|\nabla u|^{n/2}\phi^{n/2}\right)\right|^2 dv\, dt \\
&\le C\varepsilon_0^{\frac{2}{n}} \int_0^T \int_{B_{2R_0}(x_0)} \left(\left|\nabla^2 u\right|^2 |\nabla u|^{n-2}\phi^n + \frac{1}{R_0^2}|\nabla u|^n\right) dv\, dt.
\end{aligned}
$$

Integrating (2.13) in t over $[0, T]$, choosing ε_0 sufficiently small and Lemma 2.1, we have

$$
\begin{aligned}
&\int_0^T \int_{B_{R_0}(x_0)} |\nabla u|^{n+2} + \left|\nabla^2 u\right|^2 \left(\varepsilon + |\nabla u|^{n-2}\right) dv\, dt \\
&\le C \int_{B_{2R_0}(x_0)} \left(\frac{1-a+\varepsilon}{2}|\nabla u|^2 + \frac{a}{n}|\nabla u|^n\right)(x,0)\, dv \\
&\quad + C\left(1 + \frac{1}{R_0^2}\right) \int_0^T \left[E_\varepsilon\left(u; B_{2R_0}(x_0)\right) + R_0^n\right] dt \\
&\le C\left(1 + T + \frac{T}{R_0^2}\right)\left[E_\varepsilon(u_0) + R_0^n\right].
\end{aligned}
$$

This proves the claim. □

For $R > 0$ and $z_0 = (x_0, t_0) \in M \times (0, \infty)$, we denote

$$
P_R(z_0) = \left\{z = (x,t) : |x - x_0| < R, t_0 - R^2 < t \le t_0\right\}.
$$

Lemma 2.4. *Assume that* $\varepsilon > 0$ *and* $a \in [0, 1]$. *Let* u *be a smooth solution to the flow equation* (2.4) *with smooth initial value* u_0. *For any* $\beta \ge 1$, *there exists a positive constant* ε_1 *depending on* β *such that if for some* R_0 *with* $0 < R_0 < \min\{\varepsilon_1, \frac{t_0^{1/2}}{2}\}$ *the inequality*

$$
\sup_{t_0 - 4R_0^2 \le t \le t_0} \int_{B_{2R_0}(x_0)} |\nabla u|^n \, dv < \varepsilon_1,
$$

holds, then we have

$$\int_{t_0-R_0^2}^{t_0}\int_{B_{R_0}(x_0)} |\nabla u|^{n+2+\beta} + \left|\nabla^2 u\right|^2 \left(\varepsilon + |\nabla u|^{n-2+\beta}\right) dv\,dt$$
$$\leq C R_0^n + C\int_{t_0-4R_0^2}^{t_0}\int_{B_{2R_0}(x_0)} \left(1+R_0^{-2}\right)|\nabla u|^{n+\beta}\,dv\,dt, \tag{2.14}$$

where the constant C does not depend on ε, u and a, but on β.

Proof. The proof for the case of $a = 0$ is due to Hungerbuhler in [19]. However, we must prove it for $a \in (0, 1]$.

In a neighborhood of each point $x_0 \in M$, we still denote by ∇_i the first covariant derivative with respect to e_i and by $\nabla^2_{ij}u$ the second covariant derivatives of u and so on. Let $\phi = \phi(x,t)$ be a cut-off function with support in $B_{R_0}(x_0) \times [t_0 - 4R_0^2, t_0 + 4R_0^2]$ such that $\phi = 1$ in $B_{R_0}(x_0) \times [t_0 - R_0^2, t_0]$, $|\nabla\phi| \leq C/R_0$, $|\partial_t\phi| \leq \frac{1}{R_0^2}$ and $|\phi| \leq 1$ in $B_{R_0}(x_0) \times [t_0 - 4R_0^2, t_0]$.

Multiplying (2.4) by $\phi^n|\nabla u|^\beta\partial_t u$ and integrating by parts, we have

$$\begin{aligned}
&\int_{P_{2R_0}(x_0,t_0)} \left(1-a+\varepsilon+a|\nabla u|^{n-2}\right)|\nabla u|^\beta|\partial_t u|^2\phi^n\,dv\,dt\\
&= -\int_{P_{2R_0}(x_0,t_0)} \left\langle\left(\varepsilon+|\nabla u|^{n-2}\right)\nabla_k u, |\nabla u|^\beta\nabla_k(\partial_t u)\right\rangle\phi^n\,dv\,dt\\
&\quad -\int_{P_{2R_0}(x_0,t_0)} \left\langle\left(\varepsilon+|\nabla u|^{n-2}\right)\nabla_k u, \beta|\nabla u|^{\beta-1}\nabla_k(|\nabla u|)\partial_t u\right\rangle\phi^n\,dv\,dt\\
&\quad -\int_{P_{2R_0}(x_0,t_0)} \left\langle\left(\varepsilon+|\nabla u|^{n-2}\right)\nabla_k u, |\nabla u|^\beta\partial_t u\right\rangle\nabla_k\phi^n\,dv\,dt\\
&\quad +\int_{P_{2R_0}(x_0,t_0)} \left\langle\left(\varepsilon+|\nabla u|^{n-2}\right)A(u)(\nabla u,\nabla u), |\nabla u|^\beta\partial_t u\right\rangle\phi^n\,dv\,dt\\
&\leq -\int_{B_{R_0}(x_0,t_0)} \left(\frac{\varepsilon}{2+\beta}|\nabla u|^{2+\beta}+\frac{1}{n+\beta}|\nabla u|^{n+\beta}\right)\phi^n(\cdot,t_0)\,dv\\
&\quad +\int_{P_{2R_0}(x_0,t_0)} \left(\frac{\varepsilon}{2+\beta}|\nabla u|^{2+\beta}+\frac{1}{n+\beta}|\nabla u|^{n+\beta}\right)n\phi^{n-1}\partial_t\phi\,dv\,dt\\
&\quad +\frac{a}{2}\int_{P_{2R_0}(x_0,t_0)} \left(\varepsilon+|\nabla u|^{n-2}\right)|\nabla u|^\beta|\partial_t u|^2\phi^n\,dv\,dt\\
&\quad +\frac{\beta^2}{2a}\int_{P_{2R_0}(x_0,t_0)} \left(\varepsilon+|\nabla u|^{n-2}\right)|\nabla u|^{\beta-2}|\nabla_k u\nabla_k(|\nabla u|)|^2)\phi^n\,dv\,dt\\
&\quad +C\int_{P_{2R_0}(x_0,t_0)} \left(\varepsilon+|\nabla u|^{n-2}\right)|\nabla u|^{\beta+1}\phi^{n-1}(\phi|\nabla u|+|\nabla\phi|)|\partial_t u|\,dv\,dt
\end{aligned} \tag{2.15}$$

Multiplying a to both sides of (2.15) yields that

$$\begin{aligned}
& a^2 \int_{P_{2R_0}(x_0,t_0)} \left(\varepsilon + |\nabla u|^{n-2}\right) |\nabla u|^{\beta} |\partial_t u|^2 \phi^n \, dv\, dt \\
\le{} & \beta^2 \int_{P_{2R_0}(x_0,t_0)} \left(\varepsilon + |\nabla u|^{n-2}\right) |\nabla u|^{\beta-2} |\nabla_k u \nabla_k(|\nabla u|)|^2 \, dv\, dt \\
& + \frac{Ca}{2+\beta} \int_{P_{2R_0}(x_0,t_0)} \left(\varepsilon + |\nabla u|^{n-2}\right) |\nabla u|^{\beta+2} \phi^{n-1} |\partial_t \phi| \, dv\, dt \\
& + Ca \int_{P_{2R_0}(x_0,t_0)} \left(\varepsilon + |\nabla u|^{n-2}\right) |\nabla u|^{\beta} \phi^{n-1} \left(\phi |\nabla u|^2 + |\nabla u||\nabla \phi|\right) |\partial_t u| dv dt.
\end{aligned} \tag{2.16}$$

Then using Young's inequality and (2.16), we have

$$\begin{aligned}
& - a(n-2) \int_{P_{2R_0}(x_0,t_0)} \left\langle |\nabla u|^{n-3} \nabla_l(|\nabla u|) \partial_t u,\, |\nabla u|^{\beta} \nabla_l u \right\rangle \phi^n \, dv\, dt \\
\le{} & (n-2) \int_{P_{2R_0}(x_0,t_0)} |\nabla u|^{n-2+\beta} \left(\frac{a^2 |\partial_t u|^2}{2\beta} + \frac{\beta}{2} \frac{|\nabla_l(|\nabla u|) \nabla_l u|^2}{|\nabla u|^2} \right) \phi^n \, dv\, dt \\
\le{} & \beta(n-2) \int_{P_{2R_0}(x_0,t_0)} \left(\varepsilon + |\nabla u|^{n-2}\right) |\nabla u|^{\beta-2} |\nabla_l(|\nabla u|) \nabla_l u|^2 \phi^n \, dv\, dt \\
& + \frac{Ca}{\beta(2+\beta)} \int_{P_{2R_0}(x_0,t_0)} \left(\varepsilon + |\nabla u|^{n-2}\right) |\nabla u|^{\beta+2} \phi^{n-1} |\partial_t \phi| \, dv\, dt \\
& + C \frac{a}{\beta} \int_{P_{2R_0}(x_0,t_0)} \left(\varepsilon + |\nabla u|^{n-2}\right) |\nabla u|^{\beta+1} \phi^{n-1} [(\phi |\nabla u| + |\nabla \phi|) |\partial_t u|] \, dv\, dt.
\end{aligned} \tag{2.17}$$

Multiplying (2.4) by $\phi^n \nabla \cdot (|\nabla u|^{\beta} \nabla u)$ and integrating by parts, we have

$$\begin{aligned}
I_4 =: & \int_{P_{2R_0}(x_0,t_0)} \left\langle \left(1 - a + \varepsilon + a |\nabla u|^{n-2}\right) \partial_t u,\, \nabla_l \left(|\nabla u|^{\beta} \nabla_l u\right) \right\rangle \phi^n \, dv\, dt \\
= & \int_{P_{2R_0}(x_0,t_0)} \left\langle \nabla_k \left(\left(\varepsilon + |\nabla u|^{n-2}\right) \nabla_k u \right), \nabla_l \left(|\nabla u|^{\beta} \nabla_l u\right) \right\rangle \phi^n \, dv\, dt \\
& + \int_{P_{2R_0}(x_0,t_0)} \left\langle \left(\varepsilon + |\nabla u|^{n-2}\right) A(u)(\nabla u, \nabla u), \nabla_l \left(|\nabla u|^{\beta} \nabla_l u\right) \right\rangle \phi^n \, dv\, dt \\
= & \int_{P_{2R_0}(x_0,t_0)} \left\langle \nabla_k \left(\left(\varepsilon + |\nabla u|^{n-2}\right) \nabla_k u \right),\, \nabla_l \left(|\nabla u|^{\beta} \nabla_l u\right) \right\rangle \phi^n \, dv\, dt \\
& - \int_{P_{2R_0}(x_0,t_0)} \left\langle \nabla_l \left[\left(\varepsilon + |\nabla u|^{n-2}\right) A(u)(\nabla u, \nabla u) \right], |\nabla u|^{\beta} \nabla_l u \right\rangle \phi^n \, dv\, dt \\
& - \int_{P_{2R_0}(x_0,t_0)} \left\langle \left(\varepsilon + |\nabla u|^{n-2}\right) A(u)(\nabla u, \nabla u),\, |\nabla u|^{\beta} \nabla_l u \right\rangle \nabla_l(\phi^n) \, dv\, dt. \\
=: & \; I_5 + I_6 + I_7.
\end{aligned} \tag{2.18}$$

In (2.18), I_4, I_5 are main terms, , I_7 will be cancelled later and I_6 is a easy term.

To estimate the term I_4, it follows from integrating by parts that

$$\begin{aligned} I_4 = &-a(n-2)\int_{P_{2R_0}(x_0,t_0)} \left\langle |\nabla u|^{n-3}\nabla_l(|\nabla u|)\partial_t u,\, |\nabla u|^{\beta}\nabla_l u\right\rangle \phi^n \,dv\,dt \\ &-\int_{P_{2R_0}(x_0,t_0)} \left\langle\left(1-a+\varepsilon+a|\nabla u|^{n-2}\right)\nabla_l(\partial_t u),\, |\nabla u|^{\beta}\nabla_l u\right\rangle \phi^n\,dv\,dt \\ &-\int_{P_{2R_0}(x_0,t_0)} \left\langle\left(1-a+\varepsilon+a|\nabla u|^{n-2}\right)\partial_t u,\, |\nabla u|^{\beta}\nabla_l u\right\rangle \nabla_l(\phi^n)\,dv\,dt \\ =:&\, I_{4.1}+I_{4.2}+I_{4.3}. \end{aligned} \tag{2.19}$$

Note that $I_{4.1}$ is the exact term of the left-hand side of (2.17). In order to estimate $I_{4.1}$, it follows from using equation (2.4) that

$$a|\partial_t u| \le C\left(|\nabla^2 u| + |\nabla u|^2\right).$$

Combining this with (2.17) and using Young's inequality, we estimate $I_{4.1}$ to have

$$\begin{aligned} I_{4.1} \le\, &\beta(n-2)\int_{P_{2R_0}(x_0,t_0)} \left(\varepsilon+|\nabla u|^{n-2}\right)|\nabla u|^{\beta-2}|\nabla_l(|\nabla u|)\nabla_l u|^2\phi^n\,dv\,dt \\ &+\frac{Ca}{\beta(2+\beta)}\int_{P_{2R_0}(x_0,t_0)} \left(\varepsilon+|\nabla u|^{n-2}\right)|\nabla u|^{\beta+2}\phi^{n-1}|\partial_t\phi|\,dv\,dt \\ &+C\frac{1}{\beta^2}\int_{P_{2R_0}(x_0,t_0)} \left(\varepsilon+|\nabla u|^{n-2}\right)|\nabla u|^{\beta+2}\phi^{n-2}|\nabla\phi|^2\,dv\,dt \\ &+\int_{P_{2R_0}(x_0,t_0)} \left(\varepsilon+|\nabla u|^{n-2}\right)|\nabla u|^{\beta}\left(\frac14\left|\nabla^2 u\right|^2 + C|\nabla u|^4\right)\phi^n\,dv\,dt. \end{aligned} \tag{2.20}$$

To estimate $I_{4.2}$, we have

$$\begin{aligned} I_{4.2} =& -\int_{P_{2R_0}(x_0,t_0)} \left\langle\left(1-a+\varepsilon+a|\nabla u|^{n-2}\right)\nabla_l(\partial_t u),\, |\nabla u|^{\beta}\nabla_l u\right\rangle\phi^n\,dv\,dt \\ =& -\int_{B_{2R_0}(x_0,t_0)} \left(\frac{1-a+\varepsilon}{2+\beta}|\nabla u|^{2+\beta} + \frac{a}{n+\beta}|\nabla u|^{n+\beta}\right)\phi^n(\cdot,t_0)\,dv \\ &+\int_{P_{2R_0}(x_0,t_0)} \left(\frac{1-a+\varepsilon}{2+\beta}|\nabla u|^{2+\beta} + \frac{a}{n+\beta}|\nabla u|^{n+\beta}\right)\partial_t\phi^n\,dv\,dt. \end{aligned} \tag{2.21}$$

Note that the term $I_{4.3}$ will be cancelled.

To estimate the term I_5 of (2.18), integrating by parts twice and using the Ricci formula yield that

$$
\begin{aligned}
I_5 &= \int_{P_{2R_0}(x_0,t_0)} \left\langle \nabla_k \left(\left(\varepsilon + |\nabla u|^{n-2}\right) \nabla_k u\right), \nabla_l \left(|\nabla u|^{\beta} \nabla_l u\right)\right\rangle \phi^n \, dv \, dt \\
&= \int_{P_{2R_0}(x_0,t_0)} \left\langle \nabla_l \left(\left(\varepsilon + |\nabla u|^{n-2}\right) \nabla_k u\right), \nabla_k \left(|\nabla u|^{\beta} \nabla_l u\right)\right\rangle \phi^n \, dv \, dt \\
&\quad + \int_{P_{2R_0}(x_0,t_0)} \left\langle \nabla_l \left(\left(\varepsilon + |\nabla u|^{n-2}\right) \nabla_k u\right), |\nabla u|^{\beta} \nabla_l u\right\rangle \nabla_k \phi^n \, dv \, dt \\
&\quad - \int_{P_{2R_0}(x_0,t_0)} \left\langle \nabla_k \left(\left(\varepsilon + |\nabla u|^{n-2}\right) \nabla_k u\right), |\nabla u|^{\beta} \nabla_l u\right\rangle \nabla_l \phi^n \, dv \, dt \\
&\quad + \int_{P_{2R_0}(x_0,t_0)} \left\langle R_M \# \left(\left(\varepsilon + |\nabla u|^{n-2}\right) \nabla_k u\right), |\nabla u|^{\beta} \nabla_l u\right\rangle \phi^n \, dv \, dt \\
&=: I_{5.1} + I_{5.2} + I_{5.3} + I_{5.4}.
\end{aligned} \tag{2.22}
$$

In (2.22), $I_{5.1}$, $I_{5.2}$, $I_{5.3}$ are main terms to be estimated and $I_{5.4}$ is easily estimated by

$$
|I_{5.4}| \le C \int_{P_{2R_0}(x_0,t_0)} \left(\varepsilon + |\nabla u|^{n-2}\right) |\nabla u|^{2+\beta} \phi^n \, dv \, dt. \tag{2.23}
$$

To estimate $I_{5.1}$, a direct calculation gives

$$
\begin{aligned}
I_{5.1} &= \int_{P_{2R_0}(x_0,t_0)} \left\langle \nabla_l \left(\left(\varepsilon + |\nabla u|^{n-2}\right) \nabla_k u\right), \nabla_k \left(|\nabla u|^{\beta} \nabla_l u\right)\right\rangle \phi^n \, dv \, dt \\
&= \int_{P_{2R_0}(x_0,t_0)} \left(\varepsilon + |\nabla u|^{n-2}\right) |\nabla u|^{\beta} \left|\nabla^2 u\right|^2 \phi^n \, dv \, dt \\
&\quad + (n-2+\beta) \int_{P_{2R_0}(x_0,t_0)} \left(\varepsilon + |\nabla u|^{n-2}\right) |\nabla u|^{\beta} |\nabla(|\nabla u|)|^2 \phi^n \, dv \, dt \\
&\quad + \beta(n-2) \int_{P_{2R_0}(x_0,t_0)} \left(\varepsilon + |\nabla u|^{n-2}\right) |\nabla u|^{\beta-2} |\nabla_l(|\nabla u|) \nabla_l u|^2 \phi^n \, dv \, dt.
\end{aligned} \tag{2.24}
$$

To estimate $I_{5.2}$, we have

$$
\begin{aligned}
|I_{5.2}| &\le \left| \int_{P_{2R_0}(x_0,t_0)} \left\langle \nabla_l \left(|\nabla u|^{n-2}\right) \nabla_k u, |\nabla u|^{\beta} \nabla_l u\right\rangle \nabla_k \phi^n \, dv \, dt \right| \\
&\quad + \frac{1}{2} \left| \int_{P_{2R_0}(x_0,t_0)} \left(\varepsilon + |\nabla u|^{n-2}\right) |\nabla u|^{\beta} \nabla_k |\nabla u|^2 \nabla_k \phi^n \, dv \, dt \right| \\
&\le \frac{n-2+\beta}{2} \int_{P_{2R_0}(x_0,t_0)} \left(\varepsilon + |\nabla u|^{n-2}\right) |\nabla u|^{\beta} |\nabla(|\nabla u|)|^2 \phi^n \, dv \, dt \\
&\quad + \frac{C}{n-2+\beta} \int_{P_{2R_0}(x_0,t_0)} \left(\varepsilon + |\nabla u|^{n-2}\right) |\nabla u|^{\beta+2} \phi^{n-2} |\nabla \phi|^2 \, dv \, dt.
\end{aligned} \tag{2.25}
$$

To estimate $I_{5.3}$, it follows from using equation (2.4) that

$$\begin{aligned} I_{5.3} = &-\int_{P_{2R_0}(x_0,t_0)} \left\langle \left(1-a+\varepsilon+a|\nabla u|^{n-2}\right)\partial_t u,\, |\nabla u|^{\beta}\nabla_l u\right\rangle \nabla_l\phi^n\, dv\, dt \\ &+\int_{P_{2R_0}(x_0,t_0)}\left\langle\left(\varepsilon+|\nabla u|^{n-2}\right)A(u)(\nabla u,\nabla u),\, |\nabla u|^{\beta}\nabla_l u\right\rangle \nabla_l(\phi^n)\, dv\, dt \\ &= I_{4.3} - I_7. \end{aligned} \tag{2.26}$$

This means that these terms $I_{5.3}, I_{4.3}, I_7$ can be cancelled together.

For I_6, we have

$$\begin{aligned} |I_6| \le &\frac{1}{4}\int_{P_{2R_0}(x_0,t_0)} \left(\varepsilon+|\nabla u|^{n-2}\right)|\nabla u|^{\beta}\left|\nabla^2 u\right|^2\phi^n\, dv \\ &+ C\int_{P_{2R_0}(x_0,t_0)}\left(\varepsilon+|\nabla u|^{n-2}\right)|\nabla u|^{4+\beta}\phi^n\, dv. \end{aligned} \tag{2.27}$$

Substituting (2.19)-(2.27) into (2.18), we have

$$\begin{aligned} &\frac{1}{2}\int_{P_{2R_0}(x_0,t_0)}\left(\varepsilon+|\nabla u|^{n-2}\right)|\nabla u|^{\beta}\left|\nabla^2 u\right|^2\phi^n\, dv\, dt \\ &+\frac{(n-2+\beta)}{2}\int_{P_{2R_0}(x_0,t_0)}\left(\varepsilon+|\nabla u|^{n-2}\right)|\nabla u|^{\beta}|\nabla(|\nabla u|)|^2\phi^n\, dv\, dt \\ &+\int_{B_{2R_0}(x_0)}\left(\frac{1-a+\varepsilon}{2+\beta}|\nabla u|^{2+\beta}+\frac{a}{n+\beta}|\nabla u|^{n+\beta}\right)\phi^n(\cdot,t_0)\, dv \\ \le\, &C\int_{P_{2R_0}(x_0,t_0)}\left(\frac{1-a+\varepsilon}{2+\beta}|\nabla u|^{2+\beta}+\frac{a}{n+\beta}|\nabla u|^{n+\beta}\right)\phi^{n-1}|\partial_t\phi|\, dv\, dt \\ &+C\frac{1}{\beta}\int_{P_{2R_0}(x_0,t_0)}\left(\varepsilon+|\nabla u|^{n-2}\right)|\nabla u|^{\beta+2}\phi^{n-2}|\nabla\phi|^2\, dv\, dt \\ &+C\int_{P_{2R_0}(x_0,t_0)}\left(\varepsilon+|\nabla u|^{n-2}\right)\left(|\nabla u|^{2+\beta}+|\nabla u|^{4+\beta}\right)\phi^n\, dv\, dt. \end{aligned} \tag{2.28}$$

On the other hand, by using the Hölder and Sobolev inequalities, we have

$$
\begin{aligned}
&\int_{t_0-4R_0}^{t_0}\int_{B_{2R_0}(x_0)} |\nabla u|^{n+2+\beta}\phi^n \,dv\,dt \\
&\le \int_{t_0-4R_0^2}^{t_0}\left(\int_{B_{2R_0}(x_0)} |\nabla u|^n \,dv\right)^{\frac{2}{n}}\left(\int_{B_{2R_0}(x_0)} |\nabla u|^{\frac{n(n+\beta)}{n-2}}\phi^{\frac{n^2}{n-2}}\,dv\right)^{\frac{n-2}{n}} dt \\
&\le C\varepsilon_1^{\frac{2}{n}}\int_{t_0-4R_0^2}^{t_0}\int_{B_{2R_0^2}(x_0)} \left|\nabla\left(|\nabla u|^{\frac{n+\beta}{2}}\phi^{n/2}\right)\right|^2 dv\,dt \qquad (2.29)\\
&\le C\varepsilon_1^{\frac{2}{n}}\int_{t_0-4R_0^2}^{t_0}\int_{B_{2R_0^2}(x_0)} \Big((n+\beta)^2\left|\nabla^2 u\right|^2|\nabla u|^{n-2+\beta}\phi^n \\
&\qquad\qquad + |\nabla u|^{n+\beta}|\nabla\phi|^2\phi^{n-1}\Big)\,dv\,dt.
\end{aligned}
$$

Choosing ε_1 (depending on β here) sufficiently small yields

$$
\begin{aligned}
&\int_{t_0-4R_0}^{t_0}\int_{B_{2R_0}(x_0)} \left(|\nabla u|^{n+2+\beta} + \left(\varepsilon + |\nabla u|^{n-2}\right)|\nabla u|^{\beta}|\nabla^2 u|^2\right)\phi^n dv\,dt \\
&\le C\int_{P_{2R_0}(x_0,t_0)} \left(1+|\nabla\phi|^2+|\partial_t\phi|\right)|\nabla u|^{n+\beta}\,dv\,dt.
\end{aligned}
$$

This proves our claim. □

Since the constant ε_1 depends on β in Lemma 2.4, we have to get an improved estimate to obtain the gradient estimate in the following:

Lemma 2.5. *Assume that $\varepsilon > 0$ and $a \in [0, 1]$. Let u be a smooth solution to the flow equation (2.4). There exists a positive constant $\varepsilon_0 < i(M)$ such that if for some R_0 with $0 < R_0 < \min\{\varepsilon_0, \frac{t_0^{1/2}}{2}\}$ the inequality*

$$\sup_{t_0-4R_0^2\le t<t_0}\int_{B_{2R_0}(x_0)} |\nabla u|^n \,dv < \varepsilon_0$$

holds, we have

$$\sup_{P_{R_0}(x_0,t_0)} |\nabla u|^n \le C(R_0),$$

where $C(R_0)$ is a constant independent on ε, a.

Proof. Let $\phi = \phi(x,t)$ be a cut-off function with support in $B_R(x_0)\times[t_0-R_0^2, t_0+R_0^2]$ such that $\phi = 1$ in $B_R(x_0)\times[t_0-\rho^2, t_0+\rho^2]$, $|\nabla\phi| \le \frac{C}{R-\rho}$, $|\partial_t\phi| \le \frac{1}{(R-\rho)^2}$ and $|\phi| \le 1$ in $B_R(x_0)\times[t_0-R^2, t_0+R^2]$. For this new cut-off function ϕ, the same proof of (2.28) gives

$$\begin{aligned}
&\frac{1}{2}\int_{P_R(x_0,t_0)}\left(\varepsilon+|\nabla u|^{n-2}\right)|\nabla u|^{\beta}\left|\nabla^2 u\right|^2\phi^n\,dv\,dt\\
&+\frac{(n-2+\beta)}{2}\int_{P_R(x_0,t_0)}\left(\varepsilon+|\nabla u|^{n-2}\right)|\nabla u|^{\beta}|\nabla(|\nabla u|)|^2\phi^n\,dv\,dt\\
&+\sup_{t_0-R^2\le s\le t_0}\int_{B_R(x_0)}\left(\frac{1-a+\varepsilon}{2+\beta}|\nabla u|^{2+\beta}+\frac{a}{n+\beta}|\nabla u|^{n+\beta}\right)\phi^n(\cdot,s)\,dv\\
\le{}& C\int_{P_R(x_0,t_0)}\left(\frac{1-a+\varepsilon}{2+\beta}|\nabla u|^{2+\beta}+\frac{a}{n+\beta}|\nabla u|^{n+\beta}\right)\phi^{n-1}|\partial_t\phi|\,dv\,dt\\
&+C\frac{1}{\beta}\int_{P_R(x_0,t_0)}\left(\varepsilon+|\nabla u|^{n-2}\right)|\nabla u|^{\beta+2}\phi^{n-2}|\nabla\phi|^2\,dv\,dt\\
&+C\int_{P_R(x_0,t_0)}\left(\varepsilon+|\nabla u|^{n-2}\right)\left(|\nabla u|^{2+\beta}+|\nabla u|^{4+\beta}\right)\phi^n\,dv\,dt.
\end{aligned}\tag{2.30}$$

Multiplying (2.30) by β, we have

$$\begin{aligned}
&\beta^2\int_{P_R(x_0,t_0)}\left(\varepsilon+|\nabla u|^{n-2}\right)|\nabla u|^{\beta}|\nabla(|\nabla u|)|^2\phi^n\,dv\,dt\\
&+\sup_{t_0-R^2\le s\le t_0}\int_{B_R(x_0)}(1-a+\varepsilon)\left(|\nabla u|^{2+\beta}+a|\nabla u|^{n+\beta}\right)\phi^n(\cdot,s)\,dv\\
\le{}& C\int_{P_R(x_0,t_0)}\left(|\nabla u|^{2+\beta}+|\nabla u|^{n+\beta}\right)\phi^{n-2}\left(|\partial_t\phi|+|\nabla\phi|^2\right)dv\,dt\\
&+C\beta\int_{P_R(x_0,t_0)}\left(\varepsilon+|\nabla u|^{n-2}\right)\left(|\nabla u|^{2+\beta}+|\nabla u|^{4+\beta}\right)\phi^n\,dv\,dt.
\end{aligned}\tag{2.31}$$

Using Hölder's and Sobolev's inequalities with (2.31), we have

$$\begin{aligned}
&\int_{t_0-\rho^2}^{t_0}\int_{B_\rho(x_0)}|\nabla u|^{(n+\beta)(1+\frac{2}{n}\frac{\beta+2}{\beta+n})}\,dv\,dt\\
\le{}&\int_{t_0-\rho^2}^{t_0}\left(\int_{B_R(x_0)}|\nabla u|^{\beta+2}\phi^n\,dv\right)^{\frac{2}{n}}\left(\int_{B_R(x_0)}|\nabla u|^{\frac{n(n+\beta)}{n-2}}\phi^{\frac{n^2}{n-2}}\,dv\right)^{\frac{n-2}{n}}dt\\
\le{}& C\sup_{t_0-R^2\le t\le t_0}\left(\int_{B_R(x_0)}|\nabla u|^{\beta+2}\phi^n\,dv\right)^{\frac{2}{n}}\int_{t_0-R^2}^{t_0}\int_{B_R(x_0)}\left|\nabla\left(|\nabla u|^{\frac{n+\beta}{2}}\phi^{n/2}\right)\right|^2dv\,dt\\
\le{}& C\left(\int_{P_R(x_0,t_0)}\left(|\nabla u|^{2+\beta}+|\nabla u|^{n+\beta}\right)\phi^{n-2}\left(|\partial_t\phi|+|\nabla\phi|^2\right)dv\,dt\right.\\
&\left.+\beta\int_{P_R(x_0,t_0)}\left(\varepsilon+|\nabla u|^{n-2}\right)\left(|\nabla u|^{2+\beta}+|\nabla u|^{4+\beta}\right)\phi^n\,dv\,dt\right)^{1+\frac{2}{n}\gamma(a)},
\end{aligned}\tag{2.32}$$

where the index $\gamma(a)=1$ if $0\le a\le\frac{1}{2}$ and $\gamma(a)=\frac{\beta+2}{\beta+n}$ if $\frac{1}{2}\le a\le 1$.

Next, we follow [19] to process a Moser's iteration (*e.g.*, see [14]).

Set $R = R_k = R_0(1 + 2^{-k})$, $\rho = R_{k+1} = R_0(1 + 2^{-1-k})$, $\beta = \beta_k = \theta^k(d_0 - 2n) + n - 2$ and $\theta = 1 + 2/n$ with $d_0 > 2n$.

$$d_k = n + \beta_k + 2 = \theta^k(d_0 - 2n) + 2n, \quad d_{k+1} = (n + \beta_k)\left(1 + \frac{2}{n}\frac{\beta_k + 2}{\beta_k + n}\right) = \theta d_k - 4.$$

Using (2.32), we have

$$\int_{P_{k+1}} \left(1 + |\nabla u|^{d_{k+1}}\right) dv\, dt \le C 2^{k\theta} \left(\int_{P_k} \left(1 + |\nabla u|^{d_k}\right) dv\, dt\right)^{\theta},$$

where we denote $P_k = P_{R_k}(x_0, t_0)$ and use the fact that $1 + \frac{2}{n} \le 2$.

Set

$$I_k = \left(\int_{P_k} \left(1 + |\nabla u|^{d_k}\right) dv\, dt\right)^{\frac{1}{\theta^k}}.$$

Applying an iteration, we have

$$I_{k+1} \le C^{\frac{1}{\theta^{k+1}}} 2^{\frac{k}{\theta^k}} I_k \le C^{\sum_{k=1}^{\infty} \frac{1}{\theta^{k+1}}} 2^{\sum_{k=1}^{\infty} \frac{k}{\theta^k}} I_0 \le \tilde{C} I_0.$$

Therefore, noting $d_k = \theta^k(d_0 - 2n) + 2n$ for all $k \ge 1$, we have

$$\begin{aligned} &\left(\int_{P_{R_0}} |\nabla u|^{\theta^{k+1}(d_0-2n)}\, dv\, dt\right)^{\frac{1}{\theta^{k+1}(d_0-2n)}} \\ &\le \left(C \int_{P_{k+1}} \left(1 + |\nabla u|^{d_{k+1}}\right) dv\, dt\right)^{\frac{1}{\theta^{k+1}(d_0-2n)}} \\ &\le C^{\frac{1}{\theta^{k+1}(d_0-2n)}} (\tilde{C} I_0)^{\frac{1}{(d_0-2n)}} \le C(R_0). \end{aligned}$$

This implies that $|\nabla u|$ is bounded in P_{R_0}. □

Lemma 2.6. *Assume that $\varepsilon > 0$ and $a \in [0, 1]$. Let $u : M \to N$ be a smooth solution to the flow equation* (2.4). *There is a small constant $\varepsilon_0 > 0$ such that if the inequality*

$$\sup_{t_0 - T' \le t < t_0} \int_{B_{2R_0}(x_0)} |\nabla u|^n\, dv\, dt < \varepsilon_0,$$

holds for some positive R_0, then $\|u\|_{C^{0,\alpha}}(P_{R_0}((x_0, t_0)))$ is uniformly bounded in ε.

Proof. Using the above Lemma 2.5 , $|\nabla u|$ is bounded by a constant C. By a similar proof of the local energy inequality, we have

$$\begin{aligned} &\int_{P_R(z_0)} \left(\varepsilon + |\nabla u|^{n-2}\right) \left|\frac{\partial u}{\partial t}\right|^2 dv\, dt \\ &\le C \sup_{t_0 - R^2 \le t \le t_0} E_\varepsilon(u(t); B_{2R}(x_0)) \le C R^n. \end{aligned}$$

Set $u_{z_0,R} = \fint_{P_R(z_0)} u(x,t)\,dz$. By a variant of the Sobolev-Poincare inequality, we have

$$\begin{aligned}\int_{P_R(z_0)} |u-u_{z_0,R}|^2\,dv\,dt &\le C\left[R^2\int_{P_R(z_0)}|\nabla u|^2\,dv\,dt + R^4\int_{P_R(z_0)}|\partial_t u|^2\,dv\,dt\right]\\ &\le CR^{n+4}\end{aligned}$$

for all $R \le R_0/2$. This implies that $u(x,t)$ is Hölder continuous near (x_0,t_0). □

Theorem 2.7. *Let $a \in [0,1]$. For any $u_0 \in W^{1,n}(M,N)$, there exists a local solution $u : M\times[0,T_0]\to N$ of the flow equation* (1.3) *with initial value u_0 for a constant T_0 satisfying*

$$\begin{aligned}&\int_0^{T_0}\int_M\left(|\nabla u|^{n+2}+|\nabla^2 u|^2|\nabla u|^{n-2}\right)dv\,dt\\ &\le CE_n(u_0)+C\left(1+T_0R_0^{-2}\right)E_n(u_0).\end{aligned}\tag{2.33}$$

Proof. Since $u_0 \in W^{1,n}(M,N)$ can be approximated by maps in $C^\infty(M,N)$, we assume that u_0 is smooth without loss of generality. For $\varepsilon > 0$ and $a \in [0,1]$, let $u_{a,\varepsilon}$ be a solution of that equation (2.4) with smooth initial value u_0. Note that equation (2.4) is equivalent to

$$\begin{aligned}\frac{\partial u^\beta_{a,\varepsilon}}{\partial t} &= \frac{1}{\left(1-a+\varepsilon+a|\nabla u_{a,\varepsilon}|^{n-2}\right)}\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x_i}\\ &\quad\times\left[\left(\varepsilon+|\nabla u_{a,\varepsilon}|^{n-2}\right)g^{ij}\sqrt{|g|}\frac{\partial}{\partial x_j}u^\beta_{a,\varepsilon}\right]\\ &\quad+\frac{\left(\varepsilon+|\nabla u_{a,\varepsilon}|^{n-2}\right)A^\beta(u_{a,\varepsilon})(\nabla u_{a,\varepsilon},\nabla u_{a,\varepsilon})}{\left(1-a+\varepsilon+a|\nabla u_{a,\varepsilon}|^{n-2}\right)}\\ &:= \sum_{i,k,\alpha} b^{\alpha\beta}_{ij\ a}(\nabla u_{a,\varepsilon})\frac{\partial^2 u^\alpha_{a,\varepsilon}}{\partial x_i\partial x_j}+f(u_{a,\varepsilon},\nabla u_{a,\varepsilon}),\end{aligned}\tag{2.34}$$

where

$$\begin{aligned}&b^{\alpha\beta}_{ij\ a}(\nabla u_{a,\varepsilon})\\ &=\frac{\varepsilon+|\nabla u_{a,\varepsilon}|^{n-2}}{\left(1-a+\varepsilon+a|\nabla u_{a,\varepsilon}|^{n-2}\right)}\left(g^{ij}\delta^{\alpha\beta}+\frac{(n-2)|\nabla u_{a,\varepsilon}|^{n-4}\partial_{x_i}u^\alpha_{a,\varepsilon}\partial_{x_j}u^\beta_{a,\varepsilon}}{\varepsilon+|\nabla u_{a,\varepsilon}|^{n-2}}\right).\end{aligned}$$

For a fixed parameter ε, (2.34) is a parabolic system, so there is a local smooth solution $u_{a,\varepsilon}$ to the rectified gradient flow (2.4) with smooth initial value u_0 in $[0,T_{a,\varepsilon})$ for a maximal existence time $T_{a,\varepsilon}$.

For a fixed $\varepsilon > 0$, there is a constant $\tilde{T} > 0$, depending on the bound of u_0 and its higher derivatives, such that $\tilde{T} \leq T_{a,\varepsilon}$ for all $a \in [0, 1]$. In order to prove the local existence of (1.3), we need to show that there is a uniform constant $T_0 > 0$, depending only on $E_n(u_0)$, such that $T_{a,\varepsilon} \geq T_0$ for all $\varepsilon > 0$ and $a \in [0, 1]$. Since $T_{a,\varepsilon}$ is the maximal existence time of the smooth solution u_ε of the flow (2.4), it follows from using the same in [19, proof of Theorem 1 (Section 2.5)] that there is a constant $T_0 > 0$, depending only on $E_n(u_0)$, ε_0 and R_0, such that for $t \leq T_0$, we have

$$\begin{aligned}\int_{B_{R_0}(x_0)} e_\varepsilon(u_{a,\varepsilon})(\cdot, t)\,dv &\leq \int_{B_{2R_0}(x_0)} e_\varepsilon(u_0)\,dv \\ &\quad + \frac{Ct}{R_0^n}\left(\int_M e_\varepsilon(u_0)\,dv\right)^{1-\frac{1}{n}} < \varepsilon_0.\end{aligned} \tag{2.35}$$

If $\tilde{T} \leq T_0$, then it follows from using Lemma 2.5 that $\nabla u_{a,\varepsilon}$ is bounded in $M \times [0, \tilde{T}]$ by the norm $\|\nabla u_0\|_{L^n(M)}$ and hence $f(u_{a,\varepsilon}, \nabla u_{a,\varepsilon})$ is bounded. By the PDE theory, $\nabla u_{a,\varepsilon}(x,t)$ is continuous in $a \in [0, 1]$ for any $t \leq \tilde{T} < T_\varepsilon$. For any $\tilde{\delta} > 0$, there is a $\eta > 0$ such that for any two $a, a_0 \in [0, 1]$ with $|a - a_0| < \eta$, we have

$$\left| b_{ij\ a}^{\alpha\beta}(\nabla u_{a,\varepsilon})(x,t) - b_{ij\ a_0}^{\alpha\beta}(\nabla u_{a_0,\varepsilon})(x,t)\right| \leq \tilde{\delta}.$$

We assume that $\nabla u_{a_0,\varepsilon}(x,t)$ is Hölder continuous in $M \times [\frac{\tilde{T}}{4}, \tilde{T}]$, with its Hölder norm depending only on the bound of $\nabla u_{a_0,\varepsilon}(x,t)$. In fact, this is known for $a_0 = 0$ (see [19]). Noticing

$$\begin{aligned}&\frac{\partial u_{a,\varepsilon}^\beta}{\partial t} - b_{ij\ a_0}^{\alpha\beta}(\nabla u_{a_0,\varepsilon})\frac{\partial^2 u_{a,\varepsilon}^\alpha}{\partial x_i \partial x_j} \\ &= \left(b_{ij\ a}^{\alpha\beta}(\nabla u_{a,\varepsilon}) - b_{ij\ a_0}^{\alpha\beta}(\nabla u_{a_0,\varepsilon})\right)\frac{\partial^2 u_{a,\varepsilon}^\alpha}{\partial x_i \partial x_j} + f(u_{a,\varepsilon}, \nabla u_{a,\varepsilon}),\end{aligned}$$

we apply the L^p-estimate to obtain that

$$\begin{aligned}&\int_{P_{R/2}(x,\tilde{T})}\left|\frac{\partial u_{a,\varepsilon}}{\partial t}\right|^p dvdt + \int_{P_{R/2}(x,\tilde{T})}\left|\nabla^2 u_{a,\varepsilon}\right|^p dvdt \\ &\leq C\tilde{\delta}\int_{P_R(x,\tilde{T})}\left|\nabla^2 u_{a,\varepsilon}\right|^p dvdt \\ &\quad + C\int_{P_R(x,\tilde{T})}(|f(u_{a,\varepsilon}, \nabla u_{a,\varepsilon})|^p + |u_{a,\varepsilon}|^p)\,dvdt.\end{aligned} \tag{2.36}$$

By a covering argument on M and choosing $\tilde{\delta}$ sufficiently small with $C\tilde{\delta} < \frac{1}{4}$, we have

$$\begin{aligned} &\int_{M\times[\frac{1}{2}\tilde{T},\tilde{T}]}\left|\frac{\partial u_{a,\varepsilon}}{\partial t}\right|^p dxdt + \frac{1}{2}\int_{M\times[\frac{1}{2}\tilde{T},\tilde{T}]}\left|\nabla^2 u_{a,\varepsilon}\right|^p dvdt \\ \le C&\int_{M\times[\frac{1}{4}\tilde{T},\frac{1}{2}\tilde{T}]}\left(|f(u_{a,\varepsilon},\nabla u_{a,\varepsilon})|^p+|u_{a,\varepsilon}|^p + |\nabla^2 u_{a,\varepsilon}|^p\right) dvdt \le C(\tilde{T}). \end{aligned} \tag{2.37}$$

By the Sobolev imbedding theorem of parabolic version, $\nabla u_{a,\varepsilon}$ is also Hölder continuous, depending on $C(\tilde{T})$, uniformly for all $a \in [0, 1]$ and therefore $u_{a,\varepsilon}$ is smooth across to $\tilde{T} \ge T_0$ for all $a \in [0, 1]$. Therefore, for each fixed $\varepsilon > 0$, there is a smooth solution of the flow (2.4) in $[0, T_0]$ satisfying

$$\begin{aligned} &\int_0^{T_0}\int_M |\nabla u_{a,\varepsilon}|^{n+2} + \left|\nabla^2 u_{a,\varepsilon}\right|^2\left(\varepsilon + |\nabla u_{a,\varepsilon}|^{n-2}\right) dv\, dt \\ &\le C E_{n,\varepsilon}(u_0) + C\left(1 + T_0 R_0^{-2}\right) E_{n,\varepsilon}(u_0). \end{aligned} \tag{2.38}$$

As $\varepsilon \to 0$, $u_{a,\varepsilon}$ converges to a map u, which is a solution of the flow equation (1.3) satisfying (2.33) using Lemmas 2.3-2.5. □

Using the above results, we can prove Theorem 1.1.

Proof of Theorem 1.1 . By Theorem 2.7, there is a local solution to the flow equation (1.3) satisfying (2.33). Then, the solution can be extended to $M \times [0, T_1)$ for a maximal time T_1 such that as $t \to T_1$, there is a singular set

$$\Sigma_{T_1} = \bigcap_{R>0}\left\{x \in M : \limsup_{t\to T_1} E_n\left(u(x,t); B_R(x^j)\right) \ge \varepsilon_0\right\}$$

for a constant $\varepsilon_0 > 0$. For $a > 0$, we have a nice local energy inequality in Lemma 2.2. Next, we use a similar argument in [27] to prove the finiteness of the singular set Σ_{T_1}. Let $\{x_j\}_{j=1}^l$ be any finite subset of Σ_{T_1} satisfying

$$\limsup_{t\to T_1} E_n\left(u(x,t); B_R(x^j)\right) \ge \varepsilon_0, \quad \forall R > 0, 1 \le j \le l.$$

We choose $R > 0$ such that $B_{2R}(x_j)$ are disjoint. By using Lemma 2.2, we obtain

$$\begin{aligned} l\varepsilon_0 &\le \sum_{j=1}^{l}\limsup_{t\to T_1} E_n\left(u(x,t); B_R(x^j)\right) \\ &\le \sum_{j=1}^{l}\left[E_n\left(u(x,s); B_{2R}(x^j)\right) + C(a)\frac{(T_1-s)}{R^2} E_n(u_0; M)\right] \\ &\le E_n\left(u(x,t); B_R(x^j)\right) + \frac{l\varepsilon_0}{2} \end{aligned}$$

for any $s \in [T_1 - \frac{\varepsilon_0 R^2}{2C(a)E_n(u_0)}, T_1)$. This implies that $l \leq \frac{2E_n(u_0)}{\varepsilon_0}$, so the singular set Σ_{T_1} is finite. Thus, we continue the above procedure at the initial time T_1 to prove existence of a solution of the flow (1.3) in $[T_1, T_2)$ for a second singular time T_2. By induction, we complete the proof. $\square$

3. Energy identity and neck-bubble decompositions

In this section, let $u(x,t)$ be a solution of the rectified n-flow (1.3) in $M \times [0, T_1)$ in Theorem 1.1. Consider now a sequence of $\{u(x,t_i)\}$ as $t_i \to T_1 \leq \infty$. Then they have uniformly bounded energy; *i.e.*, $E_n(u(t_i); M) \leq E_n(u_0; M)$. As $t_i \to T_1$, $u(x,t_i)$ converges to a map u_{T_1} strongly in $W^{1,n+1}_{\text{loc}}(M \backslash \{x^1, \cdots, x^l\})$ with finite integer l. At each singularity x^j, there is a $R_0 > 0$ such that there is no other singularity inside $B_{R_0}(x^j)$. Moreover, there is a constant $\varepsilon_0 > 0$ such that each singular point x^j for $j = 1, \ldots, l$ is characterized by the condition

$$\limsup_{i \to \infty} E_n\left(u_i; B_R\big(x^j\big)\right) \geq \varepsilon_0,$$

for any $R \in (0, R_0]$. Then there is a $\Theta > 0$ such that as $t_i \to T_1$

$$|\nabla u(x,t_i)|^n dv \to \Theta \delta_{x^j} + |\nabla u_{T_1}|^n dv, \tag{3.1}$$

where δ_{x^j} denotes the Dirac mass at the singularity x^j.

In order to establish the energy identity of the sequence $\{u(x,t_i)\}_{i=1}^{\infty}$, we need to get the neck-bubble decomposition. We recall the removable singularity theorem of n-harmonic maps [6] and the gap theorem: there is a constant $\varepsilon_g > 0$ such that if u is a n-harmonic map on S^n satisfying $\int_{S^n} |\nabla u|^n < \varepsilon_g$, then u is a constant on S^n. For completeness, we give a detailed proof on constructing the bubble-neck decomposition by following the idea of Ding-Tian [8] (also [22]).

Step 1. To find a maximal (top) bubble at the level one (first re-scaling).

First note that $u(x,t_i) \to u_{T_1}$ regularly in $B_{R_0}(x^j)$ away from x^j, where u_{T_1} is a map in $W^{1,n}(M,N)$. Since x^j is a concentration point, we find such that as $t_i \to T_1$,

$$\max_{x \in B_{R_0}(x^j), T_1 - \delta \leq t \leq t_i} |\nabla u(x,t)| \to \infty, \quad r_{i,1} = \frac{1}{\max_{x \in B_{R_0}(x^j),\, T_1 - \delta \leq t \leq t_i} |\nabla u(x,t)|} \to 0$$

for a small $\delta > 0$. In the neighborhood of the singularity x^j, we define the rescaled map

$$\tilde{u}_i(\tilde{x}, \tilde{t}) := u_i\left(x_j + r_{i,1}\tilde{x}, t_i + (r_{i,1})^2 \tilde{t}\right).$$

Then $\tilde{u}_i(x,t)$ satisfies

$$\begin{aligned} &\left((r_{i,1})^{n-2}(1 - a + \varepsilon) + a|\nabla \tilde{u}|^{n-2}\right) \frac{\partial \tilde{u}}{\partial \tilde{t}} \\ &= \operatorname{div}\left(|\nabla \tilde{u}|^{n-2} \nabla \tilde{u}\right) + |\nabla \tilde{u}|^{n-2} A(\tilde{u})(\nabla \tilde{u}, \nabla \tilde{u}), \end{aligned} \tag{3.2}$$

in $B_{R_0 r^{-1}(0)} \times [-1, 0]$ and

$$\begin{aligned}\int_{-1}^{0}\int_{B_{R_0(r_{i,1})^{-1}}(0)} \left(r_{i,1}^{n-2}(1-a+\varepsilon)+a|\nabla\tilde{u}|^{n-2}\right)\left|\frac{\partial\tilde{u}}{\partial\tilde{t}}\right|^2 d\tilde{v}\,d\tilde{t} \\ \le \int_{t_i-(r_{i,1})^2}^{t_i}\int_M \left((1-a+\varepsilon)+a|\nabla u|^{n-2}\right)\left|\frac{\partial u}{\partial t}\right|^2 dv\,dt \to 0.\end{aligned} \tag{3.3}$$

Therefore, there is a $\tilde{t} \in (-1, 0)$ such that

$$\int_{B_{R_0(r_{i,1})^{-1}}(0)} \left(r_{i,1}^{n-2}(1-a+\varepsilon)+a|\nabla\tilde{u}|^{n-2}\right)\left|\frac{\partial\tilde{u}}{\partial t}\right|^2 (\cdot,\tilde{t}\,)\,d\tilde{v} \to 0. \tag{3.4}$$

Using Lemma 2.2, it can be shown that as $i \to \infty$,

$$\left|\nabla u(x, t_i + r_{i,1}^2\tilde{t}_i)\right|^n dv \to \Theta\delta_{x_j} + \left|\nabla u_{T_1}\right|^n dv. \tag{3.5}$$

For simplicity, we set

$$u_i(x) := u\left(x, t_i + r_{i,1}^2\tilde{t}_i\right) \text{ for } x \in B_{R_0}\left(x^j\right), \quad \tilde{u}_i(\tilde{x}) := u\left(x^j + r_{i,1}\tilde{x}, t_i + r_{i,1}^2\tilde{t}_i\right).$$

Since $|\nabla\tilde{u}_i(\tilde{x})| \le 1$ for all $\tilde{x} \in B_{R_0 r_{i,1}^{-1}}(0)$, $\tilde{u}_i$ sub-converges to an n-harmonic map $\omega_{1,j}$ locally in $C^{1,\alpha}(\mathbb{R}^n, N)$ as $i \to \infty$, and $\omega_{1,j}$ can be extended to an n-harmonic map on S^n (see [6]) and is nontrivial due to (3.5). We call $\omega_{1,j}$ the first bubble at the singularity x^j, which satisfies

$$E_n\left(\omega_{1,j}; \mathbb{R}^n\right) = \lim_{R\to\infty}\lim_{i\to\infty} E_n(\tilde{u}_i; B_R(0)) = \lim_{R\to\infty}\lim_{i\to\infty} E_n\left(u_i; B_{Rr_{i,1}}\left(x^j\right)\right). \tag{3.6}$$

Step 2. To find out new bubbles at the second level (second re-scaling).

Assume that for a fixed small constant $\varepsilon > 0$ (to be chosen later), there exist two positive constants δ_0 and R_0 with $R_0 r_{i,1} < 4\delta_0$ such that for all i sufficiently large, we have

$$\int_{B_{2r}\setminus B_r(x^j)} |\nabla u(t_i)|^n dV \le \varepsilon, \tag{3.7}$$

for all $r \in (\frac{Rr_{i,1}}{2}, 2\delta)$, and for all $R \ge R_0$ and $\delta \le \delta_0$.

By using Theorem 1.1, there are finitely many isolated singularities of u at $t = T_1$. Then for a small R_0, it follows from and (3.6) that

$$\begin{aligned}\lim_{i\to\infty} E_n\left(u_i; B_{R_0}\left(x^j\right)\right) = E_n\left(u_{T_1}; B_{R_0}\left(x^j\right)\right) + E_n\left(\omega_{1,j}; \mathbb{R}^n\right) \\ + \lim_{R\to\infty}\lim_{\delta\to 0}\lim_{i\to\infty} E_n\left(u_i; B_\delta \setminus B_{Rr_{i,1}}\left(x^j\right)\right).\end{aligned}$$

If (3.7) is true, it is done and we will prove that there is only single bubble $\omega_{1,j}$ around x^j.

If the assumption (3.7) is not true, then for any two constants R and δ with $Rr_{i,1} < 4\delta, \delta \le \delta_0$ and $R \ge R_0$, there is a number $r_i \in (\frac{Rr_{i,1}}{2}, 2\delta)$ such that

$$\lim_{i\to\infty} \int_{B_{2r_i}\setminus B_{r_i}(x^j)} |\nabla u_i|^n \, dV > \varepsilon. \tag{3.8}$$

Since there is a uniformly energy bound $K = nE_n(u_0; M)$, *i.e.*, $\int_M |\nabla u_i|^n \, dV \le K$, and ε is a fixed constant, we remark that there is no infinitely many number of above $r_i \in (0, \delta_0)$ with disjoint annuluses $B_{2r_i}\setminus B_{r_i}(x^j)$ satisfying (3.8). If $\liminf_{i\to\infty} r_i > 0$, it can be ruled out by choosing δ_0 sufficiently small, so we assume that $\lim_{i\to\infty} r_i = 0$. Similarly, if $\limsup_{i\to\infty} \frac{r_i}{r_{i,1}} < \infty$, it can be rule out choosing R_0 sufficiently large since $\tilde{u}_i$ converges regularly to $\omega_{1,\infty}$ locally in $\mathbb{R}^n$. Therefore, we can assume that $\lim_{i\to\infty} \frac{r_i}{r_{i,1}} = \infty$ up to a subsequence. Since there might be many different numbers $r_i \in (\frac{Rr_{i,1}}{2}, 2\delta)$ satisfying (3.8), we must classify these numbers. For any two numbers r_i and $\tilde{r}_i$ in $(\frac{Rr_{i,1}}{2}, 2\delta)$ satisfying (3.8), they can be classified in different classes by the following properties:

$$\lim_{i\to\infty} \frac{r_i}{\tilde{r}_i} = +\infty \quad \text{or} \quad \lim_{i\to\infty} \frac{r_i}{\tilde{r}_i} = 0; \tag{3.9}$$

$$0 < \liminf_{i\to\infty} \frac{r_i}{\tilde{r}_i} \le \limsup_{i\to\infty} \frac{r_i}{\tilde{r}_i} < \infty. \tag{3.10}$$

We say that $\{r_i\}$ and $\{\tilde{r}_i\}$ are in the same class if they satisfy (3.10). Otherwise, they are in different classes if they satisfy (3.9).

It can be seen that the number of above different classes of $\{r_i\}$ satisfying (3.8) must be finite. Let $\{\tilde{r}_i\}$ be any number satisfying (3.8) in the same class of $\{r_i\}$. Then there is an uniform positive integer N_1 such that

$$\frac{1}{N_1} \le \liminf_{i\to\infty} \frac{r_i}{\tilde{r}_i} \le \limsup_{i\to\infty} \frac{r_i}{\tilde{r}_i} \le N_1. \tag{3.11}$$

Otherwise, it will contradict with the fact that there is no infinitely number of above $r_i \in (0, \delta_0)$ with disjoint annuluses $B_{2r_i}\setminus B_{r_i}(x^j)$ satisfying (3.8). Therefore, these numbers $\tilde{r}_i$ can be ruled out by letting δ_0 sufficiently small and R_0 sufficiently large.

We say that the class of $\{r_i\}$ is smaller than the class of $\{\tilde{r}_i\}$ if $\lim_{i\to\infty} \frac{r_i}{\tilde{r}_i} = 0$, so we can give an order for such equivalent classes by $\{r_{2,i}\} \le \{r_{3,i}\} \le \cdots \le \{r_{L,i}\}$ for some positive integer $L > 0$ depending only on the energy bound K and ε. Then we can separate the neck region $B_\delta \setminus B_{Rr_{i,1}}(x^j)$ by the following finite sum:

$$\begin{aligned}
&E_n\left(u_i; B_\delta \setminus B_{Rr_{1,i}}(x^j)\right) \\
&= E_n\left(u_i; B_\delta \setminus B_{Rr_{L,i}}(x^j)\right) + E_n\left(u_i; B_{Rr_{L,i}}(x^j) \setminus B_{\delta r_{L,i}}(x^j)\right) \\
&\quad + E_n\left(u_i; B_{\delta r_{L,i}}(x^j)\setminus B_{Rr_{L-1,i}}(x^j)\right) + E_n\left(u_i; B_{Rr_{L-1,i}}(x^j)\setminus B_{\delta r_{L-1,i}}(x^j)\right) + \cdots \\
&\quad + E_n\left(u_i; B_{Rr_{2,i}}(x^j) \setminus B_{\delta r_{2,i}}(x^j)\right) + E_n\left(u_i; B_{\delta r_{2,i}}(x^j) \setminus B_{Rr_{1,i}}(x^j)\right).
\end{aligned}$$

For a sequence $\{r_{2,i}\}$ in the smallest class satisfying (3.8) with the fact that $\lim_{i\to\infty}\frac{r_{2,i}}{r_{1,i}}=\infty$ and $\lim_{i\to\infty}r_{2,i}=0$, set

$$\tilde{u}_{2,i}(\tilde{x})=u_i\left(x^j+r_{2,i}\tilde{x}\right).$$

Then we note that

$$\begin{aligned}&\lim_{R\to\infty}\lim_{\delta\to 0}\lim_{i\to\infty}E_n\left(u_i;B_{Rr_{2,i}}(x^j)\setminus B_{\delta r_{2,i}}(x^j)\right)\\ &=\lim_{R\to\infty}\lim_{\delta\to 0}\lim_{i\to\infty}E_n\left(\tilde{u}_{2,i};B_R(0)\setminus B_\delta(0)\right).\end{aligned}$$

Passing to a subsequence, $\tilde{u}_{2,i}$ converges to a ω_2 locally in $B_R(0)\setminus B_\delta(0)$ away from a finite concentration set of $\{\tilde{u}_{2,i}\}$. As $R\to\infty$ and $\delta\to 0$, ω_2 is an n-harmonic map in $\mathbb{R}^n$ by removing singularities. If ω_2 is non-trivial on $\mathbb{R}^n$, then ω_2 is a new bubble, which is different from the bubble ω_1. The above bubble connection ω_2 might be trivial. In this case, there is at least a concentration point $p\in B_2\backslash B_1$ of $\{\tilde{u}_{2,i}\}$ due to (3.8). At each concentration point p of $\tilde{u}_{2,i}$, we can repeat the procedure in Step 1; *i.e.*, at each concentration point p of $\tilde{u}_{2,i}$ in $B_R(0)\setminus B_\delta(0)$, there are sequences $x_i^p\to p$ and $\lambda_i^p\to 0$ such that

$$\tilde{u}_{2,i}\left(x_i^p+\lambda_i^p x\right)\to\omega_{2,p},$$

where $\omega_{2,p}$ is a n-harmonic map on $\mathbb{R}^n$. Note that $\tilde{u}_{2,p,\infty}$ is also a bubble for the sequence $\{u_i(x^j+r_{i,2}x_i^p+r_{i,2}\lambda_i^p x)\}$.

Set $x_i^{2,p}=x_j+r_{i,2}x_i^p$. For each $p\in B_R(0)\setminus B_\delta(0)$, we have

$$\frac{\left|x^j-x_i^{2,p}\right|}{r_i^1}=\frac{r_{i,2}}{r_{i,1}}\left|x_i^p\right|\to\infty \text{ as } i\to\infty.$$

Therefore, the bubble $\omega_{2,p}$ at $p\neq 0$ is different from the bubble ω_1. We continue the above procedure for possible new multiple bubbles at each blow-up point p again. Since there is a uniform bound K for $E_n(u_i;M)$ and each non-trivial bubble on S^n costs at least ε_g of the energy by the gap theorem, the above process must stop after finite steps.

Furthermore, we note

$$\begin{aligned}&\lim_{R\to\infty}\lim_{\delta\to 0}\lim_{i\to\infty}E_n\left(u_i;B_{\delta r_{2,i}}(x^j)\setminus B_{Rr_{1,i}}(x^j)\right)\\ &=\lim_{R\to\infty}\lim_{\delta\to 0}\lim_{i\to\infty}E_n\left(\tilde{u}_{2,i};B_\delta(0)\setminus B_{\frac{Rr_{1,i}}{r_{2,i}}}(0)\right).\end{aligned}$$

Since $\{r_{2,i}\}$ in the smallest class satisfying (3.8) with the fact that $\lim_{i\to\infty}\frac{r_{1,i}}{r_{2,i}}=0$ and $\lim_{i\to\infty}r_{2,i}=0$, we can see that u_i satisfies (3.7) on $B_{r_{2,i}\delta}(0)\setminus B_{Rr_{1,i}}(0)$.

Otherwise, there is a number $r_i \in (\frac{1}{2}Rr_{1,i}, 2\delta r_{2,i})$ satisfying (3.8), r_i must be belong to the class of $\{r_{1,i}\}$ or $\{r_{2,i}\}$. In an equivalent class, it can be ruled out by R sufficiently large or letting δ sufficiently small.

Since $\lim_{i\to\infty} \frac{r_{i,1}}{r_{i,2}} = 0$ and ω_1 is a bubble limiting map for the sequence $\{u_i(x_i^1 + r_i^1 x) = \tilde{u}_{2,i}(\frac{r_i^1}{r_i^2}x)\}$, then $p = 0$ is also a concentration point of $\tilde{u}_{2,i}$ on $\mathbb{R}^n$. Therefore the bubble $\omega_{2,0}$ must be the same bubble ω_1. Since the bubble ω_1 is produced by u_i on $B_{Rr_{1,i}}(x^j)$, we separate it from other bubbles without repeating.

Step 3. To find out all multiple bubbles.

Let $r_{i,3}$ be in the second small class of numbers satisfying (3.8) with $\lim_{i\to\infty} \frac{r_{i,3}}{r_{i,2}} = \infty$ and $\lim_{i\to\infty} r_{i,3} = 0$. Set

$$\tilde{u}_{3,i}(\tilde{x}) = u_i\left(x^j + r_{i,3}\tilde{x}\right).$$

Passing to a subsequence, $\tilde{u}_{3,i}$ converges locally to a ω_3 away from a finite concentration set of $\{\tilde{u}_{3,i}\}$ on $\mathbb{R}^n\backslash\{0\}$. Then we can repeat the argument of Steps 1-2. All bubbles produced by $\tilde{u}_{3,i}$, except for those concentrated in 0, are different from Steps 1-2. By induction, we can find out all bubbles in all cases of the finite different classes. Since there is at least one nontrivial bubble on each different classes, the total number L of equivalent classes depends only on K and ε_g. By the gap theorem of n-harmonic maps on S^n, the above process must stop after finite steps.

In summary, at each class level k, the blow-up happens, there are finitely many blow-up points and bubbles on $\mathbb{R}^n$. At each level k and each bubble point $p_{k,l}$, there are sequences $\tilde{x}_i^{k,l} \to p_{k,l}$ and $r_{i,k} \to 0$ with $\lim_{i\to\infty} \frac{r_{i,k}}{r_{i,k-1}} = \infty$ such that passing to a subsequence, $\tilde{u}_{i,k,l}(x) = u_i(x_i^{k,l} + r_{i,k}x)$ converges to $\omega_{k,l}$, where $\omega_{k,l}$ is an n-harmonic map in $\mathbb{R}^n$, where $x_i^{k,l} = x^j + r_{i,k}\tilde{x}^{k,l}$.

In conclusion, there are finite numbers $r_{i,k}$, finite points $x_i^{k,l}$, positive constants $R_{k,l}, \delta_{k,l}$ and a finite number of non-trivial n-harmonic maps $\omega_{k,l}$ on $\mathbb{R}^n$ such that

$$\begin{aligned}
&\lim_{t_i\to\infty} E_n(u_i; B_{R_0}(x_k)) \\
&= E_n(u_{T_1}; B_{R_0}(x_i)) + \sum_{k=1}^{L}\sum_{l=1}^{J_k} E_n(\omega_{k,l}; \mathbb{R}^n) \\
&\quad + \sum_{k=1}^{L}\sum_{l=1}^{J_k} \lim_{R_{k,l}\to\infty}\lim_{\delta_{k,l}\to 0}\lim_{i\to\infty} E_n\left(\tilde{u}_{k,l,i}; B_{\delta_{k,l}}\backslash B_{R_{k,l}r_{i,k}}\left(x_i^{k,l}\right)\right).
\end{aligned} \tag{3.12}$$

Moreover, at each neck region $B_{\delta_{k,l}}\backslash B_{R_{k,l}r_\alpha^k}(x_i^{k,l})$ in (3.12), for all i sufficiently large, we have

$$\int_{B_{2r}\backslash B_r(x_\alpha^{k,l})} |\nabla\tilde{u}_{k,l,i}|^n dV \le \varepsilon, \tag{3.13}$$

for all $r \in \left(\frac{R_{k,l}r_i^k}{4}, 2\delta_{k,l}\right)$, where ε is a fixed constant to be chosen sufficiently small.

Proof of Theorem 1.3. Wei-Wang [30] proved an energy identity of a sequence of regular approximated n-harmonic maps u_i in $W^{1,n}(M,N)\cap C^0(M,N)$, whose tension fields h_i are bounded in $L^{n/n-1}(M)$. Let $u_i(x)=u(x,t_i)$ satisfy the equation (1.3). In this case, $h_i := (1+a|\nabla u_i|^{n-2})\partial_t u_i$, which is bounded in $L^{n/n-1}(M)$. In fact, using Hölder's inequality, we have

$$\int_M\left(|\nabla u_i|^{n-2}\left|\frac{\partial u_i}{\partial t}\right|\right)^{\frac{n}{n-1}}\le\left(\int_M|\nabla u_i|^n\right)^{\frac{n-2}{2(n-1)}}\left(\int|\nabla u_i|^{n-2}\left|\frac{\partial u_i}{\partial t}\right|^2\right)^{\frac{n}{2(n-1)}}\le C.$$

Under the condition (3.13), we apply [30, Theorem B] to prove

$$\lim_{R_{k,l}\to\infty}\lim_{\delta_{k,l}\to 0}\lim_{i\to\infty}E_n\left(\tilde{u}_{k,l,i};\,B_{\delta_{k,l}}\backslash B_{R_{k,l}r_{i,k}}\left(x_i^{k,l}\right)\right)=0.$$

Therefore, the energy identity follows from (3.12). □

4. Minimizing the n-energy functional in homotopy classes

In this section, we will present some applications of the related n-flow to the problem of minimizing the n-energy functional in a given homotopy class and give a proof of Theorem 1.4. For a map $u: M\to N$, we recall the functional

$$E_{n,\varepsilon}(u,M)=\int_M e_{n,\varepsilon}(u)\,dv, \tag{4.1}$$

where we set $e_{n,\varepsilon}(u)=\frac{\varepsilon}{2}|\nabla u|^2+\frac{1}{n}|\nabla u|^n+\frac{\varepsilon}{n+1}|\nabla u|^{n+1}$.

Let $u_i\in C^\infty(M,N)$ be a minimizing sequence of the n-energy in a homotopy class $[u_0]$. Since a minimizing sequence u_i does not satisfy any equation, we cannot have a good tool to use. Following an idea of the α-harmonic map flow [18], we introduce a modified gradient flow for the functional (4.1) in the following:

$$\begin{aligned}&\left(1-a+\varepsilon+a|\nabla u|^{n-2}+\varepsilon|\nabla u|^{n-1}\right)\frac{\partial u}{\partial t}\\&=\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x_i}\left[\left(\varepsilon+|\nabla u|^{n-2}+\varepsilon|\nabla u|^{n-1}\right)g^{ij}\sqrt{|g|}\frac{\partial}{\partial x_j}u\right]\\&\quad+\left(\varepsilon+|\nabla u|^{n-2}+\varepsilon|\nabla u|^{n-1}\right)A(u)(\nabla u,\nabla u),\end{aligned}\tag{4.2}$$

with initial value $u(0)$ for a small constant $a>0$. Since the minimizing sequence u_i is smooth, there is a sequence ε_i with $\varepsilon_i\to 0$ such that

$$\lim_{i\to\infty}E_{n,\varepsilon_i}(u_i,M)=\lim_{i\to\infty}E_n(u_{\varepsilon_i},M)=\inf_{u\in[u_0]}E_n(u,M). \tag{4.3}$$

Choosing $u(0) = u_i$ to be initial values, there is a sequence of $\varepsilon = \varepsilon_i \to 0$ such that the flow (4.2) has a unique global smooth solution $u_{\varepsilon_i}(x,t)$ on $M \times [0,\infty)$ with $u_{\varepsilon_i}(0) = u_i$.

By (4.2), we have the energy identity

$$E_{n,\varepsilon_i}(u_{\varepsilon_i}(s), M) + \int_0^s \int_M \Big(1 - a + \varepsilon + a|\nabla u_{\varepsilon_k}|^{n-2} + \varepsilon_i |\nabla u|^{n-1}\Big) \left|\frac{\partial u_{\varepsilon_i}}{\partial t}\right|^2 dv\,dt \\ = E_{n,\varepsilon_i}(u_i, M), \tag{4.4}$$

for each $s > 0$. Since u_i is a minimizing sequence, it implies that

$$\lim_{i\to\infty} \int_M \frac{\varepsilon_i}{n} |\nabla u_{\varepsilon_i}(s)|^{n+1}\, dv = 0, \tag{4.5}$$

$$\lim_{i\to\infty} \int_0^s \int_M \left(1 - a + \varepsilon + a|\nabla u_{\varepsilon_i}|^{n-2} + \varepsilon|\nabla u_{\varepsilon_i}|^{n-1}\right) \left|\partial_t u_{\varepsilon_i}\right|^2 dv dt = 0. \tag{4.6}$$

Moreover the sequence $\{u_{\varepsilon_i}(s)\}_{i=1}^\infty$ for each $s > 0$ is also a minimizing sequence in the homotopic class $[u_0]$.

Lemma 4.1. *Let ρ, R be two constants with $\rho < R \le 2\rho$. For any x_0 with $B_{2\rho}(x_0) \subset M$ and for any two $s, \tau \in [0,T)$, we have*

$$\int_{B_\rho(x_0)} e_{n,\varepsilon_i}(u_{\varepsilon_i}(\cdot,s))\,dv - \int_{B_R(x_0)} e_{n,\varepsilon_i}(u_{\varepsilon_i})(\cdot,\tau)\,dv \\ \le C \int_s^\tau \int_M \left(1 + a|\nabla u_{\varepsilon_i}|^{n-2} + \varepsilon_i |\nabla u|^{n-1}\right) |\partial_t u_{\varepsilon_i}|^2\, dv\,dt \\ + C\bigg(\frac{(\tau - s)}{(R-\rho)^2} \int_M e_{n,\varepsilon_i}(u_i)\,dv \int_s^\tau \int_M \left(1 + a|\nabla u_{\varepsilon_i}|^{n-2} + \varepsilon_i|\nabla u_{\varepsilon_i}|^{n-1}\right) \\ |\partial_t u_{\varepsilon_i}|^2\,dv\,dt\bigg)^{1/2}.$$

Proof. Let ϕ be a cut-off function in $B_R(x_0)$ such that $\phi = 1$ in B_ρ and $|\nabla\phi| \le C/(R-\rho)$. The required result follows from multiplying (4.2) by $\phi\partial_t u_{\varepsilon_i}$. □

We can repeat the same steps of Lemma 2.5 to obtain

Lemma 4.2. *There exists a positive constant $\varepsilon_0 < i(M)$ such that if for some R_0 with $0 < R_0 < \min\{\varepsilon_0, \frac{t_0^{1/2}}{2}\}$ the inequality*

$$\sup_{t_0 - 4R_0^2 \le t < t_0} \int_{B_{2R_0}(x_0)} |\nabla u_{\varepsilon_i}|^n\, dv < \varepsilon_0$$

holds, we have

$$\|\nabla u_{\varepsilon_i}\|_{L^\infty(B_{R_0}(x_0))} \le C(R_0)$$

where C is a constant independent of ε and depends on R_0.

Now we complete the proof of Theorem 1.4.

Proof of Theorem 1.4. For a minimizing sequence u_i of the n-energy in the homotopy class, let u be the weak limit of $\{u_i\}_{i=1}^{\infty}$ in $W^{1,n}(M)$. Set

$$\Sigma_0 = \bigcap_{R>0} \left\{ x_0 \in \Omega : B_R(x_0) \subset M, \quad \limsup_{i\to\infty} \int_{B_R(x_0)} |\nabla u_i|^n \, dx \geq \varepsilon_0 \right\},$$

for a small constant $\varepsilon_0 > 0$. It is known that Σ_0 is a set of finite points. For the above sequence $\{u_{\varepsilon_i}(s)\}_{i=1}^{\infty}$, we set

$$\Sigma_s = \bigcap_{R>0} \left\{ x_0 \in \Omega : B_R(x_0) \subset M, \quad \limsup_{i\to\infty} \int_{B_R(x_0)} |\nabla u_{\varepsilon_i}(\cdot, s)|^n \, dx \geq \varepsilon_0 \right\},$$

which is also finite. Applying (4.5)-(4.6) to Lemma 4.1, we obtain that $\Sigma_0 = \Sigma_s$ for all $s > 0$ (see a similar argument to one in [16]). By using Lemmas 4.1-4.2, $|\nabla u_{\varepsilon_i}(x, s)| \leq C(R)$ on $P_R(x_0, s)$ for each $x_0 \in M \backslash \Sigma$ with $B_R(x_0) \subset M$. By this result, we know that $u(x, t)$ is a weak solution to the flow (4.2). Since $u_i(x, t)$ converges weakly to $u(x, t)$ in $W^{1,2}(M \times [0, 1])$, $u(\cdot, t) \equiv u(\cdot, 0) = u$. Then $u(x, t)$ is an n-harmonic map from M to N independent of $t \in [0, 1]$. By the regularity result on n-harmonic maps (*e.g.*, [21]), u is a $C^{1,\alpha}$-map on M.

For any $x_0 \in M \backslash \Sigma$, there is a constant $R > 0$ such that $B_R(x_0) \subset M \backslash \Sigma$. Note that $u_{\varepsilon_i}(\tau)$ converges strongly to u in $W^{1,n}(B_R(x_0))$. As $i \to \infty$, we apply Lemma 4.1 to obtain that

$$\begin{aligned} \int_{B_\rho(x_0)} \frac{1}{n} |\nabla u|^n &\leq \liminf_{i\to\infty} \int_{B_\rho(x_0)} \frac{1}{n} |\nabla u_i|^n \, dv \leq \limsup_{i\to\infty} \int_{B_\rho(x_0)} e_{\varepsilon_i}(u_i) \, dv \\ &\leq \limsup_{i\to\infty} \int_{B_R(x_0)} e_{n,\varepsilon_i}(u_{\varepsilon_i})(\cdot, \tau) \, dv = \int_{B_R(x_0)} \frac{1}{n} |\nabla u|^n \, dv \end{aligned}$$

for any R with $\rho < R$. Letting $R \to \rho$, we have

$$\int_{B_\rho(x_0)} \frac{1}{n} |\nabla u|^n = \lim_{i\to\infty} \int_{B_\rho(x_0)} \frac{1}{n} |\nabla u_i|^n \, dv.$$

This implies that u_i converges strongly to u in $W^{1,n}(B_\rho(x_0))$ and hence strongly in $W^{1,n}_{\text{loc}}(M \backslash \Sigma)$.

Next, we use a similar proof of Sacks-Uhlenbeck [24] to show that $\Sigma_0 = \Sigma_s = \emptyset$ if $\pi_n(N) = 0$. Let $\{u_{\varepsilon_i}(s)\}_{i=1}^{\infty}$ be the above sequence. it is known that $u_{\varepsilon_i}(s)$ converges to u strongly in $W^{1,n+1}_{\text{loc}}(M \backslash \Sigma_s)$. Without loss of generality, we assume that there is one singularity x^1 in Σ_s. Let $\eta(r)$ be a smooth cutoff function in $\mathbb{R}$ with the property that $\eta \equiv 1$ for $r \geq 1$ and $\eta \equiv 0$ for $r \leq 1/2$. For some $\rho > 0$, we define a new sequence of maps $v_i : M \to N$ such that v_i is the same as u_i outside $B_\rho(x_1)$, and for $x \in B_\rho(x_1)$,

$$v_i(x) = \exp_{u(x)} \left(\eta \left(\frac{|x|}{\rho} \right) \exp^{-1}_{u(x)} \circ u_{\varepsilon_i}(x, s) \right),$$

where exp is the exponential map on N. Note that $v_i \equiv u$ on $B_{\rho/2}(x_1)$ and $v_i \equiv u_{\varepsilon_i}(s)$ outside $B_\rho(x_1)$ and that $u_{\varepsilon_i}(s)$ converges to u on $B_\rho(x_1)\backslash B_{\rho/2}(x_1)$ strongly in $W^{1,n+1}$ and thus in C^β for some $\beta > 0$. Hence for sufficiently large i, $v_i(B_{\rho(x_1)} \setminus B_{\rho/2}(x_1))$ lies in a small neighborhood of $u(x_1)$, where $\exp^{-1}_{u(x)}$ is a well defined smooth map (if ρ is small). Since $F(y) = \exp_{u(x)}\left(\eta(\frac{|x|}{\rho})\exp^{-1}_{u(x)} y\right)$ is a smooth map from a neighborhood of $u(x_1)$ into itself, we have

$$\begin{aligned}\int_{B_\rho\backslash B_{\rho/2}(x_1)} |\nabla(v_i - u)|^n\,dv &= \int_{B_\rho\backslash B_{\rho/2}(x_1)} |\nabla(F\circ u_{\varepsilon_i}(s) - F\circ u)|^n\,dv\\ &\le C\int_{B_\rho\backslash B_{\rho/2}(x_1)} |\nabla(u_{\varepsilon_i}(s) - u)|^n\,dv \to 0\end{aligned}$$

as $i \to \infty$. It implies that

$$\|v_i - u\|_{W^{1,n}(M)} \to 0 \tag{4.7}$$

as $i \to \infty$.

Since $\pi_n(N)$ is trivial, v_i is in the same homotopy class as $u_{\varepsilon_i}(s)$. Since $u_{\varepsilon_i}(s)$ is a minimizing sequence of E_{n,ε_i} and $u_{\varepsilon_i}(s)$ converges weakly to u in $W^{1,n}$, we have

$$E_n(u) \le \limsup_{i\to\infty} E_{n,\varepsilon_i}(u_{\varepsilon_i}(s)) \le \limsup_{i\to\infty} E_{n,\varepsilon_i}(v_i) = E_n(u),$$

which implies that $u_{\varepsilon_i}(s)$ converges to u strongly in $W^{1,n}(M,N)$, which means that there is no energy concentration; *i.e.*, $\Sigma_0 = \Sigma_s = \emptyset$. □

5. Minimizing the *p*-energy functional in homotopy classes

For a small $\varepsilon > 0$, we introduce a perturbation of the p-energy functional by

$$E_{p,\varepsilon}(u; M) = \int_M \frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{p}|\nabla u|^p + \frac{\varepsilon}{n+1}|\nabla u|^{n+1}dv. \tag{5.1}$$

The Euler-Lagrange equation associated to this functional is

$$\begin{aligned}&\nabla\cdot\left([\varepsilon + |\nabla u|^{p-2} + \varepsilon|\nabla u|^{n-1}]\nabla u\right)\\ &+\left[\varepsilon + |\nabla u|^{p-2} + \varepsilon|\nabla u|^{n-1}\right]A(u)(\nabla u,\nabla u) = 0.\end{aligned} \tag{5.2}$$

The gradient flow for the above equation is

$$\begin{aligned}\frac{\partial u}{\partial t} &= \operatorname{div}\left[\left(\varepsilon + |\nabla u|^{p-2} + \varepsilon|\nabla u|^{n-1}\right)\nabla u\right]\\ &\quad+\left(\varepsilon + |\nabla u|^{p-2} + \varepsilon|\nabla u|^{n-1}\right)A(u)(\nabla u,\nabla u),\end{aligned} \tag{5.3}$$

with initial value $u(0) = u_0$ in M. If the initial map u_0 is smooth, there is a global smooth solution to (5.3) by using proofs in [19] and [11].

Without loss of generality, we assume $g_{ij} = \delta_{ij}$. Then we have

Lemma 5.1. *Let u be a solution of the equation* (5.3). *Then for all $\rho \le R$ with $B_R(x_0) \subset \Omega$, we have*

$$\begin{aligned}
&\rho^{p-n} \int_{B_\rho(x_0)} \left[\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{p}|\nabla u|^p + \frac{1}{n+1}\varepsilon|\nabla u|^{n+1}\right] dx \\
&\quad + \frac{n(p-2)}{2p} \int_\rho^R r^{p-1-n} \int_{B_r} \varepsilon|\nabla u|^2 \, dx\, dr \\
&\quad + \int_{B_R \setminus B_\rho(x_0)} \left[\frac{1}{2}|\partial_r u|^2 + \frac{1}{n+1}\varepsilon|\nabla u|^{n-1}|\partial_r u|^2\right] r^{p-n} \, dx \\
&= R^{p-n} \int_{B_R(x_0)} \left[\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{p}|\nabla u|^p + \frac{1}{n+1}\varepsilon|\nabla u|^{n+1}\right] dx \\
&\quad + \frac{n+1-p}{n+1} \int_\rho^R \int_{B_r(x_0)} r^{p-1-n}\varepsilon|\nabla u|^{n+1} \, dx\, dr \\
&\quad + \int_\rho^R \int_{B_r} r^{p-1-n} \left\langle \frac{\partial u}{\partial t},\, x_i \nabla_i u \right\rangle dx\, dr.
\end{aligned}$$

Proof. Without loss of generality, we assume that $x_0 = 0$. Multiplying (5.3) by $x_i \nabla_i u$, we have

$$\int_{B_r} \left\langle \frac{\partial u}{\partial t},\, x_i \nabla_i u \right\rangle - \left\langle \operatorname{div}\left(\left(\varepsilon + |\nabla u|^{p-2} + \varepsilon|\nabla u|^{n-1}\right)\nabla u\right),\, x_i \nabla_i u \right\rangle dx = 0.$$

Integration by parts yields that

$$\begin{aligned}
&\int_{B_r} \left\langle \frac{\partial u}{\partial t},\, x_i \nabla_i u \right\rangle dx - \frac{1}{r} \int_{\partial B_r} \left(\varepsilon + |\nabla u|^{p-2} + \varepsilon|\nabla u|^{n-1}\right) |x_i \nabla_i u|^2 \, d\omega \\
&= -\int_{B_r} \left(\varepsilon|\nabla u|^2 + |\nabla u|^p + \varepsilon|\nabla u|^{n+1}\right) \\
&\quad + \frac{1}{2}\left(\varepsilon + |\nabla u|^{p-2} + \varepsilon|\nabla u|^{n-1}\right) x_i \nabla_i \left(|\nabla u|^2\right) dx \\
&= \int_{B_r} \left(\frac{(n-2)\varepsilon}{2}|\nabla u|^2 + \frac{(n-p)}{p}|\nabla u|^p - \frac{\varepsilon}{n+1}|\nabla u|^{n+1}\right) dx \\
&\quad - r \int_{\partial B_r} \left(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{p}|\nabla u|^p + \frac{\varepsilon}{n+1}|\nabla u|^{n+1}\right) d\omega.
\end{aligned}$$

Multiplying by r^{p-1-n} both sides of the above identity, we have

$$\begin{aligned}
&\frac{d}{dr}\left[r^{p-n}\int_{B_r}\left(\frac{\varepsilon}{2}|\nabla u|^2+\frac{1}{p}|\nabla u|^p+\frac{\varepsilon}{n+1}|\nabla u|^{n+1}\right)dx\right]\\
&\quad-\frac{n(p-2)}{2p}r^{p-1-n}\int_{B_r}\varepsilon|\nabla u|^2\,dx+r^{p-1-n}\int_{B_r}\frac{\varepsilon(n+1-p)}{n+1}|\nabla u|^{n+1}dx\\
&=-r^{p-1-n}\int_{B_r}\left(\frac{(n-2)\varepsilon}{2}|\nabla u|^2+\frac{(n-p)}{p}|\nabla u|^p-\frac{\varepsilon}{n+1}|\nabla u|^{n+1}\right)dx\\
&\quad+r^{p-n}\int_{\partial B_r}\left(\frac{\varepsilon}{2}|\nabla u|^2+\frac{1}{p}|\nabla u|^p+\frac{\varepsilon}{n+1}|\nabla u|^{n+1}\right)d\omega\\
&=r^{p-n}\int_{\partial B_r}\left(\varepsilon+|\nabla u|^{p-2}+\varepsilon|\nabla u|^{n-1}\right)|\partial_r u|^2\,d\omega-\int_{B_r}r^{p-1-n}\left\langle\frac{\partial u}{\partial t},\,x_i\nabla_i u\right\rangle dx.
\end{aligned}$$

Then integrating with respect to r from ρ to R yields the result. □

Lemma 5.2. *Let $u_i\in C^\infty(M,N)$ be a minimizing sequence in the homotopy class $[u_0]$. Then there are a sequence of $\varepsilon_i\to 0$ and solutions u_{ε_i} of equation* (5.3) *with initial value u_i such that $u_{\varepsilon_i}(t)$ for all $t\in[0,\infty)$ is also a minimizing sequence in the same homotopy class. Moreover, there is a uniform $\tilde t\in[1/2,1]$ such that*

$$\lim_{i\to\infty}\int_M\frac{\varepsilon_i}{n+1}|\nabla u_{\varepsilon_i}|^{n+1}(\cdot,\tilde t)\,dv+\lim_{i\to\infty}\int_M\left|\partial_t u_{\varepsilon_i}(\cdot,\tilde t)\right|^2dv=0.$$

Proof. Since the minimizing sequence u_i is smooth, there is a sequence $\varepsilon_i\to 0$ such that

$$E_{p,\varepsilon_i}(u_i)\le E_p(u_{\varepsilon_i})+\frac{1}{i},$$

which implies

$$\lim_{i\to\infty}E_{p,\varepsilon_i}(u_i)=\lim_{i\to\infty}E_p(u_{\varepsilon_i})=\inf_{v\in[u_0]}E_p(v). \tag{5.4}$$

Then there is a unique solution $u_{\varepsilon_i}(x,t)$ to the flow (5.3) with initial value $u_{\varepsilon_i}(0)=u_i$. Similar to Lemma 2.1, we have

$$E_{\varepsilon_i}(u_{\varepsilon_i}(\cdot,\tau))+\int_0^\tau\int_M\left|\partial_t u_{\varepsilon_i}\right|^2dvdt=E_{\varepsilon_i}(u_{\varepsilon_i}).$$

This implies that $u_{\varepsilon_i}(x,\tau)$ for τ is a minimizing sequence of E in the homotopy class $[u_0]$, which yields

$$\lim_{i\to\infty}\int_M\frac{\varepsilon_i}{n+1}|\nabla u_{\varepsilon_i}|^{n+1}(\cdot,\tau)\,dv+\int_0^\tau\int_M\left|\partial_t u_{\varepsilon_i}\right|^2dvdt=0. \tag{5.5}$$

Then there is a uniform $\tilde t\in[1/2,1]$ such that

$$\lim_{i\to\infty}\int_M\left|\partial_t u_{\varepsilon_i}(\cdot,\tilde t)\right|^2dv=0.$$ □

Proof of Theorem 1.5. Let $u_i \in C^\infty(M, N)$ be a minimizing sequence in a homotopy class and u its weak limit. If N is a homogeneous manifold, we claim that u is a weak p-harmonic map from M and N.

Let $X = (X^1, \cdots, X^L)$ be a Killing vector on $N \subset \mathbb{R}^L$ as in Hélein [15] and $u = (u^1, \cdots, u^L) \in N$. Let φ be a cut-off function compactly supported in M. Since u_ε is a solution of (5.3), we use $\varphi X(u)$ as a testing vector to get

$$\int_M \Big\langle \nabla_k(\varphi X(u_\varepsilon)), \Big(\varepsilon + |\nabla u_\varepsilon|^{p-2} + \varepsilon\, |\nabla u_\varepsilon|^{n-1}\Big) \nabla_k u_\varepsilon \Big\rangle\, dv = -\int_M \langle \varphi X(u_\varepsilon), \partial_t u_\varepsilon\rangle.$$

Since X is a Killing vector, it implies that $\sum_{l,m=1}^L \nabla_k u_\varepsilon^m \nabla_m X^l \nabla_k u_\varepsilon^l = 0$, so

$$\begin{aligned} &\int_M \Big\langle \nabla_k \varphi\, X(u_\varepsilon), \Big(\varepsilon + |\nabla u_\varepsilon|^{p-2} + \varepsilon\, |\nabla u_\varepsilon|^{n-1}\Big) \nabla_k u_\varepsilon \Big\rangle\, dv \\ &= -\int_M \langle \varphi X(u_\varepsilon), \partial_t u_\varepsilon\rangle. \end{aligned} \tag{5.6}$$

Let u be the weak limit of u_{ε_i} in $W^{1,p}(M \times [0, 1])$ by passing to a subsequence if necessary. By a compact result in [3], $|\nabla u_{\varepsilon_i}|^{p-2}\nabla u_{\varepsilon_i}$ converges weakly to $|\nabla u|^{p-2}\nabla u$ in L^{p^*} with $\frac{1}{p} + \frac{1}{p^*} = 1$. Since u_{ε_i} converges to u strongly in L^p and X is a smooth vector on N, $X(u_{\varepsilon_i})$ converges to $X(u)$ strongly in L^p. Letting ε_i go to zero in equation (5.6) and noting (5.5), we have

$$\int_M \nabla_k\varphi \Big\langle X(u), |\nabla u|^{p-2}\nabla_k u\Big\rangle\, d\mu = 0,$$

which implies

$$\int_M \Big\langle \nabla_k(\varphi X(u)), |\nabla u|^{p-2}\nabla_k u\Big\rangle\, d\mu = 0$$

due to the fact that X is a Killing vector. Since N is a homogeneous space, we apply the construction of a Killing field $\{X_j\}$ by Helein [15] and choose φ_j to obtain that

$$\sum_j \varphi_j X_j(u)$$

is any compactly supported vector field (along u). This implies that u is a weak p-harmonic map. We know that u is a weak solution to the p-harmonic map flow. It follows from (5.5) that u is a map independent of $t \in [0, 1]$. Since $u_{\varepsilon_i}(x, t)$ converges weakly to $u(x, t)$ in $W^{1,2}(M \times [0, 1])$. Hence, $u(\cdot, t) \equiv u(\cdot, 0)$ is a (weakly) p-harmonic map from M to N.

We define

$$\Sigma = \bigcap_{R>0} \left\{ x_0 \in \Omega : B_R(x_0) \subset M, \quad \limsup_{\varepsilon_i \to 0} \frac{1}{R^{n-p}} \int_{B_R(x_0)} |\nabla u_{\varepsilon_i}|^p\, dx \geq \varepsilon_0 \right\},$$

for a sufficiently small constant ε_0. Then, $\mathcal{H}^{n-p}(\Sigma) < +\infty$. For any $x_0 \notin \Sigma$ with $B_{R_0}(x_0) \subset M \backslash \Sigma$, for each $y \in B_{R_0/2}(x_0)$ and for each $\rho \in (0, R_0/2)$, we have

$$\rho^{p-n} \int_{B_\rho(y)} |\nabla u|^p \, dM \leq \lim_{\varepsilon_i \to 0} \rho^{p-n} \int_{B_\rho(y)} |\nabla u_{\varepsilon_i}|^p \, dM < \varepsilon_0, \tag{5.7}$$

for a sufficiently small constant $\varepsilon_0 > 0$. Since u is a weakly p-harmonic map satisfying (5.7), it follows from a similar proof of stationary p-harmonic maps into homogenous manifolds (see [28]) that u belongs to $C^{1,\alpha}_{\text{loc}}(M \backslash \Sigma)$. □

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