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Morrey potentials from Campanato classes

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Abstract. This paper shows that under

$$\begin{cases} 0 < \beta, \kappa \le n; \\ -\infty < \lambda \le n; \\ 1 \le p, q < \infty; \\ p^{-1}(n-\beta) < \alpha < \min\{n, 1+p^{-1}\kappa\}; \\ \lambda = p^{-1}q(\kappa-\alpha p)+n-\beta < \begin{cases} \kappa+\varepsilon \ \forall\, \varepsilon>0 & \text{as } \alpha^{-1}\kappa \le p < \infty \\ \kappa+\varepsilon \ \forall\, \varepsilon>0 & \text{as } 1 < p < \alpha^{-1}\kappa \\ \kappa+\frac{(n-\kappa)(n-\alpha-\beta)}{n-\alpha} & \text{as } 1 = p < \alpha^{-1}\kappa, \end{cases} \end{cases}$$

if μ is a nonnegative Radon measure of finite β-variation on $\mathbb{R}^n$ then the Morrey potential class $I_\alpha L^{p,\kappa}$ embeds continuously into the Campanato class $\mathcal{L}_\mu^{q,\lambda}$, and its converse also holds with μ being admissible.

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1. Introduction

Our starting point is the following classical result on Morrey's inequality under $p \in (n, \infty)$, Poincaré's inequality under $p = n$, and Sobolev's (or Galiardo-Nirenberg-Sobolev's) inequality under $p \in [1, n)$ which plays an important role in analysis, geometry, mathematical physics, partial differential equations, and other related fields; see, *e.g.*, [9,14,15,19].

Theorem 1.1. *Let $u \in C_c^1(\mathbb{R}^n)$,* i.e., *$u$ is C^1-smooth with compact support in $\mathbb{R}^n$. Then*

$$\big\||\nabla u|\big\|_{L^p} \gtrsim \begin{cases} \|u\|_{C^{1-\frac{n}{p}}} \approx \|u\|_{\mathcal{L}^{q,\lambda}} \ \textit{as}\ (p,q)\in(n,\infty)\times[1,\infty) \ \textit{and}\ \lambda = q\left(\frac{n}{p}-1\right); \\ \|u\|_{\mathrm{BMO}} \approx \|u\|_{\mathcal{L}^{q,\lambda}} \ \textit{as}\ (p,q)\in\{n\}\times[1,\infty) \ \textit{and}\ \lambda = q\left(\frac{n}{p}-1\right); \\ \|u\|_{L^{\frac{pn}{n-p}}} \gtrsim \|u\|_{\mathcal{L}^{q,\lambda}} \ \textit{as}\ (p,q)\in[1,n)\times\left[1,\frac{pn}{n-p}\right] \ \textit{and}\ \lambda = q\left(\frac{n}{p}-1\right). \end{cases}$$

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Here and henceforth, $A \approx B$ means $A \gtrsim B \gtrsim A$; while $A \gtrsim B$ means $A \geq cB$ for a constant $c > 0$, and

$$\begin{cases} \|f\|_{C^{1-\frac{n}{p}}} = \sup_{x \neq y \text{ in } \mathbb{R}^n} |f(x) - f(y)||x - y|^{\frac{n}{p}-1} \\ \|f\|_{\mathrm{BMO}} = \sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)} \nu\big(B(x,r)\big)^{-1} \int_{B(x,r)} |f - f_{B(x,r)}|\, d\nu; \\ \|f\|_{L^{\frac{pn}{n-p}}} = \Big(\int_{\mathbb{R}^n} |f|^{\frac{pn}{n-p}}\, d\nu\Big)^{\frac{n-p}{pn}}; \\ \|f\|_{\mathcal{L}^{q,\lambda}} = \sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)} \Big(r^{\lambda-n} \int_{B(x,r)} |f - f_{B(x,r)}|^q\, d\nu\Big)^{1/q}, \end{cases}$$

express the Hölder norm; the John-Nirenberg BMO-norm (*cf.* [10]); the Lebesgue norm; the Campanato norm (*cf.* [7]), respectively, where $d\nu$ is the n-dimensional Lebesgue measure on the Euclidean space $\mathbb{R}^n$ and

$$f_{B(x,r)} = \nu\big(B(x,r)\big)^{-1} \int_{B(x,r)} f\, d\nu$$

is the ν-integral mean value of f over $B(x,r)$, the x-centred Euclidean ball with radius r.

Upon utilizing the following formula (*cf.* [14, page 58])

$$u = \frac{\Gamma\big(\frac{n+1}{2}\big)}{(n-1)\pi^{\frac{n+1}{2}}} I_1 * \left(\sum_{j=1}^{n} R_j D_j u\right) \quad \text{for all} \quad u \in C_c^1(\mathbb{R}^n),$$

where $\Gamma(\cdot)$ is the standard gamma function, I_1 is the first-order form of the $(0,n) \ni \alpha$-order Riesz integral

$$I_\alpha g(x) = (I_\alpha * g)(x) = \int_{\mathbb{R}^n} g(y)|y - z|^{\alpha-n}\, d\nu(y)$$

(whose $I_2 g$ is the Newtonian potential of g generated by the convolution of g with the fundamental gravitation potential in Newton's law of universal gravitation, see Adams [2]);

$$R_j(f) = \lim_{\epsilon\to 0} \frac{\Gamma\big(\frac{n+1}{2}\big)}{\pi^{\frac{n+1}{2}}} \int_{\mathbb{R}^n\setminus B(0,\epsilon)} y_j|y|^{-n-1} f(x-y)\, d\nu(y)$$

is the $\{1,\dots,n\} \ni j$-th Riesz transform of f (where the vector-valued operator $(R_1,\dots,R_n)$ is bounded on the Lebesgue $(1,\infty) \ni p$-space L^p on $\mathbb{R}^n$, see, *e.g.*, [8,23]), and D_j is the partial derivative with respect to x_j, Theorem 1.1 may be regarded as a consequence of the case ($\alpha = 1, \kappa = n$) of the next result due to Xiao for $\infty > p > \kappa/\alpha$ (*cf.* [24, Theorem 1]); Adams for $p = \kappa/\alpha$ (*cf.* [1, Remark 4.1]); and Adams for $1 < p < \kappa/\alpha$ (*cf.* [4, Theorem 3.2]), respectively.

Theorem 1.2. *Let $L^{p,\kappa}$ be the $(0,\infty)\times(-\infty,\infty)\ni(p,\kappa)$-Morrey space of all ν-measurable functions f on $\mathbb{R}^n$ with*

$$\|f\|_{L^{p,\kappa}}=\sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)}\left(r^{\kappa-n}\int_{B(x,r)}|f|^p\,d\nu\right)^{\frac{1}{p}}<\infty.$$

If

$$\begin{cases}1<p<\infty;\\ 1\le q<\infty;\\ 0<\kappa\le n;\\ q\left(\frac{\kappa}{p}-\alpha\right)\le\kappa;\\ 0<\alpha<\min\left\{n,1+\frac{\kappa}{p}\right\},\end{cases}\tag{†}$$

then

$$I_\alpha L^{p,\kappa}\subseteq\begin{cases}C^{\alpha-\frac{\kappa}{p}}=\mathcal{L}^{q,\lambda} & \text{as } p>\kappa/\alpha \text{ and } q\ge 1 \text{ and } \lambda=q\left(\frac{\kappa}{p}-\alpha\right);\\ \mathrm{BMO}=\mathcal{L}^{q,\lambda} & \text{as } p=\kappa/\alpha \text{ and } q\ge 1 \text{ and } \lambda=q\left(\frac{\kappa}{p}-\alpha\right);\\ L^{\frac{p\kappa}{\kappa-\alpha p},\kappa}\subset\mathcal{L}^{q,\lambda} & \text{as } p<\kappa/\alpha \text{ and } q\in\left[1,\frac{p\kappa}{\kappa-\alpha p}\right] \text{ and } \lambda=q\left(\frac{\kappa}{p}-\alpha\right).\end{cases}$$

Of course, the above linkage from the space $\mathcal{L}^{q,\lambda}$ to the three space: $C^{\alpha-\frac{n}{p}}$, BMO and $L^{\frac{p\kappa}{\kappa-\alpha p},\kappa}$ is known (*cf.*, *e.g.*, [18,22,24]). Recently, in [12] (*cf.* [3,5,6,25] for some relevant information) we established such a fundamental restriction principle that if $L^{q,\lambda}_\mu$ stands for the (q,λ)-Morrey space (based on a given nonnegative Radon measure μ on $\mathbb{R}^n$) comprising all μ-measurable functions f on $\mathbb{R}^n$ with

$$\|f\|_{L^{q,\lambda}_\mu}=\sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)}\left(r^{\lambda-n}\int_{B(x,r)}|f|^q\,d\mu\right)^{1/q}<\infty$$

then $I_\alpha:L^{p,\kappa}\to L^{q,\lambda}_\mu$ is continuous when and only when μ is of finite β-variation on $\mathbb{R}^n$, *i.e.*,

$$|\|\mu\||_\beta=\sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)}\mu\big(B(x,r)\big)r^{-\beta}<\infty$$

under

$$\begin{cases}0<\alpha<n;\\ 0<\lambda\le\kappa\le n;\\ 1<p<\frac{\kappa}{\alpha};\\ n-\alpha p<\beta\le n;\\ 0<q=\frac{p(\beta+\lambda-n)}{\kappa-\alpha p},\end{cases}\tag{††}$$

and we left the corresponding restriction problem for $\infty > p \geq \kappa/\alpha$ open. Yet, through introducing the μ-based Campanato space $\mathcal{L}_\mu^{q,\lambda}$ (under $(q,\lambda) \in (0,\infty) \times (-\infty,\infty)$) of all μ-measurable functions f on $\mathbb{R}^n$ with

$$\|f\|_{\mathcal{L}_\mu^{q,\lambda}} = \sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)} \left(r^{\lambda-n} \int_{B(x,r)} |f(y) - f_{B(x,r),\mu}|^q \, d\mu(y) \right)^{\frac{1}{q}} < \infty$$

where

$$f_{B(x,r),\mu} = \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f \, d\mu,$$

and observing Nakai's classification of $\mathcal{L}_\mu^{q,\lambda}$ as seen below (*cf.* [17]), if μ is Ahlfors β-regular for some $\beta \in (0,n]$, namely,

$$\mu\big(B(x,r)\big) \approx r^\beta \quad \text{for all} \quad (x,r) \in \mathbb{R}^n \times (0,\infty),$$

and $(q,\lambda) \in [1,\infty) \times (0,n]$, then:

- As $\beta + \lambda > n$, $\mathcal{L}_\mu^{q,\lambda}$ contains $L_\mu^{q,\lambda}$;
- As $\beta+\lambda = n$, the space $\mathcal{L}_\mu^{q,\lambda}$ is just the μ-based space of functions with bounded variation, denoted by BMO_μ, which consists of all μ-measurable functions f in $\mathbb{R}^n$, obeying

$$\|f\|_{\mathrm{BMO}_\mu} = \sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)} r^{-\beta} \int_{B(x,r)} \left| f(y) - f_{B(x,r),\mu} \right| d\mu(y) < \infty;$$

- As $n - q < \beta + \lambda < n$, the space $\mathcal{L}_\mu^{q,\lambda}$ coincides with $\mathcal{C}^{(n-\lambda-\beta)/q}$.

We recognize that it is possible to settle the previously-mentioned open problem. Below is a natural outcome (unifying and improving both (†) and (††)) which is regarded as a principle of taking the Morrey potential space $I_\alpha L^{p,\kappa}$ from the Campanato space $\mathcal{L}_\mu^{q,\lambda}$, thereby generalizing and improving Theorems 1.2 and 1.1.

Theorem 1.3. *Let μ be a non-negative Radon measure on $\mathbb{R}^n$ and*

$$\begin{cases} 0 < \beta, \kappa \leq n; \\ -\infty < \lambda \leq n \\ 1 \leq p, q < \infty; \\ p^{-1}(n-\beta) < \alpha < \min\left\{n, 1 + p^{-1}\kappa\right\}; \\ \lambda = p^{-1}q(\kappa - \alpha p) + n - \beta < \begin{cases} \kappa + \varepsilon \; \forall\, \varepsilon > 0 & \text{as } \alpha^{-1}\kappa \leq p < \infty \\ \kappa + \varepsilon \; \forall\, \varepsilon > 0 & \text{as } 1 < p < \alpha^{-1}\kappa \\ \kappa + \frac{(n-\kappa)(n-\alpha-\beta)}{n-\alpha} & \text{as } 1 = p < \alpha^{-1}\kappa. \end{cases} \end{cases}$$

The following facts hold:

(i) *If* $|||\mu|||_\beta < \infty$, *then* $I_\alpha : L^{p,\kappa} \to \mathcal{L}_\mu^{q,\lambda}$ *is continuous;*

(ii) *Conversely, if* $I_\alpha : L^{p,\kappa} \to \mathcal{L}_\mu^{q,\lambda}$ *is continuous, then* $|||\mu|||_\beta < \infty$ *under one more condition that* μ *is admissible, namely* $\mu(B_1) \approx \mu(B_2)$ *for any two balls* $B_1, B_2 \subset \mathbb{R}^n$ *with the same radius* $r > 0$ *and their Euclidean distance* $\operatorname{dist}(B_1, B_2) = 2r$.

In accordance with [14, Theorem 1.94] saying that if

$$q > n \quad \& \quad \mu\big(B(x,r)\big) \lesssim \begin{cases} \big(\ln r^{-1}\big)^{-q(1-n^{-1})} & \text{as } r \in (0, 2^{-1}) \\ r^q \ \text{ as } r \in [2^{-1}, \infty), \end{cases}$$

then

$$\left(\int_{\mathbb{R}^n} |u|^q \, d\mu\right)^{1/q} \lesssim \big\| |\nabla u| \big\|_{L^n} \quad \text{for all} \quad u \in C_c^1(\mathbb{R}^n),$$

we see that the extra hypothesis in Theorem 1.3(ii) that μ is admissible is natural. Evidently, any Ahlfors β-regular measure and any translation invariant Radon measure are admissible. Moreover, any doubling Radon measure is admissible, in fact if μ is a doubling measure on $\mathbb{R}^n$, *i.e.*, $\mu(2B) \lesssim \mu(B)$ for any ball B and its double size $2B$, then choosing $B_1 = B(x,r)$, and $B_2 = B(y,r)$ and $\operatorname{dist}(B_1, B_2) = 2r$ gives

$$|x-y| = 4r \ \text{ and } \ \mu(B_1) \le \mu\big(B(y, 8r)\big) \lesssim \mu(B_2)$$

and hence $\mu(B_1) \approx \mu(B_2)$, as required.

In order to provide a simpler and better application of the case $\alpha = 1$ of (†††) in Theorem 1.3 to the regularity of a solution to the p-Laplace equation with a Radon measure-valued being on right hand side, for an open set Ω of $\mathbb{R}^n$, denote by $W^{1,p}(\Omega)$ the space of functions f such that

$$\|f\|_{W^{1,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \|\nabla f\|_{L^p(\Omega)} = \left(\int_\Omega |f|^p \, d\nu\right)^{\frac{1}{p}} + \left(\int_\Omega |\nabla f|^p \, d\nu\right)^{\frac{1}{p}} < \infty.$$

The symbol $W^{1,p}_{\text{loc}}(\Omega)$ stands for the collection of ν-measurable functions f on $\mathbb{R}^n$ such that $f \in W^{1,p}(\Omega_1)$ for any open bounded set $\Omega_1 \subseteq \Omega$. And, the symbol $C_0^\infty(\Omega)$ represents the collection of functions with infinite differentiability and compact support in Ω.

Corollary 1.4. *Let*

$$\begin{cases} 0 < \tau < 1 < n \\ 1 < p, q < \infty \\ \max\{0, \ n-p\} < \beta \le n \\ \lambda = n - \beta - q\tau \le \kappa = p(1-\tau) < n. \end{cases} \tag{††††}$$

Suppose that the Radon measure μ is supported in a bounded open set $\Omega \subset \mathbb{R}^n$ and $u \in W^{1,p}_{\text{loc}}(\Omega)$ is a weak solution of the μ-based p-Laplace equation $-\Delta_p u = \mu$ in the sense of:

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\phi \, d\nu = \int_\Omega \phi \, d\mu \quad \textit{for all} \quad \phi \in C_0^\infty(\Omega).$$

If $|\|\mu\||_\beta < \infty$ and $u|_{\mathbb{R}^n\setminus\Omega} = 0$, then $u \in \mathcal{L}^{q,\lambda}_\mu$.

The rest of this paper is organized as follows: Section 2 comprises four technical lemmas; Section 3 is devoted to verifying Theorem 1.3 and its Corollary 1.4.

2. Four Lemmas

We now state and prove the four of rementioned technical lemmas.

Lemma 2.1. *Let $(q, \lambda) \in [1, \infty) \times (-\infty, n]$ and μ be a nonnegative Radon measure on $\mathbb{R}^n$. Then*

$$2^{-1}\|f\|_{\mathcal{L}^{q,\lambda}_\mu} \le \sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)} \inf_{c\in\mathbb{R}} \left(r^{\lambda-n}\int_{B(x,r)} |f(y)-c|^q \, d\mu(y)\right)^{1/q} \le \|f\|_{\mathcal{L}^{q,\lambda}_\mu}.$$

Proof. Note that the second inequality follows from the definition of $\|\cdot\|_{\mathcal{L}^{q,\lambda}_\mu}$. To see the first inequality, for any $(x,r) \in \mathbb{R}^n \times (0,\infty)$ and $c \in \mathbb{R}$, the Minkowski inequality and the Hölder inequality imply

$$\begin{aligned}&\left(\int_{B(x,r)} |f(y) - f_{B(x,r),\mu}|^q \, d\mu(y)\right)^{1/q}\\ &\le \left(\int_{B(x,r)} |f(y)-c|^q \, d\mu(y)\right)^{1/q} + \big(\mu(B(x,r))\big)^{1/q}|c - f_{B(x,r),\mu}|\end{aligned}$$

and

$$\begin{aligned}&(\mu(B(x,r)))^{1/q}\left|c - f_{B(x,r),\mu}\right|\\ &= (\mu(B(x,r)))^{1/q}\left|\frac{1}{\mu(B(x,r))}\int_{B(x,r)} (f(y)-c)\, d\mu(y)\right|\\ &\le \left(\int_{B(x,r)} |f(y)-c|^q \, d\mu(y)\right)^{1/q},\end{aligned}$$

which leads to

$$\begin{aligned}\|f\|_{\mathcal{L}^{q,\lambda}_\mu} &= \sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)} \left(r^{\lambda-n}\int_{B(x,r)} |f(y)-f_{B(x,r),\mu}|^q \, d\mu(y)\right)^{1/q}\\ &\le 2 \sup_{(x,r)\in\mathbb{R}^n\times(0,\infty)} \inf_{c\in\mathbb{R}} \left(r^{\lambda-n}\int_{B(x,r)} |f(y)-c|^q \, d\mu(y)\right)^{1/q}.\end{aligned}$$

This concludes the proof of Lemma 2.1. □

Lemma 2.2. *Let* $(p, \alpha, \kappa) \in [1, \infty) \times (0, n) \times (0, n]$. *The following facts hold:*

(i) *If* $\max\{0,\ n - \alpha p\} < \beta \le n$ *and* μ *is a nonnegative Radon measure on* $\mathbb{R}^n$ *with* $\||\mu\||_\beta < \infty$, *then*

$$\int_{B(x,r)} |I_\alpha(f 1_{B(x,r)})|\, d\mu \lesssim r^{\beta+\alpha-\kappa/p} \||\mu\||_\beta \, \|f\|_{L^{p,\kappa}}$$
$$\text{for all } (x, r, f) \in \mathbb{R}^n \times (0, \infty) \times L^{p,\kappa};$$

(ii) *If* $0 < \kappa < \alpha p$, *then*

$$\operatorname*{esssup}_{z \in B(x,r)} |I_\alpha(f 1_{B(x,r)})(z)| \lesssim r^{\alpha-\kappa/p} \, \|f\|_{L^{p,\kappa}} \text{ for all } (x,r,f) \in \mathbb{R}^n \times (0,\infty) \times L^{p,\kappa}.$$

Proof. See [6, Theorem 3.1] and its argument. □

Lemma 2.3. *Let*

$$\begin{cases} 0 < \alpha < n \\ 1 \le p,\ q < \infty \\ 0 < \kappa,\ \beta \le n \\ \beta + \alpha p > n \\ p \ge \kappa/\alpha. \end{cases}$$

If μ *is a nonnegative Radon measure on* $\mathbb{R}^n$ *with* $\||\mu\||_\beta < \infty$ *and* $f \in L^{p,\kappa}$ *is supported on a ball* $B(x, r)$, *then*

$$\int_{B(x,r)} |I_\alpha f|^q \, d\mu \lesssim r^{\beta+(\alpha-\kappa/p)q} \||\mu\||_\beta \|f\|^q_{L^{p,\kappa}} \text{ for all } (x, r) \in \mathbb{R}^n \times (0, \infty).$$

Proof. Denote by q' the dual exponent of q, *i.e.*, $1/q + 1/q' = 1$ and $1' = \infty$. Since $p\alpha \ge \kappa$ and $\beta + \alpha p > n$, there exists a pair $(\alpha_1,\ \alpha_2)$ such that

$$\begin{cases} \alpha_1,\ \alpha_2 \in (0, n) \\ \alpha = \frac{\alpha_1}{q} + \frac{\alpha_2}{q'} \\ \beta + \alpha_1 p > n \\ \alpha_2 p > \kappa. \end{cases}$$

Indeed, if we choose $\epsilon > 0$ small enough such that

$$\epsilon < \min\left\{ n - \alpha,\ \frac{\beta + \alpha p - n}{(q-1)p} \right\}$$

and define

$$\alpha_2 = \alpha + \epsilon \qquad \text{and} \qquad \alpha_1 = q\left(\alpha - \frac{\alpha_2}{q'}\right),$$

then it is easy to verify that the pair (α_1, α_2) fulfills all above requirements. Applying the Hölder inequality, we see that for all $y \in B(x, r)$,

$$\begin{aligned}
|I_\alpha f(y)| &\le \int_{\mathbb{R}^n} \frac{|f(z)|}{|y-z|^{n-\alpha}}\,d\nu(z) \\
&\le \left(\int_{\mathbb{R}^n} \frac{|f(z)|}{|y-z|^{n-\alpha_1}}\,d\nu(z)\right)^{1/q} \left(\int_{\mathbb{R}^n} \frac{|f(z)|}{|y-z|^{n-\alpha_2}}\,d\nu(z)\right)^{1/q'} \\
&= \Big(I_{\alpha_1}|f|(y)\Big)^{1/q}\Big(I_{\alpha_2}|f|(y)\Big)^{1/q'},
\end{aligned}$$

which together with Lemma 2.2 yields

$$\begin{aligned}
\int_{B(x,r)} |I_\alpha f|^q\,d\mu &\le \left(\int_{B(x,r)} I_{\alpha_1}(|f|)(y)d\mu(y)\right)\left(\sup_{y\in B(x,r)} I_{\alpha_2}(|f|)(y)\right)^{q/q'} \\
&\lesssim r^{\beta+\alpha_1-\kappa/p+(\alpha_2-\kappa/p)q/q'} \||\mu\||_\beta \|f\|^q_{L^{p,\kappa}} \\
&\approx r^{\beta+(\alpha-\kappa/p)q} \||\mu\||_\beta \|f\|^q_{L^{p,\kappa}}.
\end{aligned}$$

This ends the proof of Lemma 2.3. □

Lemma 2.4. *Let μ be a nonnegative Radon measure on $\mathbb{R}^n$. If μ is admissible and $f \in \mathcal{L}_\mu^{q,\lambda}$ with $(q, \lambda) \in [1, \infty) \times \mathbb{R}$, then*

$$\left(r^{\lambda-n}\int_{B_1} |f(x) - f_{B_2,\mu}|^q\,d\mu(x)\right)^{1/q} \lesssim \|f\|_{\mathcal{L}_\mu^{q,\lambda}}$$

holds for any two balls B_1 and B_2 with the same radius r and $\operatorname{dist}(B_1, B_2) = 2r$.

Proof. By the Minkowski inequality, we see

$$\begin{aligned}
&\left(r^{\lambda-n}\int_{B_1} |f(x) - f_{B_2,\mu}|^q\,d\mu(x)\right)^{1/q} \\
&\le \left(r^{\lambda-n}\int_{B_1} |f(x) - f_{B_1,\mu}|^q\,d\mu(x)\right)^{1/q} + \left(r^{\lambda-n}\mu(B_1)\right)^{1/q} |f_{B_1,\mu} - f_{B_2,\mu}|.
\end{aligned}$$

Clearly, the first term in the right hand side of the above inequality is bounded by $\|f\|_{\mathcal{L}_\mu^{q,\lambda}}$. Thus, it suffices to consider the second term in the right hand side of the above inequality.

Since B_1 and B_2 have the same radius r and $\operatorname{dist}(B_1, B_2) = 2r$, we may choose B as the ball with the same center as that of B_1 but of radius $5r$, so that both

B_1 and B_2 are contained in B. Meanwhile, the fact that μ is admissible gives us that $\mu(B_1) \approx \mu(B_2)$. Applying these facts and the Hölder inequality, we deduce

$$\begin{aligned}
&|f_{B_1,\mu} - f_{B_2,\mu}| \\
&\le |f_{B_1,\mu} - f_{B,\mu}| + |f_{B,\mu} - f_{B_2,\mu}| \\
&\le \frac{1}{\mu(B_1)} \int_{B_1} |f(x) - f_{B,\mu}|\, d\mu(x) + \frac{1}{\mu(B_2)} \int_{B_2} |f(x) - f_{B,\mu}|\, d\mu(x) \\
&\le \left(\frac{1}{\mu(B_1)} \int_{B_1} |f(x) - f_{B,\mu}|^q\, d\mu(x)\right)^{1/q} + \left(\frac{1}{\mu(B_2)} \int_{B_2} |f(x) - f_{B,\mu}|^q\, d\mu(x)\right)^{1/q} \\
&\le \left(\frac{1}{\mu(B_1)} \int_{B} |f(x) - f_{B,\mu}|^q\, d\mu(x)\right)^{1/q} + \left(\frac{1}{\mu(B_2)} \int_{B} |f(x) - f_{B,\mu}|^q\, d\mu(x)\right)^{1/q} \\
&\approx \left(\frac{1}{\mu(B_1)} \int_{B} |f(x) - f_{B,\mu}|^q\, d\mu(x)\right)^{1/q},
\end{aligned}$$

so that

$$\left(r^{\lambda-n}\mu(B_1)\right)^{1/q} \left|f_{B_1,\mu} - f_{B_2,\mu}\right| \lesssim \left(r^{\lambda-n}\int_B |f(x) - f_{B,\mu}|^q\, d\mu(x)\right)^{1/q} \lesssim \|f\|_{\mathcal{L}^{q,\lambda}_{\mu}},$$

as desired. This completes the argument for Lemma 2.4. □

3. Proofs of Theorem 1.3 and Corollary 1.4

Proof of Theorem 1.3(i). Suppose that (†††) holds. Assuming $\||\mu|\|_\beta < \infty$, we shall prove

$$\|I_\alpha f\|_{\mathcal{L}^{q,\lambda}_{\mu}} \lesssim \||\mu|\|_\beta^{1/q} \|f\|_{L^{p,\kappa}} \text{ for all } f \in L^{p,\kappa}$$

according to two cases as seen below.

Case $1 \le p < \kappa/\alpha$. If $p > 1$, then $\lambda \le \kappa$, *i.e.*,

$$\lambda = p^{-1}q(\kappa - \alpha p) + n - \beta < \kappa + \varepsilon \text{ for all } \varepsilon > 0,$$

and hence (†††) indicates that [12, Theorem 1.1] and the Hölder inequality can be used to derive

$$\|I_\alpha f\|_{\mathcal{L}^{q,\lambda}_{\mu}} \lesssim \|I_\alpha f\|_{L^{q,\lambda}_{\mu}} \lesssim \||\mu|\|_\beta^{1/q} \|f\|_{L^{p,\kappa}} \quad \forall f \in L^{p,\kappa}.$$

But, if $p = 1$, then

$$\lambda = n - \beta + q(\kappa - \alpha) < \kappa + (n - \alpha)^{-1}(n - \kappa)(n - \alpha - \beta),$$

and hence, it suffices to prove that for any given ball $B(x,r)$ there exists a constant c such that

$$\left(r^{-\beta+q(\kappa-\alpha)}\int_{B(x,r)}|I_\alpha f(y)-c|^q\,d\mu(y)\right)^{1/q}\lesssim |||\mu|||_\beta^{1/q}\|f\|_{L^{1,\kappa}}. \qquad (\ddagger)$$

To this end, we split $f=f_1+f_2$ through $f_1=f1_{B(x,4r)}$ and $f_2=f1_{\mathbb{R}^n\setminus B(x,4r)}$. In order to deal with f_1, we are partially motivated by the idea of proving [16, Lemma 9]. More precisely: for any $y\in B(x,r)$ we use Minkowski's inequality, [21, (2.4.6)] and $\beta>q(n-\alpha)\geq(n-\alpha)>0$ to obtain

$$\begin{aligned}
&\left(\int_{B(x,r)}\left(I_\alpha f_1(y)\right)^q d\mu(y)\right)^{1/q}\\
&\leq\int_{B(x,4r)}|f(z)|\left(\int_{B(z,5r)}|y-z|^{q(\alpha-n)}\,d\mu(y)\right)^{1/q}d\nu(z)\\
&\lesssim\int_{B(x,4r)}|f(z)|\left(\int_0^{5r}\left(\frac{\mu\big(B(z,t)\big)}{t^{q(n-\alpha)}}\right)\frac{dt}{t}+\frac{\mu\big(B(z,5r)\big)}{(5r)^{q(n-\alpha)}}\right)^{1/q}d\nu(z)\\
&\lesssim\int_{B(x,4r)}|f(z)|\left(\int_0^{5r}|||\mu|||_\beta t^{\beta-q(n-\alpha)}\frac{dt}{t}+|||\mu|||_\beta r^{\beta-q(n-\alpha)}\right)^{1/q}d\nu(z)\\
&\lesssim|||\mu|||_\beta^{1/q}r^{\frac{\beta}{q}+\alpha-\kappa}\|f\|_{L^{1,\kappa}},
\end{aligned}$$

thereby reaching

$$\left(r^{-\beta+q(\kappa-\alpha)}\int_{B(x,r)}|I_\alpha f_1(y)|^q\,d\mu(y)\right)^{1/q}\lesssim |||\mu|||_\beta^{1/q}\|f\|_{L^{1,\kappa}}.$$

Next, choosing

$$c=\frac{1}{\mu(B(x,r))}\int_{B(x,r)}I_\alpha f_2\,d\mu,$$

since $\alpha<1+\kappa/p=1+\kappa$ we find that the forthcoming control of $I_\alpha f_2$ in "case $\infty>p\geq\kappa/\alpha$" actually shows

$$\left(\int_{B(x,r)}\frac{|I_\alpha f_2(y)-c|^q}{r^{\beta-q(\kappa-\alpha)}}\,d\mu(y)\right)^{1/q}\lesssim |||\mu|||_\beta^{1/q}\|f\|_{L^{1,\kappa}},$$

and so that $(\ddagger)$ follows.

Case $\infty>p\geq\kappa/\alpha$. According to Lemma 2.1 and $\lambda\leq\kappa$, *i.e.*,

$$\lambda=n-\beta+q(\kappa/p-\alpha)<\kappa+\varepsilon\ \text{ for all }\ \varepsilon>0,$$

it is enough to prove that for an arbitrary ball $B(x,r)$ there exists a constant c such that

$$\left(r^{-\beta+q(\kappa/p-\alpha)}\int_{B(x,r)}|I_\alpha f(y)-c|^q\,d\mu(y)\right)^{1/q}\lesssim \||\mu\||_\beta^{1/q}\|f\|_{L^{p,\kappa}}. \tag{‡‡}$$

To validate (‡‡), we write

$$\begin{cases} f = f_1+f_2 \\ f_1 = f1_{B(x,4r)} \\ f_2 = f1_{\mathbb{R}^n\setminus B(x,4r)}. \end{cases}$$

Note that Lemma 2.3 gives us that

$$\begin{aligned}\left(\int_{B(x,r)}\frac{|I_\alpha f_1(y)|^q}{r^{\beta-q(\kappa/p-\alpha)}}\,d\mu(y)\right)^{1/q} &\lesssim \||\mu\||_\beta^{1/q}\|f_1\|_{L^{p,\kappa}} \\ &\lesssim \||\mu\||_\beta^{1/q}\|f\|_{L^{p,\kappa}}.\end{aligned} \tag{‡ ‡ ‡}$$

Again, selecting

$$c=\frac{1}{\mu(B(x,r))}\int_{B(x,r)}I_\alpha f_2\,d\mu,$$

we utilize the mean value theorem to derive that if $y\in B(x,r)$ then

$$\begin{aligned}&|I_\alpha f_2(y)-c|\\ &\le \frac{1}{\mu(B(x,r))}\int_{B(x,r)}|I_\alpha f_2(y)-I_\alpha f_2(z)|\,d\mu(z)\\ &\le \frac{1}{\mu(B(x,r))}\int_{B(x,r)}\int_{\mathbb{R}^n\setminus B(x,4r)}\Big||y-w|^{\alpha-n}-|z-w|^{\alpha-n}\Big|\,|f(w)|\,d\nu(w)\,d\mu(z)\\ &\le \frac{1}{\mu(B(x,r))}\int_{B(x,r)}\int_{\mathbb{R}^n\setminus B(x,4r)}|y-z|\sup_{\substack{\xi=\theta y+(1-\theta)z\\ \theta\in(0,1)}}|\xi-w|^{\alpha-n-1}|f(w)|\,d\nu(w)\,d\mu(z)\\ &\approx \frac{1}{\mu(B(x,r))}\int_{B(x,r)}\int_{\mathbb{R}^n\setminus B(x,4r)}|y-z||x-w|^{\alpha-n-1}|f(w)|\,d\nu(w)\,d\mu(z)\\ &\lesssim r\int_{\mathbb{R}^n\setminus B(x,4r)}|x-w|^{\alpha-n-1}|f(w)|\,d\nu(w).\end{aligned}$$

Since the Hölder inequality and $\alpha < 1 + \kappa/p$ imply

$$\begin{aligned}
&\int_{\mathbb{R}^n \setminus B(x,4r)} |x-w|^{\alpha-n-1} |f(w)|\, d\nu(w) \\
&= \sum_{k=2}^{\infty} \int_{2^k r \le |x-w| < 2^{k+1} r} |x-w|^{\alpha-n-1} |f(w)|\, d\nu(w) \\
&\approx \sum_{k=2}^{\infty} (2^k r)^{\alpha-n-1} \int_{2^k r \le |x-w| < 2^{k+1} r} |f(w)|\, d\nu(w) \\
&\lesssim \sum_{k=2}^{\infty} (2^k r)^{\alpha-1} \left((2^k r)^{-n} \int_{2^k r \le |x-w| < 2^{k+1} r} |f(w)|^p\, d\nu(w) \right)^{1/p} \\
&\lesssim \sum_{k=2}^{\infty} (2^k r)^{\alpha-1-\kappa/p} \|f\|_{L^{p,\kappa}} \\
&\lesssim r^{\alpha-1-\kappa/p} \|f\|_{L^{p,\kappa}},
\end{aligned}$$

it follows that

$$|I_\alpha f_2(y) - c| \lesssim r^{\alpha-\kappa/p} \|f\|_{L^{p,\kappa}}$$

and thus

$$\left(\int_{B(x,r)} \frac{|I_\alpha f_2(y) - c|^q}{r^{\beta - q(\kappa/p-\alpha)}}\, d\mu(y) \right)^{1/q} \lesssim \frac{\frac{\|f\|_{L^{p,\kappa}}}{\mu(B(x,r)))^{-1/q}}}{r^{\beta/q - (\kappa/p-\alpha) - (\alpha-\kappa/p)}} \lesssim \frac{\|f\|_{L^{p,\kappa}}}{\||\mu\||_\beta^{-1/q}}. \quad (\ddagger\ddagger\ddagger\ddagger)$$

Combining $(\ddagger\ddagger\ddagger)$ and $(\ddagger\ddagger\ddagger\ddagger)$ yields $(\ddagger\ddagger)$. This concludes the proof of Theorem 1.3(i). □

Proof of Theorem 1.3(ii). Assume that $I_\alpha : L^{p,\kappa} \to \mathcal{L}_\mu^{q,\lambda}$ is continuous. This assumption gives

$$\|I_\alpha f\|_{\mathcal{L}_\mu^{q,\lambda}} \lesssim \|f\|_{L^{p,\kappa}} \quad \text{for all} \quad f \in L^{p,\kappa}.$$

Moreover, suppose that μ is admissible. Given a ball $B(x,r)$ with $x \in \mathbb{R}^n$ and $r \in (0,\infty)$, let $\breve{B} = B(x,r)$ and $\tilde{B} = B(\tilde{x},r)$ such that $|x-\tilde{x}| = 4r$. In other words, $\operatorname{dist}(\breve{B}, \tilde{B}) = 2r$. Next, we let x_0 be the point on the line that connecting x and $\tilde{x}$, with $|x_0 - x| = 5r$ and $|x_0 - \tilde{x}| = 9r$. Denote by B_0 the ball with center x_0 and radius $\frac{r}{2}$. It is easy to verify that if $f_0 = 1_{B_0}$ then

$$f_0 \in L^{p,\kappa} \quad \text{with} \quad \|f_0\|_{L^{p,\kappa}} \lesssim r^{\kappa/p}.$$

Because μ is admissible, Lemma 2.4 yields

$$\left(r^{\lambda-n}\int_{\breve{B}}|I_\alpha f_0(y)-(I_\alpha f_0)_{\tilde{B},\mu}|^q\,d\mu(y)\right)^{1/q}\lesssim\|I_\alpha f_0\|_{\mathcal{L}^{q,\lambda}_\mu}\lesssim\|f_0\|_{L^{p,\kappa}}\lesssim r^{\kappa/p}.$$

Note that for any $y\in\breve{B}$, with $z\in\tilde{B}$ and $w\in B_0$, we have

$$|y-w|\le|y-x|+|x-x_0|+|x_0-w|<r+5r+\frac{r}{2}=\frac{13r}{2}$$

and

$$|z-w|\ge|\tilde{x}-x_0|-|z-\tilde{x}|-|x_0-w|>9r-r-\frac{r}{2}=\frac{15r}{2},$$

so that

$$|y-w|^{\alpha-n}-|z-w|^{\alpha-n}\ge\left(\left(\frac{13}{2}\right)^{\alpha-n}-\left(\frac{15}{2}\right)^{\alpha-n}\right)r^{\alpha-n}.$$

This in turn implies that for any $y\in\breve{B}$,

$$\begin{aligned}|I_\alpha f_0(y)-(I_\alpha f_0)_{\tilde{B},\mu}|&=\left|\frac{1}{\mu(\tilde{B})}\int_{\tilde{B}}\big(I_\alpha f_0(y)-I_\alpha f_0(z)\big)\,d\mu(z)\right|\\&=\left|\frac{1}{\mu(\tilde{B})}\int_{\tilde{B}}\left(\int_{B_0}(|y-w|^{\alpha-n}-|z-w|^{\alpha-n})\,d\nu(w)\right)d\mu(z)\right|\\&\ge\frac{1}{\mu(\tilde{B})}\int_{\tilde{B}}\left(\int_{B_0}\left(\left(\left(\frac{13}{2}\right)^{\alpha-n}-\left(\frac{15}{2}\right)^{\alpha-n}\right)r^{\alpha-n}\right)d\nu(w)\right)d\mu(z)\\&=\left(\left(\frac{13}{2}\right)^{\alpha-n}-\left(\frac{15}{2}\right)^{\alpha-n}\right)r^{\alpha}.\end{aligned}$$

Consequently, we get

$$r^{\kappa/p}\gtrsim\left(r^{\lambda-n}\int_{\breve{B}}|I_\alpha f_0(y)-(I_\alpha f_0)_{\tilde{B},\mu}|^q\,d\mu(y)\right)^{1/q}\gtrsim r^{(\lambda+\alpha q-n)/q}\mu(\breve{B})^{1/q},$$

whence reaching

$$\mu(B(x,r))=\mu(\breve{B})\lesssim r^{q\kappa/p-(\lambda+\alpha q-n)}=r^{\beta},$$

via

$$\lambda=n-\beta+q(\kappa/p-\alpha).$$

This validates $|||\mu|||_\beta<\infty$. Whence completing the argument for Theorem 1.3(ii). □

Proof of Corollary 1.4. According to the argument for [11, Theorem 1.14] (see also [20, Theorem 5.8]), we have $|\nabla u|\in L^{p,\kappa}$. This, along with the representation formula for u in terms of $(R_1,\dots,R_n)$ (which is bounded on $L^{p,p(1-\tau)}$ according to [13, Theorem 6.1(b)]) presented in Section 1 and Theorem 1.3 under (††††), implies $u\in\mathcal{L}^{q,\lambda}_\mu$. □

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