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On Lang's conjecture for some product-quotient surfaces

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Abstract. We prove effective versions of algebraic and analytic Lang's conjectures for product-quotient surfaces of general type with $P_g = 0$ and $c_1^2 = c_2$.

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1. Introduction

Lang's conjecture asserts that curves of fixed geometric genus on a surface of general type form a bounded family. An effective version of this conjecture can be stated in the following way:

Conjecture 1.1 (Lang-Vojta). Let S a smooth projective surface of general type. Then there exist real numbers A, B, and a strict subvariety $Z \subset S$ such that, for any holomorphic map $f : C \to S$ satisfying $f(C) \nsubseteq Z$, where C is a smooth projective curve, it holds

$$\deg f(C) \leq A(2g(C) - 2) + B.$$

Bogomolov proved this conjecture for minimal surfaces of general type satisfying $c_1^2 - c_2 > 0$ [8]. He actually proved that such surfaces have big cotangent bundle, and that the conjecture follows from this fact. Unfortunately, this approach does not provide effective information about A and B. However, effective results for such surfaces have been obtained more recently by Miyaoka [17].

On the other hand, the analytic version of Lang's conjecture is stated as follows:

Conjecture 1.2 (Green-Griffiths-Lang). Let S be a smooth projective surface of general type. Then there exists a strict subvariety $Z \subset S$ such that for any non constant holomorphic map $f : \mathbb{C} \to S$,

$$f(\mathbb{C}) \subset Z.$$

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Bogomolov's result has been generalized to the analytic case by McQuillan in his proof of this conjecture for minimal surfaces of general type with $c_1^2 - c_2 > 0$ given in [16].

Here, we are interested in *product-quotient surfaces*, *i.e.*, in the minimal resolutions of quotients $X := (C_1 \times C_2)/G$, where C_1 and C_2 are two smooth projective curves of respective genera $g(C_1), g(C_2) \geq 2$, and G is a finite group, acting faithfully on each of them and diagonally on the product. These surfaces generalize the so-called Beauville surfaces (the particular case where the group action is free).

The classification of product-quotient surfaces with geometric genus $p_g = 0$ was started by I. Bauer and F. Catanese in [3]; they classified the surfaces $X = (C_1 \times C_2)/G$ with G being an abelian group acting freely and $p_g(X) = 0$. Later in [4], both of them and F. Grunewald, extended this classification to the case of an arbitrary group G. Not long after, R. Pignatelli joined them, and in [5] they dropped the assumption that G acts freely on $C_1 \times C_2$; they classified product-quotient surfaces with $p_g = 0$ whose quotient model X has at most canonical singularities. Finally in [6], I. Bauer and R. Pignatelli dropped any restriction on the singularities of X and gave a complete classification of product-quotient surfaces with $p_g = 0$ and $c_1^2 > 0$. It turns out that all except one are in fact minimal surfaces (see [6, Tables 1 and 2]).

In this paper, we prove Conjectures 1.1 and 1.2, when S is a product-quotient surface of general type with geometric genus $p_g = 0$ and $c_1^2 - c_2 = 0$. Note that $p_g = 0$ implies $c_1^2 + c_2 = 12$, then the condition $c_1^2 - c_2 = 0$ is equivalent to $c_1^2 = c_2 = 6$. These surfaces are a limit case not covered by Bogomolov's theorem; however, they satisfy the criterion given in [20, Theorem 1], which ensures the bigness of their cotangent bundle.

In order to accomplish this, we start by proving the following result for product-quotient surfaces in general.

Theorem 1.3. *Let S be a product-quotient surface. If $f : C \to S$ is a holomorphic map such that $f(C) \not\subseteq E$, where C is a smooth projective curve and E is the exceptional divisor on S, then*

$$\deg f^*(K_S - E) \leq 2(2g(C) - 2).$$

Note that Theorem 1.3 is interesting only when $K_S - E$ is positive (ample or big). In Conjecture 1.1, one can take $\deg f(C) = \deg f^*\mathcal{L}$ for $\mathcal{L}$ a positive line bundle on S; hence, provided that the divisor $K_S - E$ is big, Theorem 1.3 implies Conjecture 1.1 for the case of product-quotient surfaces. However, it is not known whether $K_S - E$ is ample or even big in general and in view of the conjecture it is foreseen to be hard to determine.

We then restrict ourselves to the particular case where S is a product-quotient surface of general type with geometric genus $p_g = 0$ and $c_1^2 = 6$. In this case we prove that $K_S - E$ is big and we obtain Conjecture 1.1. We also give an alternative proof for the bigness of Ω_S, producing explicit symmetric tensors on S coming from $K_S - E$, which allows us to control rational curves on it. More precisely we prove the following theorem.

Theorem 1.4. *Let S be a product-quotient surface of general type such that $p_g(S) = 0$. If $c_1(S)^2 = 6$, then the following facts hold*

(1) *The line bundle $K_S - E$ and the cotangent bundle Ω_S are big;*
(2) *For any non constant holomorphic map $f : \mathbb{P}^1 \to S$,*

$$f\left(\mathbb{P}^1\right) \subset E \cup \mathbb{B}(K_S - E),$$

where E is the exceptional divisor on the resolution S and $\mathbb{B}(K_S - E)$ is the stable base locus of $K_S - E$.

Finally, our approach also lets us control elliptic and more generally, entire curves on S. More precisely, we prove Conjecture 1.2 in our context.

Theorem 1.5. *Let S be a product-quotient surface of general type such that $p_g(S) = 0$. If $c_1(S)^2 = 6$, then for any non constant holomorphic map $f : \mathbb{C} \to S$,*

$$f(\mathbb{C}) \subset E \cup \mathbb{B}^+(K_S - E),$$

where E is the exceptional divisor on the resolution S and $\mathbb{B}^+(K_S - E)$ is the augmented base locus of $K_S - E$.

At this point one may ask if the methods used to prove Theorems 1.4 and 1.5 could be extended to more general product-quotient surfaces than the single family with $c_1^2 = c_2$. It turns out that as c_1^2 gets smaller than c_2, the problem of determining whether $K_S - E$ is big gets more difficult. In fact, our approach still works for the case $c_1^2 = 5$ and $c_2 = 7$ but not for $c_1^2 \leq 4$.

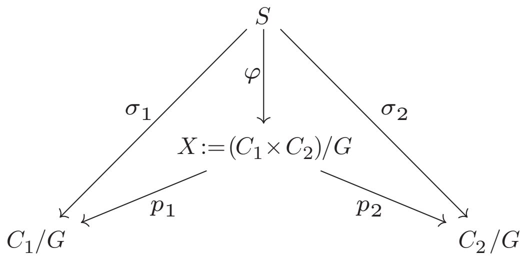
2. Preliminaries

In this section we are going to recall some definitions and results that will be used throughout this paper.

2.1. Product-quotient surfaces

Definition 2.1. A *product-quotient surface S* is the minimal resolution of the singularities of a quotient $X := (C_1 \times C_2)/G$, where C_1 and C_2 are two smooth projective curves of respective genera $g(C_1), g(C_2) \geq 2$, and G is a finite group, acting faithfully on each of them and diagonally on the product. The surface $X := (C_1 \times C_2)/G$ is called *the quotient model* of S [6].

Let S be a product-quotient surface. Let $\varphi : S \to X$ be the resolution morphism of the singularities of $X := (C_1 \times C_2)/G$, and let $p_1 : X \to C_1/G$ and $p_2 : X \to C_2/G$ be the two natural projections. Let us define $\sigma_1 : S \to C_1/G$ and $\sigma_2 : S \to C_2/G$ to be the compositions $p_1 \circ \varphi$ and $p_2 \circ \varphi$ respectively. Thus, we have the following commutative diagram encoding all this information:



The surface $X := (C_1 \times C_2)/G$ has a finite number of singularities, since there are finitely many points on $C_1 \times C_2$ with non trivial stabilizer. Moreover, since G is finite, the stabilizers are cyclic groups [12, III 7.7] and so, the singularities of X are cyclic quotient singularities. Thus, if $(x, y) \in C_1 \times C_2$ with non trivial stabilizer $H_{(x,y)}$, then, around the singularity $\overline{(x, y)} \in X := (C_1 \times C_2)/G$, X is analytically isomorphic to the quotient $\mathbb{C}^2/(\mathbb{Z}/n\mathbb{Z})$, where $n = |H_{(x,y)}|$ and the action of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ on $\mathbb{C}^2$ is defined by $\xi(z_1, z_2) = (\xi z_1, \xi^a z_2)$, where n and a are coprime integers such that $1 \leq a \leq n-1$ and $\xi = \exp(\frac{2\pi i}{n})$ is a chosen primitive n-th root of unity. In this case, the cyclic quotient singularity is called *singularity of type* $\frac{1}{n}(1, a)$. Note that singular points of type $\frac{1}{n}(1, a)$ are also of type $\frac{1}{n}(1, a')$ where a' is the multiplicative inverse of a in $(\mathbb{Z}/n\mathbb{Z})^*$ (see [2, III]).

The exceptional fiber of a cyclic quotient singularity of X of type $\frac{1}{n}(1, a)$ on the the minimal resolution S, is a *Hizerbruch-Jung string* (H-J string), that is to say, a connected union $L = \sum_{i=0}^{l} Z_i$ of smooth rationals curves $Z_1, \ldots, Z_l$ with self-intersection numbers less or equal than -2, and ordered linearly so that $Z_i Z_{i+1} = 1$ for all i and $Z_i Z_j = 0$ if $|i - j| \geq 2$ (see [2, III Theorem 5.4]) and [13]. Then, the *exceptional divisor* E on the minimal resolution S is the connected union of disjoint H-J strings each of them being the fiber of each singularity of $X := (C_1 \times C_2)/G$.

The self-intersection numbers $Z_i^2 = -b_i$ are given by the formula

$$\frac{n}{a} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\cdots - \cfrac{1}{b_l}}}$$

Slightly abusing and side notation, we denote the right of the previous formula by $[b_1, \cdots, b_l]$.

Moreover,

$$\frac{n}{a} = [b_1, \cdots, b_l] \text{ if and only if } \frac{n}{a'} = [b_l, \cdots, b_1].$$

Note that for cyclic quotient singularities of type $\frac{1}{n}(1, n-1)$ we have that all the curves Z_i have self-intersection equal to -2. These singularities are then a particular case of canonical surface singularities (the latter are also known as du Val singularities or rational double points).

On the other hand, Serrano's paper [21, Proposition 2.2] tells us that the irregularity of S, defined by $q(S) := h^1(S, \mathcal{O}_S)$, is given by the formula:

$$q(S) = g(C_1/G) + g(C_2/G).$$

Now, if S is of general type, then $q(S) \leq p_g(S)$. Therefore, we have that S is a product-quotient surface of general type with $p_g = 0$ if and only if $\chi(\mathcal{O}_S) = 1$ and $C_1/G \cong C_2/G \cong \mathbb{P}^1$. Moreover, using Noether's formula we see that the condition $p_g = 0$ also implies that $c_1^2 + c_2 = 12$.

Thanks to the work that I. Bauer, F. Catanese, F. Grunewald and R. Pignatelli, started and carried through in [3–6], we have a complete classification of product-quotient surfaces of general type with geometric genus $p_g = 0$ and $c_1^2 > 0$. Moreover, they proved that there are exactly 73 irreducible families of surfaces of this kind, and all but one of them, are in fact minimal surfaces; more precisely they proved the following result.

Theorem 2.2 ([6], Theorems 4.8, 5.1). *The following facts hold:*

1. *Minimal product-quotient surfaces of general type with $p_g = 0$ form exactly 72 irreducible families;*
2. *There is exactly one product-quotient surface with $K_S^2 > 0$ which is non minimal. It is called fake Godeaux surface. It has $K_S^2 = 1$ and its minimal model has $K^2 = 3$.*

The irreducible families mentioned in the first part of this theorem, are listed in [6, Tables 1, 2].

2.2. Isotrivial fibrations

Definition 2.3. A *fibration* is a surjective morphism from a smooth projective surface into a smooth curve, with connected fibers. A fibration is called *isotrivial fibration*, if all its smooth fibers are mutually isomorphic. A surface is called *isotrivial surface* if it admits an isotrivial fibration.

A product-quotient surface S is an example of an isotrivial surface: it admits two natural isotrivial fibrations $\sigma_1 : S \to C_1/G$ and $\sigma_2 : S \to C_2/G$ whose smooth fibers are all isomorphic to C_2 and C_1 respectively. In literature, product-quotient surfaces are also called standard isotrivial surfaces.

In Serrano's paper [21] it is proved that any isotrivial surface is birationally equivalent to a standard one, more precisely, if $\gamma : Z \to C$ is isotrivial, then there exist a quotient $(C_1 \times C_2)/G$ where C_1 is isomorphic to the general fiber of γ and G is a finite group, acting faithfully on C_1 and C_2 and diagonally on the product; such that Z is birational to $(C_1 \times C_2)/G$, the curve C is isomorphic to C_2/G, and the following diagram commutes:

$$\begin{array}{ccc} Z & \overset{\sim}{\dashrightarrow} & (C_1\times C_2)/G \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle p_2} \\ C & \xrightarrow{\quad\sim\quad} & C_2/G \end{array}$$

We also find in Serrano's paper a description of the singular fibers that can arise in a standard isotrivial surface, *i.e.*, the possible singular fibers of its natural fibrations. Namely:

Theorem 2.4 ([21], Theorem 2.1). *Let S a standard isotrivial surface and let consider the fibration $\sigma_2 : S \to C_2/G$. Let $y \in C_2$ and H_y its stabilizer. If F is the fiber of σ_2 over $\overline{y} \in C_2/G$, then:*

(1) *The reduced structure of F is the union of an irreducible smooth curve Y, called the central component of F, and either none or at least two mutually disjoint H-J strings, each one meeting Y at one point. These strings are in one-to-one correspondence with the branch points of $C_1 \to C_1/H_y$;*
(2) *The central component Y is isomorphic to C_1/H_y and it has multiplicity equal to $|H_y|$ in F. The intersection of a string with Y is transversal, and it takes place at only one of the end components of the string;*
(3) *If $L = \sum_{i=1}^{n} Z_i$ is a H-J string on F and Y' is the central component of the fiber of $\sigma_1 : S \to C_1/G$ over $\sigma_1(L)$, then L meets Y' and Y at opposite ends,* i.e., *either $Z_1Y = Z_nY' = 1$ or $Z_nY = Z_1Y' = 1$.*

If F contains exactly r H-J strings $L_1, \cdots, L_r$, where each L_i is the resolution of a cyclic quotient singularity of type $\frac{1}{n_i}(1, a_i)$, then we know that the central component Y satisfies

$$Y^2 = -\sum_{i=1}^{r} \frac{a_i}{n_i} \tag{2.1}$$

[18, Proposition 2.8]. The previous theorem holds as well for the fibration σ_1.

Finally, Serrano's paper also provides an expression for the canonical bundle of a standard isotrivial surface in terms of the fibers of the two natural fibrations. Namely, we have the following.

Theorem 2.5 ([21], Theorem 4.1). *Let S be a standard isotrivial surface with associated fibrations $\sigma_1 : S \to C_1/G$ and $\sigma_2 : S \to C_2/G$. Let $\{n_iN_i\}_{i\in I}$ and $\{m_jM_j\}_{j\in J}$ denote the components of all singular fibers of σ_1 and σ_2 respectively, with their multiplicities attached. Finally, let $\{Z_t\}_{t\in T}$ be the set of curves contracted to points by $\sigma_1 \times \sigma_2$, i.e, the exceptional locus on S. Then we have*

$$K_S = \sigma_1^*(K_{C_1/G}) + \sigma_2^*(K_{C_2/G}) + \sum_{i\in I}(n_i - 1)N_i + \sum_{j\in J}(m_i - 1)M_i + \sum_{t\in T} Z_t.$$

The fibrations $\sigma_1 : S \to C_1/G$ and $\sigma_2 : S \to C_2/G$ can be thought as foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ on S, such that Serrano's formula can be written as follows:

$$K_S = \mathcal{N}^*_{\mathcal{F}_1} \otimes \mathcal{N}^*_{\mathcal{F}_2} \otimes \mathcal{O}_S(E), \tag{2.2}$$

where $\mathcal{N}^*_{\mathcal{F}_1}$ and $\mathcal{N}^*_{\mathcal{F}_2}$ are the respective conormal line bundles, and E is the exceptional divisor on S (see [9, page 30]).

3. Proof of Theorem 1.3

The purpose of this section is to prove Theorem 1.3. Let us begin by recalling some basic facts that will be used. Let S be a product-quotient surface, C a smooth projective curve and $f : C \to S$ a holomorphic map such that $f(C) \not\subseteq E$. The differential map $df : T_C(-\log f^{-1}E) \to T_S(-\log E)$ induces a lifting $f_{[1]} : C \to \mathbb{P}(T_S(-\log E))$. Thus we get the following diagram:

$$\begin{array}{ccc} & & \mathbb{P}(TS(-\log E)) \\ & \overset{f_{[1]}}{\nearrow} & \Big\downarrow \pi \\ C & \xrightarrow{\quad f \quad} & S \end{array}$$

Moreover, we have that $\pi_*\mathcal{O}_{\mathbb{P}(T_S(-\log E))}(1) \simeq \Omega_S(\log E)$. On the other hand, recall that for each foliation $\mathcal{F}$ on S, we have the logarithmic exact sequence

$$0 \longrightarrow \mathcal{N}^*_{\mathcal{F}}(E) \longrightarrow \Omega_S(\log E) \longrightarrow \Omega_{\mathcal{F}}(\log E) \longrightarrow 0$$

and we can also define the divisor $Z := \mathbb{P}(T_{\mathcal{F}}(-\log E))$ on $\mathbb{P}(T_S(-\log E))$.

Lemma 3.1. *Let $\mathcal{F}$ be a foliation on S, C a smooth projective curve and $f : C \to S$ a holomorphic map such that $f(C) \not\subseteq E$. If f is not tangent to $\mathcal{F}$, then*

$$\deg f^*\mathcal{N}^*_{\mathcal{F}}(E) \leq 2g(C) - 2 + N_1(E),$$

where $N_1(E)$ is the number of points on $f^{-1}(E)$ counted without multiplicities.

Proof. For the sake of simplicity we denote by $\mathcal{O}(1)$ the line bundle

$$\mathcal{O}_{\mathbb{P}(T_S(-\log E))}(1).$$

Let us consider the exact sequence

$$0 \longrightarrow \mathcal{O}(1) - [Z] \longrightarrow \mathcal{O}(1) \longrightarrow \mathcal{O}(1)|_Z \longrightarrow 0$$

Now, taking the push-forwards we get

$$0 \longrightarrow \pi_*(\mathcal{O}(1) - [Z]) \longrightarrow \Omega_S(\log E) \longrightarrow \Omega_{\mathcal{F}}(\log E) \longrightarrow 0$$

and thus we obtain $\pi_*(\mathcal{O}(1) - [Z]) \simeq \mathcal{N}^*_{\mathcal{F}}(E)$. On the other hand, since f is not tangent to $\mathcal{F}$ then $f_{[1]}(C) \not\subseteq Z$, thus $f_{[1]}(C).Z \geq 0$ and hence $\deg f^*_{[1]}[Z] \geq 0$. Therefore,

$$\deg f^*\mathcal{N}^*_{\mathcal{F}}(E) \leq \deg f^*_{[1]}\mathcal{O}(1).$$

Moreover, the differential map

$$df : T_C\left(-f^{-1}(E)\right) \longrightarrow f_{[1]}^{*}\mathcal{O}(-1)$$

defines a non zero section of the line bundle $f_{[1]}^{*}\mathcal{O}(-1) \otimes K_C(f^{-1}(E))$ implying that this line bundle is effective. Then,

$$\deg f_{[1]}^{*}\mathcal{O}(1) \leq \deg K_C\left(f^{-1}(E)\right) = 2g(C) - 2 + N_1(E). \qquad \square$$

Let us recall that S admits two natural isotrivial fibrations $\sigma_1 : S \to C_1/G$ and $\sigma_2 : S \to C_2/G$ that can be thought as foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ on S.

Theorem 3.2. *Let S be a product-quotient surface. If $f : C \to S$ is a holomorphic map such that $f(C) \nsubseteq E$, with C a smooth projective curve and E the exceptional divisor on S, then*

$$\deg f^*(K_S - E) \leq 2(2g(C) - 2).$$

Proof. First, let us suppose that f is not tangent to any of the foliations $\mathcal{F}_1$, $\mathcal{F}_2$. Then by Lemma 3.1 we have that for $i = 1, 2$

$$\deg f^*\mathcal{N}_{\mathcal{F}_i}^{*}(E) \leq 2g(C) - 2 + N_1(E).$$

Using Serrano's formula for the canonical bundle (see (2.2)), we get that

$$\begin{aligned}\deg f^*(K_S + E) &= \sum_{i=1}^{2} \deg f^*\mathcal{N}_{\mathcal{F}_i}^{*}(E)\\ &\leq 2(2g(C) - 2 + N_1(E))\\ &= 2(2g(C) - 2) + 2N_1(E).\end{aligned}$$

Therefore,

$$\deg f^*(K_S - E) \leq 2(2g(C) - 2).$$

Now, we suppose that f is tangent to one of the foliations, let us say to $\mathcal{F}_1$, then $f(C)$ is contained in a fiber F of $\sigma_1 : S \to C_1/G$. Let us denote by $\deg f$ the degree of $f : C \to f(C)$.

If F is a smooth fiber we know that it is isomorphic to the curve C_2, and then,

$$\deg f^*(K_S - E) = (\deg f)(K_S - E).C_2 = (\deg f)K_{C_2}.C_2 \leq K_C.C = 2g(C) - 2.$$

If F is a singular fiber, $f(C)$ must be contained in the central component Y of the reduced structure of F and hence,

$$\deg f^*(K_S - E) = (\deg f)(K_S - E).Y = (\deg f)\left(K_Y.Y - \left(Y.E + Y^2\right)\right).$$

When F does not contain any H-J string we have $Y^2 = 0$ and $Y.E = 0$; thus,

$$\deg f^*(K_S - E) = (\deg f)K_Y.Y \leq K_C.C = 2g(C) - 2.$$

On the other hand, when F contains exactly r H-J strings, $L_1, \cdots, L_r$, where each L_i is the resolution of a cyclic quotient singularity of type $\frac{1}{n_i}(1, a_i)$, we have that

$$Y.E + Y^2 = r - \sum_{i=1}^{r} \frac{a_i}{n_i} \geq 0$$

(see formula (2.1)) and thus,

$$\deg f^*(K_S - E) \leq (\deg f)K_Y.Y \leq K_C.C = 2g(C) - 2. \qquad \square$$

4. Product-quotient surfaces with $p_g = 0$ and $c_1^2 = c_2$

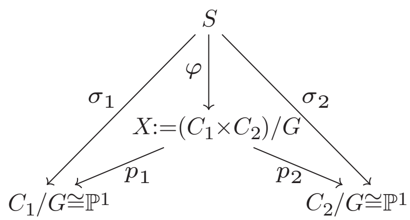
In this section, we are going to study product-quotient surfaces of general type with geometric genus $P_g = 0$ and $c_1^2 - c_2 = 0$. Recall that $P_g = 0$ implies $c_1^2 + c_2 = 12$, then the last condition is equivalent to $c_1^2 = 6$.

This kind of surfaces form exactly 8 irreducible families and we have a complete description of their quotient models. In the following table we summarize some information that might be useful.

$c_1^2(S)$	**Singularities of X**	G	$\lvert G\rvert$	$g(C_1)$	$g(C_2)$	**Number of irreducible families**
6	Two of type $\frac{1}{2}(1,1)$	$\mathbb{Z}_2 \times D_4$	16	3	7	1
		$\mathbb{Z}_2 \times \mathfrak{S}_4$	48	19	3	1
		$\mathfrak{A}_5$	60	4	16	1
		$\mathbb{Z}_2 \times \mathfrak{S}_5$	240	19	11	1
		$\mathrm{PSL}(2,7)$	168	19	8	2
		$\mathfrak{A}_6$	360	19	16	2

For even more information see [6, Table 1].

Let S be a product-quotient surface and let us suppose that S is of general type with $P_g(S) = 0$ and $c_1^2 = 6$. We recall the following commutative diagram:



Theorem 2.2 tells us that S is minimal and since S is of general type we get that K_S is nef. However, in this particular case we can prove that K_S if nef by other means, thus obtaining another argument for the minimality of S. Namely, we know that the quotient model X of S has only cyclic quotient singularities of type $\frac{1}{2}(1,1)$; since these singularities are canonical we have $K_S = \varphi^* K_X$, additionally K_X is nef because $K_{C_1 \times C_2}$ is ample, therefore K_S is nef and so S is minimal.

Since the singularities of X are of type $\frac{1}{2}(1,1)$, we have that around them, X is analytically isomorphic to the quotient $\mathbb{C}^2/(\mathbb{Z}/2\mathbb{Z})$, where the action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{C}^2$ is given by $(z_1, z_2) \to (-z_1, -z_2)$. This quotient is an affine subvariety of $\mathbb{C}^3$, with coordinates $u = z_1^2$, $v = z_1 z_2$, $w = z_2^2$, defined by the equation $uw = v^2$ [19, Proposition-Definition 1.1, Example 1.2]. Moreover, if μ_1, μ_2 are local coordinates on S, the resolution morphism φ is locally given by

$$\varphi(\mu_1, \mu_2) = \big(u = \mu_1,\ v = \mu_1\mu_2,\ w = \mu_1\mu_2^2\big) \tag{4.1}$$

[19, Example 3.1]. Therefore, we have the following relations between the local coordinates z_1, z_2 and μ_1, μ_2:

$$\begin{cases} z_1 = \mu_1^{1/2} \\ z_2 = \mu_1^{1/2}\mu_2. \end{cases} \tag{4.2}$$

On the other hand, the exceptional fiber of a cyclic quotient singularity $\frac{1}{2}(1,1)$ on the minimal resolution S, is a H-J string formed by only one smooth rational curve with self-intersection number equal to -2. Using the local coordinates μ_1, μ_2 on S we see that it is given by the set of points (μ_1, μ_2) such that $\mu_1 = 0$.

We denote by E the exceptional divisor on the minimal resolution S of X. Since X has only two cyclic quotient singularities of type $\frac{1}{2}(1,1)$, then E is the disjoint union of two rational curves with self-intersection number equal to -2. Moreover, E is locally defined by the equation $\mu_1 = 0$.

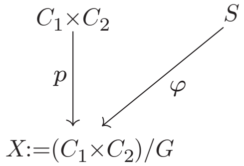
5. Proofs of Theorems 1.4 and 1.5

In this section we are going to prove Theorems 1.4 and 1.5. From now on, S will be a product-quotient surface of general type such that $p_g(S) = 0$ and $c_1(S)^2 = 6$, $X := (C_1 \times C_2)/G$ its quotient model and E the exceptional divisor.

5.1. Bigness of the cotangent bundle

We denote by Λ the set of points of $C_1 \times C_2$ with non trivial stabilizer. Recall that Λ is a finite set.

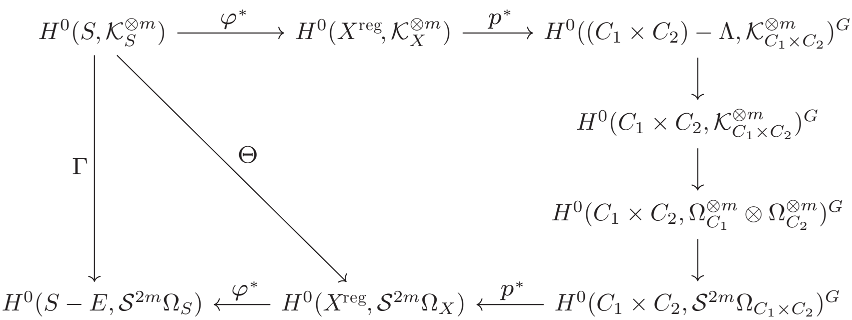
Let us first describe a natural way to produce sections of $\mathcal{S}^{2m}\Omega_X$, from sections of $K_S^{\otimes m}$ using the following diagram:



Let ω be a section of $\mathcal{K}_S^{\otimes m}$. The pushforward φ_* of ω is a section of $\mathcal{K}_X^{\otimes m}$ defined on the regular part of X, that can be lifted by the pullback p^* to a section of $(\mathcal{K}_{C_1 \times C_2}^{\otimes m})^G$ defined outside of Λ. However, since $\operatorname{codim}_{C_1 \times C_2} \Lambda = 2$, this section uniquely extends to a section defined on $C_1 \times C_2$. Moreover, the canonical isomorphism between $\mathcal{K}_{C_1 \times C_2}$ and $\lambda_1^* \Omega_{C_1} \otimes \lambda_2^* \Omega_{C_2}$ where $\lambda_1 : C_1 \otimes C_2 \to C_1$ and $\lambda_2 : C_1 \otimes C_2 \to C_2$ are the projections, allows us to identify the sections of $(\mathcal{K}_{C_1 \times C_2}^{\otimes m})^G$ with sections of $(\mathcal{S}^{2m} \Omega_{C_1 \times C_2})^G$. Therefore, we get a section of $(\mathcal{S}^{2m} \Omega_{C_1 \times C_2})^G$ which descend by p_* to a section of $\mathcal{S}^{2m} \Omega_X$ defined on the regular part of X. We denote this section by $\Theta(\omega)$.

Let us denote by $\Gamma(\omega)$ the pullback of $\Theta(\omega)$ by φ^*. Note that $\Gamma(\omega)$ is, *a priori*, a section of $\mathcal{S}^{2m} \Omega_S$ defined outside of the exceptional divisor E.

If we start with global sections of $\mathcal{K}_S^{\otimes m}$, this process is summarized in the following commutative diagram:



The following proposition ensures that taking global sections of $\mathcal{K}_S^{\otimes m}$ vanishing along E, at least with multiplicity m, is a sufficient condition to obtain global sections of $\mathcal{S}^{2m} \Omega_S$.

Proposition 5.1. *If ω is a global section of $\mathcal{O}(m(K_S - E))$, then $\Gamma(\omega)$ naturally extends to a well-defined global section of $\mathcal{S}^{2m} \Omega_S$.*

Proof. Let $\omega \in H^0(S, \mathcal{O}(m(K_S - E)))$. Following the previous diagram, we get $\Theta(\omega) \in H^0(X^{\mathrm{reg}}, \mathcal{S}^{2m} \Omega_X)$. By definition, the corresponding section on $C_1 \times C_2$ can be written locally, let us say around a fixed point, as

$$a(z_1, z_2) dz_1{}^m dz_2{}^m.$$

Using the change of coordinates $z_1 = \mu_1^{1/2}$ and $z_2 = \mu_1^{1/2}\mu_2$ given by φ at singular points of X (see Formulas (4.1) and (4.2)), we get that the pullback by φ^* of $\Theta(\omega)$, which is nothing else than $\Gamma(\omega)$, can be written locally as

$$\sum_{j=0}^{m} \binom{m}{j} \frac{\mu_2^{m-j}(a \circ \varphi)(\mu_1, \mu_2)}{2^{2m-j}\mu_1^{m-j}} d\mu_1^{2m-j} d\mu_2^{j}$$

and it naturally extends to a well-defined global section of $\mathcal{S}^{2m}\Omega_S$, since $a \circ \varphi$ vanishes along E at least with multiplicity m. □

The two following theorems constitute the proof of the first part of Theorem 1.4.

Theorem 5.2. *The line bundle $\mathcal{O}(K_S - E)$ is big.*

Proof. We know the canonical bundle $\mathcal{K}_S$ is nef and big. Then, as a consequence of Riemman-Roch theorem for surfaces and Mumford vanishing theorem [2, VI Theorem 12.1], we get that for any $m > 1$

$$h^0\big(S, \mathcal{K}_S^{\otimes m}\big) = \frac{m^2 - m}{2} c_1(S)^2 + \chi(\mathcal{O}_S).$$

Now, we have that $c_1(S)^2 = 6$ and $\chi(\mathcal{O}_S) = 1$, then we obtain

$$h^0\big(S, \mathcal{K}_S^{\otimes m}\big) = 3m^2 + 3m + 1.$$

On the other hand, let ω be a section of $\mathcal{K}_S^{\otimes m}$. The corresponding section on $C_1 \times C_2$ can be written locally, around a fixed point, as

$$a(z_1, z_2)(dz_1 \wedge dz_2)^m,$$

where a is a holomorphic function defined as

$$a(z_1, z_2) = \sum_{i,j} a_{ij} z_1^i z_2^j.$$

Using the change of coordinates $z_1 = \mu_1^{1/2}$ and $z_2 = \mu_1^{1/2}\mu_2$ given by φ at singular points of X, we see that ω vanishes along E, at least with multiplicity m if $a_{i,j} = 0$, for every i, j such that $i + j < 2m$, and this gives us, $1 + 2 + \cdots + 2m$ sufficient conditions. However, the section is invariant by the action of G, which means that around a singular point, $a(z_1, z_2)$ is invariant by the action of its stabilizer $H \simeq \mathbb{Z}_2$, therefore $a_{ij} = 0$ for all i, j such that $i + j$ is odd; thus, we just need to consider half of the conditions. Finally, since these conditions are given around one singular point, we have to multiply them by the number of singularities.

Thus, for any m we have that

$$\begin{aligned} h^0(S, \mathcal{O}(m(K_S - E))) &\geq h^0\big(S, \mathcal{K}_S^{\otimes m}\big) - \frac{2(1+2+\cdots+2m)}{2} \\ &= \big(3m^2 - 3m + 1\big) - \big(2m^2 + m\big) \\ &= m^2 - 4m + 1. \end{aligned}$$

Note for any $0 < C < 1$, we have that $m^2 - 4m + 1 \geq Cm^2$ for m large enough. Therefore,

$$h^0(S, \mathcal{O}(m(K_S - E))) \geq Cm^2$$

for m large enough, which means that $\mathcal{O}(K_S - E)$ is big [15, Lemma 2.2.3]. □

Theorem 5.3. *The cotangent bundle Ω_S is big.*

Proof. For the sake of simplicity, we are going to use the same notation to refer to a divisor and its associated line bundle.

In order to prove that Ω_S is big, we show that the line bundle $\mathcal{O}_{\mathbb{P}(T_S)}(1)$ is big, which is equivalent to see that $\mathcal{O}_{\mathbb{P}(T_S)}(k)$ is linearly equivalent to the sum of an ample divisor and an effective divisor, for a k large enough [15, Corollary 2.2.7].

Theorem 5.2 tells us that $\mathcal{O}(K_S - E)$ is big, then there exists an ample line bundle A and a positive integer m such that

$$H^0\big(S, \mathcal{O}(m(K_S - E)) \otimes A^{-1}\big) \neq 0$$

and hence

$$H^0\big(S, \mathcal{S}^{2m}\Omega_S \otimes A^{-1}\big) \neq 0.$$

However, $\mathcal{S}^{2m}\Omega_S \simeq \pi_*\mathcal{O}_{\mathbb{P}(T_S)}(2m)$ where $\pi : \mathbb{P}(T_S) \to S$ is the projective bundle associated to the tangent bundle T_S, and so we obtain that

$$H^0\big(\mathbb{P}(\Omega_S), \mathcal{O}_{\mathbb{P}(T_S)}(2m) \otimes \pi^* A^{-1}\big) \neq 0,$$

which means that $\mathcal{O}_{\mathbb{P}(T_S)}(2m) - \pi^* A$ is an effective divisor.

On the other hand, since $\mathcal{O}(1)$ is relatively ample [15, Proposition 1.2.7], we know that there exists a large enough positive integer l such that $\mathcal{O}_{\mathbb{P}(T_S)}(1) + l\pi^* A$ is an ample divisor on $\mathbb{P}(T_S)$.

Finally, taking $k = 2ml + 1$ we get

$$\mathcal{O}_{\mathbb{P}(T_S)}(2ml+1) = \underbrace{l(\mathcal{O}_{\mathbb{P}(T_S)}(2m) - \pi^* A)}_{\text{effective divisor}} + \underbrace{\mathcal{O}_{\mathbb{P}(T_S)}(1) + l\pi^* A}_{\text{ample divisor}}$$

and so $\mathcal{O}_{\mathbb{P}(T_S)}(1)$ is big. □

5.2. Rational curves

We already know that Ω_S is big and then, by Bogomolov's argument, there is only a finite number of rational curves on S; now we want to get more constraints.

Lemma 5.4. *The central component Y of any singular fiber on S, defined in proposition* 2.4*, is not rational.*

Proof. Recall that the only singularities of X are two ordinary double points, *i.e.*, two cyclic quotient singularities of type $\frac{1}{2}(1, 1)$. Then, the singular fibers on S are the union of a central component Y and either none or exactly two mutually disjoint rational curves (the exceptional divisor E) which correspond to the resolution of the two singularities.

In the first case we have that $Y^2 = 0$. In the second case, we use the formula (2.1) and we obtain that $Y^2 = -1$. On the other hand, we have that

$$2g(Y) - 2 = K_Y.Y = (K_S + Y).Y = K_S.Y + Y^2,$$

where $K_S.Y \geq 0$ since K_S is nef. Therefore, we obtain in both cases $g(Y) \geq 1$, which means that Y is not rational. □

Now, we are going to prove the second part of Theorem 1.4. We will present two proofs: the first one is an immediate consequence of Theorem 3.2 and the second one uses a local argument that will be extended to entire curves in Theorem 5.10.

Definition 5.5. Let D be a divisor on S and let $Bs(D)$ be the base locus of the linear system $|D|$. The *stable base locus* of D is defined as

$$\mathbb{B}(D) := \bigcap_{m>0} \mathrm{Bs}(mD).$$

Theorem 5.6. *Let $f : \mathbb{P}^1 \to S$ be a non constant holomorphic map. Then*

$$f\big(\mathbb{P}^1\big) \subset E \cup \mathbb{B}(K_S - E).$$

First proof. Let us suppose that $f(\mathbb{P}^1) \not\subseteq E$. We already know that $K_S - E$ is big (Theorem 5.2), so if $f(\mathbb{P}^1) \not\subseteq \mathbb{B}(K_S - E)$, it holds $\deg f^*(K_S - E) \geq 0$. However, from Theorem 3.2 we obtain $\deg f^*(K_S - E) \leq -4$, a contradiction. Thus $f(\mathbb{P}^1) \subset \mathbb{B}(K_S - E)$ and the proof is complete. □

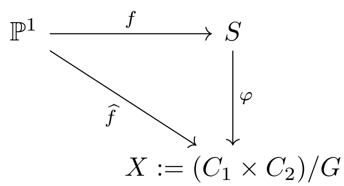
Second proof. Let $f : \mathbb{P}^1 \to S$ be a non constant holomorphic map. Then, either f is tangent to one of the foliations $\mathcal{F}_1, \mathcal{F}_2$ given by the fibrations σ_1, σ_2 respectively, or f is not.

If f is tangent to one of the foliations, let us say to $\mathcal{F}_1$, then $f(\mathbb{P}^1)$ must be contained in a singular fiber of $\sigma_1 : S \to C_1/G$. Otherwise $f(\mathbb{P}^1)$ would be contained in a smooth fiber, but smooth fibers are hyperbolic, since they are isomorphic to C_2 and $g(C_2) \geq 2$, and this is contradiction. Therefore,

$$f\left(\mathbb{P}^1\right) \subset Y \cup E,$$

where Y is the central component and E is the exceptional divisor; however, Lemma 5.4 tells us that $g(Y) \geq 1$ and hence $f(\mathbb{P}^1) \subset E$.

Now, let us suppose that f is not tangent to any of the foliations and let us consider the composition $\widehat{f} := \varphi \circ f$. So we have the following diagram:



By Theorem 5.2 there exists a positive integer m_0 such that for all $m \geq m_0$ we have a non zero section $\omega \in H^0(S, \mathcal{O}(m(K_S - E)))$. Recall the section $\Theta(\omega) \in H^0(X^{\text{reg}}, \mathcal{S}^{2m}\Omega_X)$ obtained via Θ; the section $\widehat{f}^*\Theta(\omega) = f^*\Gamma(\omega)$ vanishes because $H^0(\mathbb{P}^1, \Omega^n_{\mathbb{P}^1}) = 0$ for all n. Moreover, since $\Theta(\omega)$ is locally written as $a(z_1, z_2)dz_1{}^m dz_2{}^m$ and $\widehat{f}$ is locally given by $\widehat{f} = (\widehat{f_1}, \widehat{f_2}) = \varphi(f_1, f_2)$ where $f_1, f_2, \widehat{f_1}, \widehat{f_2}$ are holomorphic functions, the section $\widehat{f}^*\Theta(\omega)$ is locally given as

$$a\left(\widehat{f_1}, \widehat{f_2}\right)\left(\widehat{f_1}'\right)^m\left(\widehat{f_2}'\right)^m = 0.$$

Thus we obtain that $a(\widehat{f_1}, \widehat{f_2}) = (a \circ \varphi)(f_1, f_2) = 0$ since by hypothesis the other factors are not always equal to zero. This last equation means that the section ω vanishes on $f(\mathbb{P}^1)$, but this is true for any section of $\mathcal{O}(m(K_S - E))$, then $f(\mathbb{P}^1) \subset \text{Bs}(m(K_S - E))$; besides this is true for all $m \geq m_0$, therefore $f(\mathbb{P}^1) \subset \mathbb{B}(K_S - E)$. □

5.3. Entire curves

We have already seen that the central components of singular fibers on S are not rational, but we do not know yet if they can be elliptic. In the following example we will see that, in fact, for any product-quotient surface, the central components that do not intersect with the exceptional divisor, have genus bigger than one; however, in the case where the central components do intersect with the exceptional divisor, we give an example of a surface with a central component that is elliptic.

Example 5.7. Let S be a quotient-product surface and let us consider the natural fibration $\sigma_1 : S \to C_1/G$ and a point $\overline{x} \in C_1/G$ with non trivial stabilizer H_x. Recall that the fiber F of σ_1 over $\overline{x}$ is the union of a central component $Y \simeq C_1/H_x$ and either none or at least two mutually disjoint H-J strings which are in one-to-one correspondence with the branch points of $C_2 \to C_2/H_x$.

In the first case, using the Riemann-Hurwitz formula, we obtain

$$2g(C_1) - 2 = |H_x|(2g(Y) - 2),$$

but $2g(C_1) - 2 > 0$, then $g(Y) \geq 2$.

For the second case, we suppose S belongs to the first family of the table given in section 3. Since $X = (C_1 \times C_2)/G$ has only two singularities of type $\frac{1}{2}(1, 1)$, then $C_2 \to C_2/H_x$ has two branch points with multiplicity equal to 2. Thus, using the Riemann-Hurwitz formula, we get

$$2g(C_1) - 2 = |H_x|(2g(Y) - 1),$$

but $g(C_1) = 3$, then,

$$4 = |H_x|(2g(Y) - 1).$$

We easily conclude that $|H_x|$ must be equal to 4 and hence $g(Y) = 1$.

Now, we recall a well known theorem asserting that entire curves satisfy an algebraic differential equation. Namely:

Theorem 5.8 ([10, Corollary 7.9], [14]). *If there exists a non zero section $s \in H^0(S, \mathcal{S}^m \Omega_S \otimes A^{-1})$ with A an ample line bundle and m an integer, then for every entire curve $f : \mathbb{C} \to S$, it holdds $f^* s = 0$.*

Using this, we can follow the same argument used in the second proof of Theorem 5.6 to prove an analogous result for entire curves.

Definition 5.9. Let D be a divisor on S and let $Bs(D)$ be the base locus of the linear system $|D|$. The *augmented base locus* of D is defined as

$$\mathbb{B}^+(D) := \bigcap_{m>0} Bs(mD - A)$$

for any ample divisor A.

Proposition 5.10. *Let $f : \mathbb{C} \to S$ be a non constant holomorphic map. Then*

$$f(\mathbb{C}) \subset \mathbf{Y} \cup E \cup \mathbb{B}^+(K_S - E),$$

where $\mathbf{Y}$ is the union of all central components.

Proof. For any entire curve $f:\mathbb{C}\to S$ we also have the following two possibilities: either f is tangent to one of the foliations $\mathcal{F}_1,\mathcal{F}_2$, or f is not.

In the first case we have again that $f(\mathbb{C})$ must be contained in the singular fibers because the smooth ones are Brody hyperbolic. Thus, $f(\mathbb{C})\subset \mathbf{Y}\cup E$.

In the second case, let A be an ample divisor; since K_S-E is big (Theorem 5.2), there exists an infinite number of integers m such that $H^0(S,\mathcal{O}(m(K_s-E))\otimes A^{-1})\neq 0$. So, we can consider a non-zero section $\omega\in H^0(S,\mathcal{O}(m(K_s-E))\otimes A^{-1})$ and via Γ we obtain a nonzero section $\Gamma(\omega)\in H^0(S,\mathcal{S}^{2m}\Omega_S\otimes A^{-1})$. By Theorem 5.8, we have that $f^*\Gamma(\omega)=0$ and then, following the same argument than in the case of rational curves we obtain $f(\mathbb{C})\subset\mathbb{B}^+(K_s-E)$. ☐

Note that Example 5.7 shows that, *a priori*, we can not avoid the central components because they could be elliptic. However, using Theorem 3.2, we will prove that elliptic curves are contained in the the augmented base locus of K_S-E.

Proposition 5.11. *If S is a product-quotient surface of general type such that $p_g=0$ and $c_1^2=6$, and $f:C\to S$ is a holomorphic map where C is a smooth projective curve of genus $g(C)=1$, then*

$$f(C)\subset\mathbb{B}^+(K_S-E).$$

Proof. Since K_S-E is big, then it can be written as the sum of an ample divisor A and an effective divisor D. Moreover, the augmented base locus can be given in terms of all these possible sums as

$$\mathbb{B}^+(K_S-E)=\bigcap_{K_S-E=A+D}\operatorname{Supp}D$$

[11, Remark 1.3]. Now, if $f(C)\not\subseteq\mathbb{B}^+(K_S-E)$ then there is a D such that $f(C)\not\subseteq D$, thus $\deg f^*D\geq 0$ and hence,

$$\deg f^*(K_S-E)=\deg f^*A+\deg f^*D>0,$$

but note that $f(C)\not\subseteq E$, thus from Theorem 3.2 we have that

$$\deg f^*(K_S-E)\leq 0,$$

a contradiction. Therefore $f(C)\subset\mathbb{B}^+(K_S-E)$. ☐

Finally, as a consequence of Propositions 5.10 and 5.11 we obtain Theorem 1.5.

Theorem 5.12. *If S is a product-quotient surface of general type such that $p_g=0$ and $c_1^2=6$, then for any non constant holomorphic map $f:\mathbb{C}\to S$,*

$$f(\mathbb{C})\subset E\cup\mathbb{B}^+(K_S-E).$$

Proof. From Proposition 5.10 we have that $f(\mathbb{C}) \subset \mathbf{Y} \cup E \cup \mathbb{B}^+(K_S - E)$ where $\mathbf{Y}$ is the union of all central components. First, note that we can remove all components with genus greater than or equal to 2 since they are hyperbolic, and by Lemma 5.4 we know that no component can be rational. Now, by Proposition 5.11, the elliptic components must be contained in the augmented base locus of $K_S - E$. Therefore, $f(\mathbb{C}) \subset E \cup \mathbb{B}^+(K_S - E)$. □

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