Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XVIII (2018), 1459-1482

Uniqueness for the two dimensional Calderón's problem with unbounded conductivites

CĂTĂLIN I. CÂRSTEA AND JENN-NAN WANG

Abstract. In this work we consider the Calderón problem in two dimensions with conductivity $\gamma \in W^{1,2}(\Omega)$. This condition allows for the conductivity to be unbounded. We prove a uniqueness result when $||\nabla \log \gamma||_{L^2}$ is bounded by a fixed constant depending on the domain Ω.

Mathematics Subject Classification (2010): 35R30 (primary); 35J45, 35P25 (secondary).

1. Introduction

The inverse conductivity problem, first discussed by Calderón in [8], consists of determining the conductivity of the interior of an object from measurements of electrical potential and current taken on the boundary. One aspect of this problem is weather or not two different conductivity functions might give rise to the same set of boundary measurements. This question has been considered for different spatial dimensions and under various assumptions on the regularity of the conductivity function. An early result in dimension greater than two and for smooth conductivities was obtained by Sylvester and Uhlmann in their seminal paper [18]. Recently, Haberman and Tataru have shown in [12], also in dimensions higher than two, that uniqueness holds for Lipschitz conductivities γ such that $||\nabla \log \gamma||_{L^\infty}$ is small. In [11], Haberman has further improved on this result in dimensions $3 \leq n \leq 6$. In dimensions $n = 3, 4$ he proves uniqueness for conductivities in $W^{1,n}$. We want to point out that in [11] the conductivity γ satisfies $c \leq \gamma \leq c^{-1}$, in particular, $\gamma \in L^\infty$. In dimension two, Nachman proved in [14] uniqueness for conductivities in $W^{2,p}$ for some $p > 1$. Brown and Uhlmann in [7] established the uniqueness for conductivities in $W^{1,p}$ with $p > 2$. In [2], Astala and Päivärinta proved the uniqueness for conductivities in L^∞. In the recent paper [3], Astala, Lassas, and Päivärinta have proved a uniqueness result (see their Theorem 1.9) that also applies to some unbounded conductivities.

Received July 20, 2016; accepted in revised form April 18, 2017.
Published online July 2018.

In this paper we will study the two-dimensional Calderón's problem in a different class of singular conductivities. Precisely, we prove that uniqueness holds for the strictly positive conductivity $\gamma \in W^{1,2}$, with and additional constraint that $||\nabla \log \gamma||_{L^2(\Omega)} < C$ with $C = C(\Omega)$. It is important to emphasize that we do not assume $\gamma \in L^\infty$. For example, $\gamma(x) = \log|\log|ax||$ or $\gamma(x) = (\log|ax|)^s$, $0 < s < \frac{1}{2}$, have finite $||\nabla \log \gamma||_{L^2(B_{\frac{1}{2|a|}})}$, thus can be used to construct examples. The second of these two examples also falls outside the conditions required by [3, Theorem 1.9]. Therefore, our result is not a consequence of previous works. In view of the unique continuation property for the linear convection equation with L^2 coefficients in $\mathbb{R}^2$ [13], the assumption of $\gamma \in W^{1,2}$ is most likely *optimal* for the two dimensional Calderón's problem. It is not yet certain that the restriction on the L^2 norm of $\nabla \log \gamma$ is necessary.

This work is inspired by the paper of Cheng and Yamamoto in [9] where they prove a uniqueness result for the equation

$$\triangle u + \vec{b} \cdot \nabla u = 0$$

with $\vec{b} \in L^p$, $p > 2$. Here we push their result to the optimal case $p = 2$ (with an added requirement that the L^2 norm of $\vec{b}$ be bounded by a constant) and obtain the result for Calderón's problem as a corollary. The proof relies on reducing the second order equation to a first order one in the complex plane (see [1,19] for background). It also employs an inverse scattering method developed by Beals and Coifman in [4] and by Sung in [15–17]. Along the way we also prove a version of Brown's result in [6] regarding the recovery of the boundary values of the conductivity from the knowledge of the Dirichlet-Neumann map.

2. Statement of the result

We consider a domain $\Omega \subset \mathbb{R}^2$ which is open, bounded, simply connected, and with sufficiently smooth boundary. For a conductivity $\gamma : \Omega \to \mathbb{R}$ positive and bounded away from zero, we will consider the boundary value problem

$$\begin{cases} \nabla(\gamma \nabla u)(x) = 0, & x \in \Omega \\ u|_{\partial\Omega} = \omega. \end{cases} \tag{2.1}$$

The Dirichlet-Neumann map for the classical Calderon problem is then

$$\Lambda_\gamma(\omega) := \gamma|_{\partial\Omega} \frac{\partial u}{\partial \nu},$$

where u is the solution with boundary value ω. Equation (2.1) can be put in non-divergence form as

$$\begin{cases} \triangle u(x) + \vec{b}(x) \cdot \nabla u(x) = 0, & x \in \Omega \\ u|_{\partial\Omega} = \omega, \end{cases} \tag{2.2}$$

where

$$\vec{b} := \nabla \log \gamma.$$

For this equation, we consider the Dirichlet-Neumann map

$$\Lambda_{\vec{b}}(\omega) := \frac{\partial u}{\partial \nu}.$$

Throughout, we assume $\gamma \in W^{1,2}(\Omega)$ or $\vec{b} \in L^2_{\mathbb{R}}(\Omega)$, where $L^2_{\mathbb{R}}(\Omega)$ denotes the space of L^2 real-valued vectors in Ω. For both forms of the equation, we consider strong solutions $u \in W^{2,p}(\Omega)$, for some $1 < p < 2$. The Dirichlet boundary data then belong to $W^{2-\frac{1}{p},p}(\partial\Omega)$. In the appendix we give a proof of existence and uniqueness of solutions for the case when $||\vec{b}||_{L^2(\Omega)} < C$, with $C = C(p, \Omega)$.

Our main result is the following

Theorem 2.1. *There exists a constant $C = C(\Omega)$ such that if $\gamma_1, \gamma_2 \in W^{1,2}(\Omega)$, with $||\nabla \log \gamma_{1,2}||_{L^2(\Omega)} < C$ and $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$, then $\gamma_1 = \gamma_2$.*

This follows immediately from Theorem 2.2 and Proposition 2.3 stated below.

Theorem 2.2. *There exists a constant $C = C(\Omega)$ such that if $\vec{b}_1, \vec{b}_2 \in L^2_{\mathbb{R}}(\Omega)$, with $||\vec{b}_{1,2}||_{L^2(\Omega)} \leq C$ and $\Lambda_{\vec{b}_1} = \Lambda_{\vec{b}_2}$, then $\vec{b}_1 = \vec{b}_2$.*

Note that this second theorem is more general than the first. It is this result that we will prove below. We also prove the following boundary determination result.

Proposition 2.3. *If $\gamma \in W^{1,2}(\Omega)$, the trace $\gamma|_{\partial\Omega}$ can be determined from the Dirichlet-to-Neumann map Λ_γ.*

Since the proof of Proposition 2.3 is of a different nature from that of Theorem 2.2, the principal result of this paper, we have provided it as an appendix.

3. Complex variable notation and the first order form of the equation

We will identify $\mathbb{R}^2$ with $\mathbb{C}$ and use the usual notation $\mathrm{d}z = \mathrm{d}x_1 + i\,\mathrm{d}x_2$, $\mathrm{d}\bar{z} = \mathrm{d}x_1 - i\,\mathrm{d}x_2$, and $\partial_z = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})$, $\partial_{\bar{z}} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2})$. If $u \in W^{2,p}(\Omega)$ is a solution of (2.2), then $w := \partial_z u \in W^{1,p}(\Omega)$ is a solution of the equation

$$\partial_{\bar{z}} w + Bw + \bar{B}\bar{w} = 0, \tag{3.1}$$

where $B(z) := \frac{1}{4}(b_1 + ib_2)$. We can also go the other way.

Lemma 3.1 (See, *e.g.*, [19]). *Given $w \in W^{1,p}(\Omega)$ a solution to (3.1), then there exists a solution $u \in W^{2,p}(\Omega)$ of (2.2) such that $w = \partial_z u$.*

For Dirichlet boundary conditions, as we have in (2.2), w must satisfy

$$2\mathfrak{Re}\,(\dot{z}w)|_{\partial\Omega} = \frac{\partial\omega}{\partial t} \text{ (the tangential derivative along } \partial\Omega). \tag{3.2}$$

Here $\dot{z}$ is the derivative of a parametrization $z(t)$ of the boundary $\partial\Omega$. In the case of Neumann boundary conditions, we would instead need

$$2\mathfrak{Im}\,(\dot{z}w)\Big|_{\partial\Omega} = \frac{\partial u}{\partial\nu}\Big|_{\partial\Omega}. \tag{3.3}$$

For this see [1, Section 16.5]. Note that the relations (3.2) and (3.3) are valid for real solutions u of (2.2). Since $\vec{b}$ is a real vector, these relations remain true for complex-valued solutions of (2.2).

We define the Cauchy transform

$$(\mathcal{C}f)(z) := -\frac{1}{\pi}\int_\Omega \frac{f(\zeta)}{\zeta - z}\,\mathrm{d}^2\zeta.$$

It maps $\mathcal{C} : L^q(\Omega) \to W^{1,q}(\Omega)$ continuously for all $1 < q < \infty$ (again, see [1]). The operator $\mathcal{C}$ is an "inverse" to the differential operator $\partial_{\bar{z}}$. For $f \in C^\infty(\Omega) \cap C(\bar{\Omega})$

$$\begin{aligned}(\mathcal{C}\partial_{\bar{z}} f)(z) &= -\frac{1}{\pi}\lim_\epsilon \int_{\Omega - B(z,\epsilon)} \partial_{\bar{\zeta}} \frac{f(\zeta)}{\zeta - z}\frac{\mathrm{d}\bar{\zeta}\wedge\mathrm{d}\zeta}{2i}\\ &= \frac{1}{2\pi i}\left(\lim_\epsilon \int_{\partial B(z,\epsilon)} \frac{f(\zeta)-f(z)}{\zeta - z}\,\mathrm{d}\zeta + 2\pi i f(z) - \int_{\partial\Omega}\frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta\right),\end{aligned}$$

so

$$(\mathcal{C}\partial_{\bar{z}} f)(z) = f(z) - \frac{1}{2\pi i}\int_{\partial\Omega}\frac{f(\zeta)}{\zeta - z}\,\mathrm{d}\zeta. \tag{3.4}$$

Also, defining $\langle\varphi, f\rangle := \int_\Omega \varphi f$, for any $\varphi \in C_c^\infty(\Omega)$ it holds

$$\langle\varphi, \partial_{\bar{z}}\mathcal{C}f\rangle = \langle\mathcal{C}\partial_{\bar{z}}\varphi, f\rangle = \langle\varphi, f\rangle$$

and so we have

$$\partial_{\bar{z}}\mathcal{C}f = f. \tag{3.5}$$

Given the mapping properties of the Cauchy transform, the formulas (3.4), (3.5) extend to any $f \in W^{1,q}(\Omega)$, $1 < q < \infty$.

The Cauchy transform may be used to turn (3.1) into an integral equation.

Lemma 3.2. *A function $w \in W^{1,p}(\Omega)$ that satisfies* (3.1) *will also satisfy the integral equation*

$$w(z) + \mathcal{C}\big(Bw + \bar{B}\bar{w}\big)(z) = \frac{1}{2\pi i}\int_{\partial\Omega}\frac{w(\zeta)}{\zeta - z}\,d\zeta.$$

Also, if $\Phi \in \mathrm{Hol}(\Omega) \cap W^{1,p}(\Omega)$ *and* $w \in W^{1,p}(\Omega)$ *satisfies*

$$w(z) + \mathcal{C}\big(Bw + \bar{B}\bar{w}\big)(z) = \Phi(z), \tag{3.6}$$

then w *satisfies* (3.1) *and*

$$\frac{1}{2\pi i}\int_{\partial\Omega} \frac{w(\zeta)}{\zeta - z}\, d\zeta = \Phi(z).$$

Next we would like to show the existence of solutions to (3.6), with suitable Φ. A contraction principle argument of the same form as the one in the proof of Proposition A.1 would work. Instead, we present a different type of argument which doesn't require the norm of the coefficients to be small. Testing the equation (3.6) against a test function ϕ we get

$$\left\langle \phi - \left(B + \bar{B}\frac{\bar{w}}{w}\right)\mathcal{C}(\phi), w\right\rangle = \langle \phi, \Phi\rangle. \tag{3.7}$$

Let $A := B + \bar{B}\bar{w}/w \in L^2(\Omega)$, with $||A||_{L^2(\Omega)} \leq 2||B||_{L^2(\Omega)}$, and $\psi := \mathcal{C}(\phi)$. Let $1 < p < 2$ and $p^* = 2p/(2-p)$. In order to obtain an *a priori* estimate for the L^{p^*} norm of w, we would like that

$$\partial_{\bar{z}}\, \psi - A\psi = |w|^{p^*-2}\bar{w}. \tag{3.8}$$

It turns out that one solution of (3.8) is explicitly expressed by

$$\psi = \mathcal{C}\left(|w|^{p^*-2}\bar{w}\, e^{-\mathcal{C}A}\right) e^{\mathcal{C}A}.$$

The left hand side of (3.7) is just $||w||^{p^*}_{L^{p^*}(\Omega)}$. The right hand side of (3.7) consists of two terms:

$$\left\langle |w|^{p^*-2}\bar{w}, \Phi\right\rangle + \langle A\psi, \Phi\rangle =: I + II.$$

Hölder's inequality implies

$$I \leq ||w||^{p^*-1}_{L^{p^*}(\Omega)}||\Phi||_{L^{p^*}(\Omega)}.$$

To deal with the second term we begin by noticing that $|w|^{p^*-2}\bar{w} \in L^{p^*/(p^*-1)}(\Omega)$. With the help of Trudinger's inequality, we can see that $e^{\mathcal{C}A}, e^{-\mathcal{C}A} \in L^q(\Omega)$ for any $q < \infty$ (for example, see the computation in [13]). It follows that ψ is in all $L^r(\Omega)$ with $r < 2p^*/(p^*-2)$. We can then bound the second term

$$II \leq ||A||_{L^2(\Omega)}||\psi||_{L^r(\Omega)}||\Phi||_{L^{p^*+\epsilon}(\Omega)} \quad \text{with} \quad r = \frac{2(p^*+\epsilon)}{p^*+\epsilon-2}$$

with $\epsilon > 0$. Since $||\psi||_{L^r(\Omega)} \leq C||w||^{p^*-1}_{L^{p^*}(\Omega)}$, we get

$$II \leq C||w||^{p^*-1}_{L^{p^*}(\Omega)}||A||_{L^2(\Omega)}||\Phi||_{L^{p^*+\epsilon}(\Omega)}.$$

Putting these together implies that

$$\|w\|_{L^{p^*}(\Omega)} \leq C(1+\|A\|_{L^2(\Omega)})\|\Phi\|_{L^{p^*+\epsilon}(\Omega)}$$

and

$$\|w\|_{W^{1,p}(\Omega)} \leq C(1+\|A\|_{L^2(\Omega)})\|\Phi\|_{L^{p^*+\epsilon}(\Omega)}.$$

If $\Phi \in W^{1,p+\epsilon}(\Omega)$ for some $\epsilon > 0$ and $B \in C_0^\infty(\Omega)$, then a solution to (3.6) is known to exits (see, for example, the proof of [9, Lemma 3.3]). Given $B \in L^2(\Omega)$ let $B_n \in C_0^\infty(\Omega)$ be such that $||B_n - B||_{L^2} \to 0$, and let w_n be the solutions of

$$w_n + \mathcal{C}\big(B_n w_n + \bar{B}_n \bar{w}_n\big) = \Phi.$$

Then

$$||w_n||_{W^{1,p+\epsilon/2}(\Omega)} \leq C||\Phi||_{W^{1,p+\epsilon}(\Omega)}\left(1+||B||_{L^2}\right).$$

Also

$$\begin{aligned} &w_n - w_m + \mathcal{C}\left(B_n(w_n-w_m) + \overline{B_n(w_n-w_m)}\right) \\ &= -\mathcal{C}\left((B_n-B_m)w_m + \overline{(B_n-B_m)w_m}\right). \end{aligned}$$

According to the observation above, the right had side is in $W^{1,p+\epsilon/2}(\Omega)$ and

$$||RHS||_{W^{1,p+\epsilon/2}(\Omega)} \leq C||B_n-B_m||_{L^2}||\Phi||_{W^{1,p+\epsilon}(\Omega)}(1+||B||_{L^2}).$$

Applying the *a priori* estimate to the difference $w_n - w_m$ we get

$$||w_n-w_m||_{W^{1,p}(\Omega)} \leq C||B_n-B_m||_{L^2}||\Phi||_{W^{1,p+\epsilon}(\Omega)}\left(1+||B||_{L^2}\right)^2.$$

This proves that the sequence of approximate solutions is Cauchy in $W^{1,p}(\Omega)$. The fact that the limit is a solution to the equation follows from the continuity of $\mathcal{C}$ and the continuity of products with respect to strong convergence. We have therefore proven the following.

Lemma 3.3. *If* $1 < p < 2$, $\Phi \in W^{1,p+\epsilon}(\Omega)$, *and* $\epsilon > 0$ *with* $p+\epsilon < 2$, *then the equation*

$$w + \mathcal{C}\big(Bw + \bar{B}\bar{w}\big) = \Phi$$

has a unique solution $w \in W^{1,p+\epsilon/2}(\Omega)$ *and the following estimate holds*

$$\|w\|_{W^{1,p+\epsilon/2}(\Omega)} \leq C\|\Phi\|_{W^{1,p+\epsilon}(\Omega)}, \tag{3.9}$$

where $C = C(\|A\|_{L^2(\Omega)}, p, \epsilon, \Omega)$.

4. CGO solutions (introduction of the parameter k)

In this section, we want to discuss complex geometrical optics (CGO) solutions of (3.1). These special solutions are useful in Calderón's problem. Let $k = k_x + ik_y \in \mathbb{C}$. If w is a solution to (3.1) then $\alpha_k(z) := w(z)e^{-\frac{i}{2}\bar{k}z}$ satisfies the equation

$$\partial_{\bar{z}}\,\alpha_k(z) + B(z)\alpha_k(z) + e^{-\frac{i}{2}(\bar{k}z+k\bar{z})}\overline{B(z)\alpha_k(z)} = 0 \quad \text{in} \quad \Omega. \tag{4.1}$$

We will use the notation

$$e_k(z) = \exp\left(-\frac{i}{2}\big(\bar{k}z + k\bar{z}\big)\right) = e^{i(k_x x + k_y y)}.$$

Clearly, $|e_k(z)| = 1$. Analogously to the previous section, we have

Lemma 4.1. *If $\alpha_k \in W^{1,p}(\Omega)$, with $1 < p < 2$, satisfies* (4.1)*, then it also satisfies the integral equation*

$$\alpha_k(z) + \mathcal{C}\big(B\alpha_k + e_k\bar{B}\bar{\alpha}_k\big)(z) = \frac{1}{2\pi i}\int_{\partial\Omega}\frac{\alpha_k(\zeta)}{\zeta - z}\,d\zeta.$$

Conversely, the equation

$$\alpha_k(z) + \mathcal{C}\big(B\alpha_k + e_k\bar{B}\bar{\alpha}_k\big)(z) = 1 \tag{4.2}$$

has a unique solution $\alpha_k \in W^{1,p}(\Omega)$ and

$$||\alpha_k||_{W^{1,p}(\Omega)} \le C\big(||\vec{b}||_{L^2(\Omega)}, \Omega, p\big).$$

This solution will also satisfy (4.1) *and*

$$\frac{1}{2\pi i}\int_{\partial\Omega}\frac{\alpha_k(\zeta)}{\zeta - z}\,d\zeta = 1.$$

4.1. Differentiability in k

In order to investigate the differentiability with respect to k of α_k , for some fixed $k = k_x + ik_y$ we introduce the notations

$$\delta_\kappa\alpha = \frac{1}{\kappa}(\alpha_{k+\kappa}(z) - \alpha_k(z)) \quad \text{and} \quad \delta_\kappa e = \frac{1}{\kappa}(e_{k+\kappa}(z) - e_k(z)).$$

Note that this quantity satisfies the integral equation

$$\delta_\kappa\alpha + \mathcal{C}\left(B\delta_\kappa\alpha + e_{k+\kappa}\bar{B}\delta_\kappa\bar{\alpha} + \delta_\kappa e\bar{B}\bar{\alpha}_k\right) = 0.$$

We will only consider real valued κ (the case of imaginary κ will only different by a minus sign). In this case

$$\delta_\kappa\alpha + \mathcal{C}\left(B\delta_\kappa\alpha + e_{k+\kappa}\bar{B}\overline{\delta_\kappa\alpha}\right) = -\mathcal{C}\left((\delta_\kappa e)\bar{B}\bar{\alpha}_k\right). \tag{4.3}$$

Let $\epsilon > 0$ be such that $p + \epsilon < 2$, then it follows from (3.9) that

$$||\delta_\kappa\alpha||_{W^{1,p+\epsilon/2}(\Omega)} \le C\left(||\vec{b}||_{L^2(\Omega)}, p, \epsilon\right)||\delta_\kappa e||_{L^\infty(\Omega)}.$$

In order to prove that $\delta_\kappa\alpha$ is Cauchy, we need to estimate the quantity

$$\Delta_{\kappa,\kappa'}\alpha := \delta_\kappa\alpha - \delta_{\kappa'}\alpha.$$

Note that $\Delta_{\kappa,\kappa'}e \to 0$ in $C(\bar{\Omega})$. It is clear that $\Delta_{\kappa,\kappa'}\alpha$ satisfies the integral equation

$$\begin{aligned}\Delta_{\kappa,\kappa'}\alpha &+ \mathcal{C}\left(B\Delta_{\kappa,\kappa'}\alpha + e_{k+\kappa'}\bar{B}\overline{\Delta_{\kappa,\kappa'}\alpha}\right)\\ &= -\mathcal{C}\left((\Delta_{\kappa,\kappa'}e)\bar{B}\bar{\alpha}_k + (e_{k+\kappa'} - e_{k+\kappa})\bar{B}\overline{\delta_\kappa\alpha}\right).\end{aligned}$$

Applying the *a priori* estimate (3.9) again we obtain

$$\begin{aligned}&||\Delta_{\kappa,\kappa'}\alpha||_{W^{1,p}(\Omega)}\\ &\le C\left(||\vec{b}||_{L^2(\Omega)}, p, \epsilon\right)\left(||\Delta_{\kappa,\kappa'}e||_{L^\infty(\Omega)} + \kappa||\delta_\kappa e||_{L^\infty(\Omega)} + \kappa'||\delta_{\kappa'}e||_{L^\infty(\Omega)}\right).\end{aligned}$$

We have thus shown that $\delta_\kappa\alpha$ is Cauchy in $W^{1,p}(\Omega)$ as $\kappa \to 0$ for real κ. Taking the limit in (4.3) we see that $\partial_{k_x}\alpha \in W^{1,p}(\Omega)$ satisfies

$$\partial_{k_x}\alpha + \mathcal{C}\left(B\partial_{k_x}\alpha + e_k\bar{B}\overline{\partial_{k_x}\alpha}\right) = -\mathcal{C}\left((ixe_k)\bar{B}\bar{\alpha}_k\right).$$

We can again apply (3.9) to conclude that $\partial_{k_x}\alpha$ is bounded in $W^{1,p}(\Omega)$ uniformly in k. The case of imaginary κ is almost identical. We conclude that

Proposition 4.2. *The partial derivatives of α_k with respect to k exist and $\partial_{k_x}\alpha_k$, $\partial_{k_y}\alpha_k \in L^\infty(\mathbb{C}_k; W^{1,p}(\Omega))$.*

4.2. Behavior as $k \to \infty$

Because the α_k are bounded in $W^{1,p}(\Omega)$ uniformly in k, we can extract a subsequence, also denoted α_k, such that $\alpha_k \rightharpoonup \alpha_0$ in L^{p^*} and $\alpha_k \to \alpha_0$ in L^q for any $1 \le q < p^*$. Of course, $\alpha_0 \in W^{1,p}(\Omega)$.

To find the equation satisfied by α_0 we are going to integrate (4.2) against a test function $\varphi \in C_c^\infty(\Omega)$, then for the first term we have

$$\langle\varphi, \alpha_k\rangle \;\to\; \langle\varphi, \alpha_0\rangle.$$

For the second term, since $\mathcal{C}\varphi \in L^\infty(\Omega)$ and $B \in L^2(\Omega)$,

$$\langle \varphi, \mathcal{C}(B\alpha_k)\rangle = -\langle B\mathcal{C}\varphi, \alpha_k\rangle \;\to\; -\langle B\mathcal{C}\varphi, \alpha_0\rangle = \langle \varphi, \mathcal{C}(B\alpha_0)\rangle.$$

To handle the third term, we write

$$\left\langle \varphi, \mathcal{C}\left(e_k \bar{B}\bar{\alpha}_k\right)\right\rangle = -\left\langle e_k \bar{B}\mathcal{C}\varphi, \bar{\alpha}_0\right\rangle - \left\langle e_k \bar{B}\mathcal{C}\varphi, \bar{\alpha}_k - \bar{\alpha}_0\right\rangle.$$

Since $\bar{\alpha}_k \to \bar{\alpha}_0$ strongly in $L^2(\Omega)$,

$$\left\langle e_k \bar{B}\mathcal{C}\varphi, \bar{\alpha}_k - \bar{\alpha}_0\right\rangle \to 0.$$

As $\bar{\alpha}_0 \bar{B}\mathcal{C}\varphi \in L^p(\Omega)$, the Riemann-Lebesgue lemma implies

$$\left\langle e_k \bar{B}\mathcal{C}\varphi, \bar{\alpha}_0\right\rangle \to 0.$$

Putting these together we get that

$$\langle \varphi, \alpha_0 + \mathcal{C}(B\alpha_0) - 1\rangle = 0,$$

for any $\varphi \in C_c^\infty(\Omega)$ and so

$$\alpha_0(z) + \mathcal{C}(B\alpha_0)(z) = 1, \quad \text{for all} \quad z \in \Omega.$$

Of course it then follows that

$$\partial_{\bar{z}}\, \alpha_0 + B\alpha_0 = 0,$$

and applying back the Cauchy transform we need to have

$$\frac{1}{2\pi i}\int_{\partial\Omega} \frac{\alpha_0(\zeta)}{\zeta - z}\, \mathrm{d}\zeta = 1.$$

We have thus proved the following.

Lemma 4.3. *There exists a subsequence of solutions α_k of* (4.2) *such that $\alpha_k \rightharpoonup \alpha_0$ in $W^{1,p}(\Omega) \subset L^{p^*}$ and $\alpha_k \to \alpha_0$ in L^q for any $1 \le q < p^*$. The limit satisfies the integral equation*

$$\alpha_0(z) + \mathcal{C}(B\alpha_0)(z) = 1, \quad \textit{for all} \quad z \in \Omega,$$

and

$$\frac{1}{2\pi i}\int_{\partial\Omega} \frac{\alpha_0(\zeta)}{\zeta - z}\, d\zeta = 1.$$

In fact

$$\alpha_0 = e^{-\mathcal{C}(B)}. \tag{4.4}$$

Proof. We only have to show that the representation formula (4.4) holds. First note that extending B to the whole plane such that $B = B\chi_\Omega$ and using the equation to extend α_0 we have that $\alpha_0 \in 1 + W^{1,p}(\mathbb{C})$. Define $h := \alpha_0 e^{\mathcal{C}(B)}$ and note that $\partial_{\bar{z}}\, h = 0$ in $\mathbb{C}$, so it is holomorphic on $\mathbb{C}$. Since B is supported within the bonded domain Ω, then $\mathcal{C}(B)$ and $\mathcal{C}(B\alpha_0)$ are both holomorphic on $\mathbb{C} - \bar{\Omega}$. Furthermore, they both decay as $1/z$ at infinity. It follows then that $\lim_{z\to\infty} h = 1$ and by Liouville's theorem this implies that $h \equiv 1$. □

5. Cauchy transforms of B_1 and B_2

In this section, we would like to show that under the assumptions of Theorem 2.2, the Cauchy transforms of B_1 and B_2 are identical outside of the domain Ω. Let $\alpha_{1,k}, \alpha_{2,k} \in W^{1,p}(\Omega)$ be the solutions of the integral equations

$$\alpha_{j,k} + \mathcal{C}\left(B_j \alpha_{j,k} + e_k \bar{B}_j \bar{\alpha}_{j,k}\right) = 1, \quad \text{for} \quad j = 1, 2.$$

First note that, according to Lemma 4.1,

$$\frac{1}{2\pi i} \int_{\partial\Omega} \frac{\alpha_{1,k}(\zeta)}{\zeta - z} \, \mathrm{d}\zeta = 1, \quad \text{for all} \quad z \in \Omega.$$

Define $w_1(z,k) := e^{\frac{i}{2}\bar{k}z} \alpha_{1,k}(z)$, providing $w_1(\cdot,k) \in W^{1,p}(\Omega)$. It satisfies

$$\partial_{\bar{z}} w_1 + B_1 w_1 + \bar{B}_1 \bar{w}_1 = 0.$$

By Lemma 3.1, there exists a $u_1(\cdot,k) \in W^{2,p}(\Omega)$ such that $w_1 = \partial_z u_1$ and

$$\triangle u_1 + \vec{b}_1 \cdot \nabla u_1 = 0.$$

Denote $\omega := u_1|_{\partial\Omega} \in W^{2-\frac{1}{p},p}(\partial\Omega)$ and let $u_2 \in W^{2,p}(\Omega)$ be the solution of

$$\begin{cases} \triangle u_2 + \vec{b}_2 \cdot \nabla u_2 = 0 \\ u_2|_{\partial\Omega} = \omega. \end{cases}$$

The existence of u_2 with boundary condition ω is guaranteed by choosing the constant C in the statement of Theorem 2.2 such that Proposition A.1 applies. Then $w_2 := \partial_z u_2$ will satisfy

$$\partial_{\bar{z}} w_2 + B_2 w_2 + \bar{B}_2 \bar{w}_2 = 0.$$

Since u_1 and u_2 share the same Dirichlet data, $\mathfrak{Re}\,(\dot{z} w_1) = \mathfrak{Re}\,(\dot{z} w_2)$. Since we are assuming the Dirichlet-Neumann maps produced by $\vec{b}_1$ and $\vec{b}_2$ are identical, the Neumann data of u_1 must be the same as u_2, so $\mathfrak{Im}\,(\dot{z} w_1) = \mathfrak{Im}\,(\dot{z} w_2)$. It follows then that

$$w_1|_{\partial\Omega} = w_2|_{\partial\Omega}.$$

Define $\alpha'_{2,k}(z) := w_2(z,k) e^{-\frac{i}{2}\bar{k}z}$. It satisfies the differential equation

$$\partial_{\bar{z}} \alpha'_{2,k} + B_2 \alpha'_{2,k} + e_k \bar{B}_2 \bar{\alpha}'_{2,k} = 0$$

and, as $\alpha_{1,k}|_{\partial\Omega} = \alpha'_{2,k}|_{\partial\Omega}$, we see that $\alpha'_{2,k}$ satisfies the integral equation

$$\alpha'_{2,k} + \mathcal{C}\left(B_2 \alpha'_{2,k} + e_k \bar{B}_2 \bar{\alpha}'_{2,k}\right) = 1.$$

It follows the uniqueness of the solution that $\alpha'_{2,k} = \alpha_{2,k}$. Consequently, we proved that

Lemma 5.1. $\alpha_{1,k}|_{\partial\Omega} = \alpha_{2,k}|_{\partial\Omega}$.

We know from Lemma 4.3 that we can find a subsequence such that $\alpha_{j,k} \rightharpoonup \alpha_{j,0}$ in $W^{1,p}(\Omega)$, for $j = 1, 2$, and

$$\alpha_{j,0} + \mathcal{C}\big(B_j \alpha_{j,0}\big) = 1.$$

The trace operator τ maps $W^{1,p}(\Omega)$ to $W^{1-\frac{1}{p},p}(\partial\Omega) \subset L^{\frac{2p}{3-p}}(\partial\Omega)$ continuously and is compact from $W^{1,p}(\Omega)$ to $L^s(\partial\Omega)$ for $1 < s < \frac{2p}{3-p}$. Therefore

$$\alpha_{j,k}|_{\partial\Omega} \to \alpha_{j,0}|_{\partial\Omega}, \quad \text{in } L^s(\partial\Omega).$$

Lemma 5.1 then implies

$$\alpha_{1,0}|_{\partial\Omega} = \alpha_{2,0}|_{\partial\Omega}.$$

We can now prove the following.

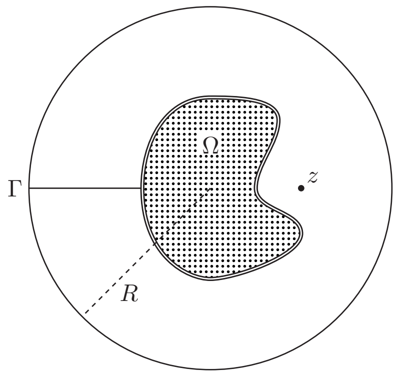

Figure 5.1. Contour for the determination of $e^{-\mathcal{C}(B_j)}(z)$.

Lemma 5.2. *If* $\Lambda_{\vec{b}_1} = \Lambda_{\vec{b}_2}$, *then* $\mathcal{C}(B_1)(z) = \mathcal{C}(B_2)(z)$ *for any* $z \in \mathbb{C} - \Omega$.

Proof. By equation (4.4), we know that $e^{-\mathcal{C}(B_1)}|_{\partial\Omega} = e^{-\mathcal{C}(B_2)}|_{\partial\Omega}$.

Let $z \in \mathbb{C} - \bar{\Omega}$. Since $e^{-\mathcal{C}(B_j)}$ is holomorphic in $C - \bar{\Omega}$, using the integration contour from the figure, we can write

$$e^{-\mathcal{C}(B_j)}(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{e^{-\mathcal{C}(B_j)}(\zeta)}{\zeta - z}\,\mathrm{d}\zeta - \frac{1}{2\pi i}\int_{|\zeta|=R} \frac{e^{-\mathcal{C}(B_j)}(\zeta)}{\zeta - z}\,\mathrm{d}\zeta.$$

As $\zeta \to \infty$, we have $e^{-\mathcal{C}(B_j)} = 1 + \mathcal{O}(\frac{1}{\zeta})$,so

$$\lim_{R\to\infty} \frac{1}{2\pi i}\int_{|\zeta|=R} \frac{e^{-\mathcal{C}(B_j)}(\zeta)}{\zeta - z}\,\mathrm{d}\zeta = 1.$$

It then follows that

$$e^{-\mathcal{C}(B_1)}(z) = e^{-\mathcal{C}(B_2)}(z),$$

for any $z \in \mathbb{C} - \bar{\Omega}$. This can happen only if $\mathcal{C}(B_1)(z) = \mathcal{C}(B_2)(z) + 2\pi n i$. Since both Cauchy transforms vanish at infinity we must have $n = 0$. □

6. A related first order system

To proceed further, we now would like to apply an inverse scattering method for a related first order system based on [4,15–17]. A similar idea was also used in [7,9]. We define

$$C_j := e^{\mathcal{C}B_j} e^{-\overline{\mathcal{C}B_j}} \bar{B}_j \quad \text{for} \quad j = 1, 2.$$

Note that, since $\mathcal{C}B_j - \overline{\mathcal{C}B_j} = 2i\mathfrak{Im}\,(\mathcal{C}B_j)$, we have $C_j \in L^2(\Omega)$ and $||C_j||_{L^2(\Omega)} = ||B_j||_{L^2(\Omega)}$. We also define the matrix

$$Q_j := \begin{pmatrix} 0 & -C_j \\ -C_j & 0 \end{pmatrix}.$$

Let $\mu_j(z)$ be the 2×2 matrix valued function that satisfies the integral equation

$$\mu_j(z, k) = I + \mathcal{C}\left(e_k Q_j \bar{\mu}_j\right) = I - \frac{1}{\pi} \int_\Omega \frac{e_k(\zeta)}{\zeta - z} Q_j(\zeta) \overline{\mu_j(\zeta, k)}\, \mathrm{d}^2\zeta. \qquad (6.1)$$

Consider the orthogonal matrix

$$\mathcal{R} := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Conjugating Q_j by $\mathcal{R}$ we get

$$\mathcal{R} Q_j \mathcal{R}^t = \begin{pmatrix} -C_j & 0 \\ 0 & C_j \end{pmatrix}.$$

Thus, conjugating the integral equation (6.1) by $\mathcal{R}$, we obtain a decoupled system of four scalar integral equations. We then can apply the same method we have used to prove the existence of solutions to the equation (3.6) to show (6.1) has unique solutions $\mu_j(\cdot, k) \in W^{1,p}(\Omega)$. These solutions satisfy the differential equation

$$\partial_{\bar{z}} \mu_j(z, k) - e_k(z) Q_j(z) \overline{\mu_j(z, k)} = 0$$

and the equality

$$I = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\mu_j(\zeta, k)}{\zeta - z}\, \mathrm{d}\zeta, \quad \text{for all} \quad z \in \Omega,\ k \in \mathbb{C}.$$

Also, just like the CGO solutions α_k, the μ_j are differentiable in k and $\partial_{k_x}\mu_j,\ \partial_{k_y}\mu_j \in W^{1,p}(\Omega)$.

Define

$$\eta_1(z,k) := e^{\frac{i}{2}\bar{k}z}\mu_1(z,k).$$

This new matrix-valued function satisfies the differential equation

$$\partial_{\bar{z}}\,\eta_1 = Q_1\bar{\eta}_1.$$

Let

$$v_1 := \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\eta_1,$$

then

$$\partial_{\bar{z}}\,v_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} Q_1 \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}\bar{v}_1 = -C_1\bar{v}_1.$$

Finally let

$$w_1 := e^{-\mathcal{C}B_1}v_1$$

and we get that w_1 satisfies the matrix differential equation

$$\partial_{\bar{z}}\,w_1(z,k) + B_1(z)w_j(z,k) + \overline{B_1(z)w_1(z,k)} = 0, \quad \text{for all} \quad z \in \Omega,\ k \in \mathbb{C}.$$

Applying Lemma 3.1 to the components of w_1, we obtain that there is a matrix-valued function $u_1 \in W^{2,p}(\Omega)$ such that $w_1 = \partial_z\,u_1$ and

$$\triangle u_1 + \vec{b}_1 \cdot \nabla u_1 = 0.$$

Let $\omega := u_1|_{\partial\Omega} \in W^{2-\frac{1}{p},p}(\partial\Omega)$ and let $u_2 \in W^{2,p}(\Omega)$ be the 2×2 matrix valued solution of the equation

$$\begin{cases} \triangle u_2 + \vec{b}_2 \cdot \nabla u_2 = 0 \\ u_2|_{\partial\Omega} = \omega. \end{cases}$$

Just as before, we know that both the Dirichlet and the Neumann data of u_1 and u_2 coincide. Then $w_2 := \partial_z\,u_2$ satisfies

$$\partial_{\bar{z}}\,w_2 + B_2w_2 + \bar{B}_2\bar{w}_2 = 0$$

and

$$w_1|_{\partial\Omega} = w_2|_{\partial\Omega}.$$

Also $v_2 := e^{\mathcal{C}B_2}w_2$ satisfies

$$\partial_{\bar{z}}\,v_2 = -C_2\bar{v}_2$$

and since, by Lemma 5.2, $e^{\mathcal{C}B_1}|_{\partial\Omega} = e^{\mathcal{C}B_2}|_{\partial\Omega}$ we have

$$v_2|_{\partial\Omega} = v_1|_{\partial\Omega}.$$

Let

$$\eta_2 := \frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} v_2, \ \textit{i.e.}, \ v_2 := \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \eta_2,$$

and finally

$$\mu_2' := e^{-\frac{i}{2}\bar{k}z}\eta_2.$$

This last matrix-valued function satisfies the differential equation

$$\partial_{\bar{z}}\, \mu_2' = e_k Q_2 \bar{\mu}_2'$$

and

$$\mu_1|_{\partial\Omega} = \mu_2'|_{\partial\Omega}.$$

Applying the Cauchy transform to the differential equation we get

$$\mu_2' = \Phi_2 + \mathcal{C}\big(e_k Q_2 \bar{\mu}_2'\big),$$

where

$$\Phi_2(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{\mu_2'(\zeta)}{\zeta - z}\, \mathrm{d}\zeta = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{\mu_1(\zeta)}{\zeta - z}\, \mathrm{d}\zeta = I.$$

So $\mu_2' \equiv \mu_2$ and we have the following.

Lemma 6.1. *It holds* $\mu_1|_{\partial\Omega} = \mu_2|_{\partial\Omega}$.

7. The $\partial_{\bar{k}}$ equation

We now extend $\mu_j(z,k)$, for $j = 1, 2$, to $z \in \mathbb{C} \setminus \Omega$ by setting

$$\mu_j(z,k) = I + \mathcal{C}\left(e_k Q_j \bar{\mu}_j\right) = I - \frac{1}{\pi}\int_{\Omega} \frac{e_k(\zeta)}{\zeta - z} Q_j(\zeta)\overline{\mu_j(\zeta,k)}\, \mathrm{d}^2\zeta, \ \text{for all } z \in \mathbb{C}\setminus\Omega.$$

Abusing notation a little, if we write $Q_j = Q_j\chi_\Omega$, then $\mu_j(z,k)$ satisfies

$$\begin{aligned} \mu_j(z,k) &= I + \mathcal{C}\left(e_k Q_j \bar{\mu}_j\right) \\ &= I - \frac{1}{\pi}\int_{\mathbb{C}} \frac{e_k(\zeta)}{\zeta - z} Q_j(\zeta)\overline{\mu_j(\zeta,k)}\, \mathrm{d}^2\zeta, \\ &\quad \text{for all } z \in \mathbb{C}, \ \text{for all } k \in \mathbb{C}. \end{aligned}$$

We can easily see that

$$\lim_{|z|\to\infty} \mu_j(z,k) = I,$$

that

$$\mu_j(\cdot,k) \in W^{1,p}(\Omega),$$

and $\mu_j(z,k) - I$ decays like $\frac{1}{z}$ as $z \to \infty$.

To simplify the notation, we suppress the index j. In other words, the computations below are carried out for μ_j, $j = 1, 2$, respectively. Define the matrix

$$\nu(z,k) := \begin{pmatrix} \overline{\mu_{11}(z,k)} & e_{-k}(z)\mu_{12}(z,k) \\ e_{-k}\mu_{21}(z,k) & \overline{\mu_{22}(z,k)} \end{pmatrix}.$$

Note that

$$\nu(z,k) = I + \frac{1}{\pi}\int_\Omega \begin{pmatrix} \frac{\overline{C(\zeta)}}{\bar\zeta-\bar z}\nu_{21}(\zeta,k) & \frac{e_k(\zeta-z)C(\zeta)}{\zeta-z}\nu_{22}(\zeta,k) \\ \frac{e_k(\zeta-z)C(\zeta)}{\zeta-z}\nu_{11}(\zeta,k) & \frac{\overline{C(\zeta)}}{\bar\zeta-\bar z}\nu_{12}(\zeta,k) \end{pmatrix} \,\mathrm{d}^2\zeta$$

and ν is the unique solution of this integral equation. Repeating the proof of Proposition 4.2, we can show that $\partial_{\bar k}\nu(z,k)$ exists in $W^{1,p}(\Omega)$ and it satisfies

$$\begin{aligned}\partial_{\bar k}\,\nu = \frac{e_{-k}(z)}{2\pi i}\int_\Omega \begin{pmatrix} 0 & e_k(\zeta)C(\zeta)\overline{\mu_{22}(\zeta,k)} \\ e_k(\zeta)C(\zeta)\overline{\mu_{11}(\zeta,k)} & 0 \end{pmatrix}\,\mathrm{d}^2\zeta \\ + \frac{e_{-k}(z)}{\pi}\int_\Omega \begin{pmatrix} \frac{\overline{C(\zeta)}}{\bar\zeta-\bar z}e_k(z)\partial_{\bar k}\,\nu_{21}(\zeta,k) & \frac{e_k(\zeta)C(\zeta)}{\zeta-z}\partial_{\bar k}\,\nu_{22}(\zeta,k) \\ \frac{e_k(\zeta)C(\zeta)}{\zeta-z}\partial_{\bar k}\,\nu_{11}(\zeta,k) & \frac{\overline{C(\zeta)}}{\bar\zeta-\bar z}e_k(z)\partial_{\bar k}\,\nu_{12}(\zeta,k) \end{pmatrix}\,\mathrm{d}^2\zeta.\end{aligned}$$

This can be rewritten as

$$\begin{aligned} e_k(z)\partial_{\bar k}\,\nu = &\begin{pmatrix} 0 & T_{12}(k) \\ T_{21}(k) & 0 \end{pmatrix} \\ &+ \frac{1}{\pi}\int_\Omega \begin{pmatrix} \frac{\overline{e_k(\zeta-z)C(\zeta)}}{\bar\zeta-\bar z}e_k(\zeta)\partial_{\bar k}\,\nu_{21}(\zeta,k) & \frac{C(\zeta)}{\zeta-z}e_k(\zeta)\partial_{\bar k}\,\nu_{22}(\zeta,k) \\ \frac{C(\zeta)}{\zeta-z}e_k(\zeta)\partial_{\bar k}\,\nu_{11}(\zeta,k) & \frac{\overline{e_k(\zeta-z)C(\zeta)}}{\bar\zeta-\bar z}e_k(\zeta)\partial_{\bar k}\,\nu_{12}(\zeta,k) \end{pmatrix}\mathrm{d}^2\zeta, \end{aligned} \tag{7.1}$$

where we define

$$\begin{aligned} T_{12}(k) &= \frac{1}{2\pi i}\int_\Omega e_k(\zeta)C(\zeta)\overline{\mu_{22}(\zeta,k)}\,\mathrm{d}^2\zeta, \\ T_{21}(k) &= \frac{1}{2\pi i}\int_\Omega e_k(\zeta)C(\zeta)\overline{\mu_{11}(\zeta,k)}\,\mathrm{d}^2\zeta. \end{aligned} \tag{7.2}$$

Comparing the integral equation (7.1) with

$$\begin{aligned}\overline{\nu(z,k)}\begin{pmatrix} 0 & T_{12}(k) \\ T_{21}(k) & 0 \end{pmatrix} = \begin{pmatrix} T_{21}(k)\overline{\nu_{12}(z,k)} & T_{12}(k)\overline{\nu_{11}(z,k)} \\ T_{21}(k)\overline{\nu_{22}(z,k)} & T_{12}(k)\overline{\nu_{21}(z,k)} \end{pmatrix} = \begin{pmatrix} 0 & T_{12}(k) \\ T_{21}(k) & 0 \end{pmatrix} \\ + \frac{1}{\pi}\int_{\mathbb{C}} \begin{pmatrix} \frac{\overline{e_k(\zeta-z)C(\zeta)}}{\bar\zeta-\bar z}T_{21}(k)\overline{\nu_{22}(\zeta,k)} & \frac{C(\zeta)}{\zeta-z}T_{12}(k)\overline{\nu_{21}(\zeta,k)} \\ \frac{C(\zeta)}{\zeta-z}T_{21}(k)\overline{\nu_{12}(\zeta,k)} & \frac{\overline{e_k(\zeta-z)C(\zeta)}}{\bar\zeta-\bar z}T_{12}(k)\overline{\nu_{11}(\zeta,k)} \end{pmatrix}\,\mathrm{d}^2\zeta,\end{aligned}$$

we deduce from the uniqueness of integral equation (7.1) that

$$\partial_{\bar{k}}\, \nu(z,k) = e_{-k}(z)\overline{\nu(z,k)} \begin{pmatrix} 0 & T_{12}(k) \\ T_{21}(k) & 0 \end{pmatrix}.$$

Since

$$\mu_{11} = 1 - \mathcal{C}(e_k C) + \mathcal{C}\big(e_k C \bar{\mathcal{C}}\big(e_{-k} \bar{C} \bar{\mu}_{11}\big)\big),$$
$$\mu_{22} = 1 - \mathcal{C}(e_k C) + \mathcal{C}\big(e_k C \bar{\mathcal{C}}\big(e_{-k} \bar{C} \bar{\mu}_{22}\big)\big),$$

it follows that $\mu_{11} = \mu_{22}$, so also $T_{12} = T_{21}$. Now let us denote

$$T(k) = \begin{pmatrix} 0 & T_{12}(k) \\ T_{12}(k) & 0 \end{pmatrix}$$

and specify $T_j(k)$ be as T above defined for C_j, for $j = 1, 2$. At this point we can also note that

$$\partial_{\bar{k}}\, \nu(z,k) = e_{-k}(z)\overline{\nu(z,k)}T(k) = e_{-k}(z)T(k)\overline{\nu(z,k)}.$$

We have the following.

Lemma 7.1. *It holds $T_1(k) = T_2(k)$ for all k.*

The proof of this is identical to the argument given for the corresponding statement of [9, (4.25)] as part of the proof of their Lemma 4.4. The result of our Lemma 6.1 is needed in the proof.

We can now use a known result of Brown on the mapping properties of the scattering map to conclude that:

Lemma 7.2. *It holds $C_1 = C_2$.*

Proof. Note that the off-diagonal entries of Q_j are zero. We use [5, Theorem 2]. There it is shown that the mapping $Q \to T$ is invertible, hence injective, provided $\|Q\|_{L^2} < \sqrt{2}$. This can be arranged by choosing the constant C in the statement of Theorem 2.2 appropriately. Since we have already determined that $T_1 = T_2$, it follows that $Q_1 = Q_2$. □

8. Conclusion of the argument

By Lemma 7.2 we have that

$$B_1 e^{\overline{\mathcal{C}B_1} - \mathcal{C}B_1} = B_2 e^{\overline{\mathcal{C}B_2} - \mathcal{C}B_2}, \text{ for all } z \in \Omega,$$

and hence

$$B_1 e^{\overline{\mathcal{C}B_1} - \mathcal{C}B_1} = B_2 e^{\overline{\mathcal{C}B_2} - \mathcal{C}B_2}, \text{ for all } z \in \mathbb{C},$$

since $B_1 = B_2 = 0$ in $\mathbb{C} \setminus \Omega$. In view of Lemma 5.2, if we let $Z := \mathcal{C}(B_2 - B_1)$, then $Z \in W^{1,2}(\mathbb{C})$ and $\operatorname{supp}(Z) \subseteq \Omega$. Denote

$$\kappa := e^{\bar{Z}-Z}.$$

Then we obtain

$$B_1 = \kappa B_2$$

and

$$\partial_{\bar{z}} Z = (1-\kappa) B_2 \text{ for } z \in \mathbb{C}.$$

By an easy computation (see [9, page 1389]), we can deduce that

$$|\kappa - 1| \leq 2|Z|$$

and thus

$$|\partial_{\bar{z}} Z| \leq 2|B_2|\,|Z|$$

holds. Applying the generalized Liouville theorem of [7, Corollary 3.11] (or [1, Theorem 5.8.3]), it follows then that $Z \equiv 0$ and so $B_1 \equiv B_2$. This ends the proof of Theorem 2.2.

Appendix

A. Existence of strong solutions for coefficients with small norm

In this appendix, we will show that if $\|\vec{b}\|_{L^2(\Omega)}$ is not too large, then (2.2) has a unique solution for any Dirichlet condition in $W^{2-\frac{1}{p},p}(\partial\Omega)$. The proof is based on a contraction mapping principle argument.

Proposition A.1. *There exists a constant* $c(p,\Omega) > 0$ *such that if* $||\vec{b}||_{L^2(\Omega)} < c(p,\Omega)$, *then* (2.2) *has a unique solution* $u \in W^{2,p}(\Omega)$.

Proof. Let $H \in W^{2,p}(\Omega)$ be the unique solution to the Dirichlet boundary value problem:

$$\begin{cases} \triangle H(x) = 0, & x \in \Omega \\ H|_{\partial\Omega} = \omega. \end{cases}$$

Defining $u_0 := u - H$, the equation becomes

$$\triangle u_0(x) + \vec{b}(x) \cdot \nabla u_0(x) = -\vec{b} \cdot \nabla H(x) \tag{A.1}$$

with $u_0 \in W_0^{1,p}(\Omega)$. Let $\mathcal{L}(f)$ be the solution of the inhomogeneous problem

$$\begin{cases} \triangle U(x) = f, & x \in \Omega \\ U|_{\partial\Omega} = 0. \end{cases}$$

Then $\mathcal{L} : L^p(\Omega) \to W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ is continuous (*e.g.*, [10, Theorem 9.15]). If (A.1) can be solved, then the solution should satisfy

$$u_0 = -\mathcal{L}\left(\vec{b} \cdot \nabla u_0 + \vec{b} \cdot \nabla H\right).$$

Seeing this, we define the operator T for functions $v \in W^{2,p}(\Omega)$ by

$$Tv := -\mathcal{L}\left(\vec{b} \cdot \nabla v + \vec{b} \cdot \nabla H\right).$$

Since $\nabla v \in W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$, with $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{2}$, it follows that $\vec{b} \cdot \nabla v \in L^p(\Omega)$. The same is true of the second term in the definition of T. Putting these together we get that there is a constant $C(p, \Omega) > 0$ such that

$$||Tv||_{W^{2,p}(\Omega)} \leq C(p, \Omega)\left(||\vec{b}||_{L^2(\Omega)}||v||_{W^{2,p}(\Omega)} + ||\vec{b}||_{L^2(\Omega)}||\omega||_{W^{2-\frac{1}{p},p}(\partial\Omega)}\right).$$

So $T : W^{2,p}(\Omega) \to W^{2,p}(\Omega)$. Moreover, for two $v_1, v_2 \in W^{2,p}(\Omega)$, it hods

$$||Tv_1 - Tv_2||_{W^{2,p}(\Omega)} \leq C(p, \Omega)||\vec{b}||_{L^2(\Omega)}||v_1 - v_2||_{W^{2,p}(\Omega)}.$$

If $||\vec{b}||_{L^2(\Omega)} < \frac{1}{C(p,\Omega)}$, the operator T is a contraction hence it will have a fixed point: the solution of (A.1). Uniqueness follows easily from these same estimates. □

B. Recovering the conductivity at the boundary

Since $\gamma \in W^{1,2}(\Omega)$ we can consider its trace $\gamma|_{\partial\Omega} \in W^{\frac{1}{2},2}(\partial\Omega)$. In this section, following the method of Brown in [6], we will prove that we can recover $\gamma|_{\partial\Omega}$ from the Dirichlet-to-Neumann map. We cannot quote the result of [6] directly since there the conductivity is assumed to be bounded.

First we need to straighten out the boundary. If $P \in \partial\Omega$, then there is a ball $B(P, R)$ and a diffeomorphism $\phi : \Omega \to \tilde{\Omega}$ such that $\phi(\Omega \cap B(P, R)) \subset \{y_2 > 0\}$ and $0 \in \phi(\partial\Omega \cap B(P, R)) \subset \{y_2 = 0\}$. In the new coordinates the equation (2.1), denoting $F := \phi^{-1}$ and $v := u \circ F$, becomes

$$\partial_k(a_{kj}\partial_j v) = 0,$$

where

$$a_{kj}(y) := \gamma(F(y))\partial_i\phi^k(F(y))\partial_i\phi^j(F(y))\det(DF)(y).$$

Hereafter, we adopt the summation convention. We can make sure that $D\phi \in C^1 \cap L^\infty$, and $DF \in C^1 \cap L^\infty$, so $a_{kj} \in W^{1,2}_{\text{loc}}$.

Choose a smooth $\eta : \mathbb{R} \to [0, \infty)$, with $\operatorname{supp} \eta \subset [-1, 1]$, $\eta|_{[-\frac{1}{2},\frac{1}{2}]} \equiv 1$. Also choose $\alpha \in \mathbb{R}^2$ such that

$$\partial_i \phi^j(P)\alpha_j \partial_i \phi^k(P)\alpha_k = \partial_i \phi^j(P)\delta_{j2}\partial_i \phi^k(P)\delta_{k2},$$
$$\partial_i \phi^j(P)\alpha_j \partial_i \phi^k(P)\delta_{k2} = 0.$$

In other words, $\mathbb{R}^2 \ni \alpha$ satisfies $|D\phi(p)\alpha| = |D\phi(P)e_2|$ and $D\phi(P)\alpha \perp D\phi(P)e_2$. Define $\mu := D\phi(P)(i\alpha - e_2) \in \mathbb{C}^2$. Then $\mu \cdot \mu = 0$, and $\mu \cdot \bar{\mu} = 2|D\phi(P)e_2|^2 > 0$. With these notations we introduce the functions

$$v_N(y) := \eta\left(N^{1/2}y_1\right)\eta\left(N^{1/2}y_2\right)e^{N(i\alpha - e_2)\cdot y} =: \psi\left(N^{1/2}y\right)E(Ny),$$

where $N \in \mathbb{N}$.

Lemma B.1. *There is a non-zero constant A such that*

$$\int_{\tilde{\Omega}} a_{ij}(y)\partial_i v_N(y)\partial_j \bar{v}_N(y)\, dy = N^{\frac{1}{2}}\gamma(F(0))A + o\left(N^{\frac{1}{2}}\right),$$

as $N \to \infty$.

Proof. Let $\tilde{\gamma}(y) := \gamma(F(y))\det(DF)(y)$. We compute

$$\begin{aligned}
&\int_{\tilde{\Omega}} a_{ij}(y)\partial_i v_N(y)\partial_j \bar{v}_N(y)\, \mathrm{d}y \\
&= N^2 \int_{\tilde{\Omega}} \tilde{\gamma}(0)\mu \cdot \bar{\mu}\psi\left(N^{1/2}y\right)^2 e^{-2Ny_2}\, \mathrm{d}y \\
&\quad + N^2 \int_{\tilde{\Omega}} (\tilde{\gamma}(y) - \tilde{\gamma}(0))\,\mu \cdot \bar{\mu}\psi\left(N^{1/2}y\right)^2 e^{-2Ny_2}\, \mathrm{d}y \\
&\quad + N \int_{\tilde{\Omega}} a_{ij}(y)(\partial_i\psi)(N^{1/2}y)(\partial_j\psi)\left(N^{1/2}y\right)e^{-2Ny_2}\, \mathrm{d}y \\
&\quad - N^{3/2} \int_{\tilde{\Omega}} a_{ij}(y)\delta_{j2}(\partial_i\psi)(N^{1/2}y)\psi\left(N^{1/2}y\right)e^{-2Ny_2}\, \mathrm{d}y \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{B.1}$$

We begin with the estimate of I_4. Note that

$$\begin{aligned}
N^{-\frac{3}{2}}|I_4| &\le \sup_{i,j} \int_{-N^{-\frac{1}{2}}}^{N^{-\frac{1}{2}}} \int_0^{N^{-\frac{1}{2}}} |a_{ij}(0)|e^{-2Ny_2}\, \mathrm{d}y_2\, \mathrm{d}y_1 \\
&\quad + \sup_{i,j} \int_{-N^{-\frac{1}{2}}}^{N^{-\frac{1}{2}}} \int_0^{N^{-\frac{1}{2}}} |a_{ij}(y_1, 0) - a_{ij}(0)|e^{-2Ny_2}\, \mathrm{d}y_2\, \mathrm{d}y_1 \\
&\quad + \sup_{i,j} \int_{-N^{-\frac{1}{2}}}^{N^{-\frac{1}{2}}} \int_0^{N^{-\frac{1}{2}}} \frac{|a_{ij}(y_1, y_2) - a_{ij}(y_1, 0)|}{y_2} y_2 e^{-2Ny_2}\, \mathrm{d}y_2\, \mathrm{d}y_1 \\
&=: J_1 + J_2 + J_3.
\end{aligned}$$

By direct computation, we have that

$$J_1 = \gamma(F(0))CN^{-\frac{3}{2}}$$

for some constant C. If 0 is a Lebesgue point for $a_{ij}|_{\partial\tilde{\Omega}}$ then

$$N^{\frac{1}{2}} \int_{-N^{-\frac{1}{2}}}^{N^{-\frac{1}{2}}} |a_{ij}(y_1, 0) - a_{ij}(0)| \, \mathrm{d}y_1 \to 0,$$

which gives that also $J_2 = o(N^{-\frac{3}{2}})$. And finally Hardy's inequality implies

$$J_3 \leq C \sup_{i,j} ||\nabla a_{ij}||_{L^2} N^{-\frac{7}{4}}.$$

Hence $I_4 = \mathcal{O}(1)$. The same type of estimates give $I_3 = \mathcal{O}(N^{-\frac{1}{2}})$.

The first term of (B.1) is easily seen by direct computation to satisfy

$$I_1 = \gamma(F(0))AN^{\frac{1}{2}} + o\left(N^{\frac{1}{2}}\right).$$

Finally, arguing in the same way as in the case of the terms J_2 and J_3 we have

$$\begin{aligned} N^{-2} I_2 = &\int_{\tilde{\Omega}} (\gamma(F(y_1, 0)) - \gamma(F(0)))\, \mu \cdot \bar{\mu} \psi \left(N^{1/2} y\right)^2 e^{-2N y_2} \, \mathrm{d}y \\ &+ \int_{\tilde{\Omega}} (\gamma(F(y_1, y_2)) - \gamma(F(y_1, 0)))\, \mu \cdot \bar{\mu} \psi \left(N^{1/2} y\right)^2 e^{-2N y_2} \, \mathrm{d}y \\ = &\, o(N^{-\frac{3}{2}}) + \mathcal{O}\left(N^{-\frac{7}{4}}\right), \end{aligned}$$

which gives $I_2 = o(N^{\frac{1}{2}})$. □

Lemma B.2. *It holds* $||\partial_i(a_{ij}\partial_j v_N)||_{H^{-1}(\tilde{\Omega})} = o(N^{\frac{1}{4}})$.

Proof. We begin by splitting the left hand side into several terms

$$\begin{aligned} \partial_i(a_{ij}\partial_j v_N)(y) = &\, a_{ij}(0) N^{3/2} (\partial_i \psi)\left(N^{1/2} y\right)(i\alpha - e_2)_j E(Ny) \\ &+ a_{ij}(0) N (\partial_i \partial_j \psi)\left(N^{1/2} y\right) E(Ny) \\ &+ \partial_i \Big((a_{ij}(y) - a_{ij}(0))\partial_j \left(\psi\left(N^{1/2} y\right) E(Ny)\right)\Big) \\ = &\, I_1 + I_2 + I_3. \end{aligned}$$

Let $\varphi \in H_0^1(\tilde{\Omega})$ (we use the norm $||\varphi||_{H_0^1} = ||\nabla\varphi||_{L^2}$) and denote $\delta(y)$ the distance between y and $\partial\tilde{\Omega}$.

$$N^{-\frac{3}{2}}\left|\int_{\tilde{\Omega}}\varphi(y)I_1(y)\ \mathrm{d}y\right| \le |a_{ij}(0)|\int_{\tilde{\Omega}}\frac{\varphi(y)}{\delta(y)}\delta(y)(\partial_i\psi)(N^{1/2}y)|(i\alpha-e_2)_j|e^{-Ny_2}\ \mathrm{d}y$$
$$\le \sup_{ij}|a_{ij}(0)|\left(\int_{\tilde{\Omega}}\frac{\varphi(y)^2}{\delta(y)^2}\ \mathrm{d}y\right)^{\frac{1}{2}}\left(\int_{-N^{-\frac{1}{2}}}^{N^{-\frac{1}{2}}}\int_0^{N^{-\frac{1}{2}}}y_2^2e^{-2Ny_2}\ \mathrm{d}y\right)^{\frac{1}{2}}$$
$$\le C||\varphi||_{H_0^1}N^{-\frac{7}{4}},$$

hence
$$||I_1||_{H^{-1}(\tilde{\Omega})} = \mathcal{O}\left(N^{-\frac{1}{4}}\right).$$

In the same way we get
$$||I_2||_{H^{-1}(\tilde{\Omega})} = \mathcal{O}\left(N^{-\frac{3}{4}}\right).$$

For the third term, we have

$$\left|\int_{\tilde{\Omega}}\varphi(y)I_3(y)\ \mathrm{d}y\right| \le C||\nabla\varphi||_{L^2}$$
$$\times\left[N^{1/2}\sup_i\left(\int_{-N^{-\frac{1}{2}}}^{N^{-\frac{1}{2}}}\int_0^{N^{-\frac{1}{2}}}(a_{ij}(y)-a_{ij}(0))^2(\partial_j\psi)^2(N^{1/2}y)e^{-2Ny_2}\ \mathrm{d}y\right)^{\frac{1}{2}}\right.$$
$$\left.+\ N\sup_{ij}\left(\int_{-N^{-\frac{1}{2}}}^{N^{-\frac{1}{2}}}\int_0^{N^{-\frac{1}{2}}}(a_{ij}(y)-a_{ij}(0))^2\psi^2(N^{1/2}y)e^{-2Ny_2}\ \mathrm{d}y\right)^{\frac{1}{2}}\right]$$
$$\le C||\nabla\varphi||_{L^2}\left[N^{1/2}J_1 + NJ_2\right].$$

We first look at J_1,

$$J_1 \le \sup_{ij}\left(\int_{-N^{-\frac{1}{2}}}^{N^{-\frac{1}{2}}}\int_0^{N^{-\frac{1}{2}}}\frac{(a_{ij}(y_1,y_2)-a_{ij}(y_1,0))^2}{y_2^2}y_2^2e^{-2Ny_2}\ \mathrm{d}y\right)^{\frac{1}{2}}$$
$$+\sup_{ij}\left(\int_{-N^{-\frac{1}{2}}}^{N^{-\frac{1}{2}}}\int_0^{N^{-\frac{1}{2}}}(a_{ij}(y_1,0)-a_{ij}(0))^2e^{-2Ny_2}\ \mathrm{d}y\right)^{\frac{1}{2}}$$
$$= K_1 + K_2.$$

Using the inequality

$$s^2 e^{-2Ns} \le N^{-2} e^{-2} \quad \text{for all} \quad s \in [0, \infty),$$

we get $K_1 = \mathcal{O}(N^{-1})$. Using the same methods as in previous estimates, we also have $K_2 = o(N^{-\frac{3}{4}})$. This gives $J_1 = o(N^{-\frac{3}{4}})$ and similarly also $J_2 = o(N^{-\frac{3}{4}})$. Then we have

$$||I_3||_{H^{-1}(\tilde{\Omega})} = o\left(N^{\frac{1}{4}}\right)$$

and the lemma is proved. □

Lemma B.3. *The boundary value problem*

$$\partial_i(a_{ij}\partial_j w_N) = \partial_i(a_{ij}\partial_j v_N), \ \textit{with}\ w_N|_{\partial\tilde{\Omega}} \equiv 0,$$

has solutions that satisfy $||w_N||_{H_0^1(\tilde{\Omega})} = o(N^{\frac{1}{4}})$.

Proof. Existence of weak solutions follows from applying the standard Lax-Milgram argument in the energy space

$$H_a := \left\{ v : \tilde{\Omega} \to \mathbb{C} : \int_{\tilde{\Omega}} a_{ij}(y)\partial_i v(y)\partial_j v(y)\,\mathrm{d}y < \infty,\ v|_{\partial\tilde{\Omega}} = 0 \right\}.$$

Recall that γ is positive and bounded away from zero. So a_{ij} is a positive-definite matrix function and we can see that $H_a \subset H_0^1(\tilde{\Omega})$. Since $H^{-1} \subset (H_a)'$ the right hand side of the equation the usual method can indeed be applied. Integrating the equation against w_N we obtain

$$||\nabla w_N||_{L^2}^2 \le ||\nabla w_N||_{L^2} ||\partial_i(a_{ij}\partial_j v_N)||_{H^{-1}}$$

and the result follows. □

We are now ready to prove the result of boundary determination.

Proof of Proposition 2.3. Let $P \in \partial\Omega, \phi, F, \tilde{\Omega}$ be as above. Assume $\phi(P) = 0$ and 0 is a Lebesgue point for a_{ij} so that our lemmas above apply. Almost every point on the boundary satisfies this, so there is no loss of generality. Define

$$f_N := A^{-\frac{1}{2}} N^{-\frac{1}{4}} v_N \circ \phi.$$

Let u_N be the solution of

$$\begin{cases} \nabla(\gamma\nabla u_N)(x) = 0, & x \in \Omega \\ u|_{\partial\Omega} = f_N|_{\partial\Omega}, \end{cases}$$

which exists, as argued in Lemma B.3. Then

$$\int_{\partial\Omega} \bar{f}_N \Lambda_\gamma f_N \, \mathrm{d}x = \int_\Omega \gamma |\nabla u_N|^2 \, \mathrm{d}y = \int_{\tilde{\Omega}} a_{ij}(y) \partial_i (u_N \circ F) \partial_j (\bar{u}_N \circ F) \, \mathrm{d}y$$
$$= A^{-1} N^{-\frac{1}{2}} \int_{\tilde{\Omega}} a_{ij} \partial_i v_N \partial_j \bar{v}_N \, \mathrm{d}y + o(1) = \gamma(P) + o(1).$$

Taking the limit as $N \to \infty$ we see that we can determine $\gamma|_{\partial\Omega}$ almost everywhere on the boundary from the Dirichlet-to-Neumann map. □

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National Center for
Theoretical Sciences Mathematics Division
Taipei 106, Taiwan
cicarstea@ncts.ntu.edu.tw

Institute of
Applied Mathematical Sciences, NCTS
National Taiwan University
Taipei 106, Taiwan
jnwang@math.ntu.edu.tw