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Definition, existence, stability and uniqueness of the solution to a semilinear elliptic problem with a strong singularity at $u = 0$

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Abstract. In this paper we consider a semilinear elliptic equation with a strong singularity at $u = 0$, namely

$$\begin{cases} u \geq 0 & \text{in } \Omega \\ -\operatorname{div} A(x)Du = F(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $F(x,s)$ is a Carathéodory function such that

$$0 \leq F(x,s) \leq \frac{h(x)}{\Gamma(s)} \text{ a.e. } x \in \Omega,\ \forall s > 0$$

with h in some $L^r(\Omega)$ and Γ a $C^1([0,+\infty[)$ function such that $\Gamma(0) = 0$ and $\Gamma'(s) > 0$ for every $s > 0$.

We introduce a notion of solution to this problem in the spirit of the solutions defined by transposition. This definition allows us to prove the existence and the stability of this solution, as well as its uniqueness when $F(x,s)$ is nonincreasing in s.

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1. Introduction

Position of the problem

In the present paper we deal with a semilinear problem with a strong singularity at $u = 0$, which consists in finding a function u which satisfies

$$\begin{cases} u \geq 0 & \text{in } \Omega \\ -\operatorname{div} A(x)Du = F(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where Ω is a bounded open set of $\mathbb{R}^N$, $N \geq 1$, A is a coercive matrix with coefficients in $L^\infty(\Omega)$, and

$$F : (x, s) \in \Omega \times [0, +\infty[\to F(x, s) \in [0, +\infty]$$

is a Carathéodory function which satisfies

$$0 \leq F(x, s) \leq \frac{h(x)}{\Gamma(s)} \text{ a.e. } x \in \Omega, \ \forall s > 0, \tag{1.2}$$

where $h : x \in \Omega \to h(x) \in [0, +\infty[$ and $\Gamma : s \in [0, +\infty[\to \Gamma(s) \in [0, +\infty[$ satisfy

$$\begin{cases} h(x) \geq 0 \text{ a.e. } x \in \Omega, \ h \in L^r(\Omega) \\ \text{with } r = \dfrac{2N}{N+2} \ \text{ if } N \geq 3, \ r > 1 \text{ if } N = 2, \ r = 1 \text{ if } N = 1 \\ \Gamma \in C^1([0, +\infty[), \ \Gamma(0) = 0, \ \Gamma'(s) > 0 \ \forall s > 0. \end{cases} \tag{1.3}$$

The function F

The basic model for the function $F(x, s)$ is given by

$$F(x, s) = \frac{g(x)}{s^\gamma} + l(x) \text{ a.e. } x \in \Omega, \ \forall s > 0 \tag{1.4}$$

where $\gamma > 0$ and where the functions g and l are nonnegative. A more general model for the function $F(x, s)$ (see an even more general model in (2.2) and Remark 2.1 viii) below) is given by

$$\begin{cases} F(x, s) = f(x) \dfrac{\left(a + \sin\left(\frac{1}{s}\right)\right)}{\exp\left(-\frac{1}{s}\right)} + g(x) \dfrac{\left(b + \sin\left(\frac{1}{s}\right)\right)}{s^\gamma} + l(x) \\ \text{a.e. } x \in \Omega, \ \forall s > 0 \end{cases} \tag{1.5}$$

where $\gamma > 0$, $a > 1$, $b > 1$ and where the functions f, g and l are nonnegative. In this model $\dfrac{\left(a + \sin\left(\frac{1}{s}\right)\right)}{\exp\left(-\frac{1}{s}\right)}$, as well as $\dfrac{\left(b + \sin\left(\frac{1}{s}\right)\right)}{s^\gamma}$, are just examples of functions which can be replaced by any singularity $\dfrac{1}{\Gamma(s)}$ with Γ satisfying (1.3). Note that the behaviour of $F(x, s)$ for $s = 0$ can be very different according to the point x.

Note also that the function $F(x, s)$ is defined only for $s \geq 0$, and that in view of (1.2) and (1.3), $F(x, s)$ is finite for almost every $x \in \Omega$ and for every $s > 0$, but that $F(x, s)$ can exhibit a singularity when $s > 0$ tends to 0 and when $h(x) > 0$. Let us emphasize the fact that the Carathéodory character of the function $F(x, s)$, which can take infinite values when $s = 0$, means in particular that for almost every $x \in \Omega$, the function $s \to F(x, s)$ is continuous not only for every $s > 0$ but also for $s = 0$.

Note finally that we do not require $F(x, s)$ to be nonincreasing in s, except when we deal with uniqueness and comparison results in the Uniqueness Theorem 4.3 and in Section 7 below.

The case of a mild singularity

In the present paper we consider the case of strong singularities, namely the case where $F(x, s)$ can have any (wild) behavior as s tends to zero, while in our previous paper [18] we restricted ourselves to the case of mild singularities, namely the case where

$$0 \leq F(x, s) \leq h(x)\left(\frac{1}{s^\gamma} + 1\right) \text{ a.e. } x \in \Omega,\ \forall s > 0,\ \text{with } 0 < \gamma \leq 1. \tag{1.6}$$

When the function F satisfies (1.6), our definition of the solution to problem (1.1) is relatively classical, because it consists in looking for a function u such that

$$\begin{cases} u \in H_0^1(\Omega) \\ u(x) \geq 0 \text{ a.e. } x \in \Omega \\ \displaystyle\int_\Omega F(x, u)\varphi < +\infty\ \forall \varphi \in H_0^1(\Omega),\ \varphi \geq 0 \\ \displaystyle\int_\Omega ADuD\varphi = \int_\Omega F(x, u)\varphi\ \forall \varphi \in H_0^1(\Omega) \end{cases} \tag{1.7 mild}$$

(see [18, Section 3]).

Definition of the solution in the case of a strong singularity

In contrast, the definition of the solution to problem (1.1) that we use in the present paper is less classical. This definition is in our opinion one of the main originalities of the present paper.

We refer the reader to Section 3 below where this definition is given in details; here we emphasize some of its main features.

When the function F satisfies (1.2) and (1.3), we will say (see Definition 3.6 below) that u is a solution to problem (1.1) if

$$\begin{cases} \text{i)} & u \in L^2(\Omega) \cap H^1_{\text{loc}}(\Omega) \\ \text{ii)} & u(x) \geq 0 \ \text{ a.e. } x \in \Omega \\ \text{iii)} & G_k(u) \in H_0^1(\Omega)\ \ \forall k > 0 \\ \text{iv)} & \varphi T_k(u) \in H_0^1(\Omega)\ \ \forall k > 0,\ \forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega), \end{cases} \tag{1.8 strong}$$

where $G_k(s) = (s-k)^+$ and $T_k(s) = \inf(s,k) = s - G_k(s)$ for $s > 0$, and if

$$\begin{cases} \forall v \in H_0^1(\Omega) \cap L^\infty(\Omega),\ v \geq 0, \\ \text{with } -\operatorname{div}\, {}^tA(x)Dv = \displaystyle\sum_{i \in I} \hat{\varphi}_i(-\operatorname{div}\, \hat{g}_i) + \hat{f} \text{ in } \mathcal{D}'(\Omega) \\ \text{where } I \text{ is finite, } \hat{\varphi}_i \in H_0^1(\Omega) \cap L^\infty(\Omega),\ \hat{g}_i \in (L^2(\Omega))^N,\ \hat{f}_i \in L^1(\Omega) \\ \text{one has} \\ \text{i) } \displaystyle\int_\Omega F(x,u)v < +\infty \\ \text{ii) } \displaystyle\int_\Omega {}^tA(x)DvDG_k(u) + \sum_{i\in I}\int_\Omega \hat{g}_i D(\hat{\varphi}_i T_k(u)) + \int_\Omega \hat{f} T_k(u) \\ \quad = \langle -\operatorname{div}\, {}^tA(x)Dv, G_k(u)\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle\langle -\operatorname{div}\, {}^tA(x)Dv, T_k(u)\rangle\rangle_\Omega \\ \quad = \displaystyle\int_\Omega F(x,u)v \ \ \forall k > 0, \end{cases} \tag{1.9 strong}$$

where $\langle\langle\ ,\ \rangle\rangle_\Omega$ is the notation for a "formal duality" introduced in (3.3) below.

Note that in this definition every term of ($1.9_{\text{ strong}}$ ii) makes sense in view of ($1.8_{\text{ strong}}$ iii) and ($1.8_{\text{ strong}}$ iv).

Definition ($1.8_{\text{ strong}}$), ($1.9_{\text{ strong}}$) strongly differs from Definition ($1.7_{\text{ mild}}$). Indeed, for instance the assertion $u \in H_0^1(\Omega)$ of ($1.7_{\text{ mild}}$) is replaced in ($1.8_{\text{ strong}}$) by $u \in H^1_{\text{loc}}(\Omega)$, $G_k(u) \in H_0^1(\Omega)$ for every $k > 0$ and $\varphi T_k(u) \in H_0^1(\Omega)$ for every $k > 0$ and for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Here the assertion $G_k(u) \in H_0^1(\Omega)$ for every $k > 0$ in particular expresses the boundary condition $u = 0$ on $\partial\Omega$ (see Remark 3.7 ii)) below.

Actually, the solution u does not in general belong to $H_0^1(\Omega)$, or in other terms u "does not belong to $H^1(\Omega)$ up to the boundary", when the function F exhibits a strong singularity at $s = 0$. It is indeed proved by A. C. Lazer and P. J. McKenna in [23, Theorem 2] that when $f \in C^\alpha(\overline{\Omega})$, $\alpha > 0$, with $f(x) \geq f_0 > 0$ in Ω, in a domain Ω with $C^{2+\alpha}$ boundary, the solution u to the equation

$$\begin{cases} -\Delta u = \dfrac{f(x)}{u^\gamma} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

does not belong to $H^1(\Omega)$ when $\gamma > 3$, even if u belongs to $C^{2+\alpha}(\Omega) \cap C^0(\overline{\Omega})$ in this case.

The "space" defined by ($1.8_{\text{ strong}}$) in which we look for a solution to (1.1) in the case of a strong singularity is therefore fairly different from the space $H_0^1(\Omega)$ in which we look for a solution to (1.1) in the case of a mild singularity. In particular this "space" is not a vectorial space.

Another important difference between the two definitions appears as far as the partial differential equation in (1.1) is concerned. Indeed, if the formulation in the last line of ($1.7_{\text{ mild}}$) is classical, with the use of test functions in $H_0^1(\Omega)$, the equation in (1.1) as formulated in (1.9 $_{\text{strong}}$) involves test functions which belong to the vectorial space described in the three first lines of (1.9 $_{\text{strong}}$), that we will denote by $\mathcal{V}(\Omega)$ in the rest of the present paper (see Definition 3.1 below). Actually this space $\mathcal{V}(\Omega)$ consists in functions v such that $-\operatorname{div} {}^tA(x)Dv$ can be put in the usual duality between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ with $G_k(u)$, and in a "formal duality" (by means of the notation $\langle\langle\ ,\ \rangle\rangle_\Omega$ introduced in Definition 3.2 below) with $T_k(u)$: indeed, when u satisfies (1.8 $_{\text{strong}}$), one writes u as the sum $u = G_k(u) + T_k(u)$, where $G_k(u) \in H_0^1(\Omega)$ and where $\varphi T_k(u) \in H_0^1(\Omega)$ for every $k > 0$ and for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$; on the other hand $-\operatorname{div} {}^tA(x)Dv$, which of course belongs to $H^{-1}(\Omega)$ when $v \in H_0^1(\Omega)$, will be assumed to be the sum of a function $\hat{f} \in L^1(\Omega)$ and of a finite sum of functions $\hat{\varphi}_i(-\operatorname{div} \hat{g}_i)$, with $\hat{\varphi}_i \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and $-\operatorname{div} \hat{g}_i \in H^{-1}(\Omega)$, a fact which allows one to correctly define the "formal duality" $\langle\langle -\operatorname{div} {}^tA(x)Dv, T_k(u)\rangle\rangle_\Omega$ (see (3.3) in Definition 3.2 below).

This is a definition of the solution by transposition in the spirit of those introduced by J.-L. Lions and E. Magenes and by G. Stampacchia. Here again things are fairly different with respect to the case of a mild singularity.

Main results of the present paper

In the framework of Definition (1.8 $_{\text{strong}}$), (1.9 $_{\text{strong}}$) (*i.e.* of Definition 3.6 below) we are able to prove that there exists at least a solution to problem (1.1) (see the Existence Theorem 4.1 below). We also prove that this solution is stable with respect to variations of the right-hand side (see the Stability Theorem 4.2 below). Finally, if besides (1.2) and (1.3) we assume that the function $F(x, s)$ is nonincreasing with respect to s, then this solution is unique (see the Uniqueness Theorem 4.3 below).

In brief Definition 3.6 below provides a framework where problem (1.1) is well posed in the sense of Hadamard.

Moreover every solution u to problem (1.1) defined in this sense satisfies the following *a priori* estimates:

- An *a priori* estimate of $G_k(u)$ in $H_0^1(\Omega)$ for every $k > 0$ (see Proposition 5.1 below), which is formally obtained by using $G_k(u)$ as test function in (1.1); this implies an *a priori* estimate of u in $L^2(\Omega)$ (see Remark 5.2 below).
- An *a priori* estimate of $\varphi DT_k(u)$ in $(L^2(\Omega))^N$ for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ (see Proposition 5.4 below), which is formally obtained by using $\varphi^2 T_k(u)$ and φ^2 as test functions in (1.1); this implies an *a priori* estimate of u in $H^1_{\text{loc}}(\Omega)$ (see Remark 5.7 below) and an a priori estimate of $\varphi T_k(u)$ in $H_0^1(\Omega)$ for every $k > 0$ and for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ (see Remark 5.5 below).
- An *a priori* estimate of the integral $\int_{\{u\le\delta\}} F(x, u)v$ for every $v \in \mathcal{V}(\Omega)$, $v \ge 0$, and every $\delta > 0$, which depends only on δ and on v (and also on u through the products $\hat{\varphi}_i Du$) by a constant which tends to zero when δ tends to zero (see Proposition 5.9 and Remark 5.10 below).

- An *a priori* estimate of $\beta(u)$ in $H_0^1(\Omega)$ (where the function β is defined from the function Γ which appears in (1.2) by $\beta(s) = \int_0^s \sqrt{\Gamma'(t)}dt$) (see Proposition 5.13 below), which is formally obtained by using $\Gamma(u)$ as test function in (1.1).

The (mathematically rigorous) proofs of all the above *a priori* estimates are based on Definition (1.8 $_{\text{strong}}$), (1.9 $_{\text{strong}}$) and on the use of convenient test functions v in (1.9 $_{\text{strong}}$).

Literature

There is a wide literature dealing with problem (1.1). We will not give here a complete list of references and we will concentrate on some papers which seem to us to be the most significant: we also refer the interested reader to the references quoted there.

The problem (1.1) was initially proposed in 1960 in the pioneering work [17] of W. Fulks and J. S. Maybe as a model for several physical situations. The problem was then studied by many authors, among which we quote the papers [1–25].

In most of the those papers the authors look for a strong solution and use sub- and super-solutions. In particular in [23] A. C. Lazer and P. J. McKenna work in $C^{2,\alpha}(\Omega)$ and $W^{2,q}(\Omega)$ and use methods of sub- and super-solutions, proving that when $F(x,s) = f(x)/s^\gamma$ with $\gamma > 0$, $f \in C^\alpha(\overline{\Omega})$ and $f(x) \geq f_0 > 0$ in a $C^{2,\alpha}$ domain Ω, then one has $c_1\phi_1(x) \leq u(x)^{\frac{\gamma+1}{2}} \leq c_2\phi_1(x)$ for two constants $0 < c_1 < c_2$, where ϕ_1 is the first (positive) eigenfunction of $-\operatorname{div} A(x)D$ in $H_0^1(\Omega)$; M. G. Crandall, P. H. Rabinowitz and L. Tartar study in [12] the behaviour of $u(x)$ and $|Du(x)|$ at the boundary. Let us finally note that C. Stuart in [25], as well as M. G. Crandall, P. H. Rabinowitz and L. Tartar in [12], do not assume that $F(x,s)$ is nonincreasing in s.

More recently L. Boccardo and L. Orsina studied in [4] the problem in the framework of weak solutions in the sense of distributions. In that paper, the authors address the problem (1.1) with $F(x,s) = f(x)/s^\gamma$, where $\gamma > 0$ and where $f \geq 0$ belongs to $L^1(\Omega)$, or to other Lebesgue spaces, or to the space of Radon measures, and they prove existence and regularity as well as non existence results. In this work the strong maximum principle and the nonincreasing character of the function $F(x,s) = f(x)/s^\gamma$ with respect to s play prominent roles. The solution u to problem (1.1) is indeed required to satisfy $u(x) \geq c(\omega, u) > 0$ in every open set ω with $\overline{\omega} \subset \Omega$, and the framework in which the solution is looked for is the Sobolev space $H_0^1(\Omega)$ or $H^1_{\text{loc}}(\Omega)$. Then L. Boccardo and J. Casado-Díaz proved in [3] the uniqueness of the solutions obtained by approximation and the stability of the solution with respect to the G-convergence of a sequence of matrices $A^\varepsilon(x)$ which are equicoercive and equibounded. The two latest papers attracted our attention on this type of semilinear singular problems.

For a variational approach to the problem and extensions to the case of a nonlinear principal part, see A. Canino and M. Degiovanni [5], L. M. De Cave [14], A. Canino, B. Sciunzi and A. Trombetta [6], and J. Casado-Díaz and F. Murat [7].

In [24], F. Oliva and F. Petitta consider the case where $F(x,s) = f(x)g(s)$ where f is a nonnegative measure and where g is a continuous function which is singular at $u = 0$; they prove in particular the existence of a solution $u \in L^1(\Omega) \cap W^{1,1}_{\text{loc}}(\Omega)$ which satisfies the equation in the sense of distributions, and, when g is nonincreasing, the uniqueness of such a solution by using convenient solutions to the adjoint problem.

Contributions of the present paper

In the present paper (as well as in [18], where the case of mild singularities is treated), we obtain in particular an *a priori* estimate of the singular term in the region where the solution is close to zero. This estimate, which is new, is an essential tool in our proof. Moreover, in the case of strong singularities, the main difficulty, with respect to the case of mild singularities, is that only local estimates in the energy space can be proved for the solutions (see [23]). In order to solve this difficulty, we prove that the solution u satisfies $G_k(u) \in H^1_0(\Omega)$ for every $k > 0$ and $\varphi Du \in (L^2(\Omega))^N$ for every $\varphi \in H^1_0(\Omega) \cap L^\infty(\Omega)$, two properties of the solution which had not been noticed before. We also introduce a convenient class of test functions, namely the class $\mathcal{V}(\Omega)$ defined in Subsection 3.1 below. The class $\mathcal{V}(\Omega)$ and the method of proof that we use seem to be rather flexible and can be adapted to other situations where other techniques would fail. This is in particular the case for the existence of solutions when the equation involves a zeroth-order term which prevents the use of the strong maximum principle (see [21, Section 5]), as well as for the homogenization of strongly singular problems in perforated domains Ω^ε obtained from Ω by removing many small holes, when the Dirichlet boundary condition on $\partial\Omega^\varepsilon$ leads to the appearance of a "strange term" μu in the homogenized equation (see [20]).

One of the strong points of the present paper is Definition 3.6. This definition makes problem (1.1) well posed in the sense of Hadamard when the function $F(x, s)$ is nonincreasing in s, and allows us to perform in a mathematically correct way all the formal computations that we want to make on problem (1.1), even when the assumption that the function $F(x, s)$ is nonincreasing in s is not made. Two other strong points are the fact that we do not assume that the function $F(x, s)$ is nonincreasing with respect to s, except as far as uniqueness is concerned, and that we do not use the strong maximum principle (see [21, Section 5] about the latest point).

A weak point of the present paper is however the fact that Definition 3.6 is a definition by transposition in the spirit of J.-L. Lions and E. Magenes and of G. Stampacchia, a feature which makes difficult (if not impossible, except maybe when $p = 2$) to extend it to the case of general nonlinear monotone operators.

Note finally that an important (and maybe insufficiently underlined) assumption made in the present paper is the fact that the function $F(x, s)$ is assumed to be nonnegative. (Note also that to the best of our kowledge this assumption is actually made in all the literature on this problem.) By the weak maximum principle, this assumption implies that the solution u is nonnegative, a fact which plays an essential role in the present paper as well as in all the literature. Progresses in the case where the function $F(x, s)$ can change sign have been very recently made by J. Casado-Díaz and F. Murat in [7].

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2. Assumptions and notation

As already said in the Introduction, we study in this paper the following singular semilinear problem

$$\begin{cases} u \geq 0 & \text{in } \Omega \\ -\operatorname{div} A(x)Du = F(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where models of the function $F(x, s)$ are given by (1.4) and by (1.5); another more general model is given by

$$\begin{cases} F(x,s) = f(x)\dfrac{(a+\sin(S(s)))}{\exp(-S(s))} + g(x)\dfrac{\left(b+\sin\left(\frac{1}{s}\right)\right)}{s^\gamma} + l(x) \\ \text{a.e. } x \in \Omega, \ \forall s > 0, \end{cases} \tag{2.2}$$

where $\gamma > 0, a > 1, b > 1$, where the function S satisfies

$$S \in C^1(]0,+\infty[), \quad S'(s) < 0 \ \forall s > 0, \quad S(s) \to +\infty \text{ as } s \to 0 \tag{2.3}$$

and where the functions f, g and l are nonnegative and belong to $L^r(\Omega)$ with r defined in (2.6 i) (see Remark 2.1 viii) below).

In this section, we give the precise assumptions that we make on the data of problem (2.1).

We assume that Ω is an open bounded set of $\mathbb{R}^N$, $N \geq 1$ (no regularity is assumed on the boundary $\partial\Omega$ of Ω), that the matrix A is bounded and coercive, *i.e.* satisfies

$$A(x) \in (L^\infty(\Omega))^{N\times N}, \ \exists\alpha > 0, \ A(x) \geq \alpha I \quad \text{a.e. } x \in \Omega \tag{2.4}$$

and that the function $F : (x, s) \in \Omega \times [0, +\infty[\to F(x, s) \in [0, +\infty]$ is a Carathéodory function, *i.e.* F satisfies

$$\begin{cases} \text{i) } \forall s \in [0, +\infty[, \ x \in \Omega \to F(x, s) \in [0, +\infty] \text{ is measurable} \\ \text{ii) for a.e. } x \in \Omega, \ s \in [0, +\infty[\to F(x, s) \in [0, +\infty] \text{ is continuous} \end{cases} \tag{2.5}$$

as well as

$$\begin{cases} \text{i) } \ \exists h, h(x) \geq 0 \text{ a.e. } x \in \Omega, h \in L^r(\Omega) \\ \quad \text{with } r = \dfrac{2N}{N+2} \text{ if } N \geq 3, \ r > 1 \text{ if } N = 2, \ r = 1 \text{ if } N = 1 \\ \text{ii) } \ \exists \Gamma : s \in [0, +\infty[\to \Gamma(s) \in [0, +\infty[\\ \quad \Gamma \in C^1([0, +\infty[), \Gamma(0) = 0, \Gamma'(s) > 0 \ \forall s > 0 \\ \text{iii) } 0 \leq F(x, s) \leq \dfrac{h(x)}{\Gamma(s)} \text{ a.e. } x \in \Omega, \forall s > 0. \end{cases} \tag{2.6}$$

Moreover, when we will prove comparison and uniqueness results (Proposition 7.1 and Theorem 4.3), we will assume that $F(x, s)$ is nonincreasing in s, *i.e.* that

$$F(x, s) \leq F(x, t) \ \text{ a.e. } \ x \in \Omega, \ \forall s, \forall t, \ 0 \leq t \leq s. \tag{2.7}$$

Remark 2.1 (About assumptions (2.5) and (2.6)).

i) If a function $\Gamma(s)$ satisfies (2.6 ii), then Γ is (strictly) increasing and satisfies $\Gamma(s) > 0$ for every $s > 0$; note that the function Γ can be either bounded or unbounded.
Observe also that if a function $F(x, s)$ satisfies (2.6) for $h(x)$ and $\Gamma(s)$, and if $\overline{\Gamma}(s)$ is a function which satisfies (2.6 ii) and $\overline{\Gamma}(s) \leq \Gamma(s)$, then $F(x, s)$ satisfies (2.6) for $h(x)$ and $\overline{\Gamma}(s)$.

ii) The function $F(x, s)$ is a nonnegative Carathéodory function with values in $[0, +\infty]$ and not only in $[0, +\infty[$. But, in view of condition (2.6 iii), for almost every $x \in \Omega$, the function $F(x, s)$ can take the value $+\infty$ only when $s = 0$ (or, in other terms, $F(x, s)$ is finite for almost every $x \in \Omega$ when $s > 0$).

iii) Note that (2.6 ii) and (2.6 iii) do not prescribe any restriction on the type of the growth of the function $F(x, s)$ as s tends to zero. Moreover it can be proved (see [19, Section 3, Proposition 1]) that (2.6) is actually equivalent to the (apparently weaker) assumption

$$\begin{cases} \forall k > 0, \ \exists h_k, \ h_k(x) \geq 0 \text{ a.e. } x \in \Omega \\ h_k \in L^r(\Omega) \text{ with } r \text{ as in (2.6 i) such that} \\ 0 \leq F(x, s) \leq h_k(x) \text{ a.e. } x \in \Omega, \ \forall s \geq k. \end{cases} \tag{2.8}$$

iv) Note that no growth condition is imposed from below on $F(x,s)$ as s tends to zero. Indeed it can easily be proved (see also [19, Section 3, Remark 2]) that for two given functions $\Gamma_1(s)$ and $\Gamma_2(s)$ which satisfy (2.6 ii) and $\Gamma_1(s) \leq \Gamma_2(s)$, there exist two sequences s_n^1 and s_n^2 which tend to zero such that $1 > \cdots > s_n^2 > s_n^1 > s_{n+1}^2 > s_{n+1}^1 > \cdots > 0$ and a function $F(s)$ which satisfies (2.6) with $h = 1$ such that $F(s_n^1) = 1/\Gamma_1(s_n^1)$, $F(s_n^2) = 1/\Gamma_2(s_n^2)$ and $F(s) \leq 1/\Gamma_1(s)$. This function $F(s)$ therefore "oscillates" between $1/\Gamma_1(s)$ and $1/\Gamma_2(s)$.
Of course the growth of $F(x,s)$ as s tends to zero can strongly depend on the point $x \in \Omega$.

v) Note that the growth condition (2.6 iii) is stated for every $s > 0$, while in (2.5) F is assumed to be a Carathéodory function for $s \geq 0$ and not only for $s > 0$. This is due to the fact that an indeterminacy $0/0$ appears in $h(x)/\Gamma(s)$ when $h(x) = 0$ and $s = 0$, while the Carathéodory and growth assumptions (2.5 ii) and (2.6 iii) imply that

$$F(x,s) = 0 \quad \forall s \geq 0 \quad \text{a.e. in } \{x \in \Omega : h(x) = 0\}.$$

In contrast, when h is assumed to satisfy $h(x) > 0$ for almost every $x \in \Omega$, it is equivalent to assume (2.6 iii) for every $s > 0$ or for every $s \geq 0$.

vi) Let us observe that the functions $F(x,s)$ given in examples (1.4), (1.5) and (2.2) satisfy assumption (2.6); indeed for these examples one has

$$0 \leq F(x,s) \leq \overline{h}(x)\left(\frac{1}{\overline{\Gamma}(s)} + 1\right)$$

for some $\overline{h}(x)$ and $\overline{\Gamma}(s)$ which satisfy (2.6 i) and (2.6 ii); defining the function Γ by $\Gamma(s) = \overline{\Gamma}(s)/(1+\overline{\Gamma}(s))$ it is clear that $\Gamma(s)$ satisfies (2.6 ii) and that $F(x,s)$ satisfies (2.6).

vii) The C^1 regularity of the function Γ which is assumed in (2.6 ii) is used to define without any technical difficulty the function β which appears in Proposition 5.13 below (see (5.56)). The (regularity) result stated in this proposition, namely $\beta(u) \in H_0^1(\Omega)$, is in turn strongly used in the proofs of the Comparison Principle of Proposition 7.1 and of the Uniqueness Theorem 4.3.
This C^1 regularity could appear as a strong restriction on the function Γ, but it can actually be proved (see [19, Section 3, proof of Proposition 1]) that, given a function $\overline{\Gamma}$ such that for some $\delta > 0$ and some $M > 0$

$$\overline{\Gamma} \in C^0([0,+\infty[),\ \overline{\Gamma}(0) = 0,\ \ \overline{\Gamma}(s) > 0\ \ \forall s > 0,\ \overline{\Gamma}(s) \geq \delta,\ \ \forall s \geq M,$$

one can construct a function Γ which satisfies (2.6 ii) as well as $\Gamma(s) \leq \overline{\Gamma}(s)$ for every $s > 0$, and then use Remark 2.1 i) above.
In the same spirit, one could also observe that in order to obtain all the results of the present paper, it would be sufficient to assume in (2.6 ii) that the function Γ is Lipschitz continuous on $[0, M]$ for every $M > 0$ (in place of belonging to $C^1([0,+\infty[)$); the price to pay for that would be a technical difficulty arising in the composition of an $H^1_{\text{loc}}(\Omega)$ function by a Lipschitz continuous function.

viii) As far as example (2.2) is concerned, note that when $a > 1$ and $b > 1$ and when the function S satisfies (2.3), the functions $\dfrac{(a + \sin(S(s)))}{\exp(-S(s))}$ and $\dfrac{\left(b + \sin(\frac{1}{s})\right)}{s^\gamma}$ are (strictly) positive and are continuous in $[0, +\infty[$, and in particular in $s = 0$ (this is no more the case when $a = 1$ or $b = 1$), and therefore satisfy (2.5). Note also that these functions are easily shown to be oscillatory (in the sense that they are not nondecreasing) when $1 < a < \sqrt{2}$ and $b > 1$. Note finally that every function Γ which satisfies (2.6 ii) can be written as $\Gamma(s) = \exp(-S(s))$ for a function S which satisfies (2.3).

Remark 2.2 (Sobolev's embedding). The function h which appears in hypothesis (2.6 i) is an element of $H^{-1}(\Omega)$. Indeed, when $N \geq 3$, the exponent $r = \frac{2N}{N+2}$ is nothing but the Hölder's conjugate $(2^*)'$ of the Sobolev's exponent 2^*, *i.e.*

$$\text{when } N \geq 3, \quad \frac{1}{r} = 1 - \frac{1}{2^*}, \text{ where } \frac{1}{2^*} = \frac{1}{2} - \frac{1}{N}. \tag{2.9}$$

Making an abuse of notation, we will set

$$\begin{cases} 2^* = \text{any } p \text{ with } 1 < p < +\infty \text{ when } N = 2 \\ 2^* = +\infty \text{ when } N = 1. \end{cases} \tag{2.10}$$

With this abuse of notation, h belongs to $L^r(\Omega) = L^{(2^*)'}(\Omega) \subset H^{-1}(\Omega)$ for all $N \geq 1$ since Ω is bounded.

This result is indeed a consequence of Sobolev, Trudinger, Moser and Morrey's inequalities, which (with the above abuse of notation) assert that

$$\|v\|_{L^{2^*}(\Omega)} \leq C_S \|Dv\|_{(L^2(\Omega))^N} \quad \forall v \in H_0^1(\Omega) \text{ when } N \geq 1, \tag{2.11}$$

where C_S is a constant which depends only on N when $N \geq 3$, which depends only on p and Q when $N = 2$, and which depends only on Q when $N = 1$, where Q is a fixed bounded open set such that $\Omega \subset Q$.

Remark 2.3 (About assumption (2.7)). Assumption (2.7), namely the fact that for almost every $x \in \Omega$ the function $s \to F(x, s)$ is nonincreasing in s, is not of the same nature as assumptions (2.5) and (2.6) on $F(x, s)$.

We will only use assumption (2.7) when proving comparison and uniqueness results, namely Proposition 7.1 and Theorem 4.3. In contrast, all the others results of the present paper, and in particular the existence and stability results stated in Theorems 4.1 and 4.2, as well as the *a priori* estimates of Section 5, do not use this assumption.

2.1. Notation

We denote by $\mathcal{D}(\Omega)$ the space of the $C^\infty(\Omega)$ functions whose support is compact and included in Ω, and by $\mathcal{D}'(\Omega)$ the space of distributions on Ω.

We denote by $\mathcal{M}_b^+(\Omega)$ the space of nonnegative bounded Radon measures on Ω.

Since Ω is bounded, $\|Dw\|_{L^2(\Omega)^N}$ is a norm equivalent to $\|w\|_{H^1(\Omega)}$ on $H_0^1(\Omega)$. We set

$$\|w\|_{H_0^1(\Omega)} = \|Dw\|_{(L^2(\Omega))^N} \quad \forall w \in H_0^1(\Omega).$$

For every $s \in]-\infty, +\infty[$ and every $k > 0$ we define as usual

$$s^+ = \max\{s, 0\}, \quad s^- = \max\{0, -s\}$$
$$T_k(s) = \max\{-k, \min\{s, k\}\}, \quad G_k(s) = s - T_k(s).$$

For every measurable function $l : x \in \Omega \to l(x) \in [0, +\infty]$ we denote

$$\{l = 0\} = \{x \in \Omega : l(x) = 0\}, \quad \{l > 0\} = \{x \in \Omega : l(x) > 0\}.$$

Finally we denote by φ and $\overline{\varphi}$ functions which belong to $H_0^1(\Omega) \cap L^\infty(\Omega)$, while we denote by ϕ and $\overline{\phi}$ functions which belong to $\mathcal{D}(\Omega)$.

3. Definition of a solution to problem (2.1)

3.1. The space $\mathcal{V}(\Omega)$ of test functions

In order to introduce the notion of solution to problem (2.1) that we will use in the present paper, we define in this subsection the following space $\mathcal{V}(\Omega)$ of test functions and a notation.

Definition 3.1 (Definition of the space $\mathcal{V}(\Omega)$). The space $\mathcal{V}(\Omega)$ is the space of the functions v which satisfy

$$v \in H_0^1(\Omega) \cap L^\infty(\Omega) \tag{3.1}$$

and which are such that there exist

$$\begin{cases} \text{a finite set } I, \\ \text{for } i \in I, \hat{\varphi}_i \in H_0^1(\Omega) \cap L^\infty(\Omega), \hat{g}_i \in \left(L^2(\Omega)\right)^N \\ \hat{f} \in L^1(\Omega), \\ \text{such that } -\text{div}\, {}^tA(x)Dv = \sum_{i \in I} \hat{\varphi}_i\left(-\text{div}\, \hat{g}_i\right) + \hat{f} \text{ in } \mathcal{D}'(\Omega). \end{cases} \tag{3.2}$$

In the definition of $\mathcal{V}(\Omega)$ we use the notation $\hat{\varphi}_i$, $\hat{g}_i$, and $\hat{f}$ to help the reader to identify the functions which enter in the definition of the functions of $\mathcal{V}(\Omega)$. Note that $\mathcal{V}(\Omega)$ is a vector space.

Definition 3.2 (Notation $\langle\langle\ ,\ \rangle\rangle_\Omega$). When $v \in \mathcal{V}(\Omega)$ with

$$-\text{div}\, {}^tA(x)Dv = \sum_{i \in I} \hat{\varphi}_i(-\text{div}\, \hat{g}_i) + \hat{f} \quad \text{in } \mathcal{D}'(\Omega),$$

where I, $\hat{\varphi}_i$, $\hat{g}_i$ and $\hat{f}$ are as in (3.2), and when y satisfies

$$y \in H^1_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega) \ \text{ with } \ \varphi y \in H^1_0(\Omega) \quad \forall \varphi \in H^1_0(\Omega) \cap L^\infty(\Omega)$$

we use the following notation:

$$\langle\langle -\operatorname{div}\, {}^tA(x)Dv, y\rangle\rangle_\Omega = \sum_{i\in I} \int_\Omega \hat{g}_i D(\hat{\varphi}_i y) + \int_\Omega \hat{f} y. \tag{3.3}$$

Remark 3.3. In (3.2), the product $\hat{\varphi}_i(-\operatorname{div}\, \hat{g}_i)$ with $\hat{\varphi}_i \in H^1_0(\Omega) \cap L^\infty(\Omega)$ and $\hat{g}_i \in (L^2(\Omega))^N$ is, as usual, the distribution on Ω defined as

$$\begin{cases} \forall \phi \in \mathcal{D}(\Omega), \\ \langle \hat{\varphi}_i(-\operatorname{div}\, \hat{g}_i), \phi\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = \langle -\operatorname{div}\, \hat{g}_i,\ \hat{\varphi}_i\phi\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} = \displaystyle\int_\Omega \hat{g}_i D(\hat{\varphi}_i\phi) \end{cases} \tag{3.4}$$

and the equality $-\operatorname{div}\, {}^tA(x)Dv = \sum_{i\in I} \hat{\varphi}_i(-\operatorname{div}\, \hat{g}_i) + \hat{f}$ holds true in $\mathcal{D}'(\Omega)$.

In notation (3.3), the right-hand side is correctly defined since $\hat{\varphi}_i y \in H^1_0(\Omega)$ and since $y \in L^\infty(\Omega)$. In contrast the left-hand side $\langle\langle -\operatorname{div}\, {}^tA(x)\, Dv, y\rangle\rangle_\Omega$ is just a notation.

Remark 3.4. If $y \in H^1_0(\Omega) \cap L^\infty(\Omega)$, then $\varphi y \in H^1_0(\Omega)$ for every $\varphi \in H^1_0(\Omega) \cap L^\infty(\Omega)$, so that for every $v \in \mathcal{V}(\Omega)$, $\langle\langle -\operatorname{div}\, {}^tA(x)\, Dv, y\rangle\rangle_\Omega$ is defined. Let us prove that actually one has

$$\begin{cases} \forall v \in \mathcal{V}(\Omega),\ \forall y \in H^1_0(\Omega) \cap L^\infty(\Omega), \\ \langle\langle -\operatorname{div}\, {}^tA(x)Dv, y\rangle\rangle_\Omega = \langle -\operatorname{div}\, {}^tA(x)Dv, y\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}. \end{cases} \tag{3.5}$$

Indeed when $-\operatorname{div}\, {}^tA(x)Dv = \sum_{i\in I} \hat{\varphi}_i(-\operatorname{div}\, \hat{g}_i) + \hat{f}$, one has for every $\phi \in \mathcal{D}(\Omega)$ (see (3.3) and (3.4)) that

$$\begin{cases} \langle\langle -\operatorname{div}\, {}^tA(x)Dv, \phi\rangle\rangle_\Omega = \displaystyle\sum_{i\in I} \int_\Omega \hat{g}_i D(\hat{\varphi}_i\phi) + \int_\Omega \hat{f}\phi \\ \qquad = \Big\langle \displaystyle\sum_{i\in I} \hat{\varphi}_i(-\operatorname{div}\, \hat{g}_i) + \hat{f}, \phi\Big\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} \\ \qquad = \langle -\operatorname{div}\, {}^tA(x)Dv, \phi\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)} = \displaystyle\int_\Omega {}^tA(x)DvD\phi \end{cases}$$

and therefore

$$\sum_{i\in I} \int_\Omega \hat{g}_i D(\hat{\varphi}_i\phi) + \int_\Omega \hat{f}\phi = \int_\Omega {}^tA\, DvD\phi \quad \forall \phi \in \mathcal{D}(\Omega). \tag{3.6}$$

For every given function $y \in H_0^1(\Omega) \cap L^\infty(\Omega)$, taking a sequence $\phi_n \in \mathcal{D}(\Omega)$ such that

$$\phi_n \to y \text{ in } H_0^1(\Omega) \text{ strongly}, \;\; \phi_n \rightharpoonup y \text{ in } L^\infty(\Omega) \text{ weakly-star}$$

and passing to the limit in (3.6) with $\phi = \phi_n$, we obtain (3.5).

Remark 3.5 (Examples of functions which belong to $\mathcal{V}(\Omega)$).

i) If $\varphi_1, \varphi_2 \in H_0^1(\Omega) \cap L^\infty(\Omega)$, then $\varphi_1\varphi_2 \in \mathcal{V}(\Omega)$. Indeed $\varphi_1\varphi_2 \in H_0^1(\Omega) \cap L^\infty(\Omega)$, and in the sense of distributions, one has

$$\begin{cases} -\operatorname{div}\, {}^tA(x)D(\varphi_1\varphi_2) = \hat{\varphi}_1(-\operatorname{div}\, \hat{g}_1) + \hat{\varphi}_2(-\operatorname{div}\, \hat{g}_2) + \hat{f} \text{ in } \mathcal{D}'(\Omega) \\ \text{with} \\ \hat{\varphi}_1 = \varphi_1 \in H_0^1(\Omega) \cap L^\infty(\Omega), \;\; \hat{\varphi}_2 = \varphi_2 \in H_0^1(\Omega) \cap L^\infty(\Omega) \\ \hat{g}_1 = {}^tA(x)D\varphi_2 \in (L^2(\Omega))^N, \;\; \hat{g}_2 = {}^tA(x)D\varphi_1 \in (L^2(\Omega))^N \\ \hat{f} = -{}^tA(x)D\varphi_2 D\varphi_1 - {}^tA(x)D\varphi_1 D\varphi_2 \in L^1(\Omega). \end{cases} \tag{3.7}$$

ii) In particular, if $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, then $\varphi^2 \in \mathcal{V}(\Omega)$, with

$$\begin{cases} -\operatorname{div}\, {}^tA(x)D\varphi^2 = \hat{\varphi}(-\operatorname{div}\, \hat{g}) + \hat{f} \text{ in } \mathcal{D}'(\Omega) \\ \text{where} \\ \hat{\varphi} = 2\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega), \;\; \hat{g} = {}^tA(x)D\varphi \in (L^2(\Omega))^N \\ \hat{f} = -2\,{}^tA(x)D\varphi D\varphi \in L^1(\Omega). \end{cases} \tag{3.8}$$

iii) If $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and has a compact support which is included in Ω, then $\varphi \in \mathcal{V}(\Omega)$ since

$$\begin{cases} -\operatorname{div}\, {}^tA(x)D\varphi = \overline{\phi}(-\operatorname{div}\, {}^tA(x)D\varphi) \;\text{ in } \mathcal{D}'(\Omega) \\ \text{for every } \overline{\phi} \in \mathcal{D}(\Omega), \;\text{ with } \overline{\phi} = 1 \text{ in the support of } \varphi. \end{cases} \tag{3.9}$$

iv) In particular every $\phi \in \mathcal{D}(\Omega)$ belongs to $\mathcal{V}(\Omega)$.

3.2. Definition of a solution to problem (2.1)

We now give the definition of a solution to problem (2.1) that we will use in the present paper.

Definition 3.6 (Definition of a solution to problem (2.1)). Assume that the matrix A and the function F satisfy (2.4), (2.5) and (2.6). We say that u is a solution to problem (2.1) if u satisfies

$$\begin{cases} \text{i)} & u \in L^2(\Omega) \cap H^1_{\text{loc}}(\Omega) \\ \text{ii)} & u(x) \geq 0 \ \text{ a.e. } x \in \Omega \\ \text{iii)} & G_k(u) \in H^1_0(\Omega) \ \ \forall k > 0 \\ \text{iv)} & \varphi T_k(u) \in H^1_0(\Omega) \ \ \forall k > 0, \ \forall \varphi \in H^1_0(\Omega) \cap L^\infty(\Omega) \end{cases} \tag{3.10}$$

$$\begin{cases} \forall v \in \mathcal{V}(\Omega), \ v \geq 0 \\ \text{with } -\text{div}\, {}^tA(x)Dv = \sum_{i \in I} \hat{\varphi}_i(-\text{div}\, \hat{g}_i) + \hat{f} \text{ in } \mathcal{D}'(\Omega) \\ \text{where } \hat{\varphi}_i \in H^1_0(\Omega) \cap L^\infty(\Omega), \hat{g}_i \in (L^2(\Omega))^N, \hat{f} \in L^1(\Omega) \\ \text{one has} \\ \text{i)} \ \displaystyle\int_\Omega F(x,u)v < +\infty \\ \text{ii)} \ \displaystyle\int_\Omega {}^tA(x)DvDG_k(u) + \sum_{i \in I} \int_\Omega \hat{g}_i D(\hat{\varphi}_i T_k(u)) + \int_\Omega \hat{f} T_k(u) \\ \quad = \langle -\text{div}\, {}^tA(x)Dv, G_k(u) \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} + \langle\langle -\text{div}\, {}^tA(x)Dv, T_k(u) \rangle\rangle_\Omega \\ \quad = \displaystyle\int_\Omega F(x,u)v \ \forall k > 0. \end{cases} \tag{3.11}$$

Remark 3.7 (About Definition 3.6).

i) Definition 3.6 is a mathematically correct framework which gives a meaning to the solution to problem (2.1); in contrast (2.1) is only formal.
In Definition 3.6, the requirement (3.10) is the "space" (which is not a vectorial space) to which the solution should belong, while the requirement (3.11), and especially (3.11 ii), expresses the partial differential equation in (2.1) in terms of (non standard) test functions in the spirit of the solutions defined by transposition introduced by J.-L. Lions and E. Magenes and by G. Stampacchia.

ii) Note that the statement (3.10 iii) formally contains the boundary condition "$u = 0$ on $\partial\Omega$". Indeed $G_k(u) \in H^1_0(\Omega)$ for every $k > 0$ formally implies that "$G_k(u) = 0$ on $\partial\Omega$", *i.e.* "$u \leq k$ on $\partial\Omega$" for every $k > 0$, which implies "$u = 0$ on $\partial\Omega$" since $u \geq 0$ in Ω by (3.10 ii) (see also the comment after Proposition 5.13 below).

iii) In Section 5 below we will obtain *a priori* estimates (and in particular *a priori* estimates in the various "spaces" which appear in (3.10)) which hold true for every solution u to problem (2.1) in the sense of Definition 3.6.

iv) Note finally that (very) formally, one has

$$\begin{cases} \text{“}\langle -\operatorname{div} {}^tA(x)Dv, G_k(u)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} &= \displaystyle\int_\Omega (-\operatorname{div} {}^tA(x)Dv)\, G_k(u) \\ &= \displaystyle\int_\Omega v\,(-\operatorname{div} A(x)DG_k(u))\text{”} \end{cases}$$

$$\begin{cases} \text{“}\langle\langle -\operatorname{div} {}^tA(x)Dv, T_k(u)\rangle\rangle_\Omega &= \displaystyle\int_\Omega (-\operatorname{div} {}^tA(x)Dv)\, T_k(u) \\ &= \displaystyle\int_\Omega v\,(-\operatorname{div} A(x)DT_k(u))\text{”} \end{cases}$$

so that (3.11 ii) formally means

$$\text{“}\int_\Omega v\,(-\operatorname{div} A(x)Du) = \int_\Omega F(x,u)v \quad \forall v \in \mathcal{V}(\Omega),\ v \geq 0\text{”}.$$

Since every v can be written as $v = v^+ - v^-$ with $v^+ \geq 0$ and $v^- \geq 0$, one has formally (since we do not know whether v^+ and v^- belong to $\mathcal{V}(\Omega)$ when v belongs to $\mathcal{V}(\Omega)$)

$$\text{“} -\operatorname{div} A(x)Du = F(x,u)\text{”}.$$

Observe that the above formal computation has no meaning in general, while (3.11 ii) has a perfectly correct mathematical sense when $v \in \mathcal{V}(\Omega)$ and when u satisfies (3.10).

The following Proposition 3.8 asserts that every solution to problem (2.1) in the sense of Definition 3.6 is a solution to (2.1) in the sense of distributions. Note that Proposition 3.8 does not say anything about the boundary conditions satisfied by u (on the latest topics, see Remark 3.7 ii) above and the comment after Proposition 5.13 below).

Proposition 3.8 (“Usual” properties of a solution). *Assume that the matrix A and the function F satisfy* (2.4), (2.5) *and* (2.6). *Then for every solution to problem* (2.1) *in the sense of Definition* 3.6 *one has*

$$u \geq 0 \text{ a.e. in } \Omega, \quad u \in H^1_{\mathrm{loc}}(\Omega), \quad F(x,u) \in L^1_{\mathrm{loc}}(\Omega) \tag{3.12}$$

$$-\operatorname{div} A(x)Du = F(x,u) \text{ in } \mathcal{D}'(\Omega). \tag{3.13}$$

Proof. Since every $\phi \in \mathcal{D}(\Omega)$ belongs to $\mathcal{V}(\Omega)$ (see Remark 3.5 iv)), assumption (3.11 i) implies that

$$\int_\Omega F(x,u)\phi < +\infty, \quad \forall \phi \in \mathcal{D}(\Omega), \quad \phi \geq 0 \tag{3.14}$$

which, together with $F(x,u) \geq 0$, implies that $F(x,u) \in L^1_{\mathrm{loc}}(\Omega)$.

Therefore (3.12) holds true when u is a solution to problem (2.1) in the sense of Definition 3.6.

As for (3.13), for every $\phi \in \mathcal{D}(\Omega)$ and $\overline{\phi} \in \mathcal{D}(\Omega)$ with $\overline{\phi} = 1$ in the support of ϕ, one has (see (3.9) with $\varphi = \phi$ as far as the second term is concerned)

$$\begin{cases} \langle -\operatorname{div} {}^tA(x)D\phi, G_k(u)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \langle\langle -\operatorname{div} {}^tA(x)D\phi, T_k(u)\rangle\rangle_\Omega \\ = \displaystyle\int_\Omega {}^tA(x)D\phi\, DG_k(u) + \int_\Omega {}^tA(x)D\phi\, D\big(\overline{\phi}T_k(u)\big) \\ = \displaystyle\int_\Omega {}^tA(x)D\phi Du = \int_\Omega A(x)DuD\phi, \end{cases}$$

where we have used the fact that $u \in H^1_{\mathrm{loc}}(\Omega)$ and that $\overline{\phi} = 1$ in the support of ϕ. This implies that when u is a solution to problem (2.1) in the sense of Definition 3.6, one has

$$\int_\Omega A(x)DuD\phi = \int_\Omega F(x,u)\phi \quad \forall \phi \in \mathcal{D}(\Omega), \quad \phi \geq 0 \tag{3.15}$$

namely

$$-\operatorname{div} A(x)Du \geq F(x,u) \text{ in } \mathcal{D}'(\Omega).$$

But (3.15) also implies that

$$\int_\Omega A(x)DuD\phi = \int_\Omega F(x,u)\phi \quad \forall \phi \in \mathcal{D}(\Omega), \quad \phi \leq 0$$

namely

$$-\operatorname{div} A(x)Du \leq F(x,u) \text{ in } \mathcal{D}'(\Omega).$$

This proves (3.13). □

Remark 3.9 ($\varphi Du \in (L^2(\Omega))^N\ \forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$). When u satisfies (3.10), then one has

$$\varphi Du \in \big(L^2(\Omega)\big)^N \quad \forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega). \tag{3.16}$$

More precisely, when u satisfies (3.10 i) and (3.10 iii), assertion (3.10 iv) is equivalent to (3.16).

Indeed, if $y \in H^1_{\mathrm{loc}}(\Omega)$ and $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, one has for every $k > 0$

$$\begin{cases} \varphi Dy = \varphi DT_k(y) + \varphi DG_k(y) & \text{in } (\mathcal{D}'(\Omega))^N \\ D(\varphi T_k(y)) = \varphi DT_k(y) + T_k(y)D\varphi & \text{in } (\mathcal{D}'(\Omega))^N \end{cases}$$

and therefore

$$\varphi Dy - D(\varphi T_k(y)) = \varphi DG_k(y) - T_k(y)D\varphi \text{ in } (\mathcal{D}'(\Omega))^N.$$

This in particular implies that $\varphi Dy - D(\varphi T_k(y)) \in (L^2(\Omega))^N$ when $y \in H^1_{\mathrm{loc}}(\Omega)$, $G_k(y) \in H_0^1(\Omega)$ and $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

Therefore if $y \in H^1_{\rm loc}(\Omega)$, $G_k(y) \in H^1_0(\Omega)$ and $\varphi \in H^1_0(\Omega) \cap L^\infty(\Omega)$, one has

$$\varphi Dy \in \left(L^2(\Omega)\right)^N \iff D(\varphi T_k(y)) \in \left(L^2(\Omega)\right)^N$$

which in view of Lemma A.1 below and of the inequality $-k\varphi \leq \varphi T_k(u) \leq +k\varphi$ implies, since $\varphi T_k(y) \in L^\infty(\Omega) \subset L^2(\Omega)$, that

$$\varphi Dy \in \left(L^2(\Omega)\right)^N \iff \varphi T_k(y) \in H^1_0(\Omega).$$

Taking $y = u$ proves the equivalence between (3.10 iv) and (3.16) when u satisfies (3.10 i) and (3.10 iii).

4. Statements of the existence, stability and uniqueness results

In this section we state results of existence, stability and uniqueness of the solution to problem (2.1) in the sense of Definition 3.6.

Theorem 4.1 (Existence). *Assume that the matrix A and the function F satisfy* (2.4)*,* (2.5) *and* (2.6)*. Then there exists at least one solution u to problem* (2.1) *in the sense of Definition* 3.6.

The proof of the Existence Theorem 4.1 will be done in Subsection 6.2. It relies on the Stability Theorem 4.2 below and on *a priori* estimates of $\|G_k(u)\|_{H^1_0(\Omega)}$ for every $k > 0$, of $\|\varphi\, DT_k(u)\|_{(L^2(\Omega))^N}$ for every $\varphi \in H^1_0(\Omega) \cap L^\infty(\Omega)$ and every $k > 0$, and of $\displaystyle\int_{\{u\leq\delta\}} F(x,u)v$ for every $\delta > 0$ and every $v \in \mathcal{V}(\Omega)$, $v \geq 0$, which are satisfied by every solution u to problem (2.1) in the sense of Definition 3.6 (see Propositions 5.1, 5.4 and 5.9 below).

Theorem 4.2 (Stability). *Assume that the matrix A satisfies assumption* (2.4)*. Let F_n be a sequence of functions and F_∞ be a function which all satisfy assumptions* (2.5) *and* (2.6) *for the same h and the same Γ. Assume moreover that*

$$a.e.\ x \in \Omega,\quad F_n(x, s_n) \to F_\infty(x, s_\infty)\ \ if\ \ s_n \to s_\infty, s_n \geq 0, s_\infty \geq 0. \tag{4.1}$$

Let u_n be any solution to problem $(2.1)_n$ in the sense of Definition 3.6*, where $(2.1)_n$ is the problem* (2.1) *with $F(x,s)$ replaced by $F_n(x,s)$.*

Then there exists a subsequence, still labelled by n, and a function u_∞, which is a solution to problem $(2.1)_\infty$ in the sense of Definition 3.6*, such that*

$$\begin{cases} u_n \to u_\infty \ \text{ in } L^2(\Omega) \text{ strongly, in } H^1_{\rm loc}(\Omega) \text{ strongly and a.e. in } \Omega \\ G_k(u_n) \to G_k(u_\infty) \ \text{ in } H^1_0(\Omega) \text{ strongly } \forall k > 0 \\ \varphi T_k(u_n) \to \varphi T_k(u_\infty) \ \text{ in } H^1_0(\Omega) \text{ strongly } \forall k > 0,\ \forall \varphi \in H^1_0(\Omega) \cap L^\infty(\Omega). \end{cases} \tag{4.2}$$

The proof of the Stability Theorem 4.2 will be done in Subsection 6.1.

Note that assumption (4.1) is in some sense an assumption of uniform convergence in s for almost every $x \in \Omega$ of the functions $F_n(x, s)$ to the function $F_\infty(x, s)$. This "uniform convergence" is nevertheless non standard since the function $F_\infty(x, s)$ can take the value $+\infty$ for $s = 0$.

Finally, the following uniqueness result is an immediate consequence of the Comparison Principle stated and proved in Section 7 below. Note that both the Uniqueness Theorem 4.3 and the Comparison Principle of Proposition 7.1 are based on the nonincreasing character of the function $F(x, s)$ with respect to s.

Theorem 4.3 (Uniqueness). *Assume that the matrix A and the function F satisfy* (2.4)*,* (2.5) *and* (2.6)*. Assume moreover that the function $F(x, s)$ is nonincreasing with respect to s,* i.e. *satisfies assumption* (2.7)*. Then the solution to problem* (2.1) *in the sense of Definition* 3.6 *is unique.*

Remark 4.4 (Well posedness). When assumptions (2.4), (2.5), (2.6) as well as (2.7) hold true, Theorems 4.1, 4.2 and 4.3 together assert that problem (2.1) is well posed in the sense of Hadamard in the framework of Definition 3.6.

5. *A priori* estimates

In this section we state and prove *a priori* estimates which are satisfied by every solution to problem (2.1) in the sense of Definition 3.6.

5.1. *A priori* estimate of $G_k(u)$ in $H_0^1(\Omega)$

Proposition 5.1 (*A priori* estimate of $G_k(u)$ in $H_0^1(\Omega)$). *Assume that the matrix A and the function F satisfy* (2.4)*,* (2.5) *and* (2.6)*. Then for every solution u to problem* (2.1) *in the sense of Definition* 3.6 *one has*

$$\|G_k(u)\|_{H_0^1(\Omega)} = \|DG_k(u)\|_{(L^2(\Omega))^N} \leq \frac{C_S}{\alpha} \frac{\|h\|_{L^r(\Omega)}}{\Gamma(k)} \ \forall k > 0, \tag{5.1}$$

where C_S is the (*generalized*) *Sobolev's constant defined in* (2.11).

Remark 5.2 (*A priori* estimate of u in $L^2(\Omega)$). Since Ω is bounded, using Poincaré's inequality

$$\|y\|_{L^2(\Omega)} \leq C_P(\Omega)\|Dy\|_{(L^2(\Omega))^N} \quad \forall y \in H_0^1(\Omega) \tag{5.2}$$

and writing $u = T_k(u) + G_k(u)$, one easily deduces from (5.1) that every solution u to problem (2.1) in the sense of Definition 3.6 satisfies the following *a priori* estimate in $L^2(\Omega)$:

$$\|u\|_{L^2(\Omega)} \leq k|\Omega|^{\frac{1}{2}} + C_P(\Omega)\frac{C_S}{\alpha}\frac{\|h\|_{L^r(\Omega)}}{\Gamma(k)} \ \forall k > 0 \tag{5.3}$$

which, taking $k = k_0$ for some k_0 fixed or minimizing the right-hand side in k, provides an *a priori* estimate of $\|u\|_{L^2(\Omega)}$ which does not depend on k; unfortunately minimizing in k does not give an explicit constant for a general function Γ.

Remark 5.3 (Formal proof of Proposition 5.1). Taking formally $G_k(u)$ as test function in (2.1) one obtains

$$\int_\Omega A(x)DG_k(u)DG_k(u) = \int_\Omega F(x,u)G_k(u) \ \forall k > 0. \tag{5.4}$$

Estimate (5.1) then follows from the growth condition (2.6 iii) and from the facts that $G_k(u) = 0$ in the set $\{u \le k\}$ and that the function $\Gamma(s)$ is nondecreasing, so that

$$0 \le F(x,u)G_k(u) \le \frac{h(x)}{\Gamma(u)}G_k(u) \le \frac{h(x)}{\Gamma(k)}G_k(u) \ \text{ a.e. } x \in \Omega,$$

and finally from the (generalized) Sobolev's inequality (2.11) and from the coerciveness (2.4) of the matrix A.

This formal computation will be made mathematically correct in the proof below.

Proof of Proposition 5.1. Define for every k and n with $0 < k < n$ the function $S_{k,n}$ as

$$S_{k,n}(s) = \begin{cases} 0 & \text{if } 0 \le s \le k \\ s-k & \text{if } k \le s \le n \\ n-k & \text{if } n \le s. \end{cases} \tag{5.5}$$

First step. In this step we will prove that

$$S_{k,n}(u) \in \mathcal{V}(\Omega). \tag{5.6}$$

Observe first that for every $n > k$ we have, since $u \in H^1_{\text{loc}}(\Omega)$,

$$0 \le S_{k,n}(u) \le G_k(u) \text{ and } |DS_{k,n}(u)| = \chi_{\{k \le u \le n\}}|Du| \le |DG_k(u)|.$$

By (3.10 iii) and Lemma A.1 of Appendix A below this implies that

$$S_{k,n}(u) \in H^1_0(\Omega) \cap L^\infty(\Omega). \tag{5.7}$$

Let us now prove that (5.6) holds true. Let $\psi_k : s \in [0, +\infty[\to \psi_k(s) \in [0, +\infty[$ be any C^1 nondecreasing function such that

$$\psi_k(s) = 0 \ \text{ for } \ 0 \le s \le \frac{k}{2} \ \text{ and } \ \psi_k(s) = 1 \ \text{ for } \ s \ge k.$$

Since $u \in H^1_{\text{loc}}(\Omega)$ one has

$$D\psi_k(u) = \psi_k'(u)Du = \psi_k'(u)DG_{\frac{k}{2}}(u) \ \text{ in } \mathcal{D}'(\Omega)$$

from which using (3.10 iii) we deduce that $D\psi_k(u) \in (L^2(\Omega))^N$. Therefore $\psi_k(u) \in H^1(\Omega) \cap L^\infty(\Omega)$, and then, using again (3.10 iii), the inequality

$$0 \le \psi_k(u) \le \frac{4}{k} G_{\frac{k}{4}}(u)$$

(which results from $\frac{4}{k} G_{\frac{k}{4}}(s) \ge 1$ when $s \ge \frac{k}{2}$) and Lemma A.1 below, we obtain

$$\psi_k(u) \in H_0^1(\Omega) \cap L^\infty(\Omega). \tag{5.8}$$

On the other hand, in view of (5.7) one has $-\operatorname{div} {}^tA(x) DS_{k,n}(u) \in H^{-1}(\Omega)$, and one easily proves that

$$-\operatorname{div} {}^tA(x) DS_{k,n}(u) = \psi_k(u)(-\operatorname{div} {}^tA(x) DS_{k,n}(u)) \quad \text{in } \mathcal{D}'(\Omega) \tag{5.9}$$

which implies (5.6) with $\hat{\varphi} = \psi_k(u)$, $\hat{g} = {}^tA(x)\, DS_{k,n}(u)$ and $\hat{f} = 0$.

Second step. Since $S_{k,n}(u) \in \mathcal{V}(\Omega)$ by (5.6) and since $S_{k,n}(u) \ge 0$, we can use $S_{k,n}(u)$ as test function in (3.11 ii). We get, using (5.9),

$$\begin{cases} \displaystyle\int_\Omega {}^tA(x) DS_{k,n}(u) DG_k(u) + \int_\Omega {}^tA(x) DS_{k,n}(u) D(\psi_k(u) T_k(u)) \\ = \langle -\operatorname{div} {}^tA(x) DS_{k,n}(u), G_k(u) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle\langle -\operatorname{div} {}^tA(x) DS_{k,n}(u), T_k(u) \rangle\rangle_\Omega \\ = \displaystyle\int_\Omega F(x,u) S_{k,n}(u), \end{cases}$$

where the second term of the left-hand side is zero since

$$D(\psi_k(u) T_k(u)) = \psi_k(u) DT_k(u) + T_k(u) D\psi_k(u) \text{ in } \mathcal{D}'(\Omega)$$

and since $DS_{k,n}(u) = 0$ in $\{u \le k\}$, while $DT_k(u) = 0$ and $D\psi_k(u) = 0$ in $\{u \ge k\}$. This gives

$$\int_\Omega A(x) DG_k(u) DS_{k,n}(u) = \int_\Omega F(x,u) S_{k,n}(u). \tag{5.10}$$

Third step. Since $S_{k,n}(s) = 0$ for $s \le k$, using the growth condition (2.6 iii) and the (generalized) Sobolev's inequality (2.11), one has

$$\begin{cases} \displaystyle\int_\Omega F(x,u) S_{k,n}(u) \le \int_\Omega \frac{h(x)}{\Gamma(u)} \chi_{\{u>k\}} S_{k,n}(u) \le \int_\Omega \frac{h(x)}{\Gamma(k)} S_{k,n}(u) \\ \displaystyle\qquad \le \frac{\|h\|_{L^r((\Omega)}}{\Gamma(k)} \|S_{k,n}(u)\|_{L^{2^*}(\Omega)} \\ \displaystyle\qquad \le \frac{\|h\|_{L^r((\Omega)}}{\Gamma(k)} C_S \|DS_{k,n}(u)\|_{(L^2(\Omega))^N} \\ \forall k > 0, \forall n > k. \end{cases} \tag{5.11}$$

With the coercivity (2.4) of the matrix A, (5.10) and (5.11) imply that

$$\alpha \|DS_{k,n}(u)\|_{(L^2(\Omega))^N} \le C_S \frac{\|h\|_{L^r((\Omega)}}{\Gamma(k)} \quad \forall k > 0, \forall n > k. \tag{5.12}$$

Therefore $S_{k,n}(u)$ is bounded in $H_0^1(\Omega)$ for $k > 0$ fixed independently of n when $n > k$, and the left-hand side (and therefore the right-hand side) of (5.10) is bounded independently of n when $n > k$. Since

$$S_{k,n}(u) \rightharpoonup G_k(u) \text{ in } H_0^1(\Omega) \text{ weakly and a.e. in } \Omega \text{ as } n \to +\infty \tag{5.13}$$

applying Fatou's lemma to the right-hand side of (5.10) one deduces that

$$\int_\Omega F(x,u)G_k(u) < +\infty \quad \forall\, k > 0. \tag{5.14}$$

Then, using (5.13) in the left-hand side and Lebesgue's dominated convergence theorem in the right-hand side of (5.10), one obtains (5.4).

Estimate (5.1) follows either from (5.12) and (5.13) or from (5.4). □

Note that in the third step of the previous proof, we have proved that the energy identity (5.4) holds true for every solution u to problem (2.1) in the sense of Definition 3.6.

5.2. *A priori* estimate of $\varphi DT_k(u)$ in $(L^2(\Omega))^N$ for $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$

Proposition 5.4 (*A priori* estimate of $\varphi DT_k(u)$ in $(L^2(\Omega))^N$ for $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$). *Assume that the matrix A and the function F satisfy* (2.4), (2.5) *and* (2.6). *Then for every solution u to problem* (2.1) *in the sense of Definition* 3.6 *one has*

$$\begin{cases} \|\varphi DT_k(u)\|^2_{(L^2(\Omega))^N} \\ \le \dfrac{32k^2}{\alpha^2}\|A\|^2_{(L^\infty(\Omega))^{N\times N}}\|D\varphi\|^2_{(L^2(\Omega))^N} + \dfrac{C_S^2}{\alpha^2}\dfrac{\|h\|^2_{L^r(\Omega)}}{\Gamma(k)^2}\|\varphi\|^2_{L^\infty(\Omega)} \\ \forall k > 0,\ \forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega), \end{cases} \tag{5.15}$$

where C_S is the (generalized) Sobolev's constant defined in (2.11).

Remark 5.5 (*A priori* estimate of $\varphi T_k(u)$ in $H_0^1(\Omega)$). From the *a priori* estimate (5.15) and using the equality $D(\varphi T_k(u)) = \varphi DT_k(u) + T_k(u)D\varphi$, one deduces that every solution u to problem (2.1) in the sense of Definition 3.6 satisfies the following *a priori* estimate of $\varphi T_k(u)$ in $H_0^1(\Omega)$

$$\begin{cases} \|\varphi T_k(u)\|^2_{H_0^1(\Omega)} = \|D(\varphi T_k(u))\|^2_{(L^2(\Omega))^N} \\ \le \left(\dfrac{64k^2}{\alpha^2}\|A\|^2_{(L^\infty(\Omega))^{N\times N}} + 2k^2\right)\|D\varphi\|^2_{(L^2(\Omega))^N} + 2\dfrac{C_S^2}{\alpha^2}\dfrac{\|h\|^2_{L^r(\Omega)}}{\Gamma(k)^2}\|\varphi\|^2_{L^\infty(\Omega)} \\ \forall k > 0,\ \forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega). \end{cases} \tag{5.16}$$

Remark 5.6 (*A priori* estimate of φDu in $(L^2(\Omega))^N$ for $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$). Using estimates (5.1) of $DG_k(u)$ and (5.15) of $\varphi DT_k(u)$ in $(L^2(\Omega))^N$, as well as

$$\varphi Du = \varphi DT_k(u) + \varphi DG_k(u),$$

and

$$|DT_k(u)||DG_k(u)| = 0 \quad \text{a.e. in } \Omega,$$

one deduces that every solution u to problem (2.1) in the sense of Definition 3.6 satisfies the following *a priori* estimate of φDu in $(L^2(\Omega))^N$

$$\begin{cases} \|\varphi Du\|^2_{(L^2(\Omega))^N} = \|\varphi DT_k(u)\|^2_{(L^2(\Omega))^N} + \|\varphi DG_k(u)\|^2_{(L^2(\Omega))^N} \\ \leq \dfrac{32k^2}{\alpha^2}\|A\|^2_{(L^\infty(\Omega))^{N\times N}}\|D\varphi\|^2_{(L^2(\Omega))^N} + 2\dfrac{C_S^2}{\alpha^2}\dfrac{\|h\|^2_{L^r(\Omega)}}{\Gamma(k)^2}\|\varphi\|^2_{L^\infty(\Omega)} \\ \forall k > 0,\ \forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega) \end{cases} \tag{5.17}$$

which, taking $k = k_0$ for some k_0 fixed or minimizing the right-hand side in k, provides an *a priori* estimate of $\|\varphi Du\|^2_{(L^2(\Omega))^N}$ which does not depend on k; unfortunately minimizing in k does not give an explicit constant for a general function Γ.

Remark 5.7 (*A priori* estimate of u in $H^1_{\mathrm{loc}}(\Omega)$). For every $\phi \in \mathcal{D}(\Omega)$ one has $D(\phi u) = \phi Du + (T_k(u) + G_k(u))D\phi$, which implies that

$$|D(\phi u)| \leq |\phi Du| + k|D\phi| + \|D\phi\|_{(L^\infty(\Omega))^N}|G_k(u)|.$$

Using the inequality $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$ and the *a priori* estimates (5.17) and (5.1) together with Poincaré's inequality (5.2), one deduces that every solution u to problem (2.1) in the sense of Definition 3.6 satisfies the following *a priori* estimate of u in $H^1_{\mathrm{loc}}(\Omega)$

$$\begin{cases} \|\phi u\|^2_{H_0^1(\Omega)} = \|D(\phi u)\|^2_{(L^2(\Omega))^N} \\ \leq 3\left(\dfrac{32k^2}{\alpha^2}\|A\|^2_{(L^\infty(\Omega))^{N\times N}}\|D\phi\|^2_{(L^2(\Omega))^N} + 2\dfrac{C_S^2}{\alpha^2}\dfrac{\|h\|^2_{L^r(\Omega)}}{\Gamma(k)^2}\|\phi\|^2_{L^\infty(\Omega)} \right. \\ \qquad \left. + k^2\|D\phi\|^2_{(L^2(\Omega))^N} + C_P^2(\Omega)\dfrac{C_S^2}{\alpha^2}\dfrac{\|h\|^2_{L^r(\Omega)}}{\Gamma(k)^2}\|D\phi\|^2_{(L^\infty(\Omega))^N}\right) \\ \forall k > 0,\ \forall \phi \in \mathcal{D}(\Omega) \end{cases} \tag{5.18}$$

which, taking $k = k_0$ for some k_0 fixed or minimizing the right-hand side in k, provides an *a priori* estimate of $\|\phi u\|^2_{H_0^1(\Omega)}$ for every fixed $\phi \in \mathcal{D}(\Omega)$, *i.e.* an *a priori* estimate of $\|u\|^2_{H^1_{\mathrm{loc}}(\Omega)}$, which does not depend on k.

Remark 5.8 (Formal proof of Proposition 5.4). The computation that we will perform in the present remark is formal. We will make it mathematically correct in the proof below.

The idea of the proof of Proposition 5.4 is to formally use $\varphi^2 T_k(u)$ as a first test function in (2.1), where $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. We formally get

$$\int_\Omega A(x) Du DT_k(u)\, \varphi^2 + 2 \int_\Omega A(x) Du D\varphi\, \varphi T_k(u) = \int_\Omega F(x,u)\varphi^2 T_k(u) \tag{5.19}$$

in the second term of which we write $Du = DT_k(u) + DG_k(u)$. Using the coercivity (2.4) of the matrix A we have

$$\begin{cases} \alpha \displaystyle\int_\Omega \varphi^2 |DT_k(u)|^2 \\ \leq 2 \left| \displaystyle\int_\Omega A(x) DT_k(u) D\varphi\, \varphi T_k(u) \right| + 2 \left| \displaystyle\int_\Omega A(x) DG_k(u) D\varphi\, \varphi T_k(u) \right| \\ \quad + \displaystyle\int_\Omega F(x,u)\varphi^2 T_k(u). \end{cases} \tag{5.20}$$

In this inequality we use the estimate

$$\begin{cases} 2 \left| \displaystyle\int_\Omega A(x) DT_k(u) D\varphi\, \varphi T_k(u) \right| + 2 \left| \displaystyle\int_\Omega A(x) DG_k(u) D\varphi\, \varphi T_k(u) \right| \\ \leq 2k \|A\|_{(L^\infty(\Omega))^{N\times N}} \|D\varphi\|_{(L^2(\Omega))^N} \|\varphi DT_k(u)\|_{(L^2(\Omega))^N} \\ \quad + 2k \|A\|_{(L^\infty(\Omega))^{N\times N}} \|D\varphi\|_{(L^2(\Omega))^N} \|\varphi\|_{L^\infty(\Omega)} \|DG_k(u)\|_{(L^2(\Omega))^N}. \end{cases} \tag{5.21}$$

On the other hand, since $0 \leq T_k(u) \leq k$, using formally φ^2 as a second test function in (2.1), we have

$$\begin{cases} 0 \leq \displaystyle\int_\Omega F(x,u)\varphi^2 T_k(u) \leq k \int_\Omega F(x,u)\varphi^2 \\ \quad = k \displaystyle\int_\Omega A(x) Du D\varphi^2 = 2k \int_\Omega A(x) Du D\varphi\, \varphi \\ \quad = 2k \displaystyle\int_\Omega A(x) DT_k(u) D\varphi\, \varphi + 2k \int_\Omega A(x) DG_k(u) D\varphi\, \varphi \\ \quad \leq 2k \|A\|_{(L^\infty(\Omega))^{N\times N}} \|D\varphi\|_{(L^2(\Omega))^N} \|\varphi DT_k(u)\|_{(L^2(\Omega))^N} \\ \qquad + 2k \|A\|_{(L^\infty(\Omega))^{N\times N}} \|D\varphi\|_{(L^2(\Omega))^N} \|\varphi\|_{L^\infty(\Omega)} \|DG_k(u)\|_{(L^2(\Omega))^N}. \end{cases} \tag{5.22}$$

Collecting together (5.20), (5.21) and (5.22) we obtain

$$\begin{cases} \alpha \displaystyle\int_\Omega \varphi^2 |DT_k(u)|^2 \\ \leq 4k\|A\|_{(L^\infty(\Omega))^{N\times N}} \|D\varphi\|_{(L^2(\Omega))^N} \|\varphi DT_k(u)\|_{(L^2(\Omega))^N} \\ \quad +4k\|A\|_{(L^\infty(\Omega))^{N\times N}} \|D\varphi\|_{(L^2(\Omega))^N} \|\varphi\|_{L^\infty(\Omega)} \|DG_k(u)\|_{(L^2(\Omega))^N} \\ \forall k > 0, \forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega). \end{cases} \tag{5.23}$$

Using Young's inequality in the first term of the right-hand side of (5.23) and the estimate (5.1) of $\|DG_k(u)\|_{(L^2(\Omega))^N}$ in the second term provides the estimate (5.15) of $\|\varphi DT_k(u)\|^2_{(L^2(\Omega))^N}$.

Proof of Proposition 5.4. In this proof $k > 0$ and $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ are fixed.

First step. By (3.10 iv) we have $\varphi T_k(u) \in H_0^1(\Omega) \cap L^\infty(\Omega)$, which implies, using Remark 3.5 i), that

$$\varphi^2 T_k(u) = \varphi\, \varphi T_k(u) \in \mathcal{V}(\Omega) \tag{5.24}$$

with

$$\begin{cases} -\operatorname{div} {}^tA(x)D(\varphi^2 T_k(u)) \\ = \varphi(-\operatorname{div} {}^tA(x)D(\varphi T_k(u))) - {}^tA(x)D(\varphi T_k(u))D\varphi \\ \quad +(\varphi T_k(u))(-\operatorname{div} {}^tA(x)D\varphi) - {}^tA(x)D\varphi D(\varphi T_k(u)) \quad \text{in } \mathcal{D}'(\Omega). \end{cases}$$

Since $\varphi^2 T_k(u) \in \mathcal{V}(\Omega)$ with $\varphi^2 T_k(u) \geq 0$ we can use $v = \varphi^2 T_k(u)$ as test function in (3.11 ii). Denoting by j the value of k which appears in (3.11 ii), we get

$$\begin{cases} \displaystyle\int_\Omega {}^tA(x)D(\varphi^2 T_k(u))DG_j(u) \\ \displaystyle + \int_\Omega {}^tA(x)D(\varphi T_k(u))D(\varphi T_j(u)) - \int_\Omega {}^tA(x)D(\varphi T_k(u))D\varphi\, T_j(u) \\ \displaystyle + \int_\Omega {}^tA(x)D\varphi\, D(\varphi T_k(u)T_j(u)) - \int_\Omega {}^tA(x)D\varphi D(\varphi T_k(u))\, T_j(u) \\ = \langle -\operatorname{div} {}^tA(x)D(\varphi^2 T_k(u)), G_j(u)\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \\ \quad + \langle\langle -\operatorname{div} {}^tA(x)D(\varphi^2 T_k(u)), T_j(u)\rangle\rangle_\Omega \\ \displaystyle = \int_\Omega F(x,u)\varphi^2 T_k(u) \quad \forall j > 0, \end{cases} \tag{5.25}$$

where we observe that the fourth term of the left-hand side makes sense due to the fact that $\varphi T_k(u) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ by (3.10 iv), which implies that $(\varphi T_k(u))T_j(u) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ again by (3.10 iv).

Since $u \in H^1_{\text{loc}}(\Omega)$, we can expand in $L^1_{\text{loc}}(\Omega)$ the integrands of the 5 terms of the left-hand side of (5.25). We obtain 13 terms whose integrands belong to $L^1(\Omega)$

since $DG_j(u)$, $\varphi DT_k(u)$ and $\varphi DT_j(u)$ belong to $(L^2(\Omega))^N$. A simple but tedious computation leads from (5.25) to

$$\begin{cases} \displaystyle\int_\Omega A(x)DT_j(u)DT_k(u)\varphi^2 + \int_\Omega A(x)DG_j(u)DT_k(u)\,\varphi^2 \\ \displaystyle +2\int_\Omega A(x)DT_j(u)D\varphi\,\varphi T_k(u) + 2\int_\Omega A(x)DG_j(u)D\varphi\,\varphi T_k(u) \\ \displaystyle = \int_\Omega F(x,u)\varphi^2 T_k(u) \quad \forall j > 0. \end{cases} \tag{5.26}$$

Taking $j = k$ gives, since $|DG_k(u)||DT_k(u)| = 0$ almost everywhere in Ω,

$$\begin{cases} \displaystyle\int_\Omega A(x)DT_k(u)DT_k(u)\varphi^2 \\ \displaystyle +2\int_\Omega A(x)DT_k(u)D\varphi\,\varphi T_k(u) + 2\int_\Omega A(x)DG_k(u)D\varphi\,\varphi T_k(u) \\ \displaystyle = \int_\Omega F(x,u)\varphi^2 T_k(u). \end{cases} \tag{5.27}$$

This result is nothing but (5.19), which had been formally obtained in Remark 5.8 by taking $\varphi^2 T_k(u)$ as test function in (2.1), but the proof that we just performed is mathematically correct. From (5.27) and the coercivity (2.4) of the matrix A we deduce that

$$\begin{cases} \displaystyle\alpha\int_\Omega \varphi^2|DT_k(u)|^2 \\ \displaystyle \le 2\left|\int_\Omega A(x)DT_k(u)D\varphi\,\varphi T_k(u)\right| + 2\left|\int_\Omega A(x)DG_k(u)D\varphi\,\varphi T_k(u)\right| \\ \displaystyle +\int_\Omega F(x,u)\varphi^2 T_k(u) \end{cases} \tag{5.28}$$

which is nothing but (5.20) of the formal proof made in Remark 5.8.

Second step. As far as the first and the second terms of the right-hand side of (5.28) are concerned, we have, as in (5.21),

$$\begin{cases} \displaystyle 2\left|\int_\Omega A(x)DT_k(u)D\varphi\,\varphi T_k(u)\right| + 2\left|\int_\Omega A(x)DG_k(u)D\varphi\,\varphi T_k(u)\right| \\ \le 2k\|A\|_{(L^\infty(\Omega))^{N\times N}}\|D\varphi\|_{(L^2(\Omega))^N}\|\varphi DT_k(u)\|_{(L^2(\Omega))^N} \\ +2k\|A\|_{(L^\infty(\Omega))^{N\times N}}\|D\varphi\|_{(L^2(\Omega))^N}\|\varphi\|_{L^\infty(\Omega)}\|DG_k(u)\|_{(L^2(\Omega))^N}. \end{cases} \tag{5.29}$$

On the other hand, since for every $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, φ^2 belongs to $\mathcal{V}(\Omega)$ (see Remark 3.5 ii)) and since $\varphi^2 \ge 0$, using φ^2 as test function in (3.11 ii) gives

$$\begin{cases} \langle -\operatorname{div}\, {}^tA(x)D\varphi^2, G_k(u)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \langle\langle -\operatorname{div}\, {}^tA(x)D\varphi^2, T_k(u)\rangle\rangle_\Omega \\ \displaystyle = \int_\Omega F(x,u)\varphi^2 \end{cases}$$

which is easily seen to yield

$$\begin{cases} \quad 2\int_\Omega A(x)DG_k(u)D\varphi\,\varphi + 2\int_\Omega A(x)DT_k(u)D\varphi\,\varphi \\ = \int_\Omega F(x,u)\varphi^2. \end{cases} \tag{5.30}$$

Therefore we have, for the last term of the right-hand side of (5.28),

$$\begin{cases} 0 \le \int_\Omega F(x,u)\varphi^2 T_k(u) \le k\int_\Omega F(x,u)\varphi^2 \\ \quad = 2k\int_\Omega A(x)DT_k(u)D\varphi\,\varphi + 2k\int_\Omega A(x)DG_k(u)D\varphi\,\varphi \\ \quad \le 2k\|A\|_{(L^\infty(\Omega))^{N\times N}}\|D\varphi\|_{(L^2(\Omega))^N}\|\varphi DT_k(u)\|_{(L^2(\Omega))^N} \\ \quad\quad +2k\|A\|_{(L^\infty(\Omega))^{N\times N}}\|D\varphi\|_{(L^2(\Omega))^N}\|\varphi\|_{L^\infty(\Omega)}\|DG_k(u)\|_{(L^2(\Omega))^N} \end{cases} \tag{5.31}$$

which is nothing but (5.22) of the formal proof made in Remark 5.8.

Third step. Collecting together (5.28), (5.29) and (5.31) we have proved that

$$\begin{cases} \quad \alpha\int_\Omega \varphi^2|DT_k(u)|^2 \\ \le 4k\|A\|_{(L^\infty(\Omega))^{N\times N}}\|D\varphi\|_{(L^2(\Omega))^N}\|\varphi DT_k(u)\|_{(L^2(\Omega))^N} \\ \quad +4k\|A\|_{(L^\infty(\Omega))^{N\times N}}\|D\varphi\|_{(L^2(\Omega))^N}\|\varphi\|_{L^\infty(\Omega)}\|DG_k(u)\|_{(L^2(\Omega))^N} \\ \forall k > 0,\ \forall \varphi \in H_0^1(\Omega)\cap L^\infty(\Omega) \end{cases} \tag{5.32}$$

which is nothing but (5.23) of the formal proof made in Remark 5.8.

Using Young's inequality $XY \le \frac{\alpha}{2}X^2 + \frac{1}{2\alpha}Y^2$ in each of the two terms of the right-hand side of (5.32) gives

$$\begin{cases} \quad \frac{\alpha}{2}\|\varphi DT_k(u)\|^2_{(L^2(\Omega)^N} \\ \le \frac{16k^2}{2\alpha}\|A\|^2_{(L^\infty(\Omega))^{N\times N}}\|D\varphi\|^2_{(L^2(\Omega))^N} \\ \quad +\frac{16k^2}{2\alpha}\|A\|^2_{(L^\infty(\Omega))^{N\times N}}\|D\varphi\|^2_{(L^2(\Omega))^N} + \frac{\alpha}{2}\|DG_k(u)\|^2_{(L^2(\Omega))^N}\|\varphi\|^2_{L^\infty(\Omega)} \end{cases} \tag{5.33}$$

in which we use the *a priori* estimate (5.1) of $DG_k(u)$ to obtain

$$\begin{cases} \quad \frac{\alpha}{2}\|\varphi DT_k(u)\|^2_{(L^2(\Omega)^N} \\ \le \frac{16k^2}{\alpha}\|A\|^2_{(L^\infty(\Omega))^{N\times N}}\|D\varphi\|^2_{(L^2(\Omega))^N} + \frac{C_S^2}{2\alpha}\frac{\|h\|^2_{L^r(\Omega)}}{\Gamma(k)^2}\|\varphi\|^2_{L^\infty(\Omega)} \end{cases} \tag{5.34}$$

i.e. the *a priori* estimate (5.15). This completes the proof of Proposition 5.4. □

5.3. Control of the integral $\int_{\{u\le\delta\}} F(x,u)v$

In this subsection we prove an *a priori* estimate (see (5.37) below) which is a key point in the proofs of our results.

For $\delta > 0$, we define the function $Z_\delta : s \in [0,+\infty[\to Z_\delta(s) \in [0,+\infty[$ by

$$Z_\delta(s) = \begin{cases} 1 & \text{if } 0 \le s \le \delta \\ -\dfrac{s}{\delta} + 2 & \text{if } \delta \le s \le 2\delta \\ 0 & \text{if } 2\delta \le s. \end{cases} \tag{5.35}$$

Proposition 5.9 (Control of the integral $\int_{\{u\le\delta\}} F(x,u)v$). *Assume that the matrix A and the function F satisfy* (2.4)*,* (2.5) *and* (2.6)*. Then for every u solution to problem* (2.1) *in the sense of Definition* 3.6 *and for every v such that*

$$\begin{cases} v \in \mathcal{V}(\Omega), \ v \ge 0 \\ \textit{with} \quad -\operatorname{div} {}^tA(x)Dv = \sum_{i\in I} \hat{\varphi}_i(-\operatorname{div} \hat{g}_i) + \hat{f} \ \textit{in } \mathcal{D}'(\Omega) \\ \textit{where} \ \ \hat{\varphi}_i \in H_0^1(\Omega)\cap L^\infty(\Omega),\, \hat{g}_i \in (L^2(\Omega))^N,\, \hat{f} \in L^1(\Omega) \end{cases} \tag{5.36}$$

one has

$$\begin{cases} \forall\, \delta > 0, \ \displaystyle\int_\Omega F(x,u)Z_\delta(u)v \\ \qquad \le \dfrac{3}{2}\left(\displaystyle\int_\Omega \left|\sum_{i\in I} \hat{g}_i D\hat{\varphi}_i + \hat{f}\right|\right)\delta + \displaystyle\int_\Omega Z_\delta(u)\sum_{i\in I}\hat{g}_i Du\,\hat{\varphi}_i. \end{cases} \tag{5.37}$$

Remark 5.10. Since $Z_\delta(s) \ge \chi_{\{s\le\delta\}}(s)$ for every $s \ge 0$, estimate (5.37) provides an estimate of the integral $\int_{\{u\le\delta\}} F(x,u)v$ as announced in the title of Subsection 5.3.

Note that the right-hand side of estimate (5.37) depends only on δ, on v, through $\sum_{i\in I}\hat{g}_i D\hat{\varphi}_i + \hat{f}$, and also on u, through $Z_\delta(u)\,\hat{g}_i\sum_{i\in I} Du\,\hat{\varphi}_i$, which belongs to $L^1(\Omega)$ in view of (3.16).

Remark 5.11 (Formal proof of Proposition 5.9). Estimate (5.37) can be formally obtained by using in (2.1) the test function $Z_\delta(u)v$, where the function Z_δ is defined by (5.35) and where v satisfies (5.36). One formally obtains

$$\int_\Omega F(x,u)Z_\delta(u)v = \int_\Omega A(x)DuDuZ'_\delta(u)v + \int_\Omega A(x)DuDvZ_\delta(u) \tag{5.38}$$

and one observes that, denoting by Y_δ the primitive of the function Z_δ (see (5.42) below), one has

$$\int_\Omega A(x)DuDuZ'_\delta(u)v \leq 0$$

$$\begin{cases} \displaystyle\int_\Omega A(x)DuDvZ_\delta(u) &= \displaystyle\int_\Omega A(x)DY_\delta(u)Dv = \int_\Omega {}^tA(x)DvDY_\delta(u) \\ &= \displaystyle\sum_{i\in I}\int_\Omega \hat{g}_i D(\hat{\varphi}_i Y_\delta(u)) + \int_\Omega \hat{f}Y_\delta(u) \\ &= \displaystyle\int_\Omega \left(\sum_{i\in I}\hat{g}_i D\hat{\varphi}_i + \hat{f}\right) Y_\delta(u) + \int_\Omega Z_\delta(u)\sum_{i\in I}\hat{g}_i Du\,\hat{\varphi}_i, \end{cases}$$

where one finally uses $0 \leq Y_\delta(s) \leq 3\delta/2$ to formally obtain estimate (5.37). These formal computations will be made mathematically correct in the proof below.

As a consequence of Proposition 5.9 we have the following result.

Proposition 5.12 ($F(x, 0) = 0$ a.e. in the set $\{u = 0\}$). *Assume that the matrix A and the function F satisfy* (2.4)*,* (2.5) *and* (2.6)*. Then for every u solution to problem* (2.1) *in the sense of Definition* 3.6 *one has*

$$\int_{\{u=0\}} F(x,u)\,v = 0 \quad \forall v \in \mathcal{V}(\Omega),\, v \geq 0 \tag{5.39}$$

or equivalently

$$F(x,0) = 0 \textit{ a.e. in the set } \{x \in \Omega : u(x) = 0\}. \tag{5.40}$$

Proof of Proposition 5.12. Observe that

$$\begin{cases} \forall\delta > 0,\ \forall v \in \mathcal{V}(\Omega),\ v \geq 0 \\ \text{one has}\ \ 0 \leq \displaystyle\int_{\{u=0\}} F(x,u)\,v \leq \int_\Omega F(x,u)Z_\delta(u)\,v. \end{cases} \tag{5.41}$$

We will use (5.37) to estimate the right-hand side of (5.41) and to prove that it tends to zero as δ tends to zero. This will imply (5.39).

When δ tends to zero, the first term of the right-hand side of (5.37) clearly tends to zero. For what concerns the second term of this rigth-hand side, we observe that the absolute value of its integrand is dominated by $\left|\sum_{i\in I}\hat{g}_i Du\,\hat{\varphi}_i\right|$, which belongs to $L^1(\Omega)$, that $Z_\delta(u)$ converges almost everywhere to $\chi_{\{u=0\}}$ as δ tends to zero, and that, since $u \in H^1_{\text{loc}}(\Omega)$, one has $Du = 0$ almost everywhere in $\{u = 0\}$; Lebesgue's dominated convergence theorem then implies that this second term satisfies

$$\int_\Omega Z_\delta(u)\sum_{i\in I}\hat{g}_i Du\,\hat{\varphi}_i \to \int_\Omega \chi_{\{u=0\}}\sum_{i\in I}\hat{g}_i Du\,\hat{\varphi}_i = \int_\Omega 0 = 0 \ \text{ as } \delta \to 0.$$

We have proved (5.39).

Let us now prove that (5.39) implies (5.40) (the converse is clear). Since every $\phi \in \mathcal{D}(\Omega)$ belongs to $\mathcal{V}(\Omega)$ in view of Remark 3.5 iv), the result (5.39) implies that one has $F(x, u(x)) = 0$ almost everywhere in the set $\{x \in \Omega : u(x) = 0\}$, or in other terms that $F(x, 0) = 0$ almost everywhere in this set, *i.e.* (5.40). □

For other comments about the set where the solution u takes the value zero, see [21, Section 4].

Proof of Proposition 5.9. In addition to the function $Z_\delta(s)$ defined by (5.35), we define for $\delta > 0$ the functions

$$Y_\delta(s) = \begin{cases} s & \text{if } 0 \le s \le \delta \\ -\dfrac{s^2}{2\delta} + 2s - \dfrac{\delta}{2} & \text{if } \delta \le s \le 2\delta \\ \dfrac{3}{2}\delta & \text{if } 2\delta \le s \end{cases} \tag{5.42}$$

$$R_\delta(s) = \begin{cases} 0 & \text{if } 0 \le s \le \delta \\ \dfrac{s^2}{2\delta} - \dfrac{\delta}{2} & \text{if } \delta \le s \le 2\delta \\ \dfrac{3}{2}\delta & \text{if } 2\delta \le s. \end{cases} \tag{5.43}$$

Observe that Z_δ, Y_δ and R_δ are Lipschitz continuous and piecewise C^1 functions with

$$\begin{cases} Y_\delta'(s) = Z_\delta(s), \;\; Z_\delta'(s) = -\dfrac{1}{\delta}\chi_{\{\delta < s < 2\delta\}}(s), \;\; R_\delta'(s) = -sZ_\delta'(s) \\ Y_\delta(s) = sZ_\delta(s) + R_\delta(s), \;\; 0 \le Y_\delta(s) \le \dfrac{3}{2}\delta, \;\; \forall s \ge 0. \end{cases} \tag{5.44}$$

First step. In this first step we will prove that for every $\delta > 0$

$$Y_\delta(u) \in H^1_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega), \;\; \varphi Y_\delta(u) \in H^1_0(\Omega) \; \forall \varphi \in H^1_0(\Omega) \cap L^\infty(\Omega) \tag{5.45}$$

$$Z_\delta(u)v \in \mathcal{V}(\Omega) \quad \forall v \in \mathcal{V}(\Omega) \tag{5.46}$$

$$R_\delta(u \in H^1_0(\Omega) \cap L^\infty(\Omega). \tag{5.47}$$

Since $u \in H^1_{\mathrm{loc}}(\Omega)$ by (3.10 i), one has $Y_\delta(u) \in H^1_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$ with

$$DY_\delta(u) = Y_\delta'(u)Du = Z_\delta(u)Du \text{ in } \mathcal{D}'(\Omega)$$

and for every $\varphi \in H^1_0(\Omega) \cap L^\infty(\Omega)$, one has $\varphi Y_\delta(u) \in H^1_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$ with

$$D(\varphi Y_\delta(u)) = \varphi Z_\delta(u)Du + Y_\delta(u)D\varphi \text{ in } \mathcal{D}'(\Omega)$$

which implies, using (3.16) and $Y_\delta(u) \in L^\infty(\Omega)$, that $D(\varphi Y_\delta(u)) \in (L^2(\Omega))^N$, and therefore that $\varphi Y_\delta(u) \in H^1(\Omega) \cap L^\infty(\Omega)$. Since $0 \le \varphi Y_\delta(u) \le 3\delta\varphi/2$, and since $\varphi \in H_0^1(\Omega)$, Lemma A.1 of Appendix A below completes the proof of (5.45).

On the other hand $Z_\delta(u) \in H^1_{\text{loc}}(\Omega)$ and

$$DZ_\delta(u) = -\frac{1}{\delta}\chi_{\{\delta<u<2\delta\}} Du \text{ in } \mathcal{D}'(\Omega).$$

In view of (3.10 iii) with $k = \delta$, this implies that $DZ_\delta(u) \in (L^2(\Omega))^N$, and therefore that

$$Z_\delta(u) \in H^1(\Omega) \cap L^\infty(\Omega), \tag{5.48}$$

which in turn implies that

$$Z_\delta(u)\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega) \quad \forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega). \tag{5.49}$$

Consider now $v \in \mathcal{V}(\Omega)$ with (5.36). Observe that $Z_\delta(u)v \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and that

$$\begin{cases} -\operatorname{div} {}^tA(x)D(Z_\delta(u)v) \\ = Z_\delta(u)(-\operatorname{div} {}^tA(x)Dv) - {}^tA(x)Dv DZ_\delta(u) \\ \quad +v(-\operatorname{div} {}^tA(x)DZ_\delta(u)) - {}^tA(x)DZ_\delta(u)Dv \\ = \sum_{i\in I} Z_\delta(u)\,\hat{\varphi}_i(-\operatorname{div} \hat{g}_i) + Z_\delta(u)\hat{f} - {}^tA(x)DvDZ_\delta(u) \\ \quad +v(-\operatorname{div} {}^tA(x)DZ_\delta(u)) - {}^tA(x)DZ_\delta(u)Dv \text{ in } \mathcal{D}'(\Omega). \end{cases} \tag{5.50}$$

This proves that $Z_\delta(u)v \in \mathcal{V}(\Omega)$, namely (5.46).

Finally, since $u \in H^1_{\text{loc}}(\Omega)$ by (3.10 i), one has

$$R_\delta(u) \in H^1_{\text{loc}}(\Omega) \cap L^\infty(\Omega)$$

with

$$DR_\delta(u) = R'_\delta(u)Du = \frac{u}{\delta}\chi_{\{\delta<u<2\delta\}} Du \text{ in } \mathcal{D}'(\Omega).$$

Therefore (3.10 iii) with $k = \delta$ implies that $DR_\delta(u) \in (L^2(\Omega))^N$, which proves that $R_\delta(u) \in H^1(\Omega) \cap L^\infty(\Omega)$. Then using the inequality $0 \le R_\delta(u) \le G_{\frac{\delta}{2}}(u)$, property (3.10 iii) with $k = \frac{\delta}{2}$ and Lemma A.1 of Appendix A below completes the proof of (5.47).

Second step. In this step we fix $\delta > 0$, $k \ge 2\delta$ and $v \in \mathcal{V}(\Omega)$, $v \ge 0$.

Since $Z_\delta(u)v \in \mathcal{V}(\Omega)$ for every $v \in \mathcal{V}(\Omega)$ (see (5.46)) with $Z_\delta(u)v \geq 0$, we can take $Z_\delta(u)v$ as test function in (3.11 ii), obtaining in view of (5.50),

$$\begin{cases} \displaystyle\int_\Omega {}^tA(x)D(Z_\delta(u)v)DG_k(u) \\ \displaystyle+\sum_{i\in I}\int_\Omega \hat{g}_i D(Z_\delta(u)\hat{\varphi}_i T_k(u)) + \int_\Omega Z_\delta(u)\hat{f}\, T_k(u) - \int_\Omega {}^tA(x)DvDZ_\delta(u)\, T_k(u) \\ \displaystyle+\int_\Omega {}^tA(x)DZ_\delta(u)D(vT_k(u)) - \int_\Omega {}^tA(x)DZ_\delta(u)Dv\, T_k(u) \\ = \langle -\operatorname{div} {}^tA(x)D(Z_\delta(u)v), G_k(u)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \\ +\langle\langle -\operatorname{div} {}^tA(x)D(Z_\delta(u)v), T_k(u)\rangle\rangle_\Omega \\ \displaystyle= \int_\Omega F(x,u)Z_\delta(u)v. \end{cases} \tag{5.51}$$

As far as the first term of the left-hand side of (5.51) is concerned, we have, since $k \geq 2\delta$,

$$\int_\Omega {}^tA(x)D(Z_\delta(u)v)DG_k(u) = 0; \tag{5.52}$$

indeed since $u \in H^1_{\mathrm{loc}}(\Omega)$ by (3.10 i) one has

$$\begin{cases} {}^tA(x)D(Z_\delta(u)v)DG_k(u) \\ \displaystyle= -{}^tA(x)DuDu\,\frac{1}{\delta}\chi_{\{\delta<u<2\delta\}}\chi_{\{u>k\}}v + {}^tA(x)DvDu\, Z_\delta(u)\chi_{\{u>k\}} \text{ in } \mathcal{D}'(\Omega), \end{cases}$$

where each term is zero almost everywhere when $k \geq 2\delta$.

As far as the fourth term of the left-hand side of (5.51) is concerned, we have, since $u \in H^1_{\mathrm{loc}}(\Omega)$ and since $k \geq 2\delta$, using (5.44),

$$DZ_\delta(u)\, T_k(u) = Du\, Z'_\delta(u)\, T_k(u) = -Du\, R'_\delta(u) = -DR_\delta(u) \text{ in } \mathcal{D}'(\Omega)$$

which implies, using the fact that $R_\delta(u) \in H_0^1(\Omega)$ (see (5.47)) and (5.36), that

$$\begin{cases} \displaystyle -\int_\Omega {}^tA(x)DvDZ_\delta(u)\, T_k(u) = \int_\Omega {}^tA(x)DvDR_\delta(u) \\ \qquad = \langle -\operatorname{div} {}^tA(x)Dv, R_\delta(u)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \\ \displaystyle\qquad = \sum_{i\in I}\int_\Omega \hat{g}_i D(\hat{\varphi}_i R_\delta(u)) + \int_\Omega \hat{f} R_\delta(u). \end{cases} \tag{5.53}$$

As far as the fifth term of the left-hand side of (5.51) is concerned, we have, since $u \in H^1_{\mathrm{loc}}(\Omega)$,

$$\begin{cases} {}^tA(x)DZ_\delta(u)D(vT_k(u)) \\ = {}^tA(x)DZ_\delta(u)Dv\, T_k(u) + {}^tA(x)DZ_\delta(u)DT_k(u)\, v \text{ in } \mathcal{D}'(\Omega), \end{cases}$$

which implies, using (5.44) and $k \geq 2\delta$, that

$$\begin{cases} \displaystyle\int_\Omega {}^tA(x)DZ_\delta(u)D(vT_k(u)) \\ \displaystyle= \int_\Omega {}^tA(x)DZ_\delta(u)Dv\,T_k(u) - \frac{1}{\delta}\int_{\{\delta<u<2\delta\}} {}^tA(x)DuDu\,v. \end{cases} \tag{5.54}$$

Collecting together (5.51), (5.52), (5.53) and (5.54), we have proved that when $k \geq 2\delta$

$$\begin{cases} \displaystyle\sum_{i\in I}\int_\Omega \hat{g}_i D\Big(\hat{\varphi}_i\big(Z_\delta(u)T_k(u) + R_\delta(u)\big)\Big) + \int_\Omega \hat{f}\,\big(Z_\delta(u)T_k(u) + R_\delta(u)\big) \\ \displaystyle= \int_\Omega F(x,u)Z_\delta(u)v + \frac{1}{\delta}\int_{\{\delta<u<2\delta\}} {}^tA(x)DuDu\,v. \end{cases} \tag{5.55}$$

When $k \geq 2\delta$ one has, using (5.44),

$$\begin{cases} Z_\delta(u)T_k(u) + R_\delta(u) = Z_\delta(u)u + R_\delta(u) = Y_\delta(u) \\ \hat{g}_i D(\hat{\varphi}_i Y_\delta(u)) = \hat{g}_i D\hat{\varphi}_i\, Y_\delta(u) + \hat{g}_i Du\, Z_\delta(u)\hat{\varphi}_i \\ {}^tA(x)DuDu\,v \geq 0,\ \ 0 \leq Y_\delta(u) \leq 3\delta/2 \end{cases}$$

so that estimate (5.37) follows from (5.55). Proposition 5.9 is proved. □

5.4. Regularity of $\beta(u)$ and *a priori* estimate of $\beta(u)$ in $H_0^1(\Omega)$

This *a priori* estimate is actually first a regularity result, since it asserts that for every u solution to problem (2.1) in the sense of Definition 3.6, a certain function $\beta(u)$ actually belongs to $H_0^1(\Omega)$. This property will be used in the proofs of the Comparison Principle 7.1 and of the Uniqueness Theorem 4.3.

Define the function $\beta : s \in [0, +\infty[\to \beta(s) \in [0, +\infty[$ by

$$\beta(s) = \int_0^s \sqrt{\Gamma'(t)}dt, \tag{5.56}$$

where Γ is the function which appears in assumption (2.6).

Proposition 5.13 (Regularity of $\beta(u)$ and *a priori* estimate of $\beta(u)$ in $H_0^1(\Omega)$). *Assume that the matrix A and the function F satisfy* (2.4), (2.5) *and* (2.6). *Then for every u solution to problem* (2.1) *in the sense of Definition* 3.6 *one has*

$$\beta(u) \in H_0^1(\Omega) \tag{5.57}$$

with the a priori *estimate*

$$\alpha\|D\beta(u)\|^2_{(L^2(\Omega))^N} \leq \|h\|_{L^1(\Omega)}. \tag{5.58}$$

Property (5.57) formally implies that $\beta(u) = 0$ on $\partial\Omega$. Since $\beta(s) = 0$ implies that $s = 0$ because of $\Gamma'(s) > 0$ for every $s > 0$, this formally implies that $u = 0$ on $\partial\Omega$ (see also Remark 3.7 ii) above).

Remark 5.14 (Formal proof of Proposition 5.13). Estimate (5.58) can be obtained formally by taking $\Gamma(u)$ as test function in equation (2.1), using the coercivity (2.4) and the growth condition (2.6 iii), which implies that

$$0 \le F(x,u)\Gamma(u) \le \frac{h(x)}{\Gamma(u)}\Gamma(u) \le h(x).$$

This formal computation will be made mathematically correct in the proof below.

Proof of Proposition 5.13. As in (5.5), we define, for every δ and k with $0 < \delta < k$, the function

$$S_{\delta,k}(s) = \begin{cases} 0 & \text{if } 0 \le s \le \delta \\ s - \delta & \text{if } \delta \le s \le k \\ k - \delta & \text{if } k \le s. \end{cases} \tag{5.59}$$

As in the first step of the proof of Proposition 5.1, one can prove that the function $\Gamma(S_{\delta,k}(u))$ belongs to $\mathcal{V}(\Omega)$ and that, if $\psi_\delta : s \in [0,+\infty[\to \psi_\delta(s) \in [0,+\infty[$ is any C^1 nondecreasing function such that

$$\psi_\delta(s) = 0 \ \text{ for } 0 \le s \le \frac{\delta}{2} \ \text{ and } \psi_\delta(s) = 1 \ \text{ for } s \ge \delta$$

then one has $\psi_\delta(u) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and

$$-\operatorname{div} {}^tA(x)D\Gamma(S_{\delta,k}(u)) = \psi_\delta(u)(-\operatorname{div} {}^tA(x)D\Gamma(S_{\delta,k}(u)) \quad \text{in } \mathcal{D}'(\Omega). \tag{5.60}$$

Since $\Gamma(S_{\delta,k}(u)) \in \mathcal{V}(\Omega)$ and since $\Gamma(S_{\delta,k}(u)) \ge 0$, we can use $\Gamma(S_{\delta,k}(u))$ as test function in (3.11 ii). In view of (5.60) we get

$$\begin{cases} \displaystyle\int_\Omega {}^tA(x)D\Gamma(S_{\delta,k}(u))DG_k(u) + \int_\Omega {}^tA(x)D\Gamma(S_{\delta,k}(u))D(\psi_\delta(u)T_k(u)) \\ = \langle -\operatorname{div} {}^tA(x)D\Gamma(S_{\delta,k}(u)), G_k(u)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \\ \quad +\langle\langle -\operatorname{div} {}^tA(x)D\Gamma(S_{\delta,k}(u)), T_k(u)\rangle\rangle_\Omega = \displaystyle\int_\Omega F(x,u)\ \Gamma(S_{\delta,k}(u)) \end{cases}$$

which can be written as

$$\begin{cases} \displaystyle\int_\Omega {}^tA(x)D\Gamma(S_{\delta,k}(u))DG_k(u) + \int_\Omega {}^tA(x)D\Gamma(S_{\delta,k}(u))D\psi_\delta(u)\ T_k(u) \\ + \displaystyle\int_\Omega {}^tA(x)D\Gamma(S_{\delta,k}(u))DT_k(u)\ \psi_\delta(u) = \int_\Omega F(x,u)\ \Gamma(S_{\delta,k}(u)). \end{cases} \tag{5.61}$$

Note that the two first integrals of (5.61) are zero, since $D\Gamma(S_{\delta,k}(u))$ is zero outside of the set $\{\delta < u < k\}$, while $DG_k(u)$ is zero outside of the set $\{u > k\}$ and $D\psi_\delta(u)$ is zero outside of the set $\{u < \delta\}$.

Note also that $\psi_\delta(u) = 1$ in the set $\{u \geq \delta\}$ while $DS_{\delta,k}(u) = 0$ outside of this set, and that $DT_k(u) = DS_{\delta,k}(u)$ in the set $\{\delta < u < k\}$. Therefore in view of (5.56) the third term of (5.61) can be written as

$$\begin{cases} \displaystyle\int_\Omega {}^tA(x)D\Gamma(S_{\delta,k}(u))DT_k(u)\psi_\delta(u) \\ \displaystyle= \int_\Omega {}^tA(x)DS_{\delta,k}(u)DT_k(u)\Gamma'(S_{\delta,k}(u))\psi_\delta(u) \\ \displaystyle= \int_\Omega {}^tA(x)DS_{\delta,k}(u)DS_{\delta,k}(u)\Gamma'(S_{\delta,k}(u)) \\ \displaystyle\geq \alpha\int_\Omega |DS_{\delta,k}(u)|^2\Gamma'(S_{\delta,k}(u)) = \alpha\int_\Omega |D\beta(S_{\delta,k}(u))|^2. \end{cases} \tag{5.62}$$

As for the right-hand side of (5.61), we have, using the growth condition (2.6 iii) and the fact that Γ is nondecreasing,

$$F(x,u)\Gamma(S_{\delta,k}(u)) \leq \frac{h(x)}{\Gamma(u)}\Gamma(S_{\delta,k}(u)) \leq h(x) \ \text{ a.e. } x \in \Omega. \tag{5.63}$$

From (5.61), (5.62) and (5.63) we get

$$\alpha\int_\Omega |D\beta(S_{\delta,k}(u))|^2 \leq \|h\|_{L^1(\Omega)} \ \ \forall\delta,\ \forall k,\ 0 < \delta < k. \tag{5.64}$$

On the other hand, the fact that for every δ with $0 < \delta < k$ one has

$$\begin{cases} |D\beta(S_{\delta,k}(u))| = |\beta'(S_{\delta,k}(u))\chi_{\{\delta<u<k\}}Du| \leq \left(\displaystyle\sup_{0\leq s\leq k} \beta'(s)\right) |DG_\delta(u)| \\ 0 \leq \beta(S_{\delta,k}(u)) \leq \beta(G_\delta(u)) \end{cases} \tag{5.65}$$

together with condition (3.10 iii) and Lemma A.1 of Appendix A below imply that $\beta(S_{\delta,k}(u)) \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Since $\beta(S_{\delta,k}(u))$ is bounded in $H_0^1(\Omega)$ independently of δ and k in view of (5.64), and since $\beta(S_{\delta,k}(u))$ tends almost everywhere to $\beta(u)$ as δ tends to zero and k tends to infinity, we have proved that $\beta(u) \in H_0^1(\Omega)$, *i.e.* (5.57). Using the weak lower semicontinuity of the norm in (5.64) then proves (5.58).

Proposition 5.13 is proved. □

6. Proofs of the Stability Theorem 4.2 and of the Existence Theorem 4.1

6.1. Proof of the Stability Theorem 4.2

First step. Since for every n the function $F_n(x,s)$ satisfies assumptions (2.5) and (2.6) for the same h and the same Γ, every solution u_n to problem $(2.1)_n$ in the sense of Definition 3.6 satisfies the *a priori* estimates (5.1), (5.3), (5.15), (5.16), (5.17), (5.18) and (5.37).

Therefore u_n is bounded in $L^2(\Omega)$ and in $H^1_{\mathrm{loc}}(\Omega)$, and there exist a subsequence, still labelled by n, and a function u_∞ such that

$$\begin{cases} u_n \rightharpoonup u_\infty \ \text{ in } \ L^2(\Omega) \ \text{ weakly, in } \ H^1_{\mathrm{loc}}(\Omega) \ \text{ weakly and a.e. in } \ \Omega \\ G_k(u_n) \rightharpoonup G_k(u_\infty) \ \text{ in } \ H^1_0(\Omega) \text{ weakly } \forall k > 0 \\ \varphi T_k(u_n) \rightharpoonup \varphi T_k(u_\infty) \ \text{ in } \ H^1_0(\Omega) \text{ weakly } \forall k>0,\ \forall \varphi \in H^1_0(\Omega) \cap L^\infty(\Omega). \end{cases} \tag{6.1}$$

Since $u_n \geq 0$, the function u_∞ satisfies $u_\infty \geq 0$ and therefore (3.10).

Fix now a function $v \in \mathcal{V}(\Omega),\ v \geq 0$, with

$$\begin{cases} -\operatorname{div} {}^tA(x)Dv = \sum_{i\in I} \hat{\varphi}_i(-\operatorname{div} \hat{g}_i) + \hat{f} \text{ in } \mathcal{D}'(\Omega) \\ \text{where } \hat{\varphi}_i \in H^1_0(\Omega) \cap L^\infty(\Omega),\ \hat{g}_i \in \left(L^2(\Omega)\right)^N,\ \hat{f} \in L^1(\Omega). \end{cases}$$

Using v as test function in (3.11 ii)$_n$, we obtain

$$\begin{cases} \displaystyle\int_\Omega {}^tA(x)DvDG_k(u_n) + \sum_{i\in I}\int_\Omega \hat{g}_i D(\hat{\varphi}_i T_k(u_n)) + \int_\Omega \hat{f} T_k(u_n) \\ = \langle -\operatorname{div} {}^tA(x)Dv, G_k(u_n)\rangle_{H^{-1}(\Omega),H^1_0(\Omega)} + \langle\langle -\operatorname{div} {}^tA(x)Dv, T_k(u_n)\rangle\rangle_\Omega \\ \displaystyle = \int_\Omega F_n(x,u_n)v. \end{cases} \tag{6.2}$$

Since the left-hand side of (6.2) is bounded independently of n for every $k > 0$ fixed in view of the estimates (5.1) and (5.16), we have

$$\int_\Omega F_n(x,u_n)v \leq C(v) < +\infty \quad \forall n$$

which gives, using the almost everywhere convergence of u_n to u_∞, assumption (4.1) on the functions F_n and Fatou's lemma,

$$\int_\Omega F_\infty(x,u_\infty)v \leq C(v) < +\infty \tag{6.3}$$

namely (3.11 i)$_\infty$.

Second step. It remains to prove that (3.11 ii)$_\infty$ holds true and that the convergences in (6.1) are strong.

For $\delta > 0$ fixed, we recall the definition (5.35) of the function Z_δ and we write the right-hand side of (6.2) as

$$\begin{cases} \displaystyle\int_\Omega {}^tA(x)DvDG_k(u_n) + \sum_{i\in I}\int_\Omega \hat{g}_i D(\hat{\varphi}_i T_k(u_n)) + \int_\Omega \hat{f} T_k(u_n) \\ \displaystyle = \int_\Omega F_n(x,u_n)Z_\delta(u_n)v + \int_\Omega F_n(x,u_n)(1 - Z_\delta(u_n))v. \end{cases} \tag{6.4}$$

Using (6.1), it is easy to pass to the limit in the left-hand side of (6.4), obtaining

$$\begin{cases} \displaystyle\int_\Omega {}^tA(x)DvDG_k(u_n) + \sum_{i\in I}\int_\Omega \hat{g}_i D(\hat{\varphi}_i T_k(u_n)) + \int_\Omega \hat{f} T_k(u_n) \\ \displaystyle\to \int_\Omega {}^tA(x)DvDG_k(u_\infty) + \sum_{i\in I}\int_\Omega \hat{g}_i D(\hat{\varphi}_i T_k(u_\infty)) + \int_\Omega \hat{f} T_k(u_\infty) \\ \text{as } n \to +\infty. \end{cases} \tag{6.5}$$

For the first term of the right-hand side of (6.4) we use the *a priori* estimate (5.37), namely

$$\begin{cases} \displaystyle\forall\,\delta > 0,\ \int_\Omega F_n(x,u_n)Z_\delta(u_n)v \\ \displaystyle\qquad\qquad \le \frac{3}{2}\left(\int_\Omega \left|\sum_{i\in I}\hat{g}_i D\hat{\varphi}_i + \hat{f}\right|\right)\delta + \int_\Omega Z_\delta(u_n)\sum_{i\in I}\hat{g}_i Du_n\,\hat{\varphi}_i \end{cases}$$

in which we pass to the limit in n for $\delta > 0$ fixed. Since $Z_\delta(u_n)\hat{g}_i$ tends strongly to $Z_\delta(u_\infty)\hat{g}_i$ in $(L^2(\Omega))^N$ while $Du_n\,\hat{\varphi}_i$ tends weakly to $Du_\infty\,\hat{\varphi}_i$ in $(L^2(\Omega))^N$ (see (5.17)), we obtain

$$\begin{cases} \displaystyle\forall\,\delta>0,\ \limsup_n \int_\Omega F_n(x,u_n)Z_\delta(u_n)v \\ \displaystyle\qquad\qquad \le \frac{3}{2}\left(\int_\Omega \left|\sum_{i\in I}\hat{g}_i D\hat{\varphi}_i + \hat{f}\right|\right)\delta + \int_\Omega Z_\delta(u_\infty)\sum_{i\in I}\hat{g}_i Du_\infty\,\hat{\varphi}_i. \end{cases} \tag{6.6}$$

Since $Z_\delta(u_\infty)$ tends to $\chi_{\{u_\infty=0\}}$ almost everywhere in Ω as δ tends to zero, and since $u_\infty \in H^1_{\mathrm{loc}}(\Omega)$ implies that $Du_\infty = 0$ almost everywhere in $\{x \in \Omega : u_\infty(x) = 0\}$, the right-hand side of (6.6) tends to 0 as δ tends to zero.

We have proved that the first term of the right-hand side of (6.4) satisfies

$$\limsup_n \int_\Omega F_n(x,u_n)Z_\delta(u_n)v \to 0 \text{ as } \delta \to 0. \tag{6.7}$$

Third step. In this step we prove that

$$\int_{\{u_\infty=0\}} F_\infty(x,u_\infty)v = 0. \tag{6.8}$$

In view of assumption (4.1) on the convergence of the functions $F_n(x,s)$, of the continuity of the function Z_δ, and of the almost everywhere convergence of u_n to u_∞, one has for every $\delta > 0$ fixed

$$F_n(x,u_n)Z_\delta(u_n)v \to F_\infty(x,u_\infty)Z_\delta(u_\infty)v \text{ a.e. in } \Omega \text{ as } n \to +\infty. \tag{6.9}$$

By Fatou's lemma this implies that

$$\int_\Omega F_\infty(x, u_\infty) Z_\delta(u_\infty) v \le \liminf_{n\to+\infty} \int_\Omega F_n(x, u_n) Z_\delta(u_n) v \ \forall \delta > 0. \tag{6.10}$$

Since $Z_\delta(s)$ tends to $\chi_{\{s=0\}}$ for every $s \ge 0$ as δ tends to zero and since $F_\infty(x, u_\infty)v$ belongs to $L^1(\Omega)$ by (6.3), the left-hand side of (6.10) tends to $\int_\Omega F_\infty(x, u_\infty)\chi_{\{u_\infty=0\}} v$ as δ tends to zero.

Combining the latest result with (6.10) and (6.7) proves (6.8).

Fourth step. In this step we pass to the limit, first as n tends to infinity for $\delta > 0$ fixed, and then as δ tends to zero, in the second term of the right-hand side of (6.4), namely in the term

$$\int_\Omega F_n(x, u_n)(1 - Z_\delta(u_n))v.$$

For that we observe that

$$0 \le 1 - Z_\delta(s) \le \chi_{\{s\ge\delta\}}(s) \ \forall s \ge 0, \ \ \forall \delta > 0$$

which combined with the growth condition (2.6 iii) and the fact that the function Γ is nondecreasing gives

$$0 \le F_n(x, u_n)(1 - Z_\delta(u_n))v \le \frac{h(x)}{\Gamma(u_n)} \chi_{\{u_n\ge\delta\}} v \le \frac{h(x)}{\Gamma(\delta)} v. \tag{6.11}$$

Since the right-hand side of (6.11) belongs to $L^1(\Omega)$, and since, for every $\delta > 0$ fixed, one has, as in (6.9),

$$F_n(x, u_n)(1 - Z_\delta(u_n))v \to F_\infty(x, u_\infty)(1 - Z_\delta(u_\infty))v \ \text{ a.e. in } \Omega \ \text{ as } n \to +\infty$$

Lebesgue's dominated convergence theorem implies that for every $\delta > 0$ fixed one has

$$\int_\Omega F_n(x, u_n)(1 - Z_\delta(u_n))v \to \int_\Omega F_\infty(x, u_\infty)(1 - Z_\delta(u_\infty))v \ \text{ as } n \to +\infty. \tag{6.12}$$

Since the right-hand side of (6.12) tends to $\int_\Omega F_\infty(x, u_\infty)(1 - \chi_{\{u_\infty=0\}})v$ as δ tends to zero, which is equal to $\int_\Omega F_\infty(x, u_\infty)v$ in view of (6.8), we have proved that

$$\lim_n \int_\Omega F_n(x, u_n)(1 - Z_\delta(u_n))v \to \int_\Omega F_\infty(x, u_\infty)v \ \text{ as } \delta \to 0. \tag{6.13}$$

Fifth step. Collecting the results obtained in (6.5), (6.7) and (6.13), we have passed to the limit in each term of (6.4), first in n for $\delta > 0$ fixed, and then in δ. This proves that for every $v \in \mathcal{V}(\Omega)$, $v \geq 0$, one has

$$\int_\Omega {}^t A(x) Dv DG_k(u_\infty) + \sum_{i \in I} \int_\Omega \hat{g}_i D(\hat{\varphi}_i T_k(u_\infty)) + \int_\Omega \hat{f} T_k(u_\infty) = \int_\Omega F_\infty(x, u_\infty) v$$

which is nothing but $(3.11\text{ ii})_\infty$.

We have thus proved that u_∞ is a solution to problem $(2.1)_\infty$ in the sense of Definition 3.6. It only remains to prove that the convergences in (6.1) are strong (see (4.2)).

To this aim it is sufficient to prove the two following strong convergences

$$G_k(u_n) \to G_k(u_\infty) \text{ in } H_0^1(\Omega) \text{ strongly } \forall k > 0 \tag{6.14}$$

$$\begin{cases} \varphi DT_k(u_n) \to \varphi DT_k(u_\infty) \text{ in } (L^2(\Omega))^N \text{ strongly} \\ \forall k > 0, \ \forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega); \end{cases} \tag{6.15}$$

indeed the strong convergence of $T_k(u_n)$ in $L^2(\Omega)$ follows from Lebesgue's dominated convergence theorem and from the almost everywhere convergence of u_n (see (6.1)); since $u_n = T_k(u_n) + G_k(u_n)$, this convergence together with (6.14) and (6.15) implies the strong convergences of u_n in $L^2(\Omega)$, of $\varphi T_k(u_n)$ in $H_0^1(\Omega)$, and of u_n in $H^1_{\text{loc}}(\Omega)$.

Sixth step. In this step we prove the strong convergence (6.14).

This strong convergence follows from the energy equality $(5.4)_n$, namely

$$\int_\Omega A(x) DG_k(u_n) DG_k(u_n) = \int_\Omega F_n(x, u_n) G_k(u_n) \quad \forall k > 0, \tag{6.16}$$

a fact that we proved in the third step of the proof of Proposition 5.1 (see the comment at the end of Subsection 5.1).

It is easy to pass to the limit in the right-hand side of (6.16). Indeed the inequality

$$0 \leq F_n(x, u_n) G_k(u_n) \leq \frac{h(x)}{\Gamma(u_n)} \chi_{\{u_n \geq k\}} G_k(u_n) \leq \frac{h(x)}{\Gamma(k)} G_k(u_n)$$

and the boundedness of $G_k(u_n)$ in $H_0^1(\Omega)$ (see (6.1)), and therefore in $L^{2^*}(\Omega)$, imply that the sequence $F_n(x, u_n) G_k(u_n)$ is equintegrable in $L^1(\Omega)$ uniformly in n; since

$$F_n(x, u_n) G_k(u_n) \to F_\infty(x, u_\infty) G_k(u_\infty) \quad \text{a.e. in } \Omega \text{ as } n \to +\infty,$$

Vitali's theorem implies that for every $k > 0$ fixed

$$\int_\Omega F_n(x, u_n) G_k(u_n) \to \int_\Omega F_\infty(x, u_\infty) G_k(u_\infty) \text{ as } n \to +\infty. \tag{6.17}$$

On the other hand, the energy equality $(5.4)_\infty$ asserts that

$$\int_\Omega A(x)DG_k(u_\infty)DG_k(u_\infty) = \int_\Omega F_\infty(x, u_\infty)G_k(u_\infty) \quad \forall k > 0. \tag{6.18}$$

Collecting together (6.16), (6.17) and (6.18), we have proved that for every $k > 0$ fixed

$$\int_\Omega A(x)DG_k(u_n)DG_k(u_n) \to \int_\Omega A(x)DG_k(u_\infty)DG_k(u_\infty) \text{ as } n \to +\infty,$$

which, with the weak convergence in $(L^2(\Omega))^N$ of $DG_k(u_n)$ to $DG_k(u_\infty)$ (see (6.1)), implies the strong convergence (6.14).

Seventh step. In this step we prove the strong convergence (6.15).

This strong convergence follows from equation $(5.27)_n$, namely

$$\begin{cases} \displaystyle\int_\Omega A(x)DT_k(u_n)DT_k(u_n)\,\varphi^2 \\ \displaystyle= -2\int_\Omega A(x)DT_k(u_n)D\varphi\,\varphi T_k(u_n) - 2\int_\Omega A(x)DG_k(u_n)D\varphi\,\varphi T_k(u_n) \\ \displaystyle\quad + \int_\Omega F_n(x, u_n)\varphi^2 T_k(u_n). \end{cases} \tag{6.19}$$

We first pass to the limit in the right-hand side of (6.19) as n tends to $+\infty$. In view of (6.1) one has, for the first two terms of this right-hand side,

$$\begin{cases} \displaystyle\quad -2\int_\Omega A(x)DT_k(u_n)D\varphi\,\varphi T_k(u_n) - 2\int_\Omega A(x)DG_k(u_n)D\varphi\,\varphi T_k(u_n) \\ \displaystyle\to -2\int_\Omega A(x)DT_k(u_\infty)D\varphi\,\varphi T_k(u_\infty) - 2\int_\Omega A(x)DG_k(u_\infty)D\varphi\,\varphi T_k(u_\infty) \\ \text{as } n \to +\infty. \end{cases} \tag{6.20}$$

As far as the last integral of the right-hand side of (6.19) is concerned, we recall the definition (5.35) of the function Z_δ and we write, for every fixed $\delta > 0$,

$$\begin{cases} \displaystyle\int_\Omega F_n(x, u_n)\varphi^2 T_k(u_n) \\ \displaystyle= \int_\Omega F_n(x, u_n)Z_\delta(u_n)\varphi^2 T_k(u_n) + \int_\Omega F_n(x, u_n)(1 - Z_\delta(u_n))\varphi^2 T_k(u_n). \end{cases} \tag{6.21}$$

For the first term of the right-hand side of (6.21), we use the *a priori* estimate (5.37) with $v = \varphi^2$; indeed $\varphi^2 \in \mathcal{V}(\Omega)$ with (see (3.8))

$$\begin{cases} -\operatorname{div} {}^tA(x)D\varphi^2 = \hat{\varphi}(-\operatorname{div} \hat{g}) + \hat{f} \text{ in } \mathcal{D}'(\Omega) \\ \text{where } \hat{\varphi} = 2\varphi,\ \hat{g} = {}^tA(x)D\varphi,\ \hat{f} = -2\,{}^tA(x)D\varphi D\varphi. \end{cases}$$

This yields, since $\hat{g} D\hat{\varphi} + \hat{f} = 0$,

$$\begin{cases} 0 \le \displaystyle\int_\Omega F_n(x, u_n) Z_\delta(u_n) \varphi^2 T_k(u_n) \le k \int_\Omega F_n(x, u_n) Z_\delta(u_n) \varphi^2 \\ \quad \le 2k \displaystyle\int_\Omega Z_\delta(u_n)\, {}^t A(x) D\varphi D u_n\, \varphi \\ \quad = 2k \displaystyle\int_\Omega Z_\delta(u_n)\, {}^t A(x) D\varphi D T_k(u_n) \varphi + 2k \int_\Omega Z_\delta(u_n)\, {}^t A(x) D\varphi D G_k(u_n)\, \varphi. \end{cases} \tag{6.22}$$

We pass to the limit in the right-hand side of (6.22), first for $\delta > 0$ fixed as n tends to infinity thanks to (6.1), and then as δ tends to 0. Since $Z_\delta(u_\infty)$ tends to $\chi_{\{u_\infty = 0\}}$ in $L^\infty(\Omega)$ weak-star as δ tends to zero, and since $Du_\infty = 0$ almost everywhere in $\{u_\infty = 0\}$, we obtain that

$$\limsup_n \int_\Omega F_n(x, u_n) Z_\delta(u_n) \varphi^2 T_k(u_n) \to 0 \ \text{ as } \delta \to 0. \tag{6.23}$$

For the second term of the right-hand side of (6.21), we repeat the proof that we performed in the fourth step above to prove (6.13), and we obtain that

$$\begin{cases} \displaystyle\lim_n \int_\Omega F_n(x, u_n)(1 - Z_\delta(u_n)) \varphi^2 T_k(u_n) \to \int_\Omega F_\infty(x, u_\infty) \varphi^2 T_k(u_\infty) \\ \text{as } \ \delta \to 0. \end{cases} \tag{6.24}$$

On the other hand, equation $(5.27)_\infty$ reads as

$$\begin{cases} \quad \displaystyle\int_\Omega A(x) D T_k(u_\infty) D T_k(u_\infty)\, \varphi^2 \\ = -2 \displaystyle\int_\Omega A(x) D T_k(u_\infty) D\varphi\, \varphi T_k(u_\infty) - 2 \int_\Omega A(x) D G_k(u_\infty) D\varphi\, \varphi T_k(u_\infty) \\ \quad + \displaystyle\int_\Omega F_\infty(x, u_\infty) \varphi^2 T_k(u_\infty). \end{cases} \tag{6.25}$$

Collecting together (6.19), (6.20), (6.21), (6.23), (6.24) and (6.25), we have proved that for every $k > 0$ fixed

$$\int_\Omega A(x) D T_k(u_n) D T_k(u_n)\, \varphi^2 \to \int_\Omega A(x) D T_k(u_\infty) D T_k(u_\infty)\, \varphi^2 \ \text{ as } n \to \infty$$

which implies, with the weak convergence in $(L^2(\Omega))^N$ of $\varphi D T_k(u_n)$ to $\varphi D T_k(u_\infty)$, the strong convergence (6.15).

This completes the proof of the Stability Theorem 4.2.

6.2. Proof of the Existence Theorem 4.1

Consider the problem

$$\begin{cases} u_n \in H_0^1(\Omega) \\ \displaystyle\int_\Omega A(x)Du_n Dw = \int_\Omega T_n(F(x,u_n^+))w \quad \forall w \in H_0^1(\Omega) \end{cases} \tag{6.26}$$

where T_n is the truncation at height n.

Since

$$T_n(F(x,s^+)) : (x,s) \in \Omega\times]-\infty,+\infty[\to T_n(F(x,s^+)) \in [0,n]$$

is a Carathéodory function which is (almost everywhere in Ω) bounded by n, Schauder's fixed point theorem implies that problem (6.26) has at least a solution. Moreover, since $T_n(F(x,s^+)) \geq 0$, every solution of (6.26) is nonnegative by the weak maximum principle, so that $u_n \geq 0$, and $T_n(F(x,u_n^+)) = T_n(F(x,u_n))$.

Define now the function F_n by

$$F_n(x,s) = T_n(F(x,s)) \ \text{ a.e. } x \in \Omega, \ \forall s \geq 0.$$

For every given n, the function F_n is bounded, and it is easy to see that every u_n solution to (6.26) is a solution to problem $(2.1)_n$ in the sense of Definition 3.6, where $(2.1)_n$ is the problem (2.1) where the function F has been replaced by F_n: indeed $u_n \geq 0$ belongs to $H_0^1(\Omega)$, and therefore satisfies $(3.10)_n$ and $(3.11\text{ i})_n$; u_n also satisfies $(3.11\text{ ii})_n$ in view of

$$\begin{cases} \langle\langle -\operatorname{div} {}^tA(x)Dv, T_k(u_n)\rangle\rangle_\Omega = \langle -\operatorname{div} {}^tA(x)Dv, T_k(u_n)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \\ \forall v \in \mathcal{V}(\Omega) \end{cases} \tag{6.27}$$

which follows from (3.5) by taking $y = T_k(u_n) \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

It is clear that the functions F_n satisfy (2.5) and (2.6) with the functions h and Γ which appear in the definition of the function F. Moreover it is not difficult (even if this is a little bit delicate in the case where $s_\infty = 0$ and where $F(x,0) = +\infty$; in this case one can use the fact that $F_n(x,s_n) = T_n(F(x,s_n)) \geq T_m(F(x,s_n))$ for every $n \geq m$, pass to the limit in n for m fixed and then to the limit in m) to verify that the functions F_n satisfy (4.1) with F_∞ given by $F_\infty(x,s) = F(x,s)$.

The Stability Theorem 4.2 then implies that there exist a subsequence and a function u_∞ which is a solution to problem (2.1) in the sense of Definition 3.6 such that the convergences (4.2) hold true.

This proves the Existence Theorem 4.1.

7. Comparison Principle and proof of the Uniqueness Theorem 4.3

In this section we state and prove a Comparison Principle which uses assumption (2.7), namely the fact that $F(x,s)$ is nonincreasing with respect to s. Note that we never use this assumption in the present paper, except in this Comparison Principle and in the Uniqueness Theorem 4.3 which is an easy consequence of it.

Proposition 7.1 (Comparison Principle). *Assume that the matrix A satisfies* (2.4). *Let $F_1(x, s)$ and $F_2(x, s)$ be two functions satisfying* (2.5) *and* (2.6) (*possibly for different functions h and Γ*). *Assume moreover that*

$$\text{either } F_1(x, s) \text{ or } F_2(x, s) \text{ is nonincreasing in } s, \text{ i.e. satisfies (2.7)} \tag{7.1}$$

and that

$$F_1(x, s) \leq F_2(x, s) \;\; a.e. \; x \in \Omega, \quad \forall s \geq 0. \tag{7.2}$$

Let u_1 and u_2 be any solutions in the sense of Definition 3.6 *to problems* $(2.1)_1$ *and* $(2.1)_2$, *where* $(2.1)_1$ *and* $(2.1)_2$ *are* (2.1) *with $F(x, s)$ replaced respectively by $F_1(x, s)$ and $F_2(x, s)$. Then one has*

$$u_1(x) \leq u_2(x) \; a.e. \; x \in \Omega. \tag{7.3}$$

Remark 7.2. The proof of the Comparison Principle of Proposition 7.1 is based on the use of the test function $\psi^2 = (B_1(T_k^+(u_1 - u_2)))^2$ (see the proof below). A similar test function has been used by L. Boccardo and J. Casado-Díaz in [3] to prove the uniqueness of the solution to problem (2.1) obtained by approximation.

Note also that the use of this test function is allowed by the regularity property (5.57) proved in Proposition 5.13.

Proof of the Uniqueness Theorem 4.3. Applying the Comparison Principle to the case where $F_1(x, s) = F_2(x, s) = F(x, s)$ with $F(x, s)$ satisfying (2.7) immediately proves the Uniqueness Theorem 4.3. □

Proof of Proposition 7.1.

First step. Let $k > 0$ be fixed. Define ψ by

$$\psi = B_1\big(T_k^+(u_1 - u_2)\big) \tag{7.4}$$

where $B_1 : s \in [0, +\infty[\to B_1(s) \in [0, +\infty[$ is the function defined by

$$B_1(s) = \int_0^s \beta_1(t) dt \;\forall s \geq 0, \quad \beta_1(t) = \int_0^t \sqrt{\Gamma_1'(r)} dr \;\; \forall t \geq 0,$$

where Γ_1 is the function for which F_1 satisfies assumption (2.6).

In this step we will prove that

$$\psi \in H_0^1(\Omega) \cap L^\infty(\Omega). \tag{7.5}$$

Since u_1 and u_2 belong to $H^1_{\text{loc}}(\Omega)$, one has $T_k^+(u_1 - u_2) \in H^1_{\text{loc}}(\Omega) \cap L^\infty(\Omega)$; since $\beta_1 \in C^0([0, +\infty[)$, the function ψ belongs to $H^1_{\text{loc}}(\Omega) \cap L^\infty(\Omega)$ and one has

$$D\psi = \beta_1\big(T_k^+(u_1 - u_2)\big)\chi_{\{0<u_1-u_2<k\}}(Du_1 - Du_2) \;\text{ in } \mathcal{D}'(\Omega). \tag{7.6}$$

Since $0 \le T_k^+(s_1 - s_2) \le T_k(s_1)$ for $s_1 \ge 0$ and $s_2 \ge 0$, and since β_1 is nondecreasing, this implies that

$$|D\psi| \le \beta_1(T_k(u_1))(|Du_1| + |Du_2|).$$

But in view of (5.57), $\beta_1(T_k(u_1)) = T_{\beta_1(k)}(\beta_1(u_1))$ belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$, and then the property (3.16) for u_1 and u_2 implies that $D\psi \in (L^2(\Omega))^N$, and therefore that $\psi \in H^1(\Omega) \cap L^\infty(\Omega)$.

Since B_1 is nondecreasing one has $0 \le \psi \le B_1(T_k(u_1))$. We now claim that

$$B_1(T_k(u_1)) \in H_0^1(\Omega) \tag{7.7}$$

which by Lemma A.1 of Appendix A below completes the proof of (7.5).

Let us now prove (7.7). For every δ with $0 < \delta < k$ and for the function $S_{\delta,k}$ defined by (5.59) one has

$$\begin{cases} |DB_1(S_{\delta,k}(u_1))| = |\beta_1(S_{\delta,k}(u_1))\chi_{\{\delta < u_1 < k\}} Du_1| \le \left(\sup\limits_{0\le s\le k} \beta_1(s)\right) |DG_\delta(u_1)| \\ 0 \le B_1(S_{\delta,k}(u_1)) \le B_1(G_\delta(u_1)). \end{cases}$$

Then (3.10 iii) and Lemma A.1 imply that

$$B_1(S_{\delta,k}(u_1)) \in H_0^1(\Omega) \cap L^\infty(\Omega). \tag{7.8}$$

On the other hand, since β_1 is nondecreasing, one has also

$$\begin{cases} \quad |DB_1(S_{\delta,k}(u_1))| = |\beta_1(S_{\delta,k}(u_1))\chi_{\{\delta < u_1 < k\}} Du_1| \\ \le \beta_1(T_k(u_1))|Du_1| = T_{\beta_1(k)}(\beta_1(u_1))|Du_1|. \end{cases} \tag{7.9}$$

In view of (5.57), $T_{\beta_1(k)}(\beta_1(u_1))$ belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$ and therefore the right-hand side of (7.9) belongs to $L^2(\Omega)$ by (3.16). Then, for k fixed, $B_1(S_{\delta,k}(u_1))$, which belongs to $H_0^1(\Omega)$ by (7.8), is bounded in $H_0^1(\Omega)$ independently of δ when $0 < \delta < k$. Since $B_1(S_{\delta,k}(u_1))$ tends to $B_1(T_k(u))$ almost everywhere as δ tends to zero, this proves that $B_1(T_k(u)) \in H_0^1(\Omega)$, *i.e.* (7.7), a fact which, as said above, implies (7.5).

Second step. Since $\psi^2 \in \mathcal{V}(\Omega)$ in view of (7.5) and of Remark 3.5 ii) and since $\psi^2 \ge 0$, we can take $v = \psi^2$ as test function in (3.11 ii)$_1$ and (3.11 ii)$_2$. Taking the difference of these two equations, we get (see (3.8))

$$\begin{cases} 2\displaystyle\int_\Omega \psi\, {}^tA(x)D\psi\, D(G_k(u_1) - G_k(u_2)) \\ +2\displaystyle\int_\Omega {}^tA(x)D\psi\, D(\psi(T_k(u_1) - T_k(u_2))) \\ -2\displaystyle\int_\Omega {}^tA(x)D\psi\, D\psi\, (T_k(u_1) - T_k(u_2)) \\ = \langle -\operatorname{div} {}^tA(x)D\psi^2, G_k(u_1) - G_k(u_2)\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \\ \quad +\langle\langle -\operatorname{div} {}^tA(x)D\psi^2, T_k(u_1) - T_k(u_2)\rangle\rangle_\Omega \\ = \displaystyle\int_\Omega (F_1(x, u_1) - F_2(x, u_2))\, \psi^2. \end{cases} \tag{7.10}$$

Expanding in $L^1_{\mathrm{loc}}(\Omega)$ the integrands of the three first lines of (7.10), one realizes that their sum is nothing but $2\psi\, {}^tA(x)D\psi\, D(u_1 - u_2)$, which belongs to $L^1(\Omega)$ in view of (3.16). Therefore (7.10) is nothing but

$$2\int_\Omega \psi\, {}^tA(x)D\psi\, D(u_1 - u_2) = \int_\Omega (F_1(x, u_1) - F_2(x, u_2))\, \psi^2 \tag{7.11}$$

which is formally easily obtained by taking ψ^2 as test function in $(2.1)_1$ and $(2.1)_2$ and making the difference.

Third step. Let us now prove that for ψ given by (7.4) one has

$$(F_1(x, u_1) - F_2(x, u_2))\, \psi^2 \le 0 \quad \text{a.e. in } \Omega. \tag{7.12}$$

Since $F_1(x, u_1)$ and $F_2(x, u_2)$ belong to $L^1_{\mathrm{loc}}(\Omega)$ (see (3.12)) and are therefore finite almost everywhere in Ω, one has

$$(F_1(x, u_1) - F_2(x, u_2))\, \psi^2 = 0 \quad \text{a.e. in the set } \{\psi^2 = 0\} \tag{7.13}$$

since there are no indeterminacies of the types $(\infty - \infty)$ and $\infty \times 0$ in the latest formula.

In the set $\{\psi^2 > 0\}$, one has $u_1 > u_2$ almost everywhere. If $F_1(x, s)$ is nonincreasing with respect to s, one has, using first this nonincreasing character and then assumption (7.2)

$$F_1(x, u_1) - F_2(x, u_2) \le F_1(x, u_2) - F_2(x, u_2) \le 0 \quad \text{a.e. in the set } \{\psi^2 > 0\} \tag{7.14}$$

since there is no indeterminacy of the type $(\infty - \infty)$ in the latest formula. Together with (7.13), the result (7.14) implies that in the case where $F_1(x, s)$ is nonincreasing with respect to s, one has (7.12).

The case where $F_2(x, s)$ is nonincreasing with respect to s is similar, using first assumption (7.2) and then this nonincreasing character.

We have proved (7.12).

Fourth step. Collecting together (7.11) and (7.12) gives

$$2\int_\Omega B_1\big(T_k^+(u_1-u_2)\big)\beta_1\big(T_k^+(u_1-u_2)\big)\,{}^tA(x)DT_k^+(u_1-u_2)D(u_1-u_2)\le 0.$$

Defining the function M_1 by

$$M_1(s)=\int_0^s\sqrt{B_1(t)\beta_1(t)}dt\quad \forall s\ge 0$$

the latest inequality together with the coercivity (2.4) of the matrix A implies that $DM_1(T_k^+(u_1-u_2))=0$ in Ω. Therefore $M_1(T_k^+(u_1-u_2))$ is a nonnegative constant in each connected component ω of Ω. Since the function M_1 is (strictly) increasing, there exists a nonnegative constant C_ω such that $T_k^+(u_1-u_2)=C_\omega$ in ω. Therefore one has $\beta_1(T_k^+(u_1-u_2))=\beta_1(C_\omega)$ in ω. Since $T_k^+(u_1-u_2)\le T_k(u_1)$ in Ω and since the function β_1 is nondecreasing, one has

$$0\le\beta_1(C_\omega)=\beta_1\big(T_k^+(u_1-u_2)\big)\le\beta_1(T_k(u_1))\text{ in }\omega.$$

By the regularity property (5.57), the function $\beta_1(T_k(u_1))$ belongs to $H_0^1(\Omega)$ and therefore to $H_0^1(\omega)$ for every connected component ω of Ω. By Lemma A.1 this implies that $\beta_1(C_\omega)\in H_0^1(\omega)$, and therefore that $C_\omega=0$. This proves that $T_k^+(u_1-u_2)=0$ in ω for each ω, and therefore in Ω. Since $k>0$, this proves (7.3). □

Appendix A. An useful lemma

In this Appendix we state and prove the following lemma which is used many times in the present paper.

Lemma A.1. *If $y\in H^1(\Omega)$ and if there exist $\underline{y}$ and $\overline{y}\in H_0^1(\Omega)$ such that*

$$\underline{y}\le y\le\overline{y}\quad a.e.\ in\ \Omega$$

then $y\in H_0^1(\Omega)$.

Remark A.2. Lemma A.1 is straightforward when $\partial\Omega$ is sufficiently smooth so that the traces of the functions of $H^1(\Omega)$ are defined. Note that we did not assume any smoothness of $\partial\Omega$ in the present paper.

Proof of Lemma A.1. Since $0\le y-\underline{y}\le\overline{y}-\underline{y}$, it is sufficient to consider the case where $\underline{y}=0$, namely the case where

$$y\in H^1(\Omega)\ \text{ with } 0\le y\le\overline{y},\ \text{ where } \overline{y}\in H_0^1(\Omega).$$

Since $\overline{y} \in H_0^1(\Omega)$, there exists a sequence $\overline{\phi}_n \in \mathcal{D}(\Omega)$, such that

$$\overline{\phi}_n \to \overline{y} \quad \text{in } H_0^1(\Omega).$$

The function y_n defined by

$$y_n = \overline{\phi_n}^+ - \left(\overline{\phi_n}^+ - y\right)^+$$

belongs to $H^1(\Omega)$ and has compact support (included in the support of ϕ_n). Therefore

$$y_n \in H_0^1(\Omega)$$

and

$$y_n \to \overline{y}^+ - (\overline{y}^+ - y)^+ = \overline{y}^+ - (\overline{y}^+ - y) = y \quad \text{in } H^1(\Omega) \quad \text{as } n \to +\infty.$$

This proves that $y \in H_0^1(\Omega)$. □

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