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Automorphisms of surfaces of general type with $q = 1$ acting trivially in cohomology

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Abstract. Let S be a complex minimal surface of general type with irregularity $q(S) = 1$ and $\mathrm{Aut}_0(S) \subset \mathrm{Aut}(S)$ the subgroup of automorphisms acting trivially on $H^*(S, \mathbb{Q})$. In this paper we show that $|\mathrm{Aut}_0(S)| \leq 4$, and if the equality holds then S is a surface isogenous to a product of unmixed type. Moreover, examples of surfaces with $|\mathrm{Aut}_0(S)| = 4$ and all possible values of the geometric genus $p_g(S)$ are provided.

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1. Introduction

In studying the automorphism group $\mathrm{Aut}(X)$ of a compact complex manifold X it is important to consider its cohomology representation, that is, its natural action on the cohomology ring, say with rational coefficients. The action of the automorphism group on the cohomology is also relevant in the construction of *fine* moduli spaces ([39, Lecture 10], see [31] for a recent treatment of the case of hypersurfaces) and in the attempt to equip Teichmüller spaces with a complex structure [21, Section 1.3]. There the faithfulness of the action seems to be a desired property. We say X is *rationally cohomologically rigidified* if the action of $\mathrm{Aut}(X)$ on $H^*(X, \mathbb{Q})$ is faithful. Obviously, it is equivalent to require that $\mathrm{Aut}(X)$ acts faithfully on $H^*(X, \mathbb{C})$, the cohomology ring with complex coefficients. In general, those automorphisms acting trivially on $H^*(X, \mathbb{Q})$ are called *numerically trivial* and they form a subgroup of the (full) automorphism group, to be denoted by $\mathrm{Aut}_0(X)$ in this paper.

It is well known that smooth projective curves are rationally cohomologically rigidified, unless the identity component of the automorphism group is nontrivial

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(in this case the genus is necessarily less than 2). The situation is more complicated in dimension two: there exist smooth projective surfaces with Kodaira dimension ranging from 0 to 2, which have automorphisms, not belonging to the identity component, acting trivially on the cohomology with rational coefficients (see [3,35–37] for surfaces of Kodaira dimension 0 and 1, and [13,14,18] for surfaces of general type).

The automorphism group of a surface S of general type is finite, and thus $\mathrm{Aut}_0(S)$ does not lie in the identity component of $\mathrm{Aut}(S)$ as soon as it is nontrivial. It turns out that nontrivial $\mathrm{Aut}_0(S)$ occurs only for those with irregularity $q(S) \leq 2$. Moreover, if $q(S) = 2$ then the order of $\mathrm{Aut}_0(S)$ is at most 2, and in case of nontrivial $\mathrm{Aut}_0(S)$ the signature of the minimal model of S vanishes [18, Theorem 1.1].

In this paper we investigate surfaces of general type with $q(S) = 1$:

Theorem 1.1. *Let S be a complex minimal projective surface of general type with $q(S) = 1$. Then we have $|\mathrm{Aut}_0(S)| \leq 4$ with equality only if S is a surface isogenous to a product of unmixed type.*

Surfaces isogenous to a product are those surfaces admitting a product of two smooth curves as an étale cover. By taking the Galois closure of the covering [20, Proposition 3.11] we may give a more restrictive definition of them, see Definition 3.4.

In view of the diversity of surfaces of general type the geometric characterization for surfaces with $\mathrm{Aut}_0(S)$ of maximal order in Theorem 1.1 seems quite satisfactory. Inspired by the results of the current paper, the second named author [34] has shown recently that surfaces of general type with $q(S) = 2$ and nontrivial $\mathrm{Aut}_0(S)$ must be isogenous to a product of curves.

The bound in Theorem 1.1 is optimal, as series of surfaces with $q(S) = 1$ and $|\mathrm{Aut}_0(S)| = 4$ are constructed in Section 5, realizing all possible values of the geometric genus. To complete the picture further we also provide examples of surfaces of general type with $q(S) = 1$ and $\mathrm{Aut}_0(S) \cong \mathbb{Z}/3\mathbb{Z}$. See also [13] and [14] for examples of surfaces of general type with $q(S) = 1$ and $\mathrm{Aut}_0(S) \cong \mathbb{Z}/2\mathbb{Z}$.

One might ask for a simple reason for the existence of nontrivial $\mathrm{Aut}_0(S)$. In fact, a diffeomorphism that is homotopic to the identity map will act trivially on the cohomology, even with integral coefficients. In particular, if an automorphism of an algebraic surface S, viewed as a diffeomorphism of the underlying differential manifold of S, comes from the identity component $\mathrm{Diff}^0(S)$ of the diffeomorphism group, then it acts trivially on the cohomology. Unfortunately, for those irregular surfaces with $|\mathrm{Aut}_0(S)| = 4$ in Theorem 1.1 this is not the case, since surfaces isogenous to a product are rigidified, that is, we have $\mathrm{Aut}(S) \cap \mathrm{Diff}^0(S) = \{\mathrm{id}_S\}$, see [18, Proposition 4.8].

Theorem 1.1 follows from Theorems 3.2, 3.3, 4.6 and 4.7 where the genus of the Albanese fibration of the surfaces with $q(S) = 1$ and $|\mathrm{Aut}_0(S)| = 4$ is also determined. The starting point of the proofs is that $\mathrm{Aut}_0(S)$ induces the trivial action on the Albanese variety, so that the Albanese map factors through the quotient map $S \to S/\mathrm{Aut}_0(S)$ (see Lemma 2.5).

The quotient map is of fundamental importance in studying automorphisms in general. In our context the quotient by $\mathrm{Aut}_0(S)$ inherits several invariants such as the geometric genus of the original surface S. This can be understood as giving certain bound on the quotient surface. Ultimately, we rely on the Bogomolov-Miyaoka-Yau inequality to conclude that $|\mathrm{Aut}_0(S)| \leq 4$. This bound has been obtained in [12] under the assumption that $\chi(\mathcal{O}_S) > 188$, where a large $\chi(\mathcal{O}_S)$ is to ensure that the canonical map is well-behaved (*cf.* [5]). We focus instead on the canonical system rather than the map it induces and it is thus possible to deal with all surfaces of general type with $q(S) = 1$ in one go, regardless of their geometric genus.

The characterization of surfaces with maximal $\mathrm{Aut}_0(S)$ is divided into two steps. First we prove the numerical equality $K_S^2 = 8\chi(\mathcal{O}_S)$, which is equivalent to the vanishing of the signature of the underlying 4-dimensional differentiable manifold of S. A key role is played by certain versions of the equivariant signature formula of Hirzebruch and Zagier ([28, page 177], see also [15, 1.6]). The second more subtle step consists in a careful analysis of the fixed loci, which a priori are a collection of scattered points and curves, to show that every singular fibre of the Albanese map is of the form $2C$ where C is a smooth curve. A lemma of Serrano [40, Lemma 5] then guarantees that the surfaces are isogenous to a product of curves of unmixed type.

Notation and conventions

We work over the complex numbers $\mathbb{C}$.

Let X be a compact complex manifold of dimension n. Then

- For a sheaf $\mathcal{F}$ on X, $h^i(X, \mathcal{F})$ is the dimension of its i-th cohomology group $H^i(X, \mathcal{F})$ and $\chi(\mathcal{F})$ the Euler characteristic;
- $q(X) := h^1(X, \mathcal{O}_X)$ and $p_g(X) := h^0(X, K_X)$ are the irregularity and the geometric genus of X respectively;
- $e(X)$ is the topological Euler characteristic;
- If X is even dimensional, $\mathrm{Sign}(X)$ denotes the signature of the intersection form on the middle cohomology $H^n(X, \mathbb{R})$;
- The Albanese torus of X is denoted by $\mathrm{Alb}(X)$ and the Albanese map by $a_X \colon X \to \mathrm{Alb}(X)$;
- The group of holomorphic automorphisms acting trivially on the cohomology ring $H^*(X, \mathbb{Q})$ will be denoted by $\mathrm{Aut}_0(X)$. For simplicity of notation we often write G_0 for $\mathrm{Aut}_0(X)$.

If $f : S \to B$ is a fibration from a smooth projective surface onto a smooth projective curve, then genus of f, denoted by $g(f)$, means the genus of a general fibre.

The symbol $\sim$ (respectively $\sim_{\mathbb{Q}}$) denotes (respectively $\mathbb{Q}$-)linear equivalence between (respectively $\mathbb{Q}$-)divisors while $\equiv$ denotes numerical equivalence.

For a finite group G we will denote its order by $|G|$. If it acts on a set X then the fixed point set of an element $\sigma \in G$ is denoted by

$$X^\sigma := \{p \in X \mid \sigma(p) = p\}.$$

For a finite abelian group G we denote by $\widehat{G}$ the character group of G. For a representation V of G and a character $\chi \in \widehat{G}$ we write

$$V^\chi = \{v \in V \mid g(v) = \chi(g)v \text{ for all } g \in G\}.$$

Note that V^χ is contained in the subspace of V that is pointwise fixed by $\ker(\chi)$.

The dihedral group of order n will be denoted by D_n and quaternion group of order 8 by Q_8.

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2. Basic properties of $\mathrm{Aut}_0(X)$

Let X be a smooth projective variety and $G \subset \mathrm{Aut}(X)$ a finite group of automorphisms. The quotient map $\pi : X \to X/G$ plays a fundamental role in studying the action of G. We make several observations about it in the case when G acts trivially on the cohomology.

We remark that the following Lemmata 2.1 and 2.5, Remark 2.2 are valid in the more general context of compact Kähler manifolds.

Lemma 2.1. *Let X be a smooth projective variety and G a finite group of automorphisms acting trivially on $H^*(X, \mathbb{C})$. Let $\lambda\colon Y \to X/G$ be a resolution of singularities. Then the following holds.*

(i) $h^i(Y, \mathcal{O}_Y) = h^i(X, \mathcal{O}_X)$ *for any* $0 \le i \le \dim X$. *As a consequence,*

$$q(Y) = q(X),\ p_g(Y) = p_g(X) \text{ and } \chi(\mathcal{O}_Y) = \chi(\mathcal{O}_X).$$

Now assume that $p_g(X) > 0$. Let $\tilde\pi : \tilde X \xrightarrow{\rho} X \dashrightarrow Y$ be a morphism eleminating the indeterminacy of the induced rational map $X \dashrightarrow Y$.

(ii) *There is an equality of complete linear systems*

$$\left|K_{\tilde X}\right| = \tilde\pi^*|K_Y| + \tilde R,$$

where $\tilde R$ is the ramification divisor of $\tilde\pi$.

(iii) *Suppose $D \subset X$ is an irreducible subvariety of codimension 1, fixed by some nontrivial element of G. Then D is contained in the base locus of the canonical system $|K_X|$.*

Proof. (i) Since G acts trivially on $H^*(X, \mathbb{C})$, so does it on the direct summands $H^i(X, \mathcal{O}_X)$ in the Hodge decomposition of $H^*(X, \mathbb{C})$. We have by the Grothendieck-Leray spectral sequence

$$H^i\big(X/G, \mathcal{O}_{X/G}\big) = H^i\Big(X/G, \pi_*^G\mathcal{O}_X\Big) = H^i\big(X, \mathcal{O}_X\big)^G = H^i(X, \mathcal{O}_X), \quad (2.1)$$

where $\pi : X \to X/G$ is the quotient map and $\pi_*^G \mathcal{O}_X$ denotes the G-invariant part of $\pi_* \mathcal{O}_X$. Since X/G has only quotient (hence rational) singularities, the cohomology groups of X/G and its resolution Y are the same:

$$H^i\big(X/G, \mathcal{O}_{X/G}\big) = H^i(Y, \mathcal{O}_Y).$$

Together with (2.1) this yields the desired conclusion.

(ii) The pull-back map $\tilde{\pi}^* : H^0(Y, K_Y) \to H^0(\tilde{X}, K_{\tilde{X}})$ is an injective linear map of vector spaces. By (i), $\tilde{\pi}^*$ is in fact an isomorphism. This proves (ii).

(iii) The subvariety D is contained in the image of $\tilde{R}$ in X, which lies in the base locus of $|K_{\tilde{X}}|$ by (ii). Since $|K_X| = \rho_* |K_{\tilde{X}}|$, where the push-forward operator ρ_* of divisors is defined as in [4, Section I], the subvariety D lies in the base locus of $|K_X|$. □

Remark 2.2. By the same proof Lemma 2.1 is also valid if we only assume that G acts trivially on $H^i(X, \mathcal{O}_X)$ for $0 \le i \le \dim X$.

For lack of an appropriate reference, we give a proof of the following version of the topological Lefschetz fixed point formula.

Theorem 2.3. *Let X be a compact differentiable manifold (respectively a compact complex space), and let $\sigma \in \mathrm{Diff}(X)$ (respectively $\sigma \in \mathrm{Aut}(X)$) be of finite order. Then one has the topological Lefschetz fixed point formula*

$$e(X^\sigma) = \sum_{0 \le i \le n} (-1)^i \mathrm{tr}\left(\sigma^* \mid H^i(X, \mathbb{C})\right), \tag{2.2}$$

where $\mathrm{tr}(\cdot)$ denotes the trace of an endomorphism of a vector space.

Proof. We first remark that X has a finite $\langle\sigma\rangle$-equivariant triangulation. Indeed, if X is a compact differentiable manifold, the existence of such a triangulation is guaranteed by [29]. If X is a compact complex space, then $X/\langle\sigma\rangle$ is again a compact complex space and we can stratify it into locally closed strata A_j such that all the points over a single A_j have the same stabilizer of the $\langle\sigma\rangle$-action. By [32, Corollary 2.2] there is a finite triangulation of $X/\langle\sigma\rangle$ such that each stratum A_j is a union of the support of (open) simplices, and one obtains a finite $\langle\sigma\rangle$-equivariant triangulation on X by [30, Theorem 5.5].

Let $C^i(X)$ be the vector space of the i-cochains over $\mathbb{C}$, with basis dual to the set of i-simplices. The action σ^* on $C^i(X)$ is induced by the permutation of σ on the set of i-simplices. Thus we have

$$\mathrm{tr}\left(\sigma^* \mid C^i(X)\right) = \dim C^i(X)^\sigma = \dim C^i(X^\sigma), \tag{2.3}$$

where $C^i(X)^\sigma$ is the σ^*-fixed part of $C^i(X)$ and it coincides with space of i-cochains supported on X^σ.

On the other hand, $H^i(X, \mathbb{C})$ are the homology groups of the cochain complex

$$0 \to C^0(X) \to C^1(X) \to \cdots \to C^{n-1}(X) \to C^n(X) \to 0,$$

where n is the dimension of X. It is not hard to see that, by (2.3),

$$\begin{aligned}\sum_{0\le i\le n} (-1)^i \operatorname{tr}\left(\sigma^* \mid H^i(X,\mathbb{C})\right) &= \sum_{0\le i\le n} (-1)^i \operatorname{tr}\left(\sigma^* \mid C^i(X)\right)\\ &= \sum_{0\le i\le n} (-1)^i \dim C^i(X^\sigma)\\ &= e(X^\sigma).\end{aligned}$$

□

The topological and holomorphic Lefschetz fixed point formulae (see [2, Theorem 4.6 and Proposition 4.8] for the later) have the following consequence.

Lemma 2.4 ([17, Lemma 2.1]). *Let S be a complex nonsingular projective surface. If there is an involution σ of S which acts trivially in $H^2(S,\mathbb{Q})$, then $K_S^2 = 8\chi(\mathcal{O}_S) + \sum_{i=1}^m D_i^2$, where $D_1, \cdots, D_m$ $(m \ge 0)$ are the σ-fixed curves.*

Lemma 2.5. *Let X be a smooth projective variety with topological Euler characteristic $e(X) \neq 0$ and G a subgroup of $\mathrm{Aut}_0(X)$. Then the Albanese map of X factors as*

$$a_X\colon X \xrightarrow{\pi} X/G \to \mathrm{Alb}(X)$$

where $\pi\colon X \to X/G$ is the quotient map.

Proof. Let $\sigma \in G$. The automorphism σ_a of $\mathrm{Alb}(X)$, induced by σ, fits into the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & X \\ \downarrow{\scriptstyle a_X} & & \downarrow{\scriptstyle a_X} \\ \mathrm{Alb}(X) & \xrightarrow{\sigma_a} & \mathrm{Alb}(X). \end{array}$$

Since σ induces the trivial action on $H^1(X,\mathbb{C})$, which can be identified with $H^1(\mathrm{Alb}(X),\mathbb{C})$, the induced map σ_a must be a translation of $\mathrm{Alb}(X)$.

On the other hand, since $e(X^\sigma) = e(X) \neq 0$ by (2.2), the fixed point set X^σ is not empty. Note that σ_a fixes the point $a_X(p)$ for any $p \in X^\sigma$, so it can only be the identity map. □

Let Y be a smooth model of X/G. By the universality of the Albanese maps and Lemma 2.5 we know that the Albanese varieties $\mathrm{Alb}(Y)$ and $\mathrm{Alb}(X)$ can be identified after fixing suitable base points for the Albanese maps. Indeed, we have a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\pi} & X/G & \longrightarrow & \mathrm{Alb}(X) \\ & & \| & & \| \\ Y & \dashrightarrow & X/G & \longrightarrow & \mathrm{Alb}(Y). \end{array} \tag{2.4}$$

From now on we focus on the case of surfaces.

Lemma 2.6. *Let S be a surface of general type with $q(S) = 1$ and G a subgroup of $\mathrm{Aut}_0(S)$. Suppose X is a smooth model of S/G. Then the Kodaira dimension $\kappa(X) \geq 1$ and the equality holds if and only if the Albanese map $a_X: X \to \mathrm{Alb}(X)$ has genus 1.*

Proof. Recall that the automorphism groups of surfaces of general type are finite. By Lemma 2.1 we have

$$p_g(X) = p_g(S) \geq 1 \text{ and } q(X) = q(S) = 1.$$

So $\kappa(X) \geq 1$ by the Kodaira-Enriques classification of algebraic surfaces.

If X is of general type then any fibration has genus at least 2.

Now suppose X has Kodaira dimension 1. We consider the m-canonical map $\varphi_m: X \to B$ of X for a sufficiently large and divisible m. Then φ_m is an elliptic fibration. Since $q(X) = 1$, the genus of B is at most 1. If $g(B) = 0$, then $q(X) = g(B) + g(\varphi_m^* b)$ where $\varphi_m^* b$ is the fiber of φ_m over a general $b \in B$. As a consequence, the minimal model of X is isogenous to a product $\tilde{B} \times \varphi_m^* b$ via a base change $\tilde{B} \to B$, so $\chi(\mathcal{O}_X) = 0$, contradicting the fact that $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_S) = 0$. This finishes the proof of the lemma. □

We end this section with a useful observation.

Lemma 2.7. *Let S be a smooth projective surface and $C \subset S$ an irreducible curve with negative self-intersection. Then the following holds.*

(i) *The curve C is invariant under the action of $\mathrm{Aut}_0(S)$;*
(ii) *Suppose furthermore that $f: S \to B$ is a fibration preserved by an automorphism $\sigma \in \mathrm{Aut}_0(S)$, that is, $f \circ \sigma = f$, and C is a section of f. Then C is fixed pointwise by σ.*

Proof. (i) Suppose on the contrary that $\gamma(C) \neq C$ for some $\gamma \in \mathrm{Aut}_0(S)$. Then $C^2 = C \cdot \gamma(C) \geq 0$, a contradiction to the assumption.

(ii) For every $p \in C$, since C is a section of $f: S \to B$, we have $\{p\} = C \cap F_b$ with $b = f(p)$. Due to (i) and the assumption, both C and F_b are preserved by σ, so p is σ-fixed. □

3. Surfaces with quotient of general type

Let S be a minimal surface of general type with $q(S) = 1$ and $G_0 := \mathrm{Aut}_0(S)$ the automorphism group acting trivially on $H^*(S, \mathbb{C})$. We know from Lemma 2.6 that $\kappa(S/G_0) \geq 1$, where $\kappa(S/G_0)$ denotes the Kodaira dimension of a smooth model of S/G_0.

Let $\lambda\colon \tilde{T} \to S/G_0$ be the minimal resolution of singularity and $\eta\colon \tilde{T} \to T$ the contraction to the minimal model of $\tilde{T}$. Then we have the following commutative diagram

$$\begin{array}{ccccc} \tilde{S} & \xrightarrow{\tilde{\pi}} & \tilde{T} & \xrightarrow{\eta} & T \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\lambda} & & \\ S & \xrightarrow{\pi} & S/G_0, & & \end{array} \tag{3.1}$$

where $\pi : S \to S/G_0$ is the quotient map and $\tilde{S}$ is the minimal resolution of singularities of the normalization of the fibre product $S \times_{S/G} \tilde{T}$.

In this section we will treat the case where T is of general type.

3.1. Bounding $|\mathrm{Aut}_0(S)|$, part I

We first bound the order of an automorphism group in terms of the volumes of the original surface and the quotient, thus improving this kind of results previously obtained by Xiao [43, Lemma 2 and Proposition 1 (i)].

Proposition 3.1. *Let S be a minimal surface of general type and G a group of its automorphisms. Assume the quotient S/G is of general type and let T be its minimal (smooth) model. Then the following hold.*

- (i) $K_S^2 \geq |G|K_T^2 + \sum(r_C - 1)K_S C$, *where the sum is taken over all irreducible curves $C \subset S$ and r_C is the order of the stabilizer at a general point of C. In particular, we have $K_S^2 \geq |G|K_T^2$;*
- (ii) $K_S^2 = |G|K_T^2$ *if and only if the following hold:*
  - (a) *The fixed point set S^γ is finite for any nontrivial $\gamma \in G$;*
  - (b) *The quotient S/G has at most canonical singularities.*

  In this case T is isomorphic to the minimal resolution of singularities of S/G.

Proof. Let $\pi : S \to S/G$ be the quotient map. For any irreducible curve $C \subset S$ we denote by $\bar{C}$ the image of C under π. There is a $\mathbb{Q}$-linear equivalence:

$$K_S \sim_{\mathbb{Q}} \pi^*\left(K_{S/G} + \sum_C \left(1 - \frac{1}{r_C}\right)\bar{C}\right), \tag{3.2}$$

where the sum is taken over all irreducible curves $C \subset S$ and r_C is the order of the stabilizer at a general point of C. Since K_S is nef and π is finite, the $\mathbb{Q}$-divisor $K_{S/G} + \sum\left(1 - \frac{1}{r_C}\right)\bar{C}$ is also nef.

Let $\lambda\colon \tilde{T} \to S/G$ be the minimal resolution of singularities and $\eta\colon \tilde{T} \to T$ the contraction to the minimal model, see the following diagram:

$$\begin{array}{ccccc} & & & \tilde{T} & \\ & & \swarrow^{\lambda} & & \searrow^{\eta} \\ S & \xrightarrow{\pi} & S/G & & T. \end{array}$$

We have

$$\lambda^* K_{S/G} \sim_{\mathbb{Q}} K_{\tilde{T}} + \sum a_i E_i \text{ and } \eta^* K_T \sim_{\mathbb{Q}} K_{\tilde{T}} - A,$$

where E_i are the exceptional divisors of λ and A is an effective divisor supported on the whole exceptional locus of η. Since the resolution of singularities $\lambda\colon \tilde{T} \to S/G$ is minimal, we have $a_i \geq 0$ in the above formula. It follows that

$$\lambda^* \left(K_{S/G} + \sum \left(1 - \frac{1}{r_C}\right) \bar{C} \right) \sim_{\mathbb{Q}} \eta^* K_T + \Omega, \tag{3.3}$$

where $\Omega := \sum \left(1 - \frac{1}{r_C}\right) \lambda^* \bar{C} + \sum a_i E_i + A$ is an effective divisor.

Since both $\lambda^* \left(K_{S/G} + \sum \left(1 - \frac{1}{r_C}\right) \bar{C} \right)$ and $\eta^* K_T$ are big and nef, the inequality in the following computation holds:

$$\begin{aligned} K_S^2 &= |G| \left(K_{S/G} + \sum \left(1 - \frac{1}{r_C}\right) \bar{C} \right)^2 \\ &= |G| \lambda^* \left(K_{S/G} + \sum \left(1 - \frac{1}{r_C}\right) \bar{C} \right)^2 \\ &= |G| \left((\eta^* K_T)^2 + \lambda^* \left(K_{S/G} + \sum \left(1 - \frac{1}{r_C}\right) \bar{C} \right) \Omega + \eta^* K_T \Omega \right) \\ &\geq |G| K_T^2 + |G| \left(K_{S/G} + \sum \left(1 - \frac{1}{r_C}\right) \bar{C} \right) \sum \left(1 - \frac{1}{r_C}\right) \bar{C} \\ &= |G| K_T^2 + \pi^* \left(K_{S/G} + \sum \left(1 - \frac{1}{r_C}\right) \bar{C} \right) \sum \left(1 - \frac{1}{r_C}\right) \pi^* \bar{C} \\ &= |G| K_T^2 + K_S \sum (r_C - 1) C, \end{aligned} \tag{3.4}$$

where the second equality descends from (3.2). This establishes (i).

Now by (3.4), $K_S^2 = |G| K_T^2$ holds if and only if

$$\lambda^* \left(K_{S/G} + \sum \left(1 - \frac{1}{r_C}\right) \bar{C} \right) \Omega = 0 \text{ and } \eta^* K_T \Omega = 0, \tag{3.5}$$

which by the Hodge index theorem (assuming (3.5) we have $\Omega^2 = 0$ by (3.3)) is in turn equivalent to $\Omega = 0$, that is, all of the three effective $\mathbb{Q}$-divisors $\sum a_i E_i$, $\sum \left(1 - \frac{1}{r_C}\right) \lambda^* \bar{C}$ and A are 0. Hence the equality $K_S^2 = |G| K_T^2$ implies (a) and (b).

Conversely, (a) and (b) imply $K_S \sim \pi^* K_{S/G}$ and $K_{\tilde{T}} \sim \lambda^* K_{S/G}$. The nefness of K_S implies the nefness of $K_{S/G}$ and in turn that of $K_{\tilde{T}}$. So $\tilde{T}$ is already minimal and we infer that

$$K_S^2 = |G|K_{S/G}^2 = |G|K_{\tilde{T}} = |G|K_{\tilde{T}}^2. \qquad \square$$

Theorem 3.2. *Let S be a minimal surface of general type with $q(S) = 1$ such that $\kappa(S/G_0) = 2$, where $G_0 := \mathrm{Aut}_0(S)$. Then $|G_0| \le 4$.*

Proof. Let T be the minimal model of S/G_0. By Lemma 2.1,

$$q(T) = q(S) > 0 \text{ and } \chi(\mathcal{O}_T) = \chi(\mathcal{O}_S),$$

and hence one has by [9, Lemma 14]

$$K_T^2 \ge 2\chi(\mathcal{O}_T) = 2\chi(\mathcal{O}_S). \tag{3.6}$$

Combined with Proposition 3.1, this implies

$$K_S^2 \ge |G_0|K_T^2 \ge 2|G_0|\chi(\mathcal{O}_S). \tag{3.7}$$

The theorem follows from (3.7) together with the Bogomolov-Miyaoka-Yau inequality. $\square$

3.2. Surfaces with $|\mathrm{Aut}_0(S)| = 4$, part I

Now we investigate the surfaces with $\mathrm{Aut}_0(S)$ of maximal order and characterize them as follows.

Theorem 3.3. *Let S be a minimal surface of general type with $q(S) = 1$ such that $\kappa(S/G_0) = 2$ and $|G_0| = 4$ where $G_0 := \mathrm{Aut}_0(S)$. Then the following hold.*

(i) *The Albanese fibration $a_S\colon S \to \mathrm{Alb}(S)$ has genus 5;*
(ii) *The group G_0 is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$;*
(iii) *S is a surface isogenous to a product of unmixed type.*

We recall the definition of surfaces isogenous to a (higher) product.

Definition 3.4. Let C, D be smooth curves of genus at least 2, and G is a finite group acting (faithfully) on C and D. If the diagonal subgroup Δ_G of $G \times G$ acts freely on $C \times D$ then the smooth quotient $S := (C \times D)/\Delta_G$ is called a surface isogenous to a product of unmixed type.

Remark 3.5. There exists also the notion of surfaces isogenous to a product of *mixed* type $S = (C \times C)/G$ where the group G acts freely and interchanges the two factors of $C \times C$.

Given a surface isogenous to a product of unmixed type $S = (C \times D)/\Delta_G$, its invariants satisfy

$$K_S^2 = 8\chi(\mathcal{O}_S) = \frac{8}{|G|}(g(C) - 1)(g(D) - 1) \text{ and } q(S) = g(C/G) + g(D/G).$$

We need two intermediate results for the proof of Theorem 3.3.

Lemma 3.6. *Let S be as in Theorem* 3.3. *Then the set S^σ is finite for any nontrivial* $\sigma \in \mathrm{Aut}_0(S)$.

Proof. We use an argument similar to the one in [18, 3.6]. Write $G_0 = \mathrm{Aut}_0(S)$ as before. Since G_0 is of order 4, it suffices to prove the lemma for $\sigma \in G_0$ that has order 2.

Suppose on the contrary that such a σ fixes curves $D_1, \dots, D_u$. Then, by Lemma 2.4,

$$K_S^2 = 8\chi(\mathcal{O}_S) + \sum_{i=1}^{u} D_i^2. \tag{3.8}$$

On the other hand, we have by Proposition 3.1 and (3.6),

$$\begin{aligned} K_S^2 &\geq 4K_T^2 + \sum_{i=1}^{u} K_S D_i \\ &\geq 8\chi(\mathcal{O}_S) + \left(K_S - \sum_{i=1}^{u} D_i\right)\sum_{i=1}^{u} D_i + \sum_{i=1}^{u} D_i^2 \\ &\geq 8\chi(\mathcal{O}_S) + 2 + \sum_{i=1}^{u} D_i^2, \end{aligned}$$

where T is the minimal model of S/G_0 and the last inequality holds since each σ-fixed curve is contained in the fixed part of $|K_S|$ (Lemma 2.1) and each effective canonical divisor of S is 2-connected (*cf.* [6, VII, Proposition 6.2]). This contradicts (3.8). □

Corollary 3.7. *Let S be as in Theorem* 3.3 *and T the minimal model of S/G_0. Then the following hold.*

(i) $K_S^2 = 8\chi(\mathcal{O}_S)$ *or, equivalently,* $e(S) = 4\chi(\mathcal{O}_S)$*;*

(ii) *The invariants of T satisfy $K_T^2 = 2\chi(\mathcal{O}_T)$ and the Albanese fibration $a_T : T \to$* $\mathrm{Alb}(T)$ *has genus 2, with singular fibres all of type (0) in the sense of Horikawa* [26]*;*

(iii) *The quotient S/G_0 has only canonical singularities and T is isomorphic to the minimal resolution of singularities of S/G_0.*

Proof. (i) The first equality follows from Lemma 3.6 and Lemma 2.4, applied to an involution from G_0. The second equality is equivalent to the first one by the Noether formula.

(ii) Since $K_S^2 = 8\chi(\mathcal{O}_S)$, the inequalities in (3.6) and (3.7) are all equalities. In particular, $K_T^2 = 2\chi(\mathcal{O}_T)$. By [27, Theorem 5.1] the Albanese map $a_T : T \to \mathrm{Alb}(T)$ is a fibration of genus 2, whose singular fibres are all of type (0) in the sense of [26].

(iii) Since $K_S^2 = |G_0|K_T^2$ holds, the assertion follows from Proposition 3.1. □

In the following lemma we record how $(\mathbb{Z}/2\mathbb{Z})^2$-actions on surfaces behave when there are no fixed curves.

Lemma 3.8. *Let $G \cong (\mathbb{Z}/2\mathbb{Z})^2$ be a group of automorphisms of a smooth surface S. Denote by $\sigma_1, \sigma_2, \sigma_3$ the three involutions in G. If σ_1 and σ_2 have only isolated singularities then the three fixed point sets S^{σ_i}, $1 \le i \le 3$ are pairwise disjoint.*

Proof. Otherwise, there is a point $p \in S$ fixed by the whole group G. Then there are local coordinates (x, y) around $p \in S$ such that each $\gamma \in G$ acts as

$$(x, y) \mapsto (\chi_1(\gamma)x, \chi_2(\gamma)y),$$

where $\chi_1, \chi_2 \in \widehat{G}$ are two distinct characters of G. The assumption implies that σ_1, $\sigma_2 \notin \ker(\chi_i)$ for $i = 1, 2$. So $\ker(\chi_i) = \langle\sigma_3\rangle$ for $i = 1, 2$, and hence σ_3 induces the trivial action on the tangent space of S at p. This implies that σ_3 is trivial, a contradiction. □

Proof of Theorem 3.3. Let T be the minimal model of S/G_0 and B the identified Albanese varieties $\mathrm{Alb}(S)$ and $\mathrm{Alb}(T)$, see (2.4).

(i) For a general $b \in B$ the fibre a_T^*b is the quotient of a_S^*b by G_0. By Lemma 3.6 the action of G_0 on a_S^*b is free and hence the quotient map $a_S^*b \to a_T^*b$ is étale. Since $g(a_T^*b) = 2$ by Corollary 3.7, one computes easily $g(a_S^*b) = 5$.

(ii) Since $|G_0| = 4$, there are two possibilities: $G_0 \cong \mathbb{Z}/4\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^2$. Suppose by absurd that $G_0 \cong \mathbb{Z}/4\mathbb{Z}$, generated by α. By Lemma 3.6 both α and α^2 have only isolated singularities. It follows that any fixed point of α is either of weight $\frac{1}{4}(1, 1)$ or of weight $\frac{1}{4}(1, 3)$. Let k_1 (respectively k_3) be the number of α-fixed points where the action is of weight $\frac{1}{4}(1, 1)$ (respectively $\frac{1}{4}(1, 3)$).

By Corollary 3.7 the quotient S/G_0 has only canonical singularities, so $k_1 = 0$. By the topological Lefschetz fixed point formula (2.2)

$$k_3 = \# S^\alpha = e(S). \tag{3.9}$$

Now we apply the equivariant signature formula to α (*cf.* [28, Equation (12), page 177], [15, 1.6], or [17, 3.3]) and obtain in our case

$$4\mathrm{Sign}(S/\alpha) = \mathrm{Sign}(S) + 2k_3. \tag{3.10}$$

Since α acts trivially on the cohomology we have $\mathrm{Sign}(S/\alpha) = \mathrm{Sign}(S) = 0$. By (3.10) we obtain $k_3 = 0$. This together with (3.9) yields $e(S) = 0$, a contradiction.

(iii) By Corollary 3.7 the Albanese map $a_T : T \to B$ is a genus 2 fibration whose singular fibres are all of type (0). Let $\Sigma := \mathrm{Proj}(a_{T*}\omega_{T/B}) \to B$ be the projectivized relative canonical bundle. By [26] the relative canonical map

$$\begin{array}{ccc} T & \xrightarrow{\quad h \quad} & \Sigma \\ & {\scriptstyle a_T}\searrow \quad \swarrow{\scriptstyle a_\Sigma} & \\ & B & \end{array}$$

is a morphism of degree 2 and its branch curve $D \subset \Sigma$ has at most simple singularities. In fact, the curve D is simple normal crossing by the forthcoming Lemma 3.10.

Since every (-2)-curve on T is contracted to a singularity of D, there is a morphism $\bar{\varphi}\colon S/G_0 \to \Sigma$ such that the morphism h factors as $\bar{\varphi} \circ \lambda$ where $\lambda\colon T \to S/G_0$ is the minimal resolution of singularities. By Lemma 3.10 the morphism $\bar{\varphi}\colon S/G_0 \to \Sigma$ is a flat double cover of Σ branched along D. The composition $\varphi = \bar{\varphi} \circ \pi\colon S \to \Sigma$ is then a finite morphism $\varphi\colon S \to \Sigma$ of degree 8, branched along D.

Over an analytic open subset U of Σ around the (nodal) singularities of D, the surface S is a disjoint union of two $(\mathbb{Z}/2\mathbb{Z})^2$-covers of U (*cf.* [6, page 102]). It is then direct to check that the singular fibres of $a_S\colon S \to B$ are of the form $2C$ with C a smooth curve of genus 3. This together with the fact that $K_S^2 = 8\chi(\mathcal{O}_S)$ guarantees that the surface S is isogenous to a product of curves of unmixed type by [40, Lemma 5]. □

Remark 3.9. Using Corollary B.4 one sees that the eight-to-one finite morphism $\varphi\colon S \to \Sigma$ in the proof of Theorem 3.3 is in fact Galois.

Lemma 3.10. *Resume the notation in the proof of Theorem* 3.3*, and write* $D = D_1 + \sum_{1\le i\le k} \Gamma_i$*, where* D_1 *is horizontal with respect to the ruling* $a_\Sigma\colon \Sigma \to B$ *and the* Γ_i*'s are* k *different fibres. Then the following hold.*

(i) *The induced morphism* $a_\Sigma|_{D_1}\colon D_1 \to B$ *is étale. As a consequence,* D *is a simple normal crossing curve;*

(ii) *The morphism* $\bar{\varphi}\colon S/G_0 \to \Sigma$ *is a* flat *double covering.*

Proof. (i) Let e be the maximal integer such that there is a section of $a_\Sigma\colon \Sigma \to B$, say Δ, with $\Delta^2 = -e$. Then $e \ge -1$ (*cf.* [25, V.2]). Numerically we can write

$$D \equiv 6\Delta + d\Gamma \text{ and } K_\Sigma \equiv -2\Delta - e\Gamma,$$

where d is some integer. The arithmetic genus of D depends only on its numerical class:

$$\begin{aligned} p_a(D) &= 1 + \frac{D(D+K_\Sigma)}{2} \\ &= 1 + \frac{(6\Delta + d\Gamma)\,(4\Delta + (d-e)\Gamma)}{2} \\ &= 1 + 5d - 15e. \end{aligned} \tag{3.11}$$

Note that

$$\chi(\mathcal{O}_S) = \frac{1}{2}K_T^2 = \left(K_\Sigma + \frac{D}{2}\right)^2 = \left(\Delta + \left(\frac{d}{2} - e\right)\Gamma\right)^2 = d - 3e,$$

where the first equality descends from Corollary 3.7, so we have

$$p_a(D) = 5\chi(S, \mathcal{O}_S) + 1. \tag{3.12}$$

Now we calculate $p_a(D)$ in another way. Let $\tilde{\Sigma} \to \Sigma$ be the minimal embedded resolution of the singularities of $D = D_1 + \sum_{1\le i\le k} \Gamma_i$ by blow-ups and $\tilde{D} \subset \tilde{\Sigma}$ the (smooth) strict transform of D. Let $\tilde{D}_1$ and $\tilde{\Gamma}_i$ be the strict transform of D_1 and Γ_i on $\tilde{\Sigma}$ respectively. Then $\tilde{D} = \tilde{D}_1 + \sum_{1\le i\le k} \tilde{\Gamma}_i$ and

$$p_a(D) = p_a(\tilde{D}) + \sum_{p\in D_{\text{sing}}} \delta_p(D), \tag{3.13}$$

where D_{sing} denotes the singular locus of the curve D and, for $p \in D_{\text{sing}}$, the number $\delta_p(D)$ is a positive integer determined locally by the type of the singularity $p \in D$.

There is a lower bound of $p_a(D)$: by (3.13) and Lemma 3.11, and since $p_a(\tilde{D}_1) \ge 1$),

$$\begin{aligned} p_a(D) &\ge p_a(\tilde{D}) + 6\chi(\mathcal{O}_S) \\ &= p_a(\tilde{D}_1) - k + 6\chi(\mathcal{O}_S) \\ &\ge 1 - k + 6\chi(\mathcal{O}_S). \end{aligned} \tag{3.14}$$

Combining (3.12) and (3.14) we see that

$$k \ge \chi(\mathcal{O}_S) = d - 3e. \tag{3.15}$$

Numerically $D_1 \equiv D - k\Gamma \equiv 6\Delta + (d-k)\Gamma$, so

$$D_1 \le_{\text{num}} 6\Delta + 3e\Gamma, \tag{3.16}$$

which means that the divisor $6\Delta + 3e\Gamma - D_1$ is numerically equivalent to an effective $\mathbb{Q}$-divisor.

There are two cases depending on whether the rank two relative canonical bundle $a_{T*}\omega_{T/B}$ is decomposable or not.

Case 1. $a_{T*}\omega_{T/B}$ is decomposable. Then $e \ge 0$. If $e > 0$ then there is no reduced curve in the linear system $|6\Delta + a_\Sigma^*\mathfrak{b}|$ for a divisor $\mathfrak{b}$ on B of degree $3e$. Since D_1 is reduced, this case does not occur. If $e = 0$ then $\Sigma = B \times \mathbb{P}^1$ and $D_1 \le_{\text{num}} 6\Delta$. Since D_1 is reduced, it is necessarily a union of disjoint 6 sections of $a_\Sigma : \Sigma \to B$. In particular, the induced morphism $a_\Sigma|_{D_1} : D_1 \to B$ is étale.

Case 2. $a_{T*}\omega_{T/B}$ is indecomposable. Then the invariant e of the corresponding ruled surface Σ is 0 or -1. As in (3.11) we compute

$$p_a(6\Delta + n\Gamma) = 1 - 15e + 5n \text{ for } n \in \mathbb{Z}, \tag{3.17}$$

which is less than 1 if $n < 3e$. Since the arithmetic genus of any horizontal reduced curve in Σ is at least 1, no such curve is numerically equivalent to $6\Delta + n\Gamma$ for $n < 3e$. As a horizontal reduced curve, D_1 must be numerically equivalent to $6\Delta + 3e\Gamma$ by (3.16), so $p_a(D_1) = 1$ by (3.17). This implies that $a_\Sigma|_{D_1} : D_1 \to B$ is étale by the Riemann-Hurwitz formula.

Having shown that the morphism $a_\Sigma|_{D_1} : D_1 \to B$ is étale, the fibres Γ_i $(1 \le i \le k)$ must intersect D_1 transversally. Therefore D is a simple normal crossing curve.

(ii) By the proof of (i) we infer that $D_1 \equiv 6\Delta + 3e\Gamma$ and the inequalities in (3.14) and (3.15) are in fact equalities. Consequently, the equality case of Lemma 3.11 is achieved, so the singular locus of S/G_0 surjects onto D_{sing}. From this the assertion of (ii) follows easily. □

Lemma 3.11. *With the same notation as in the proofs of Theorem* 3.3, (iii) *and Lemma* 3.10 *we have*

$$\sum_{p \in D_{\text{sing}}} \delta_p(D) \ge 6\chi(\mathcal{O}_S), \tag{3.18}$$

with equality only if the singular locus of S/G_0 surjects onto D_{sing}.

Proof. We know by Theorem 3.3, (ii), that $G_0 \cong (\mathbb{Z}/2\mathbb{Z})^2$. Let $\sigma_1, \sigma_2, \sigma_3$ be the three involutions of G_0. Since the sets S^{σ_i} $(1 \le i \le 3)$ are finite by Lemma 3.6, they are pairwise disjoint by Lemma 3.8.

By the topological Lefschetz fixed point formula (2.2) there are $e(S) = 4\chi(\mathcal{O}_S)$ points fixed by each involution $\sigma_i \in G_0$, $1 \le i \le 3$. The image of S^{σ_i} under the quotient map $\pi : S \to S/G_0$ consists of 2χ singularities of type A_1, where $\chi := \chi(\mathcal{O}_S)$. Resolving those singularities we obtain in total 6χ disjoint (-2)-curves $E_l^{(i)}$ on T with $1 \le i \le 3$ and $1 \le l \le 2\chi$, where $E_l^{(i)}$ $(1 \le l \le 2\chi)$ lie over the points in $\pi(S^{\sigma_i})$.

Let $\tilde{S} \to S$ be the simultaneous blow-up of S at the points of $\bigcup_{1 \le i \le 3} S^{\sigma_i}$. Then the induced morphism $\tilde{S} \to T$ is a bidouble cover, branched exactly along the $E_l^{(i)}$'s with $1 \le i \le 3$ and $1 \le l \le 2\chi$. Moreover, the stabilizer over the curves $E_l^{(i)}$ is σ_i. It follows from the theory of bidouble covers [19] that, for $1 \le i < j \le 3$, the divisor $\sum_{1 \le l \le 2\chi}(E_l^{(i)} + E_l^{(j)})$ is even, meaning that it is linearly equivalent to $2L$ for some divisor L. An even divisor has necessarily an even intersection number with each curve.

Note that each (-2)-curve $E_l^{(i)}$ is contracted to some singularity of D under the morphism $h : T \to \Sigma$. So it suffices to show that, over each $p \in D_{\text{sing}}$, we have

$$\delta_p(D) \ge \# \left\{ E_l^{(i)}, 1 \le i \le 3, 1 \le l \le 2k \;\middle|\; E_l^{(i)} \text{ is contracted to } p \right\}. \tag{3.19}$$

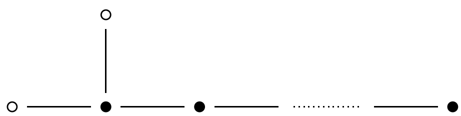
We determine $\delta_p(D)$ according to the type of the singularity $p \in D$ as in the following table:

type of $p \in D$	$A_n, n \geq 1$	$D_n, n \geq 4$	E_6	E_7	E_8
$\delta_p(D)$	$\lfloor \frac{n+1}{2} \rfloor$	$1 + \lfloor \frac{n}{2} \rfloor$	3	4	4

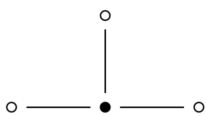
If $p \in D$ is of type A_n with n even or of type E_n with $6 \leq n \leq 8$, then there is no non-empty collection of disjoint (−2)-curves on T, which lie over p and whose sum has an even intersection number with each component over p. So in this case the right hand side of (3.19) is $0 < \delta_p(D)$.

If $p \in D$ is of type A_n with n odd then there is exactly one non-empty collection of disjoint (−2)-curves on T, which lie over p and whose sum has an even intersection number with each component over p. In terms of the following dual graph of the (−2)-curves on T lying over p, the non-empty even collection consists of the curves corresponding to the $\circ$'s and has cardinality $\lfloor \frac{n+1}{2} \rfloor = \delta_p(D)$:

Similarly, if $p \in D$ is of type D_n with $n \geq 5$ then there is exactly one non-empty collection of disjoint (−2)-curves on T, which lie over p and whose sum has an even intersection number with each component over p. In terms of the following dual graph of the (−2)-curves on T lying over p, the non-empty even collection consists of the curves corresponding to the $\circ$'s and has cardinality $2 < \delta_p(D)$:



If $p \in D$ is of type D_4 then there are three non-empty collections of disjoint (−2)-curves on T, which lie over p and whose sum has an even intersection number with each component over p. In terms of the following dual graph of the (−2)-curves on T lying over p, each of these collections consists of two of the three curves corresponding to the $\circ$'s:



so in this case

$$\#\left\{ E_l^{(i)} \,\middle|\, E_l^{(i)} \text{ is contracted to } p \right\} \leq \#\{\circ\text{'s in the dual graph}\} = 3 = \delta_p(D). \quad \square$$

4. Surfaces with quotient not of general type

Let S be a minimal surface of general type with $q(S) = 1$. In this section we assume that the Kodaira dimension of S/G_0 is 1, where we write G_0 for $\mathrm{Aut}_0(S)$. More notation is resumed from the diagram (3.1). We will let F denote a smooth fibre of the Albanese fibration $a_S\colon S \to B$, where B is the identified Albanese varieties $\mathrm{Alb}(S)$ and $\mathrm{Alb}(T)$, see (2.4).

4.1. Bounding $|\mathrm{Aut}_0(S)|$, part II

We are going to describe the canonical systems of the surfaces in the diagram (3.1). For a surface X we use ω_X and $\mathcal{O}_X(K_X)$ for the canonical sheaf interchangeably.

Since $\kappa(T) = 1$, the Albanese map $a_T\colon T \to B$ is an elliptic fibration by Lemma 2.6. The canonical bundle formula for relative minimal elliptic fibrations gives

$$\mathcal{O}_T(K_T) = a_T^*\mathfrak{b} \otimes \mathcal{O}_T\left(\sum_i (m_i - 1)F_i\right), \tag{4.1}$$

where $\mathfrak{b} = (R^1 a_{T*}\mathcal{O}_T)^\vee$ is an invertible sheaf of degree $\chi(\mathcal{O}_T) = \chi(\mathcal{O}_S)$ on B and the $m_i F_i$'s are the multiple fibres of a_T.

Since $\tilde{T}$ is obtained from T by successively blowing up smooth points, there is some effective divisor A, supported on the whole exceptional locus of $\eta\colon \tilde{T} \to T$, such that

$$\mathcal{O}_{\tilde{T}}\left(K_{\tilde{T}}\right) = \mathcal{O}_{\tilde{T}}\left(\eta^* K_T + A\right). \tag{4.2}$$

By (4.1) and (4.2) we have

$$\mathcal{O}_{\tilde{T}}\left(K_{\tilde{T}}\right) = a_{\tilde{T}}^*\mathfrak{b} \otimes \mathcal{O}_{\tilde{T}}\left(A + \sum_i (m_i - 1)\eta^* F_i\right). \tag{4.3}$$

Let $\tilde{R} = K_{\tilde{S}} - \tilde{\pi}^* K_{\tilde{T}}$ be the ramification divisor of $\tilde{\pi}$. By Lemma 2.1 there are equations between complete linear systems

$$\left|K_{\tilde{S}}\right| = \tilde{\pi}^*\left|K_{\tilde{T}}\right| + \tilde{R} = a_{\tilde{S}}^*|\mathfrak{b}| + \tilde{R} + \tilde{\pi}^* A + \sum_i (m_i - 1)\tilde{\pi}^*\eta^* F_i, \tag{4.4}$$

where $a_{\tilde{S}}\colon \tilde{S} \to B$ is the Albanese map of $\tilde{S}$.

Since $\rho\colon \tilde{S} \to S$ is a composition of blow-ups at smooth points, it is well known that $|K_S| = \rho_*|K_{\tilde{S}}|$. So (4.4) gives

$$\left|K_S\right| = \rho_*\left|K_{\tilde{S}}\right| = a_S^*|\mathfrak{b}| + \rho_*\tilde{R} + \rho_*\tilde{\pi}^* A + \sum_i (m_i - 1)\rho_*\tilde{\pi}^*\eta^* F_i. \tag{4.5}$$

Note that $\rho_*\tilde{R}$ is just the ramification divisor $R = K_S - \pi^* K_{S/G}$ of the quotient map $\pi : S \to S/G$. Every irreducible component of R is fixed (pointwise) by some nontrivial element of G, hence is smooth.

Remark 4.1. From (4.5) we see that the Albanese map $a_S\colon S\to B$ is induced by the canonical system of S if $p_g(S)>1$ and by the paracanonical system of S if $p_g(S)=1$ (*cf.* [22]).

Notation 4.2. Set $M=a_S^*\mathfrak{b}$ and $Z=\rho_*\tilde{R}+\rho_*\tilde{\pi}^*A+\sum_i(m_i-1)\rho_*\tilde{\pi}^*\eta^*F_i$.

Then (4.5) reads $|K_S|=|M|+Z$. The divisor M is algebraically equivalent to $\chi(\mathcal{O}_S)F$, and it moves if and only if $\chi(\mathcal{O}_S)>1$. On the other hand, Z always belongs to the fixed part of $|K_S|$.

We write $Z=H+V$, and $H=n_1\Gamma_1+\cdots+n_t\Gamma_t$ with $n_1\geq\cdots\geq n_t$, where H (respectively V) is the horizontal part (respectively the vertical part) of Z with respect to the Albanese fibration $a_S\colon S\to B$, and the Γ_i's are the irreducible components of H, with n_i being the multiplicity of Γ_i in H. Observe that

$$2g(F)-2=K_SF=HF=\sum_{1\leq i\leq t}n_i\Gamma_iF. \tag{4.6}$$

Obviously, the part $\rho_*\tilde{\pi}^*A+\sum_i(m_i-1)\rho_*\tilde{\pi}^*\eta^*F_i$ of Z is contained in V, so we have $H<\rho_*\tilde{R}=R$. In particular, its irreducible components Γ_i $(1\leq i\leq t)$ are all smooth. Moreover, n_i+1 is the ramification index of the quotient map $S\to S/G_0$ (equivalently, the order of the stabilizer of the G_0-action on S) at a general point of Γ_i. Since $a_{S|\Gamma_i}:\Gamma_i\to B$ is dominant we have $g(\Gamma_i)\geq g(B)=1$ for all i.

Lemma 4.3. $MH=(2g(F)-2)\chi(\mathcal{O}_S)$.

Proof. We compute

$$MH=\chi(\mathcal{O}_S)FH=\chi(\mathcal{O}_S)FK_S=(2g(F)-2)\chi(\mathcal{O}_S). \qquad\square$$

Lemma 4.4. *There are the following bounds on the K_SH, the canonical degree of H:*

$$\frac{2g(F)-2}{n_1+1}\chi(\mathcal{O}_S)+\sum_{i=1}^{t}\frac{2n_i^2}{n_1+1}(g(\Gamma_i)-1)\leq K_SH\leq(11-2g(F))\chi(\mathcal{O}_S),$$

where the second inequality is strict if $g(\Gamma_i)=1$ for some i.

Proof. Since $n_1\geq n_i$ for all $1\leq i\leq t$ by assumption, we have

$$(n_1K_S+H+V)\Gamma_i\geq(n_1K_S+n_i\Gamma_i)\Gamma_i\geq n_i(2g(\Gamma_i)-2).$$

So

$$(n_1+1)K_SH-MH=(n_1K_S+H+V)H\geq\sum_{i=1}^{t}n_i^2(2g(\Gamma_i)-2),$$

and the first inequality follows by plugging in the formula of Lemma 4.3.

We compute further

$$\begin{aligned} K_S^2 &= K_S(M + H + V) \geq K_S M + K_S H \\ &\geq MH + K_S H = (2g(F) - 2)\chi(\mathcal{O}_S) + K_S H. \end{aligned} \tag{4.7}$$

Combining this with the Bogomolov-Miyaoka-Yau inequality $K_S^2 \leq 9\chi(\mathcal{O}_S)$ we obtain $K_S H \leq (11 - 2g(F))\chi(\mathcal{O}_S)$.

Now suppose $g(\Gamma_i) = 1$ for some i. Containing an elliptic curve, the surface S cannot be a ball quotient (otherwise the elliptic curve will lift to the ball, which is absurd). Hence $K_S^2 < 9\chi(\mathcal{O}_S)$ by Yau's result [44]. By (4.7) we infer that the second inequality is strict in this case. □

The following bound on the genus of the fibration $a_S\colon S \to B$ is in the same spirit of [5, Section 2].

Corollary 4.5. *We have* $\frac{2g(F)-2}{n_1+1} < 11 - 2g(F)$. *In particular,* $g(F) \leq 5$.

By Lemma 2.5 we can analyze the action of G_0 on S by restricting to a general fibre F of the Albanese fibration $a_S\colon S \to B$, where the Riemann-Hurwitz formula applies.

Assume that the quotient map $\pi|_F\colon F \to F/G_0$ is branched at k points, over which the ramification indices are $r_1, \dots, r_k$ respectively. The following variant of the Riemann-Hurwitz formula will be used repeatedly in the proof of Theorem 4.6:

$$\frac{2g(F) - 2}{|G_0|} = \sum_{i=1}^{k} \left(1 - \frac{1}{r_i}\right), \tag{4.8}$$

the right hand side of which is at least 1 if G_0 is abelian.

Theorem 4.6. *Let S be a minimal surface of general type with* $q(S) = 1$ *such that* $\kappa(S/G_0) = 1$, *where* $G_0 := \mathrm{Aut}_0(S)$. *Then* $|G_0| \leq 4$, *and if the equality holds then the Albanese fibration of S has genus* 3.

Proof. By Corollary 4.5 we have $g(F) \leq 5$. We distinguish four cases according to the value of $g(F)$.

Case 1. $g(F) = 5$. We will show that this case does not occur.

By Corollary 4.5 we have $n_1 \geq 8$. By (4.6) it must hold $n_1 = 8$, $\Gamma_1 F = 1$ and $H = \Gamma_1$. So Γ_1 is a section of a_S. By the adjunction formula we have $K_S\Gamma_1 + \Gamma_1^2 = 0$. Since K_S is big and nef the Hodge index theorem implies that $\Gamma_1^2 < 0$. Hence Γ_1 is G_0-fixed by Lemma 2.7. Locally around a general point of Γ_1 the group action takes the form $\sigma(x, y) \mapsto (x, \chi(\sigma)y)$ for $\sigma \in G_0$, where $y = 0$ is defining equation of Γ_1 and χ is a character embedding G_0 into $\mathbb{C}^*$, so G_0 must be cyclic and its order is $n_1 + 1 = 9$ by the discussion before Lemma 4.3. This results in a contradiction to (4.8).

Case 2. $g(F) = 4$. We will show that $|G_0| \leq 3$ in this case.

By Corollary 4.5 we have $n_1 \geq 2$, so the ramification index at $\Gamma_1 \cap F$ is $r_1 = n_1 + 1 \geq 3$. If $|G_0| \geq 4$ then there are two possibilities by (4.8):

(a) $|G_0| = 8, r_1 = 4$;
(b) $|G_0| = 4, r_1 = r_2 = 4$.

In the case (a), let γ be the generator of the monodromy around the branch point q of $\pi|_F \colon F \to F/G_0$. Then γ has order $r_1 = 4$. Since $g(F/G_0) = 1$, the fundamental group $\pi_1(F/G_0 \setminus \{q\})$ has a representation $\langle a, b, c \mid aba^{-1}b^{-1}c = 1\rangle$ with a, b being generators of $\pi_1(F/G_0)$ and c a small loop around q, so that the image of c under the quotient map $\pi_1(F/G_0 \setminus \{q\}) \to G_0$ is γ. It follows that γ is a commutator of G_0. On the other hand, one sees easily that any commutator of a group of order 8 has order at most 2. So this case is excluded.

In the case (b), since G_0 contains elements of order 4, it must be isomorphic to $\mathbb{Z}/4\mathbb{Z}$. Moreover, $n_i = 3$ for all i. Then there are two possibilities for H by (4.6):

(i) $H = 3(\Gamma_1 + \Gamma_2)$ with $\Gamma_1 F = \Gamma_2 F = 1$;
(ii) $H = 3\Gamma_1$ with $\Gamma_1 F = 2$.

Let H_{red} be the reduced part of H. Since H_{red} is fixed by G_0, it is smooth and hence in the case (i) the two curves Γ_1 and Γ_2 do not intersect. We claim that

$$H_{\text{red}}^2 < 0. \tag{4.9}$$

Indeed, if H is in the case (i) then, noting that the self-intersection of a section of a_S is negative (*cf.* the proof of Case 1), we have

$$H_{\text{red}}^2 = \Gamma_1^2 + \Gamma_2^2 < 0.$$

Now assume H is in the case (ii). If $\Gamma_1^2 \geq 0$, then

$$K_S H = 3(M + 3\Gamma_1 + V)\Gamma_1 \geq 3M\Gamma_1 = 6\chi(\mathcal{O}_S),$$

a contradiction to the second inequality of Lemma 4.4. This finishes the proof of (4.9).

Let $\sigma \in G_0$ be the involution. Since each σ-fixed curve other than H_{red} is contained in fibers of a_S we have, by Lemma 2.4 and (4.9),

$$K_S^2 \leq 8\chi(\mathcal{O}_S) + H_{\text{red}}^2 < 8\chi(\mathcal{O}_S). \tag{4.10}$$

On the other hand, let X be the minimal model of S/σ and $a_X : X \to B$ the Albanese fibration. For a general $b \in B$ the fibre $a_X^* b$ is the quotient of $F_b := a_S^* b$ by σ. Since $F_b \to F_b/\sigma$ is ramified exactly at two points, the genus of a_X is 2 by the Riemann-Hurwitz formula. Thus $a_{X*}\omega_X$ is a rank two vector bundle of degree $\chi(\mathcal{O}_S)$ over B and the associated projective bundle $P := \text{Proj}(a_{X*}\omega_X)$ is a ruled surface over B. Denote by e the largest number such that there is a section Δ of $a_P: P \to B$ with $\Delta^2 = -e$. We have (*cf.* [42, page 7]):

$$e = \max\{2\deg\mathcal{L} - \deg a_{X*}\omega_X \mid \mathcal{L} \subset a_{X*}\omega_X \text{ is a sub-line-bundle}\}. \quad (4.11)$$

Since $a_X: X \to B$ is the composition of the induced rational map $X \dashrightarrow T$ with $a_T: T \to B$, the rank two vector bundle $a_{X*}\omega_X$ contains the line bundle $a_{T*}\omega_T$. Note that $\deg a_{T*}\omega_T = \chi(\mathcal{O}_T) = \chi(\mathcal{O}_S)$, so we have by (4.11)

$$e \geq 2\deg a_{T*}\omega_T - \deg a_{X*}\omega_X = \chi(\mathcal{O}_S).$$

Therefore, by [42, Theorem 2.2, (ii)], we have

$$K_X^2 \geq \chi(\mathcal{O}_X) + 3e \geq 4\chi(\mathcal{O}_X). \quad (4.12)$$

Since a_X has genus 2, it follows that X is of general type by Lemma 2.6. By Proposition 3.1 and (4.12), we obtain

$$K_S^2 \geq 2K_X^2 \geq 8\chi(\mathcal{O}_X) = 8\chi(\mathcal{O}_S),$$

which is a contradiction to (4.10). Thus we exclude the case (b).

Case 3. $g = 3$. We assume $|G_0| > 4$. Then $G_0 \cong D_6$, D_8, or Q_8 by (4.8). If $G_0 \cong D_6$ or D_8, by the proof of [12, Claim 3.8], we have

$$K_S^2 = \frac{8}{3}\chi(\mathcal{O}_S) - \frac{32}{3}(g(B) - 1) = \frac{8}{3}\chi(\mathcal{O}_S),$$

a contradiction to (4.7); if $G_0 \cong Q_8$, by the proof of [12, Claim 3.7], we have

$$K_S^2 = 3\chi(\mathcal{O}_S) + 10(g(B) - 1) = 3\chi(\mathcal{O}_S),$$

again a contradiction to (4.7).[1] So in the case $g = 3$ we have $|G_0| \leq 4$.

Case 4. $g = 2$. In this case we have $|G_0| \leq 2$ by (4.8).

This finishes the proof of Theorem 4.6. □

4.2. Surfaces with $|\mathrm{Aut}_0(S)| = 4$, part II

In this subsection we will prove the following theorem.

Theorem 4.7. *Let S be a minimal surface of general type with $q(S) = 1$ such that $\kappa(S/G_0) = 1$ and $|G_0| = 4$, where we denote by G_0 the group* $\mathrm{Aut}_0(S)$. *Then S is isogenous to a product of curves of unmixed type.*

[1] Note that in the proofs of [12, Claims 3.7 and 3.8] one does not need any condition on $\chi(\mathcal{O}_S)$ as required by the main theorem of [12].

We need some preparation for the proof of Theorem 4.7, which will be given in the end of this subsection.

By Theorem 4.6 the Albanese map $a_S : S \to B$ has genus 3. By (4.8) there are exactly 2 branch points of the quotient map $\pi|_F : F \to F/G_0$ and the ramification indices thereover are both 2. Therefore the horizontal part H of the divisors from $|K_S|$ is a reduced curve with $HF = 2g(F) - 2 = 4$.

Let γ_1, γ_2 be the stabilizers over the two branch points of $\pi|_F : F \to F/G_0$. Looking at the monodromy we see that $\gamma_1\gamma_2 = \mathrm{id}_F$ and hence $\gamma_1 = \gamma_2$. Denote by σ the $\gamma_i, i = 1, 2$. Then H is σ-fixed. It is also easy to see that

$$g(F/\sigma) = g(F/G_0) = 1. \tag{4.13}$$

Lemma 4.8. (i) $K_S^2 = 8\chi(\mathcal{O}_S) + H^2$*;* (ii) $V = 0$.

Proof. We compute

$$\begin{aligned} K_S^2 &= K_S(M + H + V) \\ &= MH + K_SH + K_SV \\ &= MH + (M + H + V)H + K_SV \\ &= 8\chi(\mathcal{O}_S) + H^2 + (H + K_S)V \qquad \text{(by Lemma 4.3)} \\ &\geq 8\chi(\mathcal{O}_S) + H^2. \end{aligned} \tag{4.14}$$

Since each σ-fixed curve other than H is contained in fibers of a_S, we have by Lemma 2.4

$$K_S^2 \leq 8\chi(\mathcal{O}_S) + H^2. \tag{4.15}$$

Combining (4.14) with (4.15) we obtain

$$K_S^2 = 8\chi(\mathcal{O}_S) + H^2, \tag{4.16}$$

$$\big(H + K_S\big)V = 0. \tag{4.17}$$

From (4.17) follows $HV = 0$ and hence $(M+H)V = 0$. This implies $V = 0$ since effective canonical divisors are 2-connected [6, VII, Proposition 6.2]. □

Corollary 4.9. *The curve H is the only curve that is fixed by a nontrivial automorphism in G_0.*

Proof. If there were another curve C that is fixed by a nontrivial automorphism in G_0 then $C \leq V$, which is a contradiction to Lemma 4.8, (ii). □

Corollary 4.10. *The minimal resolution $\tilde{T}$ of S/G_0 in* (3.1) *is minimal, that is, it does not contain any* (-1)*-curves. As a consequence $\tilde{T}$ does not contain any* (-4)*-curves.*

Proof. If $\tilde{T}$ is not minimal then there is a (-1)-curve E on it, which is necessarily not contracted by λ. The curve $\lambda(E) \subset S/G_0$ is then pulled back to be a fixed part of $|K_S|$, which is contained in V. This is a contradiction to Lemma 4.8.

For the second statement note that a relatively minimal elliptic fiberation cannot contain any (-4)-curves in the fibres for example by Kodaira's classification of singular elliptic fibres [6, V. 7]. □

Now let $\mu\colon H \to B$ be the restriction of a_S to H. Then μ is a finite map of degree 4. We remark that the degree and the ramification divisor of μ make sense even when H is not connected.

Let $R = K_H - \mu^* K_B$ be the ramification divisor of the four-to-one morphism $\mu\colon H \to B$. We may write $R = \sum_{b\in B} R_b$ where R_b is the part of R over $b \in B$. By the Riemann-Hurwitz formula and the adjunction formula,

$$\deg R = H^2 + HK_S. \tag{4.18}$$

Lemma 4.11. *For each point $p \in S$, if $p \leq R$ as effective divisors, then $F_b := a_S^* b$ is singular at p, where $b = a_S(p)$.*

Proof. There are local coordinates (x, y) such that

$$\sigma(x, y) = (x, -y).$$

Moreover, we may assume H is locally defined by $y = 0$ and F_b by

$$c_1 x + c_2 x^2 + c_3 y^2 + c_4 x y^2 + \text{higher order terms} = 0,$$

where $c_i \in \mathbb{C}$ are constants. The assumption implies that the intersection number of H and F_b at p is at least 2, so we have $c_1 = 0$ and the lemma follows. □

Lemma 4.12. *For each branch point $b \in B$ of μ, let $\epsilon(F_b) = e(F_b) + 4$ be the topological defect of the fibre $F_b := a_S^* b$, see Appendix A. Then we have $\epsilon(F_b) \geq \deg R_b$, and equality holds only if $R_b = p + q$ with $p \neq q$.*

Proof. We distinguish the two cases $G_0 \cong (\mathbb{Z}/2\mathbb{Z})^2$ and $G_0 \cong \mathbb{Z}/4\mathbb{Z}$.

Case 1. $G_0 \cong (\mathbb{Z}/2\mathbb{Z})^2$**.** We show that the morphism $\mu\colon H \to B$ is a bidouble cover. By Lemma B.1 the fibration $a_S\colon S \to B$ is hyperelliptic. Let τ be the hyperelliptic involution, which is necessarily not in G_0. Let G be the subgroup of $\mathrm{Aut}(S)$ generated by τ and G_0. Since the hyperelliptic involution of a curve of genus at least 2 commutes with all of its automorphisms, we have $G \cong (\mathbb{Z}/2\mathbb{Z})^3$.

Denote by G_H the the image of G in $\mathrm{Aut}(H)$, which is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Since $\mu\colon H \to B$ has degree 4 and factors through the quotient map $H \to H/G_H$ which also has degree 4, the two maps coincide. In particular, μ is Galois with Galois group $G_H \cong (\mathbb{Z}/2\mathbb{Z})^2$.

It follows that, for each branch point $b \in B$ of μ, the inverse image $\mu^{-1}(b)$ consists of two points, say p and q, and we have $R_b = p + q$. By Lemma 4.11, the fibre F_b is singular at both p and q. On the other hand, if $\epsilon(F_b) \le 1$ then F_b has at most one singular point by Lemma A.5 for $g = 3$. So we have $\epsilon(F_b) \ge 2 = \deg R_b$.

Case 2. $G_0 \cong \mathbb{Z}/4\mathbb{Z}$**.** Since the restriction $\alpha_H := \alpha_{|H}$ is an involution of H and $\mu : H \to B$ factors through the quotient map $H \to H/\alpha_H$, R_b is of the form either $p, p+q$ $(p \neq q)$ or $3p$. Now the lemma follows from the statements below which we will prove case by case:

(i) If $R_b = p$ then $\epsilon(F_b) > 1$;
(ii) If $R_b = p + q$ then $\epsilon(F_b) \ge 2$;
(iii) If $R_b = 3p$ then $\epsilon(F_b) > 3$.

(i) If $R_b = p$ then the point p is α-fixed. Since the curve H is σ-fixed, the action of α at $p \in S$ is of weight $\frac{1}{4}(1,2)$. By Lemmata 4.11 and A.7, p is neither a smooth point nor an ordinary node of F_b. So F_b cannot be as in Lemma A.5 and we have $\epsilon(F_b) \ge 2$.

(ii) If $R_b = p + q$ then p and q are two singular points of F_b by Lemma 4.11. Therefore F_b cannot be as in Lemma A.5 for $g = 3$ and we have $\epsilon(F_b) \ge 2 = \deg R_b$.

(iii) Since $R_b = 3p$, it follows from the fact $HF_b = 4$ that $H \cap F_b = \{p\}$ and the intersection number of H and F_b at p is 4.

We look at the action of α around $p \in S$ which is necessarily of type $\frac{1}{4}(1,2)$. There are suitable local coordinates (x, y) of S around p such that $H \subset S$ is defined by $x = 0$ and α acts as $\alpha(x, y) = (\pm\sqrt{-1}x, -y)$.

Let t be a local coordinate of B around the point b. Then, as a holomorphic function around p, the pull-back $a_S^* t$ is invariant under the action of α and takes the following form in local coordinates:

$$a_S^* t = c_1 y^2 + c_2 x^2 y + c_3 y^4 + c_4 x^4 + \text{ higher order terms}, \tag{4.19}$$

where $c_i \in \mathbb{C}$ are constants. Since the intersection number of H and F_b at p is 4, we have $c_1 = 0$ and hence the multiplicity $\mu_p(F_b) \ge 3$.

On the other hand, let F_{red} be the reduced part of F_b. Then we have by Lemma A.3

$$\epsilon(F_b) = \epsilon(F_{\text{red}}) + 2p_a(F_b) - 2p_a(F_{\text{red}}) = \epsilon(F_{\text{red}}) + 6 - 2p_a(F_{\text{red}}).$$

If $\epsilon(F_b) \le 3$ then we have either $p_a(F_{\text{red}}) = 2$ and $\epsilon(F_{\text{red}}) \le 1$ or $p_a(F_{\text{red}}) = 3$ and $\epsilon(F_{\text{red}}) \le 3$.

In the first case F_{red} cannot be smooth, otherwise $F_b = 2F_{\text{red}}$ has multiplicity two at any points, contracting the fact that $\mu_p(F_b) \ge 3$. It follows that $\epsilon(F_{\text{red}}) = 1$ and hence F_{red} has a unique node p as singularity by Lemma A.2. In particular, F_b has at most two components. We claim that $F_b = mF_{\text{red}}$ for some positive integer m. If F_b is irreducible this is clear. Otherwise F_b has two components, say C_1 and C_2. Since $F_bC_1 = F_bC_2 = 0$ and $C_1C_2 = 1$, the multiplicities of C_1 and C_2 are necessarily the same. Given $p_a(F_b) = 3$ and $p_a(F_{\mathrm{red}}) = 2$, we see that $F_b = 2F_{\mathrm{red}}$. In terms of local coordinates (x, y) around p above:

$$a_S^* t = ((ax + by)(cx + dy) + \text{terms of higher order})^2$$

for some $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$. This is a contradiction to (4.19).

In the second case $F_b = F_{\mathrm{red}}$ is reduced. Since $\mu_p(F_b) \geq 3$, we have by Lemma A.2 that $\epsilon(F_b) \geq 4 > 3$. □

Proposition 4.13. $K_S^2 = 8\chi(\mathcal{O}_S)$*, or equivalently* $e(S) = 4\chi(\mathcal{O}_S)$.

Proof. First assume $G_0 \cong (\mathbb{Z}/2\mathbb{Z})^2$. Note that the curve H is σ-fixed and the other involution σ_1 and σ_2 of G_0 do not fixed any curves (Corollary 4.9). By Lemma 2.4, applied to the involution σ_1 or σ_2, we have the equality $K_S^2 = 8\chi(\mathcal{O}_S)$.

Now assume $G_0 = \langle\alpha\rangle \cong \mathbb{Z}/4\mathbb{Z}$. Note that H is σ-fixed but not α-fixed. Applying the equivariant signature formula to α [15, 1.6], we have

$$4\mathrm{Sign}(S/\alpha) = \mathrm{Sign}(S) + H^2 + \sum_{p\in S} \mathrm{def}_p(S, \alpha), \tag{4.20}$$

where

$$\mathrm{def}_p(S, \alpha) = \begin{cases} 2 & \text{if } \alpha \text{ has weight } \frac{1}{4}(1, 3) \text{ at } p \in S \\ -2 & \text{if } \alpha \text{ has weight } \frac{1}{4}(1, 1) \text{ at } p \in S \\ 0 & \text{otherwise.} \end{cases}$$

Since σ acts trivially on $H^2(S, \mathbb{R})$, we infer that $\mathrm{Sign}(S/\sigma) = \mathrm{Sign}(S)$ and hence by (4.20)

$$3\left(K_S^2 - 8\chi(\mathcal{O}_S)\right) = 3\mathrm{Sign}(S) = H^2 + 2(k_3 - k_1), \tag{4.21}$$

where k_a $(a = 1, 3)$ is the number of isolated α-fixed points of weight $\frac{1}{4}(1, a)$.

Recall that $\lambda\colon \tilde{T} \to S/G_0$ is the minimal resolution (*cf.* (3.1)). Every fixed point of α of weight $\frac{1}{4}(1, 1)$ results in a (-4)-curve on $\tilde{T}$, which should not happen by Corollary 4.10. This implies that $k_1 = 0$. Combined with Lemma 4.8 and (4.21), we obtain

$$H^2 = k_3 \geq 0. \tag{4.22}$$

By Lemma 4.12 we have

$$\deg R = \sum_{b\in B} \deg R_b \leq \sum_{b\in B} \epsilon(F_b) = e(S), \tag{4.23}$$

where $\epsilon(F_b)$ denotes the topological defect of the fibre F_b and the last equality follows from Lemma A.4. By Lemma 4.3, (4.18) and Lemma 4.8 we have

$$\deg R = HK_S + H^2 = MH + 2H^2 = 4\chi(\mathcal{O}_S) + 2H^2 = e(S) + 3H^2. \tag{4.24}$$

Combining (4.22), (4.23) and (4.24) we obtain $H^2 = 0$. Hence $K_S^2 = 8\chi(\mathcal{O}_S)$ by Lemma 4.8.

The equivalence of the two equalities of the proposition follows from the Noether formula $12\chi(\mathcal{O}_S) = K_S^2 + e(S)$. □

Proof of Theorem 4.7. By Lemma 4.8 and Proposition 4.13, we have $H^2 = 0$ and $K_S H = (M+H)H = MH = 4\chi(\mathcal{O}_S)$. Combined with (4.18) we obtain $\deg R = 4\chi(\mathcal{O}_S) = e(S)$.

In view of (4.23) the inequality in Lemma 4.12 becomes an equality for any point $b \in B$ and in this case $\deg R_b = \epsilon(F_b) = 2$ holds for any singular fibre $F_b = a_S^* b$. Thus the singular fibres of a_S land in the list of Lemma A.6 for $g = 3$.

Write $S^\sigma = H \cup I$ where I is a finite subset of S^σ not intersecting H. Then, setting $I_b := I \cap F_b$,

$$e\left(S^\sigma\right) = e(H) + e(I) = \sum_{b\in B} (\# I_b - \deg R_b)\,. \tag{4.25}$$

Claim. For a singular fibre F_b in Lemma A.6 it holds

$$\# I_b - \deg R_b \le \deg R_b \tag{4.26}$$

with equality only if $F_b = 2C$ with C a smooth curve of genus 2.

Proof of the claim. Since $\deg R_b = 2$ it is equivalent to proving

$$\# I_b \le 4. \tag{4.27}$$

The fibre F_b is singular at the points of I_b by Lemma A.7. On the other hand, a fibre of type (ii)-(vi) in Lemma A.6 is reduced and has at most 2 singularties, so the strict inequality of (4.27) holds. If F_b is a singular fibre of type (i), *i.e.*, $F_b = 2C$ with C a smooth curve of genus 2, then $\# F_b^\sigma \le 6$ and hence

$$\# I_b = \# F_b^\sigma - \# H \cap F_b \le 4.$$

□

Plugging (4.26) into (4.25) we obtain

$$e(S) = e(S^\sigma) = \sum_{b\in B} (\# I_b - \deg R_b) \le \sum_b \deg R_b = e(S).$$

Therefore the inequality in the claim becomes an equality for any singular F_b and we infer that $F_b = 2C$ where C is a smooth curve of genus 2.

Since $K_S^2 = 8\chi(\mathcal{O}_S)$ by Proposition 4.13, we can conclude that S is a surface isogenous to a product of unmixed type by [40, Lemma 5]. □

5. Examples

In this section we construct explicitly irregular surfaces S of general type with $|\mathrm{Aut}_0(S)| = 3$ and 4. The examples of surfaces with $|\mathrm{Aut}_0(S)| = 4$ are quite exhaustive since they include (compare Theorems 3.3 and 4.6):

- Surfaces with any positive geometric genus;
- Surfaces with $g(a_S) = 5$ and $\mathrm{Aut}_0(S) \cong (\mathbb{Z}/2\mathbb{Z})^2$;
- Surfaces with $g(a_S) = 3$ and $\mathrm{Aut}_0(S) \cong (\mathbb{Z}/2\mathbb{Z})^2$;
- Surfaces with $g(a_S) = 3$ and $\mathrm{Aut}_0(S) \cong \mathbb{Z}/4\mathbb{Z}$.

For the examples of surfaces with $|\mathrm{Aut}_0(S)| = 3$ the genus of the Albanese fibration is 4.

Examples 5.2, 5.4 and 5.8 take advantage of the construction of surfaces of general type with $p_g(S) = 0$ in [10].

In Examples 5.2 and 5.4 we take the group $G \cong (\mathbb{Z}/2\mathbb{Z})^3$ together with one of the two G-coverings $C \to \bar{C} \cong \mathbb{P}^1$ in [10, 3.1] and then construct a suitable G-covering $D \to \bar{D}$ over an elliptic curve $\bar{D}$. Our surfaces are then $S = (C \times D)/\Delta_G$ and $\mathrm{Aut}_0(S)$ turns out to be a subgroup of $(G \times G)/\Delta_G$ which has an induced action on S. Here Δ_G is the diagonal of $G \times G$.

Via a similar procedure, applied to the one of the two $(\mathbb{Z}/3\mathbb{Z})^2$-coverings $C \to \bar{C} \cong \mathbb{P}^1$ in [10, 3.3] together with another $(\mathbb{Z}/3\mathbb{Z})^2$-covering $D \to \bar{D}$ with $\bar{D}$ being an elliptic curve, we construct irregular surfaces with $\mathrm{Aut}_0(S) \cong \mathbb{Z}/3\mathbb{Z}$ in Example 5.8.

It is not clear if one can use the two equivalent $(\mathbb{Z}/2\mathbb{Z})^4$-coverings in [10, 3.2] to construct irregular surfaces with $\mathrm{Aut}_0(S) \cong (\mathbb{Z}/2\mathbb{Z})^2$ in the same way. The $(\mathbb{Z}/5\mathbb{Z})^2$-coverings in [10, 3.4] do not work out, as is predicted by our bound $|\mathrm{Aut}_0(S)| \le 4$.

Example 5.6 with $\mathrm{Aut}_0(S) \cong \mathbb{Z}/4\mathbb{Z}$ does not fall into the pattern of the other examples. There the surfaces are still of the form $(C \times D)/\Delta_G$, as they should be. However, the group $\mathrm{Aut}_0(S)$ is not contained in $(G \times G)/\Delta_G$ any more.

The following result on the cohomology representation of the group of automorphisms of a curve will be used in Examples 5.2, 5.4 and 5.8.

Lemma 5.1. *Let C be a smooth curve of genus $g(C) \ge 2$ and G a finite abelian group of automorphisms of C.*

(i) *Assume $g(C/G) = 1$. Then, for any $\chi \in \widehat{G}$, $H^1(C, \mathbb{C})^\chi \neq 0$ if and only if $\chi(\sigma) \neq 1$ for some stabilizer $\langle\sigma\rangle$ over a point of C/G;*
(ii) *Assume $g(C/G) = 0$. Then, for any $\chi \in \widehat{G}$, $H^1(C, \mathbb{C})^\chi \neq 0$ if and only if there are stabilizers $\langle\sigma_1\rangle$, $\langle\sigma_2\rangle$, $\langle\sigma_3\rangle$ over 3 distinct points of C/G such that $\chi(\sigma_i) \neq 1$ for $1 \le i \le 3$.*

Proof. This is a consequence of [7, Proposition 2] or [8, page 244]. □

5.1. Examples of irregular surfaces with $|\mathrm{Aut}_0(S)| = 4$

Example 5.2 ($\mathrm{Aut}_0(S) \cong (\mathbb{Z}/2\mathbb{Z})^2$ **and** $g(a_S) = 5$)**.** We take the group $G = \langle e_1, e_2, e_3\rangle \cong (\mathbb{Z}/2\mathbb{Z})^3$. Let $\bar{C}, \bar{D}$ be two smooth curves of genera $g(\bar{C}) = 0$, $g(\bar{D}) = 1$ respectively.

By Riemann's existence theorem there is a G-covering $C \to \bar{C}$ with 6 branch points, over which the stabilizers are $\langle e_1\rangle, \langle e_1\rangle, \langle e_2\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_3\rangle$ respectively. Similarly, there is a G-covering $D \to \bar{D}$ with $2r$ branch points, over which the stabilizers are all $\langle e_1 + e_2 + e_3\rangle$.

Consider the product action of $G \times G$ on $C \times D$. Since

$$\langle e_1\rangle \cap \langle e_1 + e_2 + e_3\rangle = \langle e_2\rangle \cap \langle e_1 + e_2 + e_3\rangle = \langle e_3\rangle \cap \langle e_1 + e_2 + e_3\rangle = \{0\},$$

the induced action of the diagonal subgroup $\Delta_G \subset G \times G$ on $C \times D$ is free. Therefore $S := (C \times D)/\Delta_G$ is a surface isogenous to a product of unmixed type. One calculates easily $g(C) = 5$ and $g(D) = 4r + 1$ by Hurwitz's formula. So

$$K_S^2 = \frac{8}{|G|}(g(C) - 1)(g(D) - 1) = 16r \text{ and } \chi(\mathcal{O}_S) = \frac{1}{8}K_S^2 = 2r,$$

and our surfaces form an infinite series as r varies. The irregularity of S is $q(S) = g(\bar{C}) + g(\bar{D}) = 1$. The Albanese map of S is the induced fibration $S \to \bar{D}$ and has fibre genus $g(C) = 5$.

Consider the character χ of G such that $\chi(e_1) = \chi(e_2) = \chi(e_3) = -1$. By Lemma 5.1 this is the only character χ satisfying the following conditions:

$$H^1(C, \mathbb{C})^\chi \neq 0 \text{ and } H^1(D, \mathbb{C})^{\bar{\chi}} \neq 0.$$

Then, by the expression of $H^2(S, \mathbb{C})$ in [18, (4.5.2)], $\ker(\chi)$ acts trivially on $H^2(S, \mathbb{C})$. One also sees easily that $\ker(\chi)$ acts trivially on $H^1(S, \mathbb{C})$, so it is in fact a subgroup of $\mathrm{Aut}_0(S)$.

Now we calculate: $\ker(\chi) = \langle e_1 + e_2, e_1 + e_3\rangle \cong (\mathbb{Z}/2\mathbb{Z})^2$. On the other hand, it holds $|\mathrm{Aut}_0(S)| \leq 4$ by Theorem 1.1. Hence

$$\mathrm{Aut}_0(S) = \ker(\chi) \cong (\mathbb{Z}/2\mathbb{Z})^2.$$

Remark 5.3. As pointed out by a referee, the curve C in Example 5.2 is the so-called Kummer covering of the rational curve $\bar{C}$ of type $(2, 2, 2)$, defined by the homogeneous equations

$$z_1^2 = Q_1(x, y),\ z_2^2 = Q_2(x, y),\ z_3^2 = Q_3(x, y)$$

where $Q_i(x, y)$ are quadratic polynomials in x, y for $1 \leq i \leq 3$. Its quotient by $\ker(\chi)$ is the genus 2 curve defined by the weighted homogeneous equation

$$z^2 = Q_1(x, y)Q_2(x, y)Q_3(x, y).$$

The other curve D is the normalization of the fibre product $D_1 \times_{\bar{D}} D_2$ where $D_1 \to \bar{D}$ is an isogeny of elliptic curves of degree 2 and $D_2 \to \bar{D}$ is a double covering with the same branch locus as $D \to \bar{D}$.

Example 5.4 ($\mathrm{Aut}_0(S) \cong (\mathbb{Z}/2\mathbb{Z})^2$ **and** $g(a_S) = 3$)**.** The construction is similar to Example 5.2. Let $G = \langle e_1, e_2, e_3\rangle \cong (\mathbb{Z}/2\mathbb{Z})^3$, and let $\bar{C}$, $\bar{D}$ be two smooth curves of genera $g(\bar{C}) = 0$, $g(\bar{D}) = 1$ respectively.

We can construct by the Riemann existence theorem a G-covering $C \to \bar{C}$ with 5 branch points, over which the stabilizers are $\langle e_1\rangle, \langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_2 + e_3\rangle$ respectively and another G-covering $D \to \bar{D}$ with $2r$ branch points, over which the stabilizers are all $\langle e_1 + e_3\rangle$.

Consider the product action of $G \times G$ on $C \times D$. Since

$$\langle e_1 + e_3\rangle \cap \langle e_1\rangle = \langle e_1 + e_3\rangle \cap \langle e_2\rangle = \langle e_1 + e_3\rangle \cap \langle e_3\rangle = \langle e_1 + e_3\rangle \cap \langle e_2 + e_3\rangle = \{0\},$$

the induced action of the diagonal subgroup $\Delta_G \subset G \times G$ on $C \times D$ is free, and hence $S := (C \times D)/\Delta_G$ is a surface isogenous to a product of unmixed type. By Hurwitz's formula one computes $g(C) = 3$ and $g(D) = 4r + 1$. So

$$K_S^2 = \frac{8}{|G|}(g(C) - 1)(g(D) - 1) = 8r \text{ and } \chi(\mathcal{O}_S) = \frac{1}{8}K_S^2 = r.$$

The irregularity of S is $q(S) = g(\bar{C}) + g(\bar{D}) = 1$. The Albanese map a_S is the induced fibration $S \to \bar{D}$ and hence has fibre genus $g(C) = 3$.

The character χ of G with $\chi(e_1) = \chi(e_2) = \chi(e_1 + e_3) = -1$ is the only one satisfying the following conditions:

$$H^1(C, \mathbb{C})^{\chi} \neq 0 \text{ and } H^1(D, \mathbb{C})^{\bar{\chi}} \neq 0.$$

Using the same argument as in Example 5.2 we infer that

$$\mathrm{Aut}_0(S) = \ker(\chi) \cong (\mathbb{Z}/2\mathbb{Z})^2.$$

Remark 5.5. The genus 3 curve C in Example 5.4 is hyperelliptic by Lemma B.1. A referee writes down its affine equation as follows:

$$y^2 = (x^4 + ax^2 + 1)(x^4 + bx^2 + 1) \text{ with } a, b \in \mathbb{C} \setminus \{\pm 2\}.$$

In Examples 5.2 and 5.4, the group $\mathrm{Aut}_0(S)$ is contained in $(G \times G)/\Delta_G$, viewed as a subgroup of $\mathrm{Aut}(S)$. But this is not the case in the following example.

Example 5.6 ($\mathrm{Aut}_0(S) \cong \mathbb{Z}/4\mathbb{Z}$ **and** $g(a_S) = 3$)**.** This time take the group $G = \langle e_1, e_2\rangle \cong (\mathbb{Z}/2\mathbb{Z})^2$. Write $e_3 := e_1 + e_2$. For $j = 1, 2, 3$, let χ_j be the character of G with $\ker(\chi_j) = \langle e_j\rangle$ and χ_0 the character of the principle representation. For any $0 \leq j \leq 3$, since χ_j takes values in $\{1, -1\}$, we have $\chi_j = \bar{\chi}_j$.

Let C be a hyperelliptic curve whose affine equation is

$$y^2 = (x^4 + 1)(x^4 + a), \quad a \in \mathbb{C} \setminus \{0, 1\}.$$

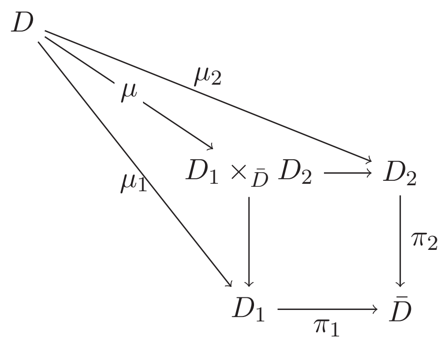
The hyperelliptic involution τ acts by $(x, y) \mapsto (x, -y)$. There is another automorphism γ of C given by $(x, y) \mapsto (\sqrt{-1}x, y)$. The 1-forms $\omega_j := \frac{x^j dx}{y}$ ($j = 0,\ 1,\ 2$) constitute a basis of $H^0(C, \Omega_C^1)$ and we have $\gamma^*\omega_j = \sqrt{-1}^{j+1}\omega_j$.

There is an action of G on C such that e_1 acts as τ and e_2 acts as γ^2. It is easy to see that e_3 acts freely on C, so $g(C/e_3) = 2$ by the Riemann-Hurwitz formula. Moreover, $g(C/e_2) = 1$ and $g(C/e_1) = 0$.

The nonzero eigenspaces of the G-action on $H^1(C, \mathbb{C})$ are as follows:

$$H^1(C, \mathbb{C})^{\chi_3} = \bigoplus_{j=2,3} \left(\mathbb{C}\omega_j \oplus \mathbb{C}\bar{\omega}_j\right), \ H^1(C, \mathbb{C})^{\chi_2} = \mathbb{C}\omega_1 \oplus \mathbb{C}\bar{\omega}_1. \tag{5.1}$$

Now let $\bar{D}$ be an elliptic curve and δ_1 and δ_2 two non-isomorphic invertible sheaves of degree r ($r > 0$) such that $\delta_1^{\otimes 2} \sim \delta_2^{\otimes 2}$. Let $B \in |\delta_1^{\otimes 2}|$ be a reduced divisor and $\pi_i \colon D_i \to \bar{D}$ the double cover defined by the data (B, δ_i). We have a commutative diagram



where μ is the normalization morphism.

For $i = 1, 2$ let β_i be the involution of D corresponding to the double cover μ_i, and write $\beta_3 = \beta_1\beta_2$. Then D is a curve of genus $g(D) = 2r + 1$ and there is an action of G on D such that e_i acts as β_i, $i = 1, 2$. By construction β_1 and β_2 act freely on D.

We have $H^1(D, \mathbb{C})^{\chi_0} = (\pi_1 \circ \mu_1)^* H^1(\bar{D}, \mathbb{C})$ and

$$\mu_j^* H^1(D_j, \mathbb{C}) = H^1(D, \mathbb{C})^{\chi_0} \oplus H^1(D, \mathbb{C})^{\chi_j} \text{ for } j = 1, 2. \tag{5.2}$$

Combining these with the equality $\sum_{\chi \in \widehat{G}} \dim_{\mathbb{C}} H^1(D, \mathbb{C})^{\chi} = \dim_{\mathbb{C}} H^1(D, \mathbb{C})$, we have $H^1(D, \mathbb{C})^{\chi_3} = 0$.

Consider the product action of $G \times G$ on $C \times D$. The induced action of diagonal subgroup Δ_G on $C \times D$ is free. So the quotient $S = (C \times D)/\Delta_G$ is a surface isogenous to a product of unmixed type, whose invariants are

$$p_g(S) = r, q(S) = 1 \text{ and } K_S^2 = 8r.$$

By the calculation of the eigenspaces of the G-actions on the cohomology groups $H^1(C,\mathbb{C})$ and $H^1(D,\mathbb{C})$ as in (5.1) and (5.2) respectively, we infer that [18, (4.5.2)]

$$\begin{aligned} H^2(S,\mathbb{C}) &= W \bigoplus \left(\bigoplus_{\chi\in\widehat{G}} H^1(C,\mathbb{C})^{\chi} \otimes H^1(D,\mathbb{C})^{\bar{\chi}} \right) \\ &= W \bigoplus H^1(C,\mathbb{C})^{\chi_2} \otimes H^1(D,\mathbb{C})^{\chi_2}, \end{aligned} \tag{5.3}$$

where $W = H^0(C,\mathbb{C}) \otimes H^2(D,\mathbb{C}) \bigoplus H^2(C,\mathbb{C}) \otimes H^0(D,\mathbb{C})$.

Let α be the automorphism of S induced by $\gamma \times \beta_3 \in \mathrm{Aut}(C \times D)$. Then α is of order 4. Note that γ and β_3 acts as $-\mathrm{id}$ on $H^1(C,\mathbb{C})^{\chi_2}$ and $H^1(D,\mathbb{C})^{\chi_2}$ respectively. Hence $\gamma \times \beta_3$ acts trivially on the right hand side of (5.3). It follows that the action of α on $H^2(S,\mathbb{C})$ is trivial. Of course, α acts trivially on $H^1(S,\mathbb{C}) = a_S^* H^1(\bar{D},\mathbb{C})$, where $a_S\colon S \to \bar{D}$ is the Albanese map. Hence α is in $\mathrm{Aut}_0(S)$. Since $|\mathrm{Aut}_0(S)| \le 4$ by Theorem 1.1, it can only happen that

$$\mathrm{Aut}_0(S) = \langle\alpha\rangle \cong \mathbb{Z}/4\mathbb{Z}.$$

Remark 5.7. (i) Example 5.4 (respectively Example 5.6) exhausts the possible values of $\chi(\mathcal{O}_S)$ and hence also of K_S^2, $p_g(S)$, $e(S)$ of irregular surfaces S with $q(S) = 1$, $g(a_S) = 3$ and $\mathrm{Aut}_0(S) \cong (\mathbb{Z}/2\mathbb{Z})^2$ (respectively $\mathrm{Aut}_0(S) \cong \mathbb{Z}/4\mathbb{Z}$).

(ii) For all of surfaces $S = (C \times D)/\Delta_G$ in the above examples the group $(G \times G)/\Delta_G \subset \mathrm{Aut}(S)$ is not contained in $\mathrm{Aut}_0(S)$. In fact, this is true more generally. Namely, let $S = (C \times D)/\Delta_G$ be a surface isogenous to a product of unmixed type with G *abelian* and $q(S) = 1$. Then the coset $(G \times G)/\Delta_G$ is well-defined as a group and has an induced action on S. The quotient of S by $(G \times G)/\Delta_G$ is isomorphic to $(C/G) \times (D/G)$ with $g(C/G) + g(D/G) = 1$ and hence has geometric genus 0. This implies that $(G \times G)/\Delta_G$ is not contained in $\mathrm{Aut}_0(S)$.

This phenomenon is reflected in the fact that the smooth fibres of the Albanese map of surfaces of general type with $q(S) = 1$ and $|\mathrm{Aut}_0(S)| = 4$ have an extra involution, see Appendix B.

All in all a remaining problem is to classify irregular surfaces of general type with $|\mathrm{Aut}_0(S)| = 4$.

5.2. Examples of irregular surfaces with $\mathrm{Aut}_0(S) \cong \mathbb{Z}/3\mathbb{Z}$

Example 5.8 ($\mathrm{Aut}_0(S) \cong \mathbb{Z}/3\mathbb{Z}$ **and** $g(a_S) = 4$)**.** Let $G = \langle e_1, e_2\rangle$ be a finite group isomorphic to $(\mathbb{Z}/3\mathbb{Z})^2$. By the Riemann existence theorem one can construct a G-covering $C \to \mathbb{P}^1$ with 4 branch points, over which the stabilizers are generated by $e_1, e_2, 2e_1, 2e_2$ respectively. By the Riemann-Hurwitz formula we have $g(C) = 4$. Similarly, one constructs another G-covering $D \to \bar{D}$ over an elliptic curve $\bar{D}$ such that there are $3r$ branch points, over which the stabilizers are all generated by $2e_1 + 2e_2$. The genus $g(D)$ is $9r + 1$ by the Riemann-Hurwitz formula.

Since the two systems of stabilizers of the coverings $C \to \mathbb{P}^1$ and $D \to \bar{D}$ as above are disjoint, the induced action of the diagonal subgroup $\Delta_G \subset G \times G$ on $C \times D$ is free. Therefore $S := (C \times D)/\Delta_G$ is a surface isogenous to a product of unmixed type with invariants

$$K_S^2 = \frac{8}{|G|}(g(C)-1)(g(D)-1) = 24r \text{ and } \chi(\mathcal{O}_S) = \frac{1}{8}K_S^2 = 3r.$$

We have $q(S) = g(\bar{D}) = 1$. The surfaces form an infinite series as r varies.

Consider the character χ such that

$$\chi(e_1) = \chi(e_2) = \exp\left(\frac{2\pi\sqrt{-1}}{3}\right).$$

By Lemma 5.1, χ and χ^2 are the only characters whose eigenspaces of the G-actions on $H^1(C, \mathbb{C})$ and $H^1(D, \mathbb{C})$ are simultaneously nonzero. By the expression of $H^2(S, \mathbb{C})$ in [18, (4.5.2)], $\ker(\chi) \cong \mathbb{Z}/3\mathbb{Z}$ acts trivially on $H^2(S, \mathbb{C})$. One also sees easily that $\ker(\chi)$ acts trivially on $H^1(S, \mathbb{C})$, so it is in fact a subgroup of $\mathrm{Aut}_0(S)$.

Since $|\mathrm{Aut}_0(S)| \leq 4$ by Theorem 1.1, it must hold

$$\mathrm{Aut}_0(S) = \ker(\chi) \cong \mathbb{Z}/3\mathbb{Z}.$$

Appendix

A. Topological defect of curves

Definition A.1. For any effective divisor D on a smooth projective surface S we define

$$\epsilon(D) = e(D) + 2p_a(D) - 2.$$

It is called the *topological defect* of D.

It is well-known that $\epsilon(D) \geq 0$ and the equality holds if and only if D is a smooth curve. If D is reduced then $\epsilon(D)$ is the sum of local contributions from the singularities.

Lemma A.2. *Let $D \subset S$ be a reduced curve on a smooth projective surface. For a point $p \in D$ denote by $\mu_p(D)$ the multiplicity of D at p. Then*

$$\epsilon(D) \geq \sum_{p \in D} (\mu_p(D) - 1)^2,$$

and the equality holds if and only if every singularity p of D is ordinary, that is, the strict transform of D in the blow-up of S at every singularity p of D contains exactly $\mu_p(D)$ points over p.

Proof. Let $\rho\colon \tilde{S} \to S$ be the simultaneous blow-up of S at all the singularities of D and E_p the exceptional divisor over a singularity p of D. Let $\tilde{D} \subset \tilde{S}$ the strict transform of D. Then $\tilde{D} = \rho^*D - \sum_p \mu_p(D)E_p$. As a set the inverse image of p in $\tilde{D}$ is $E_p \cap \tilde{D}$ and $e(\tilde{D}) = e(D) + \sum_p(\# \tilde{D} \cap E_p - 1)$. We have $\# E_p \cap \tilde{D} \le \mu_p(D) = \tilde{D}E_p$ and the equality holds if and only if $p \in D$ is an ordinary singularity. If $p \in D$ is an ordinary singularity then $\tilde{D}$ is already smooth over p.

Now we have

$$\begin{aligned}
\epsilon(D) &= e(D) + (K_S + D)D \\
&= e(D) + \big(\rho^*K_S + \rho^*D\big)\,\rho^*D \\
&= e(D) + \left(K_{\tilde{S}} + \tilde{D} + \sum_p(\mu_p(D) - 1)E_p\right)\left(\tilde{D} + \sum_p \mu_p(D)E_p\right) \\
&= e\big(\tilde{D}\big) + 2p_a\big(\tilde{D}\big) - 2 - \sum_p\left(\# \tilde{D} \cap E_p - 1\right) + \sum_p \mu_p(D)(\mu_p(D) - 1) \\
&\ge e\big(\tilde{D}\big) + 2p_a\big(\tilde{D}\big) - 2 - \sum_p(\mu_p(D) - 1) + \sum_p \mu_p(D)(\mu_p(D) - 1) \\
&= \epsilon\big(\tilde{D}\big) + \sum_p(\mu_p(D) - 1)^2 \\
&\ge \sum_p(\mu_p(D) - 1)^2.
\end{aligned}$$

If $\epsilon(D) = \sum_p(\mu_p(D) - 1)^2$ then both of the inequalities above become equalities and this is equivalent to each singularity of D being ordinary. $\square$

In case D is nonreduced the situation is more complicated. Nevertheless we will try to get a control on $\epsilon(D)$ when D is a fibre of some fibration. From the following lemma we see that the topological defect of a fibre has contributions from the singularities of the reduced part as well as the irreducible components with multiplicity.

Lemma A.3. *Let $f\colon S \to B$ be a fibration of a smooth projective surface onto a curve and F a fibre of f. Write F_{red} for the reduced part of F. Then we have*

$$\epsilon(F) \ge \epsilon(F_{\mathrm{red}}) + K_S(F - F_{\mathrm{red}}),$$

where the equality holds if and only if $F = mF_{\mathrm{red}}$ for a positive integer m.

Proof. We have

$$\begin{aligned}\epsilon(F) &= e(F_{\text{red}}) + 2p_a(F_{\text{red}}) - 2 + 2p_a(F) - 2p_a(F_{\text{red}})\\ &= \epsilon(F_{\text{red}}) + 2p_a(F) - 2p_a(F_{\text{red}})\\ &= \epsilon(F_{\text{red}}) + K_S(F - F_{\text{red}}) + F^2 - F_{\text{red}}^2\\ &= \epsilon(F_{\text{red}}) + K_S(F - F_{\text{red}}) - F_{\text{red}}^2 \qquad\qquad (\text{since } F^2 = 0)\\ &\geq \epsilon(F_{\text{red}}) + K_S(F - F_{\text{red}}),\end{aligned}$$

and the inequality becomes equality if and only if $F_{\text{red}}^2 = 0$. By Zariski's lemma [6, III, Lemma 8.2] the later is equivalent to $F = mF_{\text{red}}$ for some positive integer m. □

The usefulness of topological defects of singular fibres lies in the fact that they determine the (global) topological Euler characteristic of the fibration.

Lemma A.4 ([6, III, Proposition 11.4]). *Let $f\colon S \to B$ be a fibration from a smooth projective surface onto a smooth curve B. Let F be a smooth fibre of f and F_b a fibre over any point $b \in B$. Then*

$$e(S) = e(F)e(B) + \sum_{b\in B} \epsilon(F_b).$$

In particular, if the genus $g(B) = 1$ then $e(S) = \sum_{b\in B} \epsilon(F_b)$.

For the reader's convenience, we recall the classification of singular fibres with topological defects 1 and 2 obtained in [11].

Lemma A.5 ([11, Remark 2.6]). *Let $f\colon S \to B$ be a relatively minimal fibration of genus $g \geq 3$, and F_b a singular fibre of f. If $\epsilon(F_b) = 1$* (cf. *Definition* A.1) *then F_b belongs to one of the following types.*

(i) *An irreducible curve with exactly one node;*
(ii) *A sum of two smooth irreducible curves meeting transversally in a point.*

Lemma A.6 ([11, Lemma 2.5]). *Let f and F_b be as in Lemma A.5. If $\epsilon(F_b) = 2$ then F_b belongs to one of the following types.*

(i) *$F_b = 2C$, where C is an irreducible smooth curve of genus 2 (this case occurs only when $g = 3$);*
(ii) *F_b is an irreducible curve with exactly two nodes, and the normalization of F_b is a curve of genus $g - 2$;*
(iii) *F_b is an irreducible curve with one cusp, and the normalization of F_b is a curve of genus $g - 1$;*
(iv) *$F_b = C_1 + C_2$, where C_i are irreducible curves meeting transversally in a point, and either C_1 or C_2 (and not both) has a node;*
(v) *$F_b = C_1 + C_2$, where C_i are irreducible smooth curves meeting transversally in two points, and $g(C_1) + g(C_2) = g - 1$;*
(vi) *$F_b = C_1 + C_2 + C_3$, where C_i are irreducible smooth curves with $C_1C_2 = C_2C_3 = 1$, $C_1C_3 = 0$, and $g(C_1) + g(C_2) + g(C_3) = g$.*

We have the following description of a fibre containing an isolated fixed point of an automorphism acting on a fibration.

Lemma A.7 ([15, Lemma 1.4], [17, Lemma 2.2]). *Let $f\colon S \to B$ be a relatively minimal fibration of genus $g \geq 1$, and σ an automorphism of finite order r of S with $f \circ \sigma = f$. Let $p \in S$ be an isolated fixed point of σ and F_b the fibre containing it. Then the following holds.*

(i) *F_b is singular at p;*
(ii) *If the multiplicity $\mathrm{mult}_p F_b = 2$ and r is an odd prime, then p is a node of F_b, and the action of σ at p is of weight $\frac{1}{r}(1, r-1)$;*
(iii) *If the action of σ at p is of weight $\frac{1}{4}(1, 1)$, then $\mathrm{mult}_p F_b$ is divisible by 4;*
(iv) *If the action of σ at p is of weight $\frac{1}{4}(1, 2)$ (respectively $\frac{1}{4}(1, 3)$) and $\mathrm{mult}_p F_b = 2$, then $p \in F_b$ is not (respectively is) an ordinary node.*

B. An extra involution of curves of genera 3 and 5

Let C be a smooth projective curve and G a finite group of automorphisms. In certain situations one can lift an automorphism of the quotient C/G to C. This is the case when C is a smooth fibre of the Albanese fibration $a_S\colon S \to \mathrm{Alb}(S)$ for a surface of general type with $q(S) = 1$ and $|\mathrm{Aut}_0(S)| = 4$ and the group G is the restriction of $\mathrm{Aut}_0(S)$ to C.

In disguise the following result is contained in [38, Theorem 3.4].

Lemma B.1. *Let C be a smooth curve of genus 3. Suppose that a group $G \cong (\mathbb{Z}/2\mathbb{Z})^2$ acts faithfully on C with $g(C/G) = 1$. Then C is hyperelliptic.*

Proof. Let σ_1, σ_2 and σ_3 be the three involutions from G. Then we have $\langle\sigma_i\rangle \cap \langle\sigma_j\rangle = \{\mathrm{id}\}$ for $1 \leq i < j \leq 3$ and $G = \bigcup_{1 \leq i \leq 3} \langle\sigma_i\rangle$. The following holds by [1, Theorem. 5.9]:

$$2g(C) + 4 = \sum_{1 \leq i \leq 3} 2g(C/\sigma_i).$$

Necessarily there is an i such that $g(C/\sigma_i) = 2$. The curve C is hyperelliptic by [1, Lemma 5.10]. □

Remark B.2. Let C be a smooth curve of genus 3. Suppose that $G \cong \mathbb{Z}/4\mathbb{Z}$ acts faithfully on C with $g(C/G) = 1$. Then C also has an additional involution τ such that the group of automorphisms generated by τ and G is isomorphisc to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ if C is hyperelliptic and is isomorphic to D_8 if C is not hyperelliptic [24, Section 6.6.5].

We point out here an error in the classification of the full automorphism groups of curves of genus 3 in [33], where curves C of genus 3 with $\mathrm{Aut}(C) \cong \mathbb{Z}/4\mathbb{Z}$ and $g(C/\mathrm{Aut}(C)) = 1$ are allowed, see [33, page 295].

Lemma B.3. *Let C be a smooth curve of genus 5 and G a group of order 4 acting freely on C. Then the quotient C/G is a curve of genus 2 and the hyperelliptic involution of C/G lifts to an involution of C.*

Proof. The assertion that $g(C/G) = 2$ follows from the Riemann-Hurwitz formula. The hyperelliptic involution of C/G lifts to an involution of C by [1, Corollary 4.13 and 4.12]. □

Corollary B.4. *Let $f: S \to B$ be a relatively minimal fibration of genus 5 from a smooth projective surface S onto a smooth projective curve B. Suppose that $G \subset \mathrm{Aut}(S)$ is a finite group of automorphisms of order 4, which preserves the fibres of f and acts freely on the smooth fibres. Then there is an involution $\tau \in \mathrm{Aut}(S) \setminus G$ preserving the fibres of f.*

Proof. Let $U \subset B$ be an open subscheme, over which the fibration f is smooth. Denote $S_U = f^{-1}(U)$. Then the group G acts freely on S_U and, by Lemma B.3, the quotient fibration $h: S_U/G \to U$ has genus 2. Moreover, for any $b \in U$, the hyperelliptic involution $\bar{\tau}_b$ of the genus 2 curve f^*b/G lifts to an involution τ_b of f^*b.

Look at the following commutative diagram of fundamental groups with exact rows:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(f^*b, x) & \longrightarrow & \pi_1(S_U, x) & \longrightarrow & \pi_1(U, b) & \longrightarrow & 1 \\
 & & \big\downarrow & & \big\downarrow & & \big\| & & \\
1 & \longrightarrow & \pi_1(h^*b, \bar{x}) & \longrightarrow & \pi_1(S_U/G, \bar{x}) & \longrightarrow & \pi_1(U, b) & \longrightarrow & 1,
\end{array} \tag{B.1}$$

where $\bar{x}$ is a fixed point of the hyperelliptic involution $\bar{\tau}_b$ and $x \in f^*b$ is a point over $\bar{x}$. The fact that the hyperelliptic involution $\bar{\tau}_b$ of h^*b lifts to f^*b means that $\bar{\tau}_{b*}$ preserves the image of $\pi_1(f^*b, x) \to \pi_1(h^*b, \bar{x})$. By the commutative diagram (B.1) we infer that $\bar{\tau}_*$ preserves the image of $\pi_1(S_U, x) \to \pi_1(S_U/G, \bar{x})$, so the hyperelliptic involution $\bar{\tau}_U$ of the fibration $h: S_U/G \to U$ lifts to an automorphism τ_U of S_U, which preserves the fibres and induces the involution τ_b on f^*b. One sees immediately τ_U is an involution itself.

The relative minimality of $f: S \to B$ guarantees that the involution τ_U extends to the whole S, still preserving the fibres. □

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