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Bernstein results for symmetric minimal surfaces of controlled growth

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Abstract. We prove that there is no entire solution of the symmetric minimal surface equation which is of sublinear growth. This result is extended to parametric and non-parametric minimizers of the corresponding variational integral.

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1. Introduction

By a well known result of Bernstein [3] every entire classical solution u of the minimal surface equation

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)=0$$

in $\mathbb{R}^2$, has to be an affine-linear function. In fact this theorem was shown to hold up to dimension 7 by Fleming [17], De Giorgi [6], Almgren [1] and J. Simons [27], while there exist nonlinear entire solutions in $\mathbb{R}^n$, $n \geq 8$, as was first discovered by Bombieri-De Giorgi-Giusti [4]. Many more non-affine examples were constructed by L. Simon [25].

On the other hand Moser [21] proved that every entire solution u of the minimal surface equation in $\mathbb{R}^n$, n arbitrary, is affine linear, provided $|Du|_{0,\mathbb{R}^n}$ is finite, and it follows from the *a priori* gradient estimate of Bombieri-De Giorgi-Miranda [5] that this is already the case if u grows at most linearly, in the sense that

$$u(x) \leq C(1+|x|) \text{ for some } C>0 \text{ and all } x \in \mathbb{R}^n.$$

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Ecker and Huisken [16] extended Moser's result by requiring instead of boundedness only sublinear growth of the gradient Du, that is

$$|Du(x)| = \mathrm{o}(|x|) \text{ as } |x| \to \infty.$$

Optimal results of this type were proved by L. Simon [25,26].

In this paper we consider entire solutions of the *symmetric minimal surface equation* (in short: s.m.s.e.)

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \frac{\alpha}{u\sqrt{1+|Du|^2}}, \tag{$*$}$$

where $\alpha > 0$ denotes some positive number. $(*)$ is the Euler-equation of the variational integral

$$E(u) = \int u^\alpha \sqrt{1+|Du|^2}\, dx,$$

which, for $\alpha = m \in \mathbb{N}$ and positive $u : \Omega \to \mathbb{R}^+$, describes, up to a constant factor, the area of the rotated graph

$$\mathcal{M}_{\text{rot}} = \left\{(x, u(x)\omega) \in \mathbb{R}^n \times \mathbb{R}^{m+1}; x \in \Omega \subset \mathbb{R}^n \text{ and } \omega \in S^m\right\},$$

where $S^m \subset \mathbb{R}^{m+1}$ denotes the unit m-sphere, see, *e.g.*, the computation in [13].

A different interpretation for equation $(*)$ with $\alpha = 1$ in the two-dimensional case was already given by Poisson [23], who considered $(*)$ as a model equation for an ideal heavy surface of constant mass density which is exposed to a vertical gravitational field. Furthermore, architects consider $(*)$ as a model equation for a so called hanging roof, which is of importance for the constructions of perfect domes or cupolas, see the discussion in [22] and the literature cited therein.

The *symmetric* (or *singular*) *minimal surface equation* $(*)$ is an equation of mean curvature type, with mean curvature H given by

$$H(u, Du) = \frac{\alpha}{u\sqrt{1+|Du|^2}},$$

whence H is *a priori* not bounded, nor can a solution u of $(*)$ be of class C^2 in a neighbourhood of a point x_0 with $u(x_0) = 0$. Thus we typically consider either classical positive solutions, or weak Lipschitz solutions $u \geq 0$ of the s.m.s.e. For the existence of classical solutions of $(*)$ with prescribed boundary values we refer to the papers by Dierkes-Huisken [15] and Dierkes [12].

On the other hand, it is easily checked that the cones

$$c_n^\alpha(x) := \sqrt{\frac{\alpha}{n-1}}\left(x_1^2 + \ldots + x_n^2\right)^{\frac{1}{2}} = \sqrt{\frac{\alpha}{n-1}}\,|x|$$

are classical solutions of (∗) on $\mathbb{R}^n - \{0\}$ and weak Lipschitz-solutions on all of $\mathbb{R}^n$, for every $\alpha > 0$, $n \geq 2$. For a complete classification of these cones concerning their minimizing properties and for the construction of nonaffine entire C^∞-solution asymptotic to these cones, we refer to the papers by Dierkes [7–9].

In view of these remarks the following result is optimal.

Theorem 1.1. *There is no entire nonnegative solution* $u \in C^{0,1}(\mathbb{R}^n)$ *of the symmetric minimal surface equation* (∗) *satisfying*

$$u(x) = o(|x|) \ \text{ as } \ |x| \to \infty.$$

(*Here* $\alpha > 0, n \geq 2$ *are arbitrary*).

We also prove a version of Theorem 1.1 for less regular, local minimizers of the integral E in $\mathbb{R}^n$.

Theorem 1.2. *Let* $\alpha > 0$ *and* $u \in BV^{1+\alpha}_{+,\mathrm{loc}}(\mathbb{R}^n)$ *be a local minimizer of* E *in* $\mathbb{R}^n$ *which is of sublinear growth. Then* $u \equiv 0$.

Here the class $BV^{1+\alpha}_{+}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is open and $\alpha > 0$ is defined by

$$BV^{1+\alpha}_{+}(\Omega) := \left\{ u \in L_{1+\alpha}(\Omega) : u \geq 0 \text{ and } u^{1+\alpha} \in BV(\Omega) \right\}.$$

It is the natural function space on which the integral

$$E(u) = \int_\Omega u^\alpha \sqrt{1 + |Du|^2}\, dx$$

can be defined (as a measure) and also minimized, *cf.* the papers by Bemelmans and Dierkes [2] and [9]. Note that $\frac{1}{2}$-Hölder-continuity is the optimal regularity for minimizers of $E(\cdot)$ that can be expected in general, see the examples by Dierkes [7, 8]. Recently T. Tennstädt [28,29] proved $\frac{1}{2}$-Hölder-continuity for every minimizer in dimensions $n \leq 6$. Again, by the examples constructed in [7,8] it follows that Theorem 1.2 is optimal of its type.

Thirdly we prove an analogous result for Caccioppoli sets minimizing the parametric energy functional

$$\mathcal{E}(U) = \int |x_{n+1}|^\alpha \, |D\varphi_U| \,,$$

see Sections 3 and 4 for the pertinent definitions.

Theorem 1.3. *Let* $\alpha > 0$ *and* $U \subset \mathbb{R}^{n+1}$ *be a Caccioppoli set which locally minimizes the integral* $\mathcal{E}(\cdot)$ *in* $\mathbb{R}^{n+1}$ *and which is of sublinear growth. Then* U *is the half-space* $\{(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}\,;\ x_{n+1} \leq 0\}$ *or its complement.*

Finally we consider certain types of *exterior* solutions of the s.m.s.e. (∗) which possibly vanish on a set of positive measure.

Theorem 1.4. *Let $\alpha > 1$ and $n \geq 2$ be arbitrary. There is no non-trivial non-negative function $u \in H^1_{1,\text{loc}}(\mathbb{R}^n) \cap C^0(\mathbb{R}^n)$ which solves the symmetric minimal surface equation $(*)$ weakly in $\mathbb{R}^n - \{u = 0\}$, where the coincidence set $\{u = 0\}$ is supposed to be bounded and which is of sublinear growth in the sense that*

$$u(x) = o(|x|) \text{ as } |x| \to \infty.$$

The examples constructed in [7,8] are of class $H^1_{p,\text{loc}}(\mathbb{R}^n) \cap C^{0,\frac{1}{2}}(\mathbb{R}^n)$, for all $p < 2$, vanish on balls $\mathcal{B}_R(0) \subset \mathbb{R}^n$ and are of linear growth at infinity. Hence Theorem 1.4 is optimal.

Further Bernstein type results for stable solutions of $(*)$ in small dimensions were proved in [11].

The proofs of Theorems 1.1, 1.2, 1.3 and 1.4 follow from suitable monotonicity and area estimates given in Sections 3 and 4. The theorems are proved in Section 5.

2. Preliminaries

We here consider quite generally integer multiplicity n-rectifiable varifolds $v = v(M, \Theta)$ in $\mathbb{R}^{n+1}$ (in the sense of Allard and Simon [24]), briefly *integer n-varifolds*, that is, modulo n-dimensional Hausdorff-measure zero, a countably n-rectifiable $\mathcal{H}^n$-measurable subset M of $\mathbb{R}^{n+1}$ together with an integer valued positive and locally integrable function Θ on M. Associated to the varifold v is the Radon measure $\mu_v := \mathcal{H}^n \llcorner \Theta$, *i.e.* $\mu_v(A) = \int_A \Theta \, d\mathcal{H}^n = \int_{A \cap M} \Theta \, d\mathcal{H}^n$ for any $\mathcal{H}^n$ measurable set $A \subset \mathbb{R}^{n+1}$, where we have put $\Theta \equiv 0$ outside of M. In particular we have in mind varifolds (with multiplicity $\Theta = 1$) given by the *reduced boundary* $\partial^* E$ of a Caccioppoli set $E \subset \mathbb{R}^{n+1}$. Recall that $E \subset U \subset \mathbb{R}^{n+1}$, U open, is a set of locally *finite perimeter* (or *Caccioppoli set*) in U, if E is $\mathcal{L}^{n+1}$-measurable and if the characteristic function φ_E of E has locally finite bounded variation in U, $\varphi_E \in BV_{\text{loc}}(U)$. If $E \subset \mathbb{R}^{n+1}$ has locally finite perimeter in $U \subset \mathbb{R}^{n+1}$ there is a Radon measure $\mu_E = |D\varphi_E|$ on U and a $|D\varphi_E|$ measurable function $\nu = (\nu_1, \ldots, \nu_{n+1})$ (the generalized inward unit normal) with $\|\nu(x)\| = 1$ for $|D\varphi_E|$ a.e. $x \in U$ and such that for every $g = (g_1, \ldots, g_{n+1}) \in C^1_c(U, \mathbb{R}^{n+1})$ we have

$$\begin{aligned} \int_{E \cap U} \operatorname{div} g \, d\mathcal{L}^{n+1} &= -\int_U (g \cdot \nu) |D\varphi_E| \\ &= -\int_U g \cdot D\varphi_E, \end{aligned}$$

$D\varphi_E$ denoting the vector measure $\nu |D\varphi_E|$. Furthermore the reduced boundary $\partial^* E$ of a Caccioppoli set E is given by

$$\partial^* E = \left\{ x \in U; \quad \lim_{\rho \to 0} \frac{\int_{B_\rho(x)} \nu |D\varphi_E|}{\int_{B_\rho(x)} |D\varphi_E|} \text{ exists and has length equal to } 1 \right\}.$$

In particular we have $|D\varphi_E| = |D\varphi_E| \llcorner \partial^* E = \mathcal{H}^n \llcorner \partial^* E$, $\partial^* E$ is countably n-rectifiable and each point $x \in \partial^* E$ has an approximate tangent space T_x with multiplicity 1 given by

$$T_x = \left\{ y \in \mathbb{R}^{n+1};\ y \cdot \nu_E(x) = 0 \right\}, \quad \text{where} \quad \nu_E(x) := \lim_{\rho \to 0} \frac{\int_{B_\rho(x)} \nu |D\varphi_E|}{\int_{B_\rho(x)} |D\varphi_E|},$$

see [19] and [24] for more discussion and proofs.

Now let $v = v(M, \Theta)$ be a rectifiable n-varifold in an open set $U \subset \mathbb{R}^{n+1}$ and consider the functional

$$\mathcal{E}_\alpha(M) = \int_M |x_{n+1}|^\alpha \, d\mu_v \quad , \alpha > 0.$$

The first variation can be computed, *e.g.*, as in Simon [10,24]; for convenience we sketch the proof.

To this end consider a one parameter family Φ_t, $-1 \le t \le 1$, of diffeomorphisms of $U \subset \mathbb{R}^{n+1}$ with the following properties,

i) $\Phi_t(x) = \Phi(t, x) \in C^2\left((-1, 1) \times U, U\right)$;
ii) $\Phi_0 \equiv Id_{|U}$;
iii) $\Phi_t(x) = x$ for all $t \in [-1, 1]$ and every $x \in U - K$ for some compact set $K \subset U$.

Put $X(x) := \frac{\partial \Phi}{\partial t}(t, x)_{|t=0} \in C^1_c(U, \mathbb{R}^{n+1})$ to denote the initial velocity vector for $\Phi(t, x)$ and let $\Phi_{t\#} v$ denote the image varifold $\Phi_{t\#} v = v\big(\Phi_t(M), \Theta \circ \Phi_t^{-1}\big)$. The general area-formula ([24]) yields

$$\mathcal{E}_\alpha\left(\Phi_{t\#}(v \llcorner K)\right) = \int_{M \cap K} |\Psi_t^{n+1}|^\alpha J\Psi_t \cdot \Theta d\mathcal{H}^n,$$

where we have put $\Psi_t := \Phi_{t_{|M \cap K}}$, K compact, $K \subset U$ and $J\Psi_t$ denotes the Jacobian of Ψ_t. By definition the first variation is given by

$$\delta\mathcal{E}_\alpha(v, X) := \frac{d}{dt} \mathcal{E}_\alpha \left(\Phi_{t_\#}(v \llcorner K)\right)_{|t=0}.$$

Proposition 2.1. *Let $v = v(M, \Theta)$ be an integer n-rectifiable varifold, $\Phi_t(x) = \Phi(t, x)$ and $X(x) = \frac{\partial}{\partial t}\Phi(t, x)|_{t=0}$ be as above. Suppose either $M \subset \mathbb{R}^n \times \mathbb{R}^+$, with $\mathbb{R}^+ := \{t > 0\}$, or $\alpha > 1$, then the first variation of $\mathcal{E}_\alpha$ is given by*

$$\delta\mathcal{E}_\alpha(v) = \int_{M \cap K} |x_{n+1}|^\alpha \left(\operatorname{div}_M X(x) + \alpha \frac{X^{n+1}(x)}{x_{n+1}} \right) d\mu_v,$$

where X^{n+1} denotes the $(n+1)$-st component of the vector field $X = (X^1, \ldots, X^{n+1})$.

Proof. For convenience we sketch the argument and refer to [10,24] and [14, Chapter 3.2] for more detailed calculations. By standard arguments one finds for the Jacobian $J\Psi_t$ the development

$$J\Psi_t = 1 + t\,\mathrm{div}_M X + \mathcal{O}\big(t^2\big), \text{ also}$$
$$\left|\Psi_t^{n+1}(x)\right|^\alpha = |x_{n+1}|^\alpha \left\{1 + \alpha\, t \frac{X^{n+1}(x)}{x_{n+1}} + \mathcal{O}\big(t^2\big)\right\}.$$

The first variation formula now follows by computing the coefficient of t in the product $|\Psi_t^{n+1}(x)|^\alpha \cdot J\Psi_t$. □

Definition 2.2. The varifold $v = v(M, \Theta)$ is called *stationary* in an open set $U \subset \mathbb{R}^{n+1}$, if

$$\int_M |x_{n+1}|^\alpha \left(\mathrm{div}_M X(x) + \alpha \frac{X^{n+1}(x)}{x_{n+1}}\right) d\mu_v = 0 \tag{2.1}$$

holds for all vector fields $X(x) = \big(X^1(x), \dots, X^{n+1}(x)\big) \in C_c^1(U, \mathbb{R}^{n+1})$.

Remark 2.3. Here we either assume $\alpha > 1$ or $M \subset \mathbb{R}^n \times \mathbb{R}^+$ (or $M \subset \mathbb{R}^n \times \mathbb{R}^-$, with $\mathbb{R}^- = \{t < 0\}$).

Proposition 2.4. *Let $M \subset \mathbb{R}^{n+1}$ be a C^2-hypersurface and $U \subset \mathbb{R}^{n+1}$ be an open set, such that $M \cap U \neq \emptyset$, and $\partial M \cap U = \emptyset$ and $\mathcal{H}^n(M \cap K) < \infty$ for each compact set $K \subset U$. Then M is stationary in U if and only if the mean curvature $H = H(x)$, $x \in M \cap U$, with respect to the unit normal $\nu = (\nu_1, \dots, \nu_{n+1}) = \nu(x)$ satisfies the Euler equation*

$$|x_{n+1}|^\alpha H(x) = \alpha |x_{n+1}|^\alpha \frac{\nu_{n+1}}{x_{n+1}}. \tag{2.2}$$

Remarks 2.5.

i) Clearly, if $M \subset \mathbb{R}^n \times \mathbb{R}^+$, (2) is equivalent to $H(x) = \alpha\, \frac{\nu_{n+1}}{x_{n+1}}$, for all $x \in M$, and also, if $M = \mathrm{graph}(u)$ for some positive function $u : \Omega \to \mathbb{R}^+$, to the symmetric minimal surface equation

$$\mathrm{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right) = \frac{\alpha}{u\sqrt{1 + |Du|^2}}. \tag{2.3}$$

On the other hand, given a stationary C^2 hypersurface $M \subset \mathbb{R}^n \times \mathbb{R}$ and a point $y_0 := (\hat{y}_0, 0) \in M$, given $\hat{y}_0 \in \mathbb{R}^n$ with the property that every ball $B_\varepsilon(y_0) \subset \mathbb{R}^{n+1}$, with $\varepsilon > 0$ contains points $y_\varepsilon \in M \cap B_\varepsilon(y_0)$ with $(y_\varepsilon)_{n+1} \neq 0$ then we can conclude that

$$\lim_{\varepsilon \to 0} \left(\frac{\alpha\, \nu_{n+1}(y_\varepsilon)}{y_\varepsilon^{n+1}}\right) = H(y_0) \text{ exists;}$$

in particular $\nu_{n+1}(y_0) = 0$. Hence M intersects the coordinate plane $\{x_{n+1} = 0\}$ vertically at y_0 and can be written locally near y_0 as a graph $x_1 = f(x_2, \ldots, x_{n+1})$ say (which satisfies some singular elliptic p.d.e.);

ii) The coordinate plane $\{x_{n+1} = 0\}$ satisfies (2.2) (with $\alpha > 1$) but is not a solution of (2.3);

iii) There are Lipschitz hypersurface solutions of (2.2) given by the union of any vertical half-plane and the corresponding half-plane of the coordinate plane $\{x_{n+1} = 0\}$;

iv) There exist (Lipschitz-)continuous piecewise C^2-hypersurfaces which are $\mathcal{H}^n$-a.e. solutions of (2.2) (for $\alpha > 1$), namely the union of an n-ball $\mathcal{B}_R(0) \subset \mathbb{R}^n \times \{0\}$ and a C^2-hypersurface in $\mathbb{R}^n \times \mathbb{R}^+$ with boundary $\partial\mathcal{B}_R(0)$ given by the graph of a particular $\frac{1}{2}$-Hölder continuous function $u : \mathbb{R}^n - \mathcal{B}_R(0) \to \mathbb{R}^+ \cup \{0\}$. See the work of Dierkes [7].

Proof of Proposition 2.4. Suppose $M \subset \mathbb{R}^{n+1}$ is stationary in U and let $X(x) := \xi(x) \cdot \nu(x)$, where $\xi \in C^1_c(U, \mathbb{R})$ is arbitrary and ν is some unit normal on M. Then $\operatorname{div}_M X = \xi \operatorname{div}_M \nu = -\xi H$ and hence (2.2) follows from (2.1) and a standard device. On the other hand, if $M \in C^2$ satisfies (2.2) and $X \in C^1_c(U, \mathbb{R}^{n+1})$ is given arbitrarily, we decompose $X = X^\perp + X^\top$ into its normal part $X^\perp = (X \cdot \nu)\,\nu$ and the tangential part $X^\top \in T_x M$ respectively and compute $\operatorname{div}_M X^\perp = (X \cdot \nu) \operatorname{div}_M \nu = -H(X \cdot \nu)$. Therefore we have

$$|x_{n+1}|^\alpha \operatorname{div}_M X^\perp = -|x_{n+1}|^\alpha H(X \cdot \nu) = -\alpha |x_{n+1}|^\alpha \frac{\nu_{n+1}}{x_{n+1}} (X \cdot \nu) \tag{2.4}$$

by (2.2). Furthermore we find

$$\begin{aligned} |x_{n+1}|^\alpha \operatorname{div}_M X^\top &= \operatorname{div}_M \left(|x_{n+1}|^\alpha X^\top\right) - \nabla_M\big(|x_{n+1}|^\alpha\big)\, X^\top \\ &= \operatorname{div}_M \left\{|x_{n+1}|^\alpha X^\top\right\} - \alpha \frac{|x_{n+1}|^\alpha}{x_{n+1}} \left(\nabla_M x_{n+1} \cdot X^\top\right) \\ &= \operatorname{div}_M \left\{|x_{n+1}|^\alpha X^\top\right\} - \alpha \frac{|x_{n+1}|^\alpha}{x_{n+1}} X^{n+1} \\ &\quad + \alpha \frac{|x_{n+1}|^\alpha}{x_{n+1}} \nu_{n+1}\,(X \cdot \nu) \end{aligned} \tag{2.5}$$

where we have used the relation

$$\begin{aligned} \nabla_M x_{n+1} \cdot X^\top &= (\mathrm{e}_{n+1} - (\mathrm{e}_{n+1} \cdot \nu)\nu) \cdot X^\top \\ &= (\mathrm{e}_{n+1} - (\mathrm{e}_{n+1} \cdot \nu)\nu) \cdot X \\ &= X^{n+1} - \nu_{n+1}(X \cdot \nu), \end{aligned}$$

denoting by e_{n+1} the vector $(0, \dots, 0, 1) \in \mathbb{R}^{n+1}$. Concluding we finally obtain from (4) and (5) the identity

$$\begin{aligned}
&|x_{n+1}|^\alpha \left(\operatorname{div}_M X + \alpha \frac{X^{n+1}(x)}{x_{n+1}} \right) \\
&= \operatorname{div}_M \left\{ |x_{n+1}|^\alpha X^\top \right\} - \alpha \frac{|x_{n+1}|^\alpha}{x_{n+1}} X^{n+1} + \alpha \frac{|x_{n+1}|^\alpha}{x_{n+1}} \nu_{n+1}(X \cdot \nu) \\
&\quad - \alpha \frac{|x_{n+1}|^\alpha}{x_{n+1}} \nu_{n+1}(X \cdot \nu) + \alpha \frac{|x_{n+1}|^\alpha X^{n+1}}{x_{n+1}} \\
&= \operatorname{div}_M \left\{ |x_{n+1}|^\alpha X^\top \right\}.
\end{aligned}$$

Hence (2.1) follows from the divergence theorem since $X^\top$ has compact support on M. □

Proposition 2.6. *Let $u \in C^{0,1}(\mathbb{R}^n)$ be a weak nonnegative solution of the symmetric minimal surface equation $(*)$ in $\mathbb{R}^n$ with $\alpha > 0$. Then $M = \mathrm{graph}(u) \subset \mathbb{R}^{n+1}$ is stationary in $\mathbb{R}^{n+1}$,* i.e.

$$\int_M x_{n+1}^\alpha \left\{ \operatorname{div}_M X(x) + \alpha \frac{X^{n+1}(x)}{x_{n+1}} \right\} d\mathcal{H}^n(x) = 0$$

holds for all vector fields $X = (X^1, \dots, X^{n+1}) \in C_c^1(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$.

Remark 2.7. Note that here it is not assumed $\alpha > 1$ although the level set $\{u = 0\}$ might be nonempty. In fact we show existence of the integral in this case, even if $\alpha \in (0, 1]$.

Proof. Since $M = \{(x, u(x)) \in \mathbb{R}^n \times \mathbb{R}\}$ is the Lipschitz image of $\mathbb{R}^n$ it is countably n-rectifiable and by Schauder theory we have $u \in C^\infty(\{u > 0\})$. Whence the mean curvature of $M \cap \mathbb{R}^n \times \{t > 0\}$ is simply

$$H(x) = \alpha \frac{\nu_{n+1}}{x_{n+1}} = \frac{\alpha}{u\sqrt{1 + |Du|^2}}, \; x = (x_1, \dots, x_{n+1})$$

and by Proposition 2.4 it follows that M is stationary in $\mathbb{R}^n \times \{t > 0\}$, *i.e.*, we have the relation

$$\int_M x_{n+1}^\alpha \left\{ \operatorname{div}_M X + \alpha \frac{X^{n+1}}{x_{n+1}} \right\} d\mathcal{H}^n(x) = 0 \tag{2.6}$$

for all vector fields $X \in C_c^1(\mathbb{R}^n \times \{t > 0\}, \mathbb{R}^{n+1})$ (and, clearly, for all $X \in C_c^1(\mathbb{R}^n \times \{t \neq 0\}, \mathbb{R}^{n+1})$ since $u \geq 0$).

By assumption $u \in C^{0,1}(\mathbb{R}^n) = H^1_{\infty,\mathrm{loc}}(\mathbb{R}^n)$ is a solution of the equation

$$\int_{\mathbb{R}^n} \left\{ \frac{Du\, D\varphi}{\sqrt{1+|Du|^2}} + \frac{\alpha\varphi}{u\sqrt{1+|Du|^2}} \right\} dx = 0$$

for all $\varphi \in C^1_c(\mathbb{R}^n)$, and $|Du| \in L_{\infty,\mathrm{loc}}(\mathbb{R}^n)$ together with a standard test function argument implies that

$$\frac{1}{u} \in L_{1,\mathrm{loc}}(\mathbb{R}^n), \text{ whence also } \mathcal{L}^n(\{u=0\}) = \mathcal{H}^n(\{u=0\}) = 0.$$

For $\varepsilon > 0$ consider a smooth cutoff function $\eta_\varepsilon : \mathbb{R} \to \mathbb{R}$ given by the conditions $\eta_\varepsilon(t) = 1$, for $|t| \geq 3\varepsilon$, with $\eta_\varepsilon(t) = 0$, for $|t| \leq \varepsilon$ and $0 \leq \eta_\varepsilon \leq 1$, with $|\eta'_\varepsilon(t)| \leq \frac{1}{\varepsilon}$ for all t, hence $\eta_\varepsilon \to 1$ a.e. as $\varepsilon \to 0$. Furthermore let $X \in C^1_c(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ be an arbitrary vector field and suppose $\operatorname{supp} X \subset B_R(0) \subset \mathbb{R}^{n+1}$. The truncated vector field $X_\varepsilon(x) := \eta_\varepsilon(x_{n+1}) \cdot X(x)$ is admissible in (2.6) and since

$$\operatorname{div}_M X_\varepsilon(x) = \eta_\varepsilon(x_{n+1}) \operatorname{div}_M X + X(x) \cdot \eta'_\varepsilon(x_{n+1}) \cdot \nabla_M x_{n+1},$$

we get the relation

$$\int_{M\cap B_R} x^\alpha_{n+1} \left\{ \eta_\varepsilon(x_{n+1}) \operatorname{div}_M X + X(x)\eta'_\varepsilon(x_{n+1}) \nabla_M x_{n+1} + \alpha \frac{X^{n+1}(x)}{x_{n+1}} \eta_\varepsilon(x_{n+1}) \right\} d\mathcal{H}^n(x) = 0$$

for every $\varepsilon > 0$. The second integral can be estimated as follows

$$\begin{aligned}
&\left| \int_{M\cap B_R} x^\alpha_{n+1} \eta'_\varepsilon(x_{n+1}) X(x) \cdot \nabla_M x_{n+1} \, d\mathcal{H}^n(x) \right| \\
&\leq \sup_{M\cap B_R} |X| \int_{M\cap B_R \cap \{\varepsilon \leq x_{n+1} \leq 3\varepsilon\}} x^\alpha_{n+1} \cdot \frac{1}{\varepsilon} \, d\mathcal{H}^n(x) \\
&\leq 3 \sup_{M\cap B_R} |X| \int_{M\cap B_R \cap \{\varepsilon \leq x_{n+1} \leq 3\varepsilon\}} x^{\alpha-1}_{n+1} \, d\mathcal{H}^n(x) \\
&\leq 3\|X\|_{0,B_R} \int_{\mathcal{B}_R(0)\cap\{0\leq u\leq 3\varepsilon\}} u^{\alpha-1} \sqrt{1+|Du|^2}\, dx \\
&\leq 3\|X\|_{0,B_R} \left\{ 1 + \|Du\|^2_{0,\mathcal{B}_R} \right\}^{\frac{1}{2}} \|u^{-1}\|_{1,\mathcal{B}_R} \cdot (3\varepsilon)^\alpha \to 0, \text{ as } \varepsilon \to 0,
\end{aligned}$$

since $u^{-1} \in L_{1,\mathrm{loc}}(\mathbb{R}^n)$.

Observe in particular that the function $x^{\alpha-1}_{n+1}$ is integrable with respect to the n-dimensional Hausdorff-measure over $M \cap B_R$ for all $\alpha \geq 0$. In addition, since

$\eta_\varepsilon(x_{n+1}) \to 1$ holds $\mathcal{H}^n$-a.e. on $M \cap B_R$ (recall $\mathcal{H}^n(\{u=0\})=0$), we infer from Lebesgue's dominated convergence theorem that

$$\int_{M\cap B_R} x_{n+1}^\alpha \eta_\varepsilon(x_{n+1}) \operatorname{div}_M X(x)\, d\mathcal{H}^n(x) \to \int_{M\cap B_R} x_{n+1}^\alpha \operatorname{div}_M X(x)\, d\mathcal{H}^n(x)$$

and

$$\int_{M\cap B_R} \alpha\, x_{n+1}^{\alpha-1} X^{n+1}(x)\eta_\varepsilon(x_{n+1})\, d\mathcal{H}^n(x) \to \int_{M\cap B_R} x_{n+1}^{\alpha-1} X^{n+1}(x)\, d\mathcal{H}^n(x)$$

both as $\varepsilon \to 0$. In conclusion we have

$$\int_{M\cap B_R} x_{n+1}^\alpha \left\{ \operatorname{div}_M X(x) + \alpha \frac{X^{n+1}(x)}{x_{n+1}} \right\} d\mathcal{H}^n(x) = 0$$

for arbitrary $X \in C_c^1(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ compactly supported in the ball $B_R(0) \subset \mathbb{R}^{n+1}$. □

Similarly we prove for $\alpha > 1$.

Proposition 2.6′. *Let $\alpha > 1$ and $u : \mathbb{R}^n \to \mathbb{R}_0^+ = \{t \geq 0\}$, with $u \in H^1_{1,\mathrm{loc}}(\mathbb{R}^n) \cap C^0(\mathbb{R}^n)$, be a weak solution of the s.m.s.e. $(*)$ in $\mathbb{R}^n - \{u=0\}$. Then $M := \operatorname{graph}(u)$ is stationary in $\mathbb{R}^{n+1}$.*

Remarks 2.8.

i) Here we have in mind exterior solutions of (2.3) in $(\mathbb{R}^n - \overline{\Omega})$, where $\Omega \subset \mathbb{R}^n$ is bounded and open, which in addition satisfy $u = 0$ on $\overline{\Omega}$. Recall that there are even minima u for E of this type, where $\Omega = \mathcal{B}_R(0)$ is a ball and $u \in C^\infty(\mathbb{R}^n - \overline{B_R(0)} \cap C^{0,\frac{1}{2}}(\mathbb{R}^n) \cap H^1_{p,\mathrm{loc}}(\mathbb{R}^n)$, for all $p < 2$, see [8]. Recently, Tennstädt [28,29] proved that every local minimizer u of E is of class $H^1_{1,\mathrm{loc}} \cap C^{0,\frac{1}{2}}$, if $n \leq 6$;

ii) It was recently shown by Tennstädt [28,30] that, for minimizing functions u, the zero set $\{u=0\}$ has locally finite perimeter and is locally mean convex.

Proof. By assumption the set $\{u > 0\}$ is open and classical regularity theory implies $u \in C^2(\{u>0\})$. Furthermore $u \in H^1_{1,\mathrm{loc}}(\mathbb{R}^n) \subset BV_{\mathrm{loc}}(\mathbb{R}^n)$, whence the subgraph $U := \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : t < u(x)\}$ has locally finite perimeter given by $\int \sqrt{1+|Du|^2}\, dx$ and $M = \partial^* U = \operatorname{graph}(u)$ is n-rectifiable. Invoking Proposition 2.4 we obtain that $M = \operatorname{graph}(u)$ is stationary in $\mathbb{R}^n \times \{t \neq 0\} \subset \mathbb{R}^{n+1}$ and a similar argument as the one given in the proof of Proposition 2.6, using that now $\alpha > 1$ is assumed, finishes the proof. □

3. Monotonicity formulae

We here give two versions of the monotonicity formula; namely one for stationary varifolds and – somewhat differently – another formula for minimizing boundaries.

First assume that $v = v(M, \Theta)$ is stationary in $U \subset \mathbb{R}^{n+1}$, *i.e.* we have the identity

$$\int_M |x_{n+1}|^\alpha \left(\operatorname{div}_M X(x) + \alpha \frac{X^{n+1}(x)}{x_{n+1}} \right) d\mathcal{H}^n(x) = 0$$

for all differentiable vector fields $X = (X^1, \ldots, X^{n+1})$ with compact support in U. We choose the standard test function $X(x) := \gamma(r)(x - \xi)$, where $\xi \in U$ is fixed, $r := |x - \xi|$ and $\gamma \in C^1(\mathbb{R})$ with $\gamma'(t) \le 0$, for all $t \in \mathbb{R}$, and $\gamma(t) = 1$ for $t \le \frac{\rho}{2}$, and $\gamma(t) = 0$ for $t \ge \rho$ and $\overline{B_\rho(\xi)} \subset U$. Standard calculations (see [14,24]) yield

$$\operatorname{div}_M X(x) = \operatorname{div}_M (\gamma(r)(x - \xi)) = \gamma(r) \operatorname{div}_M (x-\xi) + \gamma'(r) \nabla_M r \cdot (x-\xi) \quad (3.1)$$

and since

$$\nabla_M r = \nabla_M |x - \xi| = \frac{(x-\xi)^\top}{|x-\xi|}$$

we have

$$\nabla_M r (x-\xi) = r \frac{(x-\xi)^\top}{|x-\xi|} \, \frac{(x-\xi)^\top}{|x-\xi|} = r \left[1 - \left(\frac{(x-\xi)^\perp}{|x-\xi|} \right)^2 \right] = r \left[1 - \left| Dr^\perp \right|^2 \right],$$

where $Dr = \frac{(x-\xi)}{|x-\xi|}$ denotes the gradient of r.

Furthermore

$$\begin{aligned} \operatorname{div}_M (x-\xi) &= \sum_{j=1}^{n+1} \mathrm{e}_j \cdot \nabla_M \big(x_j - \xi_j\big) = \sum_{j=1}^{n+1} \mathrm{e}_j \, \mathrm{e}_j^\top \\ &= \sum_{j=1}^{n+1} \mathrm{e}_j \left(\mathrm{e}_j - \mathrm{e}_j^\perp \right) = (n+1) - \sum_{j=1}^{n+1} \left(\mathrm{e}_j^\perp \right)^2 \\ &= (n+1) - \sum_{j=1}^{n+1} \left[(\nu \, \mathrm{e}_j) \cdot \nu \right]^2 = (n+1) - 1 \\ &= n, \end{aligned} \quad (3.2)$$

since $\mathrm{e}_j = \mathrm{e}_j^\top + \mathrm{e}_j^\perp$ and $\nu \, \mathrm{e}_j = \nu_j = \nu \, \mathrm{e}_j^\perp$, with $\mathrm{e}_1, \ldots, \mathrm{e}_{n+1}$ denoting the standard basis of $\mathbb{R}^{n+1}$. By (3.1), (3.2) and the first variation formula we find

$$\operatorname{div}_M X = n\gamma(r) + \gamma'(r) \, r \left(1 - \left| Dr^\perp \right|^2 \right)$$

whence

$$n\int_M |x_{n+1}|^\alpha \gamma(r)\,d\mu_v + \int_M |x_{n+1}|^\alpha \gamma'(r)\,r\left(1-\left|Dr^\perp\right|^2\right)d\mu_v + \alpha\int_M |x_{n+1}|^\alpha x_{n+1}^{-1}\gamma(r)(x_{n+1}-\xi_{n+1})\,d\mu_v = 0,$$

or

$$\begin{aligned}(n+\alpha)\int_M |x_{n+1}|^\alpha \gamma(r)\,d\mu_v + \int_M |x_{n+1}|^\alpha r\gamma'(r)\,d\mu_v \\ = \alpha\int_M |x_{n+1}|^\alpha x_{n+1}^{-1}\gamma(r)\xi_{n+1}d\mu_v + \int_M |x_{n+1}|^\alpha \gamma'(r)\,r\left|Dr^\perp\right|^2 d\mu_v.\end{aligned}\tag{3.3}$$

Now we take $\gamma(r) := \Phi\left(\frac{r}{\rho}\right)$ with $\Phi \in C^1(\mathbb{R})$ satisfying $\Phi(t) = 1$ if $t \le \frac{1}{2}$, and $\Phi(t) = 0$ if $t \ge 1$, as well as $0 \le \Phi(t) \le 1$ and $\Phi'(t) \le 0$ for all $t \in \mathbb{R}$. Then

$$r\gamma'(r) = r\Phi'\left(\frac{r}{\rho}\right)\frac{1}{\rho} = -\rho\frac{\partial}{\partial\rho}\Phi\left(\frac{r}{\rho}\right)$$

and (3.3) yields

$$\begin{aligned}(n+\alpha)\int_M |x_{n+1}|^\alpha \Phi\left(\frac{r}{\rho}\right)d\mu_v - \rho\int_M |x_{n+1}|^\alpha \frac{\partial}{\partial\rho}\Phi\left(\frac{r}{\rho}\right)d\mu_v \\ = \alpha\int_M |x_{n+1}|^\alpha x_{n+1}^{-1}\Phi\left(\frac{r}{\rho}\right)\xi_{n+1}d\mu_v - \rho\int_M |x_{n+1}|^\alpha \frac{\partial}{\partial\rho}\Phi\left(\frac{r}{\rho}\right)\left|Dr^\perp\right|^2 d\mu_v.\end{aligned}$$

Defining

$$\begin{aligned}I(\rho) &:= \int_M |x_{n+1}|^\alpha \Phi\left(\frac{r}{\rho}\right)d\mu_v \\ L(\rho) &:= \int_M |x_{n+1}|^\alpha x_{n+1}^{-1}\xi_{n+1}\Phi\left(\frac{r}{\rho}\right)d\mu_v \\ J(\rho) &:= \int_M |x_{n+1}|^\alpha \Phi\left(\frac{r}{\rho}\right)\left|Dr^\perp\right|^2 d\mu_v\end{aligned}$$

we infer the equation

$$(n+\alpha)I(\rho) - \rho I'(\rho) = \alpha L(\rho) - \rho J'(\rho)$$

and since

$$\begin{aligned}\frac{d}{d\rho}\left[\rho^{-(n+\alpha)}I(\rho)\right] &= -(n+\alpha)\rho^{-(n+\alpha+1)}I(\rho) + \rho^{-(n+\alpha)}I'(\rho) \\ &= -\rho^{-(n+\alpha+1)}\left[(n+\alpha)I - \rho I'\right]\end{aligned}$$

this implies the differential equation

$$\frac{d}{d\rho}\left(\rho^{-(n+\alpha)}I(\rho)\right) = \rho^{-(n+\alpha)}J'(\rho) - \alpha\rho^{-(n+\alpha+1)}L(\rho).$$

Integration between $0 < \sigma < \rho$ yields

$$\rho^{-(n+\alpha)}I(\rho) - \sigma^{-(n+\alpha)}I(\sigma) = \int_\sigma^\rho \tau^{-n-\alpha}J'(\tau)\,d\tau - \alpha\int_\sigma^\rho \tau^{-n-\alpha-1}L(\tau)\,d\tau$$

and upon partial integration of the first integral, then letting Φ tend to the characteristic function of the interval $(-\infty, 1)$ and finally applying Fubini's theorem, we conclude the monotonicity formula

$$\begin{aligned}\rho^{-(n+\alpha)}&\int_{B_\rho(\xi)}|x_{n+1}|^\alpha d\mu_v - \sigma^{-(n+\alpha)}\int_{B_\sigma(\xi)}|x_{n+1}|^\alpha d\mu_v \\ =&\int_{B_\rho - B_\sigma(\xi)}|x_{n+1}|^\alpha \frac{|Dr^\perp|^2}{r^{n+\alpha}}d\mu_v - \frac{\alpha\xi_{n+1}}{n+\alpha}\int_{B_\rho}\frac{|x_{n+1}|^\alpha}{x_{n+1}}\left[\frac{1}{r_\sigma^{n+\alpha}} - \frac{1}{\rho^{n+\alpha}}\right]d\mu_v\end{aligned} \tag{3.4}$$

where $r_\sigma := \max(r, \sigma)$.

In particular, if $\xi_{n+1} = 0$ we have the identity

$$\begin{aligned}\sigma^{-(n+\alpha)}\int_{B_\sigma(\xi)}|x_{n+1}|^\alpha d\mu_v = \rho^{-(n+\alpha)}&\int_{B_\rho(\xi)}|x_{n+1}|^\alpha d\mu_v \\ &- \int_{B_\rho - B_\sigma}|x_{n+1}|^\alpha \frac{|Dr^\perp|^2}{r^{n+\alpha}}d\mu_v\end{aligned} \tag{3.5}$$

and the inequality

$$\sigma^{-(n+\alpha)}\int_{B_\sigma(\xi)}|x_{n+1}|^\alpha d\mu_v \le \rho^{-(n+\alpha)}\int_{B_\rho(\xi)}|x_{n+1}|^\alpha d\mu_v, \tag{3.6}$$

holding true for all $0 < \sigma \le \rho$ with $\overline{B_\rho(\xi)} \subset U$.

We have thus proved

Proposition 3.1. *Suppose $v = v(M, \Theta)$ is stationary in $U \subset \mathbb{R}^{n+1}$ and $B_\rho(\xi) \Subset U$. Then we have the monotonicity formula* (3.4)*, and if $\xi = (\xi_1, \dots, \xi_n, 0)$ both formulae* (3.5) *or* (3.6) *hold true.*

Remark 3.2. In general we assume $\alpha > 1$ in the definition of stationarity; however if $M = \operatorname{graph} u$, where $u \ge 0$ is some Lipschitz-solution of the s.m.s.e. $(*)$ then, because of Proposition 2.6, $\alpha > 0$ is sufficient in this case. In particular we then also have the monotonicity formulae for all $\alpha > 0$ and $M =$ graph of a Lipschitz solution u. Similarly, if v is given by the reduced boundary of a minimizing set $E \subset \mathbb{R}^{n+1}$, then we conclude a monotonicity formula for all $\alpha > 0$ directly from the minimizing property of v, rather then first differentiating the functional as in Proposition 2.1, see Proposition 3.5. To show this we consider n-rectifiable varifolds $v = v(M, \Theta)$ given by the reduced boundary $\partial^* E$ of a Caccioppoli set $E \subset \mathbb{R}^{n+1}$ which locally minimizes the functional

$$\mathcal{E}(U) = \int |x_{n+1}|^\alpha \, |D\varphi_U|, \text{ for } \alpha > 0,$$

in $\mathbb{R}^{n+1}$, *i.e.*, we have

$$\int_\Omega |x_{n+1}|^\alpha \, |D\varphi_E| \le \int_\Omega |x_{n+1}|^\alpha \, |D\varphi_F|$$

for any bounded open set $\Omega \subset \mathbb{R}^{n+1}$ and all sets $F \subset \mathbb{R}^{n+1}$ with locally finite perimeter such that $F \Delta E \Subset \Omega$. In other words, if we introduce the quantities $\mathrm{N} = \mathrm{N}(E, \Omega)$ by

$$\mathrm{N}(E, \Omega) := \inf \left\{ \int_\Omega |x_{n+1}|^\alpha |D\varphi_F|;\ F \text{ has finite perimeter in } \Omega \text{ and } F\Delta E \Subset \Omega \right\}$$

and the *indicator* function $\Psi = \Psi(E, \Omega)$ by

$$\Psi(E, \Omega) := \int_\Omega |x_{n+1}|^\alpha |D\varphi_E| - \mathrm{N}(E, \Omega),$$

we consider $E \subset \mathbb{R}^{n+1}$, so that

$$\Psi(E, \Omega) = 0 \text{ for all open sets } \Omega \subset \mathbb{R}^{n+1}.$$

The following result immediately implies the monotonicity formula for minimizing boundaries, see also Giusti [19, Lemma 5.8] for a similar estimate.

Proposition 3.3. *Let $E \subset \mathbb{R}^{n+1}$ have finite perimeter in a ball $B_R(0) \subset \mathbb{R}^{n+1}$. Then for all balls $B_\sigma(0) \subset B_\rho(0) \Subset B_R(0)$ we have the estimate*

$$\begin{aligned}\left(\int_{B_\rho - B_\sigma} |x_{n+1}|^\alpha \frac{|x \cdot D\varphi_E|}{|x|^{n+\alpha+1}} \right)^2 &\le 2 \left(\int_{B_\rho - B_\sigma} |x_{n+1}|^\alpha \frac{|D\varphi_E|}{|x|^{n+\alpha}} \right) \\ &\cdot \left\{ (n+\alpha) \int_\sigma^\rho r^{-n-\alpha-1} \Psi(E, B_r)\, dr + \rho^{-n-\alpha} \int_{B_\rho} |x_{n+1}|^\alpha |D\varphi_E| \right. \\ &\left. - \sigma^{-n-\alpha} \int_{B_\sigma} |x_{n+1}|^\alpha |D\varphi_E| \right\}\end{aligned}$$

where $\alpha > 0$ and $B_\sigma = B_\sigma(0)$, $B_\rho = B_\rho(0)$.

Remark 3.4. The same result holds for arbitrary balls $B_\sigma \Subset B_\rho(\xi) \subset B_R(0)$ with center $\xi = (\xi_1, \dots, \xi_n, 0)$ lying on the coordinate hyperplane $\{x_{n+1} = 0\}$.

Proof of Proposition 3.3. Let ϕ_E^ε be a mollification of the characteristic function φ_E with the properties

$$\begin{aligned}\int_{B_r} \left|\varphi_E - \phi_E^\varepsilon\right| d\mathcal{H}^n &\to 0, \text{ as } \varepsilon \to 0,\\ \int_{B_r} |x_{n+1}|^\alpha \left|D\phi_E^\varepsilon\right| dx \to \int_{B_r} |x_{n+1}|^\alpha |D\varphi_E|&, \text{ as } \varepsilon \to 0\end{aligned} \tag{3.7}$$

for almost all $r \in [0, R]$, (see [20, Theorem 12.3]).

Define

$$\varphi_{E_{B_r}}(x) := \begin{cases}\varphi_E\left(r\dfrac{x}{|x|}\right), & \text{if } |x| \le r\\ \varphi_E(x), & \text{if } |x| > r\end{cases}$$

and

$$\eta_r^\varepsilon(x) := \phi_E^\varepsilon\left(r\frac{x}{|x|}\right).$$

First observe that

$$\begin{aligned}\int_{B_r} |\eta_r^\varepsilon - \varphi_{E_{B_r}}|\, dx &= \int_0^r \int_{\partial B_\rho} |\eta_r^\varepsilon - \varphi_{E_{B_r}}|\, d\mathcal{H}^n\, d\rho\\ &= \int_0^r \left(\frac{\rho}{r}\right)^n \int_{\partial B_r} |\eta_r^\varepsilon - \varphi_{E_{B_r}}|\, d\mathcal{H}^n d\rho\\ &= \frac{r}{n+1}\int_{\partial B_r} |\phi_E^\varepsilon - \varphi|\, d\mathcal{H}^n \to 0\end{aligned} \tag{3.8}$$

as $\varepsilon \to 0$ for almost all $r \in [0, R]$ whence by lower semicontinuity also

$$\begin{aligned}\int_{B_r} |x_{n+1}|^\alpha |D\varphi_E| - \Psi(E, B_r) &\le \int_{B_r} |x_{n+1}|^\alpha \left|D\varphi_{E_{B_r}}\right|\\ &\le \liminf_{\varepsilon\to 0} \int_{B_r} |x_{n+1}|^\alpha \left|D\eta_r^\varepsilon\right|\, dx.\end{aligned} \tag{3.9}$$

From the definition of η_r^ε we compute

$$D\eta_r^\varepsilon(x) = r\left(\frac{D\phi_E^\varepsilon\left(r\frac{x}{|x|}\right)}{|x|} - \frac{\left(D\phi_E^\varepsilon(r\frac{x}{|x|})\cdot x\right)}{|x|^3}\cdot x\right)$$

and therefore

$$\begin{aligned}
&\int_{B_r} |x_{n+1}|^\alpha \left|D\eta_r^\varepsilon\right| dx \\
&= r\int_{B_r} |x_{n+1}|^\alpha \left\{ |x|^{-2} \left| D\phi_E^\varepsilon\left(r\frac{x}{|x|}\right)\right|^2 - |x|^{-4}\left(x\cdot D\phi_E^\varepsilon\left(r\frac{x}{|x|}\right)\right)^2\right\}^{\frac{1}{2}} dx \\
&= r\int_0^r\int_{\partial B_\tau} |x_{n+1}|^\alpha |x|^{-1}\left| D\phi_E^\varepsilon\left(r\frac{x}{|x|}\right)\right| \cdot \left\{ 1 - \frac{\left(x\cdot D\phi_E^\varepsilon\left(r\frac{x}{|x|}\right)\right)^2}{|x|^2 |D\phi_E^\varepsilon\left(r\frac{x}{|x|}\right)|^2}\right\}^{\frac{1}{2}} d\mathcal{H}^n d\tau.
\end{aligned}$$

Using the transformation $x = \frac{\tau}{r} y$ we find

$$\begin{aligned}
&\int_{B_r} |x_{n+1}|^\alpha |D\eta_r^\varepsilon|\, dx \\
&= r\int_0^r\int_{\partial B_r} |y_{n+1}|^\alpha |y|^{-1}\left(\frac{\tau}{r}\right)^{\alpha-1}\left|D\phi_E^\varepsilon(y)\right|\left\{1-\frac{\left(y\cdot D\phi_E^\varepsilon(y)\right)^2}{|y|^2|D\phi_E^\varepsilon(y)|^2}\right\}^{\frac{1}{2}}\left(\frac{\tau}{r}\right)^n d\mathcal{H}^n d\tau \\
&\le r\int_0^r \left(\frac{\tau}{r}\right)^{n+\alpha-1}\int_{\partial B_r} |x_{n+1}|^\alpha r^{-1}\left|D\phi_E^\varepsilon\right|\left\{1-\frac{\left(x\cdot D\phi_E^\varepsilon(x)\right)^2}{|x|^2|D\phi_E^\varepsilon(x)|^2}\right\}^{\frac{1}{2}} d\mathcal{H}^n\, d\tau \qquad (3.10)\\
&\le \frac{r}{n+\alpha}\int_{\partial B_r} |x_{n+1}|^\alpha |D\phi_E^\varepsilon(x)| \left\{1 - \frac{1}{2}\,\frac{\left(x\cdot D\phi_E^\varepsilon(x)\right)^2}{|x|^2|D\phi_E^\varepsilon(x)|^2}\right\} d\mathcal{H}^n.
\end{aligned}$$

Now multiply (3.9) by $r^{-n-\alpha-1}$, integrate over r from σ to ρ and then employ (3.10) to obtain

$$\begin{aligned}
&\int_\sigma^\rho r^{-n-\alpha-1}\left(\int_{B_r} |x_{n+1}|^\alpha |D\varphi_E| - \Psi(E, B_r)\right) dr \\
&\le \liminf_{\varepsilon\to 0} \int_\sigma^\rho r^{-n-\alpha-1}\int_{B_r} |x_{n+1}|^\alpha \left|D\eta_r^\varepsilon\right| dx\, dr \\
&\le \liminf_{\varepsilon\to 0}\left\{\frac{1}{n+\alpha}\int_\sigma^\rho r^{-n-\alpha}\int_{\partial B_r} |x_{n+1}|^\alpha \left|D\phi_E^\varepsilon(x)\right| d\mathcal{H}^n\, dr\right. \\
&\qquad\qquad \left. - \frac{1}{2(n+\alpha)}\int_\sigma^\rho r^{-n-\alpha}\int_{\partial B_r} |x_{n+1}|^\alpha \frac{\left(x\cdot D\phi_E^\varepsilon(x)\right)^2}{|x|^2|D\phi_E^\varepsilon(x)|} d\mathcal{H}^n\, dr\right\} \\
&= \frac{1}{n+\alpha}\liminf_{\varepsilon\to 0}\left\{\rho^{-n-\alpha}\int_{B_\rho} |x_{n+1}|^\alpha |D\phi_E^\varepsilon(x)|\, dx - \sigma^{-n-\alpha}\int_{B_\sigma} |x_{n+1}|^\alpha \left|D\phi_E^\varepsilon(x)\right| dx\right. \\
&\qquad\qquad + (n+\alpha)\int_\sigma^\rho r^{-n-\alpha-1}\int_{B_r} |x_{n+1}|^\alpha |D\phi_E^\varepsilon(x)|\, dx\, dr \\
&\qquad\qquad \left. - \frac{1}{2}\int_\sigma^\rho r^{-n-\alpha}\int_{\partial B_r} |x_{n+1}|^\alpha \frac{\left(x\cdot D\phi_E^\varepsilon(x)\right)^2}{|x|^2|D\phi_E^\varepsilon(x)|} d\mathcal{H}^n\, dr\right\},
\end{aligned}$$

where in the last step we have used an integration by parts. Rearranging terms we get

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \frac{1}{2(n+\alpha)} \int_{B_\rho - B_\sigma} |x_{n+1}|^\alpha \frac{\big(x\cdot D\phi_E^\varepsilon(x)\big)^2}{|x|^{n+\alpha+2}\big|D\phi_E^\varepsilon(x)\big|}\,dx \\
&\le -\int_\sigma^\rho r^{-n-\alpha-1}\int_{B_r}|x_{n+1}|^\alpha |D\varphi_E|\,dr + \int_\sigma^\rho r^{-n-\alpha-1}\Psi(B_r)\,dr \\
&\quad + \frac{1}{(n+\alpha)}\liminf_{\varepsilon\to 0}\Big\{\rho^{-n-\alpha}\int_{B_\rho}|x_{n+1}|^\alpha\big|D\phi_E^\varepsilon(x)\big|\,dx \\
&\qquad\qquad - \sigma^{-n-\alpha}\int_{B_\sigma}|x_{n+1}|^\alpha|D\phi_E^\varepsilon(x)|\,dx \\
&\qquad\qquad + (n+\alpha)\int_\sigma^\rho r^{-n-\alpha-1}\int_{B_r}|x_{n+1}|^\alpha|D\phi_E^\varepsilon(x)|\,dx\,dr\Big\}.
\end{aligned} \tag{3.11}$$

On the other hand we apply Schwarz' inequality to obtain

$$\begin{aligned}
&\left(\int_{B_\rho-B_\sigma}|x_{n+1}|^\alpha\frac{\big|x\cdot D\phi_E^\varepsilon(x)\big|}{|x|^{n+\alpha+1}}\,dx\right)^2 \\
&\le \left(\int_{B_\rho-B_\sigma}|x_{n+1}|^\alpha\frac{\big|D\phi_E^\varepsilon(x)\big|}{|x|^{n+\alpha}}\,dx\right)\left(\int_{B_\rho-B_\sigma}|x_{n+1}|^\alpha\frac{\big(x\cdot D\phi_E^\varepsilon(x)\big)^2}{|x|^{n+\alpha+2}\big|D\phi_E^\varepsilon(x)\big|}\,dx\right)
\end{aligned}$$

and estimate the second factor with the help of (3.11). This yields the inequality

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\left(\int_{B_\rho-B_\sigma}|x_{n+1}|^\alpha\frac{\big|D\phi_E^\varepsilon(x)\cdot x\big|}{|x|^{n+\alpha+1}}\,dx\right)^2 \\
&\le \limsup_{\varepsilon\to 0} 2(n+\alpha)\int_{B_\rho-B_\sigma}|x_{n+1}|^\alpha\frac{\big|D\phi_E^\varepsilon(x)\big|}{|x|^{n+\alpha}}\,dx \\
&\quad\times\Big\{-\int_\sigma^\rho r^{-n-\alpha-1}\int_{B_r}|x_{n+1}|^\alpha|D\varphi_E|\,dr \\
&\qquad + \int_\sigma^\rho r^{-n-\alpha-1}\Psi(E,B_r)\,dr \\
&\qquad + \frac{1}{(n+\alpha)}\liminf_{\varepsilon\to 0}\Big[\rho^{-n-\alpha}\int_{B_\rho}|x_{n+1}|^\alpha\big|D\phi_E^\varepsilon(x)\big|\,dx \\
&\qquad\qquad - \sigma^{-n-\alpha}\int_{B_\sigma}|x_{n+1}|^\alpha\big|D\phi_E^\varepsilon(x)\big|\,dx \\
&\qquad\qquad + (n+\alpha)\int_\sigma^\rho r^{-n-\alpha-1}\int_{B_r}|x_{n+1}|^\alpha\big|D\phi_E^\varepsilon(x)\big|\,dx\,dr\Big]\Big\}
\end{aligned}$$

which in turn, using the approximation (3.7), proves the final estimate

$$\left(\int_{B_\rho - B_\sigma} |x_{n+1}|^\alpha \frac{|D\varphi_E \cdot x|}{|x|^{n+\alpha+1}}\right)^2 \le 2\left(\int_{B_\rho - B_\sigma} |x_{n+1}|^\alpha \frac{|D\varphi_E|}{|x|^{n+\alpha}}\right)$$
$$\cdot \Big\{(n+\alpha)\int_\sigma^\rho r^{-n-\alpha-1}\Psi(E, B_r)\,dr + \rho^{-n-\alpha}\int_{B_\rho} |x_{n+1}|^\alpha |D\varphi_E|$$
$$- \sigma^{-n-\alpha}\int_{B_\sigma} |x_{n+1}|^\alpha \big|D\varphi_E\big|\Big\}.$$
□

Proposition 3.3 immediately implies the monotonicity formula for minimizing boundaries.

Proposition 3.5. *Let $\alpha > 0$ and suppose $E \subset \mathbb{R}^{n+1}$ is a Caccioppoli set which locally minimizes $\mathcal{E}$ in $\Omega \subset \mathbb{R}^{n+1}$*, i.e. $\Psi(E, \Omega) = 0$. *Then we have the inequality*

$$\sigma^{-n-\alpha}\int_{B_\sigma} |x_{n+1}|^\alpha |D\varphi_E| \le \rho^{-n-\alpha}\int_{B_\rho} |x_{n+1}|^\alpha |D\varphi_E|$$

for all balls $B_\sigma = B_\sigma(\xi) \subset B_\rho = B_\rho(\xi) \Subset \Omega$, where $\xi = (\xi_1, \dots, \xi_n, 0) \in \mathbb{R}^n \times \{0\}$ is arbitrary.

4. Area growth

Here we suppose that $E \subset \mathbb{R}^{n+1}$ has locally finite perimeter in $\mathbb{R}^{n+1}$ and minimizes

$$\mathcal{E}(U) = \int |x_{n+1}|^\alpha |D\varphi_U| \text{ for } \alpha > 0$$

locally in $\mathbb{R}^{n+1}$ among Caccioppoli sets, *i.e.* the indicator function

$$\Psi(E, \Omega) = 0$$

for all open sets $\Omega \subset \mathbb{R}^{n+1}$. We say that E has *sublinear growth*, if there exists some nonnegative measurable function $s : \mathbb{R}^n \to \mathbb{R}^+$ such that $M = \partial^* E$ fulfills

$$M \subset \{(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R} : -s(x) \le x_{n+1} \le s(x)\} \tag{4.1}$$

and

$$\lim_{R\to\infty} \frac{|s|_{\infty,\mathcal{B}_R(0)}}{R} = 0. \tag{4.2}$$

Here $\mathcal{B}_R(0) \subset \mathbb{R}^n$ denotes the n-ball with center at $0 \in \mathbb{R}^n$ and $|s|_{\infty,\mathcal{B}_R}$ stands for the sup-norm of s on $\mathcal{B}_R$. Analogously a function $u \in BV_{\text{loc}}(\mathbb{R}^n)$ is of *sublinear growth*, if the subgraph

$$U := \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : t < u(x)\}$$

has sublinear growth.

Proposition 4.1. *Let $E \subset \mathbb{R}^{n+1}$ be a Caccioppoli set which locally minimizes $\mathcal{E}$ in $\mathbb{R}^{n+1}$ for some $\alpha > 0$ and suppose $M = \partial^* E$ is of sublinear growth. Then we have*

$$\lim_{R\to\infty} R^{-n-\alpha} \int_{B_R(0)} |x_{n+1}|^\alpha |D\varphi_E| = 0 \ \ \textit{for} \ \ B_R(0) \subset \mathbb{R}^{n+1}.$$

Remark 4.2. Proposition 4.1 is sharp as one sees by considering the cones

$$C_n^\alpha := \left\{ (x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R} : \ \ 0 < x_{n+1} < \sqrt{\frac{\alpha}{n-1}} \|x\| \right\}$$

which are of linear growth and minimize

$$\mathcal{E} = \int |x_{n+1}|^\alpha |D\varphi_U|,$$

if, for example, $n = 2$ and $\alpha \geq 6$ say, see [7,8] for more details. Also, one easily computes

$$\int_{B_R(0)} |x_{n+1}|^\alpha \left|D\varphi_{C_n^\alpha}\right| = c(n, \alpha) R^{n+\alpha}$$

for some constant $c(n, \alpha) > 0$.

Proof. Define the cylinder

$$C_R := \left\{ (x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R} : \quad |x| < R \text{ and } -|s|_{\infty, \mathcal{B}_R} < x_{n+1} < |s|_{\infty, \mathcal{B}_R} \right\}$$

where $s : \mathbb{R}^n \to \mathbb{R}^+$ is some *dominance function* with the properties (4.1) and (4.2). The minimum property of E implies for any $\varepsilon > 0$

$$\begin{aligned} \mathcal{E}(E, C_{R+\varepsilon}) :&= \int_{C_{R+\varepsilon}} |x_{n+1}|^\alpha |D\varphi_E| \leq \int_{C_{R+\varepsilon}} |x_{n+1}|^\alpha \left|D\varphi_{E-\overline{C_R}}\right| \\ &= \mathcal{E}\left(E - \overline{C_R}, C_{R+\varepsilon}\right) \end{aligned} \tag{4.3}$$

and the trace formula for BV-functions yields for almost all R, and $\varepsilon > 0$

$$\mathcal{E}\left(E - \overline{C_R}, C_{R+\varepsilon}\right) = \mathcal{E}\left(E, C_{R+\varepsilon} - \overline{C_R}\right) + \int_{\partial C_R \cap E} |x_{n+1}|^\alpha d\mathcal{H}_n \tag{4.4}$$

and similarly also

$$\begin{aligned} \mathcal{E}(E, C_{R+\varepsilon}) &\leq \int_{C_{R+\varepsilon}} |x_{n+1}|^\alpha \left|D\varphi_{E\cup\overline{C_R}}\right| \\ &= \mathcal{E}\left(E \cup \overline{C_R}, C_{R+\varepsilon}\right) \\ &= \mathcal{E}\left(E, C_{R+\varepsilon} - \overline{C_R}\right) + \int_{\partial C_R \cap (\mathbb{R}^{n+1} - E)} |x_{n+1}|^\alpha d\mathcal{H}_n. \end{aligned} \tag{4.5}$$

Formulae (4.3), (4.4) and (4.5) imply the estimate

$$\begin{aligned}\mathcal{E}(E, C_{R+\varepsilon}) &= \int_{C_{R+\varepsilon}} |x_{n+1}|^\alpha |D\varphi_E| \\ &\le \mathcal{E}\left(E, C_{R+\varepsilon} - \overline{C_R}\right) \\ &\quad + \min\left\{\int_{\partial C_R \cap E} |x_{n+1}|^\alpha d\mathcal{H}_n, \int_{\partial C_R \cap (\mathbb{R}^{n+1} - E)} |x_{n+1}|^\alpha d\mathcal{H}_n\right\}\end{aligned}$$

which in turn yields for almost all $R > 0$, as $\varepsilon \to 0$

$$\mathcal{E}(E, C_R) \le \min\left\{\int_{\partial C_R \cap E} |x_{n+1}|^\alpha d\mathcal{H}_n, \int_{\partial C_R \cap (\mathbb{R}^{n+1} - E)} |x_{n+1}|^\alpha d\mathcal{H}_n\right\}. \tag{4.6}$$

We put $\partial C_R = Z_R \cup D_R^+ \cup D_R^-$, where

$$Z_R := \left\{(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R} : |x| = R \text{ and } -|s|_{\infty,\mathcal{B}_R} \le x_{n+1} \le |s|_{\infty,\mathcal{B}_R}\right\}$$

denotes the vertical wall and

$$D_R^\pm := \left\{(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R} : |x| \le R, x_{n+1} = \pm|s|_{\infty,\mathcal{B}_R}\right\}$$

denote the top and bottom of the cylinder ∂C_R respectively. We find the estimate

$$\begin{aligned}\int_{\partial C_R} |x_{n+1}|^\alpha d\mathcal{H}_n &= \int_{D_R^+ \cup D_R^-} |x_{n+1}|^\alpha d\mathcal{H}_n + \int_{Z_R} |x_{n+1}|^\alpha d\mathcal{H}_n \\ &\le 2\omega_n R^n |s|^\alpha_{\infty,\mathcal{B}_R} + \frac{\omega_n}{1+\alpha} R^{n-1} |s|^{1+\alpha}_{\infty,\mathcal{B}_R}\end{aligned}$$

whence, by virtue of (4.6) also

$$R^{-n-\alpha} \int_{C_R} |x_{n+1}|^\alpha |D\varphi_E| \le c(n,\alpha)\left\{R^{-\alpha}|s|^\alpha_{\infty,\mathcal{B}_R} + R^{-\alpha-1}|s|^{1+\alpha}_{\infty,\mathcal{B}_R}\right\}.$$

Finally, by assumption $M = \partial^* E \subset \{(x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}; -s(x) < x_{n+1} < s(x)\}$, whence $M \cap B_R(0) \subset C_R$ and together with (4.6) and (4.2) we conclude

$$\lim_{R\to\infty} R^{-n-\alpha} \int_{B_R(0)} |x_{n+1}|^\alpha |D\varphi_E| = 0. \qquad \square$$

The proof of the following Proposition is standard, see, *e.g.*, [18, Chapter 16]. For convenience we give the argument in some detail.

Proposition 4.3. *Let $u \in H^1_{1,\mathrm{loc}}(\mathbb{R}^n - K)$, with $K \subset \mathbb{R}^n$ compact, be a weak nonnegative solution of the s.m.s.e.* (2.3) *in $(\mathbb{R}^n - K)$ and let $K \subset \mathcal{B}_{R_0}(0) \subset \mathbb{R}^n$. Then for every $\rho > R_0 + 1$ the following area estimate holds:*

$$\int_{M \cap B_\rho(0)} x^\alpha_{n+1} d\mathcal{H}_n \le c(n)\rho^n |u|^\alpha_{\infty,\mathcal{B}_\rho - \mathcal{B}_{R_0+1}} + |u|^\alpha_{\infty,\mathcal{B}_\rho - \mathcal{B}_{R_0+1}} |u|_{1,\mathcal{B}_{R_0+1} - \mathcal{B}_{R_0}},$$

where $M :=$ graph $u_{|\mathcal{B}_\rho - \mathcal{B}_{R_0+1}}$ and $|u|_{p,\Omega}$ denotes the L_p-norm of u on Ω.

Proof. Choose $\rho > R_0 + 1$ and some cut-off function $\eta \in C_c^{0,1}(\mathbb{R}^n - K)$ with the properties

$$\eta(x) = \begin{cases} 1, & \text{if } R_0 + 1 \le |x| \le \rho \\ 0, & \text{if } |x| \le R_0 \text{ or } |x| \ge 2\rho, \end{cases}$$

and such that a.e.

$$|D\eta| \le \begin{cases} 1 & \text{for } R_0 \le |x| \le R_0 + 1 \\ 0 & \text{for } R_0 + 1 < |x| < \rho \\ \frac{1}{\rho} & \text{for } \rho \le |x| \le 2\rho. \end{cases}$$

Put $\varphi := \eta \cdot u_\rho$, where u_ρ denotes the truncated function

$$u_\rho := \begin{cases} u & \text{on } \{0 \le u < \rho\} \\ \rho & \text{on } \{u \ge \rho\}. \end{cases}$$

Then it holds a.e.

$$Du_\rho := \begin{cases} Du & \text{on } \{0 \le u < \rho\} \\ 0 & \text{on } \{u \ge \rho\}, \end{cases}$$

and $\varphi \in \mathring{H}_1^1(\mathcal{B}_{2\rho} - K)$ satisfies $D\varphi = D\eta \cdot u_\rho + \eta Du_\rho$ a.e. Upon substitution of φ and $D\varphi$ into the weak formulation of (2.3)

$$\int_{\mathbb{R}^n - K} \left(\frac{Du\, D\varphi}{\sqrt{1+|Du|^2}} + \frac{\alpha\varphi}{u\sqrt{1+|Du|^2}} \right) dx = 0$$

we arrive at

$$\int_{\mathcal{B}_{2\rho} - \mathcal{B}_{R_0}} \left\{ \frac{Du\, D\eta\, u_\rho}{\sqrt{1+|Du|^2}} + \frac{Du\, Du_\rho \eta}{\sqrt{1+|Du|^2}} + \frac{\alpha \eta u_\rho}{u\sqrt{1+|Du|^2}} \right\} dx = 0.$$

Since $Du_\rho = 0$ on $\{u \ge \rho\}$ a.e. we find

$$\begin{aligned} \int_{(\mathcal{B}_{2\rho} - \mathcal{B}_{R_0}) \cap \{u < \rho\}} \frac{|Du|^2 \eta}{\sqrt{1+|Du|^2}}\, dx = &- \int_{\mathcal{B}_{2\rho} - \mathcal{B}_{R_0}} \frac{Du\, D\eta\, u_\rho}{\sqrt{1+|Du|^2}}\, dx \\ &- \alpha \int_{\mathcal{B}_{2\rho} - \mathcal{B}_{R_0}} \frac{u_\rho \eta}{u\sqrt{1+|Du|^2}}\, dx. \end{aligned}$$

In particular, because of $\eta = 1$, if $R_0 + 1 \le |x| \le \rho$, with $0 \le \eta \le 1$ and $u, u_\rho \ge 0$ we obtain

$$\int_{(\mathcal{B}_\rho - \mathcal{B}_{R_0+1}) \cap \{u < \rho\}} \frac{|Du|^2}{\sqrt{1+|Du|^2}} \le \int_{\mathcal{B}_{2\rho} - \mathcal{B}_{R_0}} \frac{u_\rho |Du|\, |D\eta|}{\sqrt{1+|Du|^2}}\, dx$$

and hence

$$\int_{(\mathcal{B}_\rho-\mathcal{B}_{R_0+1})\cap\{u<\rho\}}\sqrt{1+|Du|^2}\,dx \le \mathcal{L}^n(\mathcal{B}_\rho-\mathcal{B}_{R_0+1})+\int_{\mathcal{B}_{2\rho}-\mathcal{B}_\rho}\frac{u_\rho|Du|\,|D\eta|}{\sqrt{1+|Du|^2}}\,dx + \int_{\mathcal{B}_{R_0+1}-\mathcal{B}_{R_0}}\frac{u_\rho|Du|\,|D\eta|}{\sqrt{1+|Du|^2}}\,dx.$$

Using $0\le u_\rho\le u$, and $0\le u_\rho\le\rho$, with $|D\eta|\le\frac{1}{\rho}$ on $\{\rho\le|x|\le 2\rho\}$ and $|D\eta|\le 1$ on $\{R_0\le|x|\le R_0+1\}$ we find

$$\begin{aligned}&\int_{(\mathcal{B}_\rho-\mathcal{B}_{R_0+1})\cap\{u<\rho\}}\sqrt{1+|Du|^2}\,dx\\ &\le \mathcal{L}^n(\mathcal{B}_\rho-\mathcal{B}_{R_0+1})+\mathcal{L}^n(\mathcal{B}_{2\rho}-\mathcal{B}_\rho)+|u|_{1,\mathcal{B}_{R_0+1}-\mathcal{B}_{R_0}}\\ &\le c_1(n)\rho^n+|u|_{1,\mathcal{B}_{R_0+1}-\mathcal{B}_{R_0}}.\end{aligned}$$

Thus we have

$$\int_{(\mathcal{B}_\rho-\mathcal{B}_{R_0+1})\cap\{u<\rho\}}u^\alpha\sqrt{1+|Du|^2}\,dx\le|u|^\alpha_{\infty,\mathcal{B}_\rho-\mathcal{B}_{R_0+1}}\Big\{c_1(n)\rho^n+|u|_{1,\mathcal{B}_{R_0+1}-\mathcal{B}_{R_0}}\Big\}$$

and in particular, with $M=\operatorname{graph} u_{|\mathcal{B}_\rho-\mathcal{B}_{R_0+1}}$, it holds

$$\int_{M\cap B_\rho(0)}x^\alpha_{n+1}\,d\mathcal{H}_n\le c_1(n)\rho^n|u|^\alpha_{\infty,\mathcal{B}_\rho-\mathcal{B}_{R_0+1}}+|u|^\alpha_{\infty,\mathcal{B}_\rho-\mathcal{B}_{R_0+1}}|u|_{1,\mathcal{B}_{R_0+1}-\mathcal{B}_{R_0}}. \qquad \square$$

5. Proofs

In this section we prove the four main theorems.

Proof of Theorem 1.1. Suppose on the contrary to the statement of Theorem 1.1, there is a Lipschitz-solution $u\ge 0$ of the s.m.s.e. $(*)$ which satisfies the growth condition

$$u(x)=\mathrm{o}(|x|)\ \text{ as }\ |x|\to\infty.$$

By Propositions 2.6 and 3.1, especially formula (3.6) applied to $M=\operatorname{graph}(u)$, with $d\mu=d\mathcal{H}_n$ and $\xi=0\in\mathbb{R}^{n+1}$ we get for all $0<\sigma<\rho<\infty$ the inequality

$$\sigma^{-n-\alpha}\int_{B_\sigma(0)\cap M}x^\alpha_{n+1}\,d\mathcal{H}^n\le\rho^{-n-\alpha}\int_{B_\rho(0)\cap M}x^\alpha_{n+1}\,d\mathcal{H}^n.$$

Since $\mathcal{L}^n(\{u=0\})=0$ there is some $\sigma_0>0$ with

$$\sigma_0^{-n-\alpha}\int_{B_{\sigma_0}\cap M}x^\alpha_{n+1}\,d\mathcal{H}^n>0.$$

However, according to Proposition 4.3 we must have

$$\lim_{\rho\to\infty} \rho^{-n-\alpha} \int_{B_\rho\cap M} x_{n+1}^\alpha d\mathcal{H}^n = 0,$$

an obvious contradiction. □

Proof of Theorem 1.2. Let $u \in BV^{1+\alpha}_{+,\mathrm{loc}}(\mathbb{R}^n)$ be a local minimum of the variational integral

$$E = \int u^\alpha \sqrt{1+|Du|^2} \ \text{ for } \ \alpha > 0$$

in the class $BV^{1+\alpha}_{+}(\Omega)$, with $\Omega \subset \mathbb{R}^n$ arbitrary. Then we have $u \in BV_{\mathrm{loc}}(\mathbb{R}^n)$ (in fact $u \in H^1_{1,\mathrm{loc}}(\mathbb{R}^n)$ according to Tennstädt [28]) and the subgraph

$$U := \left\{(x,t) \in \mathbb{R}^{n+1};\, t < u(x)\right\}$$

has locally finite perimeter in $\mathbb{R}^{n+1}$. By [2, Theorem 10], the subgraph U locally minimizes

$$\mathcal{E}(U) = \int |x_{n+1}|^\alpha |D\varphi_U|$$

in $\mathbb{R}^{n+1}$. (In fact, in the paper [2] only the case $\alpha = 1$ is considered, however the generalization to arbitrary $\alpha > 0$ is straightforward!.) Now we are in the situation described in Proposition 3.5 with minimizing set U and arbitrary open set $\Omega \subset \mathbb{R}^{n+1}$. For $\xi = 0$ and $0 < \sigma < \rho < \infty$ arbitrary we get

$$\sigma^{-n-\alpha} \int_{B_\rho} |x_{n+1}|^\alpha |D\varphi_U| \le \rho^{-n-\alpha} \int_{B_\rho(0)} |x_{n+1}|^\alpha |D\varphi_U|.$$

By virtue of Proposition 4.1 and by letting $\rho \to \infty$ we finally arrive at

$$\int_{B_\sigma(0)} |x_{n+1}|^\alpha |D\varphi_U| = 0$$

for every $\sigma > 0$, hence $\partial U = \{x_{n+1} = 0\}$. □

Proof of Theorem 1.3. Theorem 1.3 follows from Propositions 3.5 and 4.1 analogously to the proof to Theorem 1.2. □

Proof of Theorem 1.4. Suppose on the contrary to the statement of Theorem 1.4, that there is a non-trivial $u \in H^1_{1,\mathrm{loc}}(\mathbb{R}^n) \cap C^0(\mathbb{R}^n)$ which solves the s.m.s.e. weakly in $\mathbb{R}^n - \{u = 0\}$ and which is of sublinear growth. By Proposition 3′ $M = \mathrm{graph}(u)$ is stationary in $\mathbb{R}^{n+1}$. Proposition 3.1, formula (3.6) with $\xi = 0$, Proposition 4.3, and the assumption of sublinear growth imply that

$$\sigma^{-n-\alpha} \int_{B_\sigma(0)\cap M} x_{n+1}^\alpha d\mathcal{H}_n = 0$$

for every $\sigma > 0$ and $M = \text{graph}(u) \subset \mathbb{R}^{n+1}$; whence we had $u = 0$ on $\mathbb{R}^n$. This contradiction concludes the proof of Theorem 1.4. □

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