Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XVIII (2018), 1125-1143

Comparison of the real and the complex Green functions, and sharp estimates of the Kobayashi distance

NIKOLAI NIKOLOV AND PASCAL J. THOMAS

Abstract. We extend the upper estimates obtained by M. Carlehed [2] and B.-Y. Chen [3] about the ratio of the classical and pluricomplex Green functions to the case of $\mathcal{C}^2$-smooth locally $\mathbb{C}$-convexifiable domains of finite type. We also give some lower estimates. In order to obtain these results, and because it is of independent interest, we refine and unify some classical estimates about the Kobayashi distance and the Lempert function in such domains.

Mathematics Subject Classification (2010): 32F45 (primary); 32U45 (secondary).

1. Introduction and results

1.1. Green functions

Two kinds of Green functions can be defined on a domain $D \subset \mathbb{C}^n \cong \mathbb{R}^{2n}$, with $n \geq 2$: the usual one, related to harmonic (or subharmonic) functions when D is seen as subdomain of $\mathbb{R}^{2n}$, and the pluricomplex Green function (see, *e.g.*, [10]), related to plurisubharmonic functions.

The pluricomplex Green function originated with the work of Lempert [11,13], Lelong [12], among others, and is the subject of many recent works, see for instance [7,20].

Let G_D stand for the usual *Green function* at a pole w in $D \subset \mathbb{R}^m$, with $m \geq 3$, given by

$$G_D(z, w) = \sup\left\{u(z) : u \in SH_-(D),\ u = |\cdot - w|^{-m+2} + O(1)\right\}.$$

Let g_D stand for the *pluricomplex Green function* at a pole w in $D \subset \mathbb{C}^n$, $n \geq 2$, given by

$$g_D(z, w) = \sup\left\{u(z) : u \in PSH_-(D),\ u = \log|\cdot - w| + O(1)\right\}.$$

Received August 24, 2016; accepted in revised form March 14, 2017.
Published online May 2018.

Here $SH_-(D)$ and $PSH_-(D)$ stand for negative subharmonic, respectively plurisubharmonic, functions on D.

Note that for $n = 1$ the second extremal problem also gives the usual Green functions for the Laplacian on $\mathbb{R}^2$.

The respective behavior of those two functions were compared by M. Carlehed [2] and B.-Y. Chen [3]. In the present paper, we extend their results to a wider class of domains, and give some improved estimates for various holomorphic invariants such as the Kobayashi distance in that class of domains.

We would like to thank the referee for his very careful reading of our manuscript and several useful suggestions.

1.2. Domains in $\mathbb{C}^n$

In order to state the results, we need to define some geometric properties of a domain in $\mathbb{C}^n$. From now on, we assume that $n \geq 2$. As usual, we say that ∂D, or D, is $\mathcal{C}^k$-smooth if $D = \{\rho < 0\}$, where ρ is a defining function of class $\mathcal{C}^k$ on $\overline{D}$ such that $\nabla\rho$ does not vanish on ∂D. A $\mathcal{C}^2$-smooth domain is *strictly pseudoconvex* if the complex Hessian of ρ restricted to the complex tangent space at every point of ∂D is positive definite.

A domain D is $\mathbb{C}$-*convex* if any non-empty intersection of D with a complex line is connected and simply connected. If D is bounded and $\mathcal{C}^1$-smooth, this is equivalent to being *lineally convex*, that is to say, for any $z \notin D$, there exists a complex hyperplane H through z such that $D \cap H = \emptyset$. For more on those two notions, see, *e.g.*, [1,8].

A domain D is $\mathbb{C}$-*convexifiable* if D is biholomorphic to a $\mathbb{C}$-convex domain.

A domain D is *locally* ($\mathbb{C}$-)*convexifiable*, if for any $a \in \partial D$, there exist a neighborhood U of a and a holomorphic embedding $\Phi : U \to \mathbb{C}^n$ such that $\Phi(D \cap U)$ is a ($\mathbb{C}$-)convex, domain.

It is well-known that any strictly pseudoconvex domain is locally convexifiable.

The *type* of a smooth boundary point a of a domain D is the supremum over the orders of contact of the one-dimensional analytic varieties through a with ∂D (possibly ∞). The type of a smooth domain D is defined as the supremum over the types of all boundary points of D. For instance, the bounded domains of type 2 are exactly the strictly pseudoconvex domains. Also, the types of the pseudoconvex domains are even numbers or ∞. If the domain is $\mathbb{C}$-convex, the type does not change considering complex lines instead of varieties (see, *e.g.*, [18, Proposition 6]).

1.3. Notations and auxiliary quantities

We will systematically use the following notations: $A \gtrsim B$ means that there is a constant $C > 0$ such that $A \geq CB$; $A \asymp B$ means that $A \gtrsim B$ and $B \gtrsim A$; and $A \sim B$ means that $A/B \to 1$. What the constants depend on, and in which sense the limit is taken, will be made clear from context.

The Green functions we consider take negative values and, when ∂D is smooth enough, tend to 0 at the boundary. A typical negative plurisubharmonic function is $\log|f|$, where f is a holomorphic function bounded by 1; so it will be convenient to consider e^{g_D}. Consideration of the Poincaré distance p in the unit disc $\mathbb{D}$, $p(w,z) = \tanh^{-1}\left|\frac{z-w}{1-\bar{z}w}\right|$, makes it expedient to consider $\tanh^{-1} e^{g_D}$.

We give a unified convention.

Definition 1.1. Given any continuous function $f : D \to (-\infty, 0)$, we write

$$f^* := e^f, \text{ so } f^* : D \to (0,1), \tag{1.1}$$

$$\tilde{f} := \tanh^{-1} f^* = \tanh^{-1} e^f = \frac{1}{2}\log\frac{1+e^f}{1-e^f}, \text{ so } \tilde{f} : D \to (0,\infty). \tag{1.2}$$

Conversely, $f^* = \tanh \tilde{f} = \frac{e^{2\tilde{f}}-1}{e^{2\tilde{f}}+1}$, and $f = \log f^*$.

Elementary calculations give:

Lemma 1.2.

(i) *Suppose that $f \to 0^-$, or equivalently $f^* \to 1^-$, or equivalently $\tilde{f} \to \infty$. Then $1 - f^* \sim -f$, $\tilde{f} \sim -\frac{1}{2}\log(-f)$, and $f \sim -2e^{-2\tilde{f}}$; in particular if $\tilde{f} = \log t$, then $f \sim -\frac{2}{t^2}$.*

(ii) *Suppose that $f \to -\infty$, or equivalently $f^* \to 0^+$, or equivalently $\tilde{f} \to 0^+$. Then $\tilde{f} \sim f^*$ and $f = \log \tilde{f} + O(1)$.*

1.4. The ratio of the Green functions

Our first main result is the extension to the case of locally $\mathbb{C}$-convexifiable domains of a theorem proved in the case of locally convexifiable domains [3, Theorem 1].

Theorem 1.3. *Let $D \subset \mathbb{C}^n$ be a bounded, smooth, locally $\mathbb{C}$-convexifiable domain of type $2m$. Then there exists $C > 0$ such that*

$$\frac{g_D(z,w)}{G_D(z,w)} \le C|z-w|^{2(n-2m)}, \quad z, w \in D,\ z \ne w.$$

For $z \in D$, let $\delta_D(z) := \min\{|z-w| : w \notin D\}$ (the distance to the boundary). Any bounded, $\mathcal{C}^{1,1}$-smooth domain D is of *positive reach*, that is to say, there exists $\delta_0 > 0$ such that for any $z \in D$ with $\delta_D(z) < \delta_0$, there exists a unique point $\pi(z) \in \partial D$ such that $|z - \pi(z)| = \delta_D(z)$.

Recall the following estimate of G_D, when D is bounded, $\mathcal{C}^{1,1}$-smooth domain in $\mathbb{R}^m$, $m \ge 3$ (see, *e.g.*, [21, (7)]):

$$c_1 G_D(z,w) \le -\min\left\{\frac{1}{|z-w|^{m-2}}, \frac{\delta_D(z)\delta_D(w)}{|z-w|^m}\right\} \le c_2 G_D(z,w), \tag{1.3}$$

where $c_1, c_2 > 0$ are constants, and $z, w \in D$.

The proof of Theorem 1.3 will rely on the second inequality in (1.3), and the following precise estimate of the pluricomplex Green function g_D which is sensitive in both extreme cases: $g_D \to 0$ and $g_D \to -\infty$.

Theorem 1.4. *Let D be as in Theorem* 1.3. *Then there exists $C > 0$ such that for any $z, w \in D$,*

$$\tilde{g}_D(z, w) \geq m \log\left(1 + C\frac{|z-w|}{\delta_D(z)^{1/2m}}\right)\left(1 + C\frac{|z-w|}{\delta_D(w)^{1/2m}}\right). \tag{1.4}$$

In the more general case of a bounded, smooth, pseudoconvex domain of finite type, a weaker estimate is proved by G. Herbort [7, Theorem 1.1].

The proof of Theorem 1.4 will be based on the respective local estimates, covering the cases where either the pole or the argument tends to a boundary point.

Theorem 1.5. *Let $D \subset \mathbb{C}^n$ be a bounded domain, which is smooth and locally $\mathbb{C}$-convexifiable near point $a \in \partial D$ of type $2m$. Then there exist a neighborhood U of a and $C > 0$ such that*

$$\tilde{g}_D(z, w) \geq m \log\left(1 + C\frac{|z-w|}{\delta_D(w)^{1/2m}}\right), \quad z \in D,\ w \in D \cap U, \tag{1.5}$$

$$\tilde{g}_D(z, w) \geq m \log\left(1 + C\frac{|z-w|}{\delta_D(z)^{1/2m}}\right), \quad z \in D \cap U,\ w \in D. \tag{1.6}$$

In the particular case when D is locally convexifiable, similar but weaker estimates than those in the above two theorems are contained in [3].

1.5. Other holomorphic invariants

We will use other holomorphically contractive functions, with notation sometimes slightly different from those of the standard reference [10], to stay in line with the convention from Definition 1.1. In particular, note that the Kobayashi pseudodistance in a domain D will be called $\tilde{k}_D$, while $k_D := \log\tanh\tilde{k}_D \in (-\infty, 0)$. This is because our main focus is on (negative-valued) Green functions.

Let $D \subset \mathbb{C}^n$, and $z, w \in D$.

The *Lempert function* is given by

$$\tilde{l}_D(z, w) := \inf\left\{p(\zeta, \omega) : \zeta, \omega \in \mathbb{D}, \exists\varphi \in \mathcal{O}(\mathbb{D}, D) : \varphi(\zeta) = z, \varphi(\omega) = w\right\}.$$

With the notation convention from Definition 1.1, this means that

$$l_D^*(z, w) := \inf\left\{\left|\frac{\zeta - \omega}{1 - \bar{\zeta}\omega}\right| : \zeta, \omega \in \mathbb{D}, \exists\varphi \in \mathcal{O}(\mathbb{D}, D) : \varphi(\zeta) = z, \varphi(\omega) = w\right\},$$

and that $l_D(z, w) = \log l_D^*(z, w) \in (-\infty, 0)$, a quantity that is easier to compare with the Green function.

The *Kobayashi-Royden (pseudo)metric* applied to a vector $X \in \mathbb{C}^n$ is given by

$$\kappa_D(z; X) := \inf\left\{\lambda > 0 : \exists\varphi \in \mathcal{O}(\mathbb{D}, D) : \varphi(0) = z, \lambda\varphi'(0) = X\right\}.$$

The *Kobayashi (pseudo)distance* is the largest pseudodistance dominated by the Lempert function. It is also given by

$$\tilde{k}_D(z, w) := \inf_\gamma \int_0^1 \kappa_D(\gamma(t); \gamma'(t))dt,$$

where the infimum is taken over all $\mathcal{C}^1$-smooth curves $\gamma : [0, 1] \to D$ with $\gamma(0) = z$ and $\gamma(1) = w$. Then $k_D(z, w) = \log\tanh\left(\tilde{k}_D(z, w)\right)$.

We have that

$$k_D \le l_D, \quad g_D \le l_D. \tag{1.7}$$

Lempert's celebrated theorem [13] implies that in the case of a convex domain, those are all equalities. This extends to the case of bounded, $\mathcal{C}^2$-smooth, $\mathbb{C}$-convex domains [9]. No inequality holds in general between $\tilde{k}_D$ and $\tilde{g}_D$; and while $\tilde{k}_D$ is symmetric in its arguments, $\tilde{g}_D$ is not always so, but we will see that under our hypotheses, they exhibit similar behavior.

1.6. Lower estimates of the Kobayashi distance

Theorem 1.6. *Let D be as in Theorem 1.3. Then there exists $C > 0$ such that for any $z, w \in D$,*

$$\tilde{k}_D(z, w) \ge m \log\left(1 + C\frac{|z - w|}{\delta_D(z)^{1/2m}}\right)\left(1 + C\frac{|z - w|}{\delta_D(w)^{1/2m}}\right). \tag{1.8}$$

This will follow from the corresponding local sharp result.

Theorem 1.7. *Let $D \subset \mathbb{C}^n$ be a domain, which is smooth and locally $\mathbb{C}$-convexifiable near a point $a \in \partial D$ of type $2m$. Then there exist a neighborhood U of a and $C > 0$ such that for any $z \in D \cap U$, $w \in D$,*

$$\tilde{k}_D(z, w) \ge m \log\left(1 + C\frac{|z - w|}{\delta_D(z)^{1/2m}}\right). \tag{1.9}$$

1.7. Upper bounds for the Lempert function and sharpness of the results

The next propositions (inspired by the examples in [2, page 404] and [3, page 35]) and (1.7) show that the exponents in all the above theorems are optimal.

Proposition 1.8. *Let $D \subset \mathbb{C}^n$ be a domain, which is smooth and $\mathbb{C}$-convex near a point $a \in \partial D$ of type $2m$. Denote by n_a the inner normal half-line to ∂D at a. If a is of type $2m$, there exist a unit vector $X \in T_a^{\mathbb{C}}\partial D$ and $C > 0$ such that for all*

$z \in n_a$, close enough to a, and all $w \in D$ such that $\frac{z-w}{|z-w|} = X$ and $C\frac{|z-w|}{\delta_D(z)^{1/2m}} < 1$, then

$$l_D^*(z, w) \leq C\frac{|z-w|}{\delta_D(z)^{1/2m}}.$$

If a is of infinite type, the last inequality holds for any $m \in \mathbb{N}$ with $C = C_m$.

We then have the following result characterizing the type of a point.

Corollary 1.9. *Let $D \subset \mathbb{C}^n$ be a domain, which is smooth and locally $\mathbb{C}$-convexifiable near a point $a \in \partial D$. Then any of the inequalities* (1.5)*,* (1.6) *and* (1.9) *holds if and only if a is of type at most $2m$.*

The next results are related to the converse of Theorem 1.4.

Proposition 1.10. *Let $D \subset \mathbb{C}^n$ be a bounded, smooth, locally $\mathbb{C}$-convexifiable domain. If D is of type $2m$, there exist sequences (z^j), $(w^j) \subset D$ and $c > 0$ such that $|z^j - w^j| \to 0$ and*

$$\frac{g_D(z^j, w^j)}{G_D(z^j, w^j)} \geq c|z^j - w^j|^{2(n-2m)}, \quad j \in \mathbb{N}.$$

If D is of infinite type, the last inequality holds for any $m \in \mathbb{N}$ with (z^j), (w^j) and c depending on m.

Theorem 1.3 and Proposition 1.10 imply the following characterizations of the type of a domain.

Corollary 1.11. *Let $D \subset \mathbb{C}^n$ be a bounded, smooth, locally $\mathbb{C}$-convexifiable domain. Then:*

(i) *there exists $C > 0$ such that*

$$\frac{g_D(z, w)}{G_D(z, w)} \leq C|z-w|^{2(n-2m)}, \quad z, w \in D,\ z \neq w.$$

If and only if D is of type at most $2m$;

(ii) *the ratio g_D/G_D is bounded from above if and only if D is of type at most n.*

If $m = 1$, the condition about $\mathbb{C}$-convexity is superfluous.

Proposition 1.12. *Let $D \subset \mathbb{C}^n$ be a bounded, $\mathcal{C}^2$-smooth domain. Then there exists $C > 0$ such that*

$$\frac{g_D(z, w)}{G_D(z, w)} \leq C|z-w|^{2n-4}, \quad z, w \in D,\ z \neq w, \tag{1.10}$$

if and only if D is strictly pseudoconvex.

In dimension 2, this proposition says that the ratio g_D/G_D is bounded from above if only if D is strictly pseudoconvex. By Corollary 1.11, this is not true if $n \geq 4$.

Proposition 1.13. *Let $D \subset \mathbb{C}^3$ be a bounded, $\mathcal{C}^3$-smooth domain. Then the ratio g_D/G_D is bounded from above if only if D is strictly pseudoconvex.*

It is natural to ask which upper bounds can be given for the functions g_D and k_D, and indeed, many results for k_D have been given in that direction, see for instance [14]. To get estimates from above, using (1.7), it will be enough to bound $\tilde{l}_D(z, w)$.

Proposition 1.14. *Let $D \subset \mathbb{C}^n$ be a bounded, $\mathcal{C}^2$-smooth, locally $\mathbb{C}$-convexifiable domain. Then there exists $C > 0$ such that*

$$\tilde{l}_D(z, w) \leq \log \left(1 + C \frac{|z - w|}{\delta_D(z)^{1/2} \delta_D(w)^{1/2}}\right), \quad z, w \in D. \tag{1.11}$$

This proposition shows that the factor m in Theorems 1.4–1.7 is sharp. On the other hand, these theorems show that the exponent $1/2$ in Proposition 1.14 is optimal.

Proposition 1.14, (1.3), (1.7), and Lemma 1.2 also imply the following:

Corollary 1.15. *Let D be as in Proposition* 1.14*. Then there exists $C > 0$ such that*

$$\frac{g_D(z, w)}{G_D(z, w)} \geq C|z - w|^{2n-2}, \quad z, w \in D,\ z \neq w. \tag{1.12}$$

We already know from [16, Theorem 2] that if D is a bounded, $\mathcal{C}^{1+\varepsilon}$-smooth domain in $\mathbb{C}^n$, then a weaker estimate than (1.11) holds:

$$\tilde{l}_D(z, w) \leq \log \frac{C}{\delta_D(z)^{1/2} \delta_D(w)^{1/2}}. \tag{1.13}$$

It would be interesting to know if (1.11) and, hence, (1.12) remain true in this general case.

The rest of the paper is organized as follows: Section 2 contains the proofs of Propositions 1.8, 1.10, 1.12, 1.13, and 1.14, Section 3 – the proofs of Theorems 1.6 and 1.7, Section 4 – the proof of Theorem 1.5, and Section 5 – the proofs of Theorem 1.3 and 1.4.

ACKNOWLEDGEMENTS. This paper was started while the first-named author was an invited professor at the Paul Sabatier University, Toulouse in May-June 2016.

2. Proofs of Propositions 1.8, 1.10, 1.12, 1.13, and 1.14

Proof of Proposition 1.8. By [18, Propositions 4 and 6], if a is of type at least $2m$, there exist a neighborhood U of a, a unit vector $X \in T_a^{\mathbb{C}}\partial D$, and $C > 0$ such that the distance $\delta_D(z; X)$ from $z \in D \cap U \cap n_a$ to ∂D in direction X verifies $\delta_D(z; X) \geq C\delta_D(z)^{1/2m}$. If a is of infinite type, the last holds for any $m \in \mathbb{N}$ with $C = C_m$. Let $D_{z,X} := \{z + tX : |t| < \delta_D(z, X)\}$. Let $w - z = \lambda X$. It remains to observe that if $r > 1$ and $r|\lambda| < \delta_D(z; X)$, then, recalling that $0 < l_D^* < 1$ with the notations from Definition 1.1,

$$l_D^*(z, z+\lambda X) \leq l_{D_{z,X}}^*(z, z+\lambda X) = \frac{|\lambda|}{\delta_D(z; X)} \leq \frac{1}{r} < 1. \qquad \square$$

Proof of Proposition 1.10. Let D be of type $2m$. Choose a point $a \in \partial D$ of type $2m$. There exist a neighborhood U_0 of a and a holomorphic embedding $\Phi : U_0 \to \mathbb{C}^n$ such that $\Omega := \Phi(D \cap U_0)$ is a $\mathbb{C}$-convex domain. Set $u' = \Phi(u)$. Since $|z' - w'| \asymp |z - w|$ and $\delta_\Omega(u') \asymp \delta_D(u)$ for $u, z, w \in U_1 \Subset U$, and $l_\Omega(z', w') \geq l_D(z, w)$, we may assume that D is $\mathbb{C}$-convex.

Let X be as in Proposition 1.8. Using, *e.g.*, a smooth defining function of D near a, one may find a neighborhood U of a and $C > 1$ such that if $z \in D \cap U \cap n_a$ and $w = z + \lambda X$, $C|\lambda| < \delta_D(z)^{1/2m}$, then $\delta_D(z) = |z - a| < C\delta_D(w)$. Changing U and C (if necessary), we may apply Proposition 1.8 to find sequences (z_j), $(w_j) \to a$ such that $\delta_D(z_j) \asymp \delta_D(w_j) \asymp |z_j - w_j|^{2m}$ and $\tilde{l}_D(z_j, w_j) \lesssim 1$.

This and the inequalities (1.3) and (1.7) imply the desired result in the finite type case.

Let D be of infinite type. Since D is locally $\mathbb{C}$-convexifiable, there exists a point $a \in \partial D$ of infinite type. Then, for any $m \in \mathbb{N}$, we may proceed as above. $\square$

Proof of Proposition 1.12. Strict pseudoconvexity implies local convexifiability and, hence, (1.10) by Theorem 1.3.

To prove the converse, we will proceed similarly to the proof of Proposition 1.10.

Assume that the ratio g_D/G_D is bounded from above, and $a \in \partial D$ is not a strictly pseudoconvex point.

After an affine change of coordinates, we may suppose that $a = 0$ and that D is defined near 0 by

$$\operatorname{Re}\left(z_1 + c_1 z_2^2\right) + c_2|z_2|^2 + o\left(|\operatorname{Im}(z_1)| + |z_2|^2 + |z''|\right) < 0$$

where $c_2 \leq 0$.

It follows by (1.3) that $g_D(z, w_0) \to 0$ as $z \to \partial D$ and hence D is a pseudoconvex domain. This implies that $c_2 = 0$.

Let $\Phi(z) = (z_1 + c_1 z_2^2, z_2, z'')$. Then $G := \Phi(D)$ is given near 0 by

$$\operatorname{Re} z_1 + o\left(|\operatorname{Im}(z_1)| + |z_2|^2 + |z''|\right) < 0.$$

Now it is easy to find sequences $\mathbb{R}_- \times \{0'\} \supset (z^j) \to 0$ and $(\lambda_j) \to \infty$ such that $G \ni w^j = z^j + \lambda_j \delta_G(z^j)^{1/2} e_2$, and $2|z^j - w^j| < \delta_G(z^j; e_2)$, where $e_2 := (0, 1, 0, \dots, 0)$.

Since the order of contact of ∂G and $\mathbb{C}e_2$ at 0 is at least 2, $|\delta_G(z^j) - \delta_G(w^j)| = O(|z^j - w^j|^2)$, so

$$\frac{\delta_G(z^j)\delta_G(w^j)}{|z^j - w^j|^4} \lesssim \frac{\delta_G(z^j)^2}{|z^j - w^j|^4} + \frac{\delta_G(z^j)}{|z^j - w^j|^2} \to 0 \quad \text{and} \quad l_G^*(z^j, w^j) < \frac{1}{2}.$$

If $\tilde{z}^j = \Phi^{-1}(z^j)$ and $\tilde{w}^j = \Phi^{-1}(w^j)$, then the inequalities $g_D \le l_D \le l_{D \cap U}$ and (1.3) easily lead to the contradiction

$$\frac{g_D(\tilde{z}^j, \tilde{w}^j)}{G_D(\tilde{z}^j, \tilde{w}^j)} |\tilde{z}^j - \tilde{w}^j|^{4-2n} \to \infty. \qquad \square$$

Proof of Proposition 1.13. As above, strict pseudoconvexity implies that

$$\frac{g_D(z, w)}{G_D(z, w)} \lesssim |z - w|^2 \lesssim 1, \quad \text{for} \quad z, w \in D,\ z \ne w.$$

For the converse, assume that the ratio g_D/G_D is bounded from above, and $a \in \partial D$ is not a strictly pseudoconvex point.

After biholomorphic changes of variables similar to that in the proof of Proposition 1.12, we may suppose that D is defined near $a = 0$ by

$$\mathrm{Re}\left(z_1 + c_3 z_2^3 + c_4 z_2^2 \overline{z_2}\right) + o\left(|\operatorname{Im}(z_1)| + |z_2|^3 + |z_3|\right) < 0.$$

Again by pseudoconvexity, $c_4 = 0$. Let $\Psi(z) = (z_1 + c_3 z_2^3, z_2, z_3)$ and Then $E := \Psi(D)$ is defined near 0 by

$$\mathrm{Re}(z_1) + o\left(|\operatorname{Im}(z_1)| + |z_2|^3 + |z_3|\right) < 0.$$

We may proceed as at the end of the proof of Proposition 1.12 to get a contradiction, finding sequences $(z^j), (w^j) \to 0$ and $(\lambda^j) \to \infty$ such that $w^j = z^j + \lambda^j \delta_E(z^j)^{1/3} e_2$, $l_E^*(z^j, w^j) < \frac{1}{2}$, and since the order of contact of ∂E at 0 and $\mathbb{C}e_2$ is at least 3, $|\delta_E(z^j) - \delta_E(w^j)| = O(|z^j - w^j|^3)$, so

$$\frac{\delta_E(z^j)\delta_E(w^j)}{|z^j - w^j|^6} \to 0 \ \text{ and } \ \frac{g_D(z^j, w^j)}{G_D(z^j, w^j)} \to \infty. \qquad \square$$

Proof of Proposition 1.14. By (1.13), for a given $\varepsilon_0 > 0$, (1.11) follows for $|z - w| \geq \varepsilon_0$. If $\min(\delta_D(w), \delta_D(z)) \geq \varepsilon_0$, (1.11) also follows, trivially. So we may assume, by symmetry of the function, that $\delta_D(z) \leq \delta_D(w) \leq 2\varepsilon_0$.

For any $a \in \partial D$, we may choose a bounded neighborhood U_0 of a such that $D \cap U_0$ is $\mathbb{C}$-convexifiable and $\mathcal{C}^2$-smooth (see [17, Proposition 3.3]), and that the projection π to ∂D is well defined on U_0. Choose neighborhoods of a, $U_2 \Subset U_1$, such that $D \cap U_1 \Subset D \cap U_0$, and $\varepsilon_1 > 0$ such that $z \in D \cap U_1$ and $\delta_D(z) \leq \varepsilon_1$ imply $\delta_{D\cap U_0}(z) = \delta_D(z)$. We can cover ∂D by a finite collection of the U_2, and choose $\varepsilon_0 > 0$ so that for any z, w such that $\delta_D(z) \leq \delta_D(w) \leq 2\varepsilon_0$ and $|z - w| \leq \varepsilon_0$, then $z \in U_2$, $w \in U_1$ (for some $a \in \partial D$) and $\delta_{D\cap U_0}(z) = \delta_D(z)$, $\delta_{D\cap U_0}(w) = \delta_D(w)$.

Given z, w as above, $\tilde{l}_D(z, w) \leq \tilde{l}_{D\cap U_0}(z, w)$.

Then, by Lempert's theorem, $\tilde{l}_{D\cap U_0} = \tilde{k}_{D\cap U_0}$, and by [14, Corollary 8],

$$\begin{aligned}\tilde{k}_{D\cap U_0}(z, w) &\leq \log\left(1 + C\frac{|z - w|}{\delta_{D\cap U_0}(z)^{1/2}\delta_{D\cap U_0}(w)^{1/2}}\right)\\ &= \log\left(1 + C\frac{|z - w|}{\delta_D(z)^{1/2}\delta_D(w)^{1/2}}\right).\end{aligned}$$ □

3. Proofs of Theorems 1.6 and 1.7

Proof of Theorem 1.6. Under the hypotheses of Theorem 1.6, Theorem 1.7 and a compactness argument show that there is $\delta_0 > 0$ such that (1.9) holds uniformly for $z, w \in D$ if $\delta_D(z) < 2\delta_0$. By symmetry, it is enough to consider three cases.

Case 1. $\delta_D(z) \geq \delta_0$, $\delta_D(w) \geq \delta_0$.

Then (1.8) follows from the inequality $\tilde{k}_D(z, w) \gtrsim |z - w|$, valid on any bounded domain.

Case 2. $\delta_D(z) < \delta_0$, $\delta_D(w) \geq 2\delta_0$.

Then $\frac{|z-w|}{\delta_D(z)^{1/2m}} \gtrsim 1 \gtrsim \frac{|z-w|}{\delta_D(w)^{1/2m}}$ and (1.8) follows by (1.9) (with bigger C).

Case 3. $\delta_D(z) < \delta_0$, $\delta_D(w) < 2\delta_0$.

For any $\varepsilon > 0$, choose a curve γ whose Kobayashi-Royden length is bounded by $(1 + \varepsilon)\tilde{k}_D(z, w)$. Choose a point $u \in \gamma$ such that $|z - u| = |u - w| \geq \frac{1}{2}|z - w|$. Then the definition of the Kobayashi distance and (1.9) applied to (z, u) and (w, u) imply that

$$\begin{aligned}(1 + \varepsilon)\tilde{k}_D(z, w) &\geq \tilde{k}_D(z, u) + \tilde{k}_D(u, w)\\ &\geq m \log\left(1 + C\frac{|z - w|}{2\delta_D(z)^{1/2m}}\right) + m \log\left(1 + C\frac{|z - w|}{2\delta_D(w)^{1/2m}}\right),\end{aligned}$$

which, replacing C by $C/2$, finishes the proof. □

Proof of Theorem 1.7. There exist a neighborhood U_0 of a and a holomorphic embedding $\Phi : U_0 \to \mathbb{C}^n$ such that $\Omega := \Phi(D \cap U_0)$ is a $\mathbb{C}$-convex domain. Let U_1 and U_2 be neighborhoods of a such that $U_1 \Subset U_2 \Subset U_0$. Let $z \in D \cap U_1$.

Case 1. $|z - w|^{2m} \leq \delta_D(z)$.

Since $\log(1 + x) \leq x$, it is enough to prove that

$$\tilde{k}_D(z, w) \gtrsim \frac{|z - w|}{\delta_D(z)^{1/2m}}. \tag{3.1}$$

Let $\tilde{k}_D(D \cap U_1, D \setminus U_2) =: C_1 > 0$. We may assume that $\tilde{k}_D(z, w) < C_1$. Then a curve connecting z and w of Kobayashi-Royden length $< C_1$ must lie inside U_2. Since

$$\kappa_D(u, X) \gtrsim \kappa_{D\cap U_0}(u, X), \quad u \in U_2, X \in \mathbb{C}^n$$

(see, *e.g.*, [10, Proposition 7.2.9]), then $\tilde{k}_D(z, w) \gtrsim \tilde{k}_{D\cap U_0}(z, w)$.

From now on, we estimate $\tilde{k}_{D\cap U_0}(z, w)$. Call L the complex line through $z' := \Phi(z)$ and $w' := \Phi(w)$. Let $z_0 \in L \cap \partial\Omega$ be such that $|z' - z_0| = \delta_{L\cap\Omega}(z')$. Let P be the linear projection from $\mathbb{C}^n$ to L, parallel to the complex tangent hyperplane to $\partial\Omega$ at z_0. Then $P(\Omega)$ is a simply connected domain (see, *e.g.*, [1, Theorem 2.3.6]), and $z_0 \in \partial P(\Omega)$. Therefore,

$$\begin{aligned}\tilde{k}_{D\cap U_0}(z, w) &= \tilde{k}_\Omega(z', w') \geq \tilde{k}_{P(\Omega)}(z', w') \\ &\geq \frac{1}{4}\log\left(1 + \frac{|z' - w'|}{\delta_{P(\Omega)}(z')}\right) = \frac{1}{4}\log\left(1 + \frac{|z' - w'|}{\delta_{L\cap\Omega}(z')}\right),\end{aligned}$$

(for the second inequality see, *e.g.*, [19, Proposition 3(ii)]). By [18, Propositions 4 and 6], $\delta_{L\cap\Omega}(z') \lesssim \delta_\Omega(z')^{1/2m}$; since Φ is a biholomorphism in a neighborhood of $\overline{D \cap U_2}$, we have $|z' - w'| \asymp |z - w|$ and $\delta_\Omega(z') = \delta_{D\cap U_0}(z)$, so we finally obtain (3.1) (the implicit constants are uniform over D by a compactness argument).

Case 2. $|z - w|^{2m} \geq \delta_D(z)$.

We may assume that $D \cap U_0$ is $\mathcal{C}^2$-smooth, and that the projection π to ∂D is well defined on U_0.

We will follow the proof of [6, Theorem 2.3]. We need to bound from below the Kobayashi-Royden length of any path γ such that $\gamma(0) = z$ and $\gamma(1) = w$. If $\gamma([0, 1]) \not\subset U_1$ (in particular if $w \notin U_1$), let $t^* := \min\{t \in [0, 1] : \gamma(t) \notin U_1\}$. It will be enough to bound below the length of $\gamma[0, t^*]$, so we can reduce ourselves to the case where $w \in \overline{U_1}$.

Let Φ be a holomorphic embedding such that $\Phi(D \cap U_0) =: \Omega$ is $\mathbb{C}$-convex.

Applying a result of K. Diederich and J. E. Fornaess about supporting functions [5] to Ω, reducing U_1 as needed, we can find neighborhoods of a, $U_1 \Subset U_2 \Subset U_0$ such that for any $a' \in U_1$, there exist $S_{\Phi(a')}$ holomorphic on $\mathbb{C}^n$, and $C, C' > 0$ such that

$$\begin{aligned}&-C'|\xi - \Phi(a')| \leq \operatorname{Re} S_{\Phi(a')}(\xi) \leq -C|\xi - \Phi(a')|^{2m}, \\ &\xi \in \Phi(U_2), \text{ and } S_{\Phi(a')}(\Phi(a')) = 0.\end{aligned} \tag{3.2}$$

We define a function P_z holomorphic on U_0 by

$$P_z(\zeta) := e^{S_{\Phi(\pi(z))}(\Phi(\zeta))}. \tag{3.3}$$

Since Φ is a uniformly bilipschitz diffeormorphism on U_2 we have, for $\zeta \in U_2$,

$$|1 - P_z(\zeta)| \lesssim |\zeta - \pi(z)| \text{ and } 1 - |P_z(\zeta)| \gtrsim |\zeta - \pi(z)|^{2m}. \tag{3.4}$$

This means in particular that [6, Lemma 2.2] can be applied, and it follows that by [6, Theorem 2.1] that there is $C_1 > 0$ such that for $z \in D \cap U_1$ and $X \in \mathbb{C}^n$,

$$\kappa_{D\cap U_0}(z; X) \geq \kappa_D(z; X) \geq (1 - C_1\delta_D(z))\kappa_{D\cap U_0}(z; X).$$

Therefore

$$\int_0^1 \kappa_D(\gamma(t), \gamma'(t))dt \geq \int_0^1 \big(1 - C_1\delta_D(\gamma(t))\big)\kappa_{D\cap U_0}\big(\gamma(t), \gamma'(t)\big)\, dt. \tag{3.5}$$

Let $\lambda := P_z \circ \gamma$. Then

$$\kappa_{D\cap U_0}\big(\gamma(t), \gamma'(t)\big) \geq \kappa_{\mathbb{D}}\big(\lambda(t), \lambda'(t)\big) \geq \frac{|\lambda'(t)|}{2(1 - |\lambda(t))|}.$$

On the other hand, by (3.4),

$$\begin{aligned} 1 - C_1\delta_D(\gamma(t)) &\geq 1 - C_1|\gamma(t) - \pi(z)| \\ &\geq 1 - C_1' \left(1 - |P_z(\gamma(t))|\right)^{1/2m} = 1 - C_1' \left(1 - |\lambda(t)|\right)^{1/2m}. \end{aligned}$$

Collecting the estimates, the right-hand side in (3.5) can be bounded below by

$$\begin{aligned} \int_0^1 \frac{1 - C_1' \left(1 - |\lambda(t)|\right)^{1/2m}}{1 - |\lambda(t)|}|\lambda'(t)|dt &\geq \int_0^1 \frac{1}{1 - |\lambda(t)|}\frac{d}{dt}|\lambda(t)|dt + O(1) \\ &= \log\frac{1 - |\lambda(1)|}{1 - |\lambda(0)|} + O(1) \\ &= \log\frac{1 - |P_z(w)|}{1 - |P_z(z)|} + O(1). \end{aligned}$$

By (3.4), $1 - |P_z(z)| \lesssim |z - \pi(z)| = \delta_D(z)$, while

$$1 - |P_z(w)| \gtrsim |w - \pi(z)|^{2m} \geq \left(|w - z| - |z - \pi(z)|\right)^{2m}.$$

Since $\delta_D(z) \leq (C_0^{-1}|w - z|)^{2m} < \frac{1}{2}|w - z|$ for C_0 large enough, we have $1 - |P_z(w)| \gtrsim |w - z|^{2m}$ and the estimate we wanted is proved. $\square$

4. Proof of Theorem 1.5

Proof of Theorem 1.5, (1.5). Choose a bounded neighborhood U_0 of a such that $D \cap U_0$ is $\mathbb{C}$-convexifiable and $\mathcal{C}^2$-smooth.

Case 1. $|z - w| \le \delta_D(w)^{1/2m}$.

We can choose a neighborhood $U \Subset U_0$ such that for any $w \in D \cap U$, then $z \in D \cap U_0$ and $\delta_{D \cap U_0}(w) = \delta_D(w)$.

By Lemma 1.2(ii), we have to prove that

$$g_D(z, w) \ge \log \frac{|z - w|}{\delta_D(w)^{1/2m}} + O(1).$$

We first reduce ourselves to the study of $g_{D \cap U_0}$ by a standard argument.

Lemma 4.1. *Shrinking U (if necessary), there is $C > 0$ such that*

$$g_D(z, w) \ge g_{D \cap U_0}(z, w) - C, \quad z \in D \cap U_0, \ w \in D \cap U. \tag{4.1}$$

Accepting this lemma, we apply Lempert's theorem to $D \cap U_0$ and obtain $g_D(z, w) \ge k_{D \cap U_0}(z, w) - C_a$. By Theorem 1.7, $\tilde{k}_{D \cap U_0}(z, w)$ satisfies (1.5) (by shrinking U once more if needed), therefore

$$k_{D \cap U_0}(z, w) \ge \log \frac{|z - w|}{\delta_D(w)^{1/2m}} + O(1),$$

and we are done.

Proof of Lemma 4.1. The proof is similar to that of [4, Theorem 1].

Let $\psi(z) = \log \frac{|z-a|}{\operatorname{diam} D}$ and $U_1 \Subset U_0 \subsetneq D$ be a neighborhood of a such that and $\inf_{D \setminus U_0} \psi > c := 1 + \sup_{D \cap \partial U_1} \psi$. Fix $w \in D \cap U_1$ and set

$$d(w) = \inf_{z \in D \cap \partial U_1} g_{D \cap U_0}(z, w), \quad u(z, w) = (c - \psi(z)) d(w), \ z \in D.$$

Since $u(z, w) \le g_{D \cap U_0}(z, w)$ for $z \in D \cap \partial U_1$, and $u(z, w) > 0 > g_{D \cap U_0}(z, w)$ for $z \in \mathcal{N} \cap (D \cap U_0)$, where $\mathcal{N}$ is a neighborhood of ∂U_0, the function

$$v(z, w) = \begin{cases} g_{D \cap U_0}(z, w) & w \in D \cap U_1 \\ \max\{g_{D \cap U_0}(z, w), u(z, w)\} & w \in D \cap U_0 \setminus U_1 \\ u(z, w) & w \in D \setminus U_0 \end{cases}$$

is a plurisubharmonic function in z with logarithmic pole at w. Also $v(z, w) < cd(w)$, so $g_D(z, w) \ge v(z, w) - cd(w)$. Now (4.1) follows by taking $U \Subset U_1$ and $C := c \inf_{w \in D \cap U} d(w)$. $\square$

Case 2. $|z-w| \geq \delta_D(w)^{1/2m}$.

By Lemma 1.2(i), we have to prove that

$$g_D(z,w) \gtrsim -\frac{\delta_D(w)}{|z-w|^{2m}}. \tag{4.2}$$

By Theorem 1.7 and Lempert's theorem,

$$g_{D\cap U_0}(z,w) \gtrsim -\frac{\delta_D(w)}{|z-w|^{2m}}, \quad z \in D\cap U_0,\ w \in D\cap U. \tag{4.3}$$

We will follow part of the proof of [3, Lemma 3]. The above inequality is analogous to [3, page 29, inequality (5)].

Denote by $B(w,r)$ the ball with center w and radius r. Set $r_0 := \frac{1}{4}\operatorname{dist}(U, D\setminus U_0)$, $\lambda := \min\{r_0, |z-w|\}$, so that

$$D\cap B(w,\lambda) \subset D\cap B(w,2r_0) \subset D\cap U_0.$$

Note that

$$\lambda \leq |z-w| \leq \frac{\operatorname{diam} D}{r_0}\lambda. \tag{4.4}$$

Finally, let

$$b := -\inf\left\{g_{D\cap U_0}(\zeta,w) : |\zeta-w| = \lambda, \zeta \in D\right\}.$$

Because of (4.3) and (4.4),

$$b \lesssim \frac{\delta_D(w)}{\lambda^{2m}} \lesssim \frac{\delta_D(w)}{|z-w|^{2m}}. \tag{4.5}$$

Let

$$v(\zeta) := b\frac{\log\frac{|\zeta-w|}{2r_0}}{\log\frac{2r_0}{\lambda}}.$$

By construction, $v(\zeta) = 0 > g_{D\cap U_0}(\zeta,w)$ when $\zeta \in D\cap\partial B(w,2r_0)$, and $v(\zeta) = -b \leq g_{D\cap U_0}(\zeta,w)$ when $\zeta \in D\cap\partial B(w,\lambda)$.

Then we construct a plurisubharmonic function u with logarithmic singularity at w by setting

$$u(\zeta) := \begin{cases} g_{D\cap U_0}(\zeta,w), & \zeta \in B(w,\lambda) \\ \max\left\{v(\zeta), g_{D\cap U_0}(\zeta,w)\right\}, & \zeta \in B(w,2r_0)\setminus B(w,\lambda) \\ v(\zeta), & \zeta \in D\setminus B(w,2r_0). \end{cases}$$

By definition of g_D, $g_D \geq u - \sup_D u$. We have

$$\sup_D u \leq \sup_D v \leq b\frac{\log\frac{\operatorname{diam} D}{2r_0}}{\log\frac{2r_0}{\lambda}} \leq b\frac{\log\frac{\operatorname{diam} D}{2r_0}}{\log 2} \lesssim \frac{\delta_D(w)}{|z-w|^{2m}},$$

by (4.5). On the other hand, if $\lambda = |z-w|$, then

$$u(z) = g_{D\cap U_0}(z,w) \gtrsim -\frac{\delta_D(w)}{|z-w|^{2m}}$$

by (4.3), while if $\lambda = r_0 < |z-w|$, then

$$u(z) \geq v(z) = b\frac{\log \frac{|z-w|}{2r_0}}{\log 2} \geq -b.$$

Collecting the estimates, $g_D(z,w) \gtrsim -\frac{\delta_D(w)}{|z-w|^{2m}}$. □

Proof of Theorem 1.5, (1.6). We choose U_1 small enough so that $\pi(z)$ is well defined whenever $z \in U_1$.

Case 1. Suppose that $z \in U$ and $|z-w| \geq \delta_D(z)^{1/2m}$.

Shrinking U_1, we may assume that $|z-w| \geq 8\delta_D(z)$.

We use the Diederich-Fornaess supporting functions [5] once again. We take $U_1 \Subset U_2 \Subset U_0$ as before. Reducing U_1 if needed, for any $a' \in U_1 \cap \partial D$, there exist $S_{\Phi(a')}$ holomorphic on Ω, and $C, C' > 0$ such that (3.2) holds.

We set $\tilde{\varphi}_z(\zeta) := \operatorname{Re} S_{\Phi(\pi(z))}(\Phi(\zeta)) \in PSH_-(D\cap U_0)$. Since Φ is a uniformly bilipschitz diffeormorphism on U_2 we have, for $\zeta \in U_2$,

$$-C'|\zeta - a'| \leq \tilde{\varphi}_z(\zeta) \leq -C|\zeta - a'|^{2m} \text{ and } \tilde{\varphi}_z(\pi(z)) = 0. \tag{4.6}$$

We need to extend $\tilde{\varphi}_z$ to a global plurisubharmonic function on D. We proceed as in [3, page 31]. Let $\eta := \sup_{z\in U_1}\sup_{\zeta\in\partial U_2}\tilde{\varphi}_z(\zeta) < 0$. We set $\varphi_z := \max(\tilde{\varphi}_z, \eta/2)$ and extend it by $\eta/2$ on the whole of D. Then $\varphi_z \in PSH_-(D)$ and satisfies the analogue of (4.6).

By the same argument as at the beginning of Case 2 of the proof of (1.5), the inequality we have to prove is the following analogue of (4.2):

$$g_D(z,w) \gtrsim -\frac{\delta_D(z)}{|z-w|^{2m}}.$$

Lemma 4.2. *Let* $w' := w + \frac{w-z}{|w-z|}$, $B_1 := B(w', 1+|w-z|/2)$, $B_2 := B(w', 1 + 3|w-z|/4)$. *Then there is* $c_0 > 0$ *so that for any* w, *there exists* $\rho_w \in \mathcal{C}^\infty(\mathbb{C}^n \setminus \{w\}, \mathbb{R}_-)$ *with logarithmic singularity at* w, *supported on* $\overline{B}_2$, *such that*

$$\partial\bar{\partial}\rho_w(\zeta) \geq -\frac{c_0}{|w-z|^2}\chi_{\overline{B}_2\setminus B_1}(\zeta)\partial\bar{\partial}\left(|\zeta|^2\right).$$

In particular, $\rho_w \in PSH\big(B_1 \cup (\mathbb{C}^n \setminus \overline{B}_2)\big)$.

This lemma is proved in [3, page 31].

We construct a function Φ with logarithmic pole at w by setting

$$\Phi(\zeta) := \frac{c_1}{|z-w|^{2m}} \left(\varphi_z(\zeta) + c_2|\zeta - \pi(z)|^{2m}\right) + \rho_w(\zeta).$$

By (4.6) and because D is bounded, we can choose $c_2 > 0$ such that $\Phi < 0$ on D.

We want to choose $c_1 > 0$ so that $\Phi \in PSH(D)$. We only need to check the case where $\zeta \in \overline{B}_2 \setminus B_1$. Then

$$|\zeta - \pi(z)| \geq |\zeta - z| - \delta_D(z) \geq \frac{1}{4}|z-w| - \delta_D(z) \geq \frac{1}{8}|z-w|.$$

By the estimate on $\partial\bar{\partial}\rho_w$ from Lemma 4.2, the fact that $\varphi_z \in PSH(D)$, and standard computations,

$$\begin{aligned}\partial\bar{\partial}\Phi(\zeta) &\geq \left(\frac{c_1}{|z-w|^{2m}} c_2 c_3 |\zeta - \pi(z)|^{2m-2} - \frac{c_0}{|w-z|^2}\right) \partial\bar{\partial}|\zeta|^2 \\ &\geq \left(\frac{c_1 c_2 c_3}{8^{2m-2}} - c_0\right) \frac{1}{|w-z|^2} \partial\bar{\partial}|\zeta|^2,\end{aligned}$$

where $c_3 > 0$ is a constant. So we can choose $c_1 > 0$ to make this form positive. With these choices, $\Phi(\zeta) \leq g_D(\zeta, w)$.

Since $\rho_w(z) = 0$, using (3.2) again,

$$\Phi(z) = \frac{c_1}{|z-w|^{2m}} \left(\varphi_z(z) + c_2 \delta_D(z)^{2m}\right) \geq -c_1 C' \frac{\delta_D(z)}{|z-w|^{2m}}.$$

Case 2. Suppose that $z \in B(a, r_1)$ and $|z-w| \leq \delta_D(z)^{1/2m}$.

Then $|w-a| \leq r_1 + r_1^{1/2m} =: r_2$. Reducing r_1 if needed, we have $B(a, r_2) \Subset U_0$, where U_0 is a bounded neighborhood of a such that $D \cap U_0$ is $\mathbb{C}$-convexifiable and $\mathcal{C}^2$-smooth. This implies that, by Lempert's theorem and (1.5),

$$\tilde{g}_{U_0 \cap D}(z, w) = \tilde{g}_{U_0 \cap D}(w, z) \geq m \log\left(1 + C \frac{|z-w|}{\delta_D(z)^{1/2m}}\right).$$

Since $\frac{|z-w|}{\delta_D(z)^{1/2m}} \leq 1$, by Lemma 1.2(ii), this is equivalent to

$$g_{U_0 \cap D}(z, w) \geq \log \frac{|z-w|}{\delta_D(z)^{1/2m}} + O(1).$$

By Lemma 4.1, the same estimate holds for $g_D(z, w)$, and we are done for this case. ∎

5. Proofs of Theorems 1.3 and 1.4

Proof of Theorem 1.3. Let

$$\Delta_D(z, w) := \frac{|z-w|^2}{\delta_D(z)^{1/2m}\delta_D(w)^{1/2m}}.$$

Using (1.3), it is enough to show that $g_D(z, w) \gtrsim -\Delta_D(z, w)^{-2m}$.

Theorem 1.4 implies that

$$\tilde{g}_D(z, w) \geq \log\left(1 + C'\Delta_D(z, w)\right)^m.$$

If $\Delta_D(z, w) \geq 1$, then $g_D(z, w) \gtrsim -\Delta_D(z, w)$ by Lemma 1.2(i).

If $\Delta_D(z, w) \leq 1$, then Lemma 1.2(ii) implies that

$$g_D(z, w) \geq \log \Delta_D(z, w) + O(1) \gtrsim -\Delta_D(z, w)^{-2m}. \qquad \square$$

Proof of Theorem 1.4. We follow an argument in [2], as adapted in [3, Proof of Proposition 2].

The hypotheses of Theorem 1.5 are met for any $a \in \partial D$. By a compactness argument, this implies that there is $K \Subset D$ such that for $z \in D \setminus K$, $w \in D$,

$$\tilde{g}_D(z, w) \geq m \log\left(1 + C\frac{|z-w|}{\delta_D(z)^{1/2m}}\right). \tag{5.1}$$

But when $z \in K$, the right-hand side of (5.1) is bounded above by $C'mC|z-w|$, while $\tilde{g}_D(z, w) \geq C''|z-w|$, so C can be chosen so that (5.1) holds for any $z, w \in D$. In the same way, changing C again if needed, we have for any $z, w \in D$,

$$\tilde{g}_D(z, w) \geq m \log\left(1 + C\frac{|z-w|}{\delta_D(w)^{1/2m}}\right). \tag{5.2}$$

If $|z-w|^{2m} \lesssim \max\{\delta_D(z), \delta_D(w)\}$, then (1.4) follows from (5.1) and (5.2) by modifying the constant C. Otherwise, by Lemma 1.2(i), (2) is equivalent to

$$g_D(z, w) \gtrsim -\frac{\delta_D(z)\delta_D(w)}{|z-w|^{4m}}.$$

We may assume that $4\max\{\delta_D(z), \delta_D(w)\} \leq |z-w|$. If $2|\zeta - \pi(z)| = |z-w|$, then

$$|\zeta - w| \geq |z-w| - |\zeta - \pi(z)| - |z - \pi(z)| \geq \frac{|z-w|}{4}.$$

Therefore, by (5.2), for those values of ζ, $g_D(\zeta, w) \gtrsim -\frac{\delta_D(w)}{|z-w|^{2m}}$. For those same ζ, the plurisubharmonic peak function φ_z from the proof of Theorem 1.5, (1.6), Case 1, verifies

$$\varphi_z(\zeta) \leq -C|\zeta - \pi(z)|^{2m} = -C2^{-2m}|z-w|^{2m},$$

so,

$$g_D(\zeta, w) \gtrsim -\frac{\delta_D(w)}{|z-w|^{4m}}\varphi_z(\zeta), \quad \zeta \in D \cap \partial B(\pi(z), |z-w|/2).$$

This inequality is trivially true on ∂D, where $g_D(\zeta, w) = 0$, and since $g_D(\cdot, w)$ is a maximal plurisubharmonic function on $D \setminus \{w\}$, it has to hold on $D \cap B(\pi(z), |z-w|/2)$, in particular at the point z, so

$$g_D(z, w) \gtrsim -\frac{\delta_D(w)}{|z-w|^{4m}}\varphi_z(z) \gtrsim -\frac{\delta_D(w)\delta_D(z)}{|z-w|^{4m}}. \qquad \square$$

References

[1] M. ANDERSSON, M. PASSARE and R. SIGURDSSON, "Complex Convexity and Analytic Functionals", Birkhäuser, Basel–Boston–Berlin, 2004.

[2] M. CARLEHED, *Comparison of the pluricomplex and the classical Green functions*, Michigan Math. J. **45** (1998), 399–407.

[3] B.-Y. CHEN, *Comparison of the pluricomplex and the classical Green functions on convex domains of finite type*, Michigan Math. J. **50** (2002), 27–36.

[4] D. COMAN, *Boundary behavior of the pluricomplex Green function*, Ark. Mat. **36** (1998), 341–353.

[5] K. DIEDERICH and J. E. FORNAESS, *Lineally convex domains of finite type: holomorphic support functions*, Manuscripta Math. **112** (2003), 403–431.

[6] F. FORSTNERIC and J.-P. ROSAY, *Localization ot the Kobayashi metric and the boundary continuity of proper holomorphic mappings*, Math. Ann. **279** (1987), 239–252.

[7] G. HERBORT, *The pluricomplex Green function on some regular pseudoconvex domains*, Ann. Pol. Math. **110** (2014), 209–226.

[8] L. HÖRMANDER, "Notions of Convexity", Birkhäuser, Boston, 1994.

[9] D. JACQUET, $\mathbb{C}$*-convex domains with* C^2 *boundary*, Complex Var. Elliptic Equ. **51** (2006), 303–312.

[10] M. JARNICKI and P. PFLUG, "Invariant Distances and Metrics in Complex Analysis", de Gruyter Exp. Math. 9, de Gruyter, Berlin, New York, 1993.

[11] M. KLIMEK, *Extremal plurisubharmonic functions and invariant pseudodistances*, Bull. Soc. Math. France **13** (1985), 123–142.

[12] P. LELONG, *Fonction de Green pluricomplexe et lemmes de Schwarz dans les espaces de Banach*, J. Math. Pures Appl. **68** (1989), 319–347.

[13] L. LEMPERT, *La métrique de Kobayashi et la représentation des domaines sur la boule*, Bull. Soc. Math. France **109** (1981), 427–474.

[14] N. NIKOLOV and L. ANDREEV, *Estimates of the Kobayashi and quasi-hyperbolic distances*, Ann. Mat. Pura Appl. **196** (2017), 43–50.

[15] N. NIKOLOV and P. PFLUG, *On the derivatives of the Lempert functions*, Ann. Mat. Pura Appl. **187** (2008), 547–553.

[16] N. NIKOLOV, P. PFLUG and P. J. THOMAS, *Upper bound for the Lempert function on smooth domains*, Math. Z. **266** (2010), 425–430.

[17] N. NIKOLOV, P. PFLUG and P. J. THOMAS, *On different extremal bases for* $\mathbb{C}$*-convex domains*, Proc. Amer. Math. Soc. **141** (2013), 3223–3230.

[18] N. NIKOLOV, P. PFLUG and W. ZWONEK, *Estimates for invariant metrics on* $\mathbb{C}$*-convex domains*, Trans. Amer. Math. Soc. **363** (2011), 6245–6256.

[19] N. NIKOLOV and M. TRYBUŁA, *The Kobayashi balls of* ($\mathbb{C}$-)*convex domains*, Monatsh. Math. **177** (2015), 627–635.

[20] A. RASHKOVSKII and P. J. THOMAS, *Powers of ideals and convergence of Green functions with colliding poles*, Int. Math. Res. Not. IMRN **2014** (2014), 1253–1272.
[21] G. SWEERS, *Positivity for a strongly coupled elliptic system by Green function estimates*, J. Geom. Anal. **4** (1994), 121–142.

Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev 8
1113 Sofia, Bulgaria
and
Faculty of Information Sciences
State University of Library Studies
and Information Technologies
Shipchenski prohod 69A
1574 Sofia, Bulgaria
nik@math.bas.bg

Institut de Mathématiques de Toulouse
UMR5219
Université de Toulouse; CNRS
UPS IMT
31062 Toulouse Cedex 9, France
pascal.thomas@math.univ-toulouse.fr