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A Hardy type inequality on fractional order Sobolev spaces on the Heisenberg group

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Abstract. In this paper, we derive a non linear Hardy type inequality on certain fractional order Sobolev spaces on the Heisenberg group. Our inequality is an analogous version of an inequality of the same name on weighted Folland-Stein spaces which had been derived in [3]. We also derive Sobolev type and Morrey type embedding to make that abstract fractional order Sobolev spaces on the Heisenberg group more familiar.

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1. Introduction

Over the last decade different types of Hardy inequalities fascinate a lot of mathematicians and physicists due to their physical significance and application to different type of PDEs. In this article, we have investigated such an inequality along with Sobolev embedding on fractional order Sobolev spaces on the Heisenberg group. Before stating the main results let us recall some relevant literature.

The classical Hardy inequality asserts that for any domain $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ with $0 \in \Omega$

$$\int_\Omega |\nabla u|^2 dx \geq \lambda_* \int_\Omega \frac{u^2}{|x|^2} dx \quad \forall u \in H_0^1(\Omega), \tag{1.1}$$

where $\lambda_* = \frac{(n-2)^2}{4}$ is the optimal constant, which is never achieved in $H_0^1(\Omega)$. The constant λ_* plays a crucial role in analysing the behaviour of heat equation with inverse square potential (See [4,29]).

Since the inequality (1.1) is strict for any $u \in H_0^1(\Omega)$, there is a purpose to improving (1.1). This opportunity has been exploited in [2,5,10] to derive various improvements of (1.1) by imposing different conditions on Ω, whereas for $\Omega = \mathbb{R}^n$ it has been shown in [15,17] that additional correction terms cannot be added.

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For $s \in (0,1)$ and $1 \leq p < \infty$, let us consider the fractional order Sobolev space

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\mathbb{R}^n) : \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy < \infty \right\},$$

endowed with the norm

$$||u||_{W^{s,p}(\Omega)} := ||u||_{L^p(\Omega)} + [u]_{W^{s,p}(\Omega)},$$

where $[u]_{W^{s,p}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy \right)^{\frac{1}{p}}$ and Ω is any open set in $\mathbb{R}^n$. In [23] Maz'ya and Shaposhnikova proved the following Hardy type inequality:

$$\int_{\mathbb{R}^n} \frac{|u|^p}{|x|^{sp}} dx \leq C(n,p) \frac{s(1-s)}{(n-sp)^p} [u]^p_{W^{s,p}(\mathbb{R}^n)}, \text{ for all } u \in W^{s,p}(\mathbb{R}^n), \tag{1.2}$$

where $sp < n$ and $C(n,p) > 0$ is a constant depending only on n, p. They used it to prove the following versions of the Sobolev inequality:

$$\text{(i)} \quad ||u||^p_{L^q(\mathbb{R}^n)} \leq C(n,p) \frac{s(1-s)}{(n-sp)^{p-1}} [u]^p_{W^{s,p}(\mathbb{R}^n)}, \text{ for all } u \in W^{s,p}(\mathbb{R}^n), \tag{1.3}$$

where $sp < n, q = \frac{np}{n-sp}$ and $C(n,p) > 0$ is a function of n, p only;

(ii) for any cube $Q \subset \mathbb{R}^n$

$$\left|\left| u - \fint_Q u \right|\right|^p_{L^q(Q)} \leq C(n,p) \frac{(1-s)}{(n-sp)^{p-1}} [u]^p_{W^{s,p}(Q)}, \text{ for all } u \in W^{s,p}(Q), \tag{1.4}$$

where $s \in (0,1)$, $1 \leq p < \infty$, $sp < n$, $q = \frac{np}{n-sp}$ and $C(n,p) > 0$ is a constant depending only on n, p.

Inequality (1.4) was established previously by Bourgain, Brezis, and Mironescu in [8] under the assumption $\frac{1}{2} \leq s < 1, 1 \leq p < \infty$ and $sp < 1$. Given the fact that, fractional Sobolev inequalities have already been proved long ago (See [1,9,22,27]), the inequalities (1.3) and (1.4) still hold strong ground because of the following two results:

(i) (See [7]) For any smooth bounded domain $\Omega \subset \mathbb{R}^n$ and for any $u \in W^{1,p}(\Omega)$

$$\lim_{s \uparrow 1} (1-s) \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy \sim ||\nabla u||^p_{L^p(\Omega)}, \tag{1.5}$$

where $1 \leq p < \infty$;

(ii) (See [23]) For any $u \in \cup_{0<s<1} W^{s,p}(\mathbb{R}^n)$

$$\lim_{s\downarrow 0} s\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}dxdy \sim ||u||^p_{L^p(\mathbb{R}^n)}\,, \tag{1.6}$$

where $1 \le p < \infty$.

Next we will recall some relevant results on the Heisenberg group. The Heisenberg group $\mathbb{H}^n$ is defined as

$$\mathbb{H}^n := \{\xi = (z,t) : z = (x,y) \in \mathbb{R}^n \times \mathbb{R}^n,\, t \in \mathbb{R},\, x = (x_1,\dots,x_n) \text{ and } y = (y_1,\dots,y_n)\}$$

with the following group operation:

$$\xi \circ \xi' = \left(x + x', y + y', t + t' + 2\langle y, x'\rangle - 2\langle x, y'\rangle\right),$$

where $\xi' = (z',t') = (x',y',t')$, $\xi \in \mathbb{H}^n$ and $\langle\cdot,\cdot\rangle$ denotes the usual Euclidean inner product in $\mathbb{R}^n$. Clearly, $0 \in \mathbb{H}^n$ is the identity element, and for any $\xi \in \mathbb{H}^n$, $\xi^{-1} = -\xi$. A basis for the left invariant vector fields is given by

$$\begin{aligned} X_j &= \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t}\,, \quad 1 \le j \le n,\\ X_{n+j} &= \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t}\,, \quad 1 \le j \le n,\\ T &= \frac{\partial}{\partial t}. \end{aligned}$$

Let $\Omega \subset \mathbb{H}^n$ be a domain. For $u \in C^1(\Omega)$ define the sub-gradient $\nabla_{\mathbb{H}^n}(u)$ by

$$\begin{aligned} \nabla_{\mathbb{H}^n}(u) &:= \left(X_1(u),\dots,X_{2n}(u)\right),\\ \text{and}\quad |\nabla_{\mathbb{H}^n}(u)|^2 &:= \sum_{j=1}^{2n}\left|X_j(u)\right|^2. \end{aligned}$$

For $\xi = (z,t) = (x,y,t) \in \mathbb{H}^n$ define the Koranyi-Folland non isotropic gauge $d(\xi) = \left(\left(|x|^2+|y|^2\right)^2+t^2\right)^{\frac{1}{4}} = \left(|z|^4+|t|^2\right)^{\frac{1}{4}}$. Having this knowledge, let us recall the sub-elliptic Sobolev embedding theorem which is due to Folland and Stein [18]. While the result is valid for any Carnot group, we will state it in the set up of the Heisenberg group.

Let $1 < p < Q$ and set $q = \frac{pQ}{Q-p}$, where $Q = 2n+2$ is the homogeneous dimension of $\mathbb{H}^n$. Then there exists $S_p(\mathbb{H}^n) > 0$ such that

$$||u||_{L^q(\mathbb{H}^n)} \le S_p(\mathbb{H}^n)\,||\nabla_{\mathbb{H}^n}u||_{L^p(\mathbb{H}^n)}\,, \text{ for all } u \in C_c^\infty(\mathbb{H}^n). \tag{1.7}$$

In this context let us mention Vassilev, who proved the existence of extremal functions of the above inequality in [28].

On the other hand in [19], Garofalo and Lanconelli proved the following Hardy type inequality

$$\int_{\mathbb{H}^n} \frac{|\nabla_{\mathbb{H}^n} d(\xi)|^2}{d(\xi)^2}|u|^2 d\xi \leq \left(\frac{2}{Q-2}\right)^2 \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n} u|^2 d\xi, \quad u \in C_c^\infty(\mathbb{H}^n \setminus \{0\}). \tag{1.8}$$

The optimality of the constant $\left(\frac{2}{Q-2}\right)^2$ is shown in [21]. Inequality (1.8) was used in [19] to establish some strong unique continuation properties for singular perturbations of the Kohn-Spencer sub-Laplacian on $\mathbb{H}^n$. Over the years, more than one generalisation of (1.8) have been derived by different authors. For example, different extensions of (1.8) have been achieved in [12,24] when $p \neq 2$. Also a sharp inequality of type (1.8) has been derived in general Carnot-Carathéodory spaces by Danielli, Garofalo and Phuc in [13, 14]. In this connection, let us mention a Hardy-Sobolev type inequality due to Adimurthi and Sekar [3] which is related to the operator L_p defined as follows. For any smooth function u, and for $1 < p < \infty$, we define

$$L_p(u) = -\sum_{j=1}^n \left[X_j \left(\left| \frac{\nabla_{\mathbb{H}^n}(u)}{|z|} \right|^{p-2} X_j(u) \right) + X_{n+j} \left(\left| \frac{\nabla_{\mathbb{H}^n}(u)}{|z|} \right|^{p-2} X_{n+j}(u) \right) \right].$$

Let $1 < p < n+2$. Then, for all $u \in FS_0^{1,p}(\mathbb{H}^n)$, the HS-type inequality asserts that

$$\int_{\mathbb{H}^n} \frac{|\nabla_{\mathbb{H}^n} u|^p}{|z|^{p-2}} - \left(\frac{2(n+2-p)}{p}\right)^p \int_{\mathbb{H}^n} \frac{|z|^2 |u|^p}{\left(|z|^2 + t^2\right)^{\frac{p}{2}}} \geq 0 \tag{1.9}$$

and $\left(\frac{2(n+2-p)}{p}\right)^p$ is the best constant, which is never achieved. Here $FS_0^{1,p}(\mathbb{H}^n)$ is defined as the completion of $C_c^\infty(\mathbb{H}^n)$ under the following norm:

$$|u|_{1,p}^p := \int_{\mathbb{H}^n} \frac{|\nabla_{\mathbb{H}^n} u|^p}{|z|^{p-2}} dx dy dt.$$

Loosely speaking, our version of the Hardy-Sobolev inequality is a fractional analog of (1.9). Before going to our results let us define the following fractional order Sobolev space on $\mathbb{H}^n$:

$$\begin{aligned} &W_0^{s,p,\alpha}(\mathbb{H}^n) \\ &= \mathrm{Cl}\left\{ f \in C_c^\infty(\mathbb{H}^n) : \int_{\mathbb{H}^n \times \mathbb{H}^n} \frac{|f(\xi') - f(\xi)|^p}{d(\xi^{-1} \circ \xi')^{Q+ps} |z' - z|^{(p-2)\alpha}} d\xi' d\xi < \infty \right\}. \end{aligned}$$

Here $0 < s < 1$, $1 \leq p < \infty$, $\alpha \in \mathbb{R}$ and as usual $\xi = (z,t)$, $\xi' = (z',t) \in \mathbb{H}^n$. For $f \in \mathbb{H}^n$ define

$$[f]_{s,p,\alpha} := \left(\int_{\mathbb{H}^n \times \mathbb{H}^n} \frac{|f(\xi') - f(\xi)|^p}{d(\xi^{-1} \circ \xi')^{Q+ps} |z' - z|^{(p-2)\alpha}} d\xi' d\xi \right)^{\frac{1}{p}} \quad \text{and}$$

$$||f||_{s,p,\alpha} := ||f||_{L^p(\mathbb{H}^n)} + [f]_{s,p,\alpha}.$$

The above closure is taken under the norm $||.||_{s,p,\alpha}$. Our first result is the Sobolev embedding which is the following.

Theorem 1.1 (Sobolev Embedding). *Let $s \in (0,1)$, $1 \leq p < \infty$ and $\alpha \in \mathbb{R}$ satisfy the following two conditions:*

(a) $(p-2)\alpha \geq 0$;
(b) $ps + (p-2)\alpha < Q$.

Then there exists a positive constant $C_{n,p,s,\alpha}$ depending only on n, p, s and α such that

$$||f||_{L^q(\mathbb{H}^n)} \leq C_{n,p,s,\alpha}[f]_{s,p,\alpha}, \text{ for all } f \in W_0^{s,p,\alpha}(\mathbb{H}^n), \tag{1.10}$$

where $q := \frac{Qp}{Q-ps-(p-2)\alpha}$.

Next we have proved the following Hardy type inequality.

Theorem 1.2. *Let $s \in (0,1)$, $1 \leq p < \infty$ and $\alpha \in \mathbb{R}$ satisfies the following three conditions:*

(a) $(p-2)\alpha \geq 0$;
(b) $ps > 2$;
(c) $ps + (p-2)\alpha < Q$.

Then there exists a positive constant $C_{n,p,s,\alpha}$ depending only on n, p, s and α such that for any $f \in W_0^{s,p,\alpha}(\mathbb{H}^n)$, the following holds true:

$$\int_{\mathbb{H}^n} \frac{|f(\xi)|^p d\xi}{d(\xi)^{ps}|z|^{(p-2)\alpha}} \leq C_{n,p,s,\alpha}[f]^p_{s,p,\alpha}, \tag{1.11}$$

where $\xi = (z,t) \in \mathbb{H}^n$.

Remark 1.3.

(i) The condition $ps > 2$ along with $s \in (0,1)$ forces us to choose $p > 2$. But the main ingredient, *i.e.* Lemma 5.1 holds true for $p = 2$. Hence, the Hardy-Sobolev type inequality (1.11) still holds for $p = 2$;
(ii) Since we are dealing with the case $p \geq 2$, the condition $(p-2)\alpha \geq 0$ is natural;
(iii) Condition (b) and (c) together imply $(p-2)\alpha < Q-2$ which is not surprising, since, as proved in Section 2, if $(p-2)\alpha \geq Q-2$ then $W_0^{s,p,\alpha}(\mathbb{H}^n)$ becomes trivial;
(iv) One can easily observe that (1.11) is an analogous version of (1.2) but with a rough constant. Because of this, our proof is more elementary than the proof of (1.2).

In Section 6 we prove the compactness of the Sobolev embedding. The theorem states as follows.

Theorem 1.4. *Let $s \in (0,1)$, $\frac{3}{2} \le p < \infty$ and $\alpha \in \mathbb{R}$ satisfy the conditions* (a) *and* (b) *mentioned in Theorem* 1.1. *Let Ω be a bounded extension domain for $W_0^{s,p,\alpha}$ and $\mathscr{F}$ be a bounded set of $L^p(\Omega)$. Suppose that*

$$\sup_{u \in \mathscr{F}} [u]_{s,p,\alpha,\Omega} < \infty.$$

Then for any $1 \le r < q$, $\mathscr{F}$ is relatively compact in $L^r(\Omega)$.

The space $W_0^{s,p,\alpha}(\Omega)$ along with the quantity $[u]_{s,p,\alpha,\Omega}$ have been defined in Section 6. Also we have recalled the definition of an extension domain in the same section. Towards the end of the article (see Section 7) we have proved the following Morrey type embedding.

Theorem 1.5. *Let $p \in [1,\infty)$, $s \in (0,1)$ and $\alpha \in \mathbb{R}$ satisfy the following two conditions:*

(i) $0 \le (p-2)\alpha < Q-2$;
(ii) $ps + (p-2)\alpha > Q$.

Then there exists a constant $C > 0$ depending only on n, s, p, α such that for any $u \in L^p(\mathbb{H}^n)$

$$||u||_{C^{0,\beta}(\mathbb{H}^n)} \le C \left(||u||^p_{L^p(\mathbb{H}^n)} + [u]^p_{s,p,\alpha} \right)^{\frac{1}{p}}, \tag{1.12}$$

where $\beta := \frac{ps+(p-2)\alpha-Q}{p}$.

We have adopted the methods introduced in [16] to prove Theorem 1.1, Theorem 1.4 and Theorem 1.5. The proofs of Theorem 1.4 and Theorem 1.5 are almost the same as the proof in [16] for the Euclidean case. But for the sake of completeness we have added it.

As far as we know, there is no work related to Hardy inequalities of type (1.11) apart form results in [11] and [25]. In both the articles the authors have derived different Hardy type inequalities for different fractional powers of the sublaplacian. In [11], the authors have derived the inequalities for general stratified Lie group whereas in [25] a completely different approach has been used. In fact, they proved a non homogeneous Hardy type inequality for the fractional sublaplacian $\mathcal{L}_s$ on the Heisenberg group (see [25, Theorem 1.1]) via the following integral representation of $\mathcal{L}_s$:

$$\langle \mathcal{L}_s f, f \rangle = \frac{2^{n-2+3s}\Gamma\left(\frac{n+1+s}{2}\right)^2}{\pi^{n+1}|\Gamma(-s)|} \int_{\mathbb{H}^n} \int_{\mathbb{H}^n} \frac{|f(\xi') - f(\xi)|^2}{d(\xi^{-1}.\xi')^{Q+2s}} d\xi' d\xi.$$

The above representation makes our definition of fractional Sobolev space more relevant. Finally, we remark that following a similar approach one can generalize all the theorems that have been proved in this article to any homogeneous Carnot group.

The outline of the article is as follows. In Section 2 we have recalled a structure theorem involving the elements of $\mathbb{H}^n$. In Section 3 we have investigated the non triviality of $W_0^{s,p,\alpha}(\mathbb{H}^n)$. In Section 4, 5, 6 and 7 we have proved Theorems 1.1, 1.2, 1.4 and 1.5 respectively.

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2. Preliminaries

Let us recall a very basic result about the structure of elements of $\mathbb{H}^n$. Although the result is true for any general Homogeneous Carnot Group, let us state it, in the set up of the Heisenberg group. The proof of this result can be found in [6, page 727, Theorem 19.2.1]. Also we have added a short proof in the Appendix.

Theorem 2.1. *Let g denote the Lie algebra of vector fields of $\mathbb{H}^n$ and* $\operatorname{Exp} : g \to \mathbb{H}^n$ *be the usual exponential map. Then there exist a constant $M \in \mathbb{N}$ (depending only on $\mathbb{H}^n$) such that for any $h \in \mathbb{H}^n$,*

$$h = h_1 \circ \ldots \circ h_M$$
$$d(h_j) \le c_0 d(h), \quad j = 1 \ldots M,$$

where $h_j = \operatorname{Exp}(t_j X_{i_j})$, for some $t_j \in \mathbb{R}$, $i_j \in \{1, \ldots, 2n\}$ and $c_0 > 0$ is a constant independent of h and h_j.

Remark 2.2. Note that, here $\operatorname{Exp}(t_j X_{i_j}) = t_j e_{i_j}$, where $\{e_i\}_{i=1}^{2n+1}$ is the standard basis on $\mathbb{R}^{2n+1}$ and hence $d(h_j) = |t_j|$.

3. Non triviality of $W_0^{s,p,\alpha}(\mathbb{H}^n)$

From the definition it is not clear whether $W_0^{s,p,\alpha}(\mathbb{H}^n)$ is non trivial or not. In fact, if $(p-2)\alpha \ge Q-2$ then $W_0^{s,p,\alpha}(\mathbb{H}^n)$ is trivial. The next proposition shows that if $(p-2)\alpha < \min\{Q-2, p(1-s)\}$ then $C_c^\infty(\mathbb{H}^n) \subset W_0^{s,p,\alpha}(\mathbb{H}^n)$. For the rest of this article we will assume the above condition on α.

Proposition 3.1. *Let $s \in (0,1)$, $1 \le p < \infty$ and $\alpha \in \mathbb{R}$ satisfy the following condition*

$$(p-2)\alpha < \min\{Q-2, p(1-s)\}. \tag{3.1}$$

Then for any $u \in C_c^\infty(\mathbb{H}^n)$, $[u]_{s,p,\alpha} < \infty$.

Proof. For any $\xi \in \mathbb{H}^n$, define $B(\xi, 1) := \{\xi' \in \mathbb{H}^n : d(\xi^{-1} \circ \xi') < 1\}$. Then $[u]^p_{s,p,\alpha} = I_1 + I_2$, where

$$I_1 := \int_{\mathbb{H}^n} \int_{B^{\mathsf{c}}(\xi,1)} \frac{|u(\xi') - u(\xi)|^p}{d(\xi^{-1} \circ \xi')^{Q+ps} |z' - z|^{(p-2)\alpha}} d\xi' d\xi,$$
$$I_2 := \int_{\mathbb{H}^n} \int_{B(\xi,1)} \frac{|u(\xi') - u(\xi)|^p}{d(\xi^{-1} \circ \xi')^{Q+ps} |z' - z|^{(p-2)\alpha}} d\xi' d\xi.$$

We will show that both I_1 and I_2 are finite. First consider I_1:

$$\begin{aligned} I_1 &\le 2^{p-1} \int_{\mathbb{H}^n} \int_{B(\xi,1)^{\mathsf{c}}} \frac{|u(\xi')|^p + |u(\xi)|^p}{d(\xi^{-1} \circ \xi')^{Q+ps} |z' - z|^{(p-2)\alpha}} d\xi' d\xi \\ &= 2^p \int_{\mathbb{H}^n} |u(\xi)|^p \int_{B(\xi,1)^{\mathsf{c}}} \frac{d\xi'}{d(\xi^{-1} \circ \xi')^{Q+ps} |z' - z|^{(p-2)\alpha}} d\xi. \end{aligned}$$

In the above integral, changing the variable ξ' by $\xi \circ \tilde{\xi}$, where $\tilde{\xi} = (\tilde{z}, \tilde{t}) \in \mathbb{R}^{2n} \times \mathbb{R}$, we obtain

$$I_1 \le 2^p \, ||u||^p_{L^p(\mathbb{H}^n)} \int_{B^{\mathsf{c}}(0,1)} \frac{d\tilde{z} d\tilde{t}}{\left(|\tilde{z}|^4 + \tilde{t}^2\right)^{\frac{Q+ps}{4}} |\tilde{z}|^{(p-2)\alpha}}. \tag{3.2}$$

Note that in view of (3.2), it is enough to show $I_3 := \int_{B^{\mathsf{c}}(0,1)} \frac{d\tilde{z} d\tilde{t}}{\left(|\tilde{z}|^4 + \tilde{t}^2\right)^{\frac{Q+ps}{4}} |\tilde{z}|^{(p-2)\alpha}} < \infty$ to establish $I_1 < \infty$. To show this, we break I_3 as $I_3 = I_4 + I_5 + I_6 + I_7$, where

$$I_4 := \int_{\{|\tilde{z}|<1\} \times \{|\tilde{t}|<1\} \setminus B(0,1)} \frac{d\tilde{z} d\tilde{t}}{\left(|\tilde{z}|^4 + \tilde{t}^2\right)^{\frac{Q+ps}{4}} |\tilde{z}|^{(p-2)\alpha}},$$
$$I_5 := \int_{\{|\tilde{z}|\ge 1\} \times \{|\tilde{t}|<1\}} \frac{d\tilde{z} d\tilde{t}}{\left(|\tilde{z}|^4 + \tilde{t}^2\right)^{\frac{Q+ps}{4}} |\tilde{z}|^{(p-2)\alpha}},$$
$$I_6 := \int_{\{|\tilde{z}|<1\} \times \{|\tilde{t}|\ge 1\}} \frac{d\tilde{z} d\tilde{t}}{\left(|\tilde{z}|^4 + \tilde{t}^2\right)^{\frac{Q+ps}{4}} |\tilde{z}|^{(p-2)\alpha}},$$
$$I_7 := \int_{\{|\tilde{z}|\ge 1\} \times \{|\tilde{t}|\ge 1\}} \frac{d\tilde{z} d\tilde{t}}{\left(|\tilde{z}|^4 + \tilde{t}^2\right)^{\frac{Q+ps}{4}} |\tilde{z}|^{(p-2)\alpha}}.$$

Clearly

$$I_4 \le 2 \int_{|\tilde{z}|<1} \frac{d\tilde{z}}{|\tilde{z}|^{(p-2)\alpha}} < \infty, \tag{3.3}$$

since $(p-2)\alpha < Q-2$. Moreover

$$I_5 \le 2\int_{|\tilde{z}|\ge 1} \frac{d\tilde{z}}{|\tilde{z}|^{Q+ps+(p-2)\alpha}} = 4nw_{2n}\int_1^{\infty} \frac{dr}{r^{3+ps+(p-2)\alpha}} < \infty. \tag{3.4}$$

Now let us estimate I_6:

$$I_6 \le 2nw_{2n}\int_{|\tilde{t}|\ge 1} \frac{d\tilde{t}}{\tilde{t}^{\frac{Q+ps}{2}}}\int_0^1 r^{Q-3-(p-2)\alpha}dr < \infty \tag{3.5}$$

since $Q+ps>2$ and $(p-2)\alpha < Q-2$. Now

$$\begin{aligned} I_7 &= 2\int_{|\tilde{z}|\ge 1}\int_{\frac{1}{|\tilde{z}|^2}}^{\infty} \frac{d\tilde{t}d\tilde{z}}{\left(1+\tilde{t}^2\right)^{\frac{Q+ps}{4}}|\tilde{z}|^{Q-2+ps+(p-2)\alpha}} \\ &\le 2\int_0^{\infty}\frac{d\tilde{t}}{\left(1+\tilde{t}^2\right)}\int_{|\tilde{z}|\ge 1}\frac{d\tilde{z}}{|\tilde{z}|^{Q-3+ps+(p-2)\alpha}} < \infty. \end{aligned} \tag{3.6}$$

Hence, from (3.3), (3.4), (3.5), (3.6) we conclude that $I_3 < \infty$ and hence from (3.2)

$$I_1 \le C_{n,s,p,\alpha}\,||u||^p_{L^p(\mathbb{H}^n)}, \tag{3.7}$$

where $C_{n,s,p,\alpha} > 0$ is a constant depending only on n, s, p, α. Now, as $u \in C_c^{\infty}(\mathbb{H}^n)$, one has $I_1 < \infty$. It remains to show that $I_2 < \infty$. Before that, let us point out the following fact, which will be used without mention. If $\xi', \xi, \tilde{\xi} \in \mathbb{H}^n$ are such that $\xi = \xi' \circ \tilde{\xi}$ then for any $u \in C^1(\mathbb{H}^n)$ we have $\nabla^{\xi'}_{\mathbb{H}^n}u(\xi'\circ\tilde{\xi}) = \nabla^{\xi}_{\mathbb{H}^n}u(\xi)$, where $\nabla^{\xi'}_{\mathbb{H}^n}$ and $\nabla^{\xi}_{\mathbb{H}^n}$ denote the sub-gradient with respect to ξ' and ξ variable respectively.

Now let us consider

$$I_2 := \int_{\mathbb{H}^n}\int_{B(\xi,1)} \frac{|u(\xi')-u(\xi)|^p}{d(\xi^{-1}\circ\xi')^{Q+ps}|z'-z|^{(p-2)\alpha}}d\xi' d\xi.$$

Changing the variable ξ' by $\xi\circ\tilde{\xi}$, $\tilde{\xi} = (\tilde{z},\tilde{t}) \in \mathbb{H}^n$ we have

$$I_2 = \int_{\mathbb{H}^n}|u(\xi\circ\tilde{\xi}) - u(\xi)|^p \int_{B(0,1)}\frac{d\tilde{\xi}}{d(\tilde{\xi})^{Q+ps}|\tilde{z}|^{(p-2)\alpha}}d\xi. \tag{3.8}$$

By Theorem 2.1, for $\tilde{\xi} \in B(0,1)$ there exists $M \in \mathbb{N}$ (depending only on $\mathbb{H}^n$) such that

$$\begin{aligned} \tilde{\xi} &= h_1 \circ \ldots \circ h_M, \\ d(h_j) &\le c_0 d(\tilde{\xi}),\ \ j = 1,\ldots,M, \end{aligned}$$

where $h_j = \mathrm{Exp}(t_j X_{i_j}) \in \mathbb{H}^n$ for some $t_j \in \mathbb{R}$ and $i_j \in \{1,\ldots,2n\}$ and $c_0 > 0$ is a constant independent of $\tilde{\xi}$ and h_j. Also $d(h_j) = |t_j|$. Using this decomposition of $\tilde{\xi}$ we will estimate $|u(\xi \circ \tilde{\xi}) - u(\xi)|$ in terms of $|\nabla_{\mathbb{H}^n} u|$. For this, we can take $t_j \geq 0$ without losing generality. Now,

$$|u(\xi \circ \tilde{\xi}) - u(\xi)| \leq \sum_{j=1}^{M} |u(\xi \circ h_1 \circ \ldots \circ h_j) - u(\xi \circ h_1 \circ \ldots \circ h_{j-1})|. \tag{3.9}$$

Here we follow the convention that at $j = 1, \xi \circ h_1 \circ \ldots \circ h_{j-1} = \xi$. Now let us consider the integral curve $\gamma(t, X_{i_j})$ of X_{i_j} starting from $\xi \circ h_1 \circ \ldots \circ h_{j-1}$. Then

$$\gamma\left(t, X_{i_j}\right) = \xi \circ h_1 \circ \ldots \circ h_{j-1} \circ \operatorname{Exp}\left(t X_{i_j}\right)$$

and

$$\gamma\left(0, X_{i_j}\right) = \xi \circ h_1 \circ \ldots \circ h_{j-1}, \ \gamma(t_j, X_{i_j}) = \xi \circ h_1 \circ \ldots \circ h_j.$$

So there holds

$$\begin{aligned}
&u\left(\gamma\left(t_j, X_{i_j}\right)\right) - u\left(\gamma\left(0, X_{i_j}\right)\right)\\
&= \int_0^{t_j} \frac{d}{dr} u\left(\gamma\left(r, X_{i_j}\right)\right) dr\\
&= \int_0^{t_j} \nabla u\left(\gamma\left(r, X_{i_j}\right)\right) \dot{\gamma}(r, X_{i_j}) dr\\
&= \int_0^{t_j} X_{i_j}^{\xi} u\left(\gamma\left(r, X_{i_j}\right)\right) dr \ \ \left(\text{since } \gamma \text{ is the integral curve of } X_{i_j}\right)\\
&= \int_0^1 t_j X_{i_j}^{\xi} u\left(\xi \circ h_1 \circ \ldots \circ h_{j-1} \circ \operatorname{Exp}(t_j r X_{i_j})\right) dr.
\end{aligned}$$

Hence using Jensen's inequality and the above in (3.9) we get

$$\left|u\left(\xi \circ \tilde{\xi}\right) - u(\xi)\right|^p \leq c_0 d(\tilde{\xi}) \sum_{j=1}^{M} \int_0^1 \left|\nabla_{\mathbb{H}^n}^{\xi} u\left(\xi \circ h_1 \circ \ldots \circ h_{j-1} \circ \operatorname{Exp}(t_j r X_{i_j})\right)\right|^p dr.$$

So, from (3.8) we have

$$\begin{aligned}
I_2 &\leq \sum_{j=1}^{M} \int_0^1 \int_{B(0,1)} \frac{1}{d(\tilde{\xi})^{Q-p(1-s)} |\tilde{z}|^{(p-2)\alpha}}\\
&\quad \cdot \int_{\mathbb{H}^n} \left|\nabla_{\mathbb{H}^n}^{\xi} u\left(\xi \circ h_1 \circ \ldots \circ h_{j-1} \circ \operatorname{Exp}(r t_j X_{i_j})\right)\right|^p d\xi d\tilde{\xi} dr\\
&= c_0 M \, ||\nabla_{\mathbb{H}^n} u||_{L^p(\mathbb{H}^n)}^p \int_{B(0,1)} \frac{d\tilde{\xi}}{d(\tilde{\xi})^{Q-p(1-s)} |\tilde{z}|^{(p-2)\alpha}}\\
&\leq c_0 M \, ||\nabla_{\mathbb{H}^n} u||_{L^p(\mathbb{H}^n)}^p \int_{|\tilde{z}|<1} \int_{|\tilde{t}|<1} \frac{d\tilde{t} d\tilde{z}}{\left(|\tilde{z}|^4 + \tilde{t}^2\right)^{\frac{Q-p(1-s)}{4}} |\tilde{z}|^{(p-2)\alpha}}.
\end{aligned} \tag{3.10}$$

Let

$$J := \int_{|\tilde{z}|<1} \int_{|\tilde{t}|<1} \frac{d\tilde{t}d\tilde{z}}{\left(|\tilde{z}|^4 + \tilde{t}^2\right)^{\frac{Q-p(1-s)}{4}} |\tilde{z}|^{(p-2)\alpha}}.$$

Then (3.10) suggests that it is enough to prove $J < \infty$ to conclude $I_2 < \infty$. To show this we will consider the following cases.

Case I: $Q - p(1-s) \leq 0$. Then, since $(p-2)\alpha < Q - 2$,

$$\int_{|\tilde{z}|<1} \frac{d\tilde{z}}{|\tilde{z}|^{(p-2)\alpha}} = 2nw_{2n} \int_0^1 r^{Q-3-(p-2)\alpha} dr < \infty.$$

Hence $J < \infty$.

Case II: $0 < Q - p(1-s) < 2$, *i.e.* $Q - 2 < p(1-s)$. Then

$$\int_0^1 \frac{d\tilde{t}}{\tilde{t}^{\frac{Q-p(1-s)}{2}}} < \infty.$$

Also, as $(p-2)\alpha < Q - 2$, $\int_{|\tilde{z}|<1} \frac{d\tilde{z}}{|\tilde{z}|^{(p-2)\alpha}} < \infty$. Hence

$$J \leq 2 \int_{|\tilde{z}|<1} \frac{d\tilde{z}}{|\tilde{z}|^{(p-2)\alpha}} \int_0^1 \frac{d\tilde{t}}{\tilde{t}^{\frac{Q-p(1-s)}{2}}} < \infty.$$

Case III: $Q - p(1-s) > 2$, *i.e.* $p(1-s) < Q - 2$. Then

$$J = 4nw_{2n} \int_0^1 \int_0^1 \frac{r^{Q-3-(p-2)\alpha} dr d\tilde{t}}{\left(r^4 + \tilde{t}^2\right)^{\frac{Q-p(1-s)}{4}}}.$$

Changing the variable $\tilde{t}$ by $r^2 t$ in the above integral we get

$$J = 4nw_{2n} \int_0^1 \int_0^{\frac{1}{r^2}} \frac{r^{-1+p(1-s)-(p-2)\alpha}}{\left(1+t^2\right)^{\frac{Q-p(1-s)}{4}}} dt dr.$$

Now, since $(p-2)\alpha < p(1-s)$ and $Q - p(1-s) > 2$ both the integrals

$$\int_0^1 r^{-1+p(1-s)-(p-2)\alpha} dr \quad \text{and} \quad \int_0^\infty \frac{d\tilde{t}}{\left(1+\tilde{t}^2\right)^{\frac{Q-p(1-s)}{4}}}$$

are finite. Hence $J < \infty$.

Case IV: $Q - p(1-s) = 2$. Then by the same change of variable as in Case III we have

$$\begin{aligned} J &= 4nw_{2n}\int_0^1\int_0^{\frac{1}{r^2}} \frac{r^{Q-3-(p-2)\alpha}}{\left(1+\tilde{t}^2\right)^{\frac{1}{2}}} d\tilde{t}dr \\ &= 4nw_{2n}\left[\int_0^1\int_0^1 \frac{r^{Q-3-(p-2)\alpha}}{\left(1+\tilde{t}^2\right)^{\frac{1}{2}}} d\tilde{t}dr + \int_0^1\int_1^{\frac{1}{r^2}} \frac{r^{Q-3-(p-2)\alpha}}{\left(1+\tilde{t}^2\right)^{\frac{1}{2}}} d\tilde{t}dr\right]. \end{aligned} \tag{3.11}$$

Now, since $(p-2)\alpha < Q-2$, the first integral of the above equality is finite. To show the second one is finite let us set

$$J_1 := \int_0^1\int_1^{\frac{1}{r^2}} \frac{r^{Q-3-(p-2)\alpha}}{\left(1+\tilde{t}^2\right)^{\frac{1}{2}}} d\tilde{t}dr.$$

Then

$$\begin{aligned} J_1 &\le \int_0^1 r^{Q-3-(p-2)\alpha}\int_1^{\frac{1}{r^2}} \frac{d\tilde{t}}{\tilde{t}} dr \\ &= 2\int_0^1 r^{Q-3-(p-2)\alpha}\log\frac{1}{r}dr. \end{aligned}$$

Put $t = \log\frac{1}{r}$ in the above integral. Then

$$J_1 \le 2\int_0^\infty te^{-(Q-2-(p-2)\alpha)t}dt < \infty,$$

since $Q-2-(p-2)\alpha > 0$. Hence from (3.11) we have $J < \infty$.

This proves the proposition. □

4. Sobolev embedding

Let $1 \le p < \infty$ and $s \in (0,1)$. We are trying to find a positive constant C independent of $f \in W_0^{s,p,\alpha}(\mathbb{H}^n)$ such that the following inequality holds:

$$||f||_{L^q(\mathbb{H}^n)} \le C[f]_{s,p,\alpha}$$

for some specific q. To find the desirable value of q let us use the following dilation argument. Let $f \in W_0^{s,p,\alpha}(\mathbb{H}^n)$; for $\lambda > 0$ we define

$$f_\lambda(\xi') = f\left(\sqrt{\lambda}z', \lambda t'\right) \quad \text{for } \xi' = (z',t') = (x',y',t') \in \mathbb{H}^n.$$

So,

$$
\begin{aligned}
||f_\lambda||_{L^q(\mathbb{H}^n)} &= \left(\int_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}} \left|f_\lambda\big(x',y',t'\big)\right|^q dx'dy'dt'\right)^{\frac{1}{q}} \\
&= \left(\int_{\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}} \left|f\big(\sqrt{\lambda}x',\sqrt{\lambda}y',\lambda t'\big)\right|^q dx'dy'dt'\right)^{\frac{1}{q}} \\
&= \frac{1}{\lambda^{\frac{Q}{2q}}}\, ||f||_{L^q(\mathbb{H}^n)}\,.
\end{aligned}
$$

For $\xi = (x,y,t) = (z,t)$ and $\xi' \in \mathbb{H}^n$ let us define

$$
\xi_\lambda = \left(\frac{x}{\sqrt{\lambda}},\frac{y}{\sqrt{\lambda}},\frac{t}{\lambda}\right), \quad \xi'_\lambda = \left(\frac{x'}{\sqrt{\lambda}},\frac{y'}{\sqrt{\lambda}},\frac{t'}{\lambda}\right).
$$

Then $d(\xi_\lambda^{-1}\circ\xi'_\lambda) = \frac{1}{\sqrt{\lambda}}d(\xi^{-1}\circ\xi')$

$$
\begin{aligned}
[f_\lambda]^p_{s,p,\alpha} &= \int_{\mathbb{H}^n\times\mathbb{H}^n} \frac{|f_\lambda(\xi')-f_\lambda(\xi)|^p}{d(\xi^{-1}\circ\xi')^{Q+ps}|z'-z|^{(p-2)\alpha}}d\xi'd\xi \\
&= \int_{\mathbb{H}^n\times\mathbb{H}^n} \frac{\lambda^{\frac{Q+ps+(p-2)\alpha}{2}}|f(\xi')-f(\xi)|^p}{\lambda^Q d(\xi^{-1}\circ\xi')^{Q+ps}|z'-z|^{(p-2)\alpha}}d\xi'd\xi \\
&= \frac{[f]^p_{s,p,\alpha}}{\lambda^{\frac{Q-ps-(p-2)\alpha}{2}}}.
\end{aligned}
$$

Consequently, to prove $||f||_{L^q(\mathbb{H}^n)} \le C[f]_{s,p,\alpha}$, it is necessary to choose q such that $\frac{Q-ps-(p-2)\alpha}{2p} = \frac{Q}{2q}$, *i.e.* $q = \frac{Qp}{Q-ps-(p-2)\alpha}$ holds. Hence for the remaining part of this section let us assume that $ps+(p-2)\alpha < Q$, $(p-2)\alpha \ge 0$ and $q = \frac{Qp}{Q-ps-(p-2)\alpha}$. From the definition of q and by our assumption it follows that $1 \le q < \infty$ and

$$
\frac{p}{q} = 1 - \frac{ps+(p-2)\alpha}{Q}.
$$

Clearly $\frac{p}{q} < 1$. Let us take $r = \frac{Q}{ps+(p-2)\alpha}$. Then $r > 1$ and $\frac{1}{r}+\frac{p}{q} = 1$.

Lemma 4.1. *Let $E \subset \mathbb{H}^n$ be any measurable set with finite Lebesgue measure. Fix $\xi = (x,y,t) = (z,t) \in \mathbb{H}^n$, $1 \le p < \infty$. Then there exists a positive constant $C_{n,p,s,\alpha}$ depending only on n, p, α, s such that*

$$
\int_{E^{\complement}} \frac{d\xi'}{d(\xi'^{-1}\circ\xi)^{Q+ps}|z-z'|^{(p-2)\alpha}} \ge C_{n,p,s,\alpha}|E|^{-\frac{ps+(p-2)\alpha}{Q}}, \tag{4.1}
$$

where $\xi' = (x',y',t') = (z',t') \in E^{\complement}$ and $|E|$ denotes the Lebesgue measure of E.

Proof. The integral of (4.1) is of the form

$$I := \int_{E^{\mathrm{c}}} \frac{dz'dt'}{\left(|z-z'|^4 + (t-t'+2 < x', y > -2 < y', x >)^2\right)^{\frac{Q+ps}{4}} |z-z'|^{(p-2)\alpha}}.$$

Put $\zeta = z - z'$, where $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^{2n}$ and $\mu = t - t' + 2 < x', y > -2 < y', x >$. Then

$$I = \int_{F^{\mathrm{c}}} \frac{d\zeta d\mu}{\left(|\zeta|^4 + \mu^2\right)^{\frac{Q+ps}{4}} |\zeta|^{(p-2)\alpha}}, \tag{4.2}$$

where F is a measurable set depending on ξ with $|F| = |E|$. Now let $R > 0$ be such that $|E| = 2w_{2n}R^Q$. Denote $B_R = B_{2n}(0, R) \times (-R^2, R^2)$, where $B_{2n}(0, R)$ is a ball in $\mathbb{R}^{2n}$ with radius R and centre at the origin. Then clearly $|E| = |F| = |B_R|$. Now (4.2) can be written as

$$\begin{aligned} I &= \int_{F^{\mathrm{c}} \cap B_R} \frac{d\zeta d\mu}{\left(|\zeta|^4 + \mu^2\right)^{\frac{Q+ps}{4}} |\zeta|^{(p-2)\alpha}} + \int_{F^{\mathrm{c}} \cap B_R^{\mathrm{c}}} \frac{d\zeta d\mu}{\left(|\zeta|^4 + \mu^2\right)^{\frac{Q+ps}{4}} |\zeta|^{(p-2)\alpha}} \\ &\geq \int_{F^{\mathrm{c}} \cap B_R} \frac{d\zeta d\mu}{2^{\frac{Q+ps}{4}} R^{Q+ps+(p-2)\alpha}} + \int_{F^{\mathrm{c}} \cap B_R^{\mathrm{c}}} \frac{d\zeta d\mu}{\left(|\zeta|^4 + \mu^2\right)^{\frac{Q+ps}{4}} |\zeta|^{(p-2)\alpha}}. \end{aligned} \tag{4.3}$$

The last inequality came from the fact that, for $(p-2)\alpha \geq 0$ and $(\zeta, \mu) \in B_R$

$$\left(|\zeta|^4 + \mu^2\right)^{\frac{Q+ps}{4}} < 2^{\frac{Q+ps}{4}} R^{Q+ps+(p-2)\alpha} \text{ and } |\zeta|^{(p-2)\alpha} \leq R^{(p-2)\alpha}.$$

Now, since $|F| = |B_R|$, we have $|B_R^{\mathrm{c}} \cap F| = |B_R \cap F^{\mathrm{c}}|$. Hence continuing from (4.3) we have

$$I \geq \int_{F \cap B_R^{\mathrm{c}}} \frac{d\zeta d\mu}{2^{\frac{Q+ps}{4}} R^{Q+ps+(p-2)\alpha}} + \int_{F^{\mathrm{c}} \cap B_R^{\mathrm{c}}} \frac{d\zeta d\mu}{\left(|\zeta|^4 + \mu^2\right)^{\frac{Q+ps}{4}} |\zeta|^{(p-2)\alpha}}. \tag{4.4}$$

Now let $\tilde{B}_R := B_{2n}^{\mathrm{c}}(0, R) \times (-R^2, R^2)^{\mathrm{c}}$. Then $\tilde{B}_R \subset B_R^{\mathrm{c}}$ and for $(\zeta, \mu) \in \tilde{B}_R$ we have

$$\left(|\zeta|^4 + \mu^2\right)^{\frac{Q+ps}{4}} |\zeta|^{(p-2)\alpha} \geq 2^{\frac{Q+ps}{4}} R^{Q+ps+(p-2)\alpha}.$$

Consequently, from (4.4) we obtain

$$\begin{aligned} I &\geq \int_{\tilde{B}_R} \frac{d\zeta d\mu}{\left(|\zeta|^4 + \mu^2\right)^{\frac{Q+ps}{4}} |\zeta|^{(p-2)\alpha}} \\ &= \int_{|\zeta|>R} \int_{|\mu|>R^2} \frac{d\mu d\zeta}{\left(|\zeta|^4 + \mu^2\right)^{\frac{Q+ps}{4}} |\zeta|^{(p-2)\alpha}}. \end{aligned} \tag{4.5}$$

Let

$$I_1 := \int_{|\mu|>R^2} \frac{d\mu}{\left(|\zeta|^4+\mu^2\right)^{\frac{Q+ps}{4}}} = 2\int_{R^2}^{\infty} \frac{d\mu}{\left(|\zeta|^4+\mu^2\right)^{\frac{Q+ps}{4}}}.$$

Put $\mu = |\zeta|^2\nu$, then I_1 becomes

$$I_1 = \frac{2}{|\zeta|^{Q+ps-2}} \int_{\frac{R^2}{|\zeta|^2}}^{\infty} \frac{d\nu}{\left(1+\nu^2\right)^{\frac{Q+ps}{4}}}. \tag{4.6}$$

Consider

$$I_2 := \int_{\frac{R^2}{|\zeta|^2}}^{\infty} \frac{d\nu}{\left(1+\nu^2\right)^{\frac{Q+ps}{4}}}.$$

Integrating by parts we get

$$\begin{aligned} I_2 &= \frac{Q+ps}{2}\int_{\frac{R^2}{|\zeta|^2}}^{\infty} \frac{\nu^2 d\nu}{\left(1+\nu^2\right)^{\frac{Q+ps}{4}+1}} - \frac{R^2}{|\zeta|^2\left(1+\frac{R^4}{|\zeta|^4}\right)^{\frac{Q+ps}{4}}} \\ &= \frac{Q+ps}{2} I_2 - \frac{Q+ps}{2}\int_{\frac{R^2}{|\zeta|^2}}^{\infty} \frac{d\nu}{\left(1+\nu^2\right)^{\frac{Q+ps}{4}+1}} - \frac{R^2}{|\zeta|^2\left(1+\frac{R^4}{|\zeta|^4}\right)^{\frac{Q+ps}{4}}}. \end{aligned}$$

Hence

$$I_2 \geq \frac{2R^2}{Q-2+ps} \frac{|\zeta|^{Q-2+ps}}{\left(|\zeta|^4+R^4\right)^{\frac{Q+ps}{4}}}.$$

So, from (4.5) and (4.6) we have

$$\begin{aligned} I &\geq \frac{4R^2}{Q+ps-2}\int_{|\zeta|>R} \frac{d\zeta}{\left(|\zeta|^4+R^4\right)^{\frac{Q+ps}{4}} |\zeta|^{(p-2)\alpha}} \\ &= \frac{8n w_{2n} R^2}{Q+ps-2}\int_R^{\infty} \frac{r^{Q-3} dr}{\left(r^4+R^4\right)^{\frac{Q+ps}{4}} r^{(p-2)\alpha}}. \end{aligned}$$

Now, in the above integral, since $r > R$, there holds $\frac{1}{r^4+R^4} \geq \frac{1}{2r^4}$. Hence from the above formula we have

$$\begin{aligned} I &\geq \frac{8n w_{2n} R^2}{(Q+ps-2)2^{\frac{Q+ps}{4}}}\int_R^{\infty} \frac{r^{Q-3-(p-2)\alpha} dr}{r^{Q+ps}} \\ &= \frac{8n w_{2n}}{(2+ps+(p-2)\alpha)(Q+ps-2)2^{\frac{Q+ps}{4}}} \frac{1}{R^{ps+(p-2)\alpha}} \\ &= C_{n,p,s,\alpha}|E|^{-\frac{ps+(p-2)\alpha}{Q}}. \end{aligned}$$

This proves the lemma. □

Lemma 4.2. *Let $s \in (0,1)$, $1 \le p < \infty$ and α be such that it satisfies $(p-2)\alpha \ge 0$. Let $ps + (p-2)\alpha < Q$. Let $\{a_k\}$, $\{d_k\}$ be two sequences of nonnegative real numbers satisfying the following properties:*

(a) *a_k is decreasing;*

(b) $a_k = \sum_{j=k}^{\infty} d_j$.

Then we have

(i) *for any fixed $T > 0$*

$$\sum_{k\in\mathbb{Z}} a_k^{\frac{Q-ps-(p-2)\alpha}{Q}} T^k \le T^{\frac{Q}{Q-ps-(p-2)\alpha}} \sum_{k\in\mathbb{Z}, a_k\neq 0} a_{k+1} a_k^{-\frac{ps+(p-2)\alpha}{Q}} T^k; \tag{4.7}$$

(ii) *moreover, if $T > 1$ then*

$$\sum_{i\in\mathbb{Z}, a_{i-1}\neq 0} \sum_{j\ge i+1} T^{pi} a_{i-1}^{-\frac{ps+(p-2)\alpha}{Q}} d_j \le \frac{1}{T^p - 1} \sum_{i\in\mathbb{Z}, a_{i-1}\neq 0} T^{pi} a_{i-1}^{-\frac{ps+(p-2)\alpha}{Q}} d_i. \tag{4.8}$$

Proof. (i) Let $\frac{1}{\eta} := \frac{ps+(p-2)\alpha}{Q}$ and $\frac{1}{\beta} := \frac{Q-ps-(p-2)\alpha}{Q}$. Then clearly $\frac{1}{\eta} + \frac{1}{\beta} = 1$. Now

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} a_k^{\frac{Q-ps-(p-2)\alpha}{Q}} T^k &= \sum_{k\in\mathbb{Z}, a_k\neq 0} a_{k+1}^{\frac{Q-ps-(p-2)\alpha}{Q}} T^{k+1} \text{ (since, } a_{k+1} = 0 \text{ if } a_k = 0) \\
&= T \sum_{k\in\mathbb{Z}, a_k\neq 0} a_{k+1}^{\frac{1}{\beta}} T^k \\
&= T \sum_{k\in\mathbb{Z}, a_k\neq 0} \left(a_k^{\frac{ps+(p-2)\alpha}{Q\beta}} T^{\frac{k}{\eta}} \right) \left(a_k^{-\frac{ps+(p-2)\alpha}{Q\beta}} a_{k+1}^{\frac{1}{\beta}} T^{\frac{k}{\beta}} \right)
\end{aligned}$$

(applying Hölder's inequality)

$$\begin{aligned}
&\le T \left(\sum_{k\in\mathbb{Z}, a_k\neq 0} a_k^{\eta\frac{ps+(p-2)\alpha}{Q\beta}} T^k \right)^{\frac{1}{\eta}} \left(\sum_{a_k\neq 0} a_k^{-\frac{ps+(p-2)\alpha}{Q}} a_{k+1} T^k \right)^{\frac{1}{\beta}} \\
&= T \left(\sum_{a_k\neq 0} a_k^{\frac{Q-ps-(p-2)\alpha}{Q}} T^k \right)^{\frac{1}{\eta}} \left(\sum_{a_k\neq 0} a_k^{-\frac{ps+(p-2)\alpha}{Q}} a_{k+1} T^k \right)^{\frac{1}{\beta}}.
\end{aligned}$$

Hence,

$$\sum_{k\in\mathbb{Z}} a_k^{\frac{Q-ps-(p-2)\alpha}{Q}} T^k \le T^\beta \sum_{k\in\mathbb{Z}, a_k\neq 0} a_k^{-\frac{ps+(p-2)\alpha}{Q}} a_{k+1} T^k.$$

This proves the first part of the lemma.

(ii) Let

$$
\begin{aligned}
I &:= \sum_{a_{i-1}\neq 0} \sum_{j\geq i+1} T^{pi} a_{i-1}^{-\frac{ps+(p-2)\alpha}{Q}} d_j \\
&= \sum_{a_{i-1}\neq 0} \sum_{j\geq i+1, d_j\neq 0} T^{pi} a_{i-1}^{-\frac{ps+(p-2)\alpha}{Q}} d_j .
\end{aligned}
$$

Now, as $j \geq i+1$ so, $d_j \leq a_j \leq a_{j-1} \leq a_i \leq a_{i-1}$. Hence

$$
\begin{aligned}
I &\leq \sum_{i\in\mathbb{Z}} \sum_{j\geq i+1, d_j a_{j-1}\neq 0} T^{pi} a_{i-1}^{-\frac{ps+(p-2)\alpha}{Q}} d_j \\
&\leq \sum_{i\in\mathbb{Z}} \sum_{j\geq i+1, d_j a_{j-1}\neq 0} T^{pi} a_{j-1}^{-\frac{ps+(p-2)\alpha}{Q}} d_j \text{ (Since } a_{j-1} \leq a_{i-1}) \\
&= \sum_{j\in\mathbb{Z}, d_j a_{j-1}\neq 0} \left(\sum_{i\leq j-1} T^{pi} \right) a_{j-1}^{-\frac{ps+(p-2)\alpha}{Q}} d_j \\
&= \left(\sum_{k=0}^{\infty} T^{-pk} \right) \sum_{j\in\mathbb{Z}, a_{j-1}\neq 0} T^{(j-1)p} a_{j-1}^{-\frac{ps+(p-2)\alpha}{Q}} d_j \\
&= \frac{1}{T^p - 1} \sum_{j\in\mathbb{Z}, a_{j-1}\neq 0} T^{jp} a_{j-1}^{-\frac{ps+(p-2)\alpha}{Q}} d_j .
\end{aligned}
$$

This proves the second part of the lemma. □

4.1. Proof of Theorem 1.1

Proof. It is enough to prove the theorem for $0 \leq f \in W_0^{s,p,\alpha}(\mathbb{H}^n)$ with compact support. For such an f let us define the following sets:

$$
\begin{aligned}
A_k &:= \left\{ x \in \mathbb{H}^n : f(x) \geq 2^k \right\}, \\
D_k &:= A_k \setminus A_{k+1} = \left\{ x \in \mathbb{H}^n : 2^k \leq f(x) < 2^{k+1} \right\}.
\end{aligned}
$$

Let $a_k = |A_k|$ and $d_k = |D_k|$. Then, clearly, a_k and d_k satisfy the conditions of Lemma 4.2. Now

$$
\begin{aligned}
||f||_{L^q(\mathbb{H}^n)}^q &= \sum_{k\in\mathbb{Z}} \int_{D_k} f^q(x)dx \\
&\leq \sum_{k\in\mathbb{Z}} 2^{q(k+1)} d_k \\
&\leq 2^q \sum_{k\in\mathbb{Z}} 2^{qk} a_k ,
\end{aligned}
$$

where, in the last inequality, we have used $d_k \le a_k$. So,

$$\begin{aligned} ||f||^p_{L^q(\mathbb{H}^n)} &\le 2^p \left(\sum_{k\in\mathbb{Z}} 2^{qk} a_k\right)^{\frac{p}{q}} \\ &\le 2^p \sum_{k\in\mathbb{Z}} 2^{pk} a_k^{\frac{p}{q}} \quad (\text{Since } p < q) \\ &= 2^p \sum_{k\in\mathbb{Z}} 2^{pk} a_k^{\frac{Q-ps-(p-2)\alpha}{Q}}. \end{aligned}$$

Using inequality (4.7) of Lemma 4.2 we obtain

$$||f||^p_{L^q(\mathbb{H}^n)} \le 2^{p+q} \sum_{k\in\mathbb{Z}, a_k\neq 0} 2^{pk} a_{k+1} a_k^{-\frac{ps+(p-2)\alpha}{Q}}. \tag{4.9}$$

Now let us estimate $[f]_{s,p,\alpha}$. Note that if $j \le i-2$, then for any $\xi' = (z', t') \in D_j$ and $\xi = (z, t) \in D_i$, we have $2^j \le f(\xi') < 2^{j+1} \le 2^{i-1} < 2^i \le f(\xi) < 2^{i+1}$. Hence, $f(\xi) - f(\xi') \ge 2^{i-1}$. So

$$\begin{aligned} [f]^p_{s,p,\alpha} &\ge \sum_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \int_{D_i\times D_j} \frac{|f(\xi') - f(\xi)|^p d\xi' d\xi}{d(\xi^{-1}\circ\xi')^{Q+ps}|z'-z|^{(p-2)\alpha}} \\ &\ge \sum_{i\in\mathbb{Z}} \sum_{j\le i-2} \int_{D_i}\int_{D_j} \frac{|f(\xi') - f(\xi)|^p d\xi' d\xi}{d(\xi^{-1}\circ\xi')^{Q+ps}|z'-z|^{(p-2)\alpha}} \\ &\ge \sum_{i\in\mathbb{Z}} \sum_{j\le i-2} 2^{p(i-1)} \int_{D_i}\int_{D_j} \frac{d\xi' d\xi}{d(\xi^{-1}\circ\xi')^{Q+ps}|z'-z|^{(p-2)\alpha}} \\ &= \sum_{i\in\mathbb{Z}} 2^{p(i-1)} \int_{D_i}\int_{\cup_{j\le i-2} D_j} \frac{d\xi' d\xi}{d(\xi^{-1}\circ\xi')^{Q+ps}|z'-z|^{(p-2)\alpha}} \\ &= \sum_{i\in\mathbb{Z}} 2^{p(i-1)} \int_{D_i}\int_{A^c_{i-1}} \frac{d\xi' d\xi}{d(\xi^{-1}\circ\xi')^{Q+ps}|z'-z|^{(p-2)\alpha}} \\ (\text{using Lemma 4.1}) &\ge C_{n,s,p,\alpha} \sum_{i\in\mathbb{Z}, a_{i-1}\neq 0} 2^{p(i-1)} a_{i-1}^{-\frac{ps+(p-2)\alpha}{Q}} d_i \qquad (4.10) \\ \left(\text{putting } d_i = a_i - \sum_{j\ge i+1} d_j\right) &= C_{n,s,p,\alpha} \sum_{a_{i-1}\neq 0} 2^{pi} a_i a_{i-1}^{-\frac{ps+(p-2)\alpha}{Q}} \\ &\quad - \sum_{a_{i-1}\neq 0} \sum_{j\ge i+1} 2^{pi} a_{i-1}^{-\frac{ps+(p-2)\alpha}{Q}} d_j. \end{aligned}$$

Hence

$$\begin{aligned}[f]_{s,p,\alpha}^{p} + C_{n,s,p,\alpha} \sum_{a_{i-1}\neq 0} \sum_{j\geq i+1} 2^{pi} a_{i-1}^{-\frac{ps+(p-2)\alpha}{Q}} d_j \\ \geq C_{n,s,p,\alpha} \sum_{a_{i-1}\neq 0} 2^{pi} a_i a_{i-1}^{-\frac{ps+(p-2)\alpha}{Q}}.\end{aligned} \tag{4.11}$$

So, by using inequality (4.8) and (4.10), we get

$$\begin{aligned}\sum_{a_{i-1}\neq 0} \sum_{j\geq i+1} 2^{pi} a_{i-1}^{-\frac{ps+(p-2)\alpha}{Q}} d_j &\leq \frac{1}{2^p - 1} \sum_{i\in\mathbb{Z}, a_{i-1}\neq 0} 2^{pi} a_{i-1}^{-\frac{ps+(p-2)\alpha}{Q}} d_i \\ &\leq C_{n,s,p,\alpha} [f]_{s,p,\alpha}^{p}.\end{aligned}$$

Hence, from (4.11) and the above inequality, we get

$$[f]_{s,p,\alpha}^{p} \geq C_{n,s,p,\alpha} \sum_{i\in\mathbb{Z}, a_{i-1}\neq 0} 2^{pi} a_i a_{i-1}^{-\frac{ps+(p-2)\alpha}{Q}}. \tag{4.12}$$

Finally, combining the above inequality with (4.9), we get the required Sobolev embedding (1.10). □

5. A Hardy type inequality

Before proving the theorem let us prove the following lemma.

Lemma 5.1. *Let $s \in (0, 1)$, $1 \leq p < \infty$ and $\alpha \in \mathbb{R}$ satisfy the following three conditions:*

(a) $(p-2)\alpha \geq 0$;
(b) $ps > 2$;
(c) $ps + (p-2)\alpha < Q$.

Then there exists a positive constant $C_{n,s,p,\alpha}$ depending only on n, s, p, α such that for any measurable set $D \subset \mathbb{R}^{2n+1}$ with $|D| < \infty$ we have

$$\int_D \frac{dz dt}{\left(|z|^4 + t^2\right)^{\frac{ps}{4}} |z|^{(p-2)\alpha}} \leq C_{n,s,p,\alpha} |D|^{1-\frac{ps+(p-2)\alpha}{Q}}. \tag{5.1}$$

Here, $z \in \mathbb{R}^{2n}$, $t \in \mathbb{R}$ and $|D|$ denotes the Lebesgue measure of D.

Proof. Let $|D| = 2w_{2n} R^Q$. Then, if we denote the set $B_{2n}(0, R) \times (-R^2, R^2)$ by B_R, where $B_{2n}(0, R)$ is the ball in $\mathbb{R}^{2n}$ centred at origin with radius R, one can easily notice that $|B_R| = |D|$. Let

$$I := \int_D \frac{dzdt}{\left(|z|^4+t^2\right)^{\frac{ps}{4}} |z|^{(p-2)\alpha}}$$
$$= \int_{D\cap B_R} \frac{dzdt}{\left(|z|^4+t^2\right)^{\frac{ps}{4}} |z|^{(p-2)\alpha}} + \int_{D\cap B_R^{\mathsf{c}}} \frac{dzdt}{\left(|z|^4+t^2\right)^{\frac{ps}{4}} |z|^{(p-2)\alpha}}.$$

Now let us define the following quantities:

$$B_{R,1} := \left\{z \in \mathbb{R}^{2n} : |z| > R\} \times \{t \in \mathbb{R} : |t| > R^2\right\},$$
$$B_{R,2} := \left\{z \in \mathbb{R}^{2n} : |z| > R\} \times \{t \in \mathbb{R} : |t| < R^2\right\},$$
$$B_{R,3} := \left\{z \in \mathbb{R}^{2n} : |z| < R\} \times \{t \in \mathbb{R} : |t| > R^2\right\}$$

and

$$I_1 := \int_{D\cap B_R} \frac{dzdt}{\left(|z|^4+t^2\right)^{\frac{ps}{4}} |z|^{(p-2)\alpha}},$$
$$I_2 := \int_{D\cap B_{R,1}} \frac{dzdt}{\left(|z|^4+t^2\right)^{\frac{ps}{4}} |z|^{(p-2)\alpha}},$$
$$I_3 := \int_{D\cap B_{R,2}} \frac{dzdt}{\left(|z|^4+t^2\right)^{\frac{ps}{4}} |z|^{(p-2)\alpha}},$$
$$I_4 := \int_{D\cap B_{R,3}} \frac{dzdt}{\left(|z|^4+t^2\right)^{\frac{ps}{4}} |z|^{(p-2)\alpha}}.$$

Then $I = I_1 + I_2 + I_3 + I_4$. We will estimate each of I_1, I_2, I_3, I_4 separately to arrive at (5.1).

Estimation of I_1:

$$I_1 \le \int_{B_R} \frac{dzdt}{\left(|z|^4+t^2\right)^{\frac{ps}{4}} |z|^{(p-2)\alpha}}$$
$$= 2\int_{|z|<R} \int_0^{R^2} \frac{dtdz}{\left(|z|^4+t^2\right)^{\frac{ps}{4}} |z|^{(p-2)\alpha}}$$
$$A = 4nw_{2n} \int_0^R \int_0^{R^2} \frac{r^{Q-3-(p-2)\alpha}}{\left(r^4+t^2\right)^{\frac{ps}{4}}} dtdr$$
$$(\text{putting } t = r^2\bar{t}) = \int_0^R \int_0^{\frac{R^2}{r^2}} \frac{r^{Q-1-(p-2)\alpha-ps}}{\left(1+\bar{t}^2\right)^{\frac{ps}{4}}} d\bar{t}dr$$
$$\le 4nw_{2n} \int_0^\infty \frac{d\bar{t}}{\left(1+\bar{t}^2\right)^{\frac{ps}{4}}} \int_0^R r^{Q-1-ps-(p-2)\alpha} dr.$$

Now, since $ps > 2$, $\int_0^\infty \frac{d\bar{t}}{(1+\bar{t}^2)^{\frac{ps}{4}}} < \infty$, we get

$$\begin{aligned} I_1 &\leq C_{n,s,p,\alpha} \frac{R^{Q-ps-(p-2)\alpha}}{Q-ps-(p-2)\alpha} \text{ (Since } ps+(p-2)\alpha < Q) \\ &= \frac{C_{n,s,p,\alpha}}{Q-ps-(p-2)\alpha} |D|^{1-\frac{ps+(p-2)\alpha}{Q}}, \end{aligned} \tag{5.2}$$

where, $C_{n,s,p,\alpha} > 0$ is a constant depending only on n, s, p, α.

Estimation of I_2:

$$\begin{aligned} I_2 &\leq \frac{|B_{R,1} \cap D|}{R^{ps+(p-2)\alpha}} \\ &\leq \frac{|D|}{R^{ps+(p-2)\alpha}} = C_{n,s,p,\alpha} |D|^{1-\frac{ps+(p-2)\alpha}{Q}}, \end{aligned} \tag{5.3}$$

where, $C_{n,s,p,\alpha}$ is a positive constant depending only on n, s, p, α.

Estimation of I_3: Let $D_1 := B_{R,2} \cap D$. Choose $R_1 > 0$ so that $|D_1| = 2nw_{2n} R_1^Q$. Define $B(0, R_1) := \{(z,t) \in \mathbb{R}^{2n} \times \mathbb{R} : \left(|z|^4 + t^2\right)^{\frac{1}{4}} < R_1\}$. Now, for $(z,t) \in D_1$ we have

$$|z|^4 + t^2 \leq |z|^4 + R^4 \leq 2|z|^4.$$

Since, $(p-2)\alpha \geq 0$ we have

$$\frac{1}{|z|^{(p-2)\alpha}} \leq \frac{2^{\frac{(p-2)\alpha}{4}}}{\left(|z|^4 + t^2\right)^{\frac{(p-2)\alpha}{4}}}.$$

So

$$\begin{aligned} \frac{I_3}{2^{\frac{(p-2)\alpha}{4}}} \leq &\int_{D_1} \frac{dz dt}{\left(|z|^4 + t^2\right)^{\frac{ps+(p-2)\alpha}{4}}} \\ = &\int_{D_1 \cap B(0,R_1)} \frac{dz dt}{\left(|z|^4 + t^2\right)^{\frac{ps+(p-2)\alpha}{4}}} + \int_{D_1 \cap B^{\mathsf{c}}(0,R_1)} \frac{dz dt}{\left(|z|^4 + t^2\right)^{\frac{ps+(p-2)\alpha}{4}}} \\ \leq &\int_{D_1 \cap B(0,R_1)} \frac{dz dt}{\left(|z|^4 + t^2\right)^{\frac{ps+(p-2)\alpha}{4}}} + \frac{|D_1 \cap B^{\mathsf{c}}(0, R_1)|}{R_1^{ps+(p-2)\alpha}} \\ = &\int_{D_1 \cap B(0,R_1)} \frac{dz dt}{\left(|z|^4 + t^2\right)^{\frac{ps+(p-2)\alpha}{4}}} + \int_{D_1^{\mathsf{c}} \cap B(0,R_1)} \frac{1}{R_1^{ps+(p-2)\alpha}} dz dt \end{aligned}$$

since $|D_1^{\mathsf{c}} \cap B(0, R_1)| = |D_1 \cap B^{\mathsf{c}}(0, R_1)|$.

Now, for $(z,t) \in B(0,R_1)$, we have $\left(|z|^4+t^2\right)^{\frac{ps+(p-2)\alpha}{4}} \le R_1^{ps+(p-2)\alpha}$. Hence

$$\begin{aligned}\frac{I_3}{2^{\frac{(p-2)\alpha}{4}}} \le & \int_{D_1\cap B(0,R_1)} \frac{dzdt}{\left(|z|^4+t^2\right)^{\frac{ps+(p-2)\alpha}{4}}} + \int_{D_1^c\cap B(0,R_1)} \frac{dzdt}{\left(|z|^4+t^2\right)^{\frac{ps+(p-2)\alpha}{4}}} \\ = & \int_{B(0,R_1)} \frac{dzdt}{\left(|z|^4+t^2\right)^{\frac{ps+(p-2)\alpha}{4}}}.\end{aligned}$$

Note that $B(0,R_1) \subset \{z \in \mathbb{R}^{2n} : |z| < R_1\} \times \{t \in \mathbb{R} : |t| < R_1^2\}$ so

$$\begin{aligned}\frac{I_3}{2^{\frac{(p-2)\alpha}{4}}} &\le \int_{|z|<R_1}\int_{|z|<R_1^2} \frac{dtdz}{\left(|z|^4+t^2\right)^{\frac{ps+(p-2)\alpha}{4}}} \\ &= 4nw_{2n}\int_0^{R_1}\int_0^{\frac{R_1^2}{r^2}} \frac{r^{Q-1-ps-(p-2)\alpha}}{\left(1+t^2\right)^{\frac{ps+(p-2)\alpha}{4}}}dtdr.\end{aligned}$$

Now $ps > 2$ implies that $ps+(p-2)\alpha > 2$, which makes $\int_0^\infty \frac{dt}{(1+t^2)^{\frac{ps+(p-2)\alpha}{4}}} < \infty$. Hence

$$\begin{aligned} I_3 &\le C_{n,s,p,\alpha}\int_0^{R_1} r^{Q-1-ps-(p-2)\alpha}dr \\ &= C_{n,s,p,\alpha}\frac{|D_1|^{1-\frac{ps+(p-2)\alpha}{Q}}}{Q-ps-(p-2)\alpha} \\ &\le C_{n,s,p,\alpha}\frac{|D|^{1-\frac{ps+(p-2)\alpha}{Q}}}{Q-ps-(p-2)\alpha} \quad \text{(Since } |D_1| \le |D|\text{)},\end{aligned} \tag{5.4}$$

where, $C_{n,s,p,\alpha} > 0$ is a constant depending only on n, s, p, α.

Estimation of I_4: Clearly,

$$I_4 \le \int_{|z|<R}\int_{|t|>R^2} \frac{dtdz}{t^{\frac{ps}{2}}|z|^{(p-2)\alpha}} = 4nw_{2n}\int_0^R r^{Q-3-(p-2)\alpha}dr\int_{R^2}^\infty \frac{dt}{t^{\frac{ps}{2}}}.$$

Now the conditions $ps > 2$ and $ps+(p-2)\alpha < Q$ together imply $(p-2)\alpha < Q-2$. Hence

$$\begin{aligned} I_4 &\le C_{n,s,p,\alpha}\frac{R^{Q-2-(p-2)\alpha}}{(Q-2-(p-2)\alpha)\,(ps-2)R^{ps-2}} \\ &= \frac{C_{n,s,p,\alpha}R^{Q-ps-(p-2)\alpha}}{(Q-2-(p-2)\alpha)\,(ps-2)} = C_{n,s,p,\alpha}|D|^{1-\frac{ps+(p-2)\alpha}{Q}},\end{aligned} \tag{5.5}$$

where, $C_{n,s,p,\alpha}$ is a positive constant depending only on n, s, p, α. Now using the relation $I = I_1+I_2+I_3+I_4$ and the inequalities (5.2), (5.3), (5.4) and (5.5) we get the required inequality (5.1). This proves the lemma. □

Remark 5.2.

(i) The condition $ps > 2$ forces us to choose $p > 2$. However, in the case of $p = 2$, the approach of estimating I_3, in the above Lemma could be followed to deduce the inequality (5.1);
(ii) In general, the inequality (5.1) is not true. For example if $0 < (p-2)\alpha < Q-2$ and $ps < \frac{2(p-2)\alpha}{Q-2}$, then one can choose $D = B_{2n}(0, R) \times (-1, 1)$ and make $R \to 0$ to show that (5.1) is false. Note that in this case $\frac{2(p-2)\alpha}{Q-2} < 2$ and so $ps < 2$.

5.1. Proof of Theorem 1.2

Proof. Note that it is enough to prove the theorem for every nonnegative compactly supported smooth function f. Let the quantities A_k, D_k, a_k, d_k be the same as defined in the proof of Theorem 1.1. Let

$$\begin{aligned} J :&= \int_{\mathbb{H}^n} \frac{|f(\xi)|^p}{d(\xi)^{ps}|z|^{(p-2)\alpha}} d\xi \\ &= \sum_{i\in\mathbb{Z}} \int_{D_i} \frac{f^p}{\left(|z|^4 + t^2\right)^{\frac{ps}{4}} |z|^{(p-2)\alpha}} dz dt \\ &\leq \sum_{i\in\mathbb{Z}} 2^{(i+1)p} \int_{D_i} \frac{dz dt}{\left(|z|^4 + t^2\right)^{\frac{ps}{4}} |z|^{(p-2)\alpha}}. \end{aligned}$$

Now we use Lemma 5.1 to obtain

$$\begin{aligned} J &\leq C_{n,s,p,\alpha} \sum_{i\in\mathbb{Z}} 2^{(i+1)p} d_i^{1-\frac{ps+(p-2)\alpha}{Q}} \\ &\leq C_{n,s,p,\alpha} \sum_{i\in\mathbb{Z}} 2^{ip} a_i^{1-\frac{ps+(p-2)\alpha}{Q}} \quad (\text{since } d_i \leq a_i). \end{aligned}$$

Using inequality (4.7) with $T = 2^p$ and combining it with (4.12) we get the required inequality (1.2). This proves the theorem. □

6. Compactness of Sobolev type embedding

In analogy with the definition of $W_0^{s,p,\alpha}(\mathbb{H}^n)$ one can define $W_0^{s,p,\alpha}(\Omega)$, where $\Omega \subset \mathbb{H}^n$ is any open set. More precisely, we define

$$W_0^{s,p,\alpha}(\Omega) = \mathrm{Cl}\left\{u \in C_c^\infty(\Omega) : \int_{\Omega\times\Omega} \frac{|u(\xi') - u(\xi)|^p}{d(\xi^{-1}.\xi')^{Q+ps}|z' - z|^{(p-2)\alpha}} d\xi' d\xi < \infty\right\}.$$

Here the closure is taken under the norm $||.||_{s,p,\alpha,\Omega} := ||.||_{L^p(\Omega)} + [.]_{s,p,\alpha,\Omega}$, where for any $u \in C_c^\infty(\Omega)$,

$$[u]_{s,p,\alpha,\Omega} := \int_{\Omega\times\Omega} \frac{|u(\xi) - u(\xi')|^p}{d(\xi^{-1}.\xi')^{Q+ps}|z' - z|^{(p-2)\alpha}} d\xi' d\xi.$$

Clearly, for $(p-2)\alpha < \min\{Q-2, p(1-s)\}$, $W_0^{s,p,\alpha}(\Omega)$ is non trivial. Let us recall the following definition.

Definition 6.1. Let Ω be any domain in $\mathbb{H}^n$. We say that Ω is an extension domain if it satisfies the following property. For any $f \in W_0^{s,p,\alpha}(\Omega)$ there exists $\bar{f} \in W_0^{s,p,\alpha}(\mathbb{H}^n)$ such that $\bar{f}|_\Omega = f$ and satisfies the following inequality

$$\|\bar{f}\|_{s,p,\alpha} \le C_{n,s,p,\alpha}(\Omega)\|f\|_{s,p,\alpha,\Omega},$$

where $C_{n,s,p,\alpha}(\Omega) > 0$ is a constant depending only on n, s, p, α and Ω.

Now if Ω is an extension domain and $s \in (0,1)$, $1 \le p < \infty$, $\alpha \in \mathbb{R}$ satisfy the following conditions:

(a) $(p-2)\alpha \ge 0$;
(b) $ps + (p-2)\alpha < Q$;

then from Theorem 1.1 it follows that there exists a constant $C_{n,s,p,\alpha} > 0$ depending only on n, s, p, α such that for any $u \in W_0^{s,p,\alpha}(\Omega)$

$$||u||_{L^q(\Omega)} \le C_{n,s,p,\alpha}\|u\|_{s,p,\alpha,\Omega}. \tag{6.1}$$

Here $q = \frac{Qp}{Q-ps-(p-2)\alpha}$. The above inequality shows that $W_0^{s,p,\alpha}(\Omega)$ is continuously embedded in $L^r(\Omega)$ for any $1 \le r < q$, if Ω is a bounded extension domain. Because of (6.1) it is relevant to prove Theorem 1.4.

6.1. Proof of Theorem 1.4

Proof.

Step 1: In this step we will show $\mathscr{F}$ is relatively compact in $L^r(\Omega)$, if $1 \le r \le \frac{2p}{3}$. Note that, it is enough to show $\mathscr{F}$ is totally bounded in $L^r(\Omega)$. Since Ω is an extension domain, there exists an extension of $u \in W_0^{s,p,\alpha}(\Omega)$, say $U \in W_0^{s,p,\alpha}(\mathbb{H}^n)$, such that $||U||_{s,p,\alpha} \le C\,||u||_{s,p,\alpha,\Omega}$. Here $C > 0$ is a constant independent of u. Let $I \subset \mathbb{R}^{2n} \times \mathbb{R}$ be a rectangle such that its projection on $\mathbb{R}^{2n}$ is a cube of side R and on $\mathbb{R}$ is an interval of length R^2. Here $R > 0$ is chosen such that $\Omega \subset I$.

$$||U||_{s,p,\alpha,I} \le ||U||_{s,p,\alpha} \le C\,||u||_{s,p,\alpha,\Omega}.$$

Also, since I is bounded, $U \in L^r(\Omega)$ for any $r \in [1, p]$. Let

$$C_0 = 1 + \sup_{u\in\mathscr{F}} ||U||_{L^r(I)} + \sup_{u\in\mathscr{F}} [U]_{s,p,\alpha,I}.$$

Now consider a collection of disjoint rectangles $I_1, \dots, I_M \subset I$ such that the projection of every rectangle on $\mathbb{R}^{2n}$ is cube of side μ and on $\mathbb{R}$ is an interval of length μ^2 and $I = \cup_{j=1}^{N} I_j$ upto a set of measure zero. We will choose μ later. For any $\xi' \in \Omega$ (excluding probably a measure zero set) we define $j(\xi')$ as the unique integer in $\{1, \dots, M\}$ for which $\xi' \in I_{j(\xi')}$ and for any $u \in \mathscr{F}$ let

$$P(u)(\xi') := \frac{1}{|I_{j(\xi')}|} \int_{I_{j(\xi')}} U(\xi) d\xi.$$

Note that P is additive and constant, say $q_j(u)$ on any I_j, for $j \in \{1, \dots, M\}$. Therefore we can define

$$R(u) := \mu^{\frac{Q}{r}} (q_1(u), \dots, q_M(u)) \in \mathbb{R}^M.$$

Consider the spatial r-norm in $\mathbb{R}^M$ as

$$||v||_r := \left(\sum_{j=1}^{M} |v_j|^r \right)^{\frac{1}{r}}, \text{ for any } v \in \mathbb{R}^M.$$

Observe that R is also additive. Moreover,

$$\begin{aligned} ||P(u)||_{L^r(\Omega)}^r &= \sum_{j=1}^{M} \int_{I_j \cap \Omega} |P(u)(\xi')|^r d\xi' \\ &\leq \sum_{j=1}^{M} |q_j(u)|^r = \frac{||R(u)||_r^r}{\mu^Q}. \end{aligned} \tag{6.2}$$

Now by Jensen's inequality

$$\begin{aligned} ||R(u)||_r^r = \mu^Q \sum_{j=1}^{M} |q_j(u)| &= \mu^Q \sum_{j=1}^{M} \left| \frac{1}{|I_j|} \int_{I_j} U(\xi) d\xi \right|^r \\ &\leq \sum_{j=1}^{M} \int_{I_j} |U(\xi)|^r d\xi = ||U||_{L^r(I)}. \end{aligned}$$

Hence,

$$\sup_{u \in \mathscr{F}} ||R(u)||_r^r \leq C_0.$$

So the set $R(\mathscr{F})$ is bounded in $\mathbb{R}^M$ and as $\mathbb{R}^M$ is finite dimensional, $R(\mathscr{F})$ is totally bounded. Therefore, for any $\eta > 0$ there exists $b_1, \dots, b_N \in \mathbb{R}^M$ such that for any $v \in R(\mathscr{F})$

$$||v - b_i||_r \leq \eta, \text{ for some } i \in \{1, \dots, N\}. \tag{6.3}$$

For any $i \in \{1, \dots, N\}$, let us write the coordinate of b_i as

$$b_i = (b_{i,1}, \dots, b_{i,M}) \in \mathbb{R}^M.$$

For any $\xi \in \Omega$ we define

$$\beta_i(\xi) := \mu^{-\frac{Q}{r}} \sum_{j=1}^{N} b_{i,j} \chi_j(\xi),$$

where χ_j denotes the characteristic function of I_j.

Note that if $\xi \in I_j$ then

$$P(\beta_i)(\xi) = \beta_i(\xi), \tag{6.4}$$

so $q_j(\beta_i) = \mu^{-\frac{Q}{r}} b_{i,j}$ and this implies

$$R(\beta_i) = b_i. \tag{6.5}$$

For any $u \in \mathscr{F}$

$$\begin{aligned}
||u - P(u)||^r_{L^r(\Omega)} &= \sum_{j=1}^{M} \int_{\Omega \cap I_j} \left| u(\xi') - P(u)(\xi') \right|^r d\xi' \\
&= \sum_{j=1}^{M} \int_{\Omega \cap I_j} \left| u(\xi') - \frac{1}{|I_j|} \int_{I_j} U(\xi) d\xi \right|^r d\xi' \\
&= \sum_{j=1}^{M} \int_{\Omega \cap I_j} \left| \frac{1}{|I_j|} \int_{I_j} u(\xi') - U(\xi) d\xi \right|^r d\xi' \qquad (6.6)\\
&\leq \sum_{j=1}^{M} \frac{1}{|I_j|^r} \int_{I_j \cap \Omega} \left[\int_{I_j} |u(\xi') - U(\xi)| d\xi \right]^r d\xi' \\
&= \frac{1}{\mu^{Qr}} \sum_{j=1}^{M} \left[\int_{I_j \cap \Omega} \int_{I_j} |u(\xi') - U(\xi)| d\xi \right]^r d\xi'.
\end{aligned}$$

Since $r < p$, using Hölder's inequality with exponents p and $\frac{p}{p-1}$ we get

$$\begin{aligned}
J :&= \frac{1}{\mu^{Qr}} \left[\int_{I_j} \left| u(\xi') - U(\xi) \right| d\xi \right]^r \\
&\leq \frac{|I_j|^{\frac{(p-1)r}{p}}}{\mu^{Qr}} \left(\int_{I_j} \left| u(\xi') - U(\xi) \right|^p d\xi \right)^{\frac{r}{p}} \qquad (6.7)\\
&= \frac{1}{\mu^{\frac{Qr}{p}}} \left[\int_{I_j} \left| u(\xi') - U(\xi) \right|^p d\xi \right]^{\frac{r}{p}}.
\end{aligned}$$

Now for $\xi' = (z', t') = (x', y', t')$ and $\xi = (z, t) = (x, y, t) \in I_j \subset I$

$$\begin{aligned} d(\xi^{-1} \circ \xi') &= \left(|z' - z|^4 + \left(t' - t + 2\left(< x, y' > - < y, x' >\right)\right)^2 \right)^{\frac{1}{4}} \\ &\leq \left(4n^2\mu^4 + 2(t' - t)^2 + 4\left(< x, y' > - < y, x' >\right)^2 \right)^{\frac{1}{4}} \\ &\leq \left((4n^2 + 2)\mu^4 + 4\left(\left\langle (y', -x'), z - z' \right\rangle\right)^2 \right)^{\frac{1}{4}} \leq C_{n,\Omega}\mu^{\frac{1}{2}}, \end{aligned}$$

where $C_{n,\Omega} > 0$ is a constant depending only on n, Ω. So from (6.7) we have

$$\begin{aligned} J &\leq C_{n,s,p,\alpha,\Omega} \frac{\mu^{\frac{r}{p}\left(\frac{Q+ps}{2} + (p-2)\alpha\right)}}{\mu^{\frac{Qr}{p}}} \left[\int_{I_j} \frac{|U(\xi') - U(\xi)|^p d\xi}{d(\xi^{-1} \circ \xi')^{Q+ps} |z' - z|^{(p-2)\alpha}} \right]^{\frac{r}{p}} \\ &= C_{n,s,p,\alpha,\Omega} \mu^{\frac{r}{p}\left(\frac{2(p-2)+ps-Q}{2}\right)} \left[\int_{I_j} \frac{|U(\xi') - U(\xi)|^p d\xi}{d(\xi^{-1} \circ \xi')^{Q+ps} |z' - z|^{(p-2)\alpha}} \right]^{\frac{r}{p}}, \end{aligned} \tag{6.8}$$

where $C_{n,s,p,\alpha,\Omega} > 0$ is a constant depending only on n, s, p, α, Ω. Hence, from (6.6) and (6.8) we get

$$\begin{aligned} &||u - P(u)||^r_{L^r(\Omega)} \\ &\leq C_{n,s,p,\alpha,\Omega} \mu^{\frac{r}{p}\left(\frac{2(p-2)\alpha+ps-Q}{2}\right)} \sum_{j=1}^{M} \int_{I_j} \left[\int_{I_j} \frac{|U(\xi') - U(\xi)|^p d\xi}{d(\xi^{-1} \circ \xi')^{Q+ps} |z' - z|^{(p-2)\alpha}} \right]^{\frac{r}{p}} d\xi' \\ &\leq C_{n,s,p,\alpha,\Omega} \mu^{\frac{r}{p}\left(\frac{2(p-2)\alpha+ps-Q}{2}\right)} \sum_{j=1}^{M} |I_j|^{1-\frac{r}{p}} \left([U]_{s,p,\alpha,I_j}\right)^{\frac{r}{p}} \text{ (by Jensen's inequality)} \\ &\leq C_{n,s,p,\alpha,\Omega} \mu^{\frac{r}{p}\left(\frac{2(p-2)\alpha+ps}{2}\right) + Q\left(1 - \frac{3r}{2p}\right)} [U]_{s,p,\alpha,I} \\ &\leq C_0 C_{n,s,p,\alpha,\Omega} \mu^{\frac{r}{p}\left(\frac{2(p-2)\alpha+ps}{2}\right) + Q\left(1 - \frac{3r}{2p}\right)}. \end{aligned}$$

Now since $r \leq \frac{2p}{3}$, for $\epsilon > 0$ small enough we can choose μ such that

$$C_0 C_{n,s,p,\alpha,\Omega} \mu^{\frac{r}{p}\left(\frac{2(p-2)\alpha+ps}{2}\right) + Q\left(1 - \frac{3r}{2p}\right)} = \frac{\epsilon^r}{2^r}.$$

As a consequence of this we have

$$||u - P(u)||_{L^r(\Omega)} \leq \frac{\epsilon}{2}. \tag{6.9}$$

Recalling the definition of P and R we have, for any $j \in \{1, \ldots, N\}$,

$$\begin{aligned} \left|\left|u - \beta_j\right|\right|_{L^r(\Omega)} &\leq ||u - P(u)||_{L^r(\Omega)} + \left|\left|P(\beta_j) - \beta_j\right|\right|_{L^r(\Omega)} \\ &+ \left|\left|P(u - \beta_j)\right|\right|_{L^r(\Omega)} \leq \frac{\epsilon}{2} + \frac{\left|\left|R(u) - R(\beta_j)\right|\right|_r}{\mu^{\frac{Q}{r}}}, \end{aligned} \tag{6.10}$$

where the last inequality is a result of the combination of (6.2) and (6.9). Now given any $u \in \mathscr{F}$, (6.3) suggests that we can choose $j \in \{1, \dots, N\}$ such that $\left|\left|R(u) - b_j\right|\right|_r < \eta$. Then from (6.5) and (6.10) we obtain

$$\left|\left|u - \beta_j\right|\right|_{L^r(\Omega)} \leq \frac{\epsilon}{2} + \frac{\eta}{\mu^{\frac{Q}{r}}}.$$

Note that, we can choose η so that $\frac{\eta}{\mu^{\frac{Q}{r}}} < \frac{\epsilon}{2}$. Hence for $\epsilon > 0$ small enough, we found $\beta_1, \dots, \beta_N \in L^r(\Omega)$ such that for any $u \in \mathscr{F}$ there exists $j \in \{1, \dots, N\}$ satisfying

$$\left|\left|u - \beta_j\right|\right|_{L^r(\Omega)} \leq \epsilon. \tag{6.11}$$

This proves Step 1.

Step 2: In this step we will show that $\mathscr{F}$ is totally bounded in $L^r(\Omega)$ for any $\frac{2p}{3} < r < q = \frac{Qp}{Q-ps-(p-2)\alpha}$ and this will complete our theorem. For $\epsilon > 0$ small enough take the same set of β_j as in the Step 1. Choose $\theta \in (0, 1)$ so that $\frac{1}{r} = \frac{3\theta}{2p} + \frac{(1-\theta)}{q}$. Then using Hölder's inequality with the exponents $\frac{2p}{3\theta r}$ and $\frac{q}{(1-\theta)r}$ we get

$$\begin{aligned}
\left|\left|u - \beta_j\right|\right|_{L^r(\Omega)} &\leq \left(\int_\Omega |u - \beta_j|^{\frac{2p}{3}} d\xi\right)^{\frac{3\theta}{2p}} \left(\int_\Omega |u - \beta_j|^q d\xi\right)^{\frac{(1-\theta)}{q}} \\
&\leq C \left|\left|u - \beta_j\right|\right|_{s,p,\alpha,\Omega}^{(1-\theta)} \left|\left|u - \beta_j\right|\right|_{L^{\frac{2p}{3}}(\Omega)}^{\theta} \quad \text{(by Theorem (1.1))} \\
&\leq C\epsilon^\theta \text{ (by (6.11))}.
\end{aligned}$$

This proves the theorem. □

7. A Morrey type inequality

In this section we prove Theorem 1.5. To do this we need the following lemma which is basically an adaptation of [20, Lemma 2.2] in our set up.

Lemma 7.1. *Let $p \in [1, \infty)$, $s \in (0, 1)$ and $\alpha \in \mathbb{R}$ satisfy $ps + (p-2)\alpha > Q$. Also, suppose that u is a real valued measurable function on $\mathbb{H}^n$ with*

$$[u]_{p,ps+(p-2)\alpha} := \left(\sup_{\xi_0 \in \mathbb{H}^n, \rho > 0} \frac{\int_{B(\xi_0,\rho)} |u - u_{\xi_0,\rho}|^p d\xi}{\rho^{ps+(p-2)\alpha}}\right)^{\frac{1}{p}} < \infty,$$

where $B(\xi_0, \rho) := \{\xi' : d(\xi_0^{-1} \circ \xi') < \rho\}$ is a ball in $\mathbb{H}^n$ with centre ξ_0 and radius ρ and $u_{\xi_0,\rho}$ denotes the average of u over $B(\xi_0, \rho)$. Then there exists a constant

$C(n) > 0$ depending only on n such that for any $\xi_0 \in \mathbb{H}^n$ and $0 < R' < R < \infty$ we have

$$\left|u_{\xi_0,R} - u_{\xi_0,R'}\right| \leq C(n)[u]_{p,ps+(p-2)\alpha}|B(\xi_0, R)|^{\frac{ps+(p-2)\alpha-Q}{Qp}}. \tag{7.1}$$

Proof. Let $0 < r < t < R$ and $\lambda := ps + (p-2)\alpha$. Then

$$\begin{aligned} |u_{\xi_0,t} - u_{\xi_0,r}| &\leq \frac{1}{|B(\xi_0, r)|}\int_{B(\xi_0,r)} |u(\xi) - u_{\xi_0,t}|d\xi \\ \text{(by Hölder's inequality)} &\leq \frac{1}{|B(\xi_0, r)|^{\frac{1}{p}}}\left(\int_{B(\xi_0,r)} |u(\xi) - u_{\xi_0,t}|^p d\xi\right)^{\frac{1}{p}} \\ &\leq C(n)\left(\frac{t}{r}\right)^{\frac{\lambda}{p}} r^{\frac{\lambda-Q}{p}}[u]_{p,\lambda} \\ &= C(n)\left(\frac{t}{r}\right)^{\frac{\lambda}{p}}[u]_{p,\lambda}|B(\xi_0, r)|^{\frac{\lambda-Q}{Qp}}. \end{aligned} \tag{7.2}$$

Putting $r = r_i = R2^{-i}$ and $s = r_{i-1}$ in the above inequality and summing over i from 0 to k (k to be chosen later) we obtain

$$|u_{\xi_0,R} - u_{\xi_0,r_k}| \leq C(n)[u]_{p,\lambda}. \tag{7.3}$$

Now choose k such that

$$r_k \leq R' < r_{k-1}.$$

Then we have from (7.2)

$$\begin{aligned} |u_{\xi_0,R'} - u_{\xi_0,r_k}| &\leq C(n)\left(\frac{R'}{r_k}\right)^{\frac{\lambda}{p}}[u]_{p,\lambda}|B(\xi_0, r_k)|^{\frac{\lambda-Q}{Qp}} \\ &\leq C(n)[u]_{p,\lambda}|B(\xi_0, R)|^{\frac{\lambda-Q}{Qp}}. \end{aligned} \tag{7.4}$$

Combining (7.3) and (7.4) we get (7.1). □

7.1. Proof of Theorem 1.5

Proof. Let $u \in L^p(\mathbb{H}^n)$. First of all note that if $[u]_{s,p,\alpha} = \infty$ then there is nothing to prove. So let $[u]_{s,p,\alpha} < \infty$. Then we claim that $[u]_{s,ps+(p-2)\alpha} \leq C(n, s, p, \alpha)[u]^p_{s,p,\alpha}$, where $C(n, s, p, \alpha) > 0$ is a constant depending only on n, s, p, α. To prove this consider $r > 0$ and $\xi_0 \in \mathbb{H}^n$ be any element. Then by Jensen's inequality we have

$$\int_{B(\xi_0,r)} |u(\xi) - u_{\xi_0,r}|^p d\xi \leq \frac{1}{|B(\xi_0, r)|}\int_{B(\xi_0,r)}\int_{B(\xi_0,r)} |u(\xi) - u(\xi')|^p d\xi d\xi'.$$

Now for $\xi, \xi' \in B(\xi_0, r)$ we have $d(\xi^{-1} \circ \xi') \leq 2r$ and $|z - z'| \leq 2r$. So from above we have

$$\begin{aligned}
&\int_{B(\xi_0,r)} |u(\xi) - u_{\xi_0,r}|^p d\xi \\
&\leq \frac{(2r)^{Q+ps+(p-2)\alpha}}{|B(\xi_0, r)|} \int_{B(\xi_0,r)} \int_{B(\xi_0,r)} \frac{|u(\xi) - u(\xi')|^p d\xi d\xi'}{d(\xi^{-1} \circ \xi')^{Q+ps} |z - z'|^{(p-2)\alpha}} \qquad (7.5)\\
&\leq C(n, s, p, \alpha) r^{ps+(p-2)\alpha} [u]^p_{s,p,\alpha}.
\end{aligned}$$

Hence

$$[u]_{p,ps+(p-2)\alpha} \leq C(n, s, p, \alpha)[u]_{s,p,\alpha} < \infty. \qquad (7.6)$$

We will use (7.6) and Lemma 7.1 to prove (1.12). Inequality (7.1) suggests that $\lim_{R\to 0} u_{\xi,R}$ exists uniformly in $\xi \in \mathbb{H}^n$ and by Lebesgue differentiation theorem, since $\mathbb{H}^n$ is a homogeneous space, so here Lebesgue differentiation theorem holds true; see [26] for details.

$$\lim_{R\to 0} u_{\xi,R} = u(\xi) \text{ almost everywhere.}$$

Since $u_{\xi,R}$ is continuous in ξ so is u. Now making $R' \to 0$ in (7.1) and taking $R = 2R$ we obtain

$$|u_{\xi,R} - u(\xi)| \leq C(n)[u]_{s,ps+(p-2)\alpha} R^{\frac{ps+(p-2)\alpha-Q}{p}}. \qquad (7.7)$$

For any $\xi, \xi' \in \mathbb{H}^n$ let $R = d(\xi^{-1} \circ \xi')$. Then

$$\left|u(\xi) - u(\xi')\right| \leq |u(\xi) - u_{\xi,2R}| + \left|u(\xi') - u_{\xi',2R}\right| + |u_{\xi,2R} - u_{\xi',2R}|. \qquad (7.8)$$

Now for any $\tilde{\xi} \in B(\xi, 2R) \cap B(\xi', 2R)$

$$|u_{\xi,2R} - u_{\xi',2R}| \leq \left|u(\tilde{\xi}) - u_{\xi,2R}\right| + \left|u(\tilde{\xi}) - u_{\xi',2R}\right|.$$

Noticing $B(\xi, R) \cup B(\xi', R) \subset \big(B(\xi, 2R) \cap B(\xi', 2R)\big)$ and integrating over $\tilde{\xi}$ we get

$$\begin{aligned}
|u_{\xi,2R} - u_{\xi',2R}| \leq &\frac{1}{|B(\xi, R)|} \int_{B(\xi,2R)} \left|u(\tilde{\xi}) - u_{\xi,2R}\right| d\tilde{\xi} \\
&+ \frac{1}{B(\xi', R)} \int_{B(\xi',2R)} \left|u(\tilde{\xi}) - u_{\xi',2R}\right| d\tilde{\xi}.
\end{aligned} \qquad (7.9)$$

By Hölder's inequality we have

$$\begin{aligned}
&\frac{1}{|B(\xi, R)|} \int_{B(\xi,2R)} \left|u(\tilde{\xi}) - u_{\xi,2R}\right| d\tilde{\xi} \\
&\leq \frac{|B(\xi, 2R)|^{\frac{p-1}{p}}}{|B(\xi, R)|} \left(\int_{|B(\xi,2R)|} |u(\tilde{\xi}) - u_{\xi,2R}|^p d\tilde{\xi}\right)^{\frac{1}{p}} \qquad (7.10)\\
&\leq C(n, s, p, \alpha)[u]_{s,ps+(p-2)\alpha} R^{\frac{ps+(p-2)\alpha-Q}{p}}.
\end{aligned}$$

Similarly,

$$\frac{1}{|B(\xi', R)|}\int_{|B(\xi',2R)|}\left|u(\tilde{\xi}) - u_{\xi',2R}\right| d\tilde{\xi} \le C(n, s, p, \alpha) R^{\frac{ps+(p-2)\alpha-Q}{p}}. \tag{7.11}$$

Using (7.7), (7.9), (7.10), and (7.11) we have from (7.8)

$$\left|u(\xi) - u(\xi')\right| \le C[u]_{s,ps+(p-2)\alpha} d\left(\xi^{-1}\circ\xi'\right)^{\frac{ps+(p-2)\alpha-Q}{p}}.$$

So using (7.6) we have

$$\sup_{\xi\neq\xi'}\frac{|u(\xi)-u(\xi')|}{d(\xi^{-1}\circ\xi')^{\beta}} \le C(n, s, p, \alpha)[u]_{s,p,\alpha}. \tag{7.12}$$

Now taking $R_0 > 0$ and using (7.6), (7.7) and Hölder's inequality we have for any $\xi \in \mathbb{H}^n$

$$|u(\xi)| \le C(n, s, p, \alpha)\left(\frac{1}{|B(\xi, R_0)|^{\frac{1}{p}}}||u||_{L^p(\mathbb{H}^n)} + [u]_{s,p,\alpha}|B(\xi, R_0)|^{\beta}\right). \tag{7.13}$$

Hence combining (7.12) and (7.13) we have (1.12). □

Appendix

A. Proof of Theorem 2.1

Proof. We will prove the theorem for $n = 1$. The same proof will work for higher values of n. Note that, it is enough to find a decomposition of type mentioned in the theorem for the elements $(0, 0, t) = \mathrm{Exp}(tT) \in \mathbb{H}^1$. Without loss generality, we can take $t > 0$. Clearly, $[X_1, X_2] = -4T$. Easy to see that

$$\mathrm{Exp}\left(X_1 + X_2 + \frac{1}{2}[X_1, X_2]\right) = \mathrm{Exp}(X_1)\circ\mathrm{Exp}(X_2). \tag{A.1}$$

Consequently,

$$\mathrm{Exp}\left([X_1, X_2]\right) = \mathrm{Exp}(X_1)\circ\mathrm{Exp}(X_2)\circ\mathrm{Exp}(-X_1)\circ\mathrm{Exp}(-X_2). \tag{A.2}$$

This implies

$$\mathrm{Exp}(tT) = \mathrm{Exp}\left(\frac{\sqrt{t}}{2}X_2\right)\circ\mathrm{Exp}\left(\frac{\sqrt{t}}{2}X_1\right)\circ\mathrm{Exp}\left(-\frac{\sqrt{t}}{2}X_2\right)\circ\mathrm{Exp}\left(-\frac{\sqrt{t}}{2}X_1\right).$$

Note that

$$d\left(\mathrm{Exp}\left(\pm\frac{\sqrt{t}}{2}X_i\right)\right) = \frac{\sqrt{t}}{2} \le \sqrt{t} = d\left(\mathrm{Exp}\left(tT\right)\right), \text{ for any } i = 1, 2.$$

This proves the theorem. □

References

[1] A. R. ADAMS, *Sobolev spaces*, In: "Pure and Applied Mathematics", Vol. 65, Academic Press (A subsidiary of Harcourt Brace Jovanovich, Publishers), New York-London, 1975.

[2] ADIMURTHI, N. CHAUDHURI and M. RAMASWAMY, *An improved Hardy-Sobolev inequality and its application*, Proc. Amer. Math. Soc. **130** (2002), 489–505.

[3] ADIMURTHI and A. SEKAR, *Role of the fundamental solution in Hardy-Sobolev-type inequalities*, Proc. Roy. Soc. Edinburgh Sect. A **136** (2006), 1111–1130.

[4] P. BARAS and J. A. GOLDSTEIN, *The heat equation with a singular potential*, Trans. Amer. Math. Soc. **284** (1984), 121–139.

[5] G. BARBATIS, S. FILIPPAS and A. TERTIKAS, *Series expansion for L^p Hardy inequalities*, Indiana Univ. Math. J. **52** (2003), 171–190.

[6] A. BONFIGLIOLI, E. LANCONELLI and F. UGUZZONI, "Stratified Lie Groups and Potential Theory for their Sub-Laplacians", Springer Monographs in Mathematics, Springer, Berlin, 2007.

[7] J. BOURGAIN, H. BREZIS and P. MIRONESCU, *Another look at Sobolev spaces*, In: "Optimal Control and Partial Differential Equations", IOS Press, 2001, 217–445. A volume in honor of A. Bensoussan's 60th birthday.

[8] J. BOURGAIN, H. BREZIS and P. MIRONESCU, *Limiting embedding theorems for $W^{s,p}$ when $s \uparrow 1$ and applications*, J. Anal. Math. **87** (2002), 77–101. Dedicated to the memory of Thomas H. Wolff.

[9] H. BREZIS and P. MIRONESCU, *Gagliardo-Nirenberg, composition and products in fractional Sobolev spaces*, J. Evol. Equ. **1** (2001) 387–404. Dedicated to the memory of Tosio Kato.

[10] H. BREZIS and J. L. VÁZQUEZ, *Blow-up solutions of some nonlinear elliptic problems*, Rev. Mat. Univ. Complut. Madrid **10** (1997), 443–469.

[11] P. CIATTI, M. G. COWLING and F. RICCI, *Hardy and uncertainty inequalities on stratified Lie groups*, Adv. Math. **277** (2015), 365–387.

[12] L. D'AMBROSIO, *Some Hardy inequalities on the Heisenberg group*, Differ. Equ. **40** (2004), 552–564.

[13] D. DANIELLI, N. GAROFALO and N. C. PHUC, *Inequalities of Hardy-Sobolev type in Carnot-Carathéodory spaces*, In: "Sobolev Spaces in Mathematics. I", Vol. 8 of Int. Math. Ser. (N. Y.), Springer, New York, 2009, 117–151.

[14] D. DANIELLI, N. GAROFALO and N. C. PHUC, *Hardy-Sobolev type inequalities with sharp constants in Carnot-Carathéodory spaces*, Potential Anal. **34** (2011), 223–242.

[15] B. DEVYVER, M. FRAAS and Y. PINCHOVER, *Optimal Hardy weight for second-order elliptic operator: an answer to a problem of Agmon*, J. Funct. Anal. **266** (2014), 4422–4489.

[16] E. DI NEZZA, G. PALATUCCI and E. VALDINOCI, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), 521–573.

[17] S. FILIPPAS and A. TERTIKAS, *Optimizing improved Hardy inequalities*, J. Funct. Anal. **192** (2002), 186–233.

[18] G. B. FOLLAND, *Subelliptic estimates and function spaces on nilpotent Lie groups*, Ark. Mat. **13** (1975), 161–207.

[19] N. GAROFALO and E. LANCONELLI, *Frequency functions on the Heisenberg group, the uncertainty principle and unique continuation*, Ann. Inst. Fourier (Grenoble) **40** (1990), 313–356.

[20] E. GIUSTI, "Direct Methods in the Calculus of Variations", World Scientific Publishing Co., Inc., River Edge, NJ, 2003.

[21] J A. GOLDSTEIN and QI S. ZHANG, *On a degenerate heat equation with a singular potential*, J. Funct. Anal. **186** (2001), 342–359.

[22] J.-L. LIONS, *Théorèmes de trace et d'interpolation. IV*, Math. Ann. **151** (1963), 42–56.

[23] V. MAZ'YA and T. SHAPOSHNIKOVA, *On the Bourgain, Brezis, and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces*, J. Funct. Anal. **195** (2002), 230–238.
[24] P. NIU, H. ZHANG and Y. WANG, *Hardy type and Rellich type inequalities on the Heisenberg group*, Proc. Amer. Math. Soc. **129** (2001), 3623–3630.
[25] L. RONCAL and S. THANGAVELU, *Hardy's inequality for fractional powers of the sublaplacian on the Heisenberg group*, Adv. Math. **302** (2016), 106–158.
[26] R. TOLEDANO, *A note on the Lebesgue differentiation theorem in spaces of homogeneous type*, Real Anal. Exchange **29** (2003/04), 335–339.
[27] H. TRIEBEL, *Theory of function spaces*, In: "Mathematik und ihre Anwendungen in Physik und Technik [Mathematics and its Applications in Physics and Technology]", Vol. 38, Akademische Verlagsgesellschaft Geest & Portig K.-G., Leipzig, 1983.
[28] D. VASSILEV, *Existence of solutions and regularity near the characteristic boundary for sub-Laplacian equations on Carnot groups*, Pacific J. Math. **227** (2006), 361–397.
[29] J. L. VAZQUEZ and E. ZUAZUA, *The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential*, J. Funct. Anal. **173** (2000), 103–153.

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