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Symplectic Wick rotations between moduli spaces of 3-manifolds

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Abstract. We describe natural maps between (parts of) $\mathcal{QF}$, the space of quasi-fuchsian hyperbolic metrics on a product 3-manifold $S \times \mathbb{R}$, and $\mathcal{GH}_{-1}$, the space of maximal globally hyperbolic anti-de Sitter metrics on the same manifold, defined in terms of special surfaces (*e.g.*, minimal/maximal surfaces, CMC surfaces, pleated surfaces) and prove that these "Wick rotations" are at least C^1 smooth and symplectic with respect to the canonical symplectic structures on both $\mathcal{QF}$ and $\mathcal{GH}_{-1}$. Similar results involving the spaces of globally hyperbolic de Sitter and Minkowski metrics are also described.

These 3-dimensional results are shown to be equivalent to purely 2-dimensional ones. Namely, consider the double harmonic map $\mathcal{H} : T^*\mathcal{T} \to \mathcal{T} \times \overline{\mathcal{T}}$, sending a conformal structure c and a holomorphic quadratic differential q on S to the pair of hyperbolic metrics (m_L, m_R) such that the harmonic maps isotopic to the identity from (S, c) to (S, m_L) and to (S, m_R) have, respectively, Hopf differentials equal to iq and $-iq$, and the double earthquake map $\mathcal{E} : \mathcal{T} \times \mathcal{ML} \to \mathcal{T} \times \overline{\mathcal{T}}$, sending a hyperbolic metric m and a measured lamination l on S to the pair $(E_L(m,l), E_R(m,l))$, where E_L and E_R denote the left and right earthquakes. We describe how such 2-dimensional double maps are related to 3-dimensional Wick rotations and prove that they are also C^1 smooth and symplectic.

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1. Introduction and results

Notation

We consider a closed, oriented surface S of genus $g \geq 2$ and the 3-dimensional product manifold $M = S \times \mathbb{R}$. The boundary of M is the disjoint union of two surfaces homeomorphic to S, which we denote by $\partial_+ M$ and $\partial_- M$.

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We denote by $\mathcal{T}$ the Teichmüller space of S, which is considered either as the space of conformal structures, the space of complex structures compatible with the orientation, or the space of hyperbolic metrics on S, all considered up to isotopy, and by $\overline{\mathcal{T}}$ the Teichmüller space of S with the opposite orientation. Recall that $\mathcal{T}$ is naturally endowed with a symplectic form ω_{WP}, called the Weil-Petersson symplectic form, and $\overline{\mathcal{T}}$ has the corresponding symplectic form $\overline{\omega_{WP}}$ (which differs from ω_{WP} by a sign).

We also denote by $\mathcal{ML}$ the space of measured laminations on S and by $\mathcal{Q}$ the bundle of holomorphic quadratic differentials on S. The space of complex projective structures on S, considered up to isotopy, will be denoted by $\mathcal{CP}$ (see Subsection 2.2). This space is enowed with a complex symplectic form ω_G, called the Goldman symplectic form. We denote by ω_G^i the imaginary part of ω_G, which defines a real symplectic structure.

1.1. Wick rotations

The heuristic idea of Wick rotation is old and quite natural. The underlying space-time of special relativity is the Minkowski space, that is, $\mathbb{R}^4$ with the Lorentzian metric $-dt^2 + dx^2 + dy^2 + dz^2$. Mathematicians (and physicists at the time) were used to the four-dimensional Euclidean space, $\mathbb{R}^4$ with the bilinear form $d\tau^2 + dx^2 + dy^2 + dz^2$. A simple way to pass from one to the other is to "complexify time", that is, write $t = i\tau$, so that the Minkowski metric is written in terms of the variables (τ, x, y, z) exactly as the Euclidean metric.

The "Wick rotations" that we consider here, following the spirit of [6], are slightly more elaborate versions of the same idea. We consider a constant curvature metric g on a 3-dimensional manifold M (homeomorphic to $S \times \mathbb{R}$) along with a surface $\Sigma \subset M$. (The metric g can be hyperbolic or Lorentzian of curvature $-1, 0$ or 1, and the surface Σ is always "special", it can be a minimal or maximal surface, a CMC surface, or a pleated surface.) We then note that under various hypothesis there is a unique metric g' on M which is also of constant curvature, but of a different type than g, containing a surface Σ' which is either isometric or conformal to Σ, and "curved" in the same way, in the sense that they have the same traceless second fundamental form or measured bending lamination, depending on the case considered.

We are thus interested in the relations between moduli spaces of geometric structures on $M = S \times \mathbb{R}$, in particular

- the space $\mathcal{QF}$ of quasifuchsian hyperbolic metrics (see Subsection 2.3), or more generally the space $\mathcal{HE}$ of hyperbolic ends (see Subsection 2.4);
- and the space $\mathcal{GH}_{-1}$ of maximal globally hyperbolic anti-de Sitter metrics (see Subsection 2.5),

but also the spaces $\mathcal{GH}_1$ and $\mathcal{GH}_0$ of maximal globally hyperbolic de Sitter and Minkowski metrics (see Subsection 2.8). We give the main definitions first for maps between quasifuchsian metrics (or more generally hyperbolic ends) and globally hyperbolic AdS metrics.

1.2. Convex pleated surfaces

Given a quasifuchsian manifold (M, h), or more generaly a hyperbolic end (E, h), let S_+ denote the upper boundary of the convex core of M (see Subsection 2.6), respectively the concave boundary component of E (see Subsection 2.4), and denote by $(m_+, l_+) = \partial_+^{\mathrm{Hyp}}(h)$ the induced metric and measured pleating lamination on S_+. The metric m_+ is hyperbolic and can be lifted to a complete hyperbolic metric $\tilde{m}_+$ on the universal cover $\widetilde{S_+}$, and l_+ to a measured geodesic lamination $\tilde{l}_+$ for $\tilde{m}_+$.

Then, the data $(\tilde{m}_+, \tilde{l}_+)$ defines a unique pleated surface $\tilde{\Sigma}$ in AdS^3 (see [6]) which by construction is invariant and cocompact under an action $\rho : \pi_1 S \to \mathrm{isom}(\mathrm{AdS}^3)$. This action extends in a properly discontinuous manner to a small tubular neighborhood of $\tilde{\Sigma}$ in AdS^3 and, taking the quotient of this tubular neighborhood by $\rho(\pi_1 S)$, defines an AdS 3-manifold (M', g') which is globally hyperbolic. Therefore (M', g') embeds isometrically in a unique GHM AdS manifold (M, g) (see [32]). Also by construction, $\Sigma/\rho(\pi_1 S)$ embeds isometrically as a pleated surface in M, which can only be the upper boundary of the convex core of M, so that (m_+, l_+) is also the data defined on the upper boundary of $C(M, g)$

$$(m_+, l_+) = \partial_+^{\mathrm{AdS}}(g) .$$

This therefore defines a *Wick rotation* map $W_\partial^{\mathrm{AdS}} : \mathcal{QF} \to \mathcal{GH}_{-1}$ associating to a quasifuchsian manifold the GHM AdS manifold with matching convex core boundary data

$$W_\partial^{\mathrm{AdS}} = (\partial_+^{\mathrm{AdS}})^{-1} \circ \partial_+^{\mathrm{Hyp}}.$$

We refer the reader to [6] for a similar construction.

The following proposition is perhaps not as obvious as it might appear at first sight. It is close in spirit to [27, Lemma 1.1].

Proposition 1.1. *The map $W_\partial^{\mathrm{AdS}} : \mathcal{QF} \to \mathcal{GH}_{-1}$ is injective and C^1-smooth.*

Note that the smooth structures on $\mathcal{QF}$ and $\mathcal{GH}_{-1}$ considered here are induced by the holonomy maps $\mathrm{hol}^{\mathrm{Hyp}} : \mathcal{QF} \to \mathcal{X}$ and $\mathrm{hol}^{\mathrm{AdS}} : \mathcal{GH}_{-1} \to \mathcal{T} \times \overline{\mathcal{T}}$. Here $\mathcal{X}$ denotes the $\mathrm{PSL}_2\mathbb{C}$-character variety of S, $\mathcal{T}$ the Teichmüller space of S and $\overline{\mathcal{T}}$ the Teichmüller space of S with the opposite orientation. Also note that $W_\partial^{\mathrm{AdS}}$ becomes one-to-one when extended to the more general setting of hyperbolic ends.

This proposition implies that we can consider the pull-back by $W_\partial^{\mathrm{AdS}}$ of the symplectic structure on the target space. We then obtain the following theorem, whose proof can be found in Section 5:

Theorem 1.2. *The map $W_\partial^{\mathrm{AdS}} : (\mathcal{QF}, \omega_G^i) \to \left(\mathcal{GH}_{-1}, \frac{1}{2}(\omega_{WP} \oplus \overline{\omega_{WP}})\right)$ is symplectic.*

The precise definition of the symplectic structures ω_G^i and ω_{WP} can be found in Section 2.

1.3. Minimal or maximal surfaces

Given a quasifuchsian manifold (M, h), it is well known (see, *e.g.*, [46]) that M contains a closed minimal surface homeomorphic to S. However this minimal surface is in general not unique. There is a specific class of quasifuchsian manifolds containing a unique closed, embedded minimal surface: they are those, called *almost-Fuchsian*, which contain a closed, embedded minimal surface with principal curvatures everywhere in $(-1, 1)$, see [46]. We call $\mathcal{AF} \subset \mathcal{QF}$ the space of almost-Fuchsian metrics on M, considered up to isotopy.

Thus, restricting our attention to $h \in \mathcal{AF}$, let $\Sigma \subset M$ be its unique closed, embedded minimal surface and consider its induced metric I and second fundamental form $I\!I$. It is well known (see, *e.g.*, [26]) that $I\!I$ is then the real part of a holomorphic quadratic differential q for the complex structure defined on S by the conformal class of I. So $([I], I\!I)$ define a point $(c, q) \in \mathcal{Q}$, and we obtain a map

$$\min : \mathcal{AF} \to \mathcal{Q},$$

sending an almost-Fuchian metric to the data on its minimal surface.

Things are somewhat simpler for GHM AdS manifolds. It is well known (see, *e.g.*, [26]) that any GHM AdS manifold contains a unique closed, space-like maximal surface. Moreover, given a complex structure c and a holomorphic quadratic differential q for c on S, there is a unique GHM AdS metric g such that the induced metric and second fundamental form on the unique maximal surface in M is $I, I\!I$ with I compatible with c and $I\!I = Re(q)$. This provides an analogous map

$$\max : \mathcal{GH}_{-1} \to \mathcal{Q}$$

sending an GHM AdS metric to the data on its maximal surface and which, by the arguments above, is one-to-one.

This defines another Wick rotation map $W_{\min} : \mathcal{AF} \to \mathcal{GH}_{-1}$ associating to an almost-Fuchian manifold the GHM AdS manifold with matching minimal/maximal surface data

$$W_{\min} : \max{}^{-1} \circ \min .$$

This map is clearly smooth and injective and we have the following result:

Theorem 1.3. *The map* $W_{\min} : (\mathcal{AF}, \omega^i_G) \to (\mathcal{GH}_{-1}, \frac{1}{2}(\omega_{WP} \oplus \overline{\omega_{WP}}))$ *is symplectic.*

1.3.1. *Constant mean curvature surfaces*

The previous picture can be extended by considering constant mean curvature (CMC) surfaces, rather than minimal or maximal surfaces. Recall that the mean curvature of a surface in a Riemannian or Lorentzian 3-manifold is given by half the trace of its second fundamental form. We will use a basic and well-known fact (see [22]): the traceless part of the second fundamental form of an oriented constant mean curvature surface in any constant curvature 3-dimensional (Riemannian or Lorentzian) manifold is the real part of a holomorphic quadratic differential, for the complex structure associated to its induced metric.

GHM AdS manifold are particularly well-behaved with respect to CMC surfaces. On one hand, any GHM AdS manifold contains a canonical foliation by CMC surfaces.

Theorem 1.4 (Barbot, Béguin, Zeghib [5]). *Any GHM AdS manifold M admits a unique foliation by closed space-like CMC surfaces, with mean curvature varying between $-\infty$ and ∞. That is, for all $H \in \mathbb{R}$, M contains a unique closed space-like CMC-H surface.*

On the other hand, one can also associate through CMC-H surfaces a GHM AdS manifold to any point in $\mathcal{Q}$, thanks to the following proposition (see [26, Lemma 3.10]).

Proposition 1.5. *Let $H \in (-\infty, \infty)$. Given c a complex structure and q a holomorphic quadratic differential for c, there is a unique GHM AdS metric g on M such that the induced metric and traceless part of the second fundamental form on the unique CMC-H surface in (M, g) is I, $I\!I_0$ with I compatible with c and $I\!I_0 = Re(q)$.*

In the quasifuchsian context, it was conjectured by Thurston that the analog of Theorem 1.4 also holds, but only for almost-Fuchsian manifolds. Lacking a proof of this fact, we introduce the following notation.

Definition 1.6. We denote by $\mathcal{AF}'$ the space of quasifuchsian metrics on M which admit a unique foliation by CMC surfaces with $H \in (-1, 1)$.

Note that the Thurston conjecture mentioned above can be reformulated as the fact that $\mathcal{AF} \subset \mathcal{AF}'$, as any closed embedded CMC surface is a leaf of the foliation by the maximum principle.

We can now construct a generalization of the map $W_{\min}$ associated to any pair of constants $H \in (-1, 1)$ and $H' \in (-\infty, \infty)$. For each $h \in \mathcal{AF}'$, let Σ_H be the unique closed CMC-H surface in (M, h), let c be the conformal class of its induced metric, and let q be the traceless part of its second fundamental form. There is then a unique GHM AdS metric g on M such that the (unique) CMC-H' surface in (M, g) has induced metric conformal to c and the traceless part of its second fundamental form is equal to q. We denote by $W^{\mathrm{AdS}}_{H,H'} : \mathcal{AF}' \to \mathcal{GH}_{-1}$ the map sending h to g.

Theorem 1.7. *For all $H \in (-1, 1)$ and $H' \in (-\infty, \infty)$, the map*

$$W^{\mathrm{AdS}}_{H,H'} : \left(\mathcal{AF}', \omega^i_G\right) \to \left(\mathcal{GH}_{-1}, \frac{1}{2}\left(\omega_{WP} \oplus \overline{\omega_{WP}}\right)\right)$$

is symplectic.

1.4. Double maps

We now translate the above stated results purely in terms of surfaces, and of maps between moduli spaces of surfaces, with no reference to 3-dimensional manifolds. We first consider harmonic maps and then earthquakes.

1.4.1. *Harmonic maps*

Recall that a map $\phi : (M, g) \to (N, h)$ between two Riemannian manifolds is harmonic if it is a critical point of the Dirichlet energy, defined as

$$E(\phi) = \int_M \|d\phi\|^2 d\,\mathrm{vol}\ .$$

If M is a surface, the Dirichlet energy is invariant under conformal deformations of the metric on M, so the notion of harmonic maps from M to N only depends on the choice of a conformal class on M (no Riemannian metric is needed).

Let us now consider the case of harmonic diffeomorphisms of a surface S. Let $c \in \mathcal{T}$ be a complex structure on S and $m \in \mathcal{T}$ a hyperbolic metric. Given a diffeomorphism $\phi : (S, c) \to (S, m)$, its Hopf differential $\mathrm{Hopf}(\phi)$ is defined as the $(2, 0)$ part of $\phi^* m$. A key relation between holomorphic quadratic differentials and harmonic diffeomorphisms is that $\mathrm{Hopf}(\phi)$ is holomorphic if and only if ϕ is harmonic. In addition, we will use the following well-know statements.

Theorem 1.8 (Eells and Sampson [15], Hartman [20], Schoen and Yau [40]). *If S is a closed surface equipped with a conformal class c and m is any hyperbolic metric on S, then there is a unique harmonic map isotopic to the identity from (S, c) to (S, m).*

Theorem 1.9 (Sampson [38], Wolf [48]). *Let $c \in \mathcal{T}$ be a complex structure on a surface S, and let q be a holomorphic quadratic differential on (S, c). There is a unique hyperbolic metric m on S, well-defined up to isotopy, such that the Hopf differential of the harmonic map $\phi : (S, c) \to (S, m)$ isotopic to the identity is equal to q.*

Together these define a map $H : \mathcal{Q} \to \mathcal{T}$, from the bundle of holomorphic quadratic differentials to Teichmüller space, associating to (c, q) the hyperbolic metric m.

Definition 1.10. We denote by $\mathcal{H} : \mathcal{Q} \to \mathcal{T} \times \overline{\mathcal{T}}$ the map defined by

$$\mathcal{H}(c, q) = (H(c, -iq), H(c, iq))\ .$$

We will call $\mathcal{H}$ the *double harmonic map*.

It is a well known fact that the bundle $\mathcal{Q}$ of holomorphic quadratic differentials can be identified with the holomorphic cotangent bundle $T^{*(1,0)}\mathcal{T}$ over Teichmüller space. We denote by ω_* the canonical complex cotangent bundle symplectic structure on $T^{*(1,0)}\mathcal{T}$ and by ω_*^r its real part, which corresponds to (half) the real canonical cotangent bundle symplectic structure on $T^*\mathcal{T}$. We then obtain the following result.

Theorem 1.11. $\mathcal{H} : (\mathcal{Q}, -\omega_*^r) \to (\mathcal{T} \times \overline{\mathcal{T}}, \frac{1}{2}(\omega_{WP} \oplus \overline{\omega_{WP}}))$ *is symplectic.*

1.4.2. *Earthquakes*

Definition 1.12. A *measured geodesic lamination* is a closed subset $l \subset S$ which is foliated by complete simple geodesics, defined with respect to a given hyperbolic metric $m \in \mathcal{T}$, together with a positive measure μ on arcs transverse to the leaves of l which is invariant under deformations among transverse arcs with fixed end-points (see, *e.g.*, [12]). We denote the space of measured geodesic laminations on S, considered up to isotopy, by $\mathcal{ML}$.

Similarly to holomorphic quadratic differentials, the definition of measured geodesic laminations depends on the choice of a point in $\mathcal{T}$ and thus determines a bundle over Teichmüller space. However, unlike $\mathcal{Q}$, there is a canonical identification between the fibres over any pair $m, m' \in \mathcal{T}$ — this extends the fact that any simple closed geodesic for m' is isotopic to a unique simple closed geodesic for m (see [12]). This justifies the notation of $\mathcal{ML}$ without any reference to a hyperbolic structure.

Given a hyperbolic metric $m \in \mathcal{T}$ and a measured geodesic lamination $l \in \mathcal{ML}$ we may define the left earthquake of m along l. This is a new hyperbolic metric on S denoted by $E^L(m, l)$. For l supported on a simple close geodesic γ with weight a, $E^L(m, l)$ is defined by cutting S along γ, rotating the left-hand side of γ by length a and then gluing it back. The operation for general laminations is then defined as certain (well-defined) limiting procedure [44]. Importantly we have the following result, which is a geometric analogue to the analytic results above.

Theorem 1.13 (Thurston [44]). *For any pair $m, m' \in \mathcal{T}$ of hyperbolic metrics on S there exists a unique measured lamination $l \in \mathcal{ML}$ such that m and m' are related by a left earthquake $m' = E^L(m, l)$.*

Theorem 1.14 (Thurston [44], Kerckhoff [25]). *The map $E^L : \mathcal{T} \times \mathcal{ML} \to \mathcal{T}$ is a homeomorphism for every fixed $m \in \mathcal{T}$ and a real analytic diffeomorphism for every fixed $l \in \mathcal{ML}$.*

The notion of right earthquake is obtained in the same way, by rotating in the other direction, so that the right earthquake along l, $E^R(l)$, is the inverse of $E^L(l)$. So we have two maps $E^L, E^R : \mathcal{T} \times \mathcal{ML} \to \mathcal{T}$.

Definition 1.15. We denote by $\mathcal{E} : \mathcal{T} \times \mathcal{ML} \to \mathcal{T} \times \overline{\mathcal{T}}$ the map defined by

$$\mathcal{E}(m, l) = \left(E^L(m, l), E^R(m, l)\right) .$$

We will call $\mathcal{E}$ the *double earthquake map*.

Note that $\mathcal{E}$ is a bijection. Indeed, from Thurston's Earthquake Theorem, given any pair $m, m' \in \mathcal{T}$, there is a unique left earthquake path going from m' to m. In other terms, there is a unique measured lamination $l \in \mathcal{ML}$ such that $E^L(l)(m') = h$. Now let $m'' = E^L(l/2)(m')$. Then clearly $(m, m') = \mathcal{E}(m'', l/2)$. Conversely, given any $(m'', l) \in \mathcal{T} \times \mathcal{ML}$ such that $(m, m') = \mathcal{E}(m'', l)$, m'' must be the midpoint of the left earthquake path from m' to m, and this path is associated to l, so the map $\mathcal{E}$ is one-to-one. However there is no reason to believe that $\mathcal{E}$ is differentiable — actually it is not even clear what it would mean, since there is no canonical differentiable structure on $\mathcal{ML}$. To deal with this differentiability issue we introduce a map $\delta : \mathcal{T} \times \mathcal{ML} \to T^*\mathcal{T}$ which sends a hyperbolic metric $m \in \mathcal{T}$ on S and a measured lamination $l \in \mathcal{ML}$ to the differential at m of the length function of l, $L(l) : \mathcal{T} \to \mathbb{R}$,

$$\delta(m, l) = d_m L(l) .$$

This map is a global homeomorphism between $\mathcal{T} \times \mathcal{ML}$ and $T^*\mathcal{T}$, see [27, Lemma 2.3].

The following can be seen as a translation of Proposition 1.1, see Section 4 for a proof.

Proposition 1.16. *The map $E^L \circ \delta^{-1} : T^*\mathcal{T} \to \mathcal{T}$ is C^1-smooth.*

Corollary 1.17. *$\mathcal{E} \circ \delta^{-1} : T^*\mathcal{T} \to \mathcal{T} \times \overline{\mathcal{T}}$ is a C^1 diffeomorphism.*

This corollary then allows to consider the following statement, whose proof can be found in Section 5

Theorem 1.18. *The map $\mathcal{E} \circ \delta^{-1} : (T^*\mathcal{T}, 2\omega^r_*) \to (\mathcal{T} \times \overline{\mathcal{T}}, \frac{1}{2}(\omega_{WP} \oplus \overline{\omega_{WP}}))$ is symplectic.*

1.5. Minkowski and de Sitter manifolds

For GHM Minkowski and de Sitter manifolds, it is also possible to define Wick rotation maps $W^{\mathrm{Mink}}_\partial : \mathcal{QF} \to \mathcal{GH}_0$, $W^{\mathrm{Mink}}_{H,H'} : \mathcal{AF}' \to \mathcal{GH}_0$ and $W^{dS}_\partial : \mathcal{QF} \to \mathcal{GH}_1$ and $W^{dS}_{H,H'} : \mathcal{AF}' \to \mathcal{GH}_1$. The main difference is that these manifolds now do not contain convex pleated surfaces nor maximal surfaces. Their relation with hyperbolic manifolds in terms of measured laminations is still possible via: (1) the inital singularity of Minkowski manifolds and (2) the projective duality between hyperbolic and de Sitter manifolds. The relation in terms of CMC foliations is also available in both cases, only with H' varying between $(-\infty, 0)$ and $(-\infty, -2)$, respectively.

We shall prove, in the de Sitter case, that the CMC Wick rotations $W^{dS}_{H,H'}$ is again symplectic, where the symplectic structure on $\mathcal{GH}_1$ is again the pull-back of the imaginary part of the Goldman symplectic structure on $\mathcal{X}$.

Theorem 1.19. *Let $H \in (-2, 2)$ and let $H' \in (-\infty, -2)$. The map $W^{dS}_{H,H'}$: $(\mathcal{AF}', \omega^i_G) \to (\mathcal{GH}_1, \omega^i_G)$ is symplectic.*

The proof can be found in Subsection 7.1.

1.6. Spaces with particles

The results above might have extensions to constant curvature 3-manifolds of various types containing "particles", that is, cone singularities of angle less than π along infinite geodesics connecting the two connected components of the boundary at infinity (in a "quasifuchsian" hyperbolic manifold) or along a maximal time-like geodesic (in a GHM AdS, dS or Minkowski spacetime).

A number of the tools needed to state the results above are known to extend to this setting. For quasifuchsian manifolds, an extension of the Bers double uniformization theorem is known in this setting [29, 34]. The Mess analog for GHM AdS manifolds also extends to this setting with "particles" [13], and the existence and uniqueness of a maximal surface (orthogonal to the particles) also holds [45]. However it is not known whether GHM AdS, dS or Minkowski space-times with particles contain a unique foliation by CMC surfaces orthogonal to the particles.

1.7. Some physical motivations

From a physical point of view there are two approaches to understand the relation between Teichmüller theory and 3d gravity which motivates the existence of the symplectic maps considered in the present work. In each approach, one rewrites the Einstein-Hilbert functional in terms of new variables as to simplify the description of the moduli space of critical points.

Recall that the Einstein-Hilbert functional on the space of 3-dimensional Lorentzian metrics on M is defined by

$$S[g] = -\int_M (R - 2\Lambda) dv$$

where dv and R are the volume form and the scalar curvature of g, $\Lambda = 0, -1, 1$ the cosmological constant. The critical points are given by solutions of Einstein's equation

$$\mathrm{Ric} - \frac{1}{2}(R - 2\Lambda) g = 0.$$

The usual approach to describe the moduli space of critical points of the Einstein-Hilbert action follows from the interpretation of Einstein's equation as a constrained dynamical system for 2-dimensional Riemannian metrics on S, see [4, 33]. One starts with the choice of a global time function on M and decomposes of the 3-dimensional metric g in terms of the induced metric I and the extrincic curvature $I\!I$ of the leaves of the constant time foliation, which are constrained by the Gauss-Codazzi equations. In terms of isothermal coordinates z on a leaf Σ, we my write

$$I = e^{2\varphi}|dz|^2, \qquad I\!I = \frac{1}{2}\left(q dz^2 + \bar{q} d\bar{z}^2\right) + e^{2\varphi} H |dz|^2,$$

and the Gauss-Codazzi equation becomes

$$4\partial_z \partial_{\bar{z}} \varphi = e^{2\varphi}\left(H^2 - \Lambda\right) - e^{-2\varphi}|q|^2, \qquad \partial_{\bar{z}} q = e^{2\varphi} \partial_{\bar{z}} H.$$

Here $e^{2\varphi}$ is the conformal factor of I, H is a the mean curvature of Σ and q is a quadratic differential.

For maximal globally hyperbolic spacetimes, the equations of motion are then uniquely solved given initial data on any Cauchy surface Σ. Also in this case it is always possible to choose a foliation containing a constant mean curvature ($H = \text{const.}$) initial surface. The constraints are then easily solved: the Codazzi constraint equation becomes a holomorphicity equation for the quadratic differential q determined by the traceless part of $I\!I$ and the Gauss constraint equation becomes an elliptic differential equation for $e^{2\varphi}$. The existence and uniqueness of solutions of the Gauss equation are guaranteed for $H^2 - \Lambda \geq 1$ (see [33]) thus showing that the inital data parameterizing the moduli space of globally hyperbolic maximal spacetimes is given by points in the cotangent bundle over Teichmüller space of the initial Cauchy surface:

$$\mathcal{GH}_\Lambda = T^*\mathcal{T} \, .$$

The symplectic structure on $\mathcal{GH}_\Lambda$ is also shown to agree, up to a multiplicative constant, with the real canonical symplectic structure ω_*^r on $T^*\mathcal{T}$, via symplectic reduction of the cotangent bundle over Riemannian metrics on S, with its canonical symplectic structure, to the constraint submanifold defined by the Gauss-Codazzi equation [33].

Another approach to describe the moduli space $\mathcal{GH}_\Lambda$ stems from the fact that all Einstein 3-manifolds have constant sectional curvature equal to the cosmological constant Λ. Thus, such manifolds can be described as quotients of appropriate domains of either Minkowski, anti-de Sitter or de Sitter 3-spacetime, in the Lorentzian setting, and Euclidean, hyperbolic or spherical 3-space, in the Riemannian setting. The study of 3d Einstein manifolds can thus be viewed in the context of locally homogeneous geometric structures, *i.e.*, flat G_Λ-bundles over spacetime. Such an approach was first suggested in the physics literature in [1,47] where the Einstein-Hilbert action is shown to be equivalent to a Chern-Simons action on the space of G-connections over the spacetime manifold. Here G_Λ is the isometry group of the relevant model spacetime, that is, $\mathrm{PSL}_2\mathbb{R} \ltimes \mathfrak{sl}_2\mathbb{R}$ for $\Lambda = 0$, $\mathrm{PSL}_2\mathbb{R} \times \mathrm{PSL}_2\mathbb{R}$ for $\Lambda = -1$, and $\mathrm{PSL}_2\mathbb{C}$ for $\Lambda = 1$.

This is obtained by first decomposing the spacetime metric g in terms of a coframe field e and spin connection ω, which are taken to be independent. By appropriately tensoring the components of e and ω with Lie algebra generators one then constructs the associated $\mathfrak{g}_\Lambda$-valued 1-form A on M. Finally, translating the Einstein-Hibert action for g in terms of A gives exactly the Chern-Simons action

$$S_{G_\Lambda}[A] = \int_M B_\Lambda \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \, ,$$

where B_Λ denotes an Ad-invariant symmetric bilinear form on $\mathfrak{g}_\Lambda$. This provides a description of the moduli space of spacetimes as a subspace of the moduli space of flat G_Λ-connections on S. In the maximal globally hyperbolic case it is possible to describe the gravitational component completely (see [32,39])

$$\mathcal{GH}_\Lambda = \begin{cases} T^*\mathcal{T} & \Lambda = 0 \\ \mathcal{T} \times \overline{\mathcal{T}} & \Lambda = -1 \\ \mathcal{CP} & \Lambda = 1. \end{cases}$$

The symplectic structure is now given by the Goldman cup product symplectic form with coefficient pairing given by B_Λ. For the isometry groups of the 3d geometric models described above, the corresponding Lie algebras are know to admit a real 2-dimensional space of such bilinear forms. Thus, there is a 2-dimensional family of real symplectic forms on the corresponding moduli spaces. In [47] Witten obtained the relevant bilinear forms for gravity, that is, the ones arrising from the Einstein-Hilbert functional. This identify the relevant symplectic forms on the moduli spaces $\mathcal{GH}_\Lambda$: for $\Lambda = 0$ the symplectic form is given by ω_*^r, the real canonical cotangent bundle symplectic form on $T^*\mathcal{T}$, for $\Lambda = -1$ it is given by $\omega_{WP} \oplus \overline{\omega_{WP}}$, the difference of Weil-Petersson symplectic forms on each copy of $\mathcal{T}$, and for $\Lambda = 1$ by ω_G^i, the imaginary part of the complex Goldman symplectic form on $\mathcal{CP}$.

1.8. Content of the paper

Section 2 contains background material on various aspects of the geometry of surfaces and 3-dimensional manifolds, which are necessary elsewhere, including the definitions and basic properties of quasifuchsian manifolds and of globally hyperbolic spacetimes of various curvatures, statements on maximal and CMC surfaces, convex cores, as well as measured laminations and transverse cocycles.

In Section 3 a more complete description of the double harmonic and double earthquake map, as well as of the Wick rotation map. We describe the precise relation between those "double" maps and the Wick rotation maps, and show the equivalence between statements on the "double" maps and statements on the Wick rotation maps. We prove that the double earthquake and double harmonic map are one-to-one.

Section 4 is mostly focused on the regularity of the double earthquake map, and therefore of the earthquake map itself. Section 5 contains the proof that the double earthquake map is symplectic, and then that the double harmonic map is symplectic — the connection between the two statements uses a volume argument that is developed in Subsection 5.3.

Section 6 is focused on CMC surfaces, while the content of Section 7 is centered on Minkowski and de Sitter manifolds.

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2. Background material

In this section we develop in greater detail a number of definitions and established results which will be needed in the later parts of this work. In particular, we will here give the definitions of the moduli spaces and symplectic structures of interest as well as some useful parametrizations of such objects in terms of measured laminations and holomorphic quadratic differentials.

2.1. Teichmüller space

Let S be a closed oriented surface of genus $g \geq 2$. We shall consider here two equivalent definitions of the Teichmüller space $\mathcal{T}$ of S.

Definition 2.1. A complex structure c on S is an atlas of $\mathbb{C}$-valued coordinate charts, whose transition functions are biholomorphic. The *Teichmüller space* $\mathcal{T}$ can be defined as the space of all complex structures on S compatible with the orientation, considered up to isotopy.

A hyperbolic metric on S is a Riemannian metric m of negative constant curvature -1. The *Teichmüller space* $\mathcal{T}$ can be equivalently defined as the space of all hyperbolic metrics on S, again considered up to isotopy.

The relation between the two definitions is given through the Riemann-Poincaré-Koebe uniformization theorem, which also identifies $\mathcal{T}$ with a connected component of the representation variety $\mathcal{R} = \mathrm{Hom}(\pi_1 S, \mathrm{PSL}_2\mathbb{R})/\mathrm{PSL}_2\mathbb{R}$, associating to each point in Teichmüller space its holonomy representation $\rho : \pi_1 S \to \mathrm{PSL}_2\mathbb{R}$. Such holonomy representations of hyperbolic surfaces are called *Fuchsian representations* and are characterized by the maximality of their Euler number [18].

2.1.1. *The Weil-Petersson symplectic structure*

The L^2-norm

$$\|q\|_{WP}^2 = \frac{1}{8}\int_S \|q\|_m^2 da_m$$

on the bundle $\mathcal{Q}$ of holomorphic quadratic differentials induces a hermitian metric on $\mathcal{T}$ via the well-known identification between $\mathcal{Q}$ and the holomorphic cotangent bundle $T^{*(1,0)}\mathcal{T}$ over Teichmüller space. The imaginary part of this hermitian metric is then a symplectic form ω_{WP} on $\mathcal{T}$, called the *Weil-Petersson symplectic form*.

This is equivalent, up to a sign, to the restriction of the Goldman symplectic structure on $\mathcal{R}$, defined via the cup product of cohomology classes with coefficients paired with (4 times) the Killing form of $\mathfrak{sl}_2\mathbb{R}$, see [16]. Specifically,

Theorem 2.2 (Goldman [16]).

$$\omega_{WP} = -\omega_G^{\mathrm{PSL}_2\mathbb{R}}.$$

2.2. Complex projective structures

We now consider another type of structure on the surface S which has many parallels with our previous considerations.

Definition 2.3. A *complex projective structure* σ on S is an atlas of $\mathbb{C}P^1$-valued coordinate charts, whose transition functions are complex projective transformations. We denote by $\mathcal{CP}$ the space of all complex projective structures on S, considered up to isotopy.

Note that there is a natural projection $p : \mathcal{CP} \to \mathcal{T}$ associating to a complex projective structure σ on S its underlying complex structure c. The space $\mathcal{CP}$ can thus be considered as the total space of a bundle over $\mathcal{T}$. There are again two possible descriptions of $\mathcal{CP}$ obtained by analytic or geometric deformations of a fixed complex projective structure. The first is related to the bundle $\mathcal{Q}$ of holomorphic quadratic differentials via the Schwarzian derivative, while the second is related to the trivial bundle $\mathcal{T} \times \mathcal{ML}$ via the operation of grafting along measured laminations.

2.2.1. *Grafting*

Given a hyperbolic metric $m \in \mathcal{T}$ and a measured geodesic lamination $l \in \mathcal{ML}$ one may define a complex projective structure via grafting of m along l as follows. For l supported on a simple close geodesic γ with weight a, $\mathcal{G}(m, l)$ is defined by cutting S along γ and inserting a Euclidean cylinder $\gamma \times [0, a]$. This defines a complex projective structure on S by complementing the Fuchsian projective structure of m by the projective structure on $\gamma \times [0, a]$ defined by its natural embedding as an annulus in $\mathbb{C}^*$, see, *e.g.*, [14]. As for earthquakes, the operation of grafting is defined for general laminations via a limiting procedure.

Theorem 2.4 (Thurston, see [23]). *The map $\mathcal{G} : \mathcal{T} \times \mathcal{ML} \to \mathcal{CP}$ is a homeomorphism.*

2.2.2. *Schwarzian derivative*

Given two complex projective structures $\sigma, \sigma' \in \mathcal{CP}$ with the same underlying complex structure $c \in \mathcal{T}$, the Schwarzian derivative of the identity map between (S, σ) and (S, σ') is a holomorphic quadratic differential $\mathcal{S}(\sigma, \sigma') \in \mathcal{Q}_c$. The composition rule satisfied by the Schwarzian derivative means that if σ, σ' and σ'' are three complex projective structures with underlying complex structure c, then $\mathcal{S}(\sigma, \sigma'') = \mathcal{S}(\sigma, \sigma') + \mathcal{S}(\sigma', \sigma'')$. This identifies $\mathcal{CP}$ with the affine bundle of holomorphic quadratic differentials on $\mathcal{T}$ (see [14, Section 3]) and we may thus write $\sigma' - \sigma \in \mathcal{Q}_c$ instead of $\mathcal{S}(\sigma, \sigma')$.

Note however that the identification $\mathcal{CP} \simeq \mathcal{Q}$ depends on the choice of a global section $\mathcal{T} \to \mathcal{CP}$, and there are distinct "natural" possible choices for such a section, which induce distinct structures on $\mathcal{CP}$. For now, let's consider the natural Fuchsian section given by the Fuchsian uniformization of Riemann surfaces. Thus, given a complex structure c on S, the Riemann Uniformization Theorem provides a unique Fuchsian complex projective structure σ_c uniformizing c. Using this canonical section we can define an identification $\mathcal{S}_F : \mathcal{CP} \to \mathcal{Q}$, sending a complex projective structure $\sigma \in \mathcal{CP}$ with underlying complex structure $c = p(\sigma)$ to $(c, \sigma - \sigma_c) \in \mathcal{Q}$. (The subscript "$F$" here reminds us that we make use of Fuchsian sections.)

2.2.3. *The Goldman symplectic structure*

For complex projective structures, unlike the case of complex structures, holonomies are not enough to parametrize the moduli space. The holonomy map $\mathrm{hol} : \mathcal{CP} \to \mathcal{X}$ gives only a local diffeomorphism between the moduli space of complex projective structures $\mathcal{CP}$ and the $\mathrm{PSL}_2\mathbb{C}$ character variety $\mathcal{X} = \mathrm{Hom}(\pi_1 S, \mathrm{PSL}_2\mathbb{C})/\mathrm{PSL}_2\mathbb{C}$, which is surjective but not injective, see, *e.g.*, [14].

On the other hand, the holonomy map can be used to pull-back to $\mathcal{CP}$ the Goldman symplectic structure $\omega_G^{\mathrm{PSL}_2\mathbb{C}}$ on $\mathcal{X}$, now obtained by taking the cup-product of the cohomology classes with coefficients paired with (4 times) the Killing form on $\mathfrak{sl}_2\mathbb{C}$, see [17]. Pulling back $\omega_G^{\mathrm{PSL}_2\mathbb{C}}$ by hol thus gives a complex symplectic structure on $\mathcal{CP}$, which we call ω_G. We will denote by ω_G^i the imaginary part of ω_G, which is a real symplectic structure and will play an important role in what follows.

Also, via the identification of the holomorphic cotangent bundle $T^{*(1,0)}\mathcal{T}$ with the bundle of holomorphic quadratic differentials $\mathcal{Q}$, we may use the Schwarzian parametrization $\mathcal{S}_F : \mathcal{CP} \to \mathcal{Q}$ to pull-back the canonical complex symplectic structure ω_* on $T^{*(1,0)}\mathcal{T}$ to another complex symplectic structure $\omega_F = \mathcal{S}_F^*\omega_*$ on $\mathcal{CP}$. We will be interested here only in the real part of ω_*, corresponding to (half) the real symplectic structure on $T^*\mathcal{T}$. We denote by ω_F^r the real part of ω_F, which is just $\mathcal{S}_F^*\omega_*^r$

The following theorem provides the relation between the Goldman symplectic structure and the pull-back of the cotangent bundle symplectic structure via the Fuchsian slice, see [30, Corollary 5.13].

Theorem 2.5 (Loustau [30]). $\omega_G^{\mathrm{PSL}_2\mathbb{C}} = p^*\omega_G^{\mathrm{PSL}_2\mathbb{R}} + i\omega_F$, *where* $p : \mathcal{CP} \to \mathcal{T}$ *is the canonical forgetful map. In particular,*

$$\omega_G^i = \omega_F^r.$$

Note that besides the Goldman symplectic structure, there are other complex symplectic structures on $\mathcal{CP}$. In fact it is known from Hitchin's work [21] that there is a hyperkähler structure defined at least on an open subset of $\mathcal{CP}$. We do not elaborate on this here, however understanding this hyperkähler structure geometrically can be one motivation for investigating the (complex) symplectic structures on $\mathcal{CP}$ in relation to other moduli spaces of geometric structures.

2.3. Quasifuchsian hyperbolic manifolds

The first moduli space of 3-dimensional geometric structures will consider here is the space of quasifuchsian hyperbolic metrics on M, which can be most simply defined in terms of convex subsets. Given a hyperbolic metric h on M, we say that a subset $K \subset M$ is convex if any geodesic segment in M with endpoints in K is contained in K.

Definition 2.6. A complete hyperbolic metric h on M is called *quasifuchsian* if (M, h) contains a non-empty compact convex subset. We denote by $\mathcal{QF}$ the space of quasifuchsian hyperbolic metrics on M, considered up to isotopy.

Note that there are other equivalent definitions of quasifuchsian manifolds, *e.g.*, as quotients of the hyperbolic 3-space by Kleinian groups whose limit set is a Jordan curve, related to quasiconformal deformations of Fuchsian representations.

Given a quasifuchsian manifold (M, h), its universal cover $\tilde{M}$ admits a developing map with values in $\mathbb{H}^3$. This then restricts to a developing map of $\widetilde{\partial_+ M}$ into $\partial_\infty \mathbb{H}^3 \simeq \mathbb{C}P^1$ and, since hyperbolic isometries act on $\partial_\infty \mathbb{H}^3$ as projective transformations, the holonomy representation of (M, h) endows $\partial_+ M$ with a complex projective structure $\sigma_+ \in \mathcal{CP}$. We thus obtain an injective map $\partial_\infty^{\mathrm{Hyp}} : \mathcal{QF} \to \mathcal{CP}$, which is however not surjective.

We will continue to denote by ω_G the pull-back to $\mathcal{QF}$ of the complex Goldman symplectic structure on $\mathcal{CP}$, and by ω_G^i its imaginary part.

2.4. Hyperbolic ends

As mentioned in Section 1 the description of quasifuchsian manifold in terms of the upper boundary of the convex core admits an extension to a more general context of hyperbolic ends, which we now describe in more details. Thus, consider a quasifuchsian manifold (M, h) homeomorphic to $S \times \mathbb{R}$, and let E_+ be the upper connected component of $M \setminus C(M, h)$. It is a non-complete hyperbolic manifold, homeomorphic to $S \times (0, \infty)$, which is complete on the side corresponding to ∞, and bounded on the side corresponding to 0 by a concave pleated surface. A hyperbolic manifold of this type is called a (non-degenerate) *hyperbolic end*. We call $\mathcal{HE}$ the space of (non-degenerate) hyperbolic ends homeomorphic to $S \times (0, \infty)$.

Given a hyperbolic end (E, h), we call $\partial_\infty E$ its "boundary at infinity" corresponding to the "complete" side, and $\partial_0 E$ its boundary component which is a concave pleated surface. The universal cover $\tilde{E}$ of E admits a developing map with values in $\mathbb{H}^3$, which restricts to a developing map of $\widetilde{\partial_\infty E}$ into $\partial_\infty \mathbb{H}^3$, which can be identified with $\mathbb{C}P^1$. Since hyperbolic isometries act on $\partial_\infty \mathbb{H}^3 \simeq \mathbb{C}P^1$ as projective transformations, $\partial_\infty E$ is endowed with a complex projective structure $\sigma \in \mathcal{CP}$. On the other hand, $\partial_0 E$ is endowed with a hyperbolic metric pleated along a measured geodesic lamination. Thus, we have a pair of maps $\partial_\infty^{\mathrm{Hyp}} : \mathcal{HE} \to \mathcal{CP}$ and $\partial_+^{\mathrm{Hyp}} : \mathcal{HE} \to \mathcal{T} \times \mathcal{ML}$, which are in fact homeomorphisms by the following result by Thurston.

Theorem 2.7 (Thurston, see [14]). *Given a pair $(m, l) \in \mathcal{T} \times \mathcal{ML}$ there is a unique non-degenerate hyperbolic end (E, h) such that $\partial_0 E$ has induced metric given by m and bending lamination given by l. Also, each $\sigma \in \mathcal{CP}$ is the complex projective structure at $\partial_\infty E$ of a unique (non-degenerate) hyperbolic end E. The* *relation between the complex projective structure σ and the pair (m, l) is given by the grafting map $\mathcal{G} : \mathcal{T} \times \mathcal{ML} \to \mathcal{CP}$ which furthermore is a homeomorphism.*

2.5. Globally hyperbolic anti-de Sitter manifolds

The second moduli space of interest in this work is that of globally hyperbolic maximal anti-de Sitter metrics on M.

The 3-dimensional anti-de Sitter space, denoted here by AdS^3, can be defined as the quadric

$$\left\{ p \in \mathbb{R}^{2,2} \mid \langle p, p \rangle = -1 \right\}$$

with the induced metric from the metric of signature (2, 2) on $\mathbb{R}^4$.

Definition 2.8. A Lorentzian metric g on M is called *globally hyperbolic maximal* (GHM) *anti-de Sitter* (AdS) if (M, g) is locally modeled on AdS^3, contains a Cauchy surface and is maximal under these conditions. We call $\mathcal{GH}_{-1}$ the space of GHM AdS metrics on M, considered up to isotopy.

We say that a surface $\Sigma \subset M$ is a Cauchy surface if it is a closed space-like surface homeomorphic to S such that any inextendible time-like curve on M intersects Σ exactly once. The maximality condition then says that any isometric embedding $(M, g) \to (M', g')$, with (M', g') also satisfying the two conditions above, is a global isometry.

The space $\mathcal{GH}_{-1}$ also carries a natural symplectic structure. First, note that the identity component $\mathrm{isom}_0(\mathrm{AdS}^3)$ of the isometry group of AdS^3 is isomorphic to $\mathrm{PSL}_2\mathbb{R} \times \mathrm{PSL}_2\mathbb{R}$. Thus, since the holonomy representation ρ of a GHM AdS metric g on M has values in $\mathrm{isom}_0(\mathrm{AdS}^3)$, it can be decomposed as $\rho = (\rho_L, \rho_R)$, where ρ_L, ρ_R are morphisms from $\pi_1 S$ to $\mathrm{PSL}_2\mathbb{R}$, well-defined up to conjugation. We will call ρ_L and ρ_R the left and right representations of g.

The following result by Mess [3, 32] provides a classification of GHM AdS manifolds in terms of their holonomy representations and can be considered as an analog of the Bers Double Uniformization Theorem.

Theorem 2.9 (Mess). *The representations ρ_L and ρ_R have maximal Euler number, so that they are by* [18] *holonomy representations of hyperbolic structures $m_L, m_R \in \mathcal{T}$. Given $(\rho_L, \rho_R) \in \mathcal{T} \times \overline{\mathcal{T}}$, there is a unique GHM AdS metric $g \in \mathcal{GH}_{-1}$ such that ρ_L and ρ_R are the left and right representations of g.*

As a consequence, we have a homeomorphism $\mathrm{hol}^{\mathrm{AdS}} : \mathcal{GH}_{-1} \to \mathcal{T} \times \overline{\mathcal{T}}$, sending g to (ρ_L, ρ_R). Moreover, $\mathcal{T}$ is equipped with a natural symplectic structure, given by the Weil-Petersson symplectic form ω_{WP}, so that $\mathcal{T} \times \overline{\mathcal{T}}$ is also equipped with a symplectic form $\omega_{WP} \oplus \overline{\omega_{WP}}$. The symplectic structure on $\mathcal{GH}_{-1}$ is then obtained by pull-back of $\omega_{WP} \oplus \overline{\omega_{WP}}$ by $\mathrm{hol}^{\mathrm{AdS}}$.

As we have seen in Section 1 it is possible to identify $\mathcal{GH}_{-1}$ with both $\mathcal{T} \times \mathcal{ML}$ and $\mathcal{T} \times \overline{\mathcal{T}}$ via homeomorphisms $\partial_+^{\mathrm{AdS}} : \mathcal{GH}_{-1} \to \mathcal{T} \times \mathcal{ML}$ and $\mathrm{hol}^{\mathrm{AdS}} : \mathcal{GH}_{-1} \to \mathcal{T} \times \overline{\mathcal{T}}$. This is analogous to the case of hyperbolic ends described above. The $\mathcal{T} \times \mathcal{ML}$ parametrization is obtained by from the upper boundary of the convex core data while the $\mathcal{T} \times \overline{\mathcal{T}}$ parametrization can be obtained by considering any "well-behaved" Cauchy surface, see [26, Lemma 3.16].

Lemma 2.10. *Let Σ be a Cauchy surface in M with principal curvatures everywhere in $(-1, 1)$. Then, up to isotopy,*

$$m_L = I\big((E + JB)\cdot, (E + JB)\cdot\big), \quad m_R = I\big((E - JB)\cdot, (E - JB)\cdot\big),$$

where I and B are the induced metric and shape operator of Σ, respectively, and E is the identity map from $T\Sigma$ to itself.

The relation between the two parametrizations is also analogous to the grafting parametrization of complex projective structures, relevant in the case of hyperbolic end. It is given by the double earthquake of the induced hyperbolic metric on the upper or lower boundary of the convex core along its bending lamination.

Theorem 2.11 (Mess [32]). *Given a pair $(m, l) \in \mathcal{T} \times \mathcal{ML}$ there is a unique GHM AdS manifold (M, g) such that $\partial_+ C(M, g)$ has induced metric given by m and bending lamination given by l. Also, each pair $(m_L, m_R) \in \mathcal{T} \times \overline{\mathcal{T}}$ are the hyperbolic metrics corresponding to the left and right Fuchsian holonomies of a unique GHM AdS manifold. The relation between the pair of hyperbolic metrics and (m_L, m_R) and the pair (m, l) is given by the double earthquake map $\mathcal{E} : \mathcal{T} \times \mathcal{ML} \to \mathcal{T} \times \overline{\mathcal{T}}$ which is a homeomorphism by the Thurston Earthquake Theorem* 1.13.

Note that one could equivalently state the above theorem in terms of the data (m_-, l_-) on the lower boundary of the convex core. The translation between the upper and lower boundary descriptions is summarized in Figure 2.1.

$$\begin{tikzcd}
 & m_+ \arrow[dl, "E^L(l_+)"'] \arrow[dr, "E^R(l_+)"] & \\
m_L & & m_R \\
 & m_- \arrow[ul, "E^R(l_-)"] \arrow[ur, "E^L(l_-)"'] &
\end{tikzcd}$$

Figure 2.1. Relation between the left/right metrics and the boundary of the convex core.

2.6. Convex cores of quasifuchsian and globally hyperbolic manifolds

Now consider a quasifuchsian metric h on M. According to the definition given above, M contains a non-empty, compact, convex subset K. It is easily seen that the intersection of two non-empty convex subsets is also convex, and it follows that M contains a unique smallest non-empty convex subset, called its convex core and denoted here by $C(M, h)$.

In some cases, $C(M, h)$ is a totally geodesic surface S. This happens exactly when M is "Fuchsian", that is, the image of its holonomy representation is conjugate to a subgroup of $\mathrm{PSL}_2\mathbb{R} \subset \mathrm{PSL}_2\mathbb{C}$. Otherwise, when M is non-Fuchsian, $C(M, h)$ has non-empty interior. Its boundary $\partial C(M, h)$ is then the disjoint union of two surfaces S_+ and S_- homeomorphic to S, facing respectively towards the upper and lower asymptotical boundaries $\partial_+ M$ and $\partial_- M$ of M. When M is Fuchsian, we set $S_- = S_+ = S$, the totally geodesic closed surface in M.

Both S_+ and S_- are locally convex surfaces with no extreme points. It follows (see [43]) that their induced metrics m_+ and m_- are hyperbolic, and that they are pleated along measured laminations l_+ and l_-. This associates to $h \in \mathcal{QF}$ a pair of hyperbolic metrics $m_+, m_- \in \mathcal{T}$ and a pair of measured laminations $l_+, l_- \in \mathcal{ML}$. These data are however not independent with, say, the pair (m_-, l_-) on the lower boundary of the convex core being completely determined by the pair (m_+, l_+) on the upper boundary. Thus, restricting our attention to the upper boundary, we obtain a map

$$\partial_+^{\mathrm{Hyp}} : \mathcal{QF} \to \mathcal{T} \times \mathcal{ML}$$

associating to a quasifuchsian metric h the data (m_+, l_+) on S_+.

2.7. Minimal and maximal surfaces

Besides the boundary of the convex core and the conformal boundary/holonomy parametrizations, quasifuchsian and GHM AdS manifolds also admit a minimal/maximal surface parametrizations. Here, in hyperbolic case, we must restrict to a subclass of quasifuchsian manifolds admiting a unique minimal surface, the so called almost-Fuchsian manifolds.

Definition 2.12. A quasifuchsian metric h on M is *almost-Fuchsian* if it contains a closed, embedded minimal surface with principal curvatures in $(-1, 1)$. We denote by $\mathcal{AF}$ the space of almost-Fuchsian metrics on M, considered up to isotopy.

It was noted by Uhlenbeck [46] that almost-Fuchsian manifolds contain only one closed, embedded minimal surface.

For AdS manifolds, there is a deep relationship between maximal surfaces, harmonic maps and minimal Lagrangian maps. A key point is the following lemma due to Ayiama, Akutagawa and Wan [2, Proposition 3.1].

Let g be a GHM AdS metric on M, and let Σ be the (unique) closed space-like maximal surface in (M, g). Let I and $I\!I$ be the induced metric and second fundamental form on Σ, and let m_L, m_R be the left and right hyperbolic metrics on Σ.

Lemma 2.13. *The identity map $f_L : (\Sigma, [I]) \to (\Sigma, m_L)$ (respectively $f_R : (\Sigma, [I]) \to (\Sigma, m_R)$) is harmonic, and the imaginary part of its Hopf differential is equal to $I\!I$ (respectively to $-I\!I$). In particular, $f_R \circ f_L^{-1} : (\Sigma, m_L) \to (\Sigma, m_R)$ is minimal Lagrangian.*

2.8. Globally hyperbolic flat and de Sitter manifolds

The 3-dimensional Minkowski space is defined as the space $\mathbb{R}^{2,1}$ with the flat Lorentzian metric of signature $(2, 1)$.

GHM flat metrics on M are defined in the same manner as in the AdS case described previously, and we denote by $\mathcal{GH}_0$ the moduli spaces of flat GHM metrics on M, considered up to isotopy. We consider only future complete spacetimes, presenting an initial singularity. Past complete spacetimes are obtained by time reversal.

The isometry group $\mathrm{isom}_0(\mathbb{R}^{2,1})$ is isomorphic to a semi-direct product $\mathrm{PSL}_2\mathbb{R} \ltimes \mathfrak{sl}_2\mathbb{R}$. Thus, the holonomy representations of GHM flat manifolds define points in the representation variety $\mathrm{Rep}(\pi_1 S, \mathrm{PSL}_2\mathbb{R} \ltimes \mathfrak{sl}_2\mathbb{R})$. A holonomy representation then decomposes as $\rho = (\rho_0, \tau)$ with linear part $\rho_0 : \pi_1(S) \to \mathrm{PSL}_2\mathbb{R}$ and a ρ_0-cocycle $\tau : \pi_1(S) \to \mathfrak{sl}_2\mathbb{R}$. The following result of Mess [3,32] provides the classification of GHM flat metrics in terms of holonomies.

Theorem 2.14 (Mess). *The linear part ρ_0 of the holonomy representations of a GHM flat metric have maximal Euler number, so that it is the holonomy representations of a hyperbolic structure $h_0 \in \mathcal{T}$. Given $\rho_0 \in \mathcal{T}$ and a ρ_0-cocycle τ, there is a unique future complete GHM Minkowski metric $h \in \mathcal{GH}_0$ such that ρ_0 and τ describes its holonomy representation.*

Adding a coboundary to τ is equivalent to conjugating the representation by a translation. Thus only the cohomology class of τ is relevant. The first cohomology group $H^1(\pi_1 S, \mathfrak{sl}_2\mathbb{R}_{\mathrm{Ad}\rho_0})$ can be seen as the fibre of the cotangent bundle $T^*\mathcal{T}$ over Teichmüller space. In fact, the embedding of $\mathcal{T}$ into the $\mathrm{PSL}_2\mathbb{R}$ representation variety parametrizes the tangent space to $\mathcal{T}$ at ρ_0 by the first cohomology group $H^1(\pi_1 S, \mathfrak{sl}_2\mathbb{R}_{\mathrm{Ad}\rho_0})$ and the non-degenerate cup product can be used as the duality pairing between $T\mathcal{T}$ and $T^*\mathcal{T}$. We thus have a one-to-one correspondence $\mathrm{hol}^{\mathrm{Mink}} : \mathcal{GH}_0 \to T^*\mathcal{T}$ sending h to (ρ_0, τ).

The 3-dimensional de Sitter space is defined as the set

$$dS^3 = \left\{ x \in \mathbb{R}^{3,1} \mid \langle x, x \rangle = 1 \right\}$$

with the induced metric from the 4-dimensional Minkowski metric.

We will denote by $\mathcal{GH}_1$ the moduli spaces of de Sitter GHM metrics on M. Again, we consider only future complete spacetimes.

The isometry group $\mathrm{isom}_0(dS^3)$ is isomorphic to $\mathrm{PSL}_2\mathbb{C}$. The holonomy representations of GHM dS manifolds therefore define points in the character variety $\mathcal{X}$. As for quasifuchsian manifolds, and more generally for hyperbolic ends, the classification of GHM de Sitter spacetimes in terms of holonomies is not possible since the map $\mathrm{hol}^{dS} : \mathcal{GH}_1 \to \mathcal{X}$ is only a local diffeomorphism (importantly it is not injective). However, similarly to hyperbolic ends, de Sitter manifolds can be understood in terms of a complex projective structure at their boundary at future infinity $\partial_+ M$. More precisely, the developing map $\mathrm{dev} : \tilde{M} \to dS^3$ restricts to a developing map dev : $\widetilde{\partial_+ M} \to \partial_+ dS^3 \simeq \mathbb{C}P^1$. The holonomy representation $\rho : \pi_1 S \to \mathrm{PSL}_2\mathbb{C}$ then endows $\partial_+ M$ with a complex projective structure. We denote the map associating to a GHM dS manifold (M, g) the corresponding complex projective structure on $\partial_+ M$ by $\partial_+^{dS} : \mathcal{GH}_1 \to \mathcal{CP}$. A result of Scannell [39] gives the converse construction of GHM dS manifolds given a complex projective structure on S. We thus obtain the following result.

Theorem 2.15 (Scannell). *GHM de Sitter spacetimes are in one-to-one correspondence with complex projective structures.*

We continue to denote by ω_G^i the symplectic form on $\mathcal{GH}_1$ obtained by pull-back of the imaginary part of the Goldman symplectic form on $\mathcal{CP}$.

3. Wick rotations and double maps

In this section we explain the relation between the three- and two-dimensional points of view developed in the introduction. More specifically, we shall see why Theorem 1.2 implies Theorem 1.18, and Theorem 1.3 is equivalent to Theorem 1.11. We then prove that the double earthquake map $\mathcal{E}$ and the double harmonic map $\mathcal{H}$ are one-to-one, leaving the discussion of the regularity properties of the earthquake map for the next section.

3.1. Earthquakes and the boundary of the convex core

Let us start considering the relations between Theorem 1.2 and Theorem 1.18. As we have seen in the introduction, the definition of the Wick rotation between hyperbolic ends and GHM AdS manifolds is given by matching the boundary data at the initial boundary of a hyperbolic ends and at the upper boundary of the convex core of a GHM AdS manifolds

$$W_\partial^{\mathrm{AdS}} = \left(\partial_+^{\mathrm{AdS}}\right)^{-1} \circ \partial_+^{\mathrm{Hyp}} : \mathcal{HE} \to \mathcal{GH}_{-1}.$$

(Recall that the maps $\partial_+^{\mathrm{AdS}}$ and $\partial_+^{\mathrm{Hyp}}$ are defined in Subsection 2.6.)

The motivation behind this definition is quite clear in terms of 3-dimensional geometry. On the other hand, due to the lack of a smooth structure on $\mathcal{T} \times \mathcal{ML}$, it is unclear how to use the Wick rotation $W_\partial^{\mathrm{AdS}}$ to relate the geometric properties of the two moduli spaces. To address this we must describe the Wick rotation in terms of better behaved (smooth) maps.

First note that by Thurston's result, Theorem 2.7, we have a relation between the complex projective data at the asymptotic boundary and the lamination data at the initial boundary of hyperbolic ends given by grafting

$$\partial_\infty^{\mathrm{Hyp}} = \mathcal{G} \circ \partial_+^{\mathrm{Hyp}} : \mathcal{HE} \to \mathcal{CP}\ .$$

The smooth and symplectic structures on $\mathcal{CP}$ can in fact be defined via pull-back the inverse of this map $\partial_\infty^{\mathrm{Hyp}}$. Analogously, by Mess' result, Theorem 2.11, the holonomy mapping can be written in terms of the upper boundary of the convex core in GHM AdS manifolds via the double earthquake map

$$\mathrm{hol}^{\mathrm{AdS}} = \mathcal{E} \circ \partial_+^{\mathrm{AdS}} : \mathcal{GH}_{-1} \to \mathcal{T} \times \overline{\mathcal{T}} \,,$$

with the smooth and symplectic structures on $\mathcal{GH}_{-1}$ also given via pull-back.

On the other hand, the composition $\mathcal{G}' = \mathcal{G} \circ \delta^{-1}$ of the grafting map $\mathcal{G}$ with the inverse of $\delta : \mathcal{T} \times \mathcal{ML} \to T^*\mathcal{T}$, the map sending (m, l) to $d_m L(l)$, is a C^1 symplectomorphism between $(T^*\mathcal{T}, 2\omega_*^r)$ and $(\mathcal{CP}, \omega_G^i)$, see [27]. This motivates us to consider the analogous composition, $\mathcal{E}' = \mathcal{E} \circ \delta^{-1}$, of the double earthquake map $\mathcal{E}$ with δ^{-1}. We then obtain be the diagram in Figure 3.1, which is shown below to be commutative.

$$\begin{tikzcd}
\mathcal{HE} \arrow[rr, "W_\partial^{\mathrm{AdS}}"] \arrow[d, "\partial_\infty^{\mathrm{Hyp}}"'] \arrow[dr, "\partial_+^{\mathrm{Hyp}}"] & & \mathcal{GH}_{-1} \arrow[dl, "\partial_+^{\mathrm{AdS}}"'] \arrow[d, "\mathrm{hol}^{\mathrm{AdS}}"] \\
(\mathcal{CP}, \omega_G^i) & \mathcal{T} \times \mathcal{ML} \arrow[l, "\mathcal{G}"'] \arrow[r, "\mathcal{E}"] \arrow[d, "\delta"] & (\mathcal{T} \times \overline{\mathcal{T}}, \frac{1}{2}(\omega_{WP} \oplus \overline{\omega_{WP}})) \\
& (T^*\mathcal{T}, 2\omega_*^r) \arrow[ul, "\mathcal{G}'"] \arrow[ur, "\mathcal{E}'"'] &
\end{tikzcd}$$

Figure 3.1. Relation between double earthquakes and Wick rotations through pleated surfaces.

Lemma 3.1. *The diagram in Figure* 3.1 *commutes.*

Proof. The commutativity of the upper triangle follows directly from the definition of $W_\partial^{\mathrm{AdS}}$, while the definitions of $\mathcal{G}'$ and $\mathcal{E}'$ provides the commutativity of the two lower triangles. The fact that the middle left triangle commutes is a translation of Thurston's Theorem 2.7, while the middle right triangle commutes by Mess' Theorem 2.11. □

This allows us to write the relation between the Wick rotation and the double earthquake map as

$$W_\partial^{\mathrm{AdS}} = \left(\mathrm{hol}^{\mathrm{AdS}}\right)^{-1} \circ \mathcal{E}' \circ (\mathcal{G}')^{-1} \circ \partial_\infty^{\mathrm{Hyp}} \,.$$

We record the following consequence for future use.

Remark 3.2. $W_\partial^{\mathrm{AdS}}$ is C^1-smooth and symplectic if and only if $\mathcal{E}'$ is C^1-smooth and symplectic.

3.2. Harmonic maps and minimal surfaces

Turning now to the relations between Theorem 1.3 and Theorem 1.11, we shall use a much simpler commutative diagram, see Figure 3.2. From the introduction, the map $W_{\min} : \mathcal{AF}' \to \mathcal{GH}_{-1}$ is defined by matching the holomorphic data of the minimal surface in an almost-Fuchsian manifold and the maximal surface of a GHM AdS manifold. More precisely, we have

$$W_{\min} = \max{}^{-1} \circ \min,$$

where $\min : \mathcal{AF}' \to T^*\mathcal{T}$ (respectively $\max : \mathcal{GH}_{-1} \to T^*\mathcal{T}$) is the map sending an almost-Fuchsian (respectively maximal globally hyperbolic AdS) metric on M to the complex structure and holomorphic quadratic differential determined on its unique minimal (respectively maximal) surface by the first and second fundamental forms.

Considering also the maps $\partial_\infty^{\mathrm{Hyp}} : \mathcal{AF}' \to \mathcal{CP}$ and $\mathrm{hol}^{\mathrm{AdS}} : \mathcal{GH}_{-1} \to \mathcal{T} \times \overline{\mathcal{T}}$ we obtain the diagram Figure 3.2, which commutes as a direct consequence of Lemma 2.13.

$$\begin{tikzcd}
\mathcal{AF}' \arrow[rrr, "W_{\min}"] \arrow[d, "\partial_\infty^{\mathrm{Hyp}}"'] \arrow[dr, "\min"] & & & \mathcal{GH}_{-1} \arrow[d, "\mathrm{hol}^{\mathrm{AdS}}"] \arrow[dl, "\max"'] \\
(\mathcal{CP}, \omega_G^i) \arrow[r, "\alpha"] & (T^*\mathcal{T}, -\omega_*^r) \arrow[rr, "\mathcal{H}"] & & (\mathcal{T} \times \overline{\mathcal{T}}, \frac{1}{2}(\omega_{WP} \oplus \overline{\omega_{WP}}))
\end{tikzcd}$$

Figure 3.2. The minimal surfaces Wick rotation

The map $\alpha = \min \circ (\partial_\infty^{\mathrm{Hyp}})^{-1}$ is symplectic up to sign, see [30, Corollary 5.29].

Theorem 3.3 (Loustau). $Re(\alpha^* \omega_*) = -\omega_G^i$.

We thus have the following remark.

Remark 3.4. $\mathcal{H}$ is symplectic (up to sign) if and only if $W_{\min}$ is symplectic.

Proof. If $\mathcal{H}$ is symplectic (up to sign), then it follows directly from the diagram in Figure 3.2 that $W_{\min}$ is symplectic, because it can be written as a composition of symplectic maps.

For the converse note that both $\mathcal{H}$ and $W_{\min}$ are real analytic. If $W_{\min}$ is symplectic, it follows from the diagram that $\mathcal{H}$ is symplectic on an open subset of $T^*\mathcal{T}$. Since the symplectic forms on both $T^*\mathcal{T}$ and $\mathcal{T} \times \overline{\mathcal{T}}$ are analytic, it follows that $\mathcal{H}$ is symplectic everywhere. □

3.3. The double maps are one-to-one and onto

This part contains (simple) proofs that the double earthquake map and the double harmonic map are one-to-one.

Lemma 3.5. *The map $\mathcal{H} : T^*\mathcal{T} \to \mathcal{T} \times \overline{\mathcal{T}}$ is bijective.*

Proof. Let $(m_L, m_R) \in \mathcal{T} \times \overline{\mathcal{T}}$. There is then a unique minimal Lagrangian diffeomorphism isotopic to the identity ϕ from (S, m_L) to (S, m_R), see [28, Corollaire 2.3.4] or [41]. If we define $m = m_L + \phi^*(m_R)$ and denote by $[m]$ its underlying conformal structure, then $\mathrm{id} : (S, c) \to (S, m_L)$ and $\phi : (S, c) \to (S, m_R)$ are harmonic with opposite Hopf differentials $-iq$ and iq. Therefore, $(m_L, m_R) = \mathcal{H}(c, q)$, where c is the complex structure on S associated to $[m]$. So $\mathcal{H}$ is onto.

Conversely, let $(m_L, m_R) \in \mathcal{T} \times \overline{\mathcal{T}}$, and let $(c, q) \in T^*\mathcal{T}$ be such that $(m_L, m_R) = \mathcal{H}(c, q)$. Then $c = [m_L + \phi^*(m_R)]$, where ϕ is the unique minimal Lagrangian diffeomorphism isotopic to the identity from (S, m_L) to (S, m_R). Moreover, the Hopf differential of the unique harmonic map isotopic to the identity from (S, c) to (S, m_L) is equal to $-iq$. This shows that (c, q) is uniquely determined by (m_L, m_R), and therefore proves that $\mathcal{H}$ is injective. (Note that another equivalent proof is obtained by noting that c must be the conformal class on the unique minimal surface in $(S \times S, m_L \oplus m_R)$ with projections on both factors diffeomorphisms, and $-iq$ must be the Hopf differential of the projection on the first factor.) □

Lemma 3.6. *The double earthquake map $\mathcal{E} : \mathcal{T} \times \mathcal{ML} \to \mathcal{T} \times \overline{\mathcal{T}}$ is bijective.*

Proof. Let $(m_L, m_R) \in \mathcal{T} \times \overline{\mathcal{T}}$. By Thurston's Earthquake Theorem (see the appendix in [24]) there exists a unique $l \in \mathcal{ML}$ such that $m_L = E^L(m_R, 2l)$. But $E^R(l) = E^L(l)^{-1}$ and $E^L(2l) = E^L(l)^2$. So, if we set $m = E^L(m_R, l)$, we have

$$m_L = E^L(m, l), \quad m_R = E^R(m, l),$$

so that $(m_L, m_R) = \mathcal{E}(m, l)$.

Conversely, if $(m_L, m_R) = \mathcal{E}(m', l')$, then $m_L = E^L(m_R, 2l')$, so it follows from the uniqueness in the Earthquake Theorem that $2l' = 2l$, so that $l = l'$ and $m = m'$. □

3.4. Wick rotations to flat and dS manifolds

We now consider analogous Wick rotations from hyperbolic ends to GHM flat and de Sitter manifolds.

3.4.1. *Hyperbolic metrics and measured laminations*

In analogy to the AdS case, we consider Wick rotations from hyperbolic ends to GHM flat manifolds $W_\partial^{\mathrm{Mink}} : \mathcal{HE} \to \mathcal{GH}_0$ given by matching the data at the inital boundary of hyperbolic ends to the pair formed by the linear holonomy and the measured lamination dual to the initial singularity of GHM flat manifolds

$$W_\partial^{\mathrm{Mink}} : \left(\partial_*^{\mathrm{Mink}}\right)^{-1} \circ \partial_+^{\mathrm{Hyp}} .$$

Again, using the fact that the cocycle part of the holonomy is related to the measured lamination via grafting, we may write

$$\mathrm{hol}^{\mathrm{Mink}} = \mathcal{G}_0 \circ \partial_*^{\mathrm{Mink}} : \mathcal{GH}_0 \to T^*\mathcal{T} \ .$$

The smooth and symplectic structures on $\mathcal{GH}_0$ are again given via pull-back. We now obtain the first diagram in Figure 3.3, where we denote $\mathcal{G}_0' = \mathcal{G}_0 \circ \delta^{-1}$.

The passage $W_\partial^{dS} : \mathcal{HE} \to \mathcal{GH}_1$ from hyperbolic ends to GHM dS manifolds is given automatically via duality, by matching the data at their common asymptotic boundary

$$W_\partial^{dS} = (\partial_\infty^{dS})^{-1} \circ \partial_\infty^{\mathrm{Hyp}} \ .$$

Here there is no problem with differentiability and the symplectic structures agree, since in both cases the smooth and symplectic structures are again given via pull-back from $\mathcal{CP}$. The second diagram in Figure 3.3 describe these relations.

$$\begin{tikzcd}
\mathcal{HE} \arrow[rr, "W_\partial^{\mathrm{Mink}}"] \arrow[d, "\partial_\infty^{\mathrm{Hyp}}"'] \arrow[dr, "\partial_+^{\mathrm{Hyp}}"] & & \mathcal{GH}_0 \arrow[dl, "\partial_*^{\mathrm{Mink}}"'] \arrow[d, "\mathrm{hol}^{\mathrm{Mink}}"] \\
(\mathcal{CP}, \omega_G^i) & \mathcal{T} \times \mathcal{ML} \arrow[l, "\mathcal{G}"'] \arrow[r, "\mathcal{G}_0"] \arrow[d, "\delta"] & (T^*\mathcal{T}, \omega_*^r) \\
& (T^*\mathcal{T}, 2\omega_*^r) \arrow[ul, "\mathcal{G}'"] \arrow[ur, "\mathcal{G}_0'"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{HE} \arrow[rr, "W_\partial^{dS}"] \arrow[d, "\partial_+^{\mathrm{Hyp}}"'] \arrow[dr, "\partial_\infty^{\mathrm{Hyp}}"] & & \mathcal{GH}_1 \arrow[dl, "\partial_\infty^{dS}"'] \\
\mathcal{T} \times \mathcal{ML} \arrow[r, "\mathcal{G}"] & (\mathcal{CP}, \omega_G^i) &
\end{tikzcd}$$

Figure 3.3. Wick rotations to flat and de Sitter manifolds.

Note that the diagrams in Figure 3.3 commute, by definition of the some of the maps used, as well as by Theorem 2.7 (for the middle left triangle of the left diagram and the lower triangle of the right diagram).

3.4.2. *CMC surfaces*

GHMC flat and de Sitter manifolds are also shown to admit a unique foliation by CMC surfaces.

Theorem 3.7 (Barbot, Béguin, Zeghib [5]). *Any GHM flat and dS manifolds admit a unique foliation by closed space-like CMC surfaces, with mean curvature in*

- $(-\infty, 0)$*, in the flat case,*
- $(-\infty, -1)$*, in the dS case.*

For every prescribed H as above, the spacetimes contain a unique closed space-like CMC-H surface.

As in the AdS case, the first and second fundamental forms of the CMC-H surface are in correspondence with a point in $T^*\mathcal{T}$ (see [33] and [26, Lemma 6.1]).

Proposition 3.8. *Let $H \in (-\infty, -1)$. Given a complex structure c and a holomorphic quadratic differential q for c on S, there is a unique GHM dS metric h on M such that the induced metric and traceless part of the second fundamental form on the unique CMC-H surface in (M, h) is I, $I\!I_0$ with I compatible with c and $I\!I_0 = Re(q)$.*

We may therefore construct as a version of the flat and de Sitter CMC-Wick rotation.

Definition 3.9. Let $H \in (-1, 1)$, $H' \in (-\infty, 0)$ and $H'' \in (-\infty, -1)$. For each $h \in \mathcal{AF}'$, let S_H be the unique closed CMC-H surface in (M, h), let c be the conformal class of its induced metric, and let q be the traceless part of its second fundamental form. There is then a unique GHM flat metric h' and a unique GHM dS metric h'' on M such that the (unique) CMC-H' surface in (M, h') and the unique CMC-H'' surface in (M, h'') have induced metric conformal to c and the traceless part of its second fundamental form is equal to q. We denote these maps respectively by $W^{\mathrm{Mink}}_{H,H'} : \mathcal{AF}' \to \mathcal{GH}_0$ and $W^{dS}_{H,H''} : \mathcal{AF}' \to \mathcal{GH}_1$.

4. Regularity of the earthquake map

We now focus on the C^1 regularity of the earthquake map, more specifically on the proof of Proposition 1.16 and of Corollary 1.17. The notations here are similar to those of [27, Section 2.5], with the relevant adaptations, further developing some of the arguments which in [27] were too elliptic. As in [27], the arguments will be based on the ideas and tools developed by Bonahon [7,8].

4.1. Maximal laminations and transverse cocycles

We first recall basic facts on transverse cocycles on a surface, which will be used to give a parametrization of both the Teichmüller space $\mathcal{T}$ and the space of measured geodesic laminations $\mathcal{ML}$, see [7].

We start with a fixed reference hyperbolic structure $m \in \mathcal{T}$ on S and a maximal geodesic lamination $\lambda \in \mathcal{L}$ on (S, m). The maximality condition here is given with respect to inclusion. Equivalently, this condition can be stated as the property that the complement of λ on S is given by finitely many disjoint ideal triangles, see [7].

Definition 4.1. A $\mathbb{R}$-*valued transverse cocycle* σ for a lamination λ is a real valued function on arcs transverse to λ which is

- additive: $\sigma(k_1 \sqcup k_2) = \sigma(k_1) + \sigma(k_2)$,
- λ-invariant: $\sigma(k_1) = \sigma(k_2)$ if k_1 and k_2 are homotopic through a family of arcs transverse to λ.

We denote $\mathcal{H}(\lambda, \mathbb{R})$ the space of all transverse cocycles for λ.

The space $\mathcal{H}(\lambda, \mathbb{R})$ has the structure of a finite dimensional vector space. In particular, if λ is a maximal lamination, its dimension is given by $\dim \mathcal{H}(\lambda, \mathbb{R}) = 6g - 6$.

Note that the notion of transverse cocycles on maximal laminations generalizes the notion of measured laminations. In fact, the support of any measured lamination $l \in \mathcal{ML}$ is contained (possibly non-uniquely) into a maximal lamination λ on S. Further, given such maximal lamination λ containing the support of l, the transverse measure of l defines uniquely a non-negative transverse cocycle μ on λ. Thus any measured lamination gives rise to a non-negative transverse cocycle on some maximal lamination on S. Conversely, a non-negative transverse cocycle can be equally seen as a transverse measure on the maximal lamination, thus defining a measured lamination. This gives a 1-to-1 correspondence between $\mathcal{ML}\big|_\lambda$, the space of measured laminations supported on λ, and $\mathcal{H}(\lambda, \mathbb{R}_+)$, the space of non-negative transverse cocycles on λ.

It is also possible to give a parametrization the Teichmüller space in terms of transverse cocycles. Given a maximal lamination λ on S, Bonahon [7] defines for each hyperbolic metric $m \in \mathcal{T}$ a transverse cocycle $\sigma_m \in \mathcal{H}(\lambda, \mathbb{R})$, assigning to each transverse arc k to λ a real number $\sigma_m(k)$ which we now define. Let $\tilde{\lambda}$ be the preimage of λ in the universal cover $\tilde{S}$ of S. The maximality condition for λ then implies that $\tilde{\lambda}$ determines a tessellation of $\tilde{S}$ by ideal triangles. For any pair P, Q of such ideal triangles we associate a real number σ_{PQ} as follows. Assuming, first, that P and Q are adjacent, we take σ_{PQ} to be the logarithm of the cross-ratio of the ideal quadrilateral defined by P and Q. Equivalently, σ_{PQ} is the signed hyperbolic distance along their common edge between the orthogonal projections of the opposite vertices to this edge. For non-adjacent ideal triangles P, Q we then define σ_{PQ} as the sum of $\sigma_{P'Q'}$ over all pairs of adjacent ideal triangles P', Q' between P and Q. Note that such sum may be an infinte sum. However, an upper bound for each of the $\sigma_{P'Q'}$, given by the distance between their outermost edges [7], implies that σ_{PQ} differs from the distance between the innermost edges of P and Q only by a finite constant, so that σ_{PQ} is indeed well defined.

The transverse cocycle $\sigma_m \in \mathcal{H}(\lambda, \mathbb{R})$ associated to the hyperbolic metric $m \in \mathcal{T}$ can now be defined. Given a transverse arc k to λ let $\tilde{k}$ be a lift of k to $\tilde{S}$. By transversality the endpoints of $\tilde{k}$ belong to the interior of ideal triangles P and Q and we can define $\sigma_m(k) = \sigma_{PQ}$.

Theorem 4.2 (Bonahon [7]). *The map $\varphi_\lambda : \mathcal{T} \to \mathcal{H}(\lambda, \mathbb{R})$ defined by*

$$\varphi_\lambda(m) = \sigma_m$$

is injective and open. Furthermore, it is real analytic into its image.

4.2. Smoothness of the double earthquake

4.2.1. *Differentiability*

We now turn to the C^1-smoothness of the double earthquake map $\mathcal{E}' = \mathcal{E} \circ \delta^{-1} : T^*\mathcal{T} \to \mathcal{T} \times \overline{\mathcal{T}}$, starting with the differentiability of $E^L \circ \delta^{-1}$. The strategy here is the same as in [27] showing that for each maximal lamination λ there is a pair of tangentiable maps $\Phi_\lambda : \mathcal{T} \times \mathcal{H}(\lambda, \mathbb{R}_+) \to \mathcal{T}$ and $\Psi_\lambda : \mathcal{T} \times \mathcal{H}(\lambda, \mathbb{R}_+) \to T^*\mathcal{T}$ such that

- The composition $\Phi_\lambda \circ \Psi_\lambda^{-1}$ agrees with $E^L \circ \delta^{-1}$ on $\delta(\mathcal{T} \times \mathcal{ML}|_\lambda) \subset T^*\mathcal{T}$;
- For two maximal laminations, λ and λ', the tangent maps of $\Phi_\lambda \circ \Psi_\lambda^{-1}$ and $\Phi_{\lambda'} \circ \Psi_{\lambda'}^{-1}$ agree on $T_{(m,u)}T^*\mathcal{T}$ for all $(m, u) \in \delta(\mathcal{T} \times \mathcal{ML}|_\lambda \cap \mathcal{ML}|_{\lambda'})$.

Start by noting that given a maximal lamination λ the notion of length of measured laminations and of earthquakes along measured laminations naturally extend to notions of length of transverse cocycles and shearings along transverse cocycles [7]. Further, such extensions are well behaved under the vector space structure of $\mathcal{H}(\lambda, \mathbb{R})$ in that the length function $L : \mathcal{T} \times \mathcal{H}(\lambda, \mathbb{R}_+) \to \mathbb{R}$ is linear in its second argument and the shear map $E : \mathcal{T} \times \mathcal{H}(\lambda, \mathbb{R}_+) \to \mathcal{T}$ satisfies the following equivariance property

$$E_{\sigma+\sigma'}(m) = E_\sigma \circ E_{\sigma'}(m).$$

It is thus natural to consider the following tangentiable maps

$$\Phi_\lambda(m, \sigma) = E_\sigma(m), \qquad \Psi_\lambda(m, \sigma) = d_m L(\sigma).$$

Given $m \in \mathcal{T}$ and $u \in T_m^*\mathcal{T}$ let $(m, l) = \delta^{-1}(m, u) \in \mathcal{T} \times \mathcal{ML}$ denote the image of (m, u) under the inverse of δ. Then, choose a maximal lamination λ containing the support of l and let $\sigma \in \mathcal{H}(\lambda, \mathbb{R}_+)$ denote the positive transverse cocycle corresponding to the measure of l. It follows directly from the definitions of length and shears that

$$\Phi_\lambda \circ \Psi_\lambda^{-1}(m, u) = \Phi_\lambda(m, \sigma) = E^L(m, l) = E^L \circ \delta^{-1}(m, u).$$

Further, from the equivariance of $E_\sigma(m)$ and the linearity of $L_m(\sigma)$, we can easily compute

$$\begin{aligned} d_{(m,\sigma)}\Phi_\lambda(\dot{m}, \dot{\sigma}) &= \frac{d}{dt}\Big|_{t=0^+} E_{t d_m\varphi_\lambda(\dot{m})} \circ E_{t\dot{\sigma}} \circ E_\sigma(m) \\ &= (e_{d_m\varphi_\lambda(\dot{m})} + e_{\dot{\sigma}})(E_\sigma(m)) = d_m E_\sigma(e_{d_m\varphi_\lambda(\dot{m})}(m) + e_{\dot{\sigma}}(m)), \end{aligned}$$

where $e_\sigma(m) \in T_m\mathcal{T}$ is the infinitesimal shearing vector at m determined by σ, and

$$d_{(m,\sigma)}\Psi_\lambda(0, \dot{\sigma}) = \frac{d}{dt}\Big|_{t=0^+} d_m L(t\dot{\sigma} + \sigma) = d_m L(\dot{\sigma}) = e_{\dot{\sigma}}^*(m),$$

where $*$ means the duality between $T_m^*\mathcal{T}$ and $T_m\mathcal{T}$ with respect to the Weil-Petersson symplectic form. Note that here $d\Phi_\lambda$ and $d\Psi_\lambda$ denote the tangent maps of Φ_λ and Ψ_λ and not their differentials.

To compute the differential of $\Phi_\lambda \circ \Psi_\lambda^{-1}$ we introduce a decomposition of the tangent space to $T^*\mathcal{T}$ at (m, u) into horizontal and vertical subspaces

$$T_{(m,u)}T^*\mathcal{T} = H_{(m,u)}T^*\mathcal{T} \oplus V_{(m,u)}T^*\mathcal{T}.$$

First note that the map δ evaluated at a fixed measured lamination l determines a section $s_l = \delta(\cdot, l) : \mathcal{T} \to T^*\mathcal{T}$ of the cotangent bundle over $\mathcal{T}$. This is in fact a smooth section since the Hessian of the length function of l depends continuously on both m and l, as follows for instance from [49, Theorem 1.1]. We can then define the horizontal and vertical subspaces as

$$H_{(m,u)}T^*\mathcal{T} = \{U^h = d_m s_l(\dot{m});\ \dot{m} \in T_m\mathcal{T}\}, \quad V_{(m,u)}T^*\mathcal{T} = \{U^v = \dot{u};\ \dot{u} \in T^*_m\mathcal{T}\}\ .$$

(Note that $V_{(m,u)}T^*\mathcal{T}$ does not appear to correspond to the vertical space defined by the Levi-Civita connection of the Weil-Petersson metric on $\mathcal{T}$.)

A simple computation now gives for a horizontal vector $U^h \in H_{(m,u)}T^*\mathcal{T}$

$$\begin{aligned} d_{(m,u)}(\Phi_\lambda \circ \Psi_\lambda^{-1})(U^h) &= \frac{d}{dt}\Big[\Phi_\lambda \circ \Psi_\lambda^{-1} \circ s_l \circ \pi(m(t), u(t))\Big] \\ &= \frac{d}{dt}\Big[\Phi_\lambda \circ \Psi_\lambda^{-1} \circ s_l(m(t))\Big] \\ &= d_m(\Phi_\lambda \circ \Psi_\lambda^{-1} \circ s_l)(\dot{m}) \\ &= d_m E_\sigma(\dot{m}) = d_m E_l^L(\dot{m}), \end{aligned}$$

with $\dot{m} = d_{(m,u)}\pi(U^h)$, and for a vertical vector $U^v \in V_{(m,u)}T^*\mathcal{T}$

$$\begin{aligned} d_{(m,u)}(\Phi_\lambda \circ \Psi_\lambda^{-1})(U^v) &= \frac{d}{dt}\Big[\Phi_\lambda \circ \Psi_\lambda^{-1}(m, u(t))\Big] = \frac{d}{dt}\Big[E_{\Psi_\lambda^{-1}(m,u(t))}(m)\Big] \\ &= d_m E_\sigma(e_{\dot{\sigma}}(m)) = d_m E_\sigma(\dot{u}^*) = d_m E_l^L(\dot{u}^*), \end{aligned}$$

with $\dot{u} = U^v$ and $\dot{\sigma} = d_{(m,u)}(pr_2 \circ \Psi_\lambda^{-1})(\dot{u})$. This shows in particular that $d(\Phi_\lambda \circ \Psi_\lambda^{-1})$ does not depend on λ, since the right-hand sides of both equations are completely independent on its choice, implying that $E^L \circ \delta^{-1}$ is differentiable at each point $(m, u) \in T^*\mathcal{T}$ with

$$d_{(m,u)}(E^L \circ \delta^{-1})(U) = d_m E_l^L(\dot{m} + \dot{u}^*)\ . \tag{4.1}$$

4.2.2. *Continuity of the differential*

To complete the argument, it now only remains to show that the differential of $E^L \circ \delta^{-1}$ is continous. Let $\alpha_{(m,l)} : T_{(m,u)}T^*\mathcal{T} \to T^*_m\mathcal{T}$ denote the projection onto the vertical subspace of $T_{(m,u)}T^*\mathcal{T}$, sending U to $\dot{u}$. To prove that $E \circ \delta^{-1}$ is C^1, it is sufficient to prove that $\alpha_{(m,l)}$ vary continuously with (m, l), since all other maps entering the right-hand side of (4.1) are clearly smooth by [25] and the analyticity of the Weyl-Petersson symplectic form.

On the other hand, the decomposition of $T_{(m,u)}T^*\mathcal{T}$ into horizontal and vertical subspaces then allows us to explicitly write $\alpha_{(m,l)}$ as

$$\alpha_{(m,l)} = \mathrm{id} - d_m s_l \circ d_{(m,u)}\pi,$$

where id is the identity map in $T_{(m,u)}T^*\mathcal{T}$, $d_m s_l$ denote the linear horizontal embedding of $T_m\mathcal{T}$ into $T_{(m,u)}T^*\mathcal{T}$ and $d_{(m,u)}\pi$ the natural projection of $T_{(m,u)}T^*\mathcal{T}$ onto $T_m\mathcal{T}$. So $\alpha_{(m,l)}$ depends continuously on (m,l) and this concludes the proof of Proposition 1.16, that $E^L \circ \delta^{-1}$ is C^1-smooth.

4.2.3. *Proof of Corollary* 1.17

The C^1-smoothness of $E^R \circ \delta^{-1}$ is proven analogously. Thus Proposition 1.16 implies that $\mathcal{E} \circ \delta^{-1}$ is C^1.

The map $\mathcal{E} : \mathcal{T} \times \mathcal{ML} \to \mathcal{T} \times \overline{\mathcal{T}}$ is clearly a bijection, because a GHM AdS manifold is uniquely determined by the induced metric and measured pleating lamination on the upper boundary of the convex core, and any hyperbolic metric and pleating lamination can be realized in this way. The map $\delta : \mathcal{T} \times \mathcal{ML} \to T^*\mathcal{T}$ is also bijective, see [27]. So $\mathcal{E} \circ \delta^{-1}$ is bijective.

It remains to prove that the differential of $\mathcal{E} \circ \delta^{-1}$ is everywhere invertible. This can be done directly from (4.1) and the corresponding expression for the differential of $E^R \circ \delta^{-1}$

$$d_{(m,u)}\left(E^R \circ \delta^{-1}\right)(U) = d_m E^R_l\left(\dot{m} - \dot{u}^*\right).$$

The extra minus sign on the RHS comes from writing right-earthquakes as the inverse of left-earthquakes, which in terms of shearing corresponds to considering the negative transverse cocycles. Thus, writing $(m_+, m_-) = \mathcal{E} \circ \delta^{-1}(m,u)$, we have

$$\begin{aligned} d_{(m,u)}(\mathcal{E} \circ \delta^{-1})(U) &= \Big(d_m E^L_l(\dot{m} + \dot{u}^*), d_m E^R_l(\dot{m} - \dot{u}^*)\Big) \\ &= (\dot{m}_+, \dot{m}_-) \in T_{(m_+,m_-)}\mathcal{T} \times \overline{\mathcal{T}}, \end{aligned}$$

and, after some simple algebra, we can solve for $(\dot{m}, \dot{u}) \in T_m\mathcal{T} \times T^*_m\mathcal{T}$ in terms of $(\dot{m}_+, \dot{m}_-) \in T_{(m_+,m_-)}\mathcal{T} \times \overline{\mathcal{T}}$:

$$\dot{m} = \frac{1}{2}\Big(d_m E^R_l(\dot{m}_+) + d_m E^L_l(\dot{m}_-)\Big), \quad \dot{u} = \frac{1}{2}\Big(d_m E^R_l(\dot{m}_+) - d_m E^L_l(\dot{m}_-)\Big)^*.$$

5. Double maps are symplectic

In this section we provide proofs for the symplecticity of the double earthquake and double harmonic maps, Theorem 1.11 and Theorem 1.18.

5.1. Train Tracks and the Thurston intersection form

We start by recalling here another set of tools that will be needed in the next part of this section. More details can be found, *e.g.*, in [35] and [42].

First let us introduce the notion of a train track carrying a lamination. A train track T on the surface S is a (regular) tubular neighborhood of an embedded smooth graph with at least 2-valent vertices. We shall consider only generic train tracks with only 3-valent vertices. The edges of T meet tangentially at vertices and, therefore, we may divide edges incident to a given vertex as incoming or outgoing according to the relative direction of their tangent vectors. We denote by e_v the incoming edge and by e_v^+, e_v^- the outgoing edges of a vertex v, where the $+$ and $-$ signs denote the order of the outgoing edges with respect to the incoming one given by a fixed choice of orientation of the surface.

An edge weight system for T is a map $a : E(T) \to \mathbb{R}$ assigning a weight $a(e) \in \mathbb{R}$ to each edge $e \in E(T)$ and satisfying the switch relation

$$a(e_v) = a(e_v^+) + a(e_v^-)$$

for each vertex $v \in V(T)$. We denote by $\mathcal{W}(T)$ the vector space of edge weight systems for T.

A lamination λ is said to be carried by a train track T if it is contained in its interior in such a way that the leaves of λ are transverse to the normal fibers of T. In the particular case of a maximal lamination λ, there is a 1-to-1 correspondence between transverse cocycles $\sigma \in \mathcal{H}(\lambda, \mathbb{R})$ and edge weight systems $a \in \mathcal{W}(T)$ obtained by assigning to each edge $e \in E(T)$ the weight

$$a(e) = \sigma(k_e)$$

where k_e is any normal fibre of T, see [42]. The swich relation is automatically satisfied due to the additivity of σ. We thus obtain a map $\mathcal{H}(\lambda, \mathbb{R}) \to \mathcal{W}(T)$ which is shown to be an isomorphism of vector spaces.

The Thurston intersection form on $\mathcal{H}(\lambda, \mathbb{R})$ defined by

$$\Omega_{\mathrm{Th}} = \sum_{v \in V(T)} da(e_v^+) \wedge da(e_v^-).$$

More precisely, given $\sigma, \sigma' \in \mathcal{H}(\lambda, \mathbb{R})$, let $a, a' \in \mathcal{W}(T)$ be the corresponding edge weight systems. Then

$$\Omega_{\mathrm{Th}}(\sigma, \sigma') = \sum_{v \in V(T)} \Big(a(e_v^+) a'(e_v^-) - a'(e_v^+) a(e_v^-) \Big).$$

This gives a non-degenerate 2-form on $\mathcal{H}(\lambda, \mathbb{R})$ which is closely related with the m-length of transverse cocycles, see [7]. Namely, given a hyperbolic metric m and σ a transverse cocycle, the m-length of σ can be computed as value of the Thurston intersection between σ_m and σ

$$L_m(\sigma) = -\Omega_{\mathrm{Th}}(\sigma_m, \sigma).$$

The main reason we consider Thurston's intersection form is due to its relation with the Weil-Petersson symplectic form.

Theorem 5.1 (Bonahon-Sözen [42]). *The map $\varphi_\lambda : (\mathcal{T}, \omega_{WP}) \to (\mathcal{H}(\lambda, \mathbb{R}), \Omega_{Th})$ is symplectic up to a sign*

$$\varphi_\lambda^* \Omega_{Th} = -\omega_{WP}.$$

Similarly, the canonical cotangent bundle symplectic structure on $T^*\mathcal{T}$ can also be related with Thurston's intersection form. First, note that the map $\varphi_\lambda : \mathcal{T}(S) \to \mathcal{H}(\lambda, \mathbb{R})$ naturally identifies the cotangent space to $\mathcal{T}(S)$ at m with the cotangent space to $\mathcal{H}(\lambda, \mathbb{R})$ at σ_m which, furthermore, is just the dual space $\mathcal{H}(\lambda, \mathbb{R})^*$ to $\mathcal{H}(\lambda, \mathbb{R})$:

$$T^*_m\mathcal{T}(S) = T^*_{\sigma_m}\mathcal{H}(\lambda, \mathbb{R}) = \mathcal{H}(\lambda, \mathbb{R})^*.$$

The total space of the cotangent bundle $T^*\mathcal{T}(S)$ over $\mathcal{T}(S)$ is then identified with a subset of $\mathcal{H}(\lambda, \mathbb{R}) \times \mathcal{H}(\lambda, \mathbb{R})^*$ by

$$\big(\varphi_\lambda, (\varphi_\lambda^{-1})^*\big) : (m, u) \mapsto \big(\varphi_\lambda(m), (\varphi_\lambda^{-1})^* u\big) = \big(\sigma_m, \sigma_u^*\big).$$

Using the Thurston intersection form we may further identify the dual space $\mathcal{H}(\lambda, \mathbb{R})^*$ with $\mathcal{H}(\lambda, \mathbb{R})$ via

$$\sigma \mapsto \sigma^* = \Omega_{\mathrm{Th}}(\,\cdot\,, \sigma)$$

so the symplectic form on $\mathcal{H}(\lambda, \mathbb{R}) \times \mathcal{H}(\lambda, \mathbb{R})^*$ can be written as

$$\Omega_*\big((\sigma_1, \tau_1^*), (\sigma_2, \tau_2^*)\big) = \Omega_{\mathrm{Th}}(\tau_1, \sigma_2) - \Omega_{\mathrm{Th}}(\tau_2, \sigma_1).$$

Proposition 5.2. *The map* $(\varphi_\lambda, (\varphi_\lambda^{-1})^*) : (T^*\mathcal{T}, \omega_*^r) \to (\mathcal{H}(\lambda, \mathbb{R}) \times \mathcal{H}(\lambda, \mathbb{R})^*, \Omega_*)$ *is a symplectomorphism*

$$\big(\varphi_\lambda, (\varphi_\lambda^{-1})^*\big)^*\Omega_* = 2\omega_*^r\,.$$

Proof. We only need to compare the canonical Liouville 1-forms θ on $T^*\mathcal{T}(S)$ and Θ on $\mathcal{H}(\lambda, \mathbb{R}) \times \mathcal{H}(\lambda, \mathbb{R})^*$

$$\theta_{(m,u)}(U) = u(\pi_* U), \quad \Theta_{(\sigma,\tau^*)}(\rho, \chi^*) = \tau^*(\rho).$$

Pulling back Θ by $(\varphi_\lambda, (\varphi_\lambda^{-1})^*)$ gives

$$\Big(\big(\varphi_\lambda, (\varphi_\lambda^{-1})^*\big)^*\Theta\Big)_{(m,u)}(U) = \big((\varphi_\lambda^{-1})^* u\big)\big((\varphi_\lambda)_* \circ \pi_* U\big) = \theta_{(m,u)}(U).$$

Thus

$$\big(\varphi_\lambda, (\varphi_\lambda^{-1})^*\big)^*\Omega_* = \big(\varphi_\lambda, (\varphi_\lambda^{-1})^*\big)^* d\Theta = d\theta = 2\omega_*^r\,. \qquad \square$$

5.2. The double earthquake map is symplectic

We now provide a proof that the double earthquake map $\mathcal{E}'$ is symplectic, up to a multiplicative factor, Theorem 1.18. First we need a description of earthquakes along measured laminations in terms of transverse cocycles for maximal laminations.

Thus, given $(m, l) \in \mathcal{T} \times \mathcal{ML}$ let $m' = E_L(m, l)$ denote the left earthquake of m along l and let λ be a maximal lamination on S containing the support of l. Denote by $\sigma = \sigma_m$ the transverse cocycles associated with m and by τ the transverse measure of l. We now compute the transverse cocycle $\sigma' = \sigma_{m'}$ corresponding to m'. Let us fix a transverse arc k to λ. Let $\tilde{k}$ be a lift of k to the universal cover of S. By transversality, the endpoints of $\tilde{k}$ lay in the interior of triangles P, Q in the triangulation of $\tilde{S}$ determined by the complement $\tilde{S} \backslash \tilde{\lambda}$ of the preimage $\tilde{\lambda}$ of λ. We only need to consider the case where P and Q are adjacent since for non-adjacent triangles the cocycles are obtained as the sum of cocycles of the intermediate pairs of triangles. The construction of the transverse cocycle associated with a hyperbolic metric is given by orthogonally projecting the third vertex of P and Q to their common edge and computing the signed hyperbolic distance between the obtained pair of points (equivalently, this is given by the logarithm of the cross-ratio of the ideal square determined by P and Q). The action of the earthquake $E_L(l)$, as viewed from P, is then to shift the projected point from Q by τ. Therefore, the transformation of the PQ-cocycle is

$$\sigma_{PQ} \mapsto \sigma'_{PQ} = \sigma_{PQ} + \tau_{PQ}$$

where τ_{PQ} is the measure of any arc transversally intersecting $\tilde{\lambda}$ a unique time at the common edge of P and Q. If P and Q are non-adjacent, the formula

$$\sigma_{PQ} \mapsto \sigma'_{PQ} = \sigma_{PQ} + \tau_{PQ}$$

is still valid, where now σ_{PQ}, τ_{PQ} are given by the sum (possibly with an infinite number of terms) over intermediate pairs of triangles. The measure of the transverse arc k is then given by

$$\sigma_{m'}(k) = \sigma_m(k) + \tau(k)$$

and we see that the transverse cocycles of m and m' are related by

$$\sigma_{m'} = \sigma_m + \tau.$$

Proof of Theorem 1.18. From the discussion above, we may write the double earthquake map $\mathcal{E} : \mathcal{T} \times \mathcal{ML} \to \mathcal{T} \times \overline{\mathcal{T}}$ in terms of transverse cocycles for λ as

$$\mathcal{E}_\lambda(\sigma, \tau) = (\varphi_\lambda, \varphi_\lambda) \circ \mathcal{E} \circ (\varphi_\lambda^{-1}, \iota_\lambda)(\sigma, \tau) = (\sigma + \tau, \sigma - \tau).$$

Here we denote by $\iota_\lambda : \mathcal{H}(\lambda, \mathbb{R}_+) \to \mathcal{ML}$ the map assigning to a non-negative transverse cocycle τ the measured lamination with support λ and transverse measure τ.

On the other hand by the relation between the m-length of measured laminations and Thurston's intersection form recalled above,

$$L_m(l) = -\Omega_{\mathrm{Th}}(\sigma_m, \sigma_l)\ ,$$

we may also describe the inverse of the map $\delta : \mathcal{T} \times \mathcal{ML} \to T^*\mathcal{T}$ in terms of cocycles by

$$\begin{aligned}\delta_\lambda(\sigma, \tau) &= (\varphi_\lambda, (\varphi_\lambda^{-1})^*) \circ \delta \circ (\varphi_\lambda^{-1}, \iota_\lambda)(\sigma, \tau) \\ &= (\varphi_\lambda, (\varphi_\lambda^{-1})^*)(\varphi_\lambda^{-1}\sigma, -\tau^* \circ (\varphi_\lambda)_*) = (\sigma, -\tau^*).\end{aligned}$$

Thus the double earthquake map $\mathcal{E}' : T^*\mathcal{T} \to \mathcal{T} \times \overline{\mathcal{T}}$ can be realized by

$$\mathcal{E}'_\lambda(\sigma, \tau^*) = \mathcal{E}_\lambda \circ \delta_\lambda^{-1}(\sigma, \tau^*) = (\sigma - \tau, \sigma + \tau)\ .$$

Now note that the map $\mathcal{E}'_\lambda : \mathcal{H}(\lambda, \mathbb{R}) \times \mathcal{H}(\lambda, \mathbb{R})^* \to \mathcal{H}(\lambda, \mathbb{R}) \times \mathcal{H}(\lambda, \mathbb{R})$ defined above is a symplectomorphism (up to a multiplicative factor) with respect to the cotangent bundle symplectic form on $\mathcal{H}(\lambda, \mathbb{R}) \times \mathcal{H}(\lambda, \mathbb{R})^*$ and the difference of Thurston intersection forms on $\mathcal{H}(\lambda, \mathbb{R}) \times \mathcal{H}(\lambda, \mathbb{R})$

$$\begin{aligned}\mathcal{E}'^*_\lambda(\Omega_{\mathrm{Th}} \oplus \overline{\Omega_{\mathrm{Th}}})\Big((\rho_1, \chi_1^*), (\rho_2, \chi_2^*)\Big) &= \Omega_{\mathrm{Th}}(\rho_1 - \chi_1, \rho_2 - \chi_2) \\ &\quad - \Omega_{\mathrm{Th}}(\rho_1 + \chi_1, \rho_2 + \chi_2) \\ &= -2\Omega_{\mathrm{Th}}(\chi_1, \rho_2) + 2\Omega_{\mathrm{Th}}(\chi_2, \rho_1) \\ &= -2\Omega_*\Big((\rho_1, \chi_1^*), (\rho_2, \chi_2^*)\Big)\ .\end{aligned}$$

Finally, restricting to the appropriate subsets, we have

$$\begin{aligned}\frac{1}{2}\mathcal{E}'^*(\omega_{WP} \oplus \overline{\omega_{WP}}) &= -\frac{1}{2}\mathcal{E}'^* \circ (\varphi_\lambda, \varphi_\lambda)^* \left(\Omega_{\mathrm{Th}} \oplus \overline{\Omega_{\mathrm{Th}}}\right) \\ &= -\frac{1}{2}\left(\varphi_\lambda, \left(\varphi_\lambda^{-1}\right)^*\right)^* \circ \mathcal{E}'^*_\lambda \left(\Omega_{\mathrm{Th}} \oplus \overline{\Omega_{\mathrm{Th}}}\right) \\ &= \left(\varphi_\lambda, \left(\varphi_\lambda^{-1}\right)^*\right)^* \Omega_* = 2\omega^r_*\ .\end{aligned}$$ □

Proof of Theorem 1.2. The proof that $W_\partial^{\mathrm{AdS}} : \mathcal{HE} \to \mathcal{GH}_{-1}$ is symplectic now follows from Theorem 1.18 and Remark 3.2. □

5.3. The dual Schläfli formula for convex cores of AdS manifolds

The main point of this section is a result on the variation, under a deformation, of the volume (or rather the dual volume) of the convex core of a globally hyperbolic AdS manifold. Although not obviously related to the main results of this paper, this formula is the key tool in proving, in the next section, that the double harmonic map is symplectic.

The result presented here should be compared with the Schläfli formula obtained by Bonahon [9] for convex cores of quasifuchsian hyperbolic manifolds, and to the dual formula, for the variation of the dual volume of quasifuchsian manifolds, used in [27]. The result we prove here (and need below) is the AdS analog of the dual Schläfli formula of [27]. We do not consider here the Schläfli formula itself for AdS convex cores, however it is possible that it could be obtained from the dual formula by a fairly direct argument (possibly similar to the argument used in the other direction in [27] in the hyperbolic setting).

Definition 5.3. Let $g \in \mathcal{GH}_{-1}$ be a GHM AdS metric on M. We denote by Ω_+ the domain of M bounded by the unique maximal surface $S \subset M$ and by the upper boundary $\partial_+ C(M, g)$ of the convex core of M, and set

$$V_+^*(g) = V(\Omega_+) - \frac{1}{2} L_{m_+}(l_+) ,$$

where $V(\Omega_+)$ is the volume of Ω_+ and m_+ and l_+ are the induced metric and the measured bending lamination on $\partial_+ C(M, g)$.

A key point of the proof of the symplecticity of the double harmonic map will be the following variation formula for the volume V_+^*.

Lemma 5.4. *The function $V_+^* : \mathcal{GH}_{-1} \to \mathbb{R}$ is tangentiable. For a first-order variation of the GHM AdS metric g, the corresponding variation of V_+^* is*

$$(V_+^*(g))' = -\frac{1}{4} \int_S \langle I', I\!I \rangle_I da_I - \frac{1}{2} d_{m_+} L(l_+)(m'_+) , \tag{5.1}$$

where I and $I\!I$ are the induced metric and second fundamental form on the unique maximal Cauchy surface S in M.

We denote by V^* the sum of V_+^* and V_-^*, analogously defined in terms of the lower boundary of the convex core, and by (m, l) the hyperbolic metric and measured lamination induced on the whole boundary of the convex core $\partial C(M, g) = \partial_+ C(M, g) \sqcup \overline{\partial_- C(M, g)}$. The dual Schläfli formula for convex core of GHM AdS manifolds now follows directly.

Proposition 5.5. *In a first-order variation of the GHM AdS metric g,*

$$(V^*(g))' = -\frac{1}{2} d_m L(l)(m') .$$

Proof. This follows directly from applying Lemma 5.4 both to V_+^* and to the corresponding quantity V_-^* for the part of M between the maximal surface S and the lower boundary of the convex core, that is, the quantity corresponding to V_+^* after changing the time orientation of M. The first term on the right-hand side of (5.1) is then exactly compensated by the corresponding term for the lower half of the convex core, and only the second term remains. □

The proof of Lemma 5.4 will basically follow from a first variation formula for the volume of AdS domains with smooth boundary. In the following statement we denote by I, $I\!I$, H the induced metric, second fundamental form and mean curvature of the boundary, with $H = \frac{1}{2}\mathrm{tr}_I(I\!I)$, and suppose that the orientation conventions are such that H is positive when the boundary is convex.

Lemma 5.6. *Let Ω be a 3-dimensional manifold with boundary, with a one-parameter family of AdS metrics $(g_t)_{t\in[0,1]}$ such that the boundary is $C^{1,1}$ smooth and space-like. Then*

$$V(\Omega)' = \int_{\partial\Omega} H' + \frac{1}{4}\langle I', I\!I\rangle da_I \ .$$

Here $V(\Omega)' = (d/dt)V(\Omega, g_t)_{|t=0}$ and similarly for the other primes.

This statement is the exact Lorentzian analog, in the 3-dimensional case, of [36, Theorem 1] (see also [37] for a complete proof). The argument there can be used almost with no modification here. We leave the details to the interested reader. (The proof can be obtained by integrating by parts the equation satisfied by a normalized deformation of the AdS structure on the convex core, considered as a symmetric 2-tensor.)

Note that Lemma 5.6 could be stated in a much more general way by considering a higher-dimensional manifold with a one-parameter family of Einstein metrics, as in [37]. The fact that the boundary is space-like is not essential. Note also that an alternate proof can be found, for Riemannian Einstein manifolds, in [19].

Corollary 5.7. *Under the same conditions as in Lemma* 5.6*, let*

$$V^*(\Omega) = V(\Omega) - \int_{\partial\Omega} H da_I \ .$$

Then

$$V^*(\Omega)' = \frac{1}{4}\int_{\partial\Omega} \langle I', I\!I - 2HI\rangle_I da_I \ .$$

Proof. This follows from Lemma 5.6 because an elementary computation shows that

$$\left(\int_{\partial\Omega} H da_I\right)' = \int_{\partial\Omega} H' + \frac{H}{2}\langle I', I\rangle_I da_I \ . \qquad \square$$

The last technical tool that will be needed in the proof of Lemma 5.4 is the description of the surfaces equidistant from a convex pleated surface in AdS^3. This description is directly analogous to what is well-known for the equidistant surfaces from a convex pleated surface in $\mathbb{H}^3$, so we give only a brief account here, leaving the details to the reader. We consider a past-convex space-like pleated surface $\Sigma \subset \mathrm{AdS}^3$, denote its induced metric by m and its measured pleating lamination by l, and will denote by Σ_r the equidistant surface at time-distance r in the past of Σ (*i.e.*, in the convex domain bounded by Σ — this contrasts with the hyperbolic situation where one typically considers the equidistant surface in the concave region).

The simplest case occurs when $l = 0$ and Σ is totally geodesic. Then a simple computation shows that Σ_r is umbilic and future-convex, with principal curvatures equal to $-\tan(r)$. If on the other hand we suppose that Σ is made of two totally geodesic half-planes P_1 and P_2 intersecting at an angle θ along their common boundary, we obtain that Σ_r has three components:

- Two umbilic surfaces $P_{1,r}$ and $P_{2,r}$, with orthogonal projection on Σ respectively on P_1 and P_2;
- A strip S of width $\theta\sin(r)$, which projects orthogonally to $\partial P_1 = \partial P_2$, where one principal direction (along the axis) is $1/\tan(r)$, while the other is $-\tan(r)$.

Suppose now that Σ is a past-convex space-like pleated surface in a GHM AdS manifold, with rational measured bending lamination l. It follows from the previous description that Σ_r has umbilic regions (projecting orthogonally to the complement of the support of l in Σ) with principal curvatures $-\tan(r)$, and "strips" projecting orthogonally to the support of l, with principal curvatures equal to $1/\tan(r)$ and to $-\tan(r)$. In particular, it will be important below to note that the area of Σ_r is

$$A(\Sigma_r) = \cos^2(r)(-2\chi(S)) + \sin(r)\cos(r)L_m(l)\ .$$

It follows by continuity that the same area formula holds for general (not rational) measured bending lamination.

We can now provide a direct proof of Lemma 5.4. Note that this contrasts with the argument given in [27], where the "dual Schläfli formula" was proved using Bonahon's Schläfli formula (see [10, 11]). It appears likely that, in the hyperbolic setting too, a direct proof of the dual Schläfli formula can be given without going through Bonahon's Schläfli formula, which is more complicated even to state since it involves the first-order variation of the measured bending lamination.

Proof of Lemma 5.4. Recall that weighted multicurves are dense in $\mathcal{ML}$. Therefore for any data (m_+, l_+) on the upper boundary of the convex core, l_+ can be approximated by a sequence of laminations supported on a disjoint union of closed curves. It is therefore sufficient to prove the lemma when l_+ is supported on a disjoint union of closed curve. We will focus on this situation in the rest of the proof.

We consider a smooth one-parameter family $(g_t)_{t\in[0,1]}$ of AdS metrics on M and the corresponding one-parameter family of hyperbolic metrics and measured laminations $(m_t, l_t)_{t\in[0,1]}$ induced on the upper boundary of the convex core. It is convenient here to choose a maximal lamination λ containing the support of l, so that l can be identified with the corresponding transverse cocycle, as outlined in Section 4.1. A first order variation $\dot g = (dg_t/dt)_{|t=0}$ of g then determines a first-order variation $\dot m \in T_m\mathcal{T}$ of m and a first-order variation $\dot l \in \mathcal{H}(\lambda, \mathbb{R})$ of l.

We can now slightly change the perspective and consider (m, l) as the main variables, with the first-order variation $\dot{g}$ of g to be determined by first-order variations $\dot{m}$ and $\dot{l}$ of m and l. Thus, given $(m, l) \in \mathcal{T} \times \mathcal{H}(\lambda, \mathbb{R})$, let (M, g) be the GHM AdS manifold whose upper boundary of the convex core has induced metric m and measured bending lamination l and denote by S the unique maximal Cauchy surface in M, and by Ω the domain in M bounded by S and by the upper boundary of the convex core $\partial_+\Omega$. We will now prove that

$$V_+^*(m, l) = V(\Omega) - \frac{1}{2} L_m(l)$$

is tangentiable with the correct derivative.

Our strategy to prove the variation formula for V_+^* will be to approximate the pleated surface $\partial_+\Omega$ by equidistant surfaces, to which we can apply the smooth dual Schläfli formula of Lemma 5.6. So, for $r > 0$, we denote by Σ_r the set of points at time distance r from $\partial_+\Omega$ in the past. If r is small enough, then $\Sigma_r \subset \Omega$. We then call Ω_r the compact domain in M bounded by S and Σ_r. So Ω_r is contained in Ω, more precisely Ω is composed of all points at time distance at most r from Ω_r in its future.

We denote by I_r, $I\!I_r$, H_r the induced metric, second fundamental form and mean curvature of Σ_r, and define

$$V_r^*(m, l) = V(\Omega_r) - \int_{\Sigma_r} H_r da_{I_r} \ .$$

The first-order variation formula for V_r^* follows from Lemma 5.6 and the proof of Corollary 5.7:

$$(V_r^*(m, l))' = -\int_S \left(H' + \frac{1}{4} \langle I', I\!I \rangle \right) da_I + \frac{1}{4} \int_{\Sigma_r} \langle I_r', I\!I_r - 2H_r I_r \rangle_{I_r} da_{I_r} \ .$$

Note that the terms corresponding to Σ_r is different from the term on S since, in the definition of $V_r^*(m, l)$, an integral mean curvature term is added but it is only an integral on Σ_r. The first integral already occurs in the statement of Lemma 5.4, and moreover $H' = 0$ since S remains a maximal surface throughout the deformation. So, to prove the statement, we need to show that

$$\int_{\Sigma_r} \langle I_r', I\!I_r - 2H_r I_r \rangle_{I_r} da_{I_r} \overset{r \to 0}{\longrightarrow} -2 d_m L(l)(\dot{m}) \ . \tag{5.2}$$

For $r > 0$ small enough, Σ_r is $C^{1,1}$ smooth — this is the Lorentzian analog of the well-known fact that the equidistant surface from a convex pleated surface in hyperbolic space, on the concave side of the complement, is $C^{1,1}$ smooth. Note that Σ_r is not convex, but this will not play any role in the argument.

There is a well-defined nearest-point projection $\rho : \Sigma_r \to \partial_+\Omega$. Therefore we can decompose Σ_r in two components:

- Σ_r^l is the inverse image by ρ of the support of l, so that is a closed subset of Σ_r,
- $\Sigma_r^f = \Sigma_r \setminus \Sigma_r^l$ is the open set of points which project to a point of $\partial_+\Omega$ which has a totally geodesic neighborhood.

Both Σ_r^l and Σ_r^f are smooth surfaces.

The area of Σ_r^f depends on the area of $\partial_+\Omega$, specifically:

$$A(\Sigma_r^f) = \cos^2(r)(-2\pi\chi(S)) .$$

Similarly, the area of Σ_r^l depends on the length of l for m:

$$A(\Sigma_r^l) = \sin(r)\cos(r)L_m(l) .$$

As a consequence, we can express the volume of Ω_r in terms of the volume of Ω:

$$V(\Omega_r) - V(\Omega) = \int_{s=0}^{r} \Big(\sin(s)\cos(s)L_m(l) + \cos^2(s)(-2\pi\chi(S))\Big)ds \xrightarrow{r\to 0} 0 .$$

Moreover

$$\int_{\Sigma_r} H_r da_{I_r} \xrightarrow{r\to 0} \frac{1}{2}L_m(l) ,$$

and it follows that $V_r^*(m,l) \to V_+^*(m,l)$ in the local C^0 sense as $r \to 0$.

Clearly, Σ_r^f is the disjoint union of open surfaces which are equidistant from a plane and therefore umbilic, with principal curvatures equal to $-\tan(r)$. The local geometry of Σ_r^l is slightly more interesting. It has a foliation Λ by geodesics, each of which project to a leaf of l. The directions parallel to Λ are principal directions, with corresponding principal curvature $-\tan(r)$, while the principal curvature corresponding to the directions orthogonal to Λ is $\operatorname{cotan}(r)$.

As a consequence, the mean curvature of Σ_r is equal to $-\tan(r)+\operatorname{cotan}(r)$ on Σ_r^l, and to $-2\tan(r)$ on Σ_r^f. It follows that $I\!I_r - 2H_r I_r$ is equal to

- $-\operatorname{cotan}(r)I_r$ on directions parallel to Λ on Σ_r^l;
- $\tan(r)I_r$ on directions orthogonal to Λ on Σ_r^l and on all directions in Σ_r^f.

To prove (5.2), we decompose the first-order variation of I_r in two terms: $dI_r(\dot m)$ corresponding to $\dot m$, and $dI_r(\dot l)$ corresponding to $\dot l$. We will compute separately the contribution of each term to the limit of the integral on the left-hand side of (5.2). For both computations, we will consider the area $A(\Sigma_r) = A(\Sigma_r^l) + A(\Sigma_r^f)$ of I_r. Similarly as in the hyperbolic setting (see, *e.g.*, [26]) we have

$$A(\Sigma_r) = -2\pi\chi(S) + \sin(r)\cos(r)L_m(l) .$$

The first-order deformation $dI_r(\dot l)$ corresponds to varying the bending on $\partial_+\Omega$ while keeping the induced metric fixed, so it vanishes in the directions parallel to Λ

on Σ_r^l (which project to directions parallel to the support of the bending lamination on $\partial_+\Omega$). So it follows from the description of $I\!I_r - 2H_r I_r$ given above that

$$\begin{aligned}\int_{\Sigma_r} \langle dI_r(\dot l), I\!I_r - 2H_r I_r\rangle_{I_r} da_{I_r} &= \int_{\Sigma_r} \langle dI_r(\dot l), \tan(r) I_r\rangle_{I_r} da_{I_r} \\ &= 2\tan(r) dA(\Sigma_r)(\dot l) \\ &= 2\sin^2(r) dL_m(\dot l) \\ &= 2\sin^2(r) L_m(\dot l) \\ &\xrightarrow{r\to 0} 0\,.\end{aligned}$$

Similarly, $dI_r(\dot m)$ is bounded on Σ_r^f, while it vanishes on Σ_r^l on directions orthogonal to Λ. It follows that

$$\begin{aligned}\int_{\Sigma_r} \langle dI_r(\dot m), I\!I_r - 2H_r I_r\rangle_{I_r} da_{I_r} = &\int_{\Sigma_r^l} \langle dI_r(\dot m), -\mathrm{cotan}(r) I_r\rangle_{I_r} da_{I_r} \\ &+ \int_{\Sigma_r^f} \langle dI_r(\dot m), \tan(r) I_r\rangle_{I_r} da_{I_r}\,.\end{aligned}$$

However

$$\int_{\Sigma_r^f} \langle dI_r(\dot m), \tan(r) I_r\rangle_{I_r} da_{I_r} \xrightarrow{r\to 0} 0\,,$$

while

$$\begin{aligned}\int_{\Sigma_r^l} \langle dI_r(\dot m), -\mathrm{cotan}(r) I_r\rangle_{I_r} da_{I_r} &= -\mathrm{cotan}(r) \int_{\Sigma_r^l} \langle dI_r(\dot m), I_r\rangle_{I_r} da_{I_r} \\ &= -2\mathrm{cotan}(r) dA(\Sigma_r^l)(\dot m) \\ &= -2\cos^2(r) d_m L(l)(\dot m) \\ &\xrightarrow{r\to 0} -2 d_m L(l)(\dot m)\,.\end{aligned}$$

Summing up, we obtain Equation (5.2).

Therefore,

$$dV_r^*(m,l) \to -\int_S \left(H' + \frac{1}{4}\langle I', I\!I\rangle\right) da_I - \frac{1}{2} d_m L(l)(m')\,.$$

pointwise as $r \to 0$. Since $V_r^*(m,l) \to V^*(m,l)$ in C^0 as $r \to 0$, the result follows. $\square$

5.4. The double harmonic map is symplectic

We turn here to the proof of Theorem 1.11: the double harmonic map $\mathcal{H} : T^*\mathcal{T} \to \mathcal{T} \times \overline{\mathcal{T}}$ is symplectic up to a factor, more precisely,

$$\mathcal{H}^*(\omega_{WP} \oplus \overline{\omega_{WP}}) = -2\omega_*^r\,,$$

where ω_*^r is the real part of the complex symplectic structure on $T^*\mathcal{T}$.

The key part of the argument is the dual Schläfli formula, more specifically Lemma 5.4 seen in the previous section. Note that a similar argument was used in the hyperbolic setting by Loustau in [31].

We will use the diagram in Figure 5.1, which is a variant of other similar (related) diagrams presented in the paper.

$$\begin{tikzcd}
\mathcal{T}\times\mathcal{ML} \arrow[d, "\delta"'] \arrow[dr, "\mathcal{E}" near start] & \mathcal{GH}_{-1} \arrow[l, "\partial_+^{\mathrm{Ads}}"'] \arrow[r, "\max"] \arrow[dl, "\partial'" near end] \arrow[d, "\mathrm{hol}^{\mathrm{Ads}}"] & (T^*\mathcal{T}, -\omega_*^r) \arrow[dl, "\mathcal{H}"] \\
(T^*\mathcal{T}, 2\omega_*^r) \arrow[r, "\mathcal{E}'"'] & (\mathcal{T}\times\overline{\mathcal{T}}, \frac{1}{2}(\omega_{WP}\oplus\overline{\omega_{WP}})) &
\end{tikzcd}$$

Figure 5.1. Earthquakes and harmonic maps associated to GHM AdS manifolds.

In this diagram we denote by $\partial' : \mathcal{GH}_{-1} \to T^*\mathcal{T}$ the composition $\partial' = \delta \circ \partial_+^{\mathrm{AdS}}$.

This diagram is commutative. The fact that the right triangle commutes is a direct translation of Lemma 2.13. In the left square, the triangles not involving the $\mathrm{hol}^{\mathrm{AdS}}$ map commute by definition, while the two triangles involving $\mathrm{hol}^{\mathrm{AdS}}$ commutes by Theorem 2.11 and Lemma 2.13.

Proposition 5.8. *The map* $\max\circ\partial'^{-1}$ *is symplectic up to a factor* -2*:* $(\max\circ\partial'^{-1})^*\omega_*^r = -2\omega_*^r$.

Proof of Proposition 5.8. Recall that the map $\delta : \mathcal{T}\times\mathcal{ML} \to T^*\mathcal{T}$ is defined as $\delta(m, l) = d_m L(l)$. Let θ denote the canonical Liouville 1-form of $T^*\mathcal{T}$, that is, the 1-form on $T^*\mathcal{T}$ defined at a point $(m, u) \in T^*\mathcal{T}$ by

$$\forall U \in T_{(m,u)}T^*\mathcal{T}, \quad \theta(U) = u(\pi_* U) ,$$

where $\pi : T^*\mathcal{T} \to \mathcal{T}$ is the canonical projection. It follows from the defintion of δ that

$$(\delta^*\theta)(\dot m, \dot l) = d(L(l))(\dot m) .$$

Pulling-back this 1-form on $\mathcal{GH}_{-1}$ by the map ∂', we obtain that

$$\left((\partial')^*\theta\right)(\dot m, \dot l) = \left((\partial\circ\delta)^*\theta\right)\left(\dot m, \dot l\right) = d(L(l))(\dot m) ,$$

where $(\dot m, \dot l)$ is now taken to define a tangent vector to $\mathcal{GH}_{-1}$, as seen at the beginning of Subsection 5.3.

A very similar argument shows that

$$(\max{}^*\theta)(\dot I, \dot{I\!I}) = \int_S \langle \dot I, I\!I\rangle da_I ,$$

where $(I, I\!I)$ determine a point in $\mathcal{GH}_{-1}$ and $(\dot I, \dot{I\!I})$ a tangent vector to $\mathcal{GH}_{-1}$ at this point.

Lemma 5.4 can therefore be stated as follows: on $\mathcal{GH}_{-1}$,

$$dV_+^* = -\frac{1}{4}\max{}^*\theta - \frac{1}{2}(\partial')^*\theta \ .$$

Taking the differential, we obtain that

$$0 = -\frac{1}{4}\max{}^*\omega_*^r - \frac{1}{2}(\partial')^*\omega_*^r \ ,$$

and therefore that

$$\left(\max \circ (\partial')^{-1}\right)^*\omega_*^r + 2\omega_*^r = 0 \ . \qquad \square$$

Proof of Theorem 1.11. The proof clearly follows from Proposition 5.8, and from the diagram in Figure 5.1, because Theorem 1.18 asserts that

$$(\mathcal{E}')^*\left(\frac{1}{2}\left(\omega_{WP} \oplus \overline{\omega_{WP}}\right)\right) = 2\omega_*^r \ . \qquad \square$$

Proof of Theorem 1.3. The proof that the map $W_{\min} : \mathcal{AF} \to \mathcal{GH}$ is symplectic follows from Theorem 1.11 and from Remark 3.4. $\square$

6. Constant mean curvature surfaces

In this section we consider the symplectic structures induced on the various moduli spaces of geometric structures in 3 dimensions ($\mathcal{AF}'$, $\mathcal{GH}_{-1}$, $\mathcal{GH}_0$ and $\mathcal{GH}_1$) by their identification with $T^*\mathcal{T}$ through constant mean curvature surfaces. We then prove Theorem 1.7.

6.1. CMC surfaces in hyperbolic manifolds

Recall that $\mathcal{AF}'$ denotes the subspace of $\mathcal{AF}$ of almost-Fuchsian metrics on $S \times \mathbb{R}$ which admit a foliation by CMC surfaces, with mean curvature going from -1 to 1. Conjecturally, $\mathcal{AF}' = \mathcal{AF}$. An elementary application of the maximum principle shows that for $h \in \mathcal{AF}'$, (M, h) contains a unique closed, embedded CMC-H surface, which is a leave of the CMC foliation.

Definition 6.1. For all $H \in (-1, 1)$, we denote by $\mathrm{CMC}_H^{\mathrm{Hyp}} : \mathcal{AF}' \to T^*\mathcal{T}$ the map sending a hyperbolic metric $h \in \mathcal{AF}'$ to $([I], I\!I_0)$, where $[I]$ is the conformal class of the induced metric and $I\!I_0$ is the traceless part of the second fundamental form of the unique closed, embedded CMC-H surface in (M, h).

A key point for us is that the symplectic form obtained on $\mathcal{AF}'$ by pulling back the cotangent symplectic structure on $T^*\mathcal{T}$ to $\mathcal{AF}'$ by all those maps is always the same. We will see below that the same result, basically with the same proof, extends to globally hyperbolic constant curvature space-times.

Proposition 6.2. *Let $H, H' \in (-1, 1)$. Then* $(\mathrm{CMC}_H^{\mathrm{Hyp}})^*\omega_*^r = (\mathrm{CMC}_{H'}^{\mathrm{Hyp}})^*\omega_*^r$.

Proof. We suppose, without loss of generality, that $H' > H$. Let Σ and Σ' be the closed, embedded surfaces with constant mean curvature H and H', respectively, and let Ω be the domain bounded by Σ and Σ'. We orient both Σ and Σ' towards increasing values of H. We define

$$V^*(\Omega) = V(\Omega) - \int_{\Sigma'} H' da_I + \int_\Sigma H da_I \ .$$

Corollary 5.7 then indicates that, in a first-order deformation of g,

$$2V^*(\Omega)' = \int_{\Sigma'} \frac{1}{2}\langle I', I\!I - 2H'I\rangle_I da_I - \int_\Sigma \frac{1}{2}\langle I', I\!I - 2HI\rangle_I da_I \ .$$

(Note that the signs are slightly different from those in Corollary 5.7 because the orientation of Σ is different, here it is towards increasing values of H and therefore towards the interior of Ω.)

Clearly we have

$$I\!I = I\!I_0 + HI \ ,$$

so that

$$I\!I - 2HI = I\!I_0 - HI \ .$$

As a consequence,

$$\begin{aligned} 2V^*(\Omega)' &= \int_{\Sigma'} \frac{1}{2}\langle I', I\!I_0\rangle_I da_I - H'\int_{\Sigma'} \frac{1}{2}\langle I', I\rangle_I da_I \\ &\quad - \int_\Sigma \frac{1}{2}\langle I', I\!I_0\rangle_I da_I + H\int_\Sigma \frac{1}{2}\langle I', I\rangle_I da_I \\ &= \frac{1}{2}\int_{\Sigma'} \langle I', I\!I_0\rangle_I da_I - \frac{1}{2}\int_\Sigma \langle I', I\!I_0\rangle_I da_I - H'A(\Sigma')' + HA(\Sigma)' \ . \end{aligned}$$

Another way to state this is that

$$2d\left(2V^*(\Omega) + H'A(\Sigma') - HA(\Sigma)\right) = (\mathrm{CMC}_{H'}^{\mathrm{Hyp}})^*\theta - (\mathrm{CMC}_H^{\mathrm{Hyp}})^*\theta \ ,$$

where θ is the Liouville form on $T^*\mathcal{T}$. It follows that

$$(\mathrm{CMC}_{H'}^{\mathrm{Hyp}})^*\omega_*^r - (\mathrm{CMC}_H^{\mathrm{Hyp}})^*\omega_*^r = d((\mathrm{CMC}_{H'}^{\mathrm{Hyp}})^*\theta - (\mathrm{CMC}_H^{\mathrm{Hyp}})^*\theta) = 0 \ . \qquad \square$$

6.2. CMC surfaces in Lorentzian space-times

Recall that, according to Theorem 1.4, any GHM AdS manifold admits a unique foliation by CMC surfaces, with mean curvature going monotonically from $-\infty$ to ∞. This makes the following definition possible.

Definition 6.3. For all $H \in \mathbb{R}$, we call $\mathrm{CMC}_H^{\mathrm{AdS}} : \mathcal{GH}_{-1} \to T^*\mathcal{T}$ the map sending a GHM AdS metric $g \in \mathcal{GH}_{-1}$ to $([I], I\!I_0)$, where $[I]$ is the conformal class of the induced metric and $I\!I_0$ is the traceless part of the second fundamental form of the unique closed, embedded CMC-H surface in (M, g).

Proposition 6.4. *Let $H, H' \in (-\infty, \infty)$. Then* $(\mathrm{CMC}_H^{\mathrm{AdS}})^*\omega_*^r = (\mathrm{CMC}_{H'}^{\mathrm{AdS}})^*\omega_*^r$.

The proof is exactly the same as in the hyperbolic setting, since the dual Schläfli formula has the same statement.

Things are similar in the de Sitter setting. According to Theorem 3.7, any GHM de Sitter manifold has a unique foliation by CMC surfaces, with mean curvature varying between $-\infty$ and -1 (with the orientation conventions used here).

Definition 6.5. For all $H \in (-\infty, -1)$, we call $\mathrm{CMC}_H^{dS} : \mathcal{GH}_1 \to T^*\mathcal{T}$ the map sending a GHM dS metric $g \in \mathcal{GH}_1$ to $([I], I\!I_0)$, where $[I]$ is the conformal class of the induced metric and $I\!I_0$ is the traceless part of the second fundamental form of the unique closed, embedded CMC-H surface in (M, g).

Proposition 6.6. *Let $H, H' \in (-\infty, -1)$. Then* $(\mathrm{CMC}_H^{dS})^*\omega_*^r = (\mathrm{CMC}_{H'}^{dS})^*\omega_*^r$.

The proof is again almost the same as for Proposition 6.2 above. The smooth Schläfli formula has a different sign in de Sitter manifolds, and it now reads:

$$V(\Omega)' = -\int_{\partial\Omega} H' + \frac{1}{4}\langle I', I\!I\rangle_I da_I \ .$$

Therefore one has to define the dual volume as

$$V^*(\Omega) = V(\Omega) + \int_{\partial\Omega} H da_I \ ,$$

and the variation formula for V^* has a minus sign compared to the hyperbolic or AdS cases. However the proof of Proposition 6.6 can be done as the proof of Proposition 6.2, with obvious sign differences.

Finally, in the Minkowski space, Theorem 3.7 indicates that any GHM Minkowski manifold has a unique foliation by CMC surfaces, with mean curvature varying between $-\infty$ and 0.

Definition 6.7. For all $H \in (-\infty, 0)$, we call $\mathrm{CMC}_H^{\mathrm{Mink}} : \mathcal{GH}_0 \to T^*\mathcal{T}$ the map sending a GHM AdS metric $g \in \mathcal{GH}_0$ to $([I], I\!I_0)$, where $[I]$ is the conformal class of the induced metric and $I\!I_0$ is the traceless part of the second fundamental form of the unique closed, embedded CMC-H surface in (M, g).

Proposition 6.8. *Let $H, H' \in (-\infty, 0)$. Then* $(\mathrm{CMC}_H^{\mathrm{Mink}})^*\omega_*^r = (\mathrm{CMC}_{H'}^{\mathrm{Mink}})^*\omega_*^r$.

The proof is again similar, but with larger differences. The smooth Schläfli formula now reads as

$$\int_{\partial\Omega} H' + \frac{1}{4}\langle I', I\!I\rangle_I da_I = 0 \ .$$

We now define

$$F(\Omega) = \int_{\partial\Omega} 2H da_I \ ,$$

and have the following variation formula for F under a first-order deformation:

$$F(\Omega)' = \int_{\partial\Omega} \frac{1}{2}\langle I', I\!I - 2HI\rangle_I da_I \ .$$

The proof of Proposition 6.8 can then proceed as the proof of Proposition 6.2, with F instead of V^*.

Proof of Theorem 1.7. Note that for all $H \in (-1, 1)$ and $H' \in (-\infty, \infty)$, we have

$$W^{\mathrm{AdS}}_{H,H'} = (\mathrm{CMC}^{\mathrm{AdS}}_{H'})^{-1} \circ \mathrm{CMC}^{\mathrm{Hyp}}_{H} \ .$$

We first consider the special case where $H = H' = 0$. With the notations used above, $\mathrm{CMC}^{\mathrm{Hyp}}_0 = \min$ while $\mathrm{CMC}^{\mathrm{AdS}}_0 = \max$. We already know by Theorem 3.3 that $\min : (\mathcal{AF}, \omega^i_G) \to (T^*\mathcal{T}, \omega^r_*)$ is symplectic up to the sign, that is

$$\min{}^*\omega^r_* = -\omega^i_G \ .$$

Moreover, $W^{\mathrm{AdS}}_{0,0} = W_{\min} : (\mathcal{AF}, \omega^i_G) \to (\mathcal{GH}_{-1}, \frac{1}{2}(\omega_{WP} \oplus \overline{\omega_{WP}}))$ is symplectic by Theorem 1.3. It follows that $\max : (\mathcal{GH}_{-1}, \frac{1}{2}(\omega_{WP} \oplus \overline{\omega_{WP}})) \to (T^*\mathcal{T}, -\omega^r_*)$ is also symplectic.

Proposition 6.2 and Proposition 6.4 therefore indicate that for all $H \in (-1, 1)$ and $H' \in (-\infty, \infty)$, $\mathrm{CMC}^{\mathrm{AdS}}_{H'}$ and $\mathrm{CMC}^{\mathrm{Hyp}}_{H}$ are symplectic. Therefore, $W^{\mathrm{AdS}}_{H,H'} : (\mathcal{AF}', \omega^i_G) \to (\mathcal{GH}_{-1}, \frac{1}{2}(\omega_{WP} \oplus \overline{\omega_{WP}}))$ is also symplectic. □

7. Minkowski and de Sitter manifolds

In this section we prove that the symplectic structure ω^i_G on the moduli space $\mathcal{GH}_1$ of globally hyperbolic de Sitter manifolds is identical (up to the sign) to the symplectic structure induced by the identification of $\mathcal{GH}_1$ with $T^*\mathcal{T}$ through CMC surfaces. The proof of Theorem 1.19 will follow.

We then describe some conjectural statements for globally hyperbolic Minkowski manifolds.

7.1. De Sitter CMC Wick rotation are symplectic

The proof of Theorem 1.19 is mostly based, in addition to the content of the previous sections, on the following proposition. We call $\Delta : \mathcal{HE} \to \mathcal{GH}_1$ the duality map, that is, the map sending a hyperbolic end E to the "dual" GHM de Sitter manifold, which has the same complex projective structure at future infinity as E. So

$\Delta = (\partial_\infty^{dS})^{-1} \circ \partial_\infty^{\mathrm{Hyp}}$ is a homeomorphism from $\mathcal{HE}$ to $\mathcal{GH}_1$, such that $\Delta^* \omega_G^i = \omega_G^i$. We also call Δ' the restriction of Δ to the space $\mathcal{AF}'$ of almost-Fuchsian metrics admitting a folation by CMC surfaces.

Proposition 7.1. *For all $H_* \in (-\infty, -1)$ and all $H \in (-1, 1)$, we have*

$$(\mathrm{CMC}_{H_*}^{dS} \circ \Delta')^* \omega_*^r = (\mathrm{CMC}_H^{\mathrm{Hyp}})^* \omega_*^r \,.$$

The proof is based on a basic differential geometry computation concerning the term which appears in the smooth Schläfli formula of Lemma 5.6.

Lemma 7.2. *Let Σ be a closed, embedded, locally convex surface with non-degenerate shape operator in a hyperbolic end E. In a first-order deformation of E and Σ, we have on Σ*

$$2H' + \frac{1}{2}\langle I', I\!I\rangle_I da_I = \frac{1}{2}\langle I\!I\!I', I\!I - 2H^* I\!I\!I\rangle_{I\!I\!I} da_{I\!I\!I} \,,$$

where $I\!I\!I$ is the third fundamental form of Σ and $H^* = H/(K+1)$ is the mean curvature of the dual surface.

Proof. By definition, we have $I\!I\!I = I(B\cdot, B\cdot)$, where B is the shape operator of Σ. Let $B^* = B^{-1}$ and let id denote the identity, then

$$I\!I - 2H^* I\!I\!I = I\!I\!I\left((B^* - \mathrm{tr}(B^*)Id)\cdot, \cdot\right) = I\!I\!I\left(\frac{B}{\det B}\cdot, \cdot\right) .$$

Let $A : T\Sigma \to T\Sigma$ be the self-adjoint (for I) bundle morphism such that $I' = I(A\cdot, \cdot)$. Then a simple computation shows that

$$I\!I\!I' = I\!I\!I\left(\left(B^{-1}AB + B^{-1}B' + \left(B^{-1}B'\right)^\dagger\right)\cdot, \cdot\right) ,$$

where the $\dagger$ is the adjoint with respect to $I\!I\!I$. Therefore

$$\langle I\!I\!I', I\!I - 2H^* I\!I\!I\rangle_{I\!I\!I} = \frac{\mathrm{tr}((B^{-1}AB + B^{-1}B' + (B^{-1}B')^\dagger)B)}{\det B} \,.$$

Since $B^\dagger = B$, it follows that

$$\langle I\!I\!I', I\!I - 2H^* I\!I\!I\rangle_{I\!I\!I} = \frac{\mathrm{tr}(AB + 2B')}{\det B} \,.$$

But $da_{I\!I\!I} = \det(B) da_I$, so it follows that

$$\langle I\!I\!I', I\!I - 2H^* I\!I\!I\rangle_{I\!I\!I} da_{I\!I\!I} = \mathrm{tr}(AB + 2B')da_I = (4H' + \langle I', I\!I\rangle_I)da_I \,,$$

as needed. ∎

Proof of Proposition 7.1. Let $E \in \mathcal{HE}$ be a hyperbolic end, and let $M \in \mathcal{GH}_1$ be the dual GHM de Sitter manifold. Thanks to Proposition 6.2 and Proposition 6.6, we only need to prove the statement for any arbitrary value of H and H_*, so we suppose (without loss of generality) that $\Sigma^*_{H_*}$ is on the positive side of Σ_H.

We denote by Ω the domain of M bounded by Σ_H and $\Sigma^*_{H_*}$. We then define

$$W = V(\Omega) + \int_{\Sigma_H} H da_I = V(\Omega) - HA(\Sigma_H)\ .$$

It then follows from Lemma 5.6 and from Corollary 5.7 that, in a first-order deformation of M,

$$2W' = \int_{\Sigma^*_{H_*}} 2H'_* + \frac{1}{2}\langle I', I\!I\rangle da_I - \int_{\Sigma_H} \frac{1}{2}\langle I', I\!I - 2HI\rangle da_I\ .$$

(The sign differs from that of Corollary 5.7 because of the orientation on Σ_H.)

Using Lemma 7.2, we can reformulate this equation as

$$2W' = \int_{\Sigma^*_{H_*}} \frac{1}{2}\langle I\!I\!I', I\!I - 2H_* I\!I\!I\rangle da_{I\!I\!I} - \int_{\Sigma_H} \frac{1}{2}\langle I', I\!I - 2HI\rangle da_I\ .$$

Now the duality between $\mathbb{H}^3$ and dS^3 exchanges the induced metric and the third fundamental forms of surfaces, and the equation becomes

$$\begin{aligned} 2W' &= \int_{\Sigma_{H_*}} \frac{1}{2}\langle I', I\!I - 2H_* I\rangle da_I - \int_{\Sigma_H} \frac{1}{2}\langle I', I\!I - 2HI\rangle da_I \\ &= \int_{\Sigma_{H_*}} \frac{1}{2}\langle I', I\!I_0\rangle da_I - H_* A(\Sigma_{H_*})' - \int_{\Sigma_H} \frac{1}{2}\langle I', I\!I_0\rangle da_I + HA(\Sigma_H)'\ . \end{aligned}$$

This means that

$$d(2W + H_* A(\Sigma_{H_*}) - HA(\Sigma_H)) = (\mathrm{CMC}^{\mathrm{Hyp}}_H)^*\theta - (\mathrm{CMC}^{\mathrm{dS}}_{H_*})^*\theta,$$

where θ denotes again the Liouville form of $T^*\mathcal{T}$. The result follows by taking the exterior differential of this last equation. ☐

We can now prove Theorem 1.19.

Proof of Theorem 1.19. Let $H \in (-1, 1)$ and $H_* \in (-\infty, -1)$, then it follows from the definition of W^{dS}_{H,H_*} that

$$W^{\mathrm{dS}}_{H,H_*} = (\mathrm{CMC}^{\mathrm{dS}}_{H_*})^{-1} \circ \mathrm{CMC}^{\mathrm{Hyp}}_H.$$

The statement therefore follows directly from Proposition 7.1, along with Theorem 3.3. ☐

7.2. Minkowski Wick rotations and Wick rotations between moduli spaces of Lorentzian space-times

We do not elaborate here on the symplectic properties of Wick rotations between quasifuchsian manifolds and GHM Minkowski manifolds. Note that there are at least two natural Wick rotations one can consider:

- The map $W^{\mathrm{Mink}}_{H,H'} : \mathcal{AF}' \to \mathcal{GH}_0$, depending on the choice of $H \in (-1, 1)$ and of $H' \in (-\infty, 0)$ sending an almost-Fuchsian manifold $M \in \mathcal{AF}'$ containing a CMC-H surface Σ_H to the unique GHM Minkowski containing a CMC-H' surface with the same data $([I], I\!I_0)$ as Σ. (This map is well-defined by [26, Lemma 6.1].)
- The map sending a hyperbolic end E with boundary data $(m, l) \in \mathcal{T} \times \mathcal{ML}$ on its pleated surface to the GHM Minkowski manifold for which (m, l) describes the initial singularity (see [32]).

It would be interesting to know whether those maps have interesting properties related to the natural symplectic structures on $\mathcal{AF}'$ (respectively $\mathcal{HE}$) and on $\mathcal{GH}_0$.

As a final note, we have considered here only Wick rotations between hyperbolic manifolds and constant curvature Lorentzian space-times — either AdS, de Sitter or Minkowski. However a number of statements on “Wick rotations” between constant curvature Lorentzian space-times of different types (AdS to Minkowski, etc.) clearly follow by composing different maps. We leave the details to the interested reader.

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