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Boundary concentrations on segments for the Lin-Ni-Takagi problem

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Abstract. We consider the following singularly perturbed Neumann problem (Lin-Ni-Takagi problem)

$$\varepsilon^2 \Delta u - u + u^p = 0\,, \quad u > 0 \quad \text{in} \quad \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega,$$

where $p > 2$ and Ω is a smooth and bounded domain in $\mathbb{R}^2$. We construct a new class of solutions that consists of a large number of spikes concentrating on a *segment* of the boundary that contains a *strict local minimum* point of the mean curvature function and has the *same* mean curvature at the two end points. We find a continuum limit of ODE systems governing the interactions of spikes and show that the derivative of the mean curvature function acts as *friction force*. Our construction is partly motivated by the construction of CMC surfaces on broken geodesics by Butscher and Mazzeo [10].

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1. Introduction and statement of main results

1.1. Introduction and Main Results

In this paper, we establish **new** concentration phenomena for the following singularly perturbed elliptic problem:

$$\begin{cases} \varepsilon^2 \Delta u - u + u^p = 0 & \text{in} \quad \Omega \\ u > 0 & \text{in} \quad \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$ with its unit outer normal ν, the exponent p is greater than 2, and $\varepsilon > 0$ is a small parameter. We prove the existence of solutions concentrating on a *segment* of $\partial\Omega$.

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This equation is known as the time-independent nonlinear Schrödinger equation:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V\psi - \tilde{\gamma}|\psi|^{p-2}\psi \tag{1.2}$$

where $\hbar$ is the Plank constant, V is the potential, and $\tilde{\gamma}, m$ are positive constants. Then, standing waves of (1.2) can be found by setting $\psi = e^{iEt/\hbar}v(x)$, where E is a constant and the real function v satisfies the elliptic equation:

$$-\hbar^2\Delta v + \tilde{V}v = |v|^{p-2}v \tag{1.3}$$

for some modified potential $\tilde{V}$. If we consider $\hbar \to 0$, the above equation becomes a singularly perturbed one.

It can also be viewed as a stationary equation of Keller-Segel system in chemotaxis [24] and the Gierer-Meinhardt biological pattern formation system [19]. In particular, Lin-Ni-Takagi [29] first derived this problem from Keller-Segel system and initiated the study of the quantitative properties of its solutions. In the literature this is also called as the Lin-Ni-Takagi problem [18].

Although problem (1.1) appears simple, it has a rich and interesting structure of solutions. For the last twenty years, it has received considerable attention. In particular, various concentration phenomena exhibited by the solutions of (1.1) seem both mathematically intriguing and scientifically useful. We refer to three survey articles [38,39], and [44] for more background and references.

In the subcritical case, problem (1.1) admits spike layer solutions, concentrating at one or multiple points of $\bar{\Omega}$. It was first established in [40,41] by Ni and Takagi the existence of least energy (mountain pass) solutions to (1.1), that is, a solution u_ϵ with minimal energy. They showed in [40,41] that, *for each $\epsilon > 0$, sufficiently small, u_ϵ has a spike at the most curved part of the boundary,* i.e.*, the region where the mean curvature attains maximum value*.

Since the publication of [41], further studies on spike-layer solutions (for the Dirichlet problem and mixed boundary problem as well) have been made. For spike solutions, solutions with multiple boundary spikes as well as multiple interior spikes and mixed interior and boundary spikes have been established (see [4,7,8,12,15–18,20–23,26,27,42,43,45,46], and the references therein). Owing to these works, the phenomenon of concentration at points is now well-understood. Necessary and sufficient conditions for the location of boundary and interior spikes are available.

In particular, with regard to the interior spike layer solutions, Lin, Ni, and Wei [28] showed that there are at least $\frac{C_N}{(\varepsilon|\log\varepsilon|)^N}$ interior spikes, and recently the first author, the third author and Zeng [5] extended their result and obtained the optimal bound of number of interior spikes, $\frac{C_N}{\varepsilon^N}$, for general smooth domain in $\mathbb{R}^N$.

A general principle is that for interior spike solutions, the distance function from the boundary $\partial\Omega$ plays an important role, while for the boundary spike solutions, the mean curvature function of the boundary plays an important role.

Besides the spike-layer solutions, it has been *conjectured* for a long time that *problem* (1.1) *should possess solutions, which have $m-$dimensional concentration* *sets for every* $0 \le m \le N-1$ (see, *e.g.* [38]). The case $m = N$ is excluded since (1.1) is not expected to exhibit phase transitions.

Under symmetry conditions, some results for problem (1.1) have been obtained in [1,2,6,9,13], as well as for the Dirichlet problem and the nonlinear Schrödinger equation.

In the general case, progresses although still limited, have also been made in [30, 31, 33–37, 47, 48]. For solutions concentrating on interior higher dimensional sets, results were first obtained in [47,48] where the third author and Yang constructed solutions concentrating on line segments in the interior of the domain Ω. For boundary concentration solutions, in a series of papers Malchiodi and Montenegro [34,36] proved the existence of solutions concentrating on the *whole boundary* or arbitrary components of $\partial\Omega$ when $\Omega \subset \mathbb{R}^N$ and solutions concentrating on closed geodesics of $\partial\Omega$, when $\Omega \subset \mathbb{R}^3$; later Mahmoudi and Malchiodi [30] extended the results and obtained the existence of solutions concentrating on the k-submanifolds of $\partial\Omega \subset \mathbb{R}^N$, provided that the sequence ε satisfies a gap condition is called *resonance*.

In [3], the first and the third authors and Musso removed the resonance condition in [47] and proved the existence of solutions concentrating on the interior straight line by putting a large number of spikes distributed along the line. It is natural to ask that whether one can remove the resonance condition for the boundary concentration solutions using similar ideas. In addition, in all the above mentioned papers, for higher dimensional boundary concentration solutions, the concentration sets were either the whole boundary or closed submanifolds of the boundary. A natural question is:

Does problem (1.1) *have solutions that concentrate on a broken segment of the boundary for all* $\varepsilon \to 0$ *?*

In this paper we provide an affirmative answer to the above question. We construct solutions concentrating on a broken segment γ of the boundary $\partial\Omega \subset \mathbb{R}^2$ for all $\varepsilon \to 0$ if γ satisfies the following condition:

($\mathbf{H_1}$). Let $\gamma = \gamma([0, b])$ be the segment parametrized by arc length, and $H(q)$ be the curvature of $\partial\Omega$ at q. Denote by

$$H'(\gamma(s)) = \frac{d}{ds} H(\gamma(s)), \;\; H''(\gamma(s)) = \frac{d^2}{ds^2} H(\gamma(s)).$$

Assume that $H''(\gamma(s)) \ge c_0 > 0$ for all $s \in [0, b]$, and $H(\gamma(0)) = H(\gamma(b))$.

Remark 1.1. From assumption ($\mathbf{H_1}$), one can see that γ must contain a non-degenerate local minimum point of the curvature H, and that the curvature at the two end points of γ must be the same.

Our main result in this paper is stated as follows:

Theorem 1.2. *Assume that* γ *satisfies* ($\mathbf{H_1}$). *Then there exists* $\varepsilon_0 > 0$ *such that, for* $\varepsilon < \varepsilon_0$, *there exists boundary spike solutions to* (1.1) *concentrating on* γ.

Remark 1.3. In the paper [23] Gui, Winter, and the third author proved the existence of multiple spike solutions concentrating at the local minimum point of the curvature $H(p)$. In this paper, we proved the existence of spike solutions concentrating on the segment that contains a local minimum of $H(p)$. Theorem 1.2 extends their result to a segment containing a local minimum point of H.

Remark 1.4. In this paper the condition $p > 2$ on the exponent is imposed for technical reasons. In the range $1 < p \leq 2$, we believe that the conclusion is also true by more refined estimates.

1.2. Description of the construction

The solutions we construct consist of a large number ($O(\frac{1}{\varepsilon \ln \varepsilon})$) of spikes distributed along the segment γ whose mutual distance is sufficiently small ($O(\varepsilon \ln \varepsilon)$).

At first glance one may discard such kind of solutions, as there seems to be no balancing force at the end points of the segment. In the following, we show that the derivative of the mean curvature function acts as *friction force*. This new phenomenon was first discovered in the work by Butscher and Mazzeo [10], in which they constructed CMC surfaces condensing to a finite geodesic segment. We comment more on this later.

In this subsection, we briefly describe the solutions to be constructed later and present the main idea in the procedure of the construction.

More precisely, let w be the unique solution of the following equation:

$$\begin{cases} \Delta w - w + w^p = 0 & \text{in } \mathbb{R}^2 \\ w > 0, \ w(0) = \max_{y \in \mathbb{R}^2} w(y) & \\ w \to 0 & \text{as } |y| \to \infty. \end{cases} \tag{1.4}$$

It is well-known (see [25]) that w is radial, *i.e.* $w = w(r)$, $w'(r) < 0$ and has the following asymptotic behavior:

$$w(y) = c_{N,p} |y|^{-\frac{N-1}{2}} e^{-|y|} (1 + o(1)) \tag{1.5}$$

and

$$w'(y) = -(1 + o(1)) w(y) \text{ as } |y| \to \infty. \tag{1.6}$$

For $q \in \partial\Omega$, we set

$$\Omega_\varepsilon = \{z : \varepsilon z \in \Omega\}, \ \Omega_{\varepsilon,q} = \{z : \varepsilon z + q \in \Omega\},$$

and

$$\mathcal{P} w_q(z) = \mathcal{P}_{\Omega_{\varepsilon,q}} w \left(z - \frac{q}{\varepsilon}\right), \, w_q(z) = w \left(z - \frac{q}{\varepsilon}\right), \ z \in \Omega_\varepsilon,$$

where $\mathcal{P}_{\Omega_{\varepsilon,q}} w(z - \frac{q}{\varepsilon})$ is defined to be the unique solution of

$$\Delta u - u + w\left(\cdot - \frac{q}{\varepsilon}\right)^p = 0 \text{ in } \Omega_{\varepsilon,q}, \ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega_{\varepsilon,q}. \tag{1.7}$$

We put a large number of boundary spikes along γ. Let $(\gamma(s_1), \cdots, \gamma(s_k))$ be the location of spikes. We define

$$U = \sum_{i=1}^{k} \mathcal{P}_{\Omega_{\varepsilon,\gamma(s_i)}} w\left(z - \frac{\gamma(s_i)}{\varepsilon}\right)$$

to be an approximate solution.

A natural and central question is how to choose s_i such that U is indeed a good approximation. By formal calculation, one has the following energy expansion for the energy functional corresponding to (1.1):

$$J(U) = \frac{k}{2} I(w) - \varepsilon\gamma_0 \sum_{i=1}^{k} H(\gamma(s_i)) - \frac{\gamma_1}{2} w\left(\frac{\gamma(s_i) - \gamma(s_j)}{\varepsilon}\right) + o(\varepsilon),$$

where γ_0, γ_1 are positive constants. One needs to find a critical point $(s_1, \cdots, s_k)$ of J in order to obtain a solution of (1.1), *i.e.*, $\frac{\partial}{\partial s_i} J = 0$ for all i. The main point in this paper is to exploit the contribution of $H'(\gamma(s))$ in $\frac{\partial J}{\partial s_i}$. The novelty of this paper is the new method of constructing balance approximate spike solutions, *i.e.*, the configuration space $\{(s_1, \cdots, s_k)\}$, such that $\frac{\partial J}{\partial s_i}$ is almost 0.

In this paper we establish a method to find such balance approximate solutions. It turns out that the number of spikes and their positions are determined by some nonlinear equations, which involve the interaction of spikes and also the effect of the boundary curvature. To explain this, we need to introduce the interaction function $\Psi(s)$ to describe the interactions of different spikes, which is defined for all $s \in \mathbb{R}$ by

$$\Psi(s) = -\int_{\mathbb{R}^2_+} w(y - (s, 0)) p w^{p-1} \frac{\partial w}{\partial y_1} dy.$$

It turns out that $\gamma(s_i)$ are determined by the following non-linear system:

$$\begin{cases} \Psi\left(\frac{|s_2 - s_1|}{\varepsilon}\right) + \varepsilon^2 H'(\gamma(s_1)) = O\left(\varepsilon^3\right) \\ \Psi\left(\frac{|s_3 - s_2|}{\varepsilon}\right) - \Psi\left(\frac{|s_2 - s_1|}{\varepsilon}\right) + \varepsilon^2 H'(\gamma(s_2)) = O\left(\varepsilon^3\right) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \Psi\left(\frac{|s_k - s_{k-1}|}{\varepsilon}\right) - \Psi\left(\frac{|s_{k-1} - s_{k-2}|}{\varepsilon}\right) + \varepsilon^2 H'(\gamma(s_{k-1})) = O\left(\varepsilon^3\right) \\ -\Psi\left(\frac{|s_k - s_{k-1}|}{\varepsilon}\right) + \varepsilon^2 H'(\gamma(s_k)) = O\left(\varepsilon^3\right), \end{cases} \tag{1.8}$$

and the number of spikes depending on ε is given by $k = k_\varepsilon = \left[\frac{b}{|\varepsilon \ln \varepsilon|}\right] + 1$.

One can see from the above equations that it is possible to balance the two end points of the segment using the derivative of the curvature function. However, in general, the above nonlinear system is difficult to solve. Our new idea is to consider this non-linear system as a discretization of its continuum limiting ODE systems (as the step size $h = \varepsilon|\ln\varepsilon|$ tends to 0):

$$\begin{cases} \dfrac{dx}{dt} = -\dfrac{1}{\ln\varepsilon}\Psi^{-1}\left(\dfrac{\varepsilon}{\ln\varepsilon}\rho(t)\right) \\ \dfrac{d\rho}{dt} = H'\big(\gamma(x(t))\big),\ 0 < t < b_\varepsilon \\ \rho(0) = 0,\ \rho(b_\varepsilon) = \rho_b \\ x'(b_\varepsilon) = -\dfrac{1}{\ln\varepsilon}\Psi^{-1}\big(\varepsilon^2 H'\big(\gamma(x(b_\varepsilon))\big)\big), \end{cases} \tag{1.9}$$

where Ψ^{-1} is the inverse function of Ψ, and $b_\varepsilon = (k_\varepsilon - 1)h = b + O(h)$ and $\rho_b < 0$ is a small constant depending on ε. The above overdetermined ODE is solvable under the assumption of the segment γ in $(\mathbf{H_1})$.

To describe the configuration space of $\gamma(s_i)$, we solve the ODE system (1.9) first and denote the solution as $x(t)$. Then, we define the positions of the spikes by midpoint approximation:

$$s_i^0 = x\left(\frac{t_i + t_{i+1}}{2}\right) \text{ for } i = 1, \cdots, k-1 \tag{1.10}$$

and

$$s_k^0 = s_{k-1}^0 + \varepsilon\Psi^{-1}\left(\varepsilon^2 H'\left(\frac{\varepsilon}{\ln\varepsilon}\rho_b\right)\right), \tag{1.11}$$

where

$$t_i = (i-1)|\varepsilon\ln\varepsilon|,\ i \geq 1. \tag{1.12}$$

The method to determine the approximate positions, *i.e.*, s_i^0, is the main contribution of this paper, which we elaborate in Section 6. The position defined in this manner is indeed an almost balanced one. We find real solutions by perturbing these spike points.

Letting $y_i \in \mathbb{R}$, we define

$$s_i = s_i^0 + y_i,\ \text{for } i = 1, \cdots, k, \tag{1.13}$$

and y_i satisfies

$$\begin{cases} |y_1| \leq C|\varepsilon\ln(-\ln\varepsilon)| \\ |(s_{i+1} - s_i) - (s_i - s_{i-1})| \leq \dfrac{C\varepsilon^3}{\min\left\{\Psi\left(\dfrac{s_i^0 - s_{i-1}^0}{\varepsilon}\right), \Psi\left(\dfrac{s_i^0 - s_{i+1}^0}{\varepsilon}\right)\right\}} \end{cases} \tag{1.14}$$

for $i = 2, \cdots, k-1$ and for some large constant $C > 0$.

With these notations, we can define the configuration space of $(s_1, \cdots, s_k)$ by:

$$\Lambda_k = \{(s_1, \cdots, s_k) \in \mathbb{R}^k | s_i \text{ is defined by (1.13) and satisfies (1.14)}\}. \quad (1.15)$$

The reason behind defining the configuration space in this manner is made clear in Section 3.

Moreover, from the analysis of the ODE (1.9) in Section 6, one can infer that

$$|s_i - s_{i-1}| \geq (1 + o(1))|\varepsilon \ln \varepsilon|, \ w\left(\frac{s_i - s_{i-1}}{\varepsilon}\right) \leq \frac{c\varepsilon}{|\ln \varepsilon|} \quad (1.16)$$

for $i = 2, \cdots, k$, and

$$|s_i - s_{i-1}| = 2(1 + o(1))|\varepsilon \ln \varepsilon|| \quad (1.17)$$

for $i = 2, k$.

We prove Theorem 1.2 by showing the following result:

Theorem 1.5. *Let γ be a segment of $\partial\Omega$ that satisfies* $(\mathbf{H_1})$. *Then, there exists ε_0 such that for $\varepsilon < \varepsilon_0$, there exists a positive number $k = k_{\varepsilon,\gamma} = \left[\frac{b}{|\varepsilon \ln \varepsilon|}\right] + 1$ and k points $(\gamma(s_1), \cdots, \gamma(s_k))$ on γ, where $(s_1, \cdots, s_k) \in \Lambda_k$ such that there exists a solution u_ε to problem* (1.1) *having the following form:*

$$u_\varepsilon(x) = \sum_{i=1}^{k} \mathcal{P}_{\Omega_{\varepsilon,\gamma(s_i)}} w\left(\frac{x - \gamma(s_i)}{\varepsilon}\right) + o(1), \quad (1.18)$$

where $o(1) \to 0$ uniformly as $\varepsilon \to 0$.

Remark 1.6. The motivation behind our construction comes from the study of the constant mean curvature surfaces. In [10], Butscher and Mazzeo constructed CMC surfaces condensing to a geodesic segment by connecting a large number $(O(\frac{1}{r}))$ of spheres of radius r distributed along the geodesic segments. Such surfaces cannot exist in Euclidean space, but they are able to show that the gradient of the ambient scalar curvature acts as a 'friction term', which permits the existence of balance surfaces. Therefore, the gradient of scalar curvature plays the same role as the gradient of the mean curvature in our case. In their paper, they require the symmetry condition on the geodesic segment. In our main Theorem 1.2, if we further require that Ω is symmetric, it is easy to see that $(\mathbf{H_1})$ can always be satisfied near the non-degenerate minimum point of the curvature $H(\gamma(s))$. Since we do not require any symmetry of the segment in Theorem 1.2, we believe that our idea can be used to construct CMC surfaces condensing to geodesic segments without the symmetry condition. This is the main contribution of our paper. We will discuss this in a forthcoming paper (A. Butscher announced this result in a preprint [11], but the full details have not been published as yet).

1.3. Sketch of the proof of Theorem 1.5

We will use the Lyapunov-Schmidt reduction method and a perturbation argument to construct the solutions to (1.1). The perturbation argument used to produce a real solution is not so different from the ones appearing elsewhere in the literature. As mentioned earlier, the main contribution of this paper is a novel approach to constructing balanced approximate solutions. In the following, we present the sketch of the proof.

We first introduce some notation. Given that, after the scaling $x = \varepsilon z$, the original problem becomes

$$\begin{cases} \Delta u - u + u^p = 0 & \text{in } \Omega_\varepsilon \\ u > 0 & \text{in } \Omega_\varepsilon \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{1.19}$$

Fixing $\mathbf{s} = (s_1, \cdots, s_k) \in \Lambda_k$, we denote by

$$\mathbf{P} = (P_1, \cdots, P_k) = \left(\frac{\gamma(s_1)}{\varepsilon}, \cdots, \frac{\gamma(s_k)}{\varepsilon} \right)$$

and define the sum of k spikes as

$$U = \sum_{i=1}^{k} \mathcal{P}_{\Omega_\varepsilon, P_i} w(z - P_i).$$

Define the operator

$$S(u) = \Delta u - u + u^p.$$

We also define the following functions as the approximate kernels

$$Z_i = \frac{\partial \mathcal{P}_{\Omega_\varepsilon, P_i} w(z - P_i)}{\partial \tau_{P_i}} \quad \text{for} \quad i = 1, \cdots, k.$$

Using U as the approximate solution, and performing the Lyapunov-Schmidt reduction, we can show that there exists ε_0 such that, for $\varepsilon < \varepsilon_0$, we can find a solution ψ of the following projected problem:

$$S(U + \psi) = \sum_{i=1}^{k} c_i Z_i, \ \int_{\Omega_\varepsilon} \psi Z_i = 0, \ i = 1, \cdots, k,$$

where c_i are constants depending on the form of ψ, Z_i.

Next, we need to solve the reduced problem

$$c_i = 0, \ i = 1, \cdots, k$$

by adjusting the points in Λ_k.

There are two main difficulties in solving the reduced problem. First, we need to control the error projection produced by ψ. In order to control this projection, we need to work in a weighted norm, which estimates ψ locally (see Section 3). Further, we need a decomposition of ψ, which is given in Section 4, from where one can see why we define the configuration space of s_i in (1.15). The reason why one needs to obtain a further decomposition of ψ is that the mutual distance of the spikes at the main order is not the same. In fact, near the two end points, the mutual distance of two consecutive spikes is $2(1 + o(1))\varepsilon|\ln\varepsilon|$, while in the more central part the mutual distance of two consecutive spikes is $(1+o(1))\varepsilon|\ln\varepsilon|$. Thus, the global estimate for ψ is not sufficient for our estimates. We need further decomposition near each spike. Second, we need to solve a non-linear system of the form (1.8), for which we use the discretization of the ODE equation (1.9).

Finally, the remainder of this paper is organized as follows. Some preliminary facts and useful estimates are explained in Section 2. Section 3 contains the standard Lyapunov-Schmidt reduction process: we study the linearized projected problem in Subsection 3.1 and then solve a non-linear projected problem in Subsection 3.2. In Section 4, we obtain further asymptotic behavior of ψ, which provides an expansion in ε. In Section 5, we derive the reduced nonlinear system of algebraic equations for the location. Section 6 is devoted to solving the reduced problem.

2. Technical analysis

In this section we introduce a projection and derive some useful estimates. Most of the results in this section are quite standard now and have been extensively used in the literature (see [21–23,40,43,45]).

Throughout this paper, we shall use the letter C to denote a generic positive constant that may vary from term to term. Through the following rescaling

$$x = \varepsilon z,\ z \in \Omega_\varepsilon := \{\varepsilon z \in \Omega\}, \tag{2.1}$$

equation (1.1) becomes

$$\begin{cases} \Delta u - u + u^p = 0, & \text{in } \Omega_\varepsilon \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{2.2}$$

We denote by $\mathbb{R}^2_+ = \{(y_1, y_2)|y_2 > 0\}$. Recall that w is the unique solution of (1.4).

Let $q \in \partial\Omega$. We can define a diffeomorphism straightening the boundary. We may assume that the inward normal to $\partial\Omega$ at q points in the direction of the positive x_2 axis. Denote $B'(R) = \{|x_1| \le R\}$, and $\Omega_1 = \Omega \cap B(q, R) = \{(x_1, x_2) \in B(q, R)|x_2 - q_2 > \rho(x_1 - q_1)\}$ where $B(q, R) = \{x \in \mathbb{R}^2||x - q| < R\}$. Then, since $\partial\Omega$ is smooth, we can find a constant R such that $\partial\Omega$ can be represented by the graph of a smooth function $\rho_q : B'(R) \to \mathbb{R}$ where $\rho_q(0) = 0$, and $\rho'_q(0) = 0$. Hereafter, we omit the use of q in ρ_q and write ρ instead, barring any confusion.

Therefore, $\partial\Omega$ can be represented near q by $(x_1, \rho(x_1))$. The curvature of $\partial\Omega$ at q is $H(q) = \rho''(0)$. After scaling, we know that near $Q = \frac{q}{\varepsilon}$, $\partial\Omega_\varepsilon$ can be represented by $(z_1, \varepsilon^{-1}\rho(\varepsilon z_1))$, where $(z_1, z_2) = \varepsilon^{-1}(x_1, x_2)$. By Taylor's expansion, we have the following:

$$\varepsilon^{-1}\rho(\varepsilon z_1) = \frac{1}{2}\rho''(0)\varepsilon z_1^2 + \frac{1}{6}\rho^{(3)}(0)\varepsilon^2 z_1^3 + O\left(\varepsilon^3 z_1^4\right). \tag{2.3}$$

Recall that for a smooth bounded domain $\mathcal{U}$, the projection $\mathcal{P}_{\mathcal{U}}$ of $H^2(\mathcal{U})$ onto $\{u \in H^2(\mathcal{U}) | \frac{\partial u}{\partial \nu} = 0 \text{ at } \partial\mathcal{U}\}$ is defined as follows: For $v \in H^2(\mathcal{U})$, let $\mathcal{P}_{\mathcal{U}} v$ be the unique solution of the boundary value problem:

$$\begin{cases} \Delta u - u + v^p = 0 & \text{in } \mathcal{U} \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\mathcal{U}. \end{cases} \tag{2.4}$$

Let $h_P(z) = w(z - P) - \mathcal{P}_{\Omega_{\varepsilon,P}} w(z - P)$. Then, h_P satisfies

$$\begin{cases} \Delta h_P(z) - h_P(z) = 0 & \text{in } \Omega_\varepsilon \\ \dfrac{\partial h_P}{\partial \nu} = \dfrac{\partial}{\partial \nu} w(z - P) & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{2.5}$$

For $z \in \Omega_{1,\varepsilon}$, for $P = (\mathbf{P}_1, \mathbf{P}_2)$, set now

$$\begin{cases} y_1 = z_1 - \mathbf{P}_1 \\ y_2 = z_2 - \mathbf{P}_2 - \varepsilon^{-1}\rho\big(\varepsilon(z_1 - \mathbf{P}_1)\big). \end{cases} \tag{2.6}$$

Under this transformation, the Laplace operator and the boundary derivative operator become

$$\Delta_z = \Delta_y + \rho(\varepsilon z_1)^2 \partial_{y_2 y_2} - 2\rho'(\varepsilon z_1)\partial_{y_1 y_2} - \varepsilon\rho''(\varepsilon z_1)\partial_{y_2},$$
$$\left(1 + \rho'(\varepsilon z_1)^2\right)^{\frac{1}{2}} \frac{\partial}{\partial \nu} = \rho'(\varepsilon z_1)\partial_{y_1} - (1 + \rho'^2(\varepsilon z_1))\partial_{y_2}.$$

First, we need to obtain the expansion of $h_P(z)$ in terms of ε, from which one can see the effect of the boundary curvature. In this paper, we need to expand it up to $O(\varepsilon^2)$. To be more specific, let $v^{(1)}$ be the unique solution of

$$\begin{cases} \Delta v - v = 0 & \text{in } \mathbb{R}^2_+ \\ \dfrac{\partial v}{\partial y_2} = \dfrac{w'}{|y|}\dfrac{\rho''(0)}{2} y_1^2 & \text{on } \partial\mathbb{R}^2_+, \end{cases} \tag{2.7}$$

where w' is the radial derivative of w, *i.e.*, $w' = w_r(r)$, and $r = |z - P|$.

Let $v^{(2)}$ be the unique solution of

$$\begin{cases} \Delta v - v - 2\rho''(0)y_1 \dfrac{\partial^2 v_1}{\partial y_1 \partial y_2} = 0 & \text{in } \mathbb{R}^2_+ \\ \dfrac{\partial v}{\partial y_2} = -\rho''(0)y_1 \dfrac{\partial v_1}{\partial y_1} & \text{on } \partial\mathbb{R}^2_+. \end{cases}$$

Let $v^{(3)}$ be the unique solution of

$$\begin{cases} \Delta v - v = 0 & \text{in } \mathbb{R}^2_+ \\ \dfrac{\partial v}{\partial y_2} = \dfrac{w'}{|y|} \frac{1}{3} \rho^{(3)}(0) y_1^3 & \text{on } \partial\mathbb{R}^2_+. \end{cases} \tag{2.8}$$

Note that $v^{(1)}$, $v^{(2)}$ are even functions in y_1 and $v^{(3)}$ is an odd function in y_1. Moreover, it is easy to see that $|v_i(y)| \le Ce^{-\mu|y|}$ for any $0 < \mu < 1$. Let $\chi(x)$ be a smooth cut-off function such that $\chi(x) = 1, x \in B(0, R_0\varepsilon|\ln\varepsilon|)$, and $\chi(x) = 0$ for $x \in B(0, 2R_0\varepsilon|\ln\varepsilon|)^c$ for R_0 large enough, and $\chi_\varepsilon(z) = \chi(\varepsilon z)$ for $z \in \Omega_\varepsilon$. In this case, one has $w(R_0|\ln\varepsilon|) = O(\varepsilon^{R_0})$. Set

$$h_P(z) = -\Big(\varepsilon v_1(y) + \varepsilon^2\big(v_2(y) + v_3(y)\big)\Big)\chi_\varepsilon(z - P) + \varepsilon^3 \xi_P(z),\ z \in \Omega_\varepsilon. \tag{2.9}$$

Then, we have the following estimate:

Proposition 2.1.

$$\|\xi(z)\|_{H^1(\Omega_\varepsilon)} \le C. \tag{2.10}$$

Proposition 2.1 was proved in [45] by Taylor expansion and a rigorous estimate for the reminder using estimates for elliptic equations. Moreover, one can check that $|\xi(z)| \le Ce^{-\mu|z-P|}$ for some $0 < \mu < 1$.

In our proof, only the evenness property in y_1 of the functions $v^{(1)}$ and $v^{(2)}$ are used. However, for the function $v^{(3)}$, both the oddness property and the equation it satisfies will be used. In fact, it is from this term that the derivative of the curvature function appears.

Similarly, we know from [45] that:

Proposition 2.2.

$$\left[\frac{\partial w}{\partial \tau_P} - \frac{\partial \mathcal{P}_{\Omega_\varepsilon, P} w}{\partial \tau_P}\right](z - P) = \varepsilon\eta(y)\chi_\varepsilon(z - P) + \varepsilon^2 \eta_1(z),\ z \in \Omega_\varepsilon, \tag{2.11}$$

where η is the unique solution of the following equation:

$$\begin{cases} \Delta\eta - \eta = 0 & \text{in } \mathbb{R}^2_+ \\ \dfrac{\partial \eta}{\partial y_2} = -\dfrac{1}{2}\left(\dfrac{w''}{|y|^2} - \dfrac{w'}{|y|^3}\right)\rho''(0)y_1^3 - \dfrac{w'}{|y|}\rho''(0)y_1 & \text{on } \partial R^2_+. \end{cases} \tag{2.12}$$

Moreover,

$$\|\eta_1\|_{H^1(\Omega_\varepsilon)} \le C. \tag{2.13}$$

One can observe that $\eta(y)$ is an odd function in y_1. It can be seen that $|\eta_i(y)| \leq Ce^{-\mu|y|}$ for some $0 < \mu < 1$.

Finally, set

$$L_0 = \Delta - 1 + pw^{p-1}(z). \tag{2.14}$$

We have the following non-degeneracy property:

Lemma 2.3.

$$\mathrm{Ker}(L_0) \cap H^2_N(\mathbb{R}^2_+) = \mathrm{span}\left\{\frac{\partial w}{\partial y_1}\right\}, \tag{2.15}$$

where $H^2_N(\mathbb{R}^2_+) = \left\{u \in H^2(\mathbb{R}^2_+), \frac{\partial u}{\partial y_2} = 0 \text{ on } \partial\mathbb{R}^2_+\right\}$.

Proof. See [40, Lemma 4.2]. □

Next, we state a useful lemma that we will frequently use:

Lemma 2.4. *If* $|q_1 - q_2| \ll |q_1|$, *we have the following estimate:*

$$\int_{\mathbb{R}^2_+} pw(y)^{p-1}\big(w(y-q_1e_1)+w(y+q_2e_1)\big)\frac{\partial w}{\partial y_1}dy = O\big(|q_1-q_2|w(|q_1|)\big) \tag{2.16}$$

as $|q_1| \to \infty$, *where* e_1 *is the unit vector* $(1,0)$.

Proof. By the oddness of $\frac{\partial w}{\partial y_1}$ in y_1, one has

$$\begin{aligned}
&\int_{\mathbb{R}^2_+} pw(y)^{p-1}\big(w(y-q_1e_1)+w(y+q_2e_1)\big)\frac{\partial w}{\partial y_1}dy\\
=\ &\int_{\mathbb{R}^2_+} pw(y)^{p-1}\big(w(y-q_1e_1)-w(y-q_2e_1)\big)\frac{\partial w}{\partial y_1}dy\\
=\ &\int_{\mathbb{R}^2} pw(y)^{p-1}\left|\frac{\partial w}{\partial y_1}\right|O\big(w'(y-q_1e_1)|q_1-q_2|\big)dy\\
=\ &O(|q_1-q_2|)w(|q_1|).
\end{aligned}$$

□

Remark 2.5. In the following sections, we will denote by $y^i = (y^i_1, y^i_2)$ the transformation defined by (2.6) centered at the point P_i and by $v_i^{(j)}$ be the corresponding solutions in the expansion of h_{P_i}.

3. Liapunov-Schmidt reduction

In this section we reduce problem (2.2) to a finite dimensional one by the Liapunov-Schmidt reduction method. The argument, thus far, is quite standard. We leave most of the proofs to the appendix. We first introduce some notation. Let $H^2_N(\Omega_\varepsilon)$ be the Hilbert space defined by

$$H^2_N(\Omega_\varepsilon) = \left\{ u \in H^2(\Omega_\varepsilon) \Big| \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega_\varepsilon \right\}. \tag{3.1}$$

Define

$$S(u) = \Delta u - u + u^p \tag{3.2}$$

for $u \in H^2_N(\Omega_\varepsilon)$. Then solving equation (2.2) is equivalent to solve

$$S(u) = 0 \quad \text{with} \quad u \in H^2_N(\Omega_\varepsilon). \tag{3.3}$$

To this end, we first study the linearized operator

$$L_\varepsilon(\psi) := \Delta\psi - \psi + p\left(\sum_{i=1}^{k} \mathcal{P}_{\Omega_{\varepsilon,P_i}} w(z - P_i)\right)^{p-1} \psi,$$

and define the approximate kernels to be

$$Z_i = \frac{\partial \mathcal{P}_{\Omega_{\varepsilon,P_i}} w(z - P_i)}{\partial \tau_{P_i}},$$

for $i = 1, \cdots, k$.

3.1. Linear projected problem

We first develop a solvability theory for the linear projected problem:

$$\begin{cases} L_\varepsilon(\psi) = h + \sum_{i=1}^{k} c_i Z_i \\ \int_{\Omega_\varepsilon} \psi Z_i dz = 0, i = 1, \cdots, k \\ \psi \in H^2_N(\Omega_\varepsilon). \end{cases} \tag{3.4}$$

Given $0 < \mu < 1$, consider the norm

$$\|h\|_* = \sup_{z \in \Omega_\varepsilon} \left| \left(\sum_j e^{-\mu|z - P_i|} \right)^{-1} h(z) \right|, \tag{3.5}$$

where $P_i \in \Lambda_k$, with Λ_k is defined in (1.15).

The proof of the following Proposition on linearized operator, which we postpone to the appendix, is, thus far, standard.

Proposition 3.1. *There exist positive numbers $\mu \in (0,1)$, ε_0 and C, such that for all $\varepsilon \le \varepsilon_0$, and for any set of points P_j, $j = 1, \dots, k$, given by* (1.15)*, there is a unique solution (ψ, c_i) to problem* (3.4)*. Furthermore,*

$$\|\psi\|_* \le C\|h\|_*. \tag{3.6}$$

In the following, if ψ is the unique solution given by Proposition 3.1, we set

$$\psi = \mathcal{A}(h). \tag{3.7}$$

Estimate (3.6) implies that

$$\|\mathcal{A}(h)\|_* \le C\|h\|_*. \tag{3.8}$$

3.2. Nonlinear projected problem

We now have sufficient context to solve the nonlinear equation:

$$\begin{cases} L_\varepsilon(\psi) + E + N(\psi) = \sum_{i=1}^{k} c_i Z_i \\ \int_{\Omega_\varepsilon} \psi Z_i = 0 \text{ for } i = 1, \cdots, k \\ \psi \in H_N^2(\Omega_\varepsilon) \end{cases} \tag{3.9}$$

where E is the error of the approximate solution U:

$$\begin{aligned} E = {} & \Delta\left(\sum_{i=1}^{k} \mathcal{P}_{\Omega_{\varepsilon,P_i}} w(z - P_i)\right) - \left(\sum_{i=1}^{k} \mathcal{P}_{\Omega_{\varepsilon,P_i}} w(z - P_i)\right) \\ & + \left(\sum_{i=1}^{k} \mathcal{P}_{\Omega_{\varepsilon,P_i}} w(z - P_i)\right)^p, \end{aligned} \tag{3.10}$$

and $N(\psi)$ is the nonlinear term:

$$\begin{aligned} N(\psi) = {} & \left(\left(\sum_{i=1}^{k} \mathcal{P}_{\Omega_{\varepsilon,P_i}} w(z - P_i)\right) + \psi\right)^p - \left(\sum_{i=1}^{k} \mathcal{P}_{\Omega_{\varepsilon,P_i}} w(z - P_i)\right)^p \\ & - p\left(\sum_{i=1}^{k} \mathcal{P}_{\Omega_{\varepsilon,P_i}} w(z - P_i)\right)^{p-1} \psi. \end{aligned} \tag{3.11}$$

By Proposition 3.1, we can rewrite (3.9) as

$$\psi = -\mathcal{A}\big(E + N(\psi)\big), \tag{3.12}$$

where $\mathcal{A}$ is the operator introduced in (3.7). In other words, ψ solves (3.9) if and only if ψ is a fixed point for the operator

$$T(\psi) := -\mathcal{A}\big(E + N(\psi)\big).$$

We show that the operator T defined above for $\psi \in H^2_N(\Omega_\varepsilon)$ is a contraction on

$$\mathcal{B} = \left\{\psi \in H^2_N(\Omega_\varepsilon) \;:\; \|\psi\|_* \le C\varepsilon,\; \int_{\Omega_\varepsilon} \psi Z_i = 0\right\}$$

for some adequately large $C > 0$.

In fact, we have the following lemma:

Lemma 3.2. *There exist $\mu \in (0, 1)$, and positive numbers ε_0, C, such that for all $\varepsilon \le \varepsilon_0$, for any set og points P_j, $j = 1, \ldots, k$, given by* (1.15)*, the following estimates hold:*

$$\|E\|_* \le C\varepsilon \tag{3.13}$$

and

$$\|N(\phi)\|_* \le C\|\phi\|_*^2. \tag{3.14}$$

Proof. We start with the proof for (3.13). Fix $j \in \{1, \ldots, k\}$ and consider the region $|z - P_j| \le \frac{\min\{|P_j - P_{j-1}|, |P_j - P_{j+1}|\}}{2}$. In this region, the error E, whose definition is given in (3.11), can be estimated in the following way:

$$\begin{aligned} |E(z)| &\le C w^{p-1}(z - P_j)\left[\sum_{P_i \ne P_j} w(z - P_i) + \sum_i h_{P_i}(z)\right] \\ &\le C(\varepsilon + \varepsilon^{\frac{p-\mu}{2}})e^{-\mu|z-P_j|} \le C\varepsilon e^{-\mu|z-P_j|}, \end{aligned} \tag{3.15}$$

if we choose an adequately small μ such that $p - \mu > 2$.

Now, consider the region $|z - P_j| > \frac{\min\{|P_j - P_{j-1}|, |P_j - P_{j+1}|\}}{2}$, for all j. From the definition of E, we obtain in the region under consideration

$$\begin{aligned} |E(z)| &\le C\left[\sum_i h_{P_i}(z) + \left(\sum_{i=1}^k P_{\Omega_{\varepsilon, P_i}} w(z - P_i)\right)^p - \sum_i w(x - P_i)^p\right] \\ &\le C\sum_i e^{-\mu|z-P_i|}\left(\varepsilon + \varepsilon^{\frac{p-\mu}{2}}\right) \\ &\le C\varepsilon \sum_i e^{-\mu|z-P_i|}. \end{aligned} \tag{3.16}$$

From (3.15) and (3.16), we obtain (3.13).

We now prove (3.14). Let $\psi \in \mathcal{B}$. Then,

$$
\begin{aligned}
|N(\psi)| \leq \Bigg| & \left(\left(\sum_{i=1}^{k} P_{\Omega_{\varepsilon,P_i}} w(z-P_i)\right)+\psi\right)^p - \left(\sum_{i=1}^{k} P_{\Omega_\varepsilon} w(z-P_i)\right)^p \\
& - p\left(\sum_{i=1}^{k} P_{\Omega_{\varepsilon,P_i}} w(z-P_i)\right)^{p-1}\psi \Bigg| \leq C\psi^2.
\end{aligned}
$$

Thus, we have

$$
\left|\left(\sum_j e^{-\mu|x-P_j|}\right)^{-1} N(\psi)\right| \leq C\|\psi\|_*^2 .
$$

This gives (3.14). □

Using the above estimates, we validate the following result:

Proposition 3.3. *There exist* $\mu \in (0,1)$ *and positive numbers* ε_0, C *such that for all* $\varepsilon \leq \varepsilon_0$, *for any set of points* P_j, $j = 1, \ldots, k$, *given by* (1.15), *there is a unique solution* (ψ, c_i) *to problem* (3.9). *This solution depends continuously on the parameters of the construction (namely* P_j, $j = 1, \ldots, k$*), and furthermore*

$$
\|\psi\|_* \leq C\varepsilon. \tag{3.17}
$$

Proof. As mentioned before, we show that the operator T is a contraction mapping in $\mathcal{B}$.

By the estimates in Lemma 3.2, (3.13), (3.14), and taking into account (3.8), we have, for any $\psi \in \mathcal{B}$,

$$
\begin{aligned}
\|T(\psi)\|_* &\leq C\,[\|E + N(\psi)\|_*] \leq C(\varepsilon+\varepsilon^2) \\
&\leq C_1\varepsilon
\end{aligned}
$$

for a proper choice of C_1 in the definition of $\mathcal{B}$. Take now ψ_1 and ψ_2 in $\mathcal{B}$. Then it is straightforward to show that

$$
\begin{aligned}
\|T(\psi_1) - T(\psi_2)\|_* &\leq C\|N(\psi_1) - N(\psi_2)\|_* \\
&\leq C\,[\|\psi_1\|_* + \|\psi_2\|_*]\ \|\psi_1 - \psi_2\|_* \\
&\leq o(1)\|\psi_1 - \psi_2\|_* .
\end{aligned}
$$

This means that T is a contraction mapping from $\mathcal{B}$ into itself.

The existence of a fixed point ψ now follows from the contraction mapping principle, and ψ is a solution of (3.9) and satisfying (3.17).

A direct consequence of the fixed point characterization of ψ given above, together with the fact that the error term E depends continuously (in the *-norm) on the parameters P_j $(j = 1, \ldots, k)$, is that the map $(P_1, \cdots, P_k) \to \psi$ into the space $C(\overline{\Omega}_\varepsilon)$ is continuous (in the $*$-norm). This concludes the proof of the Proposition. □

4. Further expansion of the error

In the previous section, we obtained a solution ψ to (3.9) which satisfies $\|\psi\|_* \leq C\varepsilon$. However, this estimate is not enough to solve the reduced problem. To proceed, we need to obtain the asymptotic behavior of the function ψ as $\varepsilon \to 0$. This is needed to compute the neighboring interactions. The idea is that although the $\|\cdot\|_*$ norm of ψ is not small enough, we can obtain a more accurate decomposition such that the projection with respect to Z_i is small enough for our purpose.

Before we state the result, we first consider the following equation:

$$\begin{cases} \Delta\phi - \phi + pw(y)^{p-1}\phi = h + d\dfrac{\partial w(y)}{\partial y_1} & \text{in } \mathbb{R}^2_+ \\ \dfrac{\partial\phi}{\partial y_2} = 0 & \text{on } \partial\mathbb{R}^2_+ \\ \displaystyle\int_{\mathbb{R}^2_+} \phi\frac{\partial w(y)}{\partial y_1}dy = 0, \end{cases} \tag{4.1}$$

where $d = -\frac{\int_{\mathbb{R}^2_+} h\frac{\partial w}{\partial y_1}}{\int_{\mathbb{R}^2_+}\left(\frac{\partial w}{\partial y_1}\right)^2}$. We consider the above equation in the space $\{\|h\|_{**} < +\infty\}$, where $\|h\|_{**} = \sup_{y\in\mathbb{R}^2_+} |e^{\mu_1|y|}h|$ for some $0 < \mu_1 < 1$. It is quite standard to show the solvability of the above equation, and the solution ϕ satisfies the following estimate:

$$\|\phi\|_{**} \leq C\|h\|_{**}. \tag{4.2}$$

Now, we decompose ψ as follows:

Proposition 4.1. *We may write*

$$\psi = \sum_{i=1}^{k} \chi_\varepsilon(z - P_i)\phi_i + \varepsilon^2\psi_1, \tag{4.3}$$

where

$$\|\psi_1\|_* \leq C, \tag{4.4}$$

and $\phi_i = \phi_i(y^i)$ *is the unique solution of*

$$\begin{cases} \Delta\phi_i - \phi_i + pw(y^i)^{p-1}\phi_i = H_i + d_i\dfrac{\partial w(y^i)}{\partial y_i} & \textit{in } \mathbb{R}^2_+ \\ \dfrac{\partial\phi_i}{\partial y_2} = 0 & \textit{on } \partial\mathbb{R}^2_+ \\ \displaystyle\int_{\mathbb{R}^2_+} \phi_i\frac{\partial w(y^i)}{\partial y_1}dy = 0, \end{cases} \tag{4.5}$$

where d_i is defined such that the right hand side of the above equation is orthogonal to $\frac{\partial w(y^i)}{\partial y_1}$ in L^2 norm, and

$$H_i = -pw(y^i)^{p-1}\left[w\left(y^i - \frac{s_{i-1}-s_i}{\varepsilon}e_1\right) + w\left(y^i - \frac{s_{i+1}-s_i}{\varepsilon}e_1\right) + \varepsilon v_i^{(1)}\right], \quad (4.6)$$

for $i = 2, \cdots, k-1$, while

$$H_1 = -pw(y^1)^{p-1}\left[w\left(y^1 - \frac{s_2-s_1}{\varepsilon}e_1\right) + \varepsilon v_1^{(1)}\right], \quad (4.7)$$

and

$$H_k = -pw(y^k)^{p-1}\left[w\left(y^k - \frac{s_{k-1}-s_k}{\varepsilon}e_1\right) + \varepsilon v_k^{(1)}\right], \quad (4.8)$$

and where we have set

$$v_i^{(1)} = v_{P_i}^{(1)}\left(y^i\right) \quad (4.9)$$

for the solutions obtained in Section 1.2 centered at the point P_i.

Proof. First, by the definition of d_i, there holds

$$d_i = -\int_{\mathbb{R}^2_+} H_i \frac{\partial w(y^i)}{\partial y_1} dy. \quad (4.10)$$

Then, from Lemma 2.4, the evenness of $v_i^{(1)}$ with respect to y_1^i, and the definition of the configuration space (1.15), we know that for $i = 2, \cdots, k-1$

$$\begin{aligned}|d_i| &\le C\varepsilon^{-1}\big||s_{i+1}-s_i| - |s_i - s_{i-1}|\big| \min\left\{w\left(\frac{s_i - s_{i+1}}{\varepsilon}\right), w\left(\frac{s_i - s_{i-1}}{\varepsilon}\right)\right\} \\ &\le C\varepsilon^2,\end{aligned} \quad (4.11)$$

and for $i = 1, k$,

$$|d_1| = O\left(w\left(\frac{s_1-s_2}{\varepsilon}\right)\right) = O\left(\varepsilon^2\right) \text{ and } |d_k| = O\left(w\left(\frac{s_k - s_{k-1}}{\varepsilon}\right)\right) = O(\varepsilon^2). \quad (4.12)$$

Moreover, from (4.1), we have the following estimate:

$$\|\phi_i\|_{**} \le C\varepsilon \text{ if } p > 2 + \mu_1. \quad (4.13)$$

Our strategy is to estimate ψ_1 in order to decompose ψ_1 into three parts and show that each of them is bounded in $\|\cdot\|_*$ as $\varepsilon \to 0$. We write ψ_1 as

$$\psi_1 = \psi_{11} + \psi_{12} + \psi_{13}, \quad (4.14)$$

where ψ_{11} satisfies

$$\begin{cases} \Delta\psi_{11} - \psi_{11} = 0 & \text{in } \Omega_\varepsilon \\ \dfrac{\partial \psi_{11}}{\partial \nu} = -\dfrac{1}{\varepsilon^2}\dfrac{\partial \sum_{i=1}^k \chi_\varepsilon(z-P_i)\phi_i}{\partial \nu} & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{4.15}$$

Define ψ_{12} by

$$\psi_{12} = \frac{1}{\varepsilon^2}\sum_{i=1}^k s_i Z_i, \tag{4.16}$$

where s_i is determined by

$$M(s_i) = -\int_{\Omega_\varepsilon}\left(\sum_{i=1}^k \chi_\varepsilon(z-P_i)\phi_i + \varepsilon^2\psi_{11}\right) Z_i. \tag{4.17}$$

Finally, define ψ_{13} to be the solution of the following equation:

$$\begin{cases} L_\varepsilon(\psi_{13}) = \dfrac{1}{\varepsilon^2} L_\varepsilon\left(\psi - \sum_{i=1}^k \chi_\varepsilon(z-P_i)\phi_i - \varepsilon^2(\psi_{11}+\psi_{12})\right) & \text{in } \Omega_\varepsilon \\ \dfrac{\partial \psi_{13}}{\partial \nu} = 0 & \text{on } \partial\Omega_\varepsilon \\ \displaystyle\int_{\Omega_\varepsilon} \psi_{13} Z_i dz = 0. \end{cases} \tag{4.18}$$

Next, we estimate $\psi_{11}, \psi_{12}, \psi_{13}$ term by term.

First, we estimate $g_{1\varepsilon} = \frac{1}{\varepsilon^2}\frac{\partial \sum_{i=1}^k \chi_\varepsilon(z-P_i)\phi_i}{\partial \nu}$. By direct calculation,

$$\begin{aligned} g_{1\varepsilon} &= \frac{1}{\varepsilon^2}\sum_{i=1}^k\left(\chi_\varepsilon(z-P_i)\frac{\partial \phi_i}{\partial \nu} + \phi_i \frac{\partial \chi_\varepsilon(z-P_i)}{\partial \nu}\right) \\ &= \frac{1}{\varepsilon^2}\sum_{i=1}^k \varepsilon e^{-\mu_1|y-\frac{s_i}{\varepsilon}|}\frac{\partial \chi_\varepsilon(z-P_i)}{\partial \nu} + O\left(\varepsilon^2\right) \\ &= O\left(\varepsilon^{-2} e^{-(\mu_1-\mu)R_0|\ln\varepsilon|}\right)\sum_{i=1}^k e^{-\mu|z-P_i|} \\ &\le C\sum_{i=1}^k e^{-\mu|z-P_i|}, \end{aligned}$$

if we choose $\mu_1 > \mu$ and the cutoff function in such a way that $(\mu_1-\mu)R_0 \ge 1$. In the above estimate, we use the definition of ϕ_i, , the Neumann boundary conditions satisfied by it, and the definition of the cut-off function χ. Thus, we have that $\|g_{1\varepsilon}\|_* \le C$. Therefore, there exists a constant $C > 0$ such that

$$\|\psi_{11}\|_* \le C. \tag{4.19}$$

By the definition of ψ_{12}, ϕ_i and the estimate on ψ_{11}, one can obtain that

$$\begin{aligned}\int_{\Omega_\varepsilon}\left(\sum_{j=1}^{k}\chi_\varepsilon(z-P_j)\phi_j+\varepsilon^2\psi_{11}\right)Z_i dz &= \int_{\Omega_\varepsilon}\chi_\varepsilon(z-P_i)\phi_i Z_i dz\\ &\quad+\sum_{j=i-1,i+1}\chi_\varepsilon(z-p_j)\phi_j Z_i dz\\ &\quad+O\left(\varepsilon^{1+(1+\mu)(1+o(1))}\right)+O\left(\varepsilon^2\right).\end{aligned}$$

In order to estimate the above term, we first consider a general function that is the solution of the following equation:

$$\begin{cases}\Delta\phi-\phi+pw(y)^{p-1}\phi=-pw(y)^{p-1}\Big(w(y-q_1e_1)\\ \qquad\qquad+w(y+q_2e_1)+\varepsilon v^{(1)}\Big)+d\dfrac{\partial w(y)}{\partial y_1} & \text{in } \mathbb{R}^2_+\\ \dfrac{\partial\phi}{\partial y_2}=0 & \text{on } \partial\mathbb{R}^2_+\\ \displaystyle\int_{\mathbb{R}^2_+}\phi\frac{\partial w(y)}{\partial y_1}dy=0.\end{cases} \tag{4.20}$$

We can decompose it as

$$\phi=\phi^1+\phi^2, \tag{4.21}$$

where

$$\begin{cases}\Delta\phi^1-\phi^1+pw(y)^{p-1}\phi^1\\ =-pw(y)^{p-1}\Big(w(y-q_1e_1)+w(y+q_1e_1)+\varepsilon v^{(1)}\Big)\\ \qquad+d_1\dfrac{\partial w(y)}{\partial y_1} & \text{in } \mathbb{R}^2_+\\ \dfrac{\partial\phi^1}{\partial y_2}=0 & \text{on } \partial\mathbb{R}^2_+\\ \displaystyle\int_{\mathbb{R}^2_+}\phi^1\frac{\partial w(y)}{\partial y_1}dy=0\end{cases} \tag{4.22}$$

and

$$\begin{cases}\Delta\phi^2-\phi^2+pw(y)^{p-1}\phi^2\\ =-pw(y)^{p-1}\Big(w(y+q_2e_1)-w(y+q_1e_1)\Big)+d_2\dfrac{\partial w(y)}{\partial y_1} & \text{in } \mathbb{R}^2_+\\ \dfrac{\partial\phi^2}{\partial y_2}=0 & \text{on } \partial\mathbb{R}^2_+\\ \displaystyle\int_{\mathbb{R}^2_+}\phi^2\frac{\partial w(y)}{\partial y_1}dy=0,\end{cases} \tag{4.23}$$

where d_i are defined such that the right hand sides of the above equations are orthogonal to $\frac{\partial w}{\partial y_1}$ in the L^2 norm. It is easy to see that ϕ^1 is even in y_1 and that, by Lemma 2.4, ϕ^2 satisfies

$$\|\phi^2\|_{**} \leq C w(q_1)|q_1 - q_2|,$$

if $|q_1 - q_2| \ll |q_1|$ and $|q_1| \to \infty$.

Using the above estimates, we can decompose ϕ_i as

$$\phi_i = \phi_{i,1} + \phi_{i,2} \tag{4.24}$$

with $\phi_{i,1}$ even in y_1^i and

$$\begin{aligned} \|\phi_{i,2}\|_{**} &\leq C \left| \left| \frac{s_i - s_{i-1}}{\varepsilon} \right| - \left| \frac{s_i - s_{i+1}}{\varepsilon} \right| \right| \min \left\{ w\left(\frac{s_i - s_{i-1}}{\varepsilon}\right), w\left(\frac{s_i - s_{i+1}}{\varepsilon}\right) \right\} \\ &\leq C\varepsilon^2. \end{aligned} \tag{4.25}$$

Then, by the above estimate and the decomposition in Proposition 2.2, we have

$$\int_{\Omega_\varepsilon} \chi_\varepsilon(z - P_i)\phi_i Z_i dz = O\left(\varepsilon^2\right), \tag{4.26}$$

and similar to the decomposition of ϕ_i, one can also decompose $\phi_{i-1} + \phi_{i+1}$ as an even function of y_1^i and an $O(\varepsilon^2)$ function; thus, we obtain

$$\begin{aligned} \sum_{j=i-1,i+1} \int_{\Omega_\varepsilon} \chi_\varepsilon(z - P_j)\phi_j Z_i dz &= \int_{\mathbb{R}^2_+} (\phi_{i-1} + \phi_{i+1}) \frac{\partial w(y^i)}{\partial y_1} dy + O\left(\varepsilon^2\right) \\ &= O\left(\varepsilon^2\right). \end{aligned}$$

Moreover, since $|s_1 - s_2| = 2(1 + o(1))|\varepsilon \ln \varepsilon|$ and $|s_{k-1} - s_k| = 2(1 + o(1))|\varepsilon \ln \varepsilon|$, one can obtain

$$\int_{\Omega_\varepsilon} \chi_\varepsilon(z - P_2)\phi_2 Z_1 dz = O(\varepsilon^2), \int_{\Omega_\varepsilon} \chi_\varepsilon(z - P_{k-1})\phi_{k-1} Z_k dz = O\left(\varepsilon^2\right). \tag{4.27}$$

Thus, we have

$$|s_i| \leq C\varepsilon^2. \tag{4.28}$$

Next, we estimate ψ_{13}. Set

$$f_\varepsilon = L_\varepsilon \left(\psi - \sum_{i=1}^{k} \chi_\varepsilon(z - p_i)\phi_i - \varepsilon^2(\psi_{11} + \psi_{12}) \right).$$

We claim that

$$\|f_\varepsilon\|_* \leq C\varepsilon^2. \tag{4.29}$$

Indeed by the definition of f_ε we have

$$
\begin{aligned}
f_\varepsilon(z) &= L_\varepsilon\left(\psi - \sum_{i=1}^{k} \chi_\varepsilon(z-P_i)\phi_i - \varepsilon^2(\psi_{11}-\psi_{12})\right)\\
&= E + N(\psi) + \sum_i c_i Z_i - \sum_i L_\varepsilon(\chi_\varepsilon(z-P_i)\phi_i) - \varepsilon^2 L(\psi_{11}+\psi_{12})\\
&= \left(\sum_i \mathcal{P}_{\Omega_{\varepsilon,P_i}} w(z-P_i)\right)^p - \sum_i w(z-P_i)^p + N(\psi) + \sum_i c_i Z_i\\
&\quad - \sum_i \chi_\varepsilon(z-P_i)\left(\Delta_y\phi_i - \phi_i + p\left(\left(\sum_i \mathcal{P}_{\Omega_\varepsilon} w(z-P_i)\right)^{p-1} + O(\varepsilon)\right)\phi_i\right)\\
&\quad + \sum_i \left(2\nabla\phi_i \nabla\left(\chi_\varepsilon(z-P_i)\right) + \phi_i \Delta\chi_\varepsilon(z-P_i)\right) - \varepsilon^2 L_\varepsilon(\psi_{11}+\psi_{12})\\
&= \left(\sum_i \mathcal{P}_{\Omega_{\varepsilon,P_i}} w(z-P_i)\right)^p - \sum_i w(z-P_i)^p + N(\psi) + \sum_i c_i Z_i\\
&\quad - \sum_i \chi_\varepsilon(z-P_i)\left(p\left(\left(\sum_i \mathcal{P}_{\Omega_{\varepsilon,P_i}} w(z-P_i)\right)^{p-1} - w(y-P_i)^{p-1}\right)\phi_i\right.\\
&\quad \left. - pw(y-P_i)^{p-1}\left(w(y-P_{i-1}) + w(y-P_{i+1}) + \varepsilon v_{1i}(y)\right) + d_i Z_i\right)\\
&\quad + \sum_i O(\varepsilon)\phi_i + \sum_i \Big(2\nabla\phi_i \nabla\left(\chi_\varepsilon(z-p_i)\right) + \phi_i \Delta\chi_\varepsilon(z-P_i)\Big)\\
&\quad - \varepsilon^2 L_\varepsilon(\psi_{11}+\psi_{12}).
\end{aligned}
$$

From the definition, and from theestimates of ϕ_i, ψ_{11}, ψ_{12}, χ, and the configuration space, we know that $|c_i| = O(\varepsilon^2)$, so

$$\|f_\varepsilon\|_* \le C\varepsilon^2.$$

By the *a priori* estimate, we know that

$$\|\psi_{13}\|_* \le C,$$

thus, we have

$$\|\psi_1\|_* \le C.$$

We thus complete the proof. □

Given points P_j defined by (1.15), Proposition 3.3 guarantees the existence (and provides estimates) of a unique solution ψ, and c_i, for $i = 1, \dots, k$, to the problem (3.9). It is clear then that the function $u = U + \psi$ is an exact solution to our problem (1.1), with the required properties stated in Theorem 1.5 if we show that there exists a configuration for the points P_j that provides all the constants c_i in (3.9) equal to zero. In order to do so, we first need to find the correct conditions on the points to obtain $c_i = 0$. This condition is naturally given by projecting in $L^2(\Omega_\varepsilon)$ the equation in (3.9) into the space spanned by Z_i, namely by multiplying the equation in (3.9) by Z_i and integrating all over Ω_ε. We do so in detail in the next section.

5. The reduced problem

In this section we keep the notation and the assumptions of the previous sections. As explained in the previous section, we have obtained a solution $u = \sum_{i=1}^k \mathcal{P}_{\Omega_\varepsilon, P_i} w(z - P_i) + \sum_{i=1}^k \chi_\varepsilon(z - P_i)\phi_i + \varepsilon^2 \psi_1$ of the following equation

$$\begin{cases} \Delta u - u + u^p = \sum_{i=1}^{k} c_i Z_i & \text{in } \Omega_\varepsilon \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{5.1}$$

In this section we solve $c_i = 0$ for all i by adjusting the position of the spikes, *i.e.*, P_i. First, multiplying the above equation (5.1) by Z_i, $i = 1, \cdots, k$ and integrating over Ω_ε, we obtain

$$M \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} \int_{\Omega_\varepsilon} (\Delta u - u + u^p) Z_1 \\ \int_{\Omega_\varepsilon} (\Delta u - u + u^p) Z_2 \\ \vdots \\ \int_{\Omega_\varepsilon} (\Delta u - u + u^p) Z_k \end{bmatrix}. \tag{5.2}$$

Recall that M is invertible, so the equation $c_i = 0$, $for all i = 1, \cdots, k$, is reduced to solve the following system:

$$\begin{bmatrix} \int_{\Omega_\varepsilon} (\Delta u - u + u^p) Z_1 \\ \int_{\Omega_\varepsilon} (\Delta u - u + u^p) Z_2 \\ \vdots \\ \int_{\Omega_\varepsilon} (\Delta u - u + u^p) Z_k \end{bmatrix} = 0. \tag{5.3}$$

We have the following estimates:

Lemma 5.1. *Under the assumption of Proposition* 3.3, *for* ε *small enough, the following expansion holds:*

$$\int_{\Omega_\varepsilon} (\Delta u - u + u^p) Z_1 dz = -\Psi\left(\frac{s_1 - s_2}{\varepsilon}\right) - \varepsilon^2 \nu_2 H'(\gamma(s_1)) + O\left(\varepsilon^3\right), \quad (5.4)$$

and moreover, for $i = 2, \cdots, k-1$, *there holds*

$$\begin{aligned} \int_{\Omega_\varepsilon} \left(\Delta u - u + u^p\right) Z_i dz = \Psi\left(\frac{s_i - s_{i-1}}{\varepsilon}\right) - \Psi\left(\frac{s_i - s_{i+1}}{\varepsilon}\right) \\ - \varepsilon^2 \nu_2 H'(\gamma(s_i)) + O\left(\varepsilon^3\right) \end{aligned} \quad (5.5)$$

and finally

$$\int_{\Omega_\varepsilon} \left(\Delta u - u + u^p\right) Z_k dz = \Psi\left(\frac{s_k - s_{k-1}}{\varepsilon}\right) - \varepsilon^2 \nu_2 H'(\gamma(s_k)) + O\left(\varepsilon^3\right), \quad (5.6)$$

where $\nu_2 > 0$ *is a constant defined in* (5.12).

Proof. First, by direct calculation, one can obtain the following expansion:

$$\begin{aligned} & \Delta u - u + u^p \\ = & \left[\Delta\left(U + \sum_{i=1}^{k} \chi_\varepsilon(z - P_i)\phi_i\right) - \left(U + \sum_{i=1}^{k} \chi_\varepsilon(z - P_i)\phi_i\right)\right. \\ & \left. + \left(U + \sum_{i} \chi_\varepsilon(z - P_i)\phi_i\right)^p\right] \\ & + \left[\varepsilon^2\left(\Delta\psi_1 - \psi_1 + p\left(U + \sum_{i=1}^{k} \chi_\varepsilon(z - P_i)\phi_i\right)^{p-1} \psi_1\right)\right] \\ & + \left[\left(U + \sum_{i=1}^{k} \chi_\varepsilon(z - P_i)\phi_i + \varepsilon^2\psi_1\right)^p - \left(U + \sum_{i=1}^{k} \chi_\varepsilon(z - P_i)\phi_i\right)^p\right. \\ & \left. - p\left(U + \sum_{i=1}^{k} \chi_\varepsilon(z - P_i)\phi_i\right)^{p-1} \varepsilon^2\psi_1\right] \\ := & I_1 + I_2 + I_3. \end{aligned}$$

Next, we calculate I_1 to I_3 term by term. First, from the estimate on ψ_1 in (4.4),

$$\begin{aligned}\int_{\Omega_\varepsilon} I_2 Z_i &= \varepsilon^2 \int_{\Omega_\varepsilon} \left(\Delta\psi_1 - \psi_1 + p\left(U + \sum_i \chi_\varepsilon(z-p_i)\phi_i\right)^{p-1} \psi_1 \right) Z_i \\ &= \varepsilon^2 \int_{\Omega_\varepsilon} -pw(z-P_i)^{p-1} \frac{\partial w(z-P_i)}{\partial \tau} \psi_1 \\ &\quad + p\left(U + \sum_i \chi_\varepsilon(z-P_i)\phi_i\right)^{p-1} Z_i \psi_1 \\ &= \varepsilon^2 \int_{\Omega_\varepsilon} p(p-1) w(z-P_i)^{p-2} \frac{\partial w(z-P_i)}{\partial \tau} \psi_1 \left(\sum_{j\neq i} \frac{\partial w(z-P_j)}{\partial \tau} + O(\varepsilon) \right) dz \\ &= O\left(\varepsilon^3\right).\end{aligned} \tag{5.7}$$

Moreover,

$$\begin{aligned}\int_{\Omega_\varepsilon} I_3 Z_i = \int_{\Omega_\varepsilon} \Bigg[&\left(U + \sum_i \chi_\varepsilon(z-P_i)\phi_i + \varepsilon^2\psi_1 \right)^p \\ &- \left(U + \sum_i \chi_\varepsilon(z-P_i)\phi_i \right)^p \\ &- p(U + \sum_i \chi_\varepsilon(z-P_i)\phi_i)^{p-1} \varepsilon^2 \psi_1 \Bigg] Z_i \\ \leq C \int_{\Omega_\varepsilon} &\varepsilon^4 |\psi_1|^2 |Z_i| = O\left(\varepsilon^3\right).\end{aligned} \tag{5.8}$$

Next, from the equation satisfied by ϕ_i and the definition of the cutoff function χ, we obtain

$$\begin{aligned}&\int_{\Omega_\varepsilon} I_1 Z_i \\ &= \int_{\Omega_\varepsilon} \left(\Delta U - U + U^p\right) Z_i + \sum_j \int_{\Omega_\varepsilon} \chi_\varepsilon(z-P_j) \left(\Delta\phi_j - \phi_j + pU^{p-1}\phi_j\right) Z_i \\ &\quad + \left[\left(U + \sum_i \chi_\varepsilon(z-P_i)\phi_i \right)^p - U^p - pU^{p-1} \sum_i \chi_\varepsilon(z-p_i)\phi_i \right] Z_i + O\left(\varepsilon^3\right) \\ &= \int_{\Omega_\varepsilon} (\Delta U - U + U^p) Z_i + I_{11} + I_{12} + O(\varepsilon^3).\end{aligned} \tag{5.9}$$

In the following, we show that, although ϕ_i is of $O(\varepsilon)$, after projection with respect to Z_i, the terms containing ϕ_i are indeed $O(\varepsilon^3)$.

Similar to the estimate in (4.26), using the equation satisfied by ϕ_i, we have for $i = 2, \cdots, k-1$

$$\begin{aligned}
&\sum_j \int_{\Omega_\varepsilon} \chi_\varepsilon(z-P_j)\left(\Delta\phi_j - \phi_j + pU^{p-1}\phi_j\right) Z_i \\
&= \int_{\Omega_\varepsilon} \chi_\varepsilon(z-P_i)\left(\Delta\phi_j - \phi_j + pU^{p-1}\phi_j\right) Z_i \\
&\quad + \sum_{j\neq i} \int_{\Omega_\varepsilon} \chi_\varepsilon(z-P_j)\left(\Delta\phi_j - \phi_j + pU^{p-1}\phi_j\right) Z_i \\
&= \int_{\Omega_\varepsilon} p\left(U^{p-1} - w_i^{p-1}\right)\phi_i Z_i + O\left(\varepsilon^3\right) \\
&\quad + \sum_{j=i-1,i+1} \int_{\Omega_\varepsilon} \chi(z-P_j)\left(\Delta\phi_j - \phi_j + pU^{p-1}\phi_j\right) Z_i + O\left(\varepsilon^3\right) \\
&= \int_{\Omega_\varepsilon} p(p-1)w_i^{p-2}(w_{i+1}+w_{i-1})\phi_i Z_i \\
&\quad + \sum_{j=i-1,i+1} \int_{\Omega_\varepsilon} \chi_\varepsilon(z-P_j)\left(\Delta\phi_j - \phi_j + pU^{p-1}\phi_j\right) Z_i + O\left(\varepsilon^3\right) \\
&= O\left(\varepsilon\left|\left|\frac{s_i-s_{i-1}}{\varepsilon}\right| - \left|\frac{s_i-s_{i+1}}{\varepsilon}\right|\right| \min\left\{w\left(\frac{s_i-s_{i-1}}{\varepsilon}\right), w\left(\frac{s_i-s_{i+1}}{\varepsilon}\right)\right\}\right) + O\left(\varepsilon^3\right) \\
&= O(\varepsilon^3)
\end{aligned}$$

and similarly, we can always decompose

$$\sum_{j=1}^k \chi_\varepsilon(z-P_j)\phi_j = \varepsilon\psi_{1,i} + O\left(\varepsilon^2\right),$$

where $\psi_{1,i}$ is a function even in y_1^i. By Proposition 2.2, we have

$$Z_i = \frac{\partial w(y^i)}{\partial y_1} + \varepsilon\eta_i + O\left(\varepsilon^2\right),$$

where η_i is odd in y_1^i. Thus, we have

$$\begin{aligned}
I_{12} &\leq C\int_{\Omega_\varepsilon} p(p-2)w_i^{p-2}\left(\sum_{j=1}^k \chi_\varepsilon(z-P_j)\phi_j\right)^2 Z_i dz + O\left(\varepsilon^3\right) \\
&\leq C\varepsilon^3.
\end{aligned}$$

For the case $i = 1, k$, recall that $w\big(\frac{s_1-s_2}{\varepsilon}\big), w\big(\frac{s_k-s_{k-1}}{\varepsilon}\big) = O(\varepsilon^2)$, one can also obtain

$$I_{11} + I_{12} = O(\varepsilon^3). \tag{5.10}$$

Thus, we have the following:

$$\int_{\Omega_\varepsilon} I_1 Z_i dz = \int_{\Omega_\varepsilon} \left(\Delta U - U + U^p\right) Z_i + O\left(\varepsilon^3\right).$$

Next, for $i = 2, \cdots, k-1$,

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \left(\Delta U - U + U^p\right) Z_i \\
&= \int_{\Omega_\varepsilon} \left[\left(\sum_i \mathcal{P}_{\Omega_\varepsilon, P_i} w(z - P_i)\right)^p - \sum_i w(z - P_i)^p\right] Z_i \\
&= \int_{\Omega_\varepsilon} \left[\left(w(z - P_i) + \varepsilon v_i^{(1)} + \varepsilon^2 \left(v_i^{(2)} + v_i^{(3)}\right)\right.\right. \\
&\quad \left.\left. + \sum_{j \neq i} \mathcal{P}_{\Omega_\varepsilon, P_i} w(z - P_j) + O(\varepsilon^3)\right)^p - \sum_i w(z - P_i)^p\right] Z_i \\
&= \int_{\Omega_\varepsilon} p w(z - P_i)^{p-1}\left(\varepsilon v_i^{(1)} + \varepsilon^2 \left(v_i^{(2)} + v_i^{(3)}\right)\right. \\
&\quad \left. + w(z - P_{i-1}) + w(z - P_{i+1})\right) \frac{\partial w(z - P_i)}{\partial \tau} + O\left(\varepsilon^3\right) \\
&= \int_{\mathbb{R}^2_+} p w(y) \left(w\left(y - \frac{s_{i-1} - s_i}{\varepsilon} e_1\right) + w\left(y - \frac{s_{i+1} - s_i}{\varepsilon} e_1\right)\right) \frac{\partial w(y)}{\partial y_1} \\
&\quad + \varepsilon^2 \int_{\mathbb{R}^2_+} p w(y)^{p-1} \frac{\partial w(y)}{\partial y_1} v_i^{(3)} + O\left(\varepsilon^3\right).
\end{aligned} \tag{5.11}$$

Similarly, one has for $i = 1, k$,

$$\begin{aligned}
\int_{\Omega_\varepsilon} \left(\Delta U - U + U^p\right) Z_1 &= \int_{\mathbb{R}^2_+} p w(y) w\left(y - \frac{s_2 - s_1}{\varepsilon} e_1\right) \frac{\partial w(y)}{\partial y_1} \\
&\quad + \varepsilon^2 \int_{\mathbb{R}^2_+} p w(y)^{p-1} \frac{\partial w(y)}{\partial y_1} v_1^{(3)} + O\left(\varepsilon^3\right),
\end{aligned}$$

and

$$\begin{aligned}
\int_{\Omega_\varepsilon} (\Delta U - U + U^p) Z_k &= \int_{\mathbb{R}^2_+} p w(y) w\left(y - \frac{s_{k-1} - s_k}{\varepsilon} e_1\right) \frac{\partial w(y)}{\partial y_1} \\
&\quad + \varepsilon^2 \int_{\mathbb{R}^2_+} p w(y)^{p-1} \frac{\partial w(y)}{\partial y_1} v_k^{(3)} + O\left(\varepsilon^3\right).
\end{aligned}$$

Next, by the definition of $v_i^{(3)}$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2_+} pw(y)^{p-1}\frac{\partial w(y)}{\partial y_1}v_i^{(3)}dy &= \int_{\mathbb{R}^2_+} -(\Delta-1)\frac{\partial w(y)}{\partial y_1}v_i^{(3)} \\ &= -\int_{\partial\mathbb{R}^2_+} \frac{\partial w(y)}{\partial y_1}\frac{\partial v_i^{(3)}}{\partial y_2} - v_i^{(3)}\frac{\partial}{\partial y_2}\frac{\partial w(y)}{\partial y_1}dy \\ &= -\frac{1}{3}\int_{\mathbb{R}}\left(\frac{w'(|y|)}{|y|}\right)^2\rho^{(3)}(P_i)y_1^4dy_1 \\ &= -\nu_2\rho^{(3)}(P_i) = -\nu_2 H'(\gamma(s_i)),\end{aligned} \tag{5.12}$$

where $\nu_2 = \frac{1}{3}\int_{\mathbb{R}}\left(\frac{w'}{|y|}\right)^2 y_1^4 > 0$ is a positive constant.

Recall that the interaction function is defined by

$$\Psi(s) = -\int_{\mathbb{R}^2_+} pw(y-(s,0))w(y)^{p-1}\frac{\partial w(y)}{\partial y_1}dy. \tag{5.13}$$

Combining (5.9), (5.10), (5.11), (5.12) and (5.13), we know that

$$\int_{\Omega_\varepsilon} I_1 Z_1 dz = -\Psi\left(\left|\frac{s_1-s_2}{\varepsilon}\right|\right) - \varepsilon^2\nu_2 H'(\gamma(s_1)) + O\left(\varepsilon^3\right), \tag{5.14}$$

that for $i = 2, \cdots, k-1$

$$\begin{aligned}\int_{\Omega_\varepsilon} I_1 Z_i dz = \Psi\left(\left|\frac{s_i-s_{i-1}}{\varepsilon}\right|\right) - \Psi\left(\left|\frac{s_i-s_{i+1}}{\varepsilon}\right|\right) - \varepsilon^2\nu_2 H'(\gamma(s_i)) \\ + O\left(\varepsilon^3\right)\end{aligned} \tag{5.15}$$

and that

$$\int_{\Omega_\varepsilon} I_1 Z_k dz = \Psi\left(\left|\frac{s_k-s_{k-1}}{\varepsilon}\right|\right) - \varepsilon^2\nu_2 H'(\gamma(s_i)) + O(\varepsilon^3). \tag{5.16}$$

The results follows from (5.7), (5.8) and (5.14)-(5.16). □

From Lemma 5.1, the problem (5.3) is reduced to the following system:

$$\begin{cases}\Psi_1\left(\left|\frac{s_1-s_2}{\varepsilon}\right|\right) + \varepsilon^2 H'\left(\gamma(s_1)\right) = O\left(\varepsilon^3\right) \\ \Psi_1\left(\left|\frac{s_3-s_2}{\varepsilon}\right|\right) - \Psi_1\left(\left|\frac{s_2-s_1}{\varepsilon}\right|\right) + \varepsilon^2 H'(\gamma(s_2)) = O\left(\varepsilon^3\right) \\ \vdots \\ \Psi_1\left(\left|\frac{s_k-s_{k-1}}{\varepsilon}\right|\right) - \Psi_1\left(\left|\frac{s_{k-1}-s_{k-2}}{\varepsilon}\right|\right) + \varepsilon^2 H'(\gamma(s_{k-1})) = O\left(\varepsilon^3\right) \\ -\Psi_1\left(\left|\frac{s_k-s_{k-1}}{\varepsilon}\right|\right) + \varepsilon^2 H'(\gamma(s_k)) = O\left(\varepsilon^3\right),\end{cases}$$

where we denote

$$\Psi_1(s) = \nu_2^{-1}\Psi(s). \tag{5.17}$$

By summing up the first i equations, one has

$$\begin{cases} \Psi_1\left(\frac{s_{i+1}-s_i}{\varepsilon}\right) + \sum_{j=1}^{i} \varepsilon^2 H'(\gamma(s_j)) = O\left(i\varepsilon^3\right) \quad \text{for } i = 1, \cdots, k-1 \\ \sum_{i=1}^{k} \varepsilon^2 H'(\gamma(s_i)) = O\left(k\varepsilon^3\right). \end{cases} \tag{5.18}$$

We need to find a solution of (5.18) in (1.15).

6. Solving the nonlinear system

Our aim in the rest of this paper is to find a solution $\{s_i\}$ to the above non-linear system (5.18) the class Λ_k defined in (1.15).

Observe that the linearized matrix of the above system at the main order is degenerate, thus the terms containing $H'(\gamma(s))$ play an important role. We explain how we solve system (5.18). The novelty of this paper is to consider the above system as a discretization of an ODE system. In order to explain this idea, we first introduce some notation.

Let

$$s = G(b)$$

be the solution of $\Psi_1(s) = b$. Since $\Psi_1(s) = C_n s^{-\frac{1}{2}} e^{-s}(1 + o(1))$ as $s \to \infty$, using this asymptotic behavior of Ψ_1, one has the following:

$$G(b) = -\left(1 + O\left(\frac{\ln(-\ln b)}{\ln b}\right)\right)\ln b, \text{ as } b \to 0. \tag{6.1}$$

Then, the above reduced system (5.18) is equivalent to the following system:

$$\begin{cases} s_{i+1}-s_i = \varepsilon G\left(-\sum_{j=1}^{i} \varepsilon^2 H'(\gamma(s_j)) + O\left(\varepsilon^3 i\right)\right) \text{ for } i = 1, \cdots, k-1 \\ s_k - s_{k-1} = \varepsilon G\left(\varepsilon^2 H'(\gamma(s_k)) + O(\varepsilon^3 k)\right). \end{cases} \tag{6.2}$$

Let $h = -\varepsilon \ln \varepsilon$ be the boot size, where we have set $s_i = x(t_i)$ and $t_i = (i-1)h$. Then from the above system (6.2),

$$\begin{cases} \frac{x(t_{i+1})-x(t_i)}{h} = -\frac{1}{\ln \varepsilon} G\left(-\frac{\varepsilon}{\ln \varepsilon}\left(-\sum_{j=1}^{i} H'(\gamma(x(t_j)))h\right) + O(\varepsilon^3 i)\right) \\ \frac{x(t_k)-x(t_{k-1})}{h} = -\frac{1}{\ln \varepsilon} G\left(\varepsilon^2 H'\left(\gamma(x(t_k))\right) + O\left(\varepsilon^3 k\right)\right). \end{cases} \tag{6.3}$$

In order to the solve the above system, we consider the limiting case of the above system, *i.e.*, view $\frac{x(t_{i+1})-x(t_i)}{h}$ as $x'(t)$ and $\sum_{j=1}^{i} H'(\gamma(x(t_j)))h$ as $\int_0^t H'(\gamma(x(t)))dt$, and introduce the following ODE:

$$\begin{cases} \dfrac{dx}{dt} = -\dfrac{1}{\ln\varepsilon} G\left(\dfrac{\varepsilon}{\ln\varepsilon}\rho(t)\right) \\ \dfrac{d\rho}{dt} = H'(\gamma(x(t))) \\ \rho(0) = 0, \ \rho(b_\varepsilon) = \rho_b \\ x'(b_\varepsilon) = -\dfrac{1}{\ln\varepsilon} G\left(\varepsilon^2 H'\big(\gamma(x(b_\varepsilon))\big)\right), \end{cases} \tag{6.4}$$

where $b_\varepsilon = (k-1)h = [\frac{b}{h}]h = b + O(h)$.

One can see that the above second order ODE has three initial conditions. Besides the two end point initial values, there is an extra condition, *i.e.*, the last equation of (6.4), which in fact comes from the last equation of (6.2). This ODE with extra initial condition is not always solvable. It turns out that this extra condition corresponds to some balancing condition of the curvature of the segment γ. In order to solve this ODE, we need assumption ($\mathbf{H_1}$) on γ. For this ODE, we have the following existence result:

Lemma 6.1. *Under the assumption* ($\mathbf{H_1}$), *there exists* $\varepsilon_0 > 0$, *such that for every* $\varepsilon < \varepsilon_0$ *there exist* $\rho_b = \rho_b(\varepsilon) < 0$, *such that the above ODE* (6.4) *is solvable. Moreover,* ρ_b *satisfies the following asymptotic behavior:*

$$\rho_b = -\left(H'(\gamma(b_\varepsilon)) + O\left(\frac{\ln(-\ln\varepsilon)}{\ln\varepsilon}\right)\right)h. \tag{6.5}$$

Proof. From the asymptotic behavior of G, we know that the first equation of (6.4) is

$$\begin{aligned} \frac{dx}{dt} &= -\frac{1}{\ln\varepsilon} G\left(\frac{\varepsilon}{\ln\varepsilon}\rho(t)\right) \\ &= \left(1 + O\left(\frac{\ln(-\ln\varepsilon)}{\ln\varepsilon}\right)\right)\big(a_1 \ln(-\rho(t)) + a_2\big), \end{aligned}$$

where

$$a_1 = \frac{1}{\ln\varepsilon}, \quad a_2 = 1 - \frac{\ln(-\ln\varepsilon)}{\ln\varepsilon}.$$

Integrating the above equation from b_ε to t, one has

$$\begin{aligned} x(t) - x(b_\varepsilon) &= \int_{b_\varepsilon}^t -\frac{1}{\ln\varepsilon} G\left(\frac{\varepsilon}{\ln\varepsilon}\rho(t)\right) dt \\ &= \left(1 + O\left(\frac{\ln(-\ln\varepsilon)}{\ln\varepsilon}\right)\right)\left[a_2(t - b_\varepsilon) + a_1 \int_{b_\varepsilon}^t \ln(-\rho(t))dt\right]. \end{aligned}$$

Plugging the expression for $x(t)$ into the second equation,

$$\rho'(t)=H'\left(\gamma(x(b_\varepsilon))+\left(1+O\left(\frac{\ln(-\ln\varepsilon)}{\ln\varepsilon}\right)\right)\left[a_2(t-b_\varepsilon)+a_1\int_{b_\varepsilon}^{t}\ln(-\rho(t))dt\right]\right). \quad (6.6)$$

By the boundary conditions $\rho(0)=0$, $\rho(b_\varepsilon)=\rho_b$, we have

$$\int_{b_\varepsilon}^{0}H'\left(\gamma\left(x(b_\varepsilon)+\left(1+O\left(\frac{\ln(-\ln\varepsilon)}{\ln\varepsilon}\right)\right)\left[a_2(t-b_\varepsilon)+a_1\int_{b_\varepsilon}^{t}\ln(-\rho(t))dt\right]\right)\right)dt=-\rho_b. \quad (6.7)$$

By Taylor's expansion,

$$\begin{aligned}&H'\left(\gamma\left(x(b_\varepsilon)+\left(1+O\left(\frac{\ln(-\ln\varepsilon)}{\ln\varepsilon}\right)\right)\left[a_2(t-b_\varepsilon)+a_1\int_{b}^{t}\ln(-\rho(t))dt\right]\right)\right)\\&=H'\left(\gamma\left(x(b_\varepsilon)+a_2(t-b)+O\left(\frac{\ln(-\ln\varepsilon)}{\ln\varepsilon}\right)\right)\right)\\&=H'\big(\gamma\big(x(b_\varepsilon)+a_2(t-b_\varepsilon)\big)\big)+O\left(\frac{\ln(-\ln\varepsilon)}{\ln\varepsilon}\right).\end{aligned}$$

So from (6.7) and the above equation, we have

$$\begin{aligned}\int_{b_\varepsilon}^{0}H'(\gamma(x(t)))dt&=\int_{b_\varepsilon}^{0}H'\big(\gamma\big(x(b_\varepsilon)+a_2(t-b)\big)\big)dt+O\left(\frac{\ln(-\ln\varepsilon)}{\ln\varepsilon}\right)\\&=H\big(\gamma\big(x(b_\varepsilon)-a_2b_\varepsilon\big)\big)-H\big(\gamma\big(x(b_\varepsilon)\big)\big)+O\left(\frac{\ln(-\ln\varepsilon)}{\ln\varepsilon}\right) \quad (6.8)\\&=\rho_b.\end{aligned}$$

Since by the third boundary condition

$$x'(b_\varepsilon)=-\frac{1}{\ln\varepsilon}G\big(\varepsilon^2H'\big(\gamma(x(b_\varepsilon))\big)\big), \quad (6.9)$$

one can obtain

$$\rho_b=H'\big(\gamma\big(x(b_\varepsilon)\big)\big)\varepsilon\ln\varepsilon. \quad (6.10)$$

We assume that

$$\rho_b=\big(H'(\gamma(b_\varepsilon))+\rho_\varepsilon\big)\varepsilon\ln\varepsilon, \quad (6.11)$$

then

$$x(b_\varepsilon)=b_\varepsilon+\frac{(1+o(1))\rho_\varepsilon}{H''(\gamma(b_\varepsilon))}. \quad (6.12)$$

Using (6.12), (6.8) is reduced to the following:

$$\begin{aligned}&H(\gamma(0))-H(\gamma(b_\varepsilon))+\frac{H'(\gamma(0))-H'(\gamma(b_\varepsilon))}{H''(\gamma(b_\varepsilon))}\rho_\varepsilon+o(\rho_\varepsilon)+O(\rho_\varepsilon^2)\\&=O\left(\frac{\ln(-\ln\varepsilon)}{\ln\varepsilon}\right).\end{aligned} \quad (6.13)$$

By the assumption ($\mathbf{H_1}$)

$$H(\gamma(0)) = H(\gamma(b)),\ H'(\gamma(0)) \neq H'(\gamma(b)), \tag{6.14}$$

and

$$H''(\gamma) \geq c_0 > 0,\ b_\varepsilon = b + O(h), \tag{6.15}$$

the above equation is uniquely solvable with

$$\rho_\varepsilon = O\left(\frac{\ln(-\ln\varepsilon)}{\ln\varepsilon}\right). \tag{6.16}$$

Therefore, there exists a unique $\rho_b = \big(H'(\gamma(b_\varepsilon)) + O\big(\frac{\ln(-\ln\varepsilon)}{\ln\varepsilon}\big)\big)\varepsilon\ln\varepsilon$ such that (6.4) is solvable, and we have

$$x(0) = O\left(\frac{\ln(-\ln\varepsilon)}{\ln\varepsilon}\right),\ x(b_\varepsilon) = b_\varepsilon + O\left(\frac{\ln(-\ln\varepsilon)}{\ln\varepsilon}\right). \tag{6.17}$$

□

We will use the solution of the ODE to approximate the solution of (6.2). In order to obtain a good approximate solution, one needs to control the error of

$$\sum_{j=1}^{i} H'\Big(\gamma(x(t_j))\Big)h - \int_0^{t_{i+1}} H'\Big(\gamma(x(t))\Big)dt.$$

Therefore, we use the midpoint Riemann sum approximation of integrals, which gives us

$$\sum_{j=1}^{i} H'\Big(\gamma(x(t_j))\Big)h - \int_0^{t_{i+1}} H'\Big(\gamma(x(t))\Big)dt = O\left(h^2\right). \tag{6.18}$$

To be more specific, we choose the approximate solution to be the following:

$$x_i^0 = x(\bar{t}_i),\ \bar{t}_i = \frac{t_i + t_{i+1}}{2}, i = 1, \cdots, k-1, \tag{6.19}$$

and

$$x_k^0 = x_{k-1}^0 + \varepsilon G\left(\frac{\varepsilon}{\ln\varepsilon}\rho_b\right), \tag{6.20}$$

where $x(t)$ is the solution determined by the ODE (6.4).

We want to find the solution to (6.2) of the form

$$s_i = x_i^0 + y_i. \tag{6.21}$$

Then, y_i satisfies the following equation:

$$\begin{cases} y_{i+1}-y_i \\ =-E_i+\varepsilon\left(G\left(-\varepsilon^2\sum_{j=1}^{i}H'\left(\gamma\left(x_j^0+y_j\right)\right)+O\left(\varepsilon^3 i\right)\right)\right. \\ \qquad\qquad \left.-G\left(-\varepsilon^2\sum_{j=1}^{i}H'\left(\gamma\left(x_j^0\right)\right)\right)\right) \quad \text{for } i=1,\cdots,k-1 \\ \varepsilon^2\sum_{j=1}^{k}H''\left(\gamma\left(x_j^0\right)\right)y_j+O\left(\varepsilon^2\right)\sum_{j=1}^{k}|y_j|^2 \\ =-E_k+O\left(\varepsilon^3 k\right), \end{cases} \tag{6.22}$$

where

$$E_i = x_{i+1}^0 - x_i^0 - \varepsilon G\left(-\varepsilon^2\sum_{j=1}^{i}H'\left(\gamma\left(x_j^0\right)\right)\right)$$

for $i = 1, \cdots, k-1$, and

$$E_k = \varepsilon^2\sum_{j=1}^{k}H'\left(\gamma\left(x_j^0\right)\right).$$

First, we show that the approximate solution we choose is indeed a good approximate solution, *i.e.*, the error E_i is small enough. In fact, we have the following error estimate:

Lemma 6.2.

$$E_i = x_{i+1}^0 - x_i^0 - \varepsilon G\left(-\varepsilon^2\sum_{j=1}^{i}H'\left(\gamma\left(x_j^0\right)\right)\right) = O(\varepsilon) \tag{6.23}$$

for $i = 1, \cdots, k-1$*, and*

$$E_k = \varepsilon^2\sum_{j=1}^{k}H'\left(\gamma\left(x_j^0\right)\right) = O\left(\varepsilon^2\frac{\ln(-\ln\varepsilon)}{\ln\varepsilon}\right). \tag{6.24}$$

Moreover, the following estimate holds:

$$\sum_{i=1}^{k-1}|E_i| = O(\varepsilon). \tag{6.25}$$

Proof. First, for $i = k-1$, we have

$$
\begin{aligned}
&x_k^0 - x_{k-1}^0 - \varepsilon G\left(-\varepsilon^2 \sum_{j=1}^{k-1} H'\left(\gamma\left(x_j^0\right)\right)\right) \\
&= \varepsilon G\left(\frac{\varepsilon}{\ln\varepsilon}\rho_b\right) - \varepsilon G\left(-\varepsilon^2 \sum_{j=1}^{k-1} H'\left(\gamma\left(x_j^0\right)\right)\right) \\
&= O\left(\frac{\varepsilon}{\rho_b}\right)\left|\rho_b - \sum_{j=1}^{k-1} H'\big(\gamma(x_j^0)\big)h\right|.
\end{aligned}
$$

Since we choose the midpoint approximation, we have, for $i = 1, \cdots, k-2$,

$$
\rho(t_{i+1}) - \sum_{j=1}^{i} H'\left(\gamma\left(x_j^0\right)\right)h = O\left(h^2\right), \tag{6.26}
$$

and

$$
\begin{aligned}
\sum_{j=1}^{k} H'\left(\gamma\left(x_j^0\right)\right)h &= \left(\sum_{j=1}^{k-1} H'\left(\gamma\left(x_j^0\right)\right)h - \rho(t_k)\right) \\
&\quad + H'\left(\gamma\left(x_k^0\right)\right)h + \rho(t_k) \\
&= O\left(h^2\right) + O\left(\frac{\ln(-\ln\varepsilon)}{\ln\varepsilon}\right)h = O\left(\frac{\ln(-\ln\varepsilon)}{\ln\varepsilon}\right)h.
\end{aligned} \tag{6.27}
$$

By (6.26) and (6.27), and recall that $\rho_b = O(h)$, one can obtain that

$$
E_{k-1} = x_k^0 - x_{k-1}^0 - \varepsilon G\left(-\varepsilon^2 \sum_{j=1}^{k-1} H'\left(\gamma\left(x_j^0\right)\right)\right) = O(\varepsilon h)
$$

and

$$
E_k = O\left(\varepsilon^2 \frac{\ln(-\ln\varepsilon)}{\ln\varepsilon}\right). \tag{6.28}
$$

Next, by the equation satisfied by $\rho(t)$, we can obtain that

$$
\rho(t_i) = O\big(\min\{i, k-i+1\}h\big), \tag{6.29}
$$

so, for $i = 1, \cdots, k-2$,

$$x_{i+1}^0 - x_i^0 - \varepsilon G\left(-\varepsilon^2 \sum_{j=1}^{i} H'\left(\gamma\left(x_j^0\right)\right)\right)$$

$$= \int_{\bar{t}_i}^{\bar{t}_{i+1}} -\frac{1}{\ln\varepsilon} G\left(\frac{\varepsilon}{\ln\varepsilon}\rho(t)\right) dt - \varepsilon G\left(-\varepsilon^2 \sum_{j=1}^{i} H'\left(\gamma\left(x_j^0\right)\right)\right)$$

$$= -\frac{1}{\ln\varepsilon} G\left(\frac{\varepsilon}{\ln\varepsilon}\rho(t_{i+1})\right) h - \varepsilon G\left(-\varepsilon^2 \sum_{j=1}^{i} H'\left(\gamma\left(x_j^0\right)\right)\right)$$
$$+ O\left(\frac{\rho''\rho - (\rho')^2}{|\ln\varepsilon|\rho^2}(t_{i+1})\right) h^3$$

$$= \varepsilon\left(G\left(\frac{\varepsilon}{\ln\varepsilon}\rho(t_{i+1})\right) - G\left(-\varepsilon^2 \sum_{j=1}^{i} H'\left(\gamma\left(x_j^0\right)\right)\right)\right)$$
$$+ O\left(\frac{\rho''\rho - (\rho')^2}{|\ln\varepsilon|\rho^2}(t_{i+1})\right) h^3$$

$$= O\left(\frac{\varepsilon}{\rho(t_{i+1})}\right)\left(\rho(t_{i+1}) - \sum_{j=1}^{i} H'\left(\gamma\left(x_j^0\right)\right) h\right) + O\left(\frac{\rho''\rho - (\rho')^2}{|\ln\varepsilon|\rho^2}(t_{i+1})\right) h^3$$

$$= O\left(\frac{\varepsilon h}{\min\{i, k-i+1\}}\right) + O(\varepsilon)\left(\frac{1}{\min\{i, k-i+1\}^2} + \frac{h}{\min\{i, k-i+1\}}\right)$$

$$= O(\varepsilon).$$

Moreover, from the above estimate, we have

$$\sum_{j=1}^{i} E_j = O(\varepsilon), \text{ for } i = 1, \cdots, k-1. \qquad \square$$

Finally, we show that equation (6.22) is solvable.

Lemma 6.3. *There exists $\varepsilon_0 > 0$, such that for $\varepsilon < \varepsilon_0$, there exists a solution $\{y_i\}_{1 \le i \le k}$ to* (6.22) *such that*

$$\|y\|_\infty \le C\varepsilon \ln(-\ln\varepsilon). \tag{6.30}$$

Proof. For $\|y\|_\infty \ll \varepsilon|\ln\varepsilon|$, we have

$$\begin{aligned}
&\varepsilon G\left(-\varepsilon^2\sum_{j=1}^{i} H'\left(\gamma\left(x_j^0+y_j\right)\right)+O\left(\varepsilon^3 i\right)\right)-\varepsilon G\left(-\varepsilon^2\sum_{j=1}^{i} H'\left(\gamma\left(x_j^0\right)\right)\right)\\
&=-\varepsilon\left(\frac{\sum_{j=1}^{i} H''\left(\gamma\left(x_j^0\right)\right)y_j}{\sum_{j=1}^{i} H'\left(\gamma\left(x_j^0\right)\right)}\right)+O\left(\frac{\varepsilon i|y|^2_{j\le i}}{\sum_{j=1}^{i} H'\left(\gamma\left(x_j^0\right)\right)}\right)\\
&\quad+O\left(\frac{\varepsilon^2 i}{\sum_{j=1}^{i} H'\left(\gamma\left(x_j^0\right)\right)}\right).
\end{aligned}$$

The equations (6.22) for y_i can be rewritten as follows:

$$\begin{cases}
y_{i+1}-y_i+\varepsilon\dfrac{\sum_{j=1}^{i} H''\left(\gamma\left(x_j^0\right)\right)y_j}{\sum_{j=1}^{i} H'\left(\gamma\left(x_j^0\right)\right)}\\
=-E_i+O\left(\dfrac{\varepsilon i|y|^2_{j\le i}}{\sum_{j=1}^{i} H'\left(\gamma\left(x_j^0\right)\right)}\right)+O\left(\dfrac{\varepsilon^2 i}{\sum_{j=1}^{i} H'\left(\gamma\left(x_j^0\right)\right)}\right)\\
\qquad\qquad\qquad\qquad\text{for } i=1,\cdots,k-1\\
\sum_{j=1}^{k} H''\left(\gamma\left(x_j^0\right)\right)y_j+\sum_{j=1}^{k} H'''\left(\gamma\left(x_j^0\right)\right)y_j^2=O(\varepsilon k)+O\left(\dfrac{\ln(-\ln\varepsilon)}{\ln\varepsilon}\right).
\end{cases} \tag{6.31}$$

We show that one can first solve y_2 to y_k in terms of y_1 from the first $k-1$ equations, and finally solve y_1 by the k-th equation of (6.31).

For $1\le l\le i_0=(1-\delta)k$, where $\delta>0$ is a small number to be determined later, we have

$$\begin{aligned}
&y_{l+1}-y_1+\varepsilon\sum_{i=1}^{l}\frac{\sum_{j=1}^{i} H''\left(\gamma\left(x_j^0\right)\right)y_j}{\sum_{j=1}^{i} H'\left(\gamma\left(x_j^0\right)\right)}\\
&=\sum_{i=1}^{l}E_i+\sum_{i=1}^{l}\frac{\varepsilon i}{\sum_{j=1}^{i} H'\left(\gamma\left(x_j^0\right)\right)}|y|^2_{i\le l}+\sum_{i=1}^{l}\frac{\varepsilon^2 i}{\sum_{j=1}^{i} H'\left(\gamma\left(x_j^0\right)\right)}\\
&=O(\varepsilon)+\sum_{i=1}^{l}\frac{\varepsilon i|y|^2_{i\le l}}{\min\{i,k-i+1\}}+\sum_{i=1}^{l}O\left(\frac{\varepsilon^2 i}{\min\{i,k-i+1\}}\right)\\
&=O(\varepsilon)+O\left(\frac{\varepsilon l}{\delta}\right)|y|^2_{i\le l},
\end{aligned}$$

where we have set

$$|y|_{i_1\le i\le i_2} = \sup_{i_1\le i\le i_2} |y_i|.$$

Moreover,

$$\varepsilon\sum_{i=1}^{l}\frac{\sum_{j=1}^{i} H''\left(\gamma\left(x_j^0\right)\right)y_j}{\sum_{j=1}^{i} H'\left(\gamma\left(x_j^0\right)\right)} = \varepsilon\sum_{i=1}^{l} O\left(\frac{i|y|_{i\le l}}{\min\{i, k-i+1\}}\right)$$
$$= O\left(\frac{\varepsilon l|y|_{i\le l}}{\delta}\right) = o(1)|y|_{i\le l}.$$

Thus, one can obtain that for $l \le i_0$

$$y_l = y_1 + o(1)|y|_{i\le i_0} + o(1)\big|y\big|^2_{i\le i_0} + O(\varepsilon). \tag{6.32}$$

Therefore, we obtain

$$y_i = \big(1+o(1)\big)y_1 + O(\varepsilon) \quad \text{for all} \quad i = 2,\cdots,i_0. \tag{6.33}$$

For $l > i_0$, we have the following:

$$\begin{aligned} y_{l+1} - y_1 &= -\varepsilon\sum_{i=i_0+1}^{l}\frac{\sum_{j=1}^{i} H''\left(\gamma\left(x_j^0\right)\right)y_j}{\sum_{j=1}^{i} H'\left(\gamma\left(x_j^0\right)\right)} \\ &\quad +O(\varepsilon) + O\left(|y|^2_{i_0<i\le l}\right) + O\left(\frac{\varepsilon l}{\delta}\right)\big|y\big|^2_{i\le i_0} + o(1)\big|y\big|_{i\le i_0} \\ &= C_0\delta|y|_{i_0<i\le l} + O\left(|y|^2_{i_0<i\le l}\right) + O\left(|y|_{i\le i_0}\right) + O(\varepsilon) \end{aligned}$$

for some C_0 independent of ε and δ. Therefore, for $i_0 < i \le k$, we have

$$y_i = O(y_1) + C_0\delta|y|_{i_0<i\le l} + O\left(\big|y\big|^2_{i_0<i\le l}\right) + O(\varepsilon). \tag{6.34}$$

If $\delta > 0$ is small such that $C_0\delta < \frac{1}{4}$, then the above system is solvable with

$$y_i = O(y_1) + O(\varepsilon). \tag{6.35}$$

From the last equation, we have

$$\begin{aligned} &\sum_{i=1}^{i_0} H''\left(\gamma\left(x_i^0\right)\right)y_i + \sum_{i=i_0+1}^{k} H''\left(\gamma\left(x_i^0\right)\right)y_i + O\left(k|y_1|^2\right) + O\left(k\varepsilon^2\right) \\ &= \sum_{i=1}^{i_0} H''\left(\gamma\left(x_i^0\right)\right)\big(1+o(1)\big)y_1 + O(\delta k|y_1|) + O\left(k\varepsilon\right) + O\left(k|y_1|^2\right) \\ &= O\left(\frac{\ln(-\ln\varepsilon)}{\ln\varepsilon}\right). \end{aligned}$$

Thus, by the assumption ($\mathbf{H_1}$), the equation is reduced to

$$y_1 = o(1)y_1 + O(\delta)|y_1| + O\left(|y_1|^2\right) + O\big(\varepsilon\ln(-\ln\varepsilon)\big).$$

If we further choose δ small enough but independent of ε such that $O(\delta)|y_1| < \frac{1}{2}|y_1|$, it is easy to see that, by the contraction mapping theorem, the above equation has a solution and satisfies

$$y_1 = O(\varepsilon\ln\big(-\ln\varepsilon)\big). \tag{6.36}$$

Thus we obtain that there exists a solution to (6.22) with

$$\|y\|_\infty \le C\varepsilon\ln(-\ln\varepsilon) \ll \varepsilon|\ln\varepsilon|.$$

Thus, we have proved the existence of solution to (6.22). □

Appendix

A. Proof of Proposition 3.1

In this appendix, we present the proof of Proposition 3.1. The proof is rather standard. It follows from the argument in [5] and [28]. It is based on Fredholm alternative theorem for compact operators and an *apriori* estimate.

First, we require an estimate on the matrix M defined by

$$M_{ij} = \int_{\Omega_\varepsilon} Z_i Z_j dz \quad \text{for all} \quad i, j = 1, \cdots, k. \tag{A.1}$$

Lemma A.1. *For ε sufficiently small, given any vector $\vec{b} \in \mathbb{R}^k$, there exists a unique vector $\vec{\beta} \in \mathbb{R}^k$, such that $M\vec{\beta} = \vec{b}$. Moreover,*

$$\|\vec{\beta}\|_\infty \le C\|\vec{b}\|_\infty \tag{A.2}$$

for some constant C independent of ε.

Proof. To prove the existence, it is sufficient to prove the *a priori* estimate (A.2). Suppose that $|\beta_i| = \|\beta\|_\infty$, we have

$$\sum_{i=1}^{k} M_{ij}\beta_j = b_i.$$

For the entries M_{ij}, from the definition of Λ_k, and the exponential decay property of Z_i, we know that

$$M_{ii} = \int_{\Omega_\varepsilon} Z_i^2 dz = \big(1 + o(1)\big)\int_{\mathbb{R}^2_+}\left(\frac{\partial w}{\partial y_1}\right)^2 dy > c_0 > 0,$$

and

$$\sum_{j\neq i}|M_{ij}|\leq C\sum_{j\neq i}e^{-\frac{|P_i-P_j|}{2}}=o(1).$$

Hence, for ε small, we have

$$\begin{aligned}c_0\|\vec{\beta}\|_\infty\leq c_0|\vec{\beta}_i|&\leq\sum_{j\neq i}|M_{ij}||\vec{\beta}_j|+|b_i|\\&\leq o(1)\|\vec{\beta}\|_\infty+\|\vec{b}\|_\infty\end{aligned}$$

from which the desired result follows. □

Next, we need the following *a priori* estimate:

Lemma A.2. *Let $h\in L^2(\Omega_\varepsilon)$ with $\|h\|_*$ bounded and assume that $(\psi,\{c_i\})$ is a solution to* (3.4). *Then, there exist positive numbers ε_0 and C, such that for all $\varepsilon\leq\varepsilon_0$, for any set of points P_i, $i=1,\ldots,k$ given by* (1.15) *, one has*

$$\|\psi\|_*\leq C\|h\|_*. \tag{A.3}$$

Proof. We argue by contradiction. Assume there exists ψ solution to (3.4) and that

$$\|h\|_*\to 0,\quad \|\psi\|_*=1.$$

We prove that

$$c_i\to 0 \text{ for } i=1,\cdots,k. \tag{A.4}$$

Multiplying the equation in (3.4) against Z_j and integrating in Ω_ε, we get

$$\int_{\Omega_\varepsilon}L_\varepsilon\psi Z_j(z)=\int_{\Omega_\varepsilon}hZ_j+M(c_j),$$

By the exponentially decay of Z_i, we first know that

$$\left|\int_{\Omega_\varepsilon}hZ_j\right|\leq C\|h\|_*.$$

Here and in what follows, C stands for a positive constant independent of ε, as $\varepsilon \to 0$. Secondly, by the equation satisfied by $\mathcal{P}_{\Omega_{\varepsilon,P_i}} w(z-P_i)$, we have

$$\int_{\Omega_\varepsilon} L_\varepsilon \psi Z_i dz = \int_{\Omega_\varepsilon} \left(\Delta\psi - \psi + p \left(\sum_{i=1}^{k} \mathcal{P}_{\Omega_{\varepsilon,P_i}} w(z-P_i) \right)^{p-1} \psi \right) Z_i dz$$

$$= \int_{\Omega_\varepsilon} \left(\Delta Z_i - Z_i + p \left(\sum_{i=1}^{k} \mathcal{P}_{\Omega_{\varepsilon,P_i}} w(z-P_i) \right)^{p-1} Z_i \right) \psi dz$$

$$= \int_{\Omega_\varepsilon} \left[p \left(\sum_{i=1}^{k} \mathcal{P}_{\Omega_{\varepsilon,P_i}} w(z-P_i) \right)^{p-1} \frac{\partial \mathcal{P}_{\Omega_{\varepsilon,P_i}} w(z-P_i)}{\partial \tau} \right.$$

$$\left. - p w(z-P_i)^{p-1} \frac{\partial w(z-P_i)}{\partial \tau} \right] \psi dz$$

$$\leq C \int_{B_{|\ln\varepsilon|}(P_i)} \left| \frac{\partial w(z-P_i)}{\partial \tau} \left\| O(\varepsilon) + w(z-P_i)^{p-2} \sum_{j\neq i} \mathcal{P}_{\Omega_{\varepsilon,P_i}} w(z-P_j) \right\| \psi \right| dz$$

$$+ \int_{\Omega_\varepsilon \setminus B_{|\ln\varepsilon|}(P_i)} \left| \frac{\partial w(z-p_i)}{\partial \tau} \right| \left[\sum_{j=1}^{k} w_j^{p-1} + O(\varepsilon) \sum_{j=1}^{k} e^{-\mu|z-P_j|} \right] |\psi| dz$$

$$\leq C\|\psi\|_* \left(O(\varepsilon) + O\left(\varepsilon^{\frac{p-\eta}{2}} \right) \right)$$

$$\leq C\varepsilon \|\psi\|_*$$

if we choose η small enough such that $p - \eta > 2$. This can be done since $p > 2$.

Since M is invertible and $\|M^{-1}\| \leq C$, we obtain

$$|c_i| \leq C(\|h\|_* + O(\varepsilon)\|\psi\|_*). \tag{A.5}$$

Thus, we obtain the validity of (A.4), since we assume $\|\psi\|_* = 1$ and $\|h\|_* \to 0$.

Now, let $\mu \in (0, 1)$. It is easy to check that the function

$$W := \sum_{i=1}^{k} e^{-\mu\,|\cdot - P_i|},$$

satisfies

$$L_\varepsilon W \leq \frac{1}{2} \left(\mu^2 - 1 \right) W ,$$

in $\Omega_\varepsilon \setminus \cup_{j=1,\ldots,k} B(P_j, R)$ provided R is fixed large enough (independently of ε). Hence, the function W can be used as a barrier to prove the pointwise estimate

$$|\phi|(x) \leq C \left(\|L_\varepsilon\, \psi\|_* + \sup_j \|\psi\|_{L^\infty(B(p_j,R)\cap\Omega_\varepsilon)} \right) W(x) , \tag{A.6}$$

for all $z \in \Omega_\varepsilon \setminus \cup_j B(P_j, R)$.

Granted these preliminary estimates, the proof of the result goes by contradiction. Let us assume there exist a sequence of $\varepsilon \to 0$ and a sequence of solutions of (3.4) for which the inequality is not true. The problem being linear, we can reduce to the case where we have a sequence $\varepsilon^{(n)}$ tending to 0 and sequences $h^{(n)}$, $\psi^{(n)}$, $c^{(n)}$ such that

$$\left\| h^{(n)} \right\|_* \to 0, \quad \text{and} \quad \left\| \psi^{(n)} \right\|_* = 1.$$

However, (A.4) implies that we also have

$$\|c^{(n)}\|_* \to 0 \, .$$

Then, (A.6) implies that there exists $P_i^{(n)}$ such that

$$\|\psi^{(n)}\|_{L^\infty(B(P_i^{(n)},R))} \geq C \tag{A.7}$$

for some fixed constant $C > 0$. Using elliptic estimates together with the Ascoli-Arzelà's theorem, we can find a sequence $P_i^{(n)}$ and we can extract, from the sequence $\psi_i^{(n)}(\cdot - P_i^{(n)})$ a subsequence that will converge (on compact) to ψ_∞, a solution of

$$\begin{cases} \left(\Delta - 1 + p\, w^{p-1}\right) \psi_\infty = 0 & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial \psi_\infty}{\partial y_2} = 0 & \text{on } \partial\mathbb{R}^2_+, \end{cases}$$

which is bounded by constant times $e^{-\mu |x|}$, with $\mu > 0$. Moreover, since $\psi_i^{(n)}$ satisfies the orthogonality conditions in (3.4), the limit function ψ_∞ also satisfies

$$\int_{\mathbb{R}^2_+} \psi_\infty \frac{\partial w}{\partial y_1} \, dx = 0 \, .$$

However, the solution w being non-degenerate, this implies that $\psi_\infty \equiv 0$, which is certainly in contradiction with (A.7), which implies that ψ_∞ is not identically equal to 0.

Having reached a contradiction, this completes the proof of the lemma. □

We can now prove Proposition 3.1.

Proof of Proposition 3.1. Consider the space

$$\mathcal{H} = \left\{ u \in H^2_N(\Omega_\varepsilon) \; : \; \int_{\Omega_\varepsilon} u Z_i = 0, \quad i = 1, \ldots, k \right\}.$$

Notice that the problem (3.4) in ψ is rewritten as

$$\psi + K(\psi) = \bar{h} \quad \text{in} \quad \mathcal{H} \tag{A.8}$$

where $\bar{h}$ is defined by duality and $K : \mathcal{H} \to \mathcal{H}$ is a linear compact operator. Using Fredholm's alternative theorem, to show that equation (A.8) has a unique solution for each $\bar{h}$ is equivalent to show that the equation has a unique solution for $\bar{h} = 0$, which in turn follows from Proposition A.2. The estimate (3.6) follows directly from Proposition A.2. This concludes the proof of Proposition (3.1). □

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