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Smooth projective varieties with a torus action of complexity 1 and Picard number 2

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Abstract. We give an explicit description of all smooth varieties with a torus action of complexity one having Picard number at most two. As a consequence, we classify in every dimension the smooth (almost) Fano varieties with a torus action of complexity one having Picard number two. It turns out that all the Fano examples are obtained via an iterated generalized cone construction from a series of smooth varieties of dimension at most seven.

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1. Introduction

A basic intention of this article is to contribute to the classification of smooth (almost) Fano varieties with a torus action. Most studied in this context are the toric Fano varieties; based on their description in terms of lattice polytopes, there are meanwhile classification results up to dimension nine [2,3,20,23,24,26]. We go one step beyond the toric case and focus on rational varieties with a torus action of complexity one, *i.e.*, an action whose general torus orbit is of dimension one less than the variety; see [25] for results on smooth Fano threefolds with an action of a two-dimensional torus.

Instead of bounding the dimension, we look here at varieties of small Picard number. Recall that for toric varieties, the projective spaces are the only smooth examples of Picard number one, and we have Kleinschmidt's description [19] of all smooth toric varieties of Picard number two, which in particular allows to figure out the (almost) Fano ones in this setting. We follow that line and study first arbitrary smooth projective rational varieties with a torus action of complexity one. The case of Picard number one is basically settled by a result of Liendo and Süß [21, Theorem 6.5]: the only non-toric examples are the smooth projective quadrics in dimensions three and four. Picard number two means to provide an analogue of Kleinschmidt's description for complexity one.

Our approach goes via the Cox ring and we use the methods developed in [1, 13,15]; the ground field $\mathbb{K}$ is algebraically closed and of characteristic zero. Recall

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that the Cox ring is graded by the divisor class group and, together with the choice of an ample class, fixes our variety up to isomorphism; we refer to [1] for the basic background. Here comes the first result.

Theorem 1.1. *Every smooth rational projective non-toric variety of Picard number two that admits a torus action of complexity one is isomorphic to precisely one of the following varieties X, specified by their Cox ring $\mathcal{R}(X)$ and an ample class $u \in \mathrm{Cl}(X)$, where we always have $\mathrm{Cl}(X) = \mathbb{Z}^2$ and the grading is fixed by the matrix $[w_1, \dots, w_r]$ of generator degrees $\deg(T_i), \deg(S_j) \in \mathrm{Cl}(X)$.*

No.	$\mathcal{R}(X)$	$[w_1, \dots, w_r]$	u	$\dim(X)$
1	$\dfrac{\mathbb{K}[T_1,\dots,T_7]}{\langle T_1T_2T_3^2+T_4T_5+T_6T_7\rangle}$	$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & a & 2-a & b & 2-b \end{bmatrix}$ $1 \le a \le b$	$\begin{bmatrix} 1 \\ 1+b \end{bmatrix}$	4
2	$\dfrac{\mathbb{K}[T_1,\dots,T_7]}{\langle T_1T_2T_3+T_4T_5+T_6T_7\rangle}$	$\begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	4
3	$\dfrac{\mathbb{K}[T_1,\dots,T_6]}{\langle T_1T_2T_3^2+T_4T_5+T_6^2\rangle}$	$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2-a & a & 1 \end{bmatrix}$ $a \ge 1$	$\begin{bmatrix} 1 \\ 1+a \end{bmatrix}$	3
4	$\dfrac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2^{l_2}+T_3T_4^{l_4}+T_5T_6^{l_6}\rangle}$ $m\ge 0$	$\left[\begin{array}{cccccc\|ccc} 0 & 1 & a & 1 & b & 1 & c_1 & \dots & c_m \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & \dots & 1 \end{array}\right]$ $0 \le a \le b,\ c_1 \le \dots \le c_m,$ $l_2 = a + l_4 = b + l_6$	$\begin{bmatrix} d+1 \\ 1 \end{bmatrix}$ $d := \max(b, c_m)$	$m+3$
5	$\dfrac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3^2T_4+T_5^2T_6\rangle}$ $m\ge 0$	$\left[\begin{array}{cccccc\|ccc} 0 & 2a+1 & a & 1 & a & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & \dots & 0 \end{array}\right]$ $a \ge 0$	$\begin{bmatrix} 2a+2 \\ 1 \end{bmatrix}$	$m+3$
6	$\dfrac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2T_6\rangle}$ $m\ge 1$	$\left[\begin{array}{cccccc\|ccc} 0 & 2c+1 & a & b & c & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & \dots & 0 \end{array}\right]$ $a, b, c \ge 0, \quad a < b,$ $a + b = 2c + 1$	$\begin{bmatrix} 2c+2 \\ 1 \end{bmatrix}$	$m+3$
7	$\dfrac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $m\ge 1$	$\left[\begin{array}{cccccc\|ccc} 0 & 0 & 0 & 0 & -1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & \dots & 0 \end{array}\right]$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	$m+3$
8	$\dfrac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $m\ge 2$	$\left[\begin{array}{cccccc\|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & a_2 & \dots & a_m \end{array}\right]$ $0 \le a_2 \le \dots \le a_m,\ a_m > 0$	$\begin{bmatrix} 1 \\ a_m+1 \end{bmatrix}$	$m+3$
9	$\dfrac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $m\ge 2$	$\left[\begin{array}{cccc\|ccc} 0 & a_2 & \dots & a_6 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 & 0 & \dots & 0 \end{array}\right]$ $0 \le a_3 \le a_5 \le a_6 \le a_4 \le a_2,$ $a_2 = a_3 + a_4 = a_5 + a_6$	$\begin{bmatrix} a_2+1 \\ 1 \end{bmatrix}$	$m+3$
10	$\dfrac{\mathbb{K}[T_1,\dots,T_5,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $m\ge 1$	$\left[\begin{array}{ccccc\|ccc} 1 & 1 & 1 & 1 & 1 & 0 & \dots & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & \dots & 1 \end{array}\right]$	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	$m+2$
11	$\dfrac{\mathbb{K}[T_1,\dots,T_5,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $m\ge 2$	$\left[\begin{array}{ccccc\|cccc} 1 & 1 & 1 & 1 & 1 & 0 & a_2 & \dots & a_m \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & \dots & 1 \end{array}\right]$ $0 \le a_2 \le \dots \le a_m,\ a_m > 0$	$\begin{bmatrix} a_m+1 \\ 1 \end{bmatrix}$	$m+2$
12	$\dfrac{\mathbb{K}[T_1,\dots,T_5,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $m\ge 2$	$\left[\begin{array}{ccccc\|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 2c & a & b & c & 1 & 1 & \dots & 1 \end{array}\right]$ $0 \le a \le c \le b,\ a + b = 2c$	$\begin{bmatrix} 1 \\ 2c+1 \end{bmatrix}$	$m+2$
13	$\dfrac{\mathbb{K}[T_1,\dots,T_8]}{\left\langle \begin{array}{c} T_1T_2+T_3T_4+T_5T_6, \\ \lambda T_3T_4+T_5T_6+T_7T_8 \end{array} \right\rangle}$ $\lambda\in\mathbb{K}^*\setminus\{1\}$	$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	4

Moreover, each of the listed data defines a smooth rational non-toric projective variety of Picard number two coming with a torus action of complexity one.

Note that by our approach we obtain the Cox ring of the respective varieties for free which in turn allows an explicit treatment of geometric questions by means of Cox ring based techniques. In particular, the canonical divisor of the varieties listed in Theorem 1.1 admits a simple description in terms of the defining data. This enables us to determine for every dimension the finitely many (families of) non-toric smooth rational Fano varieties of Picard number two that admit a torus action of complexity one; we refer to Section 6 for a geometric description of the listed varieties.

Theorem 1.2. *Every smooth rational non-toric Fano variety of Picard number two that admits a torus action of complexity one is isomorphic to precisely one of the following varieties X, specified by their Cox ring $\mathcal{R}(X)$, where the grading by $\mathrm{Cl}(X) = \mathbb{Z}^2$ is given by the matrix $[w_1, \ldots, w_r]$ of generator degrees* $\deg(T_i)$, $\deg(S_j) \in \mathrm{Cl}(X)$ *and we list the (ample) anticanonical class $-\mathcal{K}_X$.*

No.	$\mathcal{R}(X)$	$[w_1, \ldots, w_r]$	$-\mathcal{K}_X$	$\dim(X)$
1	$\frac{\mathbb{K}[T_1,\ldots,T_7]}{\langle T_1T_2T_3^2+T_4T_5+T_6T_7\rangle}$	$\begin{bmatrix} 0&0&1&1&1&1&1 \\ 1&1&0&1&1&1&1 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 4 \end{bmatrix}$	4
2	$\frac{\mathbb{K}[T_1,\ldots,T_7]}{\langle T_1T_2T_3+T_4T_5+T_6T_7\rangle}$	$\begin{bmatrix} 0&0&1&1&0&1&0 \\ 1&1&0&1&1&1&1 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 4 \end{bmatrix}$	4
3	$\frac{\mathbb{K}[T_1,\ldots,T_6]}{\langle T_1T_2T_3^2+T_4T_5+T_6^2\rangle}$	$\begin{bmatrix} 0&0&1&1&1&1 \\ 1&1&0&1&1&1 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$	3
4.A	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccc\|cccc} 0&1&0&1&0&1&c&0&\ldots&0 \\ 1&0&1&0&1&0&1&1&\ldots&1 \end{array}\right]$ $c \in \{-1, 0\}$, $c := 0$ if $m = 0$	$\begin{bmatrix} 2+c \\ 2+m \end{bmatrix}$	$m+3$
4.B	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2^2+T_3T_4+T_5T_6\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccc\|ccc} 0&1&1&1&1&1&1&\ldots&1 \\ 1&0&1&0&1&0&1&\ldots&1 \end{array}\right]$	$\begin{bmatrix} 3+m \\ 2+m \end{bmatrix}$	$m+3$
4.C	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6^2\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccc\|ccc} 0&1&0&1&0&1&0&\ldots&0 \\ 1&0&1&0&1&0&1&\ldots&1 \end{array}\right]$	$\begin{bmatrix} 1 \\ 2+m \end{bmatrix}$	$m+3$
5	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3^2T_4+T_5^2T_6\rangle}$ $m\geq 1$	$\left[\begin{array}{cccccc\|ccc} 0&2a+1&a&1&a&1&1&\ldots&1 \\ 1&1&1&0&1&0&0&\ldots&0 \end{array}\right]$ $0 \leq 2a < m$	$\begin{bmatrix} 2a+m+2 \\ 2 \end{bmatrix}$	$m+3$
6	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2T_6\rangle}$ $m\geq 1$	$\left[\begin{array}{cccccc\|ccc} 0&2c+1&a&b&c&1&1&\ldots&1 \\ 1&1&1&1&1&0&0&\ldots&0 \end{array}\right]$ $a, b, c \geq 0, \quad a < b$, $a + b = 2c + 1$, $m > 3c + 1$	$\begin{bmatrix} 3c+2+m \\ 3 \end{bmatrix}$	$m+3$
7	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $1\leq m\leq 3$	$\left[\begin{array}{cccccc\|ccc} 0&0&0&0&-1&1&1&\ldots&1 \\ 1&1&1&1&1&1&0&\ldots&0 \end{array}\right]$	$\begin{bmatrix} m \\ 4 \end{bmatrix}$	$m+3$
8	$\frac{\mathbb{K}[T_1,\ldots,T_6,S_1,\ldots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $m\geq 2$	$\left[\begin{array}{cccccc\|cccc} 0&0&0&0&0&0&1&1&\ldots&1 \\ 1&1&1&1&1&1&0&a_2&\ldots&a_m \end{array}\right]$ $0 \leq a_2 \leq \ldots \leq a_m$, $a_m \in \{1, 2, 3\}$, $4 + \sum_{k=2}^m a_k > m a_m$	$\begin{bmatrix} m \\ 4 + \sum_{k=2}^m a_k \end{bmatrix}$	$m+3$

9	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $m\geq 2$	$\left[\begin{array}{cccc\|ccc} 0 & a_2 & \dots & a_6 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 & 0 & \dots & 0 \end{array}\right]$ $0\leq a_3\leq a_5\leq a_6\leq a_4\leq a_2$, $a_2=a_3+a_4=a_5+a_6$, $2a_2<m$	$\begin{bmatrix} 2a_2+m \\ 4 \end{bmatrix}$	$m+3$
10	$\frac{\mathbb{K}[T_1,\dots,T_5,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $1\leq m\leq 2$	$\left[\begin{array}{ccccc\|ccc} 1 & 1 & 1 & 1 & 1 & 0 & \dots & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & \dots & 1 \end{array}\right]$	$\begin{bmatrix} 3 \\ m \end{bmatrix}$	$m+2$
11	$\frac{\mathbb{K}[T_1,\dots,T_5,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $m\geq 2$	$\left[\begin{array}{ccccc\|cccc} 1 & 1 & 1 & 1 & 1 & 0 & a_2 & \dots & a_m \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & \dots & 1 \end{array}\right]$ $0\leq a_2\leq\dots\leq a_m$, $a_m\in\{1,2\}$, $3+\sum_{k=2}^m a_k>ma_m$	$\begin{bmatrix} 3+\sum_{k=2}^m a_k \\ m \end{bmatrix}$	$m+2$
12	$\frac{\mathbb{K}[T_1,\dots,T_5,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $m\geq 2$	$\left[\begin{array}{ccccc\|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 2c & a & b & c & 1 & 1 & \dots & 1 \end{array}\right]$ $0\leq a\leq c\leq b$, $a+b=2c$, $3c<m$	$\begin{bmatrix} 3 \\ 3c+m \end{bmatrix}$	$m+2$
13	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\left\langle \begin{array}{c} T_1T_2+T_3T_4+T_5T_6, \\ \lambda T_3T_4+T_5T_6+T_7T_8 \end{array}\right\rangle}$ $\lambda\in\mathbb{K}^*\setminus\{1\}$	$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 2 \end{bmatrix}$	4

Moreover, each of the listed data defines a smooth rational non-toric Fano variety of Picard number two coming with a torus action of complexity one.

For $\mathbb{K}=\mathbb{C}$, the assumption of rationality can be omitted in Theorem 1.2 due to [18, Section 2.1] and [1, Remark 4.4.1.5]. A closer look to the varieties of Theorem 1.2 reveals that they all are obtained from a series of lower dimensional varieties via iterating the following procedure: we take a certain $\mathbb{P}_1$-bundle over the given variety, apply a natural series of flips and then contract a prime divisor. In terms of Cox rings, this generalized cone construction simply means duplicating a free weight, *i.e.*, given a variable not showing up in the defining relations, one adds a further one of the same degree, see Section 5. Proposition 5.4 and Theorem 5.5 then yield the following.

Corollary 1.3. *Every smooth rational non-toric Fano variety with a torus action of complexity one and Picard number two arises via iterated duplication of a free weight from a smooth rational projective (not necessarily Fano) variety with a torus action of complexity one, Picard number two and dimension at most seven.*

Note that we cannot expect such a statement in general: Remark 5.7 shows that the smooth toric Fano varieties of Picard number two do not allow a bound d such that they all arise via iterated duplication of free weights from smooth varieties of dimension at most d.

Similar to the Fano varieties, we can figure out the almost Fano varieties from Theorem 1.1, *i.e.*, those with a big and nef anticanonical divisor. In general, *i.e.*, without the assumption of a torus action, the classification of smooth almost Fano varieties of Picard number two is widely open; for the threefold case, we refer to the work of Jahnke, Peternell and Radloff [16,17]. In the setting of a torus action of complexity one, the following result together with Theorem 1.2 settles the problem in any dimension; by a *truly almost Fano variety* we mean an almost Fano variety which is not Fano.

Theorem 1.4. *Every smooth rational projective non-toric truly almost Fano variety of Picard number two that admits a torus action of complexity one is isomorphic to precisely one of the following varieties X, specified by their Cox ring $\mathcal{R}(X)$ and an ample class $u \in \mathrm{Cl}(X)$, where we always have $\mathrm{Cl}(X) = \mathbb{Z}^2$ and the grading is fixed by the matrix $[w_1, \dots, w_r]$ of generator degrees $\deg(T_i)$, $\deg(S_j) \in \mathrm{Cl}(X)$.*

No.	$\mathcal{R}(X)$	$[w_1, \dots, w_r]$	u	$\dim(X)$
4.A	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $m\geq 1$	$\left[\begin{array}{cccccc\|ccc} 0&1&0&1&0&1&c_1&\dots&c_m\\ 1&0&1&0&1&0&1&\dots&1\end{array}\right]$ $c_1 \leq \dots \leq c_m$, $d := \max(0, c_m)$, $(2+m)d = 2 + c_1 + \dots + c_m$	$\begin{bmatrix}1\\1+d\end{bmatrix}$	$m+3$
4.B	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2^2+T_3T_4+T_5T_6\rangle}$ $m\geq 1$	$\left[\begin{array}{cccccc\|cccc} 0&1&1&1&1&1&0&1&\dots&1\\ 1&0&1&0&1&0&1&1&\dots&1\end{array}\right]$	$\begin{bmatrix}1\\2\end{bmatrix}$	$m+3$
4.C	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6^2\rangle}$ $m\geq 1$	$\left[\begin{array}{cccccc\|cccc} 0&1&0&1&0&1&-1&0&\dots&0\\ 1&0&1&0&1&0&1&1&\dots&1\end{array}\right]$	$\begin{bmatrix}1\\1\end{bmatrix}$	$m+3$
4.D	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccc\|ccc} 0&1&0&1&1&1&1&\dots&1\\ 1&0&1&0&1&0&1&\dots&1\end{array}\right]$	$\begin{bmatrix}1\\2\end{bmatrix}$	$m+3$
4.E	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2^3+T_3T_4+T_5T_6\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccc\|ccc} 0&1&2&1&2&1&2&\dots&2\\ 1&0&1&0&1&0&1&\dots&1\end{array}\right]$	$\begin{bmatrix}1\\3\end{bmatrix}$	$m+3$
4.F	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2^3+T_3T_4^2+T_5T_6^2\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccc\|ccc} 0&1&1&1&1&1&1&\dots&1\\ 1&0&1&0&1&0&1&\dots&1\end{array}\right]$	$\begin{bmatrix}1\\2\end{bmatrix}$	$m+3$
5	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3^2T_4+T_5^2T_6\rangle}$ $m\geq 0$	$\left[\begin{array}{cccccc\|ccc} 0&2a+1&a&1&a&1&1&\dots&1\\ 1&1&1&0&1&0&0&\dots&0\end{array}\right]$ $m = 2a$	$\begin{bmatrix}m+2\\1\end{bmatrix}$	$m+3$
6	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2T_6\rangle}$ $m\geq 1$	$\left[\begin{array}{cccccc\|ccc} 0&2c+1&a&b&c&1&1&\dots&1\\ 1&1&1&1&1&0&0&\dots&0\end{array}\right]$ $a, b, c \geq 0, \quad a < b,$ $a + b = 2c + 1,$ $m = 3c + 1$	$\begin{bmatrix}2c+2\\1\end{bmatrix}$	$m+3$
7	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $m=4$	$\left[\begin{array}{cccccc\|cccc} 0&0&0&0&-1&1&1&1&1&1\\ 1&1&1&1&1&1&0&0&0&0\end{array}\right]$	$\begin{bmatrix}1\\2\end{bmatrix}$	7
8	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $m\geq 2$	$\left[\begin{array}{cccccc\|cccc} 0&0&0&0&0&0&1&1&\dots&1\\ 1&1&1&1&1&1&0&a_2&\dots&a_m\end{array}\right]$ $0 \leq a_2 \leq \dots \leq a_m,\ a_m > 0,$ $4 + a_2 + \dots + a_m = ma_m$	$\begin{bmatrix}1\\a_m+1\end{bmatrix}$	$m+3$
9	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$ $m\geq 2$	$\left[\begin{array}{cccc\|ccc} 0&a_2&\dots&a_6&1&\dots&1\\ 1&1&\dots&1&0&\dots&0\end{array}\right]$ $0 \leq a_3 \leq a_5 \leq a_6 \leq a_4 \leq a_2,$ $a_2 = a_3 + a_4 = a_5 + a_6,$ $m = 2a_2$	$\begin{bmatrix}a_2+1\\1\end{bmatrix}$	$m+3$

10	$\frac{\mathbb{K}[T_1,\dots,T_5,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $m=3$	$\left[\begin{array}{rrrrr\|rrr} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]$	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	5
11	$\frac{\mathbb{K}[T_1,\dots,T_5,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $m\geq 2$	$\left[\begin{array}{ccccc\|cccc} 1 & 1 & 1 & 1 & 1 & 0 & a_2 & \dots & a_m \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & \dots & 1 \end{array}\right]$ $0 \leq a_2 \leq \dots \leq a_m,\ a_m > 0,$ $3 + a_2 + \dots + a_m = ma_m$	$\begin{bmatrix} 1 \\ a_m + 1 \end{bmatrix}$	$m+2$
12	$\frac{\mathbb{K}[T_1,\dots,T_5,S_1,\dots,S_m]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$ $m\geq 3$	$\left[\begin{array}{ccccc\|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 2c & a & b & c & 1 & 1 & \dots & 1 \end{array}\right]$ $0 \leq a \leq c \leq b,\ a+b=2c,$ $m = 3c$	$\begin{bmatrix} 1 \\ 2c + 1 \end{bmatrix}$	$m+2$

Moreover, each of the listed data defines a smooth rational non-toric truly almost Fano variety of Picard number two coming with a torus action of complexity one.

The article is organized as follows. In Section 2, we briefly present the necessary background on rational varieties X with a torus action of complexity one. In Section 3, we derive first constraints on the defining data for smooth X of Picard number two. Section 4 is devoted to proving the main results. In Section 5, we introduce and discuss duplication of free weights and show how to obtain the Fano varieties of Theorem 1.2 via this procedure from lower dimensional varieties. Finally, in Section 6, we describe the Fano varieties of Theorem 1.2 in more geometric terms.

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2. Varieties with torus action of complexity one

We recall from [1,13,15] the Cox ring-based approach to (irreducible) normal projective rational varieties X with a torus action of complexity one and thereby fix the notation used throughout the article. The first step is to describe the possible Cox rings $\mathcal{R}(X)$; they are encoded by a pair (A, P) of matrices of the following shape.

Notation 2.1. Fix $r \in \mathbb{Z}_{\geq 1}$, a sequence $n_0, \dots, n_r \in \mathbb{Z}_{\geq 1}$, set $n := n_0 + \dots + n_r$, and fix integers $m \in \mathbb{Z}_{\geq 0}$ and $0 < s < n + m - r$. A pair (A, P) of *defining matrices* consists of

- a matrix $A := [a_0, \dots, a_r]$ with pairwise linearly independent column vectors $a_0, \dots, a_r \in \mathbb{K}^2$;
- an integral block matrix P of size $(r + s) \times (n + m)$, the columns of which are pairwise different primitive vectors generating $\mathbb{Q}^{r+s}$ as a cone:

$$P = \begin{bmatrix} L & 0 \\ d & d' \end{bmatrix},$$

where d is an $(s \times n)$-matrix, d' an $(s \times m)$-matrix and L an $(r \times n)$-matrix built from tuples $l_i := (l_{i1}, \dots, l_{in_i}) \in \mathbb{Z}^{n_i}_{\geq 1}$ as follows:

$$L = \begin{bmatrix} -l_0 & l_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -l_0 & 0 & \dots & l_r \end{bmatrix}.$$

Denote by v_{ij}, where $0 \leq i \leq r$ and $1 \leq j \leq n_i$, the first n columns of P and by v_k, where $1 \leq k \leq m$, the last m ones. Moreover, $e_{ij}, e_k \in \mathbb{Z}^{n+m}$ are the canonical basis vectors indexed accordingly, *i.e.*, P sends e_{ij} to v_{ij} and e_k to v_k.

Construction 2.2. Fix (A, P) as in 2.1. Consider the polynomial ring $\mathbb{K}[T_{ij}, S_k]$ in the variables T_{ij}, where $0 \leq i \leq r$, $1 \leq j \leq n_i$, and S_k, where $1 \leq k \leq m$. For every $0 \leq i \leq r$, define a monomial

$$T_i^{l_i} := T_{i1}^{l_{i1}} \cdots T_{in_i}^{l_{in_i}} \in \mathbb{K}[T_{ij}, S_k].$$

Denote by $\mathfrak{I}$ the set of all triples $I = (i_1, i_2, i_3)$ with $0 \leq i_1 < i_2 < i_3 \leq r$ and define for any $I \in \mathfrak{I}$ a trinomial

$$g_I := g_{i_1,i_2,i_3} := \det \begin{bmatrix} T_{i_1}^{l_{i_1}} & T_{i_2}^{l_{i_2}} & T_{i_3}^{l_{i_3}} \\ a_{i_1} & a_{i_2} & a_{i_3} \end{bmatrix}.$$

Let P^* denote the transpose of P, consider the factor group $K := \mathbb{Z}^{n+m}/\mathrm{im}(P^*)$ and the projection $Q\colon \mathbb{Z}^{n+m} \to K$. We define a K-grading on $\mathbb{K}[T_{ij}, S_k]$ by setting

$$\deg(T_{ij}) := w_{ij} := Q(e_{ij}), \qquad \deg(S_k) := w_k := Q(e_k).$$

Then the trinomials g_I just introduced are K-homogeneous, all of the same degree. In particular, we obtain a K-graded factor ring

$$R(A, P) := \mathbb{K}[T_{ij}, S_k;\ 0 \leq i \leq r,\ 1 \leq j \leq n_i, 1 \leq k \leq m] \,/\, \langle g_I;\ I \in \mathfrak{I} \rangle.$$

The rings $R(A, P)$ are precisely those which occur as Cox rings of normal rational projective (or, more generally, complete A_2-) varieties with a torus action of complexity one; see [13, Theorem 1.5]. We recall basic properties.

Remark 2.3. The K-graded ring $R(A, P)$ of Construction 2.2 is a complete intersection: setting $g_i := g_{i,i+1,i+2}$ we have

$$\langle g_I;\ I \in \mathfrak{I} \rangle = \langle g_0, \dots, g_{r-2} \rangle, \qquad \dim(R(A, P)) = n + m - (r - 1).$$

Remark 2.4. The following operations on the columns and rows of the defining matrix P do not change the isomorphy type of the graded ring $R(A, P)$; we call them *admissible operations*:

(i) swap two columns inside a block $v_{ij_1}, \dots, v_{ij_{n_i}}$;
(ii) swap two whole column blocks $v_{ij_1}, \dots, v_{ij_{n_i}}$ and $v_{i'j_1}, \dots, v_{i'j_{n_{i'}}}$;

(iii) add multiples of the upper r rows to one of the last s rows;
(iv) any elementary row operation among the last s rows;
(v) swap two columns inside the d' block.

The operations of type (iii) and (iv) do not even change $R(A, P)$, whereas types (i), (ii), (v) correspond to certain renumberings of the variables of $R(A, P)$ keeping the (graded) isomorphy type.

Remark 2.5. If we have $n_i = 1$ and $l_{i1} = 1$ in a defining matrix P, then we may eliminate the variable T_{i1} in $R(A, P)$ by modifying P appropriately. This can be repeated until P is *irredundant* in the sense that $l_{i1} + \ldots + l_{in_i} \geq 2$ holds for all $i = 0, \ldots, r$.

We come to the construction of all normal projective varieties sharing a given $R(A, P)$ as their Cox ring. By $K_{\mathbb{Q}} := K \otimes_{\mathbb{Z}} \mathbb{Q}$ we denote the rational vector space associated to an abelian group K. We shortly write w for $w \otimes 1 \in K_{\mathbb{Q}}$ and, similarly, we keep the symbols when passing from homomorphisms $K \to K'$ to the associated linear maps $K_{\mathbb{Q}} \to K'_{\mathbb{Q}}$. Moreover, when we speak of a cone $\tau \subseteq K_{\mathbb{Q}}$, then we mean a convex, polyhedral cone in $K_{\mathbb{Q}}$. The relative interior of τ is denoted by τ°.

Definition 2.6. The *moving cone* in $K_{\mathbb{Q}}$ of the K-graded ring $R(A, P)$ from Construction 2.2 is the

$$\mathrm{Mov}(A, P) := \bigcap_{i,j} \mathrm{cone}(Q(e_{uv}, e_t;\ (u,v) \neq (i,j))) \cap \bigcap_{k} \mathrm{cone}(Q(e_{uv}, e_t;\ t \neq k)).$$

Construction 2.7. Take $R(A, P)$ as in Construction 2.2 and fix $u \in \mathrm{Mov}(A, P)^\circ$. The K-grading on $\mathbb{K}[T_{ij}, S_k]$ defines an action of the quasitorus $H := \mathrm{Spec}\ \mathbb{K}[K]$ on $\overline{Z} := \mathbb{K}^{n+m}$ leaving $\overline{X} := V(g_I;\ I \in \mathfrak{I}) \subseteq \overline{Z}$ invariant. Consider

$$\widehat{Z} := \{z \in \overline{Z};\ f(z) \neq 0 \text{ for some } f \in \mathbb{K}[T_{ij}, S_k]_{\nu u},\ \nu \in \mathbb{Z}_{>0}\} \subseteq \overline{Z},$$

the set of H-semistable points with respect to the weight u. Then $\widehat{X} := \overline{X} \cap \widehat{Z}$ is an open H-invariant set in $\overline{X}$ and we have a commutative diagram

$$\begin{tikzcd} \widehat{X} \arrow[r] \arrow[d, "/\!\!/ H"', "\pi"] & \widehat{Z} \arrow[d, "/\!\!/ H"] \\ X(A, P, u) \arrow[r] & Z \end{tikzcd}$$

where $X = X(A, P, u)$ is a variety with torus action of complexity one, $Z := \widehat{Z} /\!\!/ H$ is a toric variety, the downward maps are characteristic spaces and the lower horizontal arrow is a closed embedding. We have

$$\dim(X) = s + 1, \qquad \mathrm{Cl}(X) \cong K, \qquad \mathcal{R}(X) \cong R(A, P).$$

Moreover, for an irredundant defining matrix P, the variety $X = X(A, P)$ is non-toric if and only if $r \geq 2$ holds.

See [1,13] for the proof that this construction yields indeed all normal rational projective varieties with a torus action of complexity one. We will make intensive use of the machinery developed in [1,7,12]. Let us briefly summarize the necessary notions and statements in a series of remarks adapted to our needs.

Remark 2.8. Fix defining matrices (A, P) and let $\gamma \subseteq \mathbb{Q}^{n+m}$ be the positive orthant, spanned by the canonical basis vectors $e_{ij}, e_k \in \mathbb{Z}^{n+m}$. Every face $\gamma_0 \preceq \gamma$ defines a toric orbit in $\overline{Z} = \mathbb{K}^{n+m}$:

$$\overline{Z}(\gamma_0) \ := \ \{z \in \overline{Z};\ z_{ij} \neq 0 \Leftrightarrow e_{ij} \in \gamma_0 \text{ and } z_k \neq 0 \Leftrightarrow e_k \in \gamma_0\} \ \subseteq \ \overline{Z}.$$

We say that $\gamma_0 \preceq \gamma$ is an $\mathfrak{F}$-*face* (for (A, P)) if the associated toric orbit meets the total coordinate space $\overline{X} = V(g_I;\ I \in \mathfrak{I}) \subseteq \overline{Z}$, that means if we have

$$\overline{X}(\gamma_0) \ := \ \overline{X} \cap \overline{Z}(\gamma_0) \ \neq \ \emptyset.$$

In particular, $\overline{X}$ is the disjoint union of the locally closed pieces $\overline{X}(\gamma_0)$ associated to the $\mathfrak{F}$-faces.

Remark 2.9. Fix $u \in \mathrm{Mov}(A, P)^\circ$. Then, for the ambient toric variety Z and $X = X(A, P, u)$ of Construction 2.7, we have the collections of *relevant faces*:

$$\begin{aligned} \mathrm{rlv}(Z) \ &:= \ \big\{\gamma_0 \preceq \gamma;\ u \in Q(\gamma_0)^\circ\big\}, \\ \mathrm{rlv}(X) \ &:= \ \big\{\gamma_0 \in \mathrm{rlv}(Z);\ \gamma_0 \text{ is an } \mathfrak{F}\text{-face}\big\}. \end{aligned}$$

Let $\gamma_0^* := \gamma_0^\perp \cap \gamma \preceq \gamma$ denote the complementary face of $\gamma_0 \preceq \gamma$. Then there is a bijection between $\mathrm{rlv}(Z)$ and the fan Σ of the toric variety Z:

$$\mathrm{rlv}(Z) \ \to \ \Sigma, \qquad \qquad \gamma_0 \ \mapsto \ P(\gamma_0^*).$$

The toric orbits of Z correspond to the cones of the fan Σ and thus to the cones of $\mathrm{rlv}(Z)$. Concretely, the toric orbit of Z associated with $\gamma_0 \in \mathrm{rlv}(Z)$ is

$$Z(\gamma_0) \ = \ \pi\big(\overline{Z}(\gamma_0)\big).$$

The relevant faces $\mathrm{rlv}(X)$ of X define exactly the toric orbits of Z that intersect $X \subseteq Z$ non-trivially and thus give a locally closed decomposition

$$X \ = \bigcup_{\gamma_0 \in \mathrm{rlv}(X)} X(\gamma_0), \qquad \qquad X(\gamma_0) \ := \ X \cap Z(\gamma_0) \ = \ \pi\big((\overline{X}(\gamma_0)\big).$$

The fan Σ_X generated by the cones $\sigma = P(\gamma_0^*)$, where $\gamma_0 \in \mathrm{rlv}(X)$, defines the minimal toric open subset $Z_X \subseteq Z$ containing X. For the set of rays we have

$$\Sigma_X^{(1)} \ = \ \Sigma^{(1)} \ = \ \big\{\varrho_{ij}, \varrho_k;\ 0 \le i \le r,\ 1 \le j \le n_i,\ 1 \le k \le m\big\},$$

where the $\varrho_{ij} := \mathrm{cone}(v_{ij})$ and $\varrho_k := \mathrm{cone}(v_k)$ are the rays through the columns v_{ij} and v_k of the defining matrix P.

Remark 2.10. Let $X = X(A, P, u)$ arise from Construction 2.7. Then the cones of effective, movable, semiample and ample divisor classes are given as

$$\mathrm{Eff}(X) = Q(\gamma), \qquad \mathrm{Mov}(X) = \mathrm{Mov}(A, P) = \bigcap_{\gamma_0 \text{ facet of } \gamma} Q(\gamma_0),$$

$$\mathrm{SAmple}(X) = \bigcap_{\gamma_0 \in \mathrm{rlv}(X)} Q(\gamma_0), \qquad \mathrm{Ample}(X) = \bigcap_{\gamma_0 \in \mathrm{rlv}(X)} Q(\gamma_0)^\circ.$$

In particular, the GIT-fan of the H-action on $\overline{X}$ induces the Mori chamber decomposition, *i.e.*, it subdivides $\mathrm{Mov}(X)$ into the nef cones of the small birational relatives of X.

Remark 2.11. Let $X = X(A, P, u)$ arise from Construction 2.7. Consider $\gamma_0 \in \mathrm{rlv}(X)$ and $x \in X(\gamma_0)$. Then the following statements hold:

(i) x is $\mathbb{Q}$-factorial if and only if $Q(\gamma_0)$ is full-dimensional;
(ii) x is factorial if and only if Q maps $\mathrm{lin}(\gamma_0) \cap \mathbb{Z}^{n+m}$ onto $\mathrm{Cl}(X)$;
(iii) x is smooth if and only if x is factorial and all $z \in \pi^{-1}(x)$ are smooth in $\overline{X}$.

Remark 2.12. Let $X = X(A, P, u)$ arise from Construction 2.7. The anticanonical class of X does not depend on u and is given by

$$-\mathcal{K}_X = \kappa(A, P) := \sum_{i,j} Q(e_{ij}) + \sum_k Q(e_k) - (r-1)\sum_{j=0}^{n_0} l_{0j} Q(e_{0j}) \in K.$$

In particular, a K-graded ring $R(A, P)$ is the Cox ring of a Fano variety if and only if $\kappa(A, P)$ belongs to the relative interior of $\mathrm{Mov}(A, P)$.

Remark 2.13. Consider $X \subseteq Z$, where $X = X(A, P, u)$ and Z are as in Construction 2.7. Then, setting $\lambda := 0 \times \mathbb{Q}^s \subseteq \mathbb{Q}^{r+s}$, the canonical basis vectors $e_1, \dots, e_r \in \mathbb{Z}^{r+s}$ and $e_0 := -e_1 - \dots - e_r$, the associated tropical variety is

$$\mathrm{trop}(X) = \lambda_0 \cup \dots \cup \lambda_r \subseteq \mathbb{Q}^{r+s}, \qquad \text{where} \quad \lambda_i := \lambda + \mathrm{cone}(e_i).$$

Note that this defines the coarsest possible quasifan structure on $\mathrm{trop}(X)$, and the lineality space of this quasifan is λ. Moreover, a cone $\sigma \in \Sigma$ corresponds to $\gamma_0 \in \mathrm{rlv}(X)$ if and only if $\sigma^\circ \cap \mathrm{trop}(X) \neq \emptyset$ holds.

Definition 2.14. Consider $X \subseteq Z$, where $X = X(A, P, u)$ and Z are as in Construction 2.7. A cone $\sigma \in \Sigma_X$ is called:

(i) *big*, if $\sigma \cap \lambda_i^\circ \neq \emptyset$ holds for each $i = 0, \dots, r$.
(ii) *elementary big* if it is big, has no rays inside λ and precisely one inside λ_i for each $i = 0, \dots, r$.
(iii) a *leaf cone* if $\sigma \subseteq \lambda_i$ holds for some i.

We say that the variety X is *weakly tropical*, if the fan Σ_X is supported on the tropical variety $\mathrm{trop}(X)$.

Remark 2.15. Let $X = X(A, P, u)$ arise from Construction 2.7. Then the following holds:

(i) The fan Σ_X is generated by big cones and leaf cones;
(ii) Every big cone of Σ_X is of the form $P(\gamma_0^*)$ with a $\gamma_0 \in \mathrm{rlv}(X)$;
(iii) The tropical variety $\mathrm{trop}(X)$ is contained in the support of Σ_X;
(iv) X is weakly tropical if and only Σ_X consists of leaf cones;
(v) If X is weakly tropical, then $\lambda \subseteq \mathrm{trop}(X)$ is a union of cones of Σ_X.

3. First structural constraints

We derive first constraints on the defining matrices of smooth rational varieties with a torus action of complexity one having Picard number two. We work in the notation of Section 2. The aim is to show the following:

Proposition 3.1. *Let X be a non-toric smooth rational projective variety with a torus action of complexity one and Picard number $\rho(X)=2$. Then $X \cong X(A, P, u)$, where P is irredundant and fits into one of the following cases:*

(I) *We have $r = 2$ and one of the following constellations:*

(a) $m \geq 0$ *and* $n = 4 + n_0$, *where* $n_0 \geq 3$, $n_1 = n_2 = 2$;
(b) $m = 0$ *and* $n = 6$, *where* $n_0 = 3$, $n_1 = 2$, $n_2 = 1$;
(c) $m = 0$ *and* $n = 5$, *where* $n_0 = 3$, $n_1 = 1$, $n_2 = 1$;
(d) $m \geq 0$ *and* $n = 6$, *where* $n_0 = n_1 = n_2 = 2$;
(e) $m \geq 0$ *and* $n = 5$, *where* $n_0 = n_1 = 2$, $n_2 = 1$;
(f) $m \geq 1$ *and* $n = 4$, *where* $n_0 = 2$, $n_1 = n_2 = 1$;

(II) *We have $r = 3$ and one of the following constellations:*

(a) $m = 0$ *and* $n = 8$, *where* $n_0 = n_1 = n_2 = n_3 = 2$;
(b) $m = 0$ *and* $n = 7$, *where* $n_0 = n_1 = n_2 = 2$, $n_3 = 1$;
(c) $m = 0$ *and* $n = 6$, *where* $n_0 = n_1 = 2$, $n_2 = n_3 = 1$.

The statement is an immediate consequence of Propositions 3.12 and 3.13; see the end of this section. Throughout the whole section, the defining matrix P is irredundant. In particular, $X(A, P, u)$ is non-toric if and only if $r \geq 2$ holds, *i.e.*, we have a relation in the Cox ring. During our considerations, we will freely use the Remarks 2.8 to 2.15.

We first study the impact of $X = X(A, P, u)$ being locally factorial on the defining matrix P, where locally factorial means that the local rings of the points $x \in X$ are unique factorization domains.

Lemma 3.2. *Let $X = X(A, P, u)$ be non-toric and locally factorial. If X is weakly tropical, then $n_i \geq 2$ holds for all $i = 0, \dots, r$.*

Proof. Assume that $n_i = 1$ holds for some i. Since X is weakly tropical, there exists a cone $\sigma \in \Sigma_X$ of dimension $s+1$ contained in the leaf λ_i. Since $n_i = 1$, we have $\sigma = \varrho_{i1} + \tau$ with a face $\tau \preceq \sigma$ such that $\tau \subseteq \lambda$. Now, $\sigma = P(\gamma_0^*)$ holds for some $\gamma_0 \subseteq \mathrm{rlv}(X)$. Since the points of $X(\gamma_0)$ are factorial, σ is a regular cone. Thus, also $\tau \subseteq \lambda$ must be regular. This implies $l_{i1} = 1$, contradicting the irredundancy of P. □

Lemma 3.3. *Let $X = X(A, P, u)$ be non-toric and locally factorial. If X is weakly tropical, then $\rho(X) \geq r+3$ holds.*

Proof. Lemma 3.2 ensures that $n_i \geq 2$ for all $i = 1, \dots, r$, hence $n \geq 2 \cdot (r+1)$. The s-dimensional lineality space $\lambda = \{0\} \times \mathbb{Q}^s \subseteq \mathrm{trop}(X)$ is a union of cones of Σ_X. Thus P must have at least $s+1$ columns v_k which means $m \geq s+1$. Together the following yields:

$$\rho(X) = n + m - (r-1) - (s+1) \geq r+3. \qquad \square$$

Lemma 3.4. *Let $X = X(A, P, u)$ be non-toric and not weakly tropical. If X is $\mathbb{Q}$-factorial, then there is an elementary big cone in Σ_X.*

Proof. Since X is not weakly tropical, there exists a big cone $\sigma \in \Sigma_X$. We have $\sigma = P(\gamma_0^*)$ with $\gamma_0 \in \mathrm{rlv}(X)$. Since the points of $X(\gamma_0)$ are $\mathbb{Q}$-factorial, the cone σ is simplicial. For every $i = 0 \dots, r$ choose a ray $\varrho_i \preceq \sigma$ with $\varrho_i \in \lambda_i$. Then $\sigma_0 := \varrho_0 + \dots + \varrho_r \preceq \sigma$ is as wanted. □

Corollary 3.5. *Let $X = X(A, P, u)$ be non-toric and locally factorial. If $\rho(X) \leq 4$ holds, then there exists an elementary big cone $\sigma \in \Sigma_X$.*

Next we investigate the effect of quasismoothness on the defining matrix P, where we call $X = X(A, P, u)$ *quasismooth* if $\widehat{X}$ is smooth. Thus, quasismoothness means that X has at most quotient singularities by quasitori. The smoothness of $\widehat{X}$ will lead to conditions on P via the Jacobian of the defining relations of $\overline{X}$.

Remark 3.6. Let (A, P) be defining matrices. Then the Jacobian J_g of the defining relations $g_0, \dots, g_{r-2}$ from Remark 2.3 is of the shape $J_g = (J, 0)$ with a zero block of size $(r-1) \times m$ corresponding to the variables $S_1, \dots, S_m$ and a block

$$J := \begin{bmatrix} \delta_{10} & \delta_{11} & \delta_{12} & 0 & & & & \\ 0 & \delta_{21} & \delta_{22} & \delta_{23} & 0 & & & \\ & & & & \vdots & & & \\ & & & & \delta_{r-2,r-3} & \delta_{r-2,r-2} & \delta_{r-2,r-1} & 0 \\ & & & & 0 & \delta_{r-1,r-2} & \delta_{r-1,r-1} & \delta_{r-1,r} \end{bmatrix}$$

of size $(r-1) \times n$, where each vector $\delta_{a,i}$ is a nonzero multiple of the gradient of the monomial $T_i^{l_i}$:

$$\delta_{a,i} = \alpha_{a,i} \left(l_{i1} \frac{T_i^{l_i}}{T_{i1}}, \ \dots, \ l_{in_i} \frac{T_i^{l_i}}{T_{in_i}} \right), \qquad \alpha_{a,i} \in \mathbb{K}^*.$$

For given $1 \le a, b \le r-1, 0 \le i \le r$ and $z \in \overline{X}$, we have $\delta_{a,i}(z) = 0$ if and only if $\delta_{b,i}(z) = 0$. Moreover, the Jacobian $J_g(z)$ of a point $z \in \overline{X}$ is of full rank if and only if $\delta_{a,i}(z) = 0$ holds for at most two different $i = 0, \dots, r$.

Lemma 3.7. *Assume that $X = X(A, P, u)$ is non-toric and that there is an elementary big cone $\sigma = \varrho_{0j_0} + \dots + \varrho_{rj_r} \in \Sigma_X$. If X is quasismooth, then $l_{ij_i} \ge 2$ holds for at most two $i = 0, \dots, r$.*

Proof. We have $\sigma = P(\gamma_0^*)$ with a relevant face $\gamma_0 \in \mathrm{rlv}(X)$. Since X is quasismooth, any $z \in \overline{X}(\gamma_0)$ is a smooth point of $\overline{X}$. Thus, $J_g(z)$ is of full rank $r-1$. Consequently, $\delta_{a,i}(z) = 0$ holds for at most two different i. This means $l_{ij_i} \ge 2$ for at most two different i. □

Corollary 3.8. *Let $X = X(A, P, u)$ be non-toric and quasismooth. If there is an elementary big cone in Σ_X, then $n_i = 1$ holds for at most two different $i = 0, \dots, r$.*

Lemma 3.9. *Let (A, P) be defining matrices. Consider the rays $\gamma_k := \mathrm{cone}(e_k)$ and $\gamma_{ij} := \mathrm{cone}(e_{ij})$ of the orthant $\gamma \subseteq \mathbb{Q}^{r+s}$ and the two-dimensional faces*

$$\gamma_{k_1,k_2} := \gamma_{k_1} + \gamma_{k_2}, \quad \gamma_{ij,k} := \gamma_{ij} + \gamma_k, \quad \gamma_{i_1j_1,i_2j_2} := \gamma_{i_1j_1} + \gamma_{i_2j_2}.$$

Then the following hold:

(i) *All γ_k, respectively γ_{k_1,k_2}, are $\mathfrak{F}$-faces and each $\overline{X}(\gamma_k)$, respectively $\overline{X}(\gamma_{k_1,k_2})$, consists of singular points of $\overline{X}$;*

(ii) *A given γ_{ij}, respectively $\gamma_{ij,k}$, is an $\mathfrak{F}$-face if and only if $n_i \ge 2$ holds. In that case, $\overline{X}(\gamma_{ij})$, respectively $\overline{X}(\gamma_{ij,k})$, consists of smooth points of $\overline{X}$ if and only if $r = 2$, $n_i = 2$ and $l_{i,3-j} = 1$ hold;*

(iii) *A given γ_{ij_1,ij_2} with $j_1 \ne j_2$ is an $\mathfrak{F}$-face if and only if $n_i \ge 3$ holds. In that case, $\overline{X}(\gamma_{ij_1,ij_2})$ consists of smooth points of $\overline{X}$ if and only if $r = 2$, $n_i = 3$ and $l_{ij} = 1$ for the $j \ne j_1, j_2$ hold;*

(iv) *A given $\gamma_{i_1j_1,i_2j_2}$ with $i_1 \ne i_2$ is an $\mathfrak{F}$-face if and only if we have $n_{i_1}, n_{i_2} \ge 2$ or $n_{i_1} = n_{i_2} = 1$ and $r = 2$. In the former case, $\overline{X}(\gamma_{i_1j_1,i_2j_2})$ consists of smooth points of $\overline{X}$ if and only if one of the following holds:*
   - *$r = 2$, $n_{i_t} = 2$ and $l_{i_t,3-j_t} = 1$ for a $t \in \{1, 2\}$;*
   - *$r = 3$, $n_{i_1} = n_{i_2} = 2$, $l_{i_1,3-j_1} = l_{i_2,3-j_2} = 1$.*

Proof. The statements follow directly from the structure of the defining relations $g_0, \dots, g_{r-2}$ of $R(A, P)$ and the shape of the Jacobian J_g. □

We now restrict to the case that the rational divisor class group $\mathrm{Cl}(X)_{\mathbb{Q}} = K_{\mathbb{Q}}$ of $X = X(A, P, u)$ is of dimension two. Set $\tau_X := \mathrm{Ample}(X)$. Then the effective cone $\mathrm{Eff}(X)$ is of dimension two and is uniquely decomposed into three convex sets

$$\mathrm{Eff}(X) = \tau^+ \cup \tau_X \cup \tau^-,$$

such that τ^+, τ^- do not intersect the ample cone τ_X and $\tau^+ \cap \tau^-$ consists of the origin (Figure 3.1). Recall that $u \in \tau_X$ holds and that, due to $\tau_X \subseteq \mathrm{Mov}(X)$, each of τ^+ and τ^- contains at least two of the weights w_{ij}, w_k.

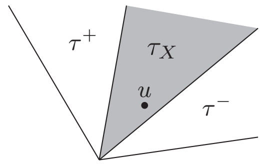

Fig. 3.1.

Remark 3.10. Consider $X = X(A, P, u)$ such that $\mathrm{Cl}(X)_{\mathbb{Q}}$ is of dimension two. Then, for every $\mathfrak{F}$-face $\{0\} \neq \gamma_0 \preceq \gamma$ precisely one of the following inclusions holds

$$Q(\gamma_0) \subseteq \tau^+, \qquad \tau_X \subseteq Q(\gamma_0)^\circ, \qquad Q(\gamma_0) \subseteq \tau^-.$$

The $\mathfrak{F}$-faces $\gamma_0 \preceq \gamma$ satisfying the second inclusion are exactly those with $\gamma_0 \in \mathrm{rlv}(X)$, *i.e.*, the relevant ones.

Lemma 3.11. *Let $X = X(A, P, u)$ be non-toric with* $\mathrm{rk}\,(\mathrm{Cl}(X)) = 2$. *Then the following hold:*

(i) *Suppose that X is $\mathbb{Q}$-factorial. Then $w_k \notin \tau_X$ holds for all $1 \leq k \leq m$ and for all $0 \leq i \leq r$ with $n_i \geq 2$ we have $w_{ij} \notin \tau_X$, where $1 \leq j \leq n_i$;*
(ii) *Suppose that X is quasismooth, $m > 0$ holds and there is $0 \leq i_1 \leq r$ with $n_{i_1} \geq 3$. Then the w_{ij}, w_k with $n_i \geq 3$, $j = 1, \dots, n_i$ and $k = 1, \dots, m$ lie either all in τ^+ or all in τ^-;*
(iii) *Suppose that X is quasismooth and there is $0 \leq i_1 \leq r$ with $n_{i_1} \geq 4$. Then the w_{ij} with $n_i \geq 4$ and $j = 1, \dots, n_i$ lie either all in τ^+ or all in τ^-;*
(iv) *Suppose that X is quasismooth and there exist $0 \leq i_1 < i_2 \leq r$ with $n_{i_1}, n_{i_2} \geq 3$. Then the w_{ij} with $n_i \geq 3$, $j = 1, \dots, n_i$ lie either all in τ^+ or all in τ^-;*
(v) *Suppose that X is quasismooth. Then $w_1, \dots, w_m$ lie either all in τ^+ or all in τ^-.*

Proof. We prove (i). By Lemma 3.9 (i) and (ii), the rays $\gamma_k, \gamma_{ij} \preceq \gamma$ with $n_i \geq 2$ are $\mathfrak{F}$-faces. Since X is $\mathbb{Q}$-factorial, the ample cone $\tau_X \subseteq K_{\mathbb{Q}}$ of X is of dimension two and thus $\tau_X \subseteq Q(\gamma_{ij})^\circ$ or $\tau_X \subseteq Q(\gamma_k)^\circ$ is not possible. Remark 3.10 yields the assertion.

We turn to (ii). By Lemma 3.9 (i) and (ii), all $\gamma_k, \gamma_{ij}, \gamma_{ij,k} \preceq \gamma$ in question are $\mathfrak{F}$-faces and the corresponding pieces in $\overline{X}$ consist of singular points. Because X is quasismooth, none of these $\mathfrak{F}$-faces is relevant. Thus, Remark 3.10 gives $w_{i_11} \in \tau^+$ or $w_{i_11} \in \tau^-$; say we have $w_{i_11} \in \tau^+$. Then, applying again Remark 3.10, we obtain $w_k, w_{ij} \in \tau^+$ for $k = 1, \dots, m$, all i with $n_i \geq 3$ and $j = 1, \dots, n_i$.

Assertion (iii) is proved analogously: treat first γ_{i_11,i_12} with Lemma 3.9 (iii), then $\gamma_{i_11,ij}$ with Lemma 3.9 (iii) and (iv). Similarly, we obtain (iv) by treating first γ_{i_11,i_21} and then all $\gamma_{i_11,ij}$ and $\gamma_{i_21,ij}$ with Lemma 3.9 (iii) and (iv). Finally, we obtain (v) using Lemma 3.9 (i). □

Proposition 3.12. *Let $X = X(A, P, u)$ be non-toric, quasismooth and $\mathbb{Q}$-factorial with $\rho(X) = 2$. Assume that there is an elementary big cone in Σ_X and that we* *have $n_0 \geq \ldots \geq n_r$. If $m > 0$ holds, then there is a $\gamma_{ij,k} \in \mathrm{rlv}(X)$, we have $r = 2$ and the constellation of the n_i is $(n_0, 2, 2)$, $(2, 2, 1)$ or $(2, 1, 1)$.*

Proof. According to Lemma 3.11 (v), we may assume $w_1, \ldots, w_m \in \tau^+$. We claim that there is a $w_{i_1 j_1} \in \tau^-$ with $n_{i_1} \geq 2$. Otherwise, use Corollary 3.8 to see that there exist w_{ij} with $n_i \geq 2$ and Lemma 3.11 (i) to see that they all lie in τ^+. Since all monomials $T_i^{l_i}$ have the same degree in K, we obtain in addition $w_{i1} \in \tau^+$ for all i with $n_i = 1$. But then no weights w_{ij}, w_k are left to lie in τ^-, a contradiction.

Having verified the claim, we may take a $w_{i_1 j_1} \in \tau^-$ with $n_{i_1} \geq 2$. Then $\gamma_{i_1 j_1, 1} \in \mathrm{rlv}(X)$ is as desired. Moreover, Lemma 3.9 (ii) yields $r = 2$ and $n_{i_1} = 2$. If $n_0 \geq 3$ holds, then Lemma 3.11 (ii) gives $w_{ij} \in \tau^+$ for all i with $n_i \geq 3$. Moreover, as all $T_i^{l_i}$ share the same K-degree, we have $w_{i1} \in \tau^+$ for all i with $n_i = 1$. By the same reason, one of the $w_{i_1 1}, w_{i_1 2}$ must lie in τ^+. As τ^- contains at least two weights, there is a $w_{i_2 j_2} \in \tau^-$ with $n_{i_2} = 2$ and $i_1 \neq i_2$. Thus, the constellation of $n_0 \geq n_1 \geq n_2$ is as claimed. □

Proposition 3.13. *Let $X = X(A, P, u)$ be non-toric, quasismooth and $\mathbb{Q}$-factorial with $\rho(X) = 2$. Assume that there is an elementary big cone in Σ_X and that we have $n_0 \geq \ldots \geq n_r$. If $m = 0$ holds, then there is a $\gamma_{i_1 j_1, i_2 j_2} \in \mathrm{rlv}(X)$, we have $r \leq 3$ and the constellation of the n_i is one of the following:*

- $r = 2$: $(n_0, 2, 2)$, $(3, 2, 1)$, $(3, 1, 1)$, $(2, 2, 2)$, $(2, 2, 1)$,
- $r = 3$: $(2, 2, 2, 2)$, $(2, 2, 2, 1)$, $(2, 2, 1, 1)$.

Proof. We first show that $n_1 \leq 2$. Otherwise we had $n_1 \geq 3$. Then, according to Lemma 3.11 (iv), we may assume that all the w_{ij} with $n_i \geq 3$ lie in τ^+. In particular, w_{11} lies in τ^+. Because all monomials $T_i^{l_i}$ have the same degree in K, also $w_{i1} \in \tau^+$ holds for all i with $n_i = 1$. At least two weights $w_{i_1 j_1}$ and $w_{i_2 j_2}$ must belong to τ^-. For these, only $n_{i_1} = n_{i_2} = 2$ and $i_1 \neq i_2$ is possible. Applying Lemma 3.9 (iv) to $\gamma_{11, i_1 j_1} \in \mathrm{rlv}(X)$ gives $r = 2$, contradicting the fact that $n_0 \geq n_1 \geq 3$ and $n_{i_1} = n_{i_2} = 2$.

We treat the case $n_0 \geq 4$. By Lemma 3.11 (iii), we can assume that $w_{01}, \ldots, w_{0 n_0} \in \tau^+$. As before, we obtain that $w_{i1} \in \tau^+$ for all i with $n_i = 1$, and we find two weights $w_{i_1 j_1}, w_{i_2 j_2} \in \tau^-$ with $n_{i_1} = n_{i_2} = 2$ and $i_1 \neq i_2$. Then we have $\gamma_{01, i_1 j_1} \in \mathrm{rlv}(X)$ is as wanted. Lemma 3.9 (iv) gives $r = 2$ and we end up with $(n_0, 2, 2)$.

Now let $n_0 = 3$. Lemma 3.11 (i) guarantees that no w_{0j} lies in τ_X. If weights w_{0j} occur in both cones τ^+ and τ^-, say $w_{01} \in \tau^+$ and $w_{02} \in \tau^-$, then $\gamma_{01,02}$ is as wanted. Lemma 3.9 (iii) yields $r = 2$ and we obtain the constellations $(n_0, 2, 2)$, $(3, 2, 1)$ and $(3, 1, 1)$. So, assume that all weights w_{0j} lie in one of τ^+ and τ^-, say in τ^+. Then we proceed as in the case $n_0 \geq 4$ to obtain a $\gamma_{01, i_1 j_1} \in \mathrm{rlv}(X)$ and $r = 2$ with the constellation $(3, 2, 2)$.

Finally, let $n_0 \leq 2$. Corollary 3.8 yields $n_0 = 2$. According to Lemma 3.11 (i) no w_{ij} with $n_i = 2$ lies in τ_X. So, we may assume $w_{01} \in \tau^+$. Moreover, all w_{ij} with $n_i = 1$ lie together in one τ^+, τ_X or in τ^-. Since each of τ^+ and τ^- contains two weights, we obtain $n_1 = 2$ and some $\gamma_{0j_1,1j_2}$ is as wanted. Lemma 3.9 (iv) shows $r \le 3$. $\square$

We retrieve a special case of [11, Corollary 4.18].

Corollary 3.14. *Let $X = X(A, P, u)$ be smooth with $\rho(X) = 2$. Then the divisor class group* $\mathrm{Cl}(X)$ *is torsion-free.*

Proof. By Corollary 3.5, there is an elementary big cone in Σ_X. Thus, Propositions 3.12 and 3.13 deliver a two-dimensional $\gamma_0 \in \mathrm{rlv}(X)$. The corresponding weights generate K as a group. This gives $\mathrm{Cl}(X) \cong K \cong \mathbb{Z}^2$. $\square$

Proof of Proposition 3.1. The variety X is isomorphic to some $X(A, P, u)$, where after suitable admissible operations we may assume $n_0 \ge \ldots \ge n_r$. Thus, Propositions 3.12 and 3.13 apply. $\square$

4. Proof of Theorems 1.1, 1.2 and 1.4

We prove Theorems 1.1, 1.2 and 1.4 by going through the cases established in Proposition 3.1. The notation is the same as in Sections 2 and 3. So, we deal with a smooth projective variety $X = X(A, P, u)$ of Picard number $\rho(X) = 2$ coming with an effective torus action of complexity one.

From Corollary 3.14 we know that $\mathrm{Cl}(X) = K = \mathbb{Z}^2$ holds. With $w_{ij} = Q(e_{ij})$ and $w_k = Q(e_k)$, the columns of the $2 \times (n+m)$ degree matrix Q will be written as

$$w_{ij} = (w_{ij}^1, w_{ij}^2) \in \mathbb{Z}^2, \qquad w_k = (w_k^1, w_k^2) \in \mathbb{Z}^2.$$

Recall that all relations $g_0, \ldots, g_{r-2}$ of $R(A, P)$ have the same degree in $K = \mathbb{Z}^2$; we set for short

$$\mu = (\mu^1, \mu^2) := \deg(g_0) \in \mathbb{Z}^2.$$

We will frequently work with the faces of the orthant $\gamma = \mathbb{Q}_{\ge 0}^{n+m}$ introduced in Lemma 3.9:

$$\gamma_{ij,k} = \mathrm{cone}(e_{ij}, e_k) \preceq \gamma, \qquad \gamma_{i_1 j_1, i_2 j_2} = \mathrm{cone}(e_{i_1 j_1}, e_{i_2 j_2}) \preceq \gamma.$$

Remark 4.1. Consider a face $\gamma_0 \preceq \gamma$ of type $\gamma_{ij,k}$ or $\gamma_{i_1 j_1, i_2 j_2}$. Write e', e'' for the two generators of γ_0 and $w' = Q(e')$, $w'' = Q(e'')$ for the corresponding columns of the degree matrix Q such that (w', w'') is positively oriented in $\mathbb{Z}^2$. Then Remark 2.11 tells us

$$\gamma_0 \in \mathrm{rlv}(X) \Rightarrow \det(w', w'') = 1.$$

So, if $\gamma_0 \in \mathrm{rlv}(X)$, then we may multiply Q from the left with a unimodular 2×2 matrix transforming w' and w'' into $(1, 0)$ and $(0, 1)$. This change of coordinates on $\mathrm{Cl}(X)$ does not affect the defining data (A, P). If $w' = (1, 0)$ and $w'' = (0, 1)$ hold and $e \in \gamma$ is a canonical basis vector with corresponding column $w = Q(e)$, then we have

$$\begin{aligned} \mathrm{cone}(e', e) \in \mathrm{rlv}(X) \quad &\Rightarrow \quad w = (w^1, 1), \\ \mathrm{cone}(e'', e) \in \mathrm{rlv}(X) \quad &\Rightarrow \quad w = (1, w^2). \end{aligned}$$

We are ready to go through the cases of Proposition 3.1; we keep the numbering introduced there.

Case (I) (a). We have $r = 2, m \geq 0$ and the list of n_i is $(n_0, 2, 2)$, where $n_0 \geq 3$. This leads to the first two cases of Theorems 1.1 and 1.2.

Proof. In a first step we show that there occur weights w_{0j} in each of τ^+ and τ^-. Otherwise, we may assume that all w_{0j} lie in τ^+, see Lemma 3.11 (i). Then Lemma 3.11 (ii) says that also all w_k lie in τ^+. Moreover, we have $\deg(T_i^{l_i}) \in \tau^+$ for $i = 0, 1, 2$. Thus, we may assume $w_{11}, w_{21} \in \tau^+$ and obtain $w_{12}, w_{22} \in \tau^-$, as there must be at least two weights in τ^-. Finally, we may assume that $\mathrm{cone}(w_{01}, w_{12})$ contains $w_{02}, \ldots, w_{0n_0}$ and w_{22}. Applying Remark 4.1 first to $\gamma_{01,12}$, then to all $\gamma_{0j,12}, \gamma_{12,k}$ and $\gamma_{01,22}, \gamma_{12,21}$ yields

$$Q = \left[\begin{array}{cccc|cc|cc|ccc} 0 & w_{02}^1 & \ldots & w_{0n_0}^1 & w_{11}^1 & 1 & w_{21}^1 & 1 & w_1^1 & \ldots & w_m^1 \\ 1 & 1 & \ldots & 1 & w_{11}^2 & 0 & 1 & w_{22}^2 & 1 & \ldots & 1 \end{array}\right],$$

where $w_{0j}^1 \geq 0$ and $w_{22}^2 \geq 0$. Since $\gamma_{01,12}, \gamma_{01,22} \in \mathrm{rlv}(X)$ holds, Lemma 3.9 (iv) implies $l_{11} = l_{21} = 1$. Applying $P \cdot Q^t = 0$ to the first row of P and the second row of Q gives

$$0 < 3 \leq n_0 \leq l_{01} + \ldots + l_{0n_0} = w_{11}^2 = 1 + w_{22}^2 w_{11}^1,$$

where the last equality is due to $\gamma_{11,22} \in \mathrm{rlv}(X)$, and thus $\det(w_{22}, w_{11}) = 1$. We conclude that $w_{22}^2 > 0$ and $w_{11}^1 > 0$. Because of $\gamma_{0j,22} \in \mathrm{rlv}(X)$, we obtain $\det(w_{22}, w_{0j}) = 1$. This implies $w_{0j}^1 = 0$ for all $j = 2, \ldots, n_0$. Applying $P \cdot Q^t = 0$ to the first row of P and the first row of Q gives $w_{11}^1 + l_{12} = 0$: a contradiction.

Knowing that each of τ^+ and τ^- contains weights w_{0j}, we can assume $w_{01}, w_{02} \in \tau^+$ and $w_{03} \in \tau^-$. Lemma 3.11 (ii) and (iii) show $n_0 = 3$ and $m = 0$. There is at least one other weight in τ^-, say $w_{11} \in \tau^-$. Applying Lemma 3.9 (iii) to $\gamma_{0j,03} \in \mathrm{rlv}(X)$ for $j = 1, 2$ and (iv) to suitable $\gamma_{0j_1,i_2j_2} \in \mathrm{rlv}(X)$, we obtain

$$l_{01} = l_{02} = 1, \qquad l_{11} = l_{12} = 1, \qquad l_{21} = l_{22} = 1.$$

Moreover, Remark 4.1 applied to $\gamma_{01,03}$ as well as $\gamma_{02,03}$ and $\gamma_{01,11}$ brings the matrix Q into the shape

$$Q = \left[\begin{array}{ccc|cc|cc} 0 & w_{02}^1 & 1 & 1 & w_{12}^1 & w_{21}^1 & w_{22}^1 \\ 1 & 1 & 0 & w_{11}^2 & w_{12}^2 & w_{21}^2 & w_{22}^2 \end{array}\right].$$

Observe that the second component of the degree of the relation is $\mu^2 = 2$. The possible positions of the weights w_{2j} define three subcases (Figure 4.1):

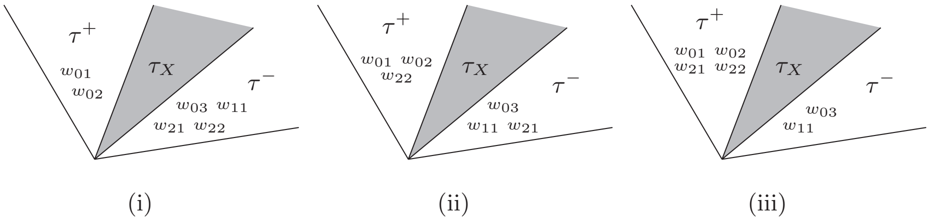

Fig. 4.1.

We will see that cases (i) and (ii) give the first two cases of Theorem 1.1 respectively and case (iii) will not provide any smooth variety.

In (i) we assume $w_{21}, w_{22} \in \tau^-$. Then $\gamma_{01,21}, \gamma_{01,22} \in \mathrm{rlv}(X)$ holds and Remark 4.1 shows $w_{21}^1 = w_{22}^1 = 1$. This implies $\mu^1 = 2$. Similarly, considering $\gamma_{02,21}, \gamma_{02,22} \in \mathrm{rlv}(X)$, we obtain $w_{02}^1 = 0$ or $w_{21}^2 = w_{22}^2 = 0$. The latter contradicts $\mu^2 = 2$ and thus $w_{02}^1 = 0$ holds. We conclude $l_{03} = \mu^1 = 2$. Furthermore $w_{12}^1 = \mu^1 - w_{11}^1 = 1$. Together, we have

$$g_0 = T_{01}T_{02}T_{03}^2 + T_{11}T_{12} + T_{21}T_{22}, \qquad Q = \left[\begin{array}{ccc|cc|cc} 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & a & 2-a & b & 2-b \end{array}\right],$$

where $a, b \in \mathbb{Z}$. Observe that $w_{12} \in \tau^-$ must hold; otherwise, $\gamma_{03,12} \in \mathrm{rlv}(X)$ and Remark 4.1 yields $w_{12}^2 = 1$, contradicting $w_{12} = (1, 1) = w_{11} \in \tau^-$. The semiample cone is $\mathrm{SAmple}(X) = \mathrm{cone}((0, 1), (1, d))$, where $d = \max(a, 2 - a, b, 2 - b)$. The anticanonical class is $-\mathcal{K}_X = (3, 4)$. Hence X is an almost Fano variety if and only if $d = 1$, which is equivalent to $a = b = 1$. In this situation X is already a Fano variety.

In (ii) we assume $w_{21} \in \tau^-$ and $w_{22} \in \tau^+$. Remark 4.1, applied to $\gamma_{01,21}, \gamma_{03,22} \in \mathrm{rlv}(X)$ shows $w_{21}^1 = w_{22}^2 = 1$. The latter implies $w_{21}^2 = \mu^2 - w_{22}^2 = 1$. We claim $w_{11}^2 \neq 0$. Otherwise, we have $w_{12}^2 = \mu^2 = 2$. This gives $\det(w_{03}, w_{12}) = 2$. We conclude $\gamma_{03,12} \notin \mathrm{rlv}(X)$ and $w_{12} \in \tau^-$. Then $\gamma_{01,12} \in \mathrm{rlv}(X)$ implies $w_{12}^1 = 1$. Thus, $w_{22} = (1, 1)$ and $w_{12} = (1, 2)$ hold, contradicting $w_{22} \in \tau^+$ and $w_{12} \in \tau^-$. Now, $\gamma_{11,22} \in \mathrm{rlv}(X)$ yields $w_{11}^2 w_{22}^1 = 0$ and thus $w_{22}^1 = 0$. We obtain $\mu^1 = 1$ and, as a consequence $l_{03} = 1$, $w_{02}^1 = 0$ and $w_{12}^1 = 0$. Therefore $w_{12} \in \tau^+$ holds. Now $\gamma_{03,12} \in \mathrm{rlv}(X)$ implies $w_{12}^2 = 1$ and $w_{11}^2 = \mu^2 - w_{12}^2 = 1$. We arrive at

$$g_0 = T_{01}T_{02}T_{03} + T_{11}T_{12} + T_{21}T_{22}, \qquad Q = \left[\begin{array}{ccc|cc|cc} 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{array}\right].$$

The anticanonical class is $-\mathcal{K}_X = (2, 4)$ and the semiample cone is $\mathrm{SAmple}(X) = \mathrm{cone}((0, 1), (1, 1))$. In particular X is Fano.

We turn to (iii), where both w_{21} and w_{22} lie in τ^+. The homogeneity of g_0 yields $w_{12} \in \tau^+$. Thus, $\gamma_{03,12}, \gamma_{03,21}, \gamma_{03,22} \in \mathrm{rlv}(X)$ holds and Remark 4.1

implies $w_{12}^2 = w_{21}^2 = w_{22}^2 = 1$. We conclude $w_{11}^2 = \mu^2 - w_{12}^2 = 1$. Similarly, $\gamma_{02,11}, \gamma_{11,21}, \gamma_{11,22} \in \mathrm{rlv}(X)$ yields $w_{02}^1 = w_{21}^1 = w_{22}^1 = 0$. This gives $0 \neq l_{03} = \mu^1 = w_{21}^1 + w_{22}^1 = 0$ which is not possible. □

Case (I) (b). We have $r = 2, m = 0, n = 6$ and the list of n_i is $(3, 2, 1)$. This leads to the third case Theorems 1.1 and 1.2.

Proof. Since there are at least two weights in τ^+ and two more in τ^-, we can assume $w_{01}, w_{02} \in \tau^+$ and $w_{03}, w_{12} \in \tau^-$. By Lemma 3.9 (iii) and (iv) we obtain $l_{01} = l_{02} = l_{11} = l_{12} = 1$. We may assume that $\mathrm{cone}(w_{01}, w_{03})$ contains w_{02}. Applying Remark 4.1 firstly to $\gamma_{01,03}$, then to $\gamma_{02,03}$ and $\gamma_{01,12}$, we obtain

$$Q = \left[\begin{array}{ccc|cc|c} 0 & w_{02}^1 & 1 & w_{11}^1 & 1 & w_{21}^1 \\ 1 & 1 & 0 & w_{11}^2 & w_{12}^2 & w_{21}^2 \end{array}\right],$$

where $w_{02}^1 \geq 0$. For the degree μ of g_0, we have $\mu^2 = 2$. We conclude $w_{11}^2 = 2 - w_{12}^2$ and $l_{21} w_{21}^2 = 2$ which in turn implies $l_{21} = 2$ and $w_{21}^2 = 1$. For $\gamma_{02,12} \in \mathrm{rlv}(X)$, Remark 4.1 gives $\det(w_{12}, w_{02}) = 1$ and thus $w_{02}^1 = 0$ or $w_{12}^2 = 0$ must hold.

We treat the case $w_{02}^1 = 0$. Then $\mu = (l_{03}, 2)$ holds. We conclude $w_{11}^1 = l_{03} - 1$ and $w_{21}^1 = l_{03}/2$. With $c := l_{03}/2 \in \mathbb{Z}_{\geq 1}$ and $a := w_{12}^2 \in \mathbb{Z}$, we obtain the degree matrix

$$Q = \left[\begin{array}{ccc|cc|c} 0 & 0 & 1 & 2c-1 & 1 & c \\ 1 & 1 & 0 & 2-a & a & 1 \end{array}\right].$$

We show that $w_{11} \in \tau^-$. Otherwise, $w_{11} \in \tau^+$ holds, we have $\gamma_{03,11} \in \mathrm{rlv}(X)$ and Remark 4.1 yields $a = 1$. But then $w_{01} = (0, 1) \in \tau^+$ and $w_{11} = (2c-1, 1) \in \tau^+$ imply $w_{12} = (1, 1) \in \tau^+$; a contradiction. So we have $w_{11} \in \tau^-$. Then $\gamma_{01,11} \in \mathrm{rlv}(X)$ holds. Remark 4.1 gives $\det(w_{11}, w_{01}) = 1$ which means $c = 1$ and, as a consequence, $l_{03} = 2$. Together, we have

$$g_0 = T_{01} T_{02} T_{03}^2 + T_{11} T_{12} + T_{21}^2, \qquad Q = \left[\begin{array}{ccc|cc|c} 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2-a & a & 1 \end{array}\right],$$

where we may assume $a \geq 2 - a$ that means $a \in \mathbb{Z}_{\geq 1}$. The semiample cone is $\mathrm{SAmple}(X) = \mathrm{cone}((0, 1), (1, a))$, and the anticanonical class is $-\mathcal{K}_X = (2, 3)$. In particular, X is an almost Fano variety if and only $a = 1$ holds. In this situation X is already a Fano variety.

We turn to the case $w_{12}^2 = 0$. Here, $w_{11}^2 = \mu^2 = 2$ leads to $\det(w_{03}, w_{11}) = 2$ and thus the $\mathfrak{F}$-face $\gamma_{03,11}$ does not belong to $\mathrm{rlv}(X)$; see Remark 4.1. Hence $w_{11} \in \tau^-$ and thus $\gamma_{01,11} \in \mathrm{rlv}(X)$. This gives $w_{11}^1 = 1$ and thus $w_{11} = (1, 2)$. Because of $w_{02} = (w_{02}, 1) \in \tau^+$, we must have $w_{02}^1 = 0$ and the previous consideration applies. □

Case (I) (c). We have $r = 2, m = 0, n = 5$ and the list of n_i is $(3, 1, 1)$. This case does not provide smooth varieties.

Proof. Each of τ^+ and τ^- contains at least two weights. We may assume $w_{01}, w_{02} \in \tau^+$ and $w_{03}, w_{11}, w_{21} \in \tau^-$. Then $\gamma_{01,03}, \gamma_{02,03} \in \mathrm{rlv}(X)$ holds and Lemma 3.9 (iii) yields $l_{01} = l_{02} = 1$. By Remark 4.1 we can assume $w_{03} = (1, 0)$ and $w_{01}^2 = w_{02}^2 = 1$. This implies $\mu^2 = 2$ and, as a consequence, $l_{11} = l_{21} = 2$. By [13, Theorem 1.1], we have torsion in $\mathrm{Cl}(X)$; a contradiction to Corollary 3.14. □

Case (I) (d). We have $r = 2, m \geq 0, n = 6$ and the list of n_i is $(2, 2, 2)$. Suitable admissible operations lead to one of the following configurations for the weights w_{ij} (Figure 4.2):

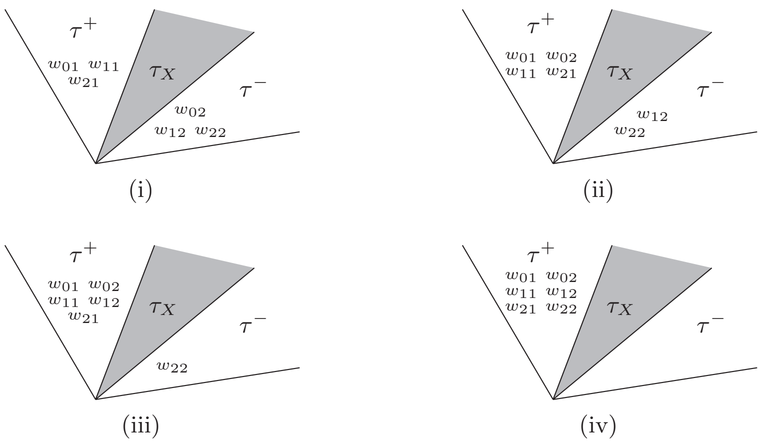

Fig. 4.2.

Configuration (i) amounts to No. 4 in Theorems 1.1, 1.2 and 1.4, configuration (ii) to No. 5, configuration (iii) to Nos. 6 and 7, and configuration (iv) to Nos. 8 and 9.

Proof for configuration (i). We have $w_{01}, w_{11}, w_{21} \in \tau^+$ and $w_{02}, w_{12}, w_{22} \in \tau^-$. We may assume $w_k \in \tau^+$ for all $k = 1, \dots, m$. If $m > 0$, we have $\gamma_{i2,1} \in \mathrm{rlv}(X)$ and Lemma 3.9 (ii) gives $l_{i1} = 1$ for $i = 0, 1, 2$. If $m = 0$, we use $\gamma_{i_1 1, i_2 2} \in \mathrm{rlv}(X)$ and Lemma 3.9 (iv) to obtain $l_{i_1 2} = 1$ or $l_{i_2 1} = 1$ for all $i_1 \neq i_2$. Thus, for $m = 0$, we may assume $l_{01} = l_{11} = 1$ and are left with $l_{21} = 1$ or $l_{22} = 1$.

We treat the case $m \geq 0$ and $l_{01} = l_{11} = l_{21} = 1$. Here we may assume $w_{11}, w_{21}, w_{22} \in \mathrm{cone}(w_{01}, w_{12})$. Applying Remark 4.1 firstly to $\gamma_{01,12}$ and then to $\gamma_{01,22}, \gamma_{12,21}$ and all $\gamma_{12,k}$ gives

$$Q = \left[\begin{array}{cc|cc|cc|ccc} 0 & w_{02}^1 & w_{11}^1 & 1 & w_{21}^1 & 1 & w_1^1 & \dots & w_m^1 \\ 1 & w_{02}^2 & w_{11}^2 & 0 & 1 & w_{22}^2 & 1 & \dots & 1 \end{array}\right].$$

Using $w_{11}, w_{21}, w_{22} \in \mathrm{cone}(w_{01}, w_{12})$ and the fact that the determinants of (w_{02}, w_{01}), (w_{12}, w_{11}) and (w_{22}, w_{21}) are positive, we obtain

$$w_{11}^1,\ w_{21}^1,\ w_{22}^2 \geq 0, \qquad w_{02}^1,\ w_{11}^2 > 0, \qquad 1 > w_{22}^2 w_{21}^1.$$

The degree μ of the relation satisfies

$$0 \;<\; \mu^1 \;=\; l_{02}w_{02}^1 \;=\; w_{11}^1 + l_{12} \;=\; w_{21}^1 + l_{22},$$
$$0 \;<\; \mu^2 \;=\; 1 + l_{02}w_{02}^2 \;=\; w_{11}^2 \;=\; 1 + l_{22}w_{22}^2.$$

In particular, $w_{02}^2 \geq 0$ holds and thus all components of the w_{ij} are non-negative. With $\gamma_{02,11}, \gamma_{02,21}, \in \mathrm{rlv}(X)$ and Remark 4.1, we obtain

$$w_{02}^1 w_{11}^2 \;=\; 1 + w_{02}^2 w_{11}^1, \qquad\qquad w_{02}^1 - 1 \;=\; w_{02}^2 w_{21}^1.$$

We show $w_{22}^2 = 0$. Otherwise, because of $1 > w_{22}^2 w_{21}^1$, we have $w_{21}^1 = 0$. This implies $w_{02}^1 = 1$ and thus

$$w_{11}^2 \;=\; 1 + w_{02}^2 w_{11}^1 \;=\; 1 + l_{02} w_{02}^2.$$

This gives $w_{02}^2 = 0$ or $w_{11}^1 = l_{02}$. The first is impossible because of $l_{02}w_{02}^2 = l_{22}w_{22}^2$ and the second because of $l_{02} = l_{02}w_{02}^1 = w_{11}^1 + l_{12}$.

Knowing $w_{22}^2 = 0$, we directly conclude $w_{11}^2 = 1$ and $w_{02}^2 = 0$ from $\mu^2 = 1$. This gives $w_{02}^1 = 1$. With $a := w_{11}^1 \in \mathbb{Z}_{\geq 0}, b := w_{21}^1 \in \mathbb{Z}_{\geq 0}$ and $c_k := w_k^1 \in \mathbb{Z}$ we are in the situation

$$g_0 \;=\; T_{01}T_{02}^{l_{02}} + T_{11}T_{12}^{l_{12}} + T_{21}T_{22}^{l_{22}}, \qquad Q \;=\; \left[\begin{array}{cc|cc|cc|ccc} 0 & 1 & a & 1 & b & 1 & c_1 & \dots & c_m \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & \dots & 1 \end{array}\right],$$

where we may assume $0 \leq a \leq b$ and $c_1 \leq \ldots \leq c_m$. Observe $l_{02} = a + l_{12} = b + l_{22}$. The anticanonical class and the semiample cone of X are given by

$$-\mathcal{K}_X \;=\; (3 + b + c_1 + \ldots + c_m - l_{12},\; 2 + m),$$
$$\mathrm{SAmple}(X) \;=\; \mathrm{cone}((1, 0), (d, 1)),$$

where $d := \max(b, c_m)$. Consequently, X is a Fano variety if and only if the following inequality holds:

$$3 + b + c_1 + \ldots + c_m - l_{12} \;>\; (2 + m)d.$$

A necessary condition for this is $0 \leq d \leq 1$ with $l_{12} = 1$ if $d = 1$ and $l_{12} \leq 2$ if $d = 0$ The sextuples $(a, b, d, l_{02}, l_{12}, l_{22})$ fulfilling that condition are

$$(0, 0, 0, 2, 2, 2), \qquad (0, 0, 0, 1, 1, 1), \qquad (1, 1, 1, 2, 1, 1).$$

Each of these three tuples leads indeed to a Fano variety X; the respectively possible choices of the c_k lead to Nos. 4.A, 4.B and 4.C of Theorem 1.2 and are as follows:

$$c_1 = \ldots = c_m = 0, \qquad -1 \leq c_1 \leq 0 = c_2 = \ldots = c_m, \qquad c_1 = \ldots = c_m = 1.$$

Moreover X is a truly almost Fano variety if and only if the following equality holds

$$3 + b + c_1 + \ldots + c_m - l_{12} \;=\; (2 + m)d.$$

This implies $0 \le d \le 2$ and the only possible parameters fulfilling that condition are listed as Nos. 4.A to 4.F in the table of Theorem 1.4.

We turn to the case $m = 0$, $l_{01} = l_{11} = 1$ and $l_{21} \ge 2$. Lemma 3.9 (iv) applied to $\gamma_{01,22}, \gamma_{11,22} \in \mathrm{rlv}(X)$ gives $l_{02} = l_{12} = 1$. If $l_{22} = 1$, then suitable admissible operations bring us to the previous case. So, let $l_{22} \ge 2$. We may assume $w_{11} \in \mathrm{cone}(w_{01}, w_{12})$. We apply Remark 4.1 firstly to $\gamma_{01,12}$, then to $\gamma_{01,22}, \gamma_{12,21}$ and arrive at

$$g_0 = T_{01}T_{02} + T_{11}T_{12} + T_{21}^{l_{21}}T_{22}^{l_{22}}, \qquad Q = \left[\begin{array}{cc|cc|cc} 0 & w_{02}^1 & w_{11}^1 & 1 & w_{21}^1 & 1 \\ 1 & w_{02}^2 & w_{11}^2 & 0 & 1 & w_{22}^2 \end{array}\right],$$

where $w_{11}^1 \ge 0$ and $w_{11}^2 = \det(w_{12}, w_{11}) > 0$. We have $\mu = w_{02} + w_{01} = w_{11} + w_{12}$ and thus $w_{02} = w_{11} + w_{12} - w_{01}$. Because of $\gamma_{02,11} \in \mathrm{rlv}(X)$, we obtain

$$1 = \det(w_{02}, w_{11}) = \det(w_{12} - w_{01}, w_{11}) = w_{11}^1 + w_{11}^2.$$

We conclude $w_{11} = (0, 1)$ and $\mu = (1, 1)$. Using $\mu = l_{21}w_{21} + l_{22}w_{22}$ and $l_{21}, l_{22} \ge 2$ we see $w_{21}^1, w_{22}^2 < 0$. On the other hand, $0 < \det(w_{22}, w_{21}) = 1 - w_{21}^1 w_{22}^2$, a contradiction. Thus $l_{22} \ge 2$ does not occur. □

Proof for configuration (ii). We have $w_{01}, w_{02}, w_{11}, w_{21} \in \tau^+$ and $w_{12}, w_{22} \in \tau^-$. We may assume that $w_{02}, w_{12} \in \mathrm{cone}(w_{01}, w_{22})$ holds. Applying Remark 4.1 first to $\gamma_{01,22} \in \mathrm{rlv}(X)$ and then to $\gamma_{01,12}, \gamma_{02,22}, \gamma_{11,22} \in \mathrm{rlv}(X)$ we obtain

$$Q = \left[\begin{array}{cc|cc|cc|ccc} 0 & w_{02}^1 & w_{11}^1 & 1 & w_{21}^1 & 1 & w_1^1 & \dots & w_m^1 \\ 1 & 1 & 1 & w_{12}^2 & w_{21}^2 & 0 & w_1^2 & \dots & w_m^2 \end{array}\right],$$

where we have $w_{02}^1, w_{12}^2 \ge 0$ due to $w_{02}, w_{12} \in \mathrm{cone}(w_{01}, w_{22})$. Moreover, $w_{21}^2 > 0$ holds, as we infer from the conditions

$$\begin{aligned} 0 \le \mu^1 &= l_{02}w_{02}^1 = l_{11}w_{11}^1 + l_{12} = l_{21}w_{21}^1 + l_{22}, \\ 0 < \mu^2 &= l_{01} + l_{02} = l_{11} + l_{12}w_{12}^2 = l_{21}w_{21}^2. \end{aligned}$$

We show $l_{11} \ge 2$. Otherwise, the above conditions give $l_{12}w_{12}^2 > 0$ and thus $w_{12}^2 > 0$. For $\gamma_{02,12} \in \mathrm{rlv}(X)$, Remark 4.1 gives $\det(w_{12}, w_{02}) = 1$ which means $w_{12}^2 w_{02}^1 = 0$ and thus $w_{02}^1 = 0$. This implies $l_{21}w_{21}^1 + l_{22} = 0$ and thus $w_{21}^1 < 0$; a contradiction to $1 = \det(w_{12}, w_{21}) = w_{21}^2 - w_{12}^2 w_{21}^1$ which in turn holds due to $\gamma_{12,21} \in \mathrm{rlv}(X)$ and Remark 4.1.

Lemma 3.9 (iv) applied to $\gamma_{02,12}, \gamma_{01,12}, \gamma_{21,12} \in \mathrm{rlv}(X)$ shows that $l_{01} = l_{02} = l_{22} = 1$. Putting together $\mu^2 = 2 = l_{11} + l_{12}w_{12}^2$ and $l_{11} \ne 1$, we conclude $l_{11} = 2$ and $w_{12}^2 = 0$. With $\gamma_{12,21} \in \mathrm{rlv}(X)$ and Remark 4.1 we obtain $w_{21}^2 = 1$ and hence $l_{21} = \mu^2 = 2$. From

$$0 \le \mu^1 = w_{02}^1 = 2w_{11}^1 + 1 = 2w_{21}^1 + 1$$

we conclude $w_{11}^1 = w_{21}^1 \geq 0$ and thus $w_{02}^1 > 0$. Lemma 3.9 (ii) implies that the possible weights of type w_k lie in τ^-. Thus Remark 4.1 and $\gamma_{01,k}$ imply $w_k^1 = 1$ for all k. Moreover, since $\gamma_{02,k} \in \mathrm{rlv}(X)$, the latter implies $w_k^2 = 0$. All in all, we arrive at

$$g_0 = T_{01}T_{02} + T_{11}^2T_{12} + T_{21}^2T_{22}, \quad Q = \left[\begin{array}{cc|cc|cc|ccc} 0 & 2a+1 & a & 1 & a & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & \dots & 0 \end{array}\right],$$

where $a \in \mathbb{Z}_{\geq 0}$. The anticanonical class is $-\mathcal{K}_X = (2a + 2 + m, 2)$ and the semiample cone is $\mathrm{SAmple}(X) = \mathrm{cone}((1, 0), (2a + 1, 1))$. Hence X is an almost Fano variety if and only if $m \geq 2a$ holds and X is a Fano variety if and only if $m > 2a$ holds. □

Proof for configuration (iii). We have $w_{01}, w_{02}, w_{11}, w_{12}, w_{21} \in \tau^+$ and $w_{22} \in \tau^-$. As there must be another weight in τ^-, we obtain $m > 0$. Lemma 3.11 (v) yields $w_1, \dots, w_m \in \tau^-$. We may assume $w_{02}, w_{11}, w_{12}, w_k \in \mathrm{cone}(w_{01}, w_1)$, where $k = 2, \dots, m$. Applying Remark 4.1 firstly to $\gamma_{01,1} \in \mathrm{rlv}(X)$ and then to the remaining faces $\gamma_{01,22}, \gamma_{01,k}, \gamma_{ij,1}$ from $\mathrm{rlv}(X)$ leads to the degree matrix

$$Q = \left[\begin{array}{cc|cc|cc|cccc} 0 & w_{02}^1 & w_{11}^1 & w_{12}^1 & w_{21}^1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 1 & w_{22}^2 & 0 & w_2^2 & \dots & w_m^2 \end{array}\right],$$

with at most w_{21}^1, w_{22}^2 negative. We infer $l_{01} = l_{02} = l_{11} = l_{12} = l_{22} = 1$ from Lemma 3.9 (ii). For $\gamma_{02,22}, \gamma_{11,22}, \gamma_{12,22} \in \mathrm{rlv}(X)$ Remark 4.1 tells us

$$w_{22}^2 = 0 \qquad \text{or} \qquad w_{02}^1 = w_{11}^1 = w_{12}^1 = 0.$$

We treat the case $w_{22}^2 = 0$. Here $l_{21} = \mu^2 = 2$ holds. Thus $\mu^1 = w_{02}^1 = 2w_{21}^1 + 1$ holds. Because of $w_{02}^1 \geq 0$, we conclude $w_{02}^1 > 0$ and $w_{21}^1 \geq 0$. Remark 4.1 applied to $\gamma_{02,k} \in \mathrm{rlv}(X)$ gives $w_k^2 = 0$ for all $k = 2, \dots, m$. We arrive at

$$g_0 = T_{01}T_{02} + T_{11}T_{12} + T_{21}^2T_{22}, \quad Q = \left[\begin{array}{cc|cc|cc|ccc} 0 & 2c+1 & a & b & c & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & \dots & 0 \end{array}\right],$$

where $a, b, c \in \mathbb{Z}_{\geq 0}$ and $a + b = 2c + 1$. Furthermore, the anticanonical class is $-\mathcal{K}_X = (3c + 2 + m, 3)$ and we have $\mathrm{SAmple}(X) = \mathrm{cone}((1, 0), (2c + 1, 1))$. In particular, X is an almost Fano variety if and only if $3c + 1 \leq m$ holds and a Fano variety if and only if the corresponding strict inequality holds.

Now we consider the case $w_{02}^1 = w_{11}^1 = w_{12}^1 = 0$. We have $\mu^1 = 0$, which implies $l_{21} = 1$, $w_{21}^1 = -1$. Consequently, $\mu^2 = 2$ gives $w_{22}^2 = 1$. Since $\gamma_{21,k} \in \mathrm{rlv}(X)$ for $2 \leq k \leq m$, we conclude $w_k^2 = 0$ for all k. Therefore we obtain

$$g_0 = T_{01}T_{02} + T_{11}T_{12} + T_{21}T_{22}, \quad Q = \left[\begin{array}{cc|cc|cc|ccc} 0 & 0 & 0 & 0 & -1 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & \dots & 0 \end{array}\right].$$

Finally, we have $-\mathcal{K}_X = (m, 4)$ and $\mathrm{SAmple}(X) = \mathrm{cone}((1,1),(0,1))$. Thus, X is a Fano variety if and only if $m < 4$ holds. Moreover, X is an almost Fano variety if and only if $m \le 4$ holds. □

Proof for configuration (iv). All w_{ij} lie in τ^+. Then we have $m \ge 2$, and one, and hence all w_k, lie in τ^-, see Lemma 3.11 (v). Applying Lemma 3.9 (ii) to $\gamma_{ij,1} \in \mathrm{rlv}(X)$, we conclude $l_{ij} = 1$ for all i, j. Thus we have the relation

$$g_0 = T_{01}T_{02} + T_{11}T_{12} + T_{21}T_{22}.$$

We may assume that $\mathrm{cone}(w_{01}, w_1)$ contains all w_{ij}, w_k. Remark 4.1 applied to $\gamma_{01,1} \in \mathrm{rlv}(X)$ leads to $w_1 = (1,0)$ and $w_{01} = (0,1)$. All other weights lie in the positive orthant. For $\gamma_{ij,1}, \gamma_{01,k} \in \mathrm{rlv}(X)$ Remark 4.1 shows $w_{ij}^2 = w_k^1 = 1$ for all i, j, k. Consider the case that all w_k^2 vanish. Then the degree matrix is of the form

$$Q = \left[\begin{array}{cc|cc|cc|ccc} 0 & a_2 & a_3 & a_4 & a_5 & a_6 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & \dots & 0 \end{array}\right],$$

where $a_i \in \mathbb{Z}_{\ge 0}$ and $a_2 = a_3 + a_4 = a_5 + a_6$. We have $-\mathcal{K}_X = (2a_2 + m, 4)$ and $\mathrm{SAmple}(X) = \mathrm{cone}((1,0),(a_2,1))$. Hence X is a Fano variety if and only if $2a_2 < m$ holds and an almost Fano variety if and only if $2a_2 \le m$ holds.

Finally, let $w_k^2 > 0$ for some k. Note that we may assume $0 \le w_2^2 \le \dots \le w_m^2$; in particular $w_m^2 > 0$. Since $\gamma_{ij,m} \in \mathrm{rlv}(X)$ for all i, j, Remark 4.1 yields $w_{ij}^1 = 0$ for all i, j. Thus we obtain the degree matrix

$$Q = \left[\begin{array}{cc|cc|cc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & \dots & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & a_2 & \dots & a_m \end{array}\right],$$

where $0 \le a_2 \le \dots \le a_m$ and $a_m > 0$. The anticanonical class and the semiample cone are given as

$$-\mathcal{K}_X = (m,\, 4 + a_2 + \dots + a_m), \qquad \mathrm{SAmple}(X) = \mathrm{cone}((0,1),(1,a_m)).$$

In particular, X is a Fano variety if and only if $4 + a_2 + \dots + a_m > m a_m$ holds. Note that for the latter $a_m \le 3$ is necessary. Moreover, X is a truly almost Fano variety if and only if the equality $4 + a_2 + \dots + a_m = m a_m$ holds. □

Case (I) (e). We have $r = 2, m \ge 0, n = 5$ and the list of n_i is $(2,2,1)$. This leads to Nos. 10, 11 and 12 in Theorems 1.1, 1.2 and 1.4.

Proof. We divide this case into the following three configurations, according to the way some weights lie with respect to τ_X (Figure 4.3).

We show that configuration (i) does not provide any smooth variety, (ii) delivers No. 10 of Theorem 1.1 and (iii) delivers Nos. 11 and 12.

In configuration (i) we have $w_{01}, w_{11} \in \tau^-$ and $w_{02}, w_{12} \in \tau^+$. We may assume $w_{11} \in \mathrm{cone}(w_{01}, w_{12})$. Remark 4.1 applied to $\gamma_{01,12} \in \mathrm{rlv}(X)$ leads to

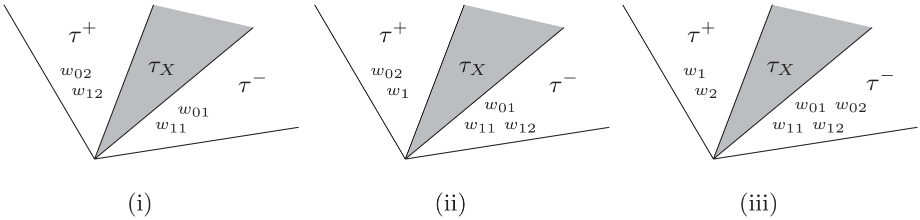

Fig. 4.3.

$w_{01} = (1, 0)$ and $w_{12} = (0, 1)$. Observe $w_{11}^1, w_{11}^2 \geq 0$. Due to $\det(w_{11}, w_{12}) > 0$, we even have $w_{11}^1 > 0$ and $\det(w_{01}, w_{02}) > 0$ gives $w_{02}^2 > 0$. Since $T_0^{l_0}$ and $T_1^{l_1}$ share the same degree, we have

$$l_{01}w_{01} + l_{02}w_{02} \;=\; l_{11}w_{11} + l_{12}w_{12}.$$

Lemma 3.9 (iv) says $l_{02} = 1$ or $l_{11} = 1$, which allows us to resolve for w_{02} or for w_{11} in the above equation. Using $\gamma_{02,11} \in \mathrm{rlv}(X)$, we obtain

$$l_{02} = 1 \implies 1 = \det(w_{11}, w_{02}) = \det(w_{11}, l_{12}w_{12} - l_{01}w_{01}) = l_{12}w_{11}^1 + l_{01}w_{11}^2,$$

$$l_{11} = 1 \implies 1 = \det(w_{11}, w_{02}) = \det(l_{01}w_{01} - l_{12}w_{12}, w_{02}) = l_{01}w_{02}^2 + l_{12}w_{02}^1.$$

We show $l_{02} > 1$. Otherwise, $l_{02} = 1$ holds. The above consideration shows $w_{11}^2 = 0$ and $l_{12} = w_{11}^1 = 1$. Thus, $l_{21}w_{21}^2 = l_{12} = 1$ holds and we obtain $l_{21} = 1$; a contradiction to P being irredundant. Thus, $l_{02} > 1$ and $l_{11} = 1$ must hold. Because of $w_{02}^2 > 0$, we must have $w_{02}^1 \leq 0$. With

$$1 \;=\; \det(w_{11}, w_{02}) \;=\; w_{11}^1 w_{02}^2 - w_{11}^2 w_{02}^1$$

we see that $w_{11}^2 w_{02}^1 = 0$ and $w_{11}^1 = w_{02}^2 = 1$. But then we arrive at $1 = l_{11}w_{11}^1 = l_{21}w_{21}^1$. Again this means $l_{21} = 1$; a contradiction to P being irredundant.

In configuration (ii) we have $w_{01}, w_{11}, w_{12} \in \tau^-$ and $w_{02}, w_1 \in \tau^+$. In particular $m \geq 1$. Lemma 3.11 (v) yields $w_2, \ldots, w_m \in \tau^+$. Applying Remark 4.1 first to $\gamma_{11,1} \in \mathrm{rlv}(X)$ an then to $\gamma_{01,1}, \gamma_{12,1}, \gamma_{02,11}, \gamma_{11,k} \in \mathrm{rlv}(X)$ leads to

$$Q \;=\; \left[\begin{array}{cc|cc|c|cccc} 1 & w_{02}^1 & 1 & 1 & w_{21}^1 & 0 & w_2^1 & \ldots & w_m^1 \\ w_{01}^2 & 1 & 0 & w_{12}^2 & w_{21}^2 & 1 & 1 & \ldots & 1 \end{array}\right].$$

Applying Lemma 3.9 (ii) to $\gamma_{01,1}, \gamma_{12,1}, \gamma_{11,1} \in \mathrm{rlv}(X)$ we obtain $l_{02} = l_{11} = l_{12} = 1$. For the degree μ of the relation g_0 we note

$$\mu^1 \;=\; l_{01} + w_{02}^1 \;=\; 2 \;=\; l_{21}w_{21}^1, \qquad \mu^2 \;=\; l_{01}w_{01}^2 + 1 \;=\; w_{12}^2 \;=\; l_{21}w_{21}^2.$$

From $\mu^1 = 2$ we infer $l_{21} = 2$ and $w_{21}^1 = 1$. Consequently, μ^2 is even and both l_{01}, w_{01}^2 are odd. Using again $\mu^1 = 2$ gives $w_{02}^1 \neq 0$. For $\gamma_{02,12} \in \mathrm{rlv}(X)$

Remark 4.1 yields $\det(w_{12}, w_{02}) = 1$ which means $w_{02}^1 w_{12}^2 = 0$. We conclude $w_{12}^2 = 0 = \mu^2$. This implies $w_{21}^2 = 0$, $w_{01}^2 = -1$, $l_{01} = 1$ and $w_{02}^1 = 1$. We obtain

$$g_0 = T_{01}T_{02} + T_{11}T_{12} + T_{21}^2, \qquad Q = \left[\begin{array}{cc|cc|c|ccc} 1 & 1 & 1 & 1 & 1 & 0 & \dots & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 & \dots & 1 \end{array}\right],$$

where $w_2^1 = \dots = w_m^1 = 0$ follows from Remark 4.1 applied to $\gamma_{01,k} \in \mathrm{rlv}(X)$. The semiample cone is given as $\mathrm{SAmple}(X) = \mathrm{cone}((1,0),(1,1))$ and the anticanonical class as $-\mathcal{K}_X = (3, m)$. Therefore X is a Fano variety if and only if $m < 3$, i.e $m = 1, 2$. Moreover, X is an almost Fano variety if and only if $m \le 3$.

In configuration (iii) we have $w_{01}, w_{02}, w_{11}, w_{12} \in \tau^-$ and $w_1, w_2 \in \tau^+$. In particular $m \ge 2$. Lemma 3.11 (v) ensures $w_3, \dots, w_m \in \tau^+$. We can assume that all w_{ij}, w_k lie in $\mathrm{cone}(w_{01}, w_1)$. Applying Remark 4.1, firstly to $\gamma_{01,1}$ and then to all relevant faces of the types $\gamma_{ij,1}$ and $\gamma_{01,k}$, we achieve

$$w_{01} = (1,0), \quad w_1 = (0,1), \quad w_{02}^1 = w_{11}^1 = w_{12}^1 = 1, \quad w_2^2 = \dots = w_m^2 = 1.$$

Lemma 3.9 (ii) applied to all $\gamma_{ij,1}$ shows $l_{ij} = 1$ for all i, j. We conclude $\mu^1 = 2$ which in turn implies $l_{21} = 2$ and $w_{21}^1 = 1$. In particular, we have the relation

$$g_0 = T_{01}T_{02} + T_{11}T_{12} + T_{21}^2.$$

We treat the case in which $w_1^1 = \dots = w_m^1 = 0$ holds. All columns of the degree matrix lie in $\mathrm{cone}(w_{01}, w_1)$ and thus Q is of the form

$$Q = \left[\begin{array}{cc|cc|c|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 2c & a & b & c & 1 & 1 & \dots & 1 \end{array}\right],$$

where $a, b, c \in \mathbb{Z}_{\ge 0}$ and $a + b = 2c$. The anticanonical class is $-\mathcal{K} = (3, m + 3c)$ and we have $\mathrm{SAmple}(X) = \mathrm{cone}((0,1),(1,2c))$. Therefore X is a Fano variety if and only if $m > 3c$. Moreover, X is an almost Fano variety if and only if $m \ge 3c$.

We treat the case that $w_k^1 > 0$ holds for some k. Then we obtain $w_{02}^2 = 0$ by applying Remark 4.1 to $\gamma_{02,k}$. This yields $\mu^2 = 0$ and thus $w_{ij}^2 = 0$ for all i, j. Consequently, the degree matrix is given as

$$Q = \left[\begin{array}{cc|cc|c|cccc} 1 & 1 & 1 & 1 & 1 & 0 & w_2^1 & \dots & w_m^1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & \dots & 1 \end{array}\right],$$

where we can assume $0 \le w_2^1 \le \dots \le w_m^1$. The semiample cone and the anticanonical divisor are given as

$$\mathrm{SAmple}(X) = \mathrm{cone}((1,0),(w_m^1, 1)), \qquad -\mathcal{K} = (3 + w_2^1 + \dots + w_m^1, m).$$

We see that X is an almost Fano variety if and only if $m w_m^1 \le 3 + w_2^1 + \dots + w_m^1$ and that X is a Fano variety if and only if the corresponding strict inequality holds. $\square$

Case (I) (f). We have $r = 2, m \geq 1, n = 4$ and the list of n_i is $(2, 1, 1)$. This case does not provide any smooth variety.

Proof. We can assume $w_{01} \in \tau^-$ and $w_1 \in \tau^+$. Lemma 3.11 (v) ensures $w_2, \dots, w_m \in \tau^+$. Applying Remark 4.1 first to $\gamma_{01,1} \in \mathrm{rlv}(X)$ and then to the remaining $\gamma_{01,k} \in \mathrm{rlv}(X)$, we achieve

$$Q = \left[\begin{array}{cc|c|c|cccc} 1 & w_{02}^1 & w_{11}^1 & w_{21}^1 & 0 & w_2^1 & \dots & w_m^1 \\ 0 & w_{02}^2 & w_{11}^2 & w_{21}^2 & 1 & 1 & \dots & 1 \end{array}\right].$$

Moreover $\gamma_{01,1} \in \mathrm{rlv}(X)$ implies $l_{02} = 1$ by Lemma 3.9 (ii). Recall from Corollary 3.14 that $\mathrm{Cl}(X)$ is torsion-free. Thus [13, Theorem 1.1] implies that l_{11} and l_{21} are coprime.

Consider the case $w_{02} \in \tau^-$. Then $\gamma_{02,1} \in \mathrm{rlv}(X)$ holds, Lemma 3.9 (ii) yields $l_{01} = 1$ and Remark 4.1 shows $w_{02}^1 = 1$. We conclude $\mu^1 = 2$ and thus obtain $l_{11} = l_{21} = 2$: a contradiction.

Now let $w_{02} \in \tau^+$, which implies $\gamma_{01,02,11} \in \mathrm{rlv}(X)$. Since X is locally factorial, Remark 2.11 (ii) shows that w_{02}^2 and w_{11}^2 are coprime. Now we look at

$$\mu^2 = w_{02}^2 = l_{11} w_{11}^2 = l_{21} w_{21}^2.$$

We infer that l_{21} divides w_{02}^2 and w_{11}^2. This contradicts the coprimeness of w_{02}^2 and w_{11}^2, because by irredundancy of P we have $l_{21} \geq 2$. □

Case (II). We have $r = 3, m = 0$ and $2 = n_0 = n_1 \geq n_2 \geq n_3 \geq 1$. This leads to No. 13 in Theorems 1.1 and 1.2.

Proof. We treat the constellations (a), (b) and (c) at once. First observe that for every $w_{i_1 j_1}$ with $n_{i_1} = 2$, there is at least one $w_{i_2 j_2}$ with $n_{i_2} = 2$ and $i_1 \neq i_2$ such that $\tau_X \subseteq Q(\gamma_{i_1 j_1, i_2 j_2})^\circ$ and thus $\gamma_{i_1 j_1, i_2 j_2} \in \mathrm{rlv}(X)$. Since $r = 3$, we conclude $l_{ij} = 1$ for all i with $n_i = 2$; see Lemma 3.9 (iv).

We can assume $w_{01}, w_{11} \in \tau^-$ and $w_{02}, w_{12} \in \tau^+$ as well as $w_{11} \in \mathrm{cone}(w_{01}, w_{12})$. Applying Remark 4.1 to $\gamma_{01,12}, \in \mathrm{rlv}(X)$, we obtain $w_{01} = (1, 0)$ and $w_{12} = (0, 1)$. Moreover $w_{11}^1, w_{11}^2 \geq 0$ holds and, because of $w_{11} \notin \tau^+$, we even have $w_{11}^1 > 0$. For the degree μ of g_0 and g_1 we note that

$$\mu^1 = w_{02}^1 + 1 = w_{11}^1, \qquad \mu^2 = w_{02}^2 = w_{11}^2 + 1.$$

Thus, we can express w_{02} in terms of w_{11}. Remark 4.1 applied to $\gamma_{02,11} \in \mathrm{rlv}(X)$ gives $1 = \det(w_{11}, w_{02}) = w_{11}^1 + w_{11}^2$. We conclude $w_{11} = (1, 0)$ and $w_{02} = (0, 1)$. In particular, the degree of the relations g_0 and g_1 is $\mu = (1, 1)$.

In constellations (b) and (c), we have $n_3 = 1$ and $\mu = (1, 1)$. This implies $l_{31} = 1$, a contradiction to P being irredundant. Thus, constellations (b) and (c) do not occur.

We are left with constellation (a), that means that we have $n_0 = \ldots = n_3 = 2$. As seen before, $l_{ij} = 2$ for all i, j. Thus, the relations are

$$g_0 = T_{01}T_{02} + T_{11}T_{12} + T_{21}T_{22}, \qquad g_1 = \lambda T_{11}T_{12} + T_{21}T_{22} + T_{31}T_{32},$$

where $\lambda \in \mathbb{K}^* \setminus \{1\}$. In this situation, we may assume $w_{21}, w_{31} \in \tau^-$. Applying Remark 4.1 to the relevant faces $\gamma_{02,21}, \gamma_{02,31}$, we conclude $w_{21}^1 = w_{31}^1 = 1$. Since $\mu^1 = 1$ and $l_{ij} = 1$, we obtain $w_{22}^1 = w_{32}^1 = 0$. Thus, w_{22} and w_{32} lie in τ^+. Again Remark 4.1, this time applied to $\gamma_{01,22}, \gamma_{01,32} \in \mathrm{rlv}(X)$, yields $w_{22}^2 = w_{32}^2 = 1$. Since $\mu^2 = 1$ and $l_{ij} = 1$, we obtain $w_{21}^2 = w_{31}^2 = 0$. Hence we obtain the degree matrix

$$Q = \left[\begin{array}{cc|cc|cc|cc} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right].$$

The semiample cone is $\mathrm{SAmple}(X) = (\mathbb{Q}_{\geq 0})^2$ and the anticanonical divisor is $-\mathcal{K}_X = (2, 2)$. In particular, X is a Fano variety. □

Proof of Theorems 1.1*,* 1.2 *and* 1.4. The preceding analysis of the cases of Proposition 3.1 shows that every smooth rational non-toric projective variety of Picard number two coming with a torus action of complexity one occurs in Theorem 1.1 and, among these, the Fano ones in Theorem 1.2 and the truly almost Fano ones in Theorem 1.4. Comparing the defining data, one directly verifies that any two different listed varieties are not isomorphic to each other. Finally, using Remark 2.11 one explicitly checks that indeed all varieties listed in Theorem 1.1 are smooth. □

5. Duplicating free weights

As mentioned in the introduction, there are (up to isomorphism) just two smooth non-toric projective varieties with a torus action of complexity one and Picard number one, namely the smooth projective quadrics in dimensions three and four. In Picard number two we obtained examples in every dimension and this even holds when we restrict to the Fano case. Nevertheless, also in Picard number two we will observe a certain finiteness feature: each Fano variety listed in Theorem 1.2 arises from a smooth, but not necessarily Fano, variety of dimension at most seven via an iterated generalized cone construction. In terms of the Cox ring the generalized cone construction simply means *duplicating a free weight*.

For the precise treatment, the setting of bunched rings $(R, \mathfrak{F}, \Phi)$ is most appropriate. Recall from [1, Section 3.2] that R is an integral normal almost freely factorially K-graded $\mathbb{K}$-algebra, $\mathfrak{F}$ a system of pairwise non-associated K-prime generators for R and Φ a certain collection of polyhedral cones in $K_{\mathbb{Q}}$ defining an open set $\widehat{X} \subseteq \overline{X} = \mathrm{Spec}\, R$ with a good quotient $X = \widehat{X} /\!\!/ H$ by the action of the quasitorus $H = \mathrm{Spec}\, \mathbb{K}[K]$ on $\overline{X}$. Dimension, divisor class group and Cox ring of X are given by

$$\dim(X) = \dim(R) - \dim(K_{\mathbb{Q}}), \qquad \mathrm{Cl}(X) = K, \qquad \mathcal{R}(X) = R.$$

We call $X = X(R, \mathfrak{F}, \Phi)$ the variety associated with the bunched ring $(R, \mathfrak{F}, \Phi)$. This construction yields for example all normal complete A_2-varieties with a finitely generated Cox ring, *e.g.*, Mori dream spaces. Observe that our Construction 2.7 presented earlier is a special case; it yields precisely the rational projective varieties with a torus action of complexity one. The approach via bunched rings allows in particular an algorithmic treatment [14].

Construction 5.1. Let $R = \mathbb{K}[T_1, \dots, T_r]/\langle g_1, \dots, g_s \rangle$ a K-graded algebra presented by K-homogeneous generators T_i and relations $g_j \in \mathbb{K}[T_1, \dots, T_{r-1}]$. By *duplicating the free weight* $\deg(T_r)$ we mean passing from R to the K-graded algebra

$$R' := \mathbb{K}[T_1, \dots, T_r, T_{r+1}]/\langle g_1, \dots, g_s \rangle, \qquad \deg(T_{r+1}) := \deg(T_r) \in K,$$

where $g_j \in \mathbb{K}[T_1, \dots, T_{r-1}] \subseteq \mathbb{K}[T_1, \dots, T_r, T_{r+1}]$. If in this situation $(R, \mathfrak{F}, \Phi)$ is a bunched ring with $\mathfrak{F} = (T_1, \dots, T_r)$, then $(R', \mathfrak{F}', \Phi)$ is a bunched ring with $\mathfrak{F}' = (T_1, \dots, T_r, T_{r+1})$.

Proof. The $\mathbb{K}$-algebra R' is integral normal and, by [4, Theorem 1.4], factorially K-graded. Obviously, the K-grading is almost free in the sense of [1, Definition 3.2.1.1]. Moreover, $(R, \mathfrak{F})$ and $(R', \mathfrak{F}')$ have the same sets of generator weights in the common grading group K and the collection of projected $\mathfrak{F}'$-faces equals the collection of projected $\mathfrak{F}$-faces. We conclude that Φ is a true $\mathfrak{F}'$-bunch in the sense of [1, Definition 3.2.1.1] and thus $(R', \mathfrak{F}', \Phi)$ is a bunched ring. □

The word "free" in Construction 5.1 indicates that the variable T_r does not occur in the relations g_j. In the above setting, we say that R is a complete intersection, for short c.i., if R is of dimension $r - s$. Here are the basic features of the procedure.

Proposition 5.2. *Let $(R', \mathfrak{F}', \Phi)$ arise from the bunched ring $(R, \mathfrak{F}, \Phi)$ via Construction 5.1. Set $X' := X(R', \mathfrak{F}', \Phi)$ and $X := X(R, \mathfrak{F}, \Phi)$. Then:*

(i) *We have* $\dim(X') = \dim(X) + 1$*;*
(ii) *The cones of semiample divisor classes satisfy* $\mathrm{SAmple}(X') = \mathrm{SAmple}(X)$*;*
(iii) *The variety X' is smooth if and only if X is smooth;*
(iv) *The ring R' is a c.i. if and only if R is a c.i.;*
(v) *If R is a c.i.,* $\deg(T_r)$ *semiample and X Fano, then X' is Fano.*

Proof. By construction, $\dim(R') = \dim(R) + 1$ holds. Since R and R' have the same grading group K, we obtain (i). Moreover, R and R' have the same defining relations g_j, hence we have (iv). According to [1, Proposition 3.3.2.9], the semiample cone is the intersection of all elements of Φ and thus (ii) holds.

To obtain the third assertion, we show first that $\widehat{X}'$ is smooth if and only if $\widehat{X}$ is smooth. For every relevant $\mathfrak{F}$-face $\gamma_0 \preceq \mathbb{Q}^r_{\geq 0}$ consider

$$\gamma_0' := \gamma_0 + \mathrm{cone}(e_{r+1}), \qquad \gamma_0'' := \mathrm{cone}(e_i;\ 1 \leq i < r,\ e_i \in \gamma_0) + \mathrm{cone}(e_{r+1}).$$

Then $\gamma_0, \gamma_0', \gamma_0'' \preceq \mathbb{Q}^{r+1}_{\ge 0}$ are relevant $\mathfrak{F}'$-faces and, in fact, all relevant $\mathfrak{F}'$-faces are of this form. Since the variables T_r and T_{r+1} do not appear in the relations g_j, we see that a stratum $\overline{X}(\gamma_0)$ is smooth if and only if the strata $\overline{X}'(\gamma_0)$, $\overline{X}'(\gamma_0')$ and $\overline{X}'(\gamma_0'')$ are smooth. Now [1, Corollary 3.3.1.11] gives (iii).

Finally, we show (v). As we have complete intersection Cox rings, [1, Proposition 3.3.3.2] applies and we obtain

$$-\mathcal{K}_{X'} \;=\; \sum_{i=1}^{r+1} \deg(T_i) - \sum_{j=1}^{s} \deg(g_j) \;=\; -\mathcal{K}_X + \deg(T_{r+1}).$$

Since X and X' share the same ample cone, we conclude that ampleness of $-\mathcal{K}_X$ implies ampleness of $-\mathcal{K}_{X'}$. □

We interprete the duplication of free weights in terms of birational geometry: it turns out to be a composition of a contraction of fiber type, a series of flips and a divisorial contraction, where all contractions are elementary, *i.e.*, of relative Picard number one; see [8] for a detailed study of the latter type of maps in the context of general smooth Fano 4-folds.

Proposition 5.3. *Let $(R', \mathfrak{F}', \Phi)$ arise from the bunched ring $(R, \mathfrak{F}, \Phi)$ via Construction 5.1. Set $X' := X(R', \mathfrak{F}', \Phi)$ and $X := X(R, \mathfrak{F}, \Phi)$. Assume that X is $\mathbb{Q}$-factorial. Then there is a sequence*

$$X \;\longleftarrow\; \widetilde{X}_1 \;\dashrightarrow\; \dots \;\dashrightarrow\; \widetilde{X}_t \;\longrightarrow\; X',$$

where $\widetilde{X}_1 \to X$ is a contraction of fiber type with fibers $\mathbb{P}_1$, every $\widetilde{X}_i \dashrightarrow \widetilde{X}_{i+1}$ is a flip and $\widetilde{X}_t \to X'$ is the contraction of a prime divisor. If $\deg(T_r) \in K$ is Cartier, then $\widetilde{X}_1 \to X$ is the $\mathbb{P}_1$-bundle associated with the divisor on X corresponding to T_r.

Proof. In order to define $\widetilde{X}_1$, we consider the canonical toric embedding $X \subseteq Z$ in the sense of [1, Construction 3.2.5.3]. Let Σ be the fan of Z and $P = [v_1, \dots, v_r]$ be the matrix having the primitive generators $v_i \in \mathbb{Z}^n$ of the rays of Σ as its columns. Define a further matrix

$$\widetilde{P} \;:=\; \begin{bmatrix} v_1 & \dots & v_{r-1} & v_r & 0 & 0 \\ 0 & \dots & 0 & -1 & 1 & -1 \end{bmatrix}.$$

We denote the columns of $\widetilde{P}$ by $\widetilde{v}_1, \dots, \widetilde{v}_r, \widetilde{v}_+, \widetilde{v}_- \in \mathbb{Z}^{n+1}$, write ϱ_+, ϱ_- for the rays through $\widetilde{v}_+, \widetilde{v}_-$ and define a fan

$$\widetilde{\Sigma}_1 \;:=\; \{\widetilde{\sigma} + \varrho_+,\ \widetilde{\sigma} + \varrho_-,\ \widetilde{\sigma};\ \sigma \in \Sigma\}, \qquad \widetilde{\sigma} \;:=\; \operatorname{cone}(\widetilde{v}_i;\ v_i \in \sigma).$$

The projection $\mathbb{Z}^{n+1} \to \mathbb{Z}^n$ is a map of fans $\widetilde{\Sigma}_1 \to \Sigma$. The associated toric morphism $\widetilde{Z}_1 \to Z$ has fibers $\mathbb{P}_1$. If the toric divisor D_r corresponding to the ray through v_r is Cartier, then $\widetilde{Z}_1 \to Z$ is the $\mathbb{P}_1$-bundle associated with D_r. We define $\widetilde{X}_1 \subseteq \widetilde{Z}_1$ to be the preimage of $X \subseteq Z$. Then $\widetilde{X}_1 \to X$ has fibers $\mathbb{P}_1$. If $\deg(T_r)$ is Cartier, then so is D_r and hence $\widetilde{X}_1 \to X$ inherits the $\mathbb{P}_1$-bundle structure.

Now we determine the Cox ring of the variety $\widetilde{X}_1$. For this, observe that the projection $\mathbb{Z}^{r+2} \to \mathbb{Z}^r$ defines a lift of $\widetilde{Z}_1 \to Z$ to the toric characteristic spaces and thus leads to the commutative diagram

$$\begin{tikzcd}
\widetilde{\pi}^\sharp(\widetilde{X}_1) \arrow[d, "\widetilde{\pi}"'] & \subseteq & \widetilde{W}_1 \arrow[r] \arrow[d, "\widetilde{\pi}"'] & W \arrow[d, "\pi"] & \supseteq & \pi^\sharp(X) \arrow[d, "\pi"] \\
\widetilde{X}_1 & \subseteq & \widetilde{Z}_1 \arrow[r] & Z & \supseteq & X
\end{tikzcd}$$

where $\widetilde{\pi}^\sharp(\widetilde{X}_1)$ and $\pi^\sharp(X)$ denote the proper transforms with respect to the downwards toric morphisms. Pulling back the defining equations of $\pi^\sharp(X) \subseteq W$, we see that $\widetilde{\pi}^\sharp(\widetilde{X}_1) \subseteq \widetilde{W}_1$ has coordinate algebra $\widetilde{R} := R[S^+, S^-]$ graded by $\widetilde{K} := K \times \mathbb{Z}$ via

$$\begin{aligned}
\deg(T_i) &:= (w_i, 0), \\
w^+ := \deg(S^+) &:= (w_r, 1), \\
w^- := \deg(S^-) &:= (0, 1),
\end{aligned}$$

where $w_i := \deg(T_i) \in K$. The $\mathbb{K}$-algebra $\widetilde{R}$ is normal and, by [4, Theorem 1.4], factorially $\widetilde{K}$-graded. Moreover the $\widetilde{K}$-grading is almost free, as the K-grading of R has this property and $\widetilde{\mathfrak{F}} = (T_1, \dots, T_r, S^+, S^-)$ is a system of pairwise non-associated $\widetilde{K}$-prime generators. We conclude that $\widetilde{R}$ is the Cox ring of $\widetilde{X}_1$.

Next we look for the defining bunch of cones for $\widetilde{X}_1$. Observe that K sits inside $\widetilde{K}$ as $K \times \{0\}$. With $\theta := \mathrm{SAmple}(X) \times \{0\}$ we obtain a GIT-cone $\theta_1 := \mathrm{cone}(\theta, w^+) \cap \mathrm{cone}(\theta, w^-)$ of the $\widetilde{K}$-graded ring $\widetilde{R}$. The associated bunch $\widetilde{\Phi}_1$ consists of all cones of the form

$$\widetilde{\tau} + \mathrm{cone}(w^+), \qquad \widetilde{\tau} + \mathrm{cone}(w^-), \qquad \widetilde{\tau} + \mathrm{cone}(w^+, w^-),$$

where $\widetilde{\tau} = \tau \times \{0\}$, $\tau \in \Phi$. Since Φ is a true bunch, so is $\widetilde{\Phi}_1$. Together we obtain a bunched ring $(\widetilde{R}, \widetilde{\mathfrak{F}}, \widetilde{\Phi}_1)$. By construction, the fan corresponding to $\widetilde{\Phi}_1$ via Gale duality is $\widetilde{\Sigma}_1$. We conclude that $\widetilde{X}_1$ is the variety associated with $(\widetilde{R}, \widetilde{F}, \widetilde{\Phi}_1)$ and $\widetilde{X}_1 \subseteq \widetilde{Z}_1$ is the canonical toric embedding.

Observe that $\widetilde{X}_1 \to X$ corresponds to the passage from the GIT-cone θ_1 to the facet θ. In particular, we see that $\widetilde{X}_1 \to X$ is a Mori fiber space. To obtain the flips and the final divisorial contraction, we consider the full GIT-fan (Figure 5.1).

The GIT-cones inside $\theta + \mathrm{cone}(w^-)$ are the important ones. There we have the facet θ and the semiample cone θ_1 of $\widetilde{X}_1$. Proceeding in the direction of w^-, we come across other full-dimensional GIT-cones, say $\theta_2, \dots, \theta_{t+1}$. This gives a sequence of flips $\widetilde{X}_1 \dashrightarrow \dots \dashrightarrow \widetilde{X}_t$, where $\widetilde{X}_i$ is the variety with semiample cone θ_i. Passing from θ_t to θ_{t+1} gives a morphism $\widetilde{X}_t \to \widetilde{X}_{t+1}$ contracting the prime divisor corresponding to the variable S^- of the Cox ring $\widetilde{R}$ of $\widetilde{X}_t$. Note that $\widetilde{X}_{t+1}$ is $\mathbb{Q}$-factorial, as it is the GIT-quotient associated with a full-dimensional chamber.

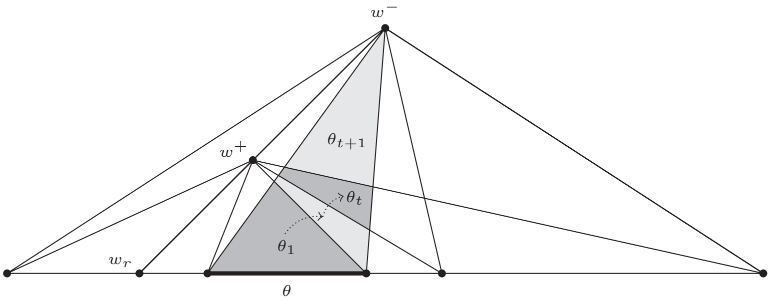

Fig. 5.1.

We show $\widetilde{X}_{t+1} \cong X'$. Recall that X' arises from X by duplicating the weight $\deg(T_r)$. We have $\mathrm{Cl}(X') = K$ and the Cox ring $R' = R[T_{r+1}]$ of X' is K-graded via $\deg(T_i) = w_i$ for $i = 1, \ldots, r$ and $\deg(T_{r+1}) = w_r$. In particular, the fan of the canonical toric ambient variety of X' has as its primitive ray generators the columns of the matrix

$$P' = \begin{bmatrix} v_1 & \ldots & v_{r-1} & v_r & 0 \\ 0 & \ldots & 0 & -1 & 1 \end{bmatrix}.$$

On the other hand, the canonical toric ambient variety $\widetilde{Z}_{t+1}$ of $\widetilde{X}_{t+1}$ is obtained from $\widetilde{Z}_t$ by contracting the divisor corresponding to the ray ϱ_-. Hence P' is as well the primitive generator matrix for the fan of $\widetilde{Z}_{t+1}$. We conclude that

$$\mathrm{Cl}(\widetilde{X}_{t+1}) = \mathbb{Z}^{r+1}/\operatorname{im}((P')^*) = \mathrm{Cl}(X') = K.$$

Similarly, we compare the Cox rings of $\widetilde{X}_{t+1}$ and X'. Let $\widetilde{Z}_t$ denote the canonical toric ambient variety of $\widetilde{X}_t$. Then the projection $\mathbb{Z}^{r+2} \to \mathbb{Z}^{r+1}$ defines a lift of $\widetilde{Z}_t \to \widetilde{Z}_{t+1}$ to the toric characteristic spaces and thus leads to the commutative diagram

$$\begin{array}{ccccccc} \widetilde{\pi}^\sharp(\widetilde{X}_t) & \subseteq & \widetilde{W}_t & \longrightarrow & \widetilde{W}_{t+1} & \supseteq & \pi^\sharp(\widetilde{X}_{t+1}) \\ \downarrow{\scriptstyle\widetilde{\pi}} & & \downarrow{\scriptstyle\widetilde{\pi}} & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ \widetilde{X}_t & \subseteq & \widetilde{Z}_t & \longrightarrow & \widetilde{Z}_{t+1} & \supseteq & \widetilde{X}_{t+1} \end{array}$$

where the proper transforms $\widetilde{\pi}^\sharp(\widetilde{X}_t)$ and $\pi^\sharp(\widetilde{X}_{t+1})$ are the characteristic spaces of $\widetilde{X}_t$ and $\widetilde{X}_{t+1}$ respectively and the first is mapped onto the second one. We conclude that the Cox ring of $\widetilde{X}_{t+1}$ is $R[S^+]$ graded by $\deg(T_i) = w_i$ for $i = 1, \ldots, r$ and $\deg(S^+) = w_r$ and thus is isomorphic to the Cox ring R' of X'.

The final step is to compare the defining bunches of cones $\widetilde{\Phi}_{t+1}$ of $\widetilde{X}_{t+1}$ and Φ' of X'. For this, observe that the fan of the toric ambient variety $\widetilde{Z}_{t+1}$ contains the cones $\widetilde{\sigma} + \varrho_+$, where $\sigma \in \Sigma$. Thus, every $\tau \in \Phi'$ belongs to $\widetilde{\Phi}_{t+1}$. We conclude that

$$\mathrm{SAmple}(\widetilde{X}_{t+1}) \subseteq \mathrm{SAmple}(X').$$

Since $\widetilde{X}_{t+1}$ is $\mathbb{Q}$-factorial, its semiample cone is of full dimension. Both cones belong to the GIT-fan, hence we see that the above inclusion is in fact an equality. Thus $\widetilde{\Phi}_{t+1}$ equals Φ'. ☐

We return to the Fano varieties of Theorem 1.2. We first list the (finitely many) examples which do not allow duplication of a free weight and then present the starting models for constructing the Fano varieties via duplication of weights.

Proposition 5.4. *The varieties of Theorem* 1.2 *containing no divisors with infinite general isotropy are precisely the following ones.*

No.	$\mathcal{R}(X)$	$[w_1, \dots, w_r]$	$-\mathcal{K}_X$	$\dim(X)$
1	$\frac{\mathbb{K}[T_1,\dots,T_7]}{\langle T_1T_2T_3^2+T_4T_5+T_6T_7\rangle}$	$\begin{bmatrix} 0&0&1&1&1&1&1 \\ 1&1&0&1&1&1&1 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 4 \end{bmatrix}$	4
2	$\frac{\mathbb{K}[T_1,\dots,T_7]}{\langle T_1T_2T_3+T_4T_5+T_6T_7\rangle}$	$\begin{bmatrix} 0&0&1&1&0&1&0 \\ 1&1&0&1&1&1&1 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 4 \end{bmatrix}$	4
3	$\frac{\mathbb{K}[T_1,\dots,T_6]}{\langle T_1T_2T_3^2+T_4T_5+T_6^2\rangle}$	$\begin{bmatrix} 0&0&1&1&1&1 \\ 1&1&0&1&1&1 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$	3
4.A	$\frac{\mathbb{K}[T_1,\dots,T_6]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\begin{bmatrix} 0&1&0&1&0&1 \\ 1&0&1&0&1&0 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 2 \end{bmatrix}$	3
4.B	$\frac{\mathbb{K}[T_1,\dots,T_6]}{\langle T_1T_2^2+T_3T_4+T_5T_6\rangle}$	$\begin{bmatrix} 0&1&1&1&1&1 \\ 1&0&1&0&1&0 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$	3
4.C	$\frac{\mathbb{K}[T_1,\dots,T_6]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6^2\rangle}$	$\begin{bmatrix} 0&1&0&1&0&1 \\ 1&0&1&0&1&0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	3
13	$\frac{\mathbb{K}[T_1,\dots,T_8]}{\left\langle \begin{smallmatrix} T_1T_2+T_3T_4+T_5T_6, \\ \lambda T_3T_4+T_5T_6+T_7T_8 \end{smallmatrix} \right\rangle}$, $\lambda\in\mathbb{K}^*\setminus\{1\}$	$\begin{bmatrix} 1&0&1&0&1&0&1&0 \\ 0&1&0&1&0&1&0&1 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 2 \end{bmatrix}$	4

Proof. For a T-variety $X = X(A, P, u)$, the divisors having infinite general T-isotropy are precisely the vanishing sets of the variable S_k. Thus we just have to pick out the cases with $m = 0$ from Theorem 1.2. ☐

Theorem 5.5. *Let X be a smooth rational Fano variety with a torus action of complexity one and Picard number two. If there is a prime divisor with infinite general isotropy on X, then X arises via iterated duplication of the free weight w_r from one of the following varieties Y.*

No.	$\mathcal{R}(Y)$	$[w_1, \dots, w_r]$	u	$\dim(Y)$
4.A	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cccccc\|c} 0&1&0&1&0&1&0 \\ 1&0&1&0&1&0&1 \end{array}\right]$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	4
4.A	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,S_2]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cccccc\|cc} 0&1&0&1&0&1&-1&0 \\ 1&0&1&0&1&0&1&1 \end{array}\right]$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	5

4.B	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1]}{\langle T_1T_2^2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cccccc\|c} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right]$	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	4
4.C	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1]}{\langle T_1T_2^2+T_3T_4^2+T_5T_6^2\rangle}$	$\left[\begin{array}{cccccc\|c} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right]$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	4
5	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1]}{\langle T_1T_2+T_3^2T_4+T_5^2T_6\rangle}$	$\left[\begin{array}{cccccc\|c} 0 & 2a+1 & a & 1 & a & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \end{array}\right]$ $a \ge 0$	$\begin{bmatrix} 2a+2 \\ 1 \end{bmatrix}$	4
6	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1]}{\langle T_1T_2+T_3T_4+T_5^2T_6\rangle}$	$\left[\begin{array}{cccccc\|c} 0 & 2c+1 & a & b & c & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{array}\right]$ $a, b, c \ge 0, \quad a < b,$ $a+b = 2c+1$	$\begin{bmatrix} 2c+2 \\ 1 \end{bmatrix}$	4
7	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cccccc\|c} 0 & 0 & 0 & 0 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{array}\right]$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	4
8	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,S_2]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cccccc\|cc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & a \end{array}\right]$ $a \in \{1, 2, 3\}$	$\begin{bmatrix} 1 \\ a+1 \end{bmatrix}$	5
8	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,S_2,S_3]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cccccc\|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & a-1 & a \end{array}\right]$ $a \in \{1, 2\}$	$\begin{bmatrix} 1 \\ a+1 \end{bmatrix}$	6
8	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,\dots,S_4]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cccccc\|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	7
9	$\frac{\mathbb{K}[T_1,\dots,T_6,S_1,S_2]}{\langle T_1T_2+T_3T_4+T_5T_6\rangle}$	$\left[\begin{array}{cccc\|cc} 0 & a_2 & \dots & a_6 & 1 & 1 \\ 1 & 1 & \dots & 1 & 0 & 0 \end{array}\right]$ $0 \le a_3 \le a_5 \le a_6 \le a_4 \le a_2,$ $a_2 = a_3 + a_4 = a_5 + a_6$	$\begin{bmatrix} a_2+1 \\ 1 \end{bmatrix}$	5
10	$\frac{\mathbb{K}[T_1,\dots,T_5,S_1]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{ccccc\|c} 1 & 1 & 1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	3
11	$\frac{\mathbb{K}[T_1,\dots,T_5,S_1,S_2]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{ccccc\|cc} 1 & 1 & 1 & 1 & 1 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$ $a \in \{1, 2\}$	$\begin{bmatrix} a+1 \\ 1 \end{bmatrix}$	4
11	$\frac{\mathbb{K}[T_1,\dots,T_5,S_1,S_2,S_3]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{ccccc\|ccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]$	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	5
12	$\frac{\mathbb{K}[T_1,\dots,T_5,S_1,S_2]}{\langle T_1T_2+T_3T_4+T_5^2\rangle}$	$\left[\begin{array}{ccccc\|cc} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2c & a & b & c & 1 & 1 \end{array}\right]$ $0 \le a \le c \le b,\ a+b = 2c$	$\begin{bmatrix} 1 \\ 2c+1 \end{bmatrix}$	4

For Nos. 4, 8 *and* 11, *the variety Y is Fano and any iterated duplication of w_r produces a Fano variety X. For the remaining cases, the following table tells which Y are Fano and gives the characterizing condition when an iterated duplication of w_r produces a Fano variety X:*

No.	5	6	7	9	10	12
Y Fano	$a = 0$	$c = 0$	✓	$a_2 = 0$	✓	$c = 0$
X Fano	$m > 2a$	$m > 3c + 1$	$m \le 3$	$m > 2a_2$	$m \le 2$	$m > 3c$

Proof. A T-variety $X = X(A, P, u)$ has a divisor with infinite general T-isotropy if and only if $m \ge 1$ holds. In the cases 4.A, 4.B, 4.C, 5, 6, 7, 9, 10 and 12 we directly infer from Theorem 1.2 that the examples with higher m arise from those listed in the table above via iterated duplication of w_r.

We still have to consider Nos. 8 and 11. If X is a variety of type 8, then the condition for X to be a Fano variety is

$$4 + a_2 + \ldots, +a_m > ma_m,$$

where $a_m = 1, 2, 3$ and $0 \leq a_2 \leq \ldots \leq a_m$. This is satisfied if and only if one of the following conditions holds:

(i) $a_2 = \ldots = a_m \in \{1, 2, 3\}$;
(ii) $a_2 + 1 = a_3 \ldots = a_m \in \{1, 2\}$, with $m \geq 3$;
(iii) $a_2 = a_3 = 0$ and $a_4 = \ldots = a_m = 1$, with $m \geq 4$.

Similarly for No. 11 the Fano condition in the table of Theorem 1.2 is equivalent to the fulfillment of one of the following:

(i) $a_2 = \ldots = a_m \in \{1, 2\}$;
(ii) $a_2 = 0$ and $a_3 = \ldots = a_m = 1$, with $m \geq 3$.

In both cases this explicit characterization makes clear that we are in the setting of the duplication of a free weight. □

Remark 5.6. Consider iterated duplication of w_r for a variety $X = X(A, P, u)$ as in Theorem 5.5. Recall that the effective cone of X is decomposed as $\tau^+ \cup \tau_X \cup \tau^-$, where $\tau_X = \mathrm{Ample}(X)$. Lemma 3.11 (i) says $w_r \notin \tau_X$ and thus we have a unique $\kappa \in \{\tau^+, \tau^-\}$ with $w_r \notin \kappa$. Then the number of flips per duplication step equals

$$|\{\mathrm{cone}(w_{ij}), \mathrm{cone}(w_k);\ \ w_{ij}, w_k \in \kappa\}| - 1.$$

In particular, for Nos. 4.A, 4.B, 4.C, 8, 11, 9 with $a_i = 0$, 12 with $b = 0$ the duplications steps require no flips.

Remark 5.7. For toric Fano varieties, there is no statement like Corollary 1.3. Recall from [6] that all smooth projective toric varieties Z with $\mathrm{Cl}(Z) = \mathbb{Z}^2$ admit a description via the following data:

- Weight vectors $w_1 := (1, 0)$ and $w_i := (b_i, 1)$ with $0 = b_n < b_{n-1} < \ldots < b_2$;
- Multiplicities $\mu_i := \mu(w_i) \geq 1$, where $\mu_1 \geq 2$ and $\mu_2 + \ldots + \mu_n \geq 2$.

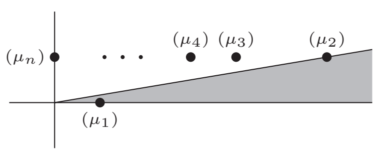

Fig. 5.2.

The variety Z arises from the bunched polynomial ring $(R, \mathfrak{F}, \Phi)$, where R equals $\mathbb{K}[S_{ij};\ 1 \leq i \leq n, 1 \leq j \leq \mu_i]$ with the system of generators $\mathfrak{F} = (S_{11}, \ldots, S_{n\mu_n})$

and the bunch $\Phi = \{\mathrm{cone}(w_1, w_i);\ i = 2, \dots, n\}$. In this setting Z is Fano if and only if

$$b_2(\mu_3 + \dots + \mu_n) < \mu_1 + \mu_3 b_3 + \dots + \mu_{n-1} b_{n-1}.$$

For any $n \in \mathbb{Z}_{\geq 4}$ and $i = 2, \dots, n$ set $\mu_i := 1$ and $w_i := (n-i, 1)$. Then, with $\mu_1 := 2$ we obtain a smooth (non-Fano) toric variety Z_n' of Picard number two and dimension $n-1$. Moreover, for $\mu_1 := 1 + (n-2)(n-1)/2$ we obtain a smooth toric Fano variety Z_n of Picard number two that is Fano and is obtained from Z_n' via iterated duplication of w_1 but cannot be constructed from any lower dimensional smooth variety this way.

6. Geometry of the Fano varieties

We take a closer look at the Fano varieties X listed in Theorem 1.2 and describe explicitly their Mori fibre spaces and their divisorial contractions. The approach uses suitable toric ambient varieties. The following Remark can be found, at least partially, for example in [9, Section 7.3].

Remark 6.1. Let Z be a smooth projective toric variety of Picard number 2, given by weight vectors $w_1 := (1, 0)$ and $w_i := (b_i, 1)$ with $0 = b_n < b_{n-1} < \dots < b_2$, and multiplicities $\mu_i := \mu(w_i) \geq 1$, where $\mu_1 \geq 2$ and $\mu_2 + \dots + \mu_n \geq 2$ as in Remark 5.7. Then the toric variety Z is a projectivized split vector bundle of rank r over a projective space $\mathbb{P}_s$, where $s := \mu_1 - 1$ and $r := \mu_2 + \dots + \mu_n - 1$. More precisely, we have

$$Z \ \cong\ \mathbb{P}\left(\bigoplus_{i=1}^{\mu_n} \mathcal{O}_{\mathbb{P}_s} \oplus \bigoplus_{i=1}^{\mu_{n-1}} \mathcal{O}_{\mathbb{P}_s}(b_{n-1}) \oplus \dots \oplus \bigoplus_{i=1}^{\mu_2} \mathcal{O}_{\mathbb{P}_s}(b_2)\right).$$

The bundle projection $Z \to \mathbb{P}_s$ is the elementary contraction associated to the divisor class $w_1 \in \mathbb{Z}^2 = \mathrm{Cl}(Z)$. If $n = 2$ holds, then we have $Z \cong \mathbb{P}_s \times \mathbb{P}_r$. If $n = 3$ and $\mu_3 = 1$ hold, then the class $w_3 \in \mathbb{Z}^2 = \mathrm{Cl}(Z)$ gives rise to a divisorial contraction onto a weighted projective space:

$$Z \ \to\ Z' \ :=\ \mathbb{P}(\underbrace{1, \dots, 1}_{\mu_1}, \underbrace{b_2, \dots, b_2}_{\mu_2}).$$

The exceptional divisor $E_Z \subseteq Z$ is isomorphic to $\mathbb{P}_s \times \mathbb{P}_{\mu_2 - 1}$ and the center $C(Z') \subseteq Z'$ of the contraction is isomorphic to $\mathbb{P}_{\mu_2 - 1}$. In particular, for $\mu_2 = 1$, we have $E_Z \cong \mathbb{P}_s$ and $C(Z')$ is a point.

From the explicit description of the Cox ring of our Fano variety X, we obtain via Construction 2.7 a closed embedding $X \to Z$ into a toric variety Z. As a byproduct of our classification, it turns out that, whenever X admits a elementary contraction, then X inherits all its elementary contractions from Z. Remark 6.1 together with the explicit equations for X in Z will then allow us to study the situation in detail. We now present the results. The cases are numbered according to the table of Theorem 1.2. Moreover, we denote by $Q_3 \subseteq \mathbb{P}_4$ and $Q_4 \subseteq \mathbb{P}_5$ the three and four-dimensional smooth projective quadrics and we write $\mathbb{P}(a_1^{\mu_1}, \dots, a_r^{\mu_r})$ for the weighted projective space, where the superscript μ_i indicates that the weight a_i occurs μ_i times.

No. 1. The variety X is of dimension four and admits two elementary contractions, $Q_4 \leftarrow X \rightarrow \mathbb{P}_1$. The morphism $X \rightarrow Q_4$ is a divisorial contraction with exceptional divisor isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$ and center isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$. The morphism $X \rightarrow \mathbb{P}_1$ is a Mori fiber space with general fiber isomorphic to Q_3 and singular fibers over $[0, 1]$ and $[1, 0]$ each isomorphic to the singular quadric $V(T_2T_3 + T_4T_5) \subseteq \mathbb{P}_4$.

No. 2. The variety X is of dimension four and admits two elementary contractions, $Q_4 \leftarrow X \rightarrow \mathbb{P}_3$. The morphism $X \rightarrow Q_4$ is a divisorial contraction with exceptional divisor isomorphic to a hypersurface of bidegree $(1, 1)$ in $\mathbb{P}_1 \times \mathbb{P}_3$ and center isomorphic to $\mathbb{P}_1$. The morphism $X \rightarrow \mathbb{P}_3$ is a Mori fiber space with fibers isomorphic to $\mathbb{P}_1$.

No. 3. The variety X is of dimension three and occurs as No. 2.29 in the Mori-Mukai classification [22]. Moreover, X admits two elementary contractions, $Q_3 \leftarrow X \rightarrow \mathbb{P}_1$. The morphism $X \rightarrow Q_3$ is a divisorial contraction with exceptional divisor isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$ and center isomorphic to $\mathbb{P}_1$. The morphism $X \rightarrow \mathbb{P}_1$ is a Mori fiber space with general fiber isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$ and singular fibers over $[0, 1]$ and $[1, 0]$ each isomorphic to $V(T_1T_2 + T_3^2) \subseteq \mathbb{P}_3$.

No. 4A. CASE 1: we have $c = -1$. Then X admits two elementary contractions $Y \leftarrow X \rightarrow \mathbb{P}_2$, where $Y := V(T_1T_2 + T_3T_4 + T_5T_6) \subseteq \mathbb{P}_{m+4}$ is a terminal factorial Fano variety which is smooth if and only if $m = 1$ holds. The morphism $X \rightarrow Y$ is a divisorial contraction with exceptional divisor isomorphic to a hypersurface of bidegree $(1, 1)$ in $\mathbb{P}_2 \times \mathbb{P}_{m+1}$ and center isomorphic to $\mathbb{P}_{m+1}$. The morphism $X \rightarrow \mathbb{P}_2$ is a Mori fiber space with fibers isomorphic to $\mathbb{P}_{m+1}$.

CASE 2: we have $c = 0$. Then X is a hypersurface of bidegree $(1, 1)$ in $\mathbb{P}_2 \times \mathbb{P}_{m+2}$. Moreover, X admits two Mori fiber spaces $\mathbb{P}_{m+2} \leftarrow X \rightarrow \mathbb{P}_2$. The Mori fiber space $X \rightarrow \mathbb{P}_2$ has fibers isomorphic to $\mathbb{P}_{m+1}$, whereas the Mori fiber space $X \rightarrow \mathbb{P}_{m+1}$ has general fiber isomorphic to $\mathbb{P}_1$ and special fibers over $V(T_1, T_2, T_3) \subseteq \mathbb{P}_{m+2}$ isomorphic to $\mathbb{P}_2$. For $m = 0$, we have $\dim(X) = 3$ and X is the variety No. 2.32 in [22].

No. 4B. The variety X admits two elementary contractions $Y \leftarrow X \rightarrow \mathbb{P}_2$, where $Y := V(T_1^2 + T_2T_3 + T_4T_5) \subseteq \mathbb{P}_{m+4}$ is a terminal factorial Fano variety. The variety Y is smooth if and only if $m = 0$ holds and in this case X occurs as No. 2.31 in [22]. The morphism $X \rightarrow Y$ is a divisorial contraction with exceptional divisor isomorphic to a hypersurface of bidegree $(1, 1)$ in $\mathbb{P}_2 \times \mathbb{P}_{m+1}$ and center isomorphic to $\mathbb{P}_{m+1}$. The morphism $X \rightarrow \mathbb{P}_2$ is a Mori fiber space with fibers isomorphic to $\mathbb{P}_{m+1}$.

No. 4C. The variety X is a hypersurface of bidegree $(2,1)$ in $\mathbb{P}_2\times\mathbb{P}_{m+2}$; for $m=0$ we have $\dim(X)=3$ and X is No. 2.24 in [22]. Moreover, X admits two Mori fiber spaces $\mathbb{P}_{m+2}\leftarrow X\to\mathbb{P}_2$. The morphism $X\to\mathbb{P}_2$ has fibers isomorphic to $\mathbb{P}_{m+1}$. To describe the fibers of $\varphi\colon X\to\mathbb{P}_{m+2}$, set $Y_i:=V_{\mathbb{P}_{m+2}}(T_i)$, $Y_{ij}:=V_{\mathbb{P}_{m+2}}(T_i,T_j)$ and $Y_{123}:=V_{\mathbb{P}_{m+2}}(T_1,T_2,T_3)$. Then we have

$$\varphi^{-1}(z)\ \cong\ \begin{cases}\mathbb{P}_2 & \text{if } z\in Y_{123}\\ \mathbb{P}_1 & \text{if } z\in(Y_{12}\cup Y_{13}\cup Y_{23})\setminus Y_{123},\\ V_{\mathbb{P}_2}(T_1T_2) & \text{if } z\in(Y_1\cup Y_2\cup Y_3)\setminus(Y_{12}\cup Y_{13}\cup Y_{23})\\ \mathbb{P}_1 & \text{otherwise.}\end{cases}$$

No. 5. The variety X admits a Mori fiber space $\varphi\colon X\to\mathbb{P}_{m+1}$, whose general fiber is isomorphic to $\mathbb{P}_1\times\mathbb{P}_1$. More precisely, with $Y_1:=V_{\mathbb{P}_{m+1}}(T_1)$ and $Y_2:=V_{\mathbb{P}_{m+1}}(T_2)$, we have

$$\varphi^{-1}(z)\ \cong\ \begin{cases}V_{\mathbb{P}_3}(T_1T_2) & \text{if } z\in Y_1\cap Y_2\\ V_{\mathbb{P}_3}(T_1T_2+T_3^2) & \text{if } z\in Y_1\setminus Y_2 \text{ or } z\in Y_2\setminus Y_1\\ \mathbb{P}_1\times\mathbb{P}_1 & \text{otherwise.}\end{cases}$$

No. 6. The variety X admits a Mori fiber space $X\to\mathbb{P}_m$, with general fiber isomorphic to Q_3 and singular fibers over $V(T_1)\subseteq\mathbb{P}_m$ each isomorphic to $V(T_1T_2+T_3T_4)\subseteq\mathbb{P}_4$.

No. 7. The variety X admits a divisorial contraction $X\to\mathbb{P}_{m+3}$ with exceptional divisor isomorphic to the projectivized split bundle

$$\mathbb{P}\left(\bigoplus_{i=1}^{m}\mathcal{O}_{\mathbb{P}_1\times\mathbb{P}_1}\oplus\mathcal{O}_{\mathbb{P}_1\times\mathbb{P}_1}(1,1)\right)$$

and center isomorphic to $\mathbb{P}_1\times\mathbb{P}_1$. Moreover, if $m=1$ holds, X admits a further divisorial contraction $X\to Q_4$ with exceptional divisor isomorphic to $\mathbb{P}_3$ and center a point.

No. 8. Here we have $X=\mathbb{P}(\mathcal{O}_{Q_4}\oplus\mathcal{O}_{Q_4}(a_2)\ldots\oplus\mathcal{O}_{Q_4}(a_m))$. Thus, there is a Mori fiber space $X\to Q_4$ with fibers isomorphic to $\mathbb{P}_{m-1}$. If $a_2=\ldots=a_m>0$ holds, then X admits in addition a divisorial contraction $X\to Y$, where $Y:=V(T_1T_2+T_3T_4+T_5T_6)\subseteq\mathbb{P}(1^6,a_2^{m-1})$. The exceptional divisor is isomorphic to $Q_4\times\mathbb{P}_{m-2}$ and the center to $\mathbb{P}_{m-2}$.

No. 9. The variety X is a bundle over $\mathbb{P}_{m-1}$ with fibers isomorphic to Q_4. In particular, if $a_i=0$ holds for all $2\le i\le 6$, then $X\cong Q_4\times\mathbb{P}_{m-1}$.

No. 10. The variety X admits a divisorial contraction $X \to \mathbb{P}_{m+2}$ with exceptional divisor isomorphic to the projectivized split bundle

$$\mathbb{P}\left(\bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(1)\right)$$

and center isomorphic to $\mathbb{P}_1$. For $m = 1$, we have $\dim(X) = 3$ and X is No. 2.30 from [22]; in this case it admits a further divisorial contraction $X \to Q_3$ with exceptional divisor isomorphic to $\mathbb{P}_2$ and center a point.

No. 11. Here $X = \mathbb{P}(\mathcal{O}_{Q_3} \oplus \mathcal{O}_{Q_3}(a_2) \ldots \oplus \mathcal{O}_{Q_3}(a_m))$ holds. Thus, there is a Mori fiber space $X \to Q_3$ with fibers isomorphic to $\mathbb{P}_{m-1}$. If $a_2 = \ldots = a_m > 0$ holds, then X admits a divisorial contraction $X \to Y$, where the variety Y equals $V(T_1T_2 + T_3T_4 + T_5^2) \subseteq \mathbb{P}(1^5, a_2^{m-1})$. The exceptional divisor is isomorphic to $Q_3 \times \mathbb{P}_{m-2}$ and the center to $\mathbb{P}_{m-2}$.

No. 12. The variety X is a bundle over $\mathbb{P}_{m-1}$ with fibers isomorphic to Q_3. In particular, if $a = b = c = 0$ holds, then $X \cong Q_3 \times \mathbb{P}_{m-1}$.

No. 13. This case presents a one-parameter family of varieties X_λ, with parameter $\lambda \in \mathbb{K}^* \setminus \{1\}$. They are generally non-isomorphic to each other, except for the pairs $X_\lambda \cong X_{\lambda^{-1}}$ for all λ. The variety X_λ is the intersection of two hypersurfaces

$$D_1 = V(T_1S_1 + T_2S_2 + T_3S_3), \qquad D_2 = V(\lambda T_2S_2 + T_3S_3 + T_4S_4),$$

both of bidegree (1,1) in $\mathbb{P}_3 \times \mathbb{P}_3$, where the T_i are the coordinates of the first $\mathbb{P}_3$ and the S_j those of the second. Note that each D_i has an isolated singularity, which is not contained in the other hypersurface. Both D_1, D_2 are terminal and factorial. Moreover, X admits two Mori fiber spaces $\mathbb{P}_3 \leftarrow X \to \mathbb{P}_3$, both with typical fiber $\mathbb{P}_1$ and having four special fibers, all isomorphic to $\mathbb{P}_2$ and lying over the points $[1, 0, 0, 0]$, $[0, 1, 0, 0]$, $[0, 0, 1, 0]$ and $[0, 0, 0, 1]$.

Remark 6.2. In contrast to the toric case, a smooth projective variety of Picard number 2 with torus action of complexity one need not admit a non-trivial Mori fiber space. For example, in Theorem 1.2, this happens in precisely two cases, namely No. 7 and No. 10, both with $m = 1$.

Remark 6.3. In the list of Theorem 1.2 there are several examples, where the effective cone coincides with the cone of movable divisor classes: No. 4A with $c = 0$, No. 4C, No. 5 with $a = 0$, No. 6 with $a = 0$, No. 8 with $a_2 = 0$, No. 9 with $a_3 = 0$, No. 11 with $a_2 = 0$, No. 12 with $a = 0$ and No. 13. Thus, these varieties admit no divisorial contraction.

Remark 6.4. In Theorem 1.1 it is possible that non-isomorphic varieties share the same Cox ring and thus differ from each other by a small quasimodification, *i.e.* only by the choice of the ample class. This happens exactly in the following cases:

(i) No. 4 with $l_2 = l_4 = 2, l_6 = 1, a = 0, b = 1, c_i = 0$ for all $i = 1, \dots, m$ has the same Cox ring as No. 5 with $a = 0$. Note that for $m = 0$ both varieties are truly almost Fano, whereas for $m \geq 1$ No. 5 is Fano;
(ii) For $m \geq 1$, No. 4 with $l_2 = 2, l_4 = l_6 = 1, a = b = 1, c_i = 0$ for all $i = 1, \dots, m$ has the same Cox ring as No. 6 with $a = c = 0$ and $b = 1$. Note that for $m = 1$ both varieties are truly almost Fano, whereas for $m \geq 2$ No. 6 is Fano;
(iii) For $m \geq 2$, No. 7 has the same Cox ring as No. 9 with $a_2 = 2$ and $a_3 = \ldots = a_6 = 1$. Note that for $m = 2, 3$ No. 7 is Fano, for $m = 4$ both varieties are truly almost Fano, whereas for $m \geq 5$ No. 9 is Fano;
(iv) For $m \geq 2$, No. 10 has the same Cox ring as No. 12 with $a = b = c = 1$. Note that for $m = 2$ No. 10 is Fano, for $m = 3$ both varieties are truly almost Fano, whereas for $m \geq 4$ No. 12 is Fano.

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