Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XVIII (2018), 565-591

Robin Laplacian in the large coupling limit: convergence and spectral asymptotic

FATEN BELGACEM, HICHEM BELHADJALI, ALI BENAMOR AND AMINA THABET

Abstract. We study convergence modes as well as their respective rates for the resolvent difference of Robin and Dirichlet Laplacian on bounded smooth domains in the large coupling limit. Asymptotic expansions for the resolvent, the eigenprojections and the eigenvalues of the Robin Laplacian are performed. Finally we apply our results to the case of the unit disc.

Mathematics Subject Classification (2010): 35P05 (primary); 47A10 (secondary).

1. Introduction

Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain with C^∞ boundary Γ (a smooth domain for short) and σ the surface measure on Γ.

We consider the bilinear symmetric form defined in $L^2(\Omega) := L^2(\Omega, dx)$ by

$$D(\mathcal{E}^\beta) = H^1(\Omega),\ \mathcal{E}^\beta(u,v) := \int_\Omega \nabla u \cdot \nabla v\, dx + \beta \int_\Gamma uv\, d\sigma,\ \beta \geq 0. \quad (1.1)$$

Owing to the continuity of the trace operator from $H^1(\Omega)$ into $L^2(\Gamma, \sigma)$, the form $\mathcal{E}^\beta$ is closed. Let us denote by H_β the self-adjoint operator associated with $\mathcal{E}^\beta$ via the Kato representation theorem. The operator H_β is commonly named the Laplacian with Robin boundary conditions and is characterized by

$$D(H_\beta) = \left\{u \in H^2(\Omega),\ \frac{\partial u}{\partial \nu} + \beta u = 0, \quad \text{on } \Gamma\right\}, \quad H_\beta u = -\Delta u \text{ in } \Omega, \quad (1.2)$$

where ν is the outer normal unit vector on Γ.

Received January 17, 2016; accepted in revised form December 19, 2016.
Published online April 2018.

By Kato's monotone convergence theorem for sesquilinear forms (see [21, Theorem 3.13a, page 461]), the bilinear symmetric forms $\mathcal{E}^\beta$ increase, as β increases to infinity, to the closed bilinear symmetric form $\mathcal{E}^\infty$, defined by

$$D(\mathcal{E}^\infty) = \Big\{u \in H^1(\Omega),\ u = 0,\ \text{on } \Gamma\Big\},\ \mathcal{E}^\infty(u, v) = \int_\Omega \nabla u \cdot \nabla v\, dx. \quad (1.3)$$

Thus $D(\mathcal{E}^\infty) = H_0^1(\Omega)$ and $\mathcal{E}^\infty$ is nothing else but the quadratic form associated with the Dirichlet Laplacian in $L^2(\Omega)$, which we denote by $-\Delta_D$. Thereby we obtain the strong convergence

$$\lim_{\beta\to\infty} (H_\beta + 1)^{-1} = (-\Delta_D + 1)^{-1} \text{ strongly.} \quad (1.4)$$

In a wide variety of applications it turns out that it is more easy to analyze the limit than the approximating operators $(H_\beta + 1)^{-1}$. For this reason one might use the following strategy in order to investigate the operator H_β for large β: one studies the limit of the operators $(H_\beta + 1)^{-1}$ and estimates the error one makes when replacing $(H_\beta + 1)^{-1}$ by the limit. Hence we are led to know how fast do the operators $(H_\beta + 1)^{-1}$ converge. It is also important to determine the kind of convergence. For instance convergence with respect to the operator norm admits much stronger conclusions about the spectral properties than strong convergence; compare, *e.g.*, the discussion of this point in [23, Chapter VIII.7].

In this spirit it is also interesting and practical to write down explicit asymptotic expansions for the operator $(H_\beta + 1)^{-1}$ and possibly for the eigenprojections and eigenvalues of the operator H_β for large β.

On the light of these motivations we shall establish, in these notes, operator norm convergence as well as convergence within Schatten-von Neumann ideals of $(H_\beta + 1)^{-1}$ towards $(-\Delta_D + 1)^{-1}$ as $\beta \to \infty$. The speed of convergence for both convergence modes will be determined as well. Furthermore large coupling asymptotic for spectral objects is performed.

An aspect of novelty at this stage, among others, is that we shall establish a second order asymptotic for the eigenvalues, which coefficients are explicitly computed. In its own this expansion generalizes and pushes forward the expansion given in [12] where the Neumann Laplacian with high conductivity inside Ω is studied.

Let us emphasize that although we shall consider bounded domains with C^∞ boundary, our method (which basically rests on the theory elaborated in [8–11, 13]) still works for exterior domains with C^∞ boundary, regarding convergence of resolvents differences.

Physically the Laplacian with Robin boundary conditions describes the interaction of a particle inside Ω with a potential of strength β concentrated on the boundary Γ. Thus, for large β, it describes the motion of a particle inside a set with high conductivity on the boundary (superconductivity on the boundary). We shall show, among others, that this phenomenon is completely different from the case of having conductivity inside Ω, concerning convergence modes and convergence rates, and hence spectral asymptotic.

The outline of this paper is as follows: in Section 2 we give some preliminaries, whereas in Section 3 we prove uniform convergence as well as convergence with respect to the Schatten-von Neumann norm of $(H_\beta+1)^{-1}-(-\Delta_D+1)^{-1}$. The rate of convergence for both convergence types is also discussed in this section. Sections 4 and 5 are devoted to establish the asymptotic expansions for the resolvent, the projection and the eigenvalues of Laplacian with Robin boundary conditions for large coupling constant. In the last section we work out the case where Ω is the unit disc.

ACKNOWLEDGEMENTS. The authors are grateful to the referee for his/her careful reading of the manuscript.

2. Preliminary

Along the paper we adopt the following notations:

- $K_1 = (-\Delta_N + 1)^{-1}$, where $-\Delta_N$ is the Neumann Laplacian on Ω;
- H_β is the self-adjoint operator in $L^2(\Omega)$ associated with $\mathcal{E}^\beta$;
- $D_\beta = K_1 - (H_\beta + 1)^{-1}$;
- D_∞ is the strong limit $\lim_{\beta\to\infty} D_\beta = K_1 - (-\Delta_D + 1)^{-1}$;
- $\mathcal{E}[u] = \mathcal{E}(u, u) = \int_\Omega |\nabla u|^2\, dx,\ \forall u \in H^1(\Omega)$;
- $\mathcal{E}_1 = \mathcal{E} + (\cdot, \cdot)_{L^2(\Omega)}$.

We designate by J the operator *trace on the boundary* of functions from $H^1(\Omega)$:

$$J : (H^1(\Omega), \mathcal{E}_1) \to L^2(\Gamma) := L^2(\Gamma, \sigma),\ Ju = \operatorname{tr} u. \tag{2.1}$$

Lemma 2.1. *The operator J is compact.*

Although the result is known we shall give an alternative proof.

Proof. Owing to the smoothness of Ω and precisely to the fact that

$$\sigma(B_r(x) \cap \Gamma) \sim r^{d-1},\ \forall x \in \Gamma,\ 0 < r < 1, \tag{2.2}$$

the following known trace inequality holds true (see [2, Theorem 5.36, page 164]): for $d \geq 3,\ 2 \leq p \leq \frac{2(d-1)}{d-2}$, there is a constant c such that

$$\left(\int_\Gamma |Ju|^p\, d\sigma\right)^{2/p} \leq c\mathcal{E}_1[u],\ \forall u \in H^1(\Omega), \tag{2.3}$$

whereas the latter inequality holds true for every $2 \leq p < \infty$, for $d = 2$.

Now the compactness of J follows from [7, Theorem 4.1]. □

As Γ is C^∞, it is known that Ran $J = H^{1/2}(\Gamma)$, which is dense in $L^2(\Gamma)$. Hence the operator JJ^* is invertible. We set

$$\check{H} := (JJ^*)^{-1}. \tag{2.4}$$

Obviously $\check{H}$ is a non-negative self-adjoint operator in $L^2(\Gamma)$ and has, by Lemma 2.1, a compact resolvent.

We shall make extensive use of the following formulae, established in [9, Lemma 2.3]:

$$D_\beta = J^*\left(\frac{1}{\beta} + JJ^*\right)^{-1} JK_1 = (JK_1)^*\left(\frac{1}{\beta} + JJ^*\right)^{-1}(JK_1),\ \beta > 0 \tag{2.5}$$

and [9, Lemma 2.4]

$$D_\infty := \lim_{\beta\to\infty} D_\beta = \left(\check{H}^{1/2}JK_1\right)^* \check{H}^{1/2}JK_1. \tag{2.6}$$

Let us also recall that (see [2, Theorem 5.37, page 165])

$$\ker(J) = H_0^1(\Omega). \tag{2.7}$$

Let $H_0^1(\Omega)^\perp$ be the $\mathcal{E}_1$-orthogonal of $H_0^1(\Omega)$ and P be the $\mathcal{E}_1$-orthogonal projection of $H^1(\Omega)$ into $H_0^1(\Omega)^\perp$. Then

$$J|_{H_0^1(\Omega)^\perp} : H_0^1(\Omega)^\perp \to H^{1/2}(\Gamma)$$

is an isomorphism. Its inverse operator, which we denote by $\mathcal{R}$, is given by

$$\mathcal{R} : H^{1/2}(\Gamma) \to H^1(\Omega),\ \psi \mapsto Pv,\ Jv = \psi. \tag{2.8}$$

The operator $\mathcal{R}$ is well defined. Indeed, $\mathcal{R}\psi$ is the unique solution in $H^1(\Omega)$ of the boundary value problem

$$\begin{cases} -\Delta v + v = 0 & \text{in } \Omega \\ v = \psi & \text{on } \Gamma. \end{cases} \tag{2.9}$$

Of major importance for our method is the operator $\check{H}$ introduced via formula (2.4), for which we list the relevant properties, and give precise description.

It is hard in general to give a clear description of the domain of $\check{H}$. To overcome this difficulty we shall associate to $\check{H}$ a bilinear symmetric form, whose domain is well known as well as its features.

Let us introduce the quadratic form $\check{\mathcal{E}}_1$ in $L^2(\Gamma)$, as follows:

$$D(\check{\mathcal{E}}_1) = \mathrm{Ran}(J),\ \check{\mathcal{E}}_1(Ju, Jv) = \mathcal{E}_1(Pu, Pv)\ \forall u, v \in H^1(\Omega). \tag{2.10}$$

The quadratic form $\check{\mathcal{E}}_1$ is called the trace of the Dirichlet form $\mathcal{E}_1$ with respect to the measure σ (see [17, Chapter 6]). In Proposition 2.2, we will show that $\check{H}$ is in fact the celebrated Dirichlet-to-Neumann operator. This operator was studied by many authors, for instance, we refer the reader to [4–6, 14, 16] and references therein.

According to [11, Theorem 1.1], the quadratic form $\check{\mathcal{E}}_1$ is closed and is associated, in the sense of Kato's representation theorem, to the self-adjoint operator $\check{H}$. In this special context we shall collect some properties of the operator $\check{H}$.

Proposition 2.2. *The following statements hold true:*

(1) *Let $\psi \in H^{1/2}(\Gamma)$ and $u \in H^1(\Omega)$ be the unique solution of the boundary value problem*

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega \\ u = \psi & \text{on } \Gamma. \end{cases} \tag{2.11}$$

Then $\check{\mathcal{E}}_1[\psi] = \mathcal{E}_1[u]$. Furthermore for every $\psi \in D(\check{H})$, $\check{H}\psi = \frac{\partial u}{\partial \nu}$;

(2) *(Dirichlet principle). For every $\psi \in H^{1/2}(\Gamma)$, we have*

$$\check{\mathcal{E}}_1[\psi] = \inf\left\{\mathcal{E}_1[v]:\ v \in H^1(\Omega),\ Jv = \psi\right\}. \tag{2.12}$$

It follows that $\check{\mathcal{E}}_1$ is a Dirichlet form;

(3) *For every $\psi \in L^2(\Gamma)$, set $U_1^\sigma \psi$ the 1-potential of the signed measure $\psi\sigma$. Then $\check{H}^{-1}\psi = JU_1^\sigma \psi$.*

Proof. Assertion (1) follows from the very construction of $\check{\mathcal{E}}_1$ and the use of Green's formula.

(2) Clearly the left-hand-side of (2.12) is bigger than its right-hand side. The reversed inequality follows from the existence of a minimizer together with the continuity of J.

(3) Let us first observe that for every fixed $\psi \in L^2(\Gamma)$ the signed measure $\psi\sigma$ has finite energy integral, *i.e.*,

$$\left|\int_\Gamma Jv \cdot \psi \, d\sigma\right| \le c(\mathcal{E}_1[v])^{1/2},\ \forall\, v \in H^1(\Omega). \tag{2.13}$$

Thus the 1-potential of $\psi\sigma$ is well defined and is characterized as being the unique element of $H^1(\Omega)$ such that

$$\mathcal{E}_1(U_1^\sigma \psi, v) = \int_\Gamma Jv \cdot \psi \, d\sigma,\ \forall\, v \in H^1(\Omega). \tag{2.14}$$

Hence, making use of the first part of (2.10) together with the latter identity we achieve

$$\begin{aligned} \check{\mathcal{E}}_1(JU_1^\sigma\psi, Jv) = \mathcal{E}_1(U_1^\sigma\psi, Pv) = \int_\Gamma JPv\cdot\psi\,d\sigma = \int_\Gamma Jv\cdot\psi\,d\sigma \\ = (\psi, Jv)_{L^2(\Gamma)}, \forall\, v\in H^1(\Omega),\ \psi\in L^2(\Gamma). \end{aligned} \tag{2.15}$$

Thus $JU_1^\sigma\psi\in D(\check{H})$ and $\check{H}JU_1^\sigma\psi=\psi$, which was to be proved. □

Henceforth we denote by $e^{-tT},\ t>0$, respectively $\check{T}_t,\ t>0$, the semigroup associated to $-\Delta_N+1$, respectively to $\check{H}$.

Remark 2.3. From potential theoretical results relating properties of $(T_t)_{t>0}$ to those of $(\check{T}_t)_{t>0}$, we conclude on the light of the latter proposition that $(\check{T}_t)_{t>0}$ is Markovian and transient, however it is not conservative, *i.e.*,

$$0\le \check{T}_t1\ne 1,\quad \forall t>0 \tag{2.16}$$

3. Uniform and trace class convergence

In this section we shall concentrate on various types of convergence of D_β to D_∞ as well as their rates. These types are precisely convergence with respect to the operator norm and the norms of Schatten-von Neumann ideals.

Let us first quote that $\lim_{\beta\to\infty}\|D_\beta-D_\infty\|=0$. Indeed, we already mentioned that D_β increases strongly to D_∞, which is compact. Thus, using [9, Theorem 2.6], we get uniform convergence.

Theorem 3.1. *The operator* $\check{H}JK_1$ *is bounded. Consequently* $(H_\beta+1)^{-1}$ *converges in the operator norm to* $(-\Delta_D+1)^{-1}$*, with maximal rate proportional to* β^{-1}*. Moreover,*

$$\lim_{\beta\to\infty}\beta\|D_\beta-D_\infty\| = \|\check{H}JK_1\|^2. \tag{3.1}$$

Proof. Let $u\in H^2(\Omega)$. We claim that $Pu\in H^2(\Omega)$. Indeed, $JPu=Ju$. Therefore Pu is the unique solution in $H^1(\Omega)$ of the boundary problem:

$$\begin{cases} -\Delta v+v=0 & \text{in } \Omega\\ v=u & \text{on } \Gamma. \end{cases} \tag{3.2}$$

From elliptic regularity (see [18, Theorem 8.13]), we get that $Pu\in H^2(\Omega)$ and the claim is proved.

Let $u \in L^2(\Omega)$ and $v \in H^1(\Omega)$. Then

$$\begin{aligned}\check{\mathcal{E}}_1(JK_1u, Jv) &= \mathcal{E}_1(PK_1u, Pv)\\ &= \int_\Omega (\nabla PK_1u)\nabla Pv + \int_\Omega (PK_1u)Pv.\end{aligned} \tag{3.3}$$

As $K_1u \in H^2(\Omega)$ (see [20, Theorem 5.31-ii, page 143]) then $PK_1u \in H^2(\Omega)$ as well. Thus by Green's formula one obtains

$$\begin{aligned}\check{\mathcal{E}}_1(JK_1u, Jv)) &= -\int_\Omega \Delta PK_1uPv + \int_\Gamma \frac{\partial PK_1u}{\partial\nu} JPv + \int_\Omega PK_1uPv\\ &= \int_\Gamma \frac{\partial PK_1u}{\partial\nu} Jv = \left(\frac{\partial PK_1u}{\partial\nu}, Jv\right)_{L^2(\Gamma)}.\end{aligned} \tag{3.4}$$

It follows that $JK_1u \in D(\check{H})$ and

$$\check{H}JK_1u = \frac{\partial PK_1u}{\partial\nu}. \tag{3.5}$$

Thereby $\check{H}JK_1$ is a closed everywhere defined operator on $L^2(\Omega)$, and hence is bounded.

Finally, utilizing [9, Theorem 2.7], we conclude that $(H_\beta + 1)^{-1}$ converges uniformly to $(-\Delta_D + 1)^{-1}$, with maximal rate proportional to $\frac{1}{\beta}$, and that formula (3.1) holds true. □

Remark 3.2. Here we observe a qualitative difference between inner superconductivity and boundary superconductivity: whereas in our setting uniform convergence is as fast as possible, it occurs for $-\Delta + \beta 1_{\Omega_1}$, where Ω_1 is open and $\overline{\Omega}_1 \subset \Omega$, with a rate which is $O(\beta^{-1/2})$, according to [12, Theorem 1.1] and [3, Theorem 3.1].

For further investigations concerning convergence of resolvent differences, as well as spectral asymptotic, one needs strengthened regularizing properties of the operator JK_1. To that end we establish:

Lemma 3.3. *The operator $\check{H}^{3/2}JK_1$ is bounded.*

Proof. Let $u \in L^2(\Omega)$. We have already proved that $\check{H}JK_1u = \dfrac{\partial PK_1u}{\partial\nu}$, which, by elliptic regularity, lies in the space $H^{1/2}(\Gamma) = D(\check{H}^{1/2})$.

Thus $\check{H}^{3/2}JK_1$ is a closed everywhere defined operator on $L^2(\Omega)$, and then it is bounded. □

Before dealing with convergence within Schatten-von Neumann operator ideals, let us introduce a few notations.

Let $1 \le p < \infty$ and $\mathcal{H}_i$ be Hilbert spaces, $i = 1, 2$ and $K : \mathcal{H}_1 \to \mathcal{H}_2$ be a compact operator. Then $\mathcal{H}_2$ has an orthonormal basis $(e_i)_{i \in I}$ such that, with $|K| := \sqrt{KK^*}$, we have

$$|K|e_i = \lambda_i e_i,\ \forall\, i \in I,$$

for some suitably chosen family $(\lambda_i)_{i\in I} \subset [0, \infty)$, which is unique up to permutations. We set

$$\|K\|_{S_p} := \left(\sum_{i\in I} \lambda_i^p\right)^{1/p}.$$

The ideal $S_p(\mathcal{H}_1, \mathcal{H}_2)$, ($S_p$ for short) denotes the set of compact operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ such that $\|K\|_p < \infty$. It is called the Schatten-von Neumann class of order p.

On the light of Lemma 3.3, we are able to prove convergence with respect to the S_p norm.

Theorem 3.4. *For every $p > \frac{d-1}{2}$ we have*

$$\lim_{\beta\to\infty} \|D_\beta - D_\infty\|_{S_p} = 0. \tag{3.6}$$

In particular trace class convergence holds true for $d = 2$.

Proof. First we recall that owing to [9, Corollary 2.20], S_p-convergence holds true whenever $D_\infty \in S_p$.

Having in mind that $D_\infty = (\check{H}^{1/2} JK_1)^* \check{H}^{1/2} JK_1$, we get that it lies in S_p if and only if $\check{H}^{1/2} JK_1$ lies in S_{2p}. On the other hand, from the boundedness of $\check{H}^{3/2} JK_1$, we obtain

$$\left\|\check{H}^{1/2} JK_1\right\|_{S_{2p}} \le \left\|\check{H}^{-1}\right\|_{S_{2p}} \left\|\check{H}^{3/2} JK_1\right\|. \tag{3.7}$$

Thus we are led to prove that $\check{H}^{-1} \in S_q$ for $q > d - 1$.

To that end we shall use the trace inequality (2.3) to obtain the thesis.

For $d \ge 3$, we know that, from the construction of $\check{\mathcal{E}}_1$, the following Sobolev type inequality holds true:

$$\left(\int_\Gamma |\psi|^{\frac{2(d-1)}{d-2}}\, d\sigma\right)^{\frac{d-2}{d-1}} \le C\check{\mathcal{E}}_1[\psi],\ \forall\, \psi \in H^{1/2}(\Gamma). \tag{3.8}$$

Now Sobolev inequality in conjunction with Hölder inequality lead to the Faber–Krahn inequality for the smallest eigenvalue of $\check{H}$:

$$\check{\lambda}_1 \ge C\big(\sigma(\Gamma)\big)^{-\frac{1}{d-1}}, \tag{3.9}$$

which in turn (see [19, Theorem 3.1]) leads to the following lower bound for the eigenvalues $\check{\lambda}_k$ (repeated as many times as their multiplicity in an increasing way) of $\check{H}$:

$$\check{\lambda}_k \geq Ck^{\frac{1}{d-1}}. \tag{3.10}$$

Thus $\check{H}^{-1} \in S_q$ for every $q > d-1$, which was to be proved for $d \geq 3$.

For $d = 2$, the proof is similar, so we omit it. □

By the end of this section we shall discuss the rate of convergence in S_1 in two dimensions. It was proved in [10, Theorem 2.3] that the maximal rate of convergence in S_1 is proportional to $1/\beta$ and that trace-class convergence with maximal rate holds true if and only if the operator $\check{H}JK_1$ is a Hilbert-Schmidt operator. However, according to [10, Proposition 2.4], if for some $r \in (0,1)$ the operator, $\check{H}^{\frac{1+r}{2}}JK_1$ is a Hilbert-Schmidt operator then one has trace-class convergence with rate $O(1/\beta^r)$.

Proposition 3.5. *In the case $d = 2$ it holds*

$$\lim_{\beta\to\infty} \beta^r \|D_\beta - D_\infty\|_{S_1} < \infty, \ \forall r \in (0,1). \tag{3.11}$$

Proof. For $d = 2$ we have the lower bound

$$\check{\lambda}_k \geq Ck^s, \ \forall s \in (0,1).$$

Thus, if for a given $r \in (0,1)$, we choose $\frac{1}{2-r} < s < 1$, we get $(2-r)s > 1$. Thus $\check{H}^{\frac{r-2}{2}}$ is a Hilbert-Schmidt operator, and so is $\check{H}^{\frac{1+r}{2}}JK_1$. □

Remark 3.6. We shall show in Theorem 6.3 that the limit exponent $r = 1$ is excluded!

4. Asymptotic expansions for the resolvents and the eigenprojections

Asymptotic expansions are theoretically and numerically interesting in the sense that they offers 'good' approximations for the studied objects. How "good" is the approximation depends on its order and on the computation of its coefficients. In operator theory there are two types of asymptotic: uniform, *i.e.*, the rest is small with respect the operator norm, and strong asymptotic, *i.e.*, the rest is small for every fixed element from the domain of the operator.

Although we shall give lower order asymptotic (of second order) of the spectral objects related to Robin Laplacian, we shall write explicitly the coefficients of the asymptotic which is new to the best of our knowledge for such problems. In particular we shall show that the coefficients involved in the asymptotic depend solely on the Neumann Laplacian, the Dirichlet–to–Neumann operator, the Dirichlet Laplacian and the trace operator.

We start by giving an asymptotic expansion for $(H_\beta + 1)^{-1}$.

Theorem 4.1. *The following first order uniform expansion holds true:*

$$(H_\beta + 1)^{-1} = (-\Delta_D + 1)^{-1} + \frac{1}{\beta} K + \frac{1}{\beta^2} K', \tag{4.1}$$

where $K = (\check{H} J K_1)^* \check{H} J K_1 = \mathcal{R} \frac{\partial P K_1}{\partial \nu}$ *and* $\|K'\| \le \|\check{H}^{3/2} J K_1\|^2$.

Proof. From the construction of $\check{\mathcal{E}}_1$ we derive

$$\check{\mathcal{E}}_1(J K_1 u, J v) = \mathcal{E}_1(K_1 u, P v) = (u, P v)_{L^2(\Omega)}, \forall u \in L^2(\Omega), \forall v \in H^1(\Omega). \tag{4.2}$$

It follows that

$$\left(\check{H} J K_1 u, J v\right)_{L^2(\Gamma)} = (u, P v)_{L^2(\Omega)} \quad \text{and} \quad \left(\check{H} J K_1\right)^* J v = P v. \tag{4.3}$$

Then,

$$\left(\check{H} J K_1\right)^* \check{H} J K_1 u = P \mathcal{R} \frac{\partial P K_1 u}{\partial \nu} = \mathcal{R} \frac{\partial P K_1 u}{\partial \nu}. \tag{4.4}$$

On the other hand, relying on the resolvent formula (2.5) we obtain

$$\begin{aligned} D_\infty - D_\beta &= \left(\check{H}^{1/2} J K_1\right)^* \check{H}^{1/2} J K_1 - (J K_1)^* \left(\frac{1}{\beta} + \check{H}^{-1}\right)^{-1} J K_1 \\ &= \left(\check{H}^{1/2} J K_1\right)^* \check{H}^{1/2} J K_1 - \left(\check{H}^{1/2} J K_1\right)^* \left(1 + \frac{1}{\beta} \check{H}\right)^{-1} \check{H}^{1/2} J K_1 \\ &= \frac{1}{\beta} \left(\check{H} J K_1\right)^* \check{H} J K_1 - \frac{1}{\beta^2} \left(\check{H}^{3/2} J K_1\right)^* \left(1 + \frac{1}{\beta} \check{H}\right)^{-1} \check{H}^{3/2} J K_1. \end{aligned} \tag{4.5}$$

To conclude, it suffices to note that $0 \le \left(1 + \frac{1}{\beta} \check{H}\right)^{-1} \le 1$, and the proof is finished. □

Henceforth, $o_s\left(\frac{1}{\beta^2}\right)$ (respectively $o_u\left(\frac{1}{\beta^2}\right)$) denotes an operator-valued function such that $\beta^2 o_s\left(\frac{1}{\beta^2}\right) f \to 0,\ \forall f$ (respectively $\beta^2 \|o_u\left(\frac{1}{\beta^2}\right)\| \to 0$) as $\beta \to \infty$.

The latter theorem yields automatically the second order strong asymptotic expansion for large β.

Corollary 4.2. *For large β the following strong asymptotic formula holds true:*

$$\begin{aligned} (H_\beta + 1)^{-1} = (-\Delta_D + 1)^{-1} &+ \frac{1}{\beta} \left(\check{H} J K_1\right)^* \check{H} J K_1 \\ &- \frac{1}{\beta^2} (\check{H}^{3/2} J K_1)^* \check{H}^{3/2} J K_1 + o_s\left(\frac{1}{\beta^2}\right). \end{aligned} \tag{4.6}$$

We turn our attention now to give the expansions of the eigenprojections. To that end we need an expansion for $(H_\beta - z)^{-1}$ for z in the resolvent set $\rho(H_\beta)$.

Since $\{(H_\beta + 1)^{-1}\}$ converges in norm to $(-\Delta_D + 1)^{-1}$ when $\beta \to \infty$, it follows that if $z \in \rho(-\Delta_D)$, then $z \in \rho(H_\beta)$ for β sufficiently large and $\{(H_\beta - z)^{-1}\}$ converge in norm to $(-\Delta_D - z)^{-1}$ uniformly in any compact subset of $\rho(-\Delta_D)$ as β goes to infinity. In particular the family of the resolvents $\{(H_\beta - z)^{-1}\}$ is bounded uniformly in β and z in any compact subset of $\rho(-\Delta_D)$ (for large β). Moreover, one has:

Proposition 4.3. *For large β, the resolvent $(H_\beta - z)^{-1}$ admits the second order strong asymptotic expansion uniformly in any compact subset of $\rho(-\Delta_D)$:*

$$(H_\beta - z)^{-1} = (-\Delta_D - z)^{-1} + \frac{1}{\beta} LKL - \frac{1}{\beta^2}\big(LRL - (1+z)LKLKL\big) + o_s\left(\frac{1}{\beta^2}\right), \tag{4.7}$$

where K is the operator given by Theorem 4.1 *and*

$$L = L(z) := \left(1 + (1+z)(-\Delta_D - z)^{-1}\right), \; R := \left(\check{H}^{3/2} J K_1\right)^* \check{H}^{3/2} J K_1. \tag{4.8}$$

Proof. Let $z \in \rho(-\Delta_D)$. Then, for large β, one has

$$(H_\beta - z)^{-1} - (-\Delta_D - z)^{-1} = \left(1 + (1+z)(H_\beta - z)^{-1}\right)(D_\infty - D_\beta) \cdot \left(1 + (1+z)(-\Delta_D - z)^{-1}\right). \tag{4.9}$$

By formula (4.6), it follows that

$$u - \lim_{\beta\to\infty} \beta\left((H_\beta - z)^{-1} - (-\Delta_D - z)^{-1}\right) = LKL, \tag{4.10}$$

uniformly in any compact subset of $\rho(-\Delta_D)$.

Thus, one writes

$$(H_\beta - z)^{-1} = (-\Delta_D - z)^{-1} + \frac{1}{\beta} LKL + o_u\left(\frac{1}{\beta}\right). \tag{4.11}$$

Finally, we substitute $(D_\infty - D_\beta)$ and $(H_\beta - z)^{-1}$ by the corresponding terms given by formulae (4.6) and (4.11) respectively in the equation (4.9) to obtain the desired result. □

Since the Dirichlet Laplacian has compact resolvent, it follows that for each eigenvalue λ_∞ of $-\Delta_D$ there exists $\epsilon > 0$ such that $\mathrm{spec}(-\Delta_D) \cap B(\lambda_\infty, \epsilon) = \{\lambda_\infty\}$, where $B(\lambda_\infty, \epsilon) = \{z \in \mathbb{C}, |z - \lambda_\infty| < \epsilon\}$.

In the following we set:

- $E_\infty = \ker(-\Delta_D - \lambda_\infty)$, the eigenspace of λ_∞, and P_∞ the spectral projection onto E_∞. It is known that

$$P_\infty = -\frac{1}{2i\pi}\int_{C(\lambda_\infty,\epsilon)} (-\Delta_D - z)^{-1}\, dz, \tag{4.12}$$

  where $C(\lambda_\infty, \epsilon)$ is the circle of center λ_∞ and radius ϵ;
- E_β is the direct sum of the eigenspaces associated to the eigenvalues of H_β contained in $B(\lambda_\infty, \epsilon)$ and P_β is the spectral projection onto E_β given by:

$$P_\beta = -\frac{1}{2i\pi}\int_{C(\lambda_\infty,\epsilon)} (H_\beta - z)^{-1}\, dz.$$

Proposition 4.4. *The spectral projection P_β admits a strong asymptotic expansion of the form*

$$P_\beta = P_\infty + \frac{1}{\beta}\,Q - \frac{1}{\beta^2}Q_1 + o_s\left(\frac{1}{\beta^2}\right). \tag{4.13}$$

Moreover, $P_\infty Q P_\infty = 0$.

Proof. Setting

$$\begin{aligned} Q &= -\frac{1}{2i\pi}\int_{C(\lambda_\infty,\epsilon)} LKL\, dz,\\ Q_1 &= -\frac{1}{2i\pi}\int_{C(\lambda_\infty,\epsilon)} \big(LRL - (1+z)LKLKL\big)\, dz, \end{aligned} \tag{4.14}$$

the first identity is immediate by integrating formula (4.7) along the circle C.

For the second identity, since $\lim_{\beta\to\infty}\beta(P_\beta - P_\infty) = Q$, we obtain

$$P_\infty Q P_\infty = \lim_{\beta\to\infty}\beta\, P_\infty(P_\beta - P_\infty)P_\beta = \lim_{\beta\to\infty}\beta\, P_\infty(1 - P_\infty)P_\beta = 0. \tag{4.15}$$

□

5. Asymptotic expansion for the eigenvalues

Next we shall improve the asymptotic expansion of eigenvalues developed in [12, Theorem 1.2] and extend it to our context which deals with singular perturbations. The novelty at this stage is that we give a second order asymptotic expansion which coefficients are given by the eigenvalues of a matrix depending only on the Dirichlet Laplacian.

To that end we need some intermediate results.

Proposition 5.1. *The following formulae hold true:*

(1)

$$PK_1 = (-\Delta_N + 1)^{-1} - (-\Delta_D + 1)^{-1} = D_\infty. \tag{5.1}$$

In particular,

$$\frac{\partial PK_1}{\partial \nu} = -\frac{\partial(-\Delta_D + 1)^{-1}}{\partial \nu}; \tag{5.2}$$

(2) *Let* $(f_1, \cdots, f_m)$ *be an orthonormal basis of* E_∞*. Then for every* $z \in \rho(-\Delta_D)$ *it holds*

$$(P_\infty LKLP_\infty f_i, f_j) = \frac{1}{(\lambda_\infty - z)^2}\left(\frac{\partial f_i}{\partial \nu}, \frac{\partial f_j}{\partial \nu}\right)_{L^2(\Gamma)}. \tag{5.3}$$

Proof. For every $u \in L^2(\Omega)$, PK_1u is the unique solution of the boundary value problem

$$\begin{cases} -\Delta v + v = 0 & \text{on } \Omega \\ v = K_1 u & \text{in } \Gamma. \end{cases} \tag{5.4}$$

Let v_0 be the unique solution of the equation $-\Delta v + v = -u$ in $H^2(\Omega) \cap H_0^1(\Omega)$, that is $v_0 = (-\Delta_D + 1)^{-1}(-u)$. Then PK_1u is given by:

$$PK_1u = v_0 + K_1u = (-\Delta_N + 1)^{-1}u - (-\Delta_D + 1)^{-1}u, \tag{5.5}$$

yielding the first assertion.

Let f_i, f_j be eigenfunctions associated to the eigenvalue λ_∞ of $-\Delta_D$. Since $L(z)f_i = (\frac{\lambda_\infty+1}{\lambda_\infty-z})\, f_i$, a straightforward computation yields

$$\begin{aligned}(P_\infty LKLP_\infty f_i, f_j) &= \big(KL(z)f_i, L(\bar{z})f_j\big)_{L^2(\Omega)} \\ &= \left(\frac{\lambda_\infty + 1}{\lambda_\infty - z}\right)^2 \left(\check{H}JK_1 f_i, \check{H}JK_1 f_j\right)_{L^2(\Gamma)} \\ &= \left(\frac{\lambda_\infty+1}{\lambda_\infty-z}\right)^2 \left(\frac{\partial(-\Delta_D+1)^{-1} f_i}{\partial\nu}, \frac{\partial(-\Delta_D+1)^{-1} f_j}{\partial\nu}\right)_{L^2(\Gamma)} \\ &= \frac{1}{(\lambda_\infty - z)^2}\left(\frac{\partial f_i}{\partial\nu}, \frac{\partial f_j}{\partial\nu}\right)_{L^2(\Gamma)},\end{aligned} \tag{5.6}$$

and the proof is done. □

Proposition 5.2. *Let (f_k) be an orthonormal basis of Dirichlet eigenfunctions,*

$$-\Delta_D f_k = \lambda_k f_k,$$

and Q be the operator given by Proposition 4.4.

For f_i and f_j in E_∞ we set,

$$a_{i,j,k} := \left(\frac{\partial f_i}{\partial \nu}, \frac{\partial f_k}{\partial \nu}\right)_{L^2(\Gamma)} \left(\frac{\partial f_j}{\partial \nu}, \frac{\partial f_k}{\partial \nu}\right)_{L^2(\Gamma)}. \tag{5.7}$$

Then, for every $z \in \rho(-\Delta_D)$, we obtain:

(1) $(P_\infty LRLP_\infty f_i, f_j) = \dfrac{1}{(\lambda_\infty - z)^2}\left(\dfrac{\partial}{\partial \nu}\mathcal{R}\left(\dfrac{\partial f_i}{\partial \nu}\right), \dfrac{\partial f_j}{\partial \nu}\right)_{L^2(\Gamma)}$;

(2) $(P_\infty LKLKLP_\infty f_i, f_j) = \dfrac{1}{(z-\lambda_\infty)^2}\displaystyle\sum_k \dfrac{1}{(\lambda_k - z)(1+\lambda_k)} a_{i,j,k}$;

(3) $(P_\infty LKLQP_\infty f_i, f_j) = \displaystyle\sum_{f_k \in E_\infty^\perp} \dfrac{1}{(\lambda_\infty - z)(\lambda_k - z)(\lambda_k - \lambda_\infty)} a_{i,j,k}$.

Proof.

(1) We first prove that for any Dirichlet eigenfunction f on Ω it holds

$$\frac{\partial f}{\partial \nu} \in D(\check{H}). \tag{5.8}$$

According to the relationship between forms and operators with the help of Green's formula, the domain of $\check{H}$ consists of functions $\psi \in H^{1/2}(\Gamma)$ such that

$$\frac{\partial \mathcal{R}\psi}{\partial \nu} \in L^2(\Gamma). \tag{5.9}$$

As Ω is C^∞ it is well known that $f \in C^\infty(\overline{\Omega})$. In particular, $f \in H^3(\Omega)$ and hence by [20, Theorem 4.24-i, page 105] (where the notations $W_2^k(\Omega)$ and $W_2^s(\Gamma)$ stand for the spaces $H^k(\Omega)$ and $H^s(\Gamma)$ respectively), $\frac{\partial f}{\partial \nu} \in H^{3/2}(\Gamma) \subset H^{1/2}(\Gamma)$.

The fact that $\frac{\partial f}{\partial \nu} \in H^{3/2}(\Gamma)$, in conjunction with [20, Theorem 5.39, page 152], yields $\mathcal{R}(\frac{\partial f}{\partial \nu}) \in H^2(\Omega)$. Accordingly we obtain once again by [20, Theorem 4.24-ii, page 105] that

$$\frac{\partial}{\partial \nu}\mathcal{R}\left(\frac{\partial f}{\partial \nu}\right) \in H^{1/2}(\Gamma) \subset L^2(\Gamma), \tag{5.10}$$

and the claim is proved.

Let us turn our attention now to the rest of the proof.
Having (5.8) in mind, a straightforward computation yields

$$\begin{aligned}(P_\infty LRLP_\infty f_i, f_j) &= (RL(z)f_i, L(\bar{z})f_j)\\ &= \left(\frac{1+\lambda_\infty}{\lambda_\infty - z}\right)^2 (Rf_i, f_j)\\ &= \left(\frac{1+\lambda_\infty}{\lambda_\infty - z}\right)^2 \left(\check{H}^{3/2}JK_1 f_i, \check{H}^{3/2}JK_1 f_j\right)_{L^2(\Gamma)}\\ &= \frac{1}{(\lambda_\infty - z)^2}\left(\check{H}^{1/2}\frac{\partial f_i}{\partial\nu}, \check{H}^{1/2}\frac{\partial f_j}{\partial\nu}\right)_{L^2(\Gamma)}\\ &= \frac{1}{(\lambda_\infty - z)^2}\left(\check{H}\frac{\partial f_i}{\partial\nu}, \frac{\partial f_j}{\partial\nu}\right)_{L^2(\Gamma)}\\ &= \frac{1}{(\lambda_\infty - z)^2}\left(\frac{\partial}{\partial\nu}\mathcal{R}\left(\frac{\partial f_i}{\partial\nu}\right), \frac{\partial f_j}{\partial\nu}\right)_{L^2(\Gamma)}.\end{aligned} \tag{5.11}$$

In the last step we used the fact that for $\varphi \in D(\check{H})$ we have $\check{H}\varphi = \frac{\partial u}{\partial\nu}$ where $u = \mathcal{R}\varphi$. Indeed, by the definition of $\mathcal{R}$, $\mathcal{R}\varphi$ solves

$$\begin{cases} -\Delta u + u = 0 & \text{on } \Omega\\ u = \varphi & \text{in } \Gamma;\end{cases}$$

(2) Making use of Proposition 5.1, an elementary computation yields

$$\begin{aligned}(Kf_i, f_k) &= \left(\check{H}JK_1 f_i, \check{H}JK_1 f_k\right)\\ &= (1+\lambda_i)^{-1}(1+\lambda_k)^{-1}\left(\frac{\partial f_i}{\partial\nu}, \frac{\partial f_k}{\partial\nu}\right)_{L^2(\Gamma)}.\end{aligned} \tag{5.12}$$

Thus

$$Kf_i = \sum_k (1+\lambda_i)^{-1}(1+\lambda_k)^{-1}\left(\frac{\partial f_i}{\partial\nu}, \frac{\partial f_k}{\partial\nu}\right)_{L^2(\Gamma)} f_k \tag{5.13}$$

and

$$L(z)Kf_i = \sum_k (1+\lambda_i)^{-1}(\lambda_k - z)^{-1}\left(\frac{\partial f_i}{\partial\nu}, \frac{\partial f_k}{\partial\nu}\right)_{L^2(\Gamma)} f_k. \tag{5.14}$$

Set $B = P_\infty L(z)KL(z)KL(z)P_\infty$. Then

$$\begin{aligned}(Bf_i, f_j) &= \left(\frac{1+\lambda_\infty}{z-\lambda_\infty}\right)^2 (KLKf_i, f_j) = \left(\frac{1+\lambda_\infty}{z-\lambda_\infty}\right)^2 (LKf_i, Kf_j) \\ &= \frac{1}{(z-\lambda_\infty)^2} \sum \frac{1}{(\lambda_k - z)(1+\lambda_k)} \\ &\quad \cdot \left(\frac{\partial f_i}{\partial \nu}, \frac{\partial f_k}{\partial \nu}\right)_{L^2(\Gamma)} \left(\frac{\partial f_j}{\partial \nu}, \frac{\partial f_k}{\partial \nu}\right)_{L^2(\Gamma)};\end{aligned} \tag{5.15}$$

(3) Finally, setting $A(z,s) = P_\infty L(z)KL(z)L(s)KL(s)P_\infty$, we obtain

$$\begin{aligned}(Af_i, f_j) &= \frac{(1+\lambda_\infty)^2}{(\lambda_\infty - z)(\lambda_\infty - s)} (L(s)Kf_i, L(\bar{z})Kf_j) \\ &= \frac{1}{(\lambda_\infty - z)(\lambda_\infty - s)} \sum_k \frac{1}{(\lambda_k - z)(\lambda_k - s)} \\ &\quad \cdot \left(\frac{\partial f_i}{\partial \nu}, \frac{\partial f_k}{\partial \nu}\right)_{L^2(\Gamma)} \left(\frac{\partial f_j}{\partial \nu}, \frac{\partial f_k}{\partial \nu}\right)_{L^2(\Gamma)}.\end{aligned} \tag{5.16}$$

Regarding the definition of Q, we achieve

$$\begin{aligned}(P_\infty L(z)KL(z)QP_\infty f_i, f_j) &= -\frac{1}{2i\pi} \int_{C(\lambda_\infty, \epsilon)} (A(z,s)f_i, f_j)\, ds \\ &= \sum_{f_k \in E_\infty^\perp} \frac{(\lambda_k - \lambda_\infty)^{-1}}{(\lambda_\infty - z)(\lambda_k - z)} \\ &\quad \cdot \left(\frac{\partial f_i}{\partial \nu}, \frac{\partial f_k}{\partial \nu}\right)_{L^2(\Gamma)} \left(\frac{\partial f_j}{\partial \nu}, \frac{\partial f_k}{\partial \nu}\right)_{L^2(\Gamma)}.\end{aligned} \tag{5.17}$$

□

We are now in position to establish the asymptotic of the eigenvalue of the Robin Laplacian.

Theorem 5.3. *Let λ_∞ be an eigenvalue of $-\Delta_D$ with multiplicity m and eigenspace E_∞. Then, for sufficiently large β, the operator H_β has exactly m eigenvalues, counted according to their multiplicities, in $B(\lambda_\infty, \epsilon)$. These eigenvalues admit the asymptotic expansion*

$$\lambda_{i,j,\beta} = \lambda_\infty - \frac{1}{\beta}\alpha_i + \frac{1}{\beta^2}\mu_{i,j} + o\left(\frac{1}{\beta^2}\right), \tag{5.18}$$

where (α_i) are the repeated eigenvalues of the matrix

$$M := \left(\int_\Gamma \frac{\partial f_i}{\partial \nu}\frac{\partial f_j}{\partial \nu}\right)_{1 \le i,j \le m}$$

in an orthonormal basis $(f_1, \cdots, f_m)$ of the eigenspace E_∞.

Moreover, setting P_i for the eigenprojection associated to the eigenvalue α_i and N for the matrix given by

$$N := \Bigg(\left(\frac{\partial}{\partial \nu} \mathcal{R} \left(\frac{\partial f_i}{\partial \nu} \right), \frac{\partial f_j}{\partial \nu} \right)_{L^2(\Gamma)} + \frac{1}{1+\lambda_\infty} \sum_{f_k \in E_\infty} a_{i,j,k} + \sum_{f_k \in E_\infty^\perp} \frac{(1+\lambda_\infty)}{(1+\lambda_k)(\lambda_\infty - \lambda_k)} a_{i,j,k} \Bigg)_{1 \le i,j \le m}, \tag{5.19}$$

then $\mu_{i,j}$, for all $1 \le j \le \dim P_i$, are the repeated eigenvalues of $P_i N P_i$ in the subspace $P_i E_\infty$.

Proof. We shall follow Kato's method (see [21, the proof of Theorem VIII.2.9, pages 450–453]), which consists in transforming the problem into a finite-dimensional one.

As a first step, we introduce the operator

$$A_\beta := 1 - P_\infty + P_\beta P_\infty = 1 - (P_\infty - P_\beta) P_\infty.$$

For large β the operator A_β is invertible and maps E_∞ onto E_β, since $\|P_\beta - P_\infty\|$ is small, and leaves the orthogonal of E_∞ invariant.

Using Proposition 4.4, we obtain the asymptotic expansion for A_β:

$$A_\beta = 1 + \frac{1}{\beta} Q P_\infty - \frac{1}{\beta^2} Q_1 P_\infty + o_s\left(\frac{1}{\beta^2}\right) \tag{5.20}$$

Since $P_\infty Q P_\infty = 0$ it follows that,

$$\begin{aligned} A_\beta^{-1} &= 1 - \left(\frac{1}{\beta} Q P_\infty - \frac{1}{\beta^2} Q_1 P_\infty \right) + \left(\frac{1}{\beta} Q P_\infty - \frac{1}{\beta^2} Q_1 P_\infty \right)^2 \\ &\quad + o_s\left(\frac{1}{\beta^2}\right) \\ &= 1 - \frac{1}{\beta} Q P_\infty + \frac{1}{\beta^2} Q_1 P_\infty + o_s\left(\frac{1}{\beta^2}\right) \end{aligned} \tag{5.21}$$

Now we define the operator B_β as

$$B_\beta := P_\infty A_\beta^{-1} H_\beta A_\beta P_\infty.$$

Obviously B_β is bounded on E_∞ and has finite rank. Furthermore, the repeated eigenvalues of H_β considered in the m-dimensional subspace E_β are equal to the eigenvalues of $H_\beta P_\beta$ in E_β and therefore also to those of $A_\beta^{-1} H_\beta A_\beta$ which is similar to $H_\beta P_\beta$ in E_β.

Thus, taking into account that $P_\infty Q P_\infty = 0$ and that P_∞ commutes with $(-\Delta_D - z)^{-1}$, we obtain the asymptotic expansion:

$$\begin{aligned}
P_\infty A_\beta^{-1}(H_\beta - z)^{-1} A_\beta P_\infty = &\left(P_\infty + \frac{1}{\beta^2} P_\infty Q_1 P_\infty + o_s\left(\frac{1}{\beta^2}\right)\right) \\
&\cdot \left((-\Delta_D - z)^{-1} + \frac{1}{\beta} LKL - \frac{1}{\beta^2}\left(LRL - (1+z)LKLKL\right) + o_s\left(\frac{1}{\beta^2}\right)\right) \\
&\cdot \left(1 + \frac{1}{\beta} QP_\infty - \frac{1}{\beta^2} Q_1 P_\infty + o_s\left(\frac{1}{\beta^2}\right)\right) P_\infty \\
= &P_\infty(-\Delta_D - z)^{-1} P_\infty + \frac{1}{\beta} P_\infty LKLP_\infty \\
&- \frac{1}{\beta^2}\left(P_\infty LRLP_\infty - (1+z)P_\infty LKLKLP_\infty\right) \\
&+ \frac{1}{\beta^2} P_\infty Q_1(-\Delta_D - z)^{-1} P_\infty - \frac{1}{\beta^2} P_\infty(-\Delta_D - z)^{-1} Q_1 P_\infty \\
&+ \frac{1}{\beta^2} P_\infty LKLQP_\infty + o_u\left(\frac{1}{\beta^2}\right) \\
= &(\lambda_\infty - z)^{-1} P_\infty + \frac{1}{\beta}(\lambda_\infty - z)^{-2} MP_\infty - \frac{1}{\beta^2} P_\infty LRLP_\infty \\
&+ \frac{1}{\beta^2}\left((1+z)P_\infty LKLKLP_\infty + P_\infty LKLQP_\infty\right) + o_u\left(\frac{1}{\beta^2}\right).
\end{aligned} \tag{5.22}$$

Here we have used the fact that $o_s\left(\frac{1}{\beta^2}\right) P_\infty = o_u\left(\frac{1}{\beta^2}\right)$ because P_∞ has finite rank.

Since

$$H_\beta P_\beta = -\frac{1}{2\pi i} \int_{C(\lambda_\infty, \epsilon)} z\,(H_\beta - z)^{-1}\, dz, \tag{5.23}$$

integration of (5.22) along the circle $C(\lambda_\infty, \epsilon)$ after multiplication by $(-z/2\pi i)$ and an elementary calculation of residues at the singularity λ_∞ lead to

$$B_\beta = P_\infty A_\beta^{-1} H_\beta P_\beta A_\beta P_\infty = \lambda_\infty P_\infty - \frac{1}{\beta} MP_\infty + \frac{1}{\beta^2} NP_\infty + o_u\left(\frac{1}{\beta^2}\right). \tag{5.24}$$

Theorem 5.3 is then a consequence [21, Theorem II.5.11, page 115] concerning asymptotic expansion in finite dimensions. □

6. Example: the case of the unit disc in $\mathbb{R}^2$

In this section we shall apply the theory developed in the previous paragraphs to the special case where Ω is the unit disc, which we denote by D, and whose boundary we will denote by C (the unit circle).

First, we study the solutions of the eigenvalue problem $-\Delta f = \lambda f,\ \lambda \geq 0$, with either Dirichlet or Neumann boundary conditions on D.

By separating variables it turns out that the solutions of the equation $-\Delta f = \lambda f$ are given by (see [15, page 304])

$$J_{|n|}(\sqrt{\lambda}r)e^{in\theta},\ n \in \mathbb{Z}, \tag{6.1}$$

where the J_n's are Bessel functions of the first kind. Note that the latter formula still holds for $\lambda < 0$, and will be used to establish (6.16).

For $n \in \mathbb{N}$, if $J_n(\sqrt{\lambda}) = 0$, then λ is an eigenvalue of the Dirichlet Laplacian on D, with eigenfunctions $J_n(\sqrt{\lambda}r)e^{\pm in\theta}$. As every J_n has infinitely many positive zeros, we shall order them as follows: $0 < k_{n,1} < k_{n,2} < \cdots < k_{n,m} < \ldots \quad n \in \mathbb{N}$.

Therefore the eigenvalues of the Dirichlet Laplacian on the unit disc are given by

$$\lambda_{n,m} = k^2_{|n|,m},\ \ n \in \mathbb{Z},\ m \geq 1, \tag{6.2}$$

with associated eigenfunctions

$$\varphi_{n,m}(r,\theta) = J_{|n|}(k_{|n|,m}r)e^{in\theta},\ \ n \in \mathbb{Z},\ m \geq 1. \tag{6.3}$$

The Neumann eigenvalues are characterized by the equation $\sqrt{\lambda}J'_n(\sqrt{\lambda}) = 0,\ \lambda \geq 0$. As before we order the zeros of each J'_n in an increasing order:

$$0 < k'_{n,1} < k'_{n,2} < \ldots < k'_{n,m} < \ldots,\ \ n \geq 1 \tag{6.4}$$

$$0 = k'_{0,1} < k'_{0,2} < \ldots < k'_{0,m} < \ldots \tag{6.5}$$

Thus the eigenvalues of the Neumann Laplacian on the unit disc are given by

$$\mu_{n,m} = k'^2_{|n|,m},\ n \in \mathbb{Z},\ \ ,m \geq 1, \tag{6.6}$$

with associated eigenfunctions,

$$\psi_{n,m}(r,\theta) = J_{|n|}(k'_{|n|,m}r)e^{in\theta},\ \ n \in \mathbb{Z},\ m \geq 1\,. \tag{6.7}$$

Making use of the formula (see [15, page 306])

$$\int_0^1 J_n^2(cr)r\,dr = \frac{1}{2}J'^2_n(c) + \frac{1}{2}\left(1 - \frac{n^2}{c^2}\right)J_n^2(c), \tag{6.8}$$

we obtain the following expression for the normalized Neumann eigenfunctions associated to the eigenvalues $\mu_{n,m} = k'^2_{|n|,m}$:

$$\Psi_{n,m}(r,\theta) = \pi^{-1/2}\left(1 - \frac{n^2}{k'^2_{|n|,m}}\right)^{-1/2}\frac{J_{|n|}(k'_{|n|,m}r)}{J_{|n|}(k'_{|n|,m})}\,e^{in\theta},\ \ n \in \mathbb{Z},\ m \geq 1. \tag{6.9}$$

Thus, by the spectral calculus, we obtain

$$K_1 = (-\Delta_N + 1)^{-1} = \sum_{n\in\mathbb{Z}, m\geq 1} \left(1 + k'^2_{|n|,m}\right)^{-1} (\Psi_{n,m}, \cdot)\Psi_{n,m}, \tag{6.10}$$

$$JK_1 = \sum_{n,m} \pi^{-1/2} \left(1 + k'^2_{|n|,m}\right)^{-1} \left(1 - \frac{n^2}{k'^2_{|n|,m}}\right)^{-1/2} (\Psi_{n,m}, \cdot)e^{in\theta}, \tag{6.11}$$

$$(JK_1)^* = \sum_{n,m} \pi^{-1/2} \left(1 + k'^2_{|n|,m}\right)^{-1} \left(1 - \frac{n^2}{k'^2_{|n|,m}}\right)^{-1/2} \left(e^{in\theta}, \cdot\right) \Psi_{n,m}, \tag{6.12}$$

yielding

$$(JK_1)^* e^{in\theta} = \sum_{m\geq 1} 2\pi^{1/2} \left(1 + k'^2_{|n|,m}\right)^{-1} \left(1 - \frac{n^2}{k'^2_{|n|,m}}\right)^{-1/2} \Psi_{n,m}, \tag{6.13}$$

$$\|(JK_1)^* e^{in\theta}\|^2_{L^2(D)} = \sum_{m\geq 1} 4\pi \left(1 + k'^2_{|n|,m}\right)^{-2} \left(1 - \frac{n^2}{k'^2_{|n|,m}}\right)^{-1}. \tag{6.14}$$

Let us now determine the operator $\check{H}$.

An elementary computation yields that the solution of the boundary value problem

$$\begin{cases} -\Delta u + u = 0 & \text{in } D \\ u = e^{in\theta} & \text{on } C \end{cases} \tag{6.15}$$

is given by

$$u_n(r,\theta) = \frac{J_{|n|}(ir)}{J_{|n|}(i)}\, e^{in\theta}. \tag{6.16}$$

Hence the functions $e^{in\theta}$, $n \in \mathbb{Z}$, belong to the domain of $\check{H}$, and

$$\check{H} e^{in\theta} = \frac{\partial u_n}{\partial \nu} = \frac{\partial u_n(r,\theta)}{\partial r}\Big|_{r=1} = i\frac{J'_{|n|}(i)}{J_{|n|}(i)}\, e^{in\theta}. \tag{6.17}$$

That is, the eigenvalues of $\check{H}$ are

$$\check{\lambda}_n = i\frac{J'_{|n|}(i)}{J_{|n|}(i)} \tag{6.18}$$

with respective associated eigenfunctions $e^{in\theta}$, $\forall n \in \mathbb{Z}$.

Observe that each eigenvalue is a double eigenvalue except $\check{\lambda}_0$.

Set $L^2(C) := L^2([0, 2\pi], d\theta)$; then

$$D(\check{H}) = \left\{ \varphi \in L^2(C) \colon \sum_{n \in \mathbb{Z}} \check{\lambda}_n^2 |(\varphi, e^{in\theta})_{L^2(C)}|^2 < \infty \right\}, \\ \check{H}\varphi = \sum_{n \in \mathbb{Z}} \frac{1}{2\pi} \check{\lambda}_n \left(\varphi, e^{in\theta}\right)_{L^2(C)} e^{in\theta}, \ \forall \varphi \in D\left(\check{H}\right). \tag{6.19}$$

In other words, if we set $(c_n)_{n \in \mathbb{Z}}$ for the Fourier coefficients of $\varphi \in L^2(C)$, then $\varphi \in D(\check{H})$ if and only if

$$\sum_{n \in \mathbb{Z}} \check{\lambda}_n^2 |c_n|^2 < \infty. \tag{6.20}$$

This observation leads to a full description of $D(\check{H})$:

Proposition 6.1.

(1) *For each $n \in \mathbb{Z}$, we have $|n| < \check{\lambda}_n < |n| + 1/2$;*
(2) *It follows that $\varphi \in L^2(C)$ belongs to $D(\check{H})$ if and only if*

$$\sum_{n \in \mathbb{Z}} n^2 |c_n|^2 < \infty.$$

Proof. The second assertion follows from the first one, which we proceed to prove.

From the recursion relations between Bessel functions and their derivatives one has

$$\check{\lambda}_n = i \frac{J'_{|n|}(i)}{J_{|n|}(i)} = |n| - i \frac{J_{|n|+1}(i)}{J_{|n|}(i)} \quad \forall n \in \mathbb{Z}. \tag{6.21}$$

Since $J_n(i) = \left(\frac{i}{2}\right)^n \sum_{k=0}^{\infty} \frac{1}{2^{2k} k!(n+k)!} \ \ \forall n \in \mathbb{N}$, it follows that

$$|n| < \check{\lambda}_n < |n| + \frac{1}{2}, \quad \forall n \in \mathbb{Z}, \tag{6.22}$$

which finishes the proof. □

Now we turn our attention to compute explicitly the operators $\check{H}^s J K_2$, as they are involved in the trace-class convergence as well as in the asymptotic developments. We shall especially prove that the limiting exponent $r = 1$ in Proposition 3.5 is excluded.

Let $s \in (0, 3/2]$. Relying on formulae (6.11), (6.19), and owing to the fact that $\check{H}^{3/2} J K_1$ is bounded, we obtain

$$\begin{aligned}\check{H}^s J K_1 &= \sum_{n\in\mathbb{Z}, m\geq 1} \pi^{-1/2} \check{\lambda}_n^s \left(1 + k'^2_{|n|,m}\right)^{-1} \left(1 - \frac{n^2}{k'^2_{|n|,m}}\right)^{-1/2} (\Psi_{n,m}, \cdot) e^{in\theta} \\ &= \sum_{n\in\mathbb{Z}, m\geq 1} \check{\lambda}_n^s \theta_{n,m} (\Psi_{n,m}, \cdot) e^{in\theta} = \sum_{n\in\mathbb{Z}} \check{\lambda}_n^s (\tilde{\Psi}_n, \cdot) e^{in\theta},\end{aligned} \tag{6.23}$$

where

$$\theta_{n,m} := \pi^{-1/2} \left(1 + k'^2_{|n|,m}\right)^{-1} \left(1 - \frac{n^2}{k'^2_{|n|,m}}\right)^{-1/2}, \ \tilde{\Psi}_n := \sum_{m\geq 1} \theta_{n,m} \Psi_{n,m}. \tag{6.24}$$

Let us note that the family $\tilde{\Psi}_n$ is orthogonal in $L^2(D)$. Hence, setting

$$\gamma_n^2 := \left\|\tilde{\Psi}_n\right\|^2_{L^2(D)} = \sum_{m\geq 1} \theta^2_{n,m}, \ \phi_n := \gamma_n^{-1} \tilde{\Psi}_n, \tag{6.25}$$

we obtain

$$\begin{aligned}\check{H}^s J K_1 &= \sum_{n\in\mathbb{Z}} \check{\lambda}_n^s \gamma_n (\phi_n, .) e^{in\theta}, \ \left(\check{H}^s J K_1\right)^* \check{H}^s J K_1 \\ &= \sum_{n\in\mathbb{Z}} 2\pi \check{\lambda}_n^{2s} \gamma_n^2 (\phi_n, .)\phi_n.\end{aligned} \tag{6.26}$$

In particular we derive:

Proposition 6.2.

(1) *The following representation for D_∞ holds true:*

$$D_\infty = (-\Delta_N + 1)^{-1} - (-\Delta_D + 1)^{-1} = \sum_{n\in\mathbb{Z}} 2\pi \check{\lambda}_n \gamma_n^2 (\phi_n, .)\phi_n; \tag{6.27}$$

(2) $\lim_{\beta\to\infty} \beta \|D_\beta - D_\infty\| = \max_{n\in\mathbb{N}} 2\pi \check{\lambda}_n^2 \gamma_n^2$.

Proof. Claim (1) is a consequence of formulae (2.6)-(6.26), whereas claim (2) comes from Theorem 3.1 together with (6.26), in conjunction with the fact that $(\check{H} J K_1)^* \check{H} J K_1$ is compact, self-adjoint, and positive. □

Now we proceed to prove that trace-class convergence with maximal rate, *i.e.*, with rate proportional to $1/\beta$, does not hold true.

Theorem 6.3. *The operator $\check{H}JK_1$ is not a Hilbert-Schmidt operator. Consequently*

$$\lim_{\beta\to\infty} \beta\|D_\beta - D_\infty\|_{S_1} = \infty. \tag{6.28}$$

Proof. By [10, Theorem 2.3-b], trace-class convergence with maximal rate holds true if and only if the operator $\check{H}JK_1$ is a Hilbert-Schmidt operator. Hence we are led to prove that $\|\check{H}JK_1\|_{S_2} = \infty$.

Let (f_j) be an orthonormal basis of $L^2(D)$. As $\big(\frac{e^{in\theta}}{\sqrt{2\pi}}\big)_{n\in\mathbb{Z}}$ is an orthonormal basis of $L^2(C)$, we achieve

$$\begin{aligned}\check{H}JK_1 f_j &= \sum_{n\in\mathbb{Z}} \frac{1}{2\pi}\left(e^{in\theta}, \check{H}JK_1 f_j\right)_{L^2(C)} e^{in\theta} \\ &= \sum_{n\in\mathbb{Z}} \frac{1}{2\pi}\left(\check{H}e^{in\theta}, JK_1 f_j\right)_{L^2(C)} e^{in\theta} \\ &= \sum_{n\in\mathbb{Z}} \frac{1}{2\pi}\check{\lambda}_n\left(e^{in\theta}, JK_1 f_j\right)_{L^2(C)} e^{in\theta}.\end{aligned} \tag{6.29}$$

This yields to

$$\|\check{H}JK_1 f_j\|^2_{L^2(C)} = \sum_{n\in\mathbb{Z}} \frac{1}{2\pi}\check{\lambda}_n^2 \left|(e^{in\theta}, JK_1 f_j)_{L^2C)}\right|^2. \tag{6.30}$$

Thus we have:

$$\begin{aligned}\|\check{H}JK_1\|^2_{S_2} &= \sum_j \left\|\check{H}JK_1 f_j\right\|^2_{L^2(C)} = \sum_j\sum_{n\in\mathbb{Z}} \frac{1}{2\pi}\check{\lambda}_n^2 \left|\left(e^{in\theta}, JK_1 f_j\right)_{L^2(C)}\right|^2 \\ &= \sum_{n\in\mathbb{Z}} \frac{1}{2\pi}\check{\lambda}_n^2 \sum_j \left|\left((JK_1)^* e^{in\theta}, f_j\right)_{L^2(D)}\right|^2 \\ &= \sum_{n\in\mathbb{Z}} \frac{1}{2\pi}\check{\lambda}_n^2 \left\|(JK_1)^* e^{in\theta}\right\|^2_{L^2(D)}.\end{aligned} \tag{6.31}$$

Having formula (6.14) in mind we get

$$\begin{aligned}2\pi\left\|\check{H}JK_1\right\|^2_{S_2} &= \check{\lambda}_0^{\,2}\left\|(JK_1)^*1\right\|^2_{L^2(D)} + 2\sum_{n\geq1}\check{\lambda}_n^2\left\|(JK_1)^* e^{in\theta}\right\|^2_{L^2(D)} \\ &= \check{\lambda}_0^2\left\|(JK_1)^*1\right\|^2_{L^2(D)} + \sum_{n\geq1,\, m\geq1} \frac{8\pi\,\check{\lambda}_n^2\,k'^2_{n,m}}{(1+k'^2_{n,m})^2(k'^2_{n,m}-n^2)}.\end{aligned} \tag{6.32}$$

Now we have to investigate the behavior of $k'_{n,m}$ for large n and m.

According to [22], one has, for $n, m \geq 1$,

$$n + 2^{-1/3}a_m n^{1/3} < k_{n,m} < n + 2^{-1/3}a_m n^{1/3} + \frac{3}{10}a_m^2 n^{-1/3}, \tag{6.33}$$

where a_m is the m^{th} positive root of the equation

$$\mathrm{Ai}(-x) = \frac{1}{3}\sqrt{x}\left(J_{1/3}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + J_{-1/3}\left(\frac{2}{3}x^{\frac{3}{2}}\right)\right) = 0 \tag{6.34}$$

and Ai is the Airy function.

In the following, c denotes different positive constants.

For large m one has $a_m \sim c\, m^{2/3}$ (see [1, page 450]). Accordingly, there exists a positive constant c such that for $n, m \geq 1$,

$$n + cm^{2/3}n^{1/3} < k_{n,m} < n + cm^{2/3}n^{1/3} + cm^{4/3}n^{-1/3}. \tag{6.35}$$

On the other hand it is known that the zeroes of J_n and J'_n are interlaced in the following manner:

$$n \leq \ldots < k'_{n,m} < k_{n,m} < k'_{n,m+1} < k_{n,m+1} < \ldots \tag{6.36}$$

Hence for $n \geq 1$, $m \geq 2$, one has

$$n + c(m-1)^{2/3}n^{1/3} < k'_{n,m} < n + cm^{2/3}n^{1/3} + cm^{4/3}n^{-1/3}. \tag{6.37}$$

Thus, relying on the comparison (6.37), we get

$$\begin{aligned}
&\sum_{m\geq 2} \frac{1}{k'^2_{n,m}(k'^2_{n,m} - n^2)} \\
&\geq \sum_{m\geq 2} \frac{1}{(2n + cm^{2/3}n^{1/3} + cm^{4/3}n^{-1/3})^3(cm^{2/3}n^{1/3} + cm^{4/3}n^{-1/3})} \\
&= \sum_{m\geq 2} \frac{1}{cn^{10/3}m^{2/3}(2 + cm^{2/3}n^{-2/3} + cm^{4/3}n^{-4/3})^3(1 + m^{2/3}n^{-2/3})} \\
&\geq \frac{1}{cn^{10/3}} \int_2^\infty \frac{1}{x^{2/3}(2 + cx^{2/3}n^{-2/3} + cx^{4/3}n^{-4/3})^3(1 + x^{2/3}n^{-2/3})}dx \\
&= \frac{1}{cn^3} \int_{2/n}^\infty \frac{1}{u^{2/3}(2 + cu^{2/3} + cu^{4/3})^3(1 + u^{2/3})}du \\
&\sim \frac{1}{cn^3} \int_0^\infty \frac{1}{u^{2/3}(2 + cu^{2/3} + cu^{4/3})^3(1 + u^{2/3})}du = \frac{c}{n^3}.
\end{aligned}$$

Therefore $\sum_{n\geq 1,\, m\geq 2} \frac{n^2}{k'^2_{n,m}(k'^2_{n,m} - n^2)} = \infty$ and $\|\check{H} J K_1\|_{S_2} = \infty$, which finishes the proof. □

By the end of this section we shall utilize Theorem 5.3 to perform second order asymptotic for the eigenvalues of H_β. Accordingly for $n \geq 1$ (respectively $n = 0$), $m \geq 1$ there exist $\epsilon > 0$ and $\beta_0 > 0$ such that the Laplacian with Robin boundary conditions H_β, $\beta > \beta_0$ has exactly two (respectively only one) eigenvalues counted according to their multiplicities in the ball $B(k_{n,m}^2, \epsilon)$.

Theorem 6.4. *Set* $\lambda_{n,m}^{(\beta)}$, $n \in \mathbb{N}$, $m \geq 1$ *the eigenvalues of* H_β. *Then*

$$\lambda_{n,m}^{(\beta)} = k_{n,m}^2 - \frac{2k_{n,m}^2}{\beta} + \frac{\alpha_{n,m}}{\beta^2} + o\left(\frac{1}{\beta^2}\right), \; n \in \mathbb{N}, \; m \geq 1 \text{ for large } \beta, \tag{6.38}$$

where

$$\alpha_{n,m} = 2ik_{n,m}^2 \frac{J_n'(i)}{J_n(i)} + \frac{4k_{n,m}^4}{1 + k_{n,m}^2} + \sum_{q \neq m} \frac{4(1 + k_{n,m}^2)k_{n,m}^2 k_{n,q}^2}{(1 + k_{n,q}^2)(k_{n,m}^2 - k_{n,q}^2)}. \tag{6.39}$$

Proof. Using formulae (6.3), (6.8) and the recursion relation (see [1, page 361]) $J_n'(z) = \frac{n}{z} J_n(z) - J_{n+1}(z)$, we obtain that the normalized Dirichlet eigenfunctions associated to the eigenvalue $\lambda_{n,m} = k_{n,m}^2$ are given by

$$f_1(r, \theta) = \pi^{-1/2} \frac{J_n(k_{n,m} r)}{J_{n+1}(k_{n,m})} e^{in\theta}, \quad f_2(r, \theta) = \pi^{-1/2} \frac{J_n(k_{n,m} r)}{J_{n+1}(k_{n,m})} e^{-in\theta}. \tag{6.40}$$

In particular, we get

$$\begin{aligned} \frac{\partial f_{1,2}}{\partial r} &:= \frac{\partial f_{1,2}(r, \theta)}{\partial r} \rfloor_{r=1} \\ &= \pi^{-1/2} k_{n,m} \frac{J_n'(k_{n,m})}{J_{n+1}(k_{n,m})} e^{\pm in\theta} = -\pi^{-1/2} k_{n,m} e^{\pm in\theta}. \end{aligned} \tag{6.41}$$

Then, for $p, q \in \{1, 2\}$

$$\left(\frac{\partial f_p}{\partial r}, \frac{\partial f_q}{\partial r} \right)_{L^2(C)} = 2k_{n,m}^2 \delta_{p,q}. \tag{6.42}$$

Moreover

$$\mathcal{R}\left(\frac{\partial f_{1,2}}{\partial r} \right) = -\pi^{-1/2} k_{n,m} \frac{J_n(ir)}{J_n(i)} e^{\pm in\theta} \tag{6.43}$$

$$\left(\frac{\partial}{\partial r} \mathcal{R}(\frac{\partial f_p}{\partial r}), \frac{\partial f_q}{\partial r} \right)_{L^2(C)} = 2ik_{n,m}^2 \frac{J_n'(i)}{J_n(i)} \delta_{p,q}. \tag{6.44}$$

On the other hand, setting $E_{n,m}$ for the eigenspace associated to the eigenvalue $k_{n,m}^2$ we get $E_{n,m} = \text{Vect}(f_1, f_2)$, and

$$E_{n,m}^\perp = \overline{\text{Vect}} \left(\varphi_{p,q}(r, \theta) = \pi^{-1/2} \frac{J_{|p|}(k_{|p|,q} r)}{J_{|p|+1}(k_{|p|,q})} e^{ip\theta}, \; (p, q) \neq (\pm n, m) \right).$$

Consequently

$$a_{i,j,k} = \left(\frac{\partial f_i}{\partial r}, \frac{\partial f_k}{\partial r}\right)_{L^2(C)} \left(\frac{\partial f_j}{\partial r}, \frac{\partial f_k}{\partial r}\right)_{L^2(C)} = 4k_{n,m}^4 \delta_{i,k}\delta_{j,k}, \tag{6.45}$$

$$\frac{1}{1+k_{n,m}^2} \sum_{f_k \in E_{n,m}} a_{i,j,k} = \frac{4k_{n,m}^4}{1+k_{n,m}^2} \delta_{i,j}, \tag{6.46}$$

$$\left(\frac{\partial f_i}{\partial r}, \frac{\partial \varphi_{p,q}}{\partial r}\right)_{L^2(C)} = 2k_{n,m}k_{|p|,q}\delta_{\pm n,p}, \tag{6.47}$$

and

$$\sum_{\varphi_{p,q}\in E_{n,m}^{\perp}} \frac{(1+k_{n,m}^2)}{(1+k_{|p|,q}^2)(k_{n,m}^2 - k_{|p|,q}^2)} \left(\frac{\partial f_i}{\partial r}, \frac{\partial \varphi_{p,q}}{\partial r}\right)_{L^2(C)} \left(\frac{\partial f_j}{\partial r}, \frac{\partial \varphi_{p,q}}{\partial r}\right)_{L^2(C)} = \sum_{q\neq m} \frac{4(1+k_{n,m}^2)k_{n,m}^2 k_{n,q}^2}{(1+k_{n,q}^2)(k_{n,m}^2 - k_{n,q}^2)} \delta_{i,j}. \tag{6.48}$$

Finally, the desired asymptotic expansion (6.38) is immediate from Theorem 5.3 and formulae (6.42), (6.44), (6.46), (6.48). □

References

[1] M. ABRAMOWITZ and I. A. STEGUN (eds.), "Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables", A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York; National Bureau of Standards, Washington, DC, 1984. Reprint of the 1972 edition, Selected Government Publications.

[2] A. ADAMS ROBERT and J. J. F. FOURNIER, "Sobolev Spaces", Pure and Applied Mathematics (Amsterdam), Vol. 140, Elsevier/Academic Press, Amsterdam, second edition, 2003.

[3] I. AGBANUSI, *Pseudo-differential Operators, Transmission Problems and the Large Coupling Limit*, ArXiv e-prints, September 2015.

[4] W. ARENDT and R. MAZZEO, *Friedlander's eigenvalue inequalities and the Dirichlet-to-Neumann semigroup*, Commun. Pure Appl. Anal. **11** (2012), 2201–2212.

[5] W. ARENDT, A. F. M. TER ELST, J. B. KENNEDY and M. SAUTER, *The Dirichlet-to-Neumann operator via hidden compactness*, J. Funct. Anal. **266** (2014), 1757–1786.

[6] G. AUCHMUTY, *Steklov eigenproblems and the representation of solutions of elliptic boundary value problems*, Numer. Funct. Anal. Optim. **25** (2004), 321–348.

[7] A. BEN AMOR, *Sobolev-Orlicz inequalities, ultracontractivity and spectra of time changed Dirichlet forms*, Math. Z. **255** (2007), 627–647.

[8] A. BEN AMOR and J. F. BRASCHE, *Sharp estimates for large coupling convergence with applications to Dirichlet operators*, J. Funct. Anal. **254** (2008), 454–475.

[9] H. BELHADJALI, A. BEN AMOR and J. F. BRASCHE, *Large coupling convergence: Overview and new results*, In: "Partial Differential Equations and Spectral Theory", M. Demuth, B.-W. Schulze and I. Witt (eds.), Operator Theory: Advances and Applications, Vol. 211, Springer Basel, 2011, 73–117.

[10] H. BELHADJALI, A. B. AMOR and J. F. BRASCHE, *On trace and Hilbert-Schmidt norm estimates*, Bull. London Math. Soc. **44** (2012), 661–674.

[11] H. BELHADJALI, A. BENAMOR and J. F. BRASCHE, *Large coupling convergence with negative perturbations*, J. Math. Anal. Appl. **409** (2014), 582–597.
[12] V. BRUNEAU and G. CARBOU, *Spectral asymptotic in the large coupling limit*, Asymptot. Anal. **29** (2002), 91–113.
[13] J. BRASCHE and M. DEMUTH, *Dynkin's formula and large coupling convergence*, J. Funct. Anal. **219** (2005), 34–69.
[14] Z.-Q. CHEN and M. FUKUSHIMA "Symmetric Markov Processes, Time Change, and Boundary Theory", London Mathematical Society Monographs Series, Vol. 35, Princeton University Press, Princeton, NJ, 2012.
[15] R. COURANT and D. HILBERT, "Methods of Mathematical Physics", Vol. I, Interscience Publishers, Inc., New York, N.Y., 1953.
[16] D. DANERS, *Non-positivity of the semigroup generated by the Dirichlet-to-Neumann operator*, Positivity **18** (2014), 235–256.
[17] M. FUKUSHIMA, Y. OSHIMA and M. TAKEDA, "Dirichlet Forms and Symmetric Markov Processes", de Gruyter Studies in Mathematics, Vol. 19, Walter de Gruyter & Co., Berlin, extended edition, 2011.
[18] D. GILBARG and N. S. TRUDINGER, "Elliptic Partial Differential Equations of Second Order", Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.
[19] A. GRIGOR'YAN and S.-T. YAU, *Isoperimetric properties of higher eigenvalues of elliptic operators*, Amer. J. Math. **125** (2003), 893–940.
[20] D. D. HAROSKE and H. TRIEBEL "Distributions, Sobolev Spaces, Elliptic Equations", EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2008.
[21] T. KATO, "Perturbation Theory for Linear Operators", Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.
[22] C. K. QU and R. WONG, *Best possible" upper and lower bounds for the zeros of the Bessel function* $J_\nu(x)$, Trans. Amer. Math. Soc. **351** (1999), 2833–2859.
[23] M. REED and B. SIMON, "Methods of Modern Mathematical Physics. I", Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, second edition, 1980. Functional analysis.

Department of Mathematics
Faculty of Sciences of Gabès
Uni. Gabès, Tunisia
fatenbelgacem1@hotmail.com
ali.benamor@ipeit.rnu.tn
maino88@live.com

Department of Mathematics
I.P.E.I.N.
Uni. Carthage, Tunisia
hichem.belhadjali@ipein.rnu.tn