Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XVIII (2018), 537-549

A complete Riemannian manifold whose isoperimetric profile is discontinuous

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Abstract. The first known example of a complete Riemannian manifold whose isoperimetric profile is discontinuous is given.

Mathematics Subject Classification (2010): 53C20 (primary); 49Q20 (secondary).

1. Introduction

1.1. The problem

Let M be a Riemannian manifold. Given $0 < v < \mathrm{vol}(M)$, consider all domains, *i.e.* smooth compact codimensional 0 submanifolds in M with volume v. Define $I_M(v)$ as the greatest lower bound of the boundary areas of such domains. In this way, one gets a function $I_M : (0, \mathrm{vol}(M)) \to \mathbf{R}_+$ called the *isoperimetric profile* of M.

Question 1.1. When is the isoperimetric profile a continuous function?

The answer is affirmative when M is compact [7, Lemma 6.2]. S. Gallot's proof uses techniques of metric geometry. In the compact case alternative proofs, based on the direct method of the calculus of variations, can be found in books like [1,11,12]. The finite volume case is similar, see [13, Corollary 2.4].

There are positive results for special classes of Riemannian manifolds: homogeneous spaces [9, Lemma 3, Theorem 6], complete Riemannian manifolds possessing a strictly convex Lipschitz continuous exhaustion function, [15] (Hadamard manifolds and complete non-compact manifolds with strictly positive sectional curvature belong to the latter class), unbounded convex Euclidean domains [10]. For more informations about the literature on the continuity of the isoperimetric profile, the reader should consult the introductions of [15] and [10], and references therein.

A general belief is that the answer should again be affirmative under bounded geometry assumptions. The case of complete manifolds with $C^{2,\alpha}$-locally asymp-

P. Pansu is supported by MAnET Maria Curie Initial Training Network, and by Agence Nationale de la Recherche, ANR-10-BLAN 116-01 GGAA.

Received January 13, 2016; accepted in revised form December 12, 2016.
Published online April 2018.

totic bounded geometry is covered by [6, Theorem 1] and [5, Theorem 2.2]. To some extent, their argument mimics the compact case.

If one assumes existence of isoperimetric regions of every volume, one can weaken bounded geometry assumptions. It suffices to assume a lower bound on the Ricci curvature and on the volumes of balls of radius 1, see [6, Theorem 4.1]. In our opinion, it remains an open question whether the noncollapsing assumption that is a lower bound on the volumes of balls can be removed or not (see Question 1.4 below).

An example of a manifold with density with discontinuous isoperimetric profile has been described by Adams, Morgan and Nardulli in [2, Proposition 2]. This has triggered our interest in this question.

1.2. The result

Theorem 1.2. *There exists a complete connected noncompact* 3*-dimensional Riemannian manifold M such that I_M is a discontinuous function.*

The proof is a modification of the treatment of Riemannian manifolds with density by Adams, Morgan and Nardulli, an account of which can be found in Frank Morgan's blog [2].

In our example, the isoperimetric profile fails to be lower semi-continuous. Here is a typical way of proving lower semi-continuity: given an almost minimizing sequence of domains Ω_j, *i.e.*, with $\mathrm{vol}(\Omega_j)$ decreasing to 1 and area equal to $I_M(\mathrm{vol}(\Omega_j)) + \nu_j$ satisfying the condition $\nu_j \to 0$, modify Ω_j locally in order to decrease volume substantially without increasing area too much. This can be done when Ω_j intersects parts of the manifold where ambient geometry stays bounded. In our construction, we arrange so that almost minimizing sequences escape to infinity, encountering higher and higher curvatures and lower and lower injectivity radii.

Start with a disjoint union of compact Riemannian manifolds $N = \coprod_n M_n$ such that $\mathrm{vol}(M_n) = 1 + \tau_n$ where $\tau_n > 0$ tends to 0. Then $I_N(1 + \tau_n) = 0$. Assume that, for all n, $I_{M_n}(1) = I_{M_n}(\tau_n) \geq 1$. Then it is not too hard to show that $I_N(1) \geq 1$. Connecting each M_n to M_{n+1} with a very thin tube produces a connected Riemannian manifold M for which $I_M(1 + \alpha_n)$ tends to 0, where α_n is another sequence tending to 0. Again, it is not too hard to show that $I_M(1) > 0$. Therefore I_M is discontinuous.

Thus the key input is the sequence of Riemannian manifolds M_n with $\mathrm{vol}(M_n)$ bounded and $I_{M_n}(\tau_n)$ bounded below. Adams, Morgan and Nardulli indulged themselves in introducing densities. They took for M_n a tiny round sphere with a high constant density. Since volumes and boundary areas rescale differently, one can achieve $I_{M_n}(\epsilon_n) \geq 1$. Instead, we use nilmanifolds equipped with metrics which converge (up to rescaling) to a single Carnot-Carathéodory metric. The Carnot-Carathéodory isoperimetric inequality established in [14] gives a uniform lower bound for the isoperimetric profiles of such metrics.

Question 1.3. Does there exist a 2-dimensional Riemannian manifold whose isoperimetric profile is discontinuous?

In [10], it is proven that, for unbounded convex Euclidean domains, the (relative) isoperimetric profile is either identically zero, or positive and continuous, the latter case arising if and only if the volumes of unit balls are bounded below. This suggests that non-collapsing might be needed merely to garantee existence of isoperimetric minimizers, and raises the following question.

Question 1.4. Does a manifold with Ricci curvature bounded below and admitting isoperimetric regions in every volume, have a continuous isoperimetric profile?

ACKNOWLEDGEMENTS. We thank the referee for numerous helpful suggestions.

2. Isoperimetry in nilmanifolds

2.1. Isoperimetry in the Heisenberg group

The Heisenberg group $\mathbf{H}$ is the group of real upper triangular unipotent 3×3 matrices,

$$\mathbf{H} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \; ; \; x,\; y,\; z \in \mathbf{R} \right\}.$$

Putting integer entries produces the discrete subgroup $\mathbf{H_Z} \subset \mathbf{H}$. Let $dx, dy, \theta = dz - xdy$ be a basis of left-invariant forms. Let

$$g_\epsilon = dx^2 + dy^2 + \frac{1}{\epsilon^2}\theta^2.$$

Our notation differs from [4, page 25], in the sense that our g_ϵ coincides with their g_L with $L = 1/\epsilon^2$. This is a left-invariant Riemannian metric on $\mathbf{H}$. As ϵ tends to 0, the distance d_ϵ associated to g_ϵ converges to the *Carnot-Carathéodory distance*

$$d_c(p,q) = \inf \left\{ \text{length}(\gamma) \; ; \; \gamma(0) = p,\; \gamma(1) = q,\; \gamma^*\theta = 0 \right\}.$$

The volume element of g_ϵ is $\frac{1}{\epsilon} dx \wedge dy \wedge \theta$. Next we investigate perimeters. Let $(X_1 = \frac{\partial}{\partial x}, X_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, X_3 = \frac{\partial}{\partial z})$ denote the basis of left-invariant vector fields dual to (dx, dy, θ). Let $\mathcal{F}$ denote the space of pairs of smooth functions $\phi : \mathbf{H} \to \mathbf{R}^2$ having compact support and whose Euclidean norm satisfies $|\phi| \leq 1$ pointwise. In [4, page 96], the (horizontal) *perimeter* of a subset $E \subset \mathbf{H}$ is defined by

$$P_{\mathbf{H}}(E) = \sup_{\phi \in \mathcal{F}} \int_E (X_1\phi_1 + X_2\phi_2)\, dx \wedge dy \wedge \theta.$$

Assume that E has a smooth boundary. Since $X_1\phi_1 + X_2\phi_2$ is the divergence of the vector field $\phi_1 X_1 + \phi_2 X_2$ (independently of the choice of ϵ), an integration by parts gives

$$\int_E (X_1\phi_1 + X_2\phi_2)\, dx \wedge dy \wedge \theta = \epsilon \int_{\partial E} \langle \phi_1 X_1 + \phi_2 X_2, n_\epsilon \rangle_\epsilon \, d\mathrm{area}_\epsilon,$$

where n_ϵ is the unit outward normal and $d\mathrm{area}_\epsilon$ denotes Riemannian area relative to the Riemannian metric g_ϵ. Therefore

$$P_{\mathbf{H}}(E) = \int_{\partial E} \epsilon |n_\epsilon^h|_\epsilon \, d\mathrm{area}_\epsilon,$$

where n_ϵ^h is the *horizontal projection* of n_ϵ, *i.e.*, n_ϵ^h is the orthogonal projection of n_ϵ onto the horizontal distribution $\mathrm{span}(X_1, X_2)$ generated by the vector fields X_1, X_2 with respect to the metric g_ε. Note that $|n_\epsilon^h|_\epsilon \leq 1$, so

$$P_{\mathbf{H}}(E) \leq \epsilon\, \mathrm{area}_\epsilon(\partial E). \tag{2.1}$$

As ϵ tends to 0, the *vertical component* $n_\varepsilon^{vert} := n_\varepsilon - n_\epsilon^h$ of n_ϵ with respect to g_ε tends to 0, therefore $|n_\epsilon^h|$ converges uniformly on ∂E, and

$$P_{\mathbf{H}}(E) = \lim_{\epsilon \to 0} \epsilon\, \mathrm{area}_\epsilon(\partial E), \tag{2.2}$$

compare with [4, page 99]. It turns out that, for smooth domains, perimeter coincides with Hausdorff 3-dimensional measure of boundary. By convention, we define the Heisenberg volume element as $V_{\mathbf{H}} = dx \wedge dy \wedge \theta$.

The Heisenberg isoperimetric inequality [14] states that for all smooth domains $\Omega \subset \mathbf{H}$,

$$P_{\mathbf{H}}(\Omega) \geq \left(\frac{\pi}{12}\right)^{\frac{1}{4}} V_{\mathbf{H}}(\Omega)^{3/4} \tag{2.3}$$

(the unsharp numerical constant is irrelevant here).

With inequality (2.1), the Heisenberg isoperimetric inequality (2.3) implies a lower bound on the isoperimetric profile of $(\mathbf{H}, g_\epsilon)$ for all $\epsilon > 0$:

$$I_{(\mathbf{H}, g_\epsilon)}(v) \geq \left(\frac{\pi}{12}\right)^{\frac{1}{4}} \frac{v^{\frac{3}{4}}}{\epsilon^{1/4}}. \tag{2.4}$$

This is asymptotically sharp for large volumes, but not for small volumes, where the correct asymptotics is $v^{2/3}$. However it is the dependency on ϵ which is most important here.

We shall not directly use inequality (2.4). Instead, we shall rely on inequality (2.3) to study the Carnot-Carathéodory isoperimetric profile of a quotient of $\mathbf{H}$. Only at the very end we shall return to Riemannian geometry, using inequality (2.1).

2.2. Nilmanifolds

H possesses group automorphisms $\delta_t(x, y, z) = (tx, ty, t^2z)$. Let $\Gamma_t = \delta_t(\mathbf{H_Z})$, and let $N_t = \Gamma_t \setminus \mathbf{H}$ be the quotient manifold. It inherits quotient metrics g_ϵ, yielding Riemannian nilmanifolds $N_{t,\epsilon}$ of total volume equal to $\frac{t^4}{\epsilon}$. But it also inherits a Carnot-Carathéodory metric that depends only on t. Our first goal is to show that the Carnot-Carathéodory isoperimetric profile of N_t satisfies an inequality similar to (2.3). Note that δ_t induces a homothetic map of N_1 onto N_t, volumes $V_\mathbf{H}$ are multiplied by t^4 and perimeters $P_\mathbf{H}$ by t^3 (see for instance [4, pages 22 and 96]), so it suffices to work with one single compact space N_1. The volume of N_1 is $V_\mathbf{H}(N_1) = 1$.

Theorem 2.1. *There exists a constant c such that the Carnot-Carathéodory isoperimetric profile of* N_1 *satisfies* $I_{(N_1,d_c)}(v) \geq c \min\{v, 1 - v\}^{3/4}$. *In other words, if* $\Omega \subset N_1$ *is a smooth domain of volume less that* $1/2$, *then*

$$P_\mathbf{H}(\Omega) \geq c\, V_\mathbf{H}(\Omega)^{3/4}.$$

The method, inspired by [3], consists in cutting domains of N_1 into pieces that lift to covering spaces. Ultimately, pieces lift to **H** where one can apply (2.3). This covers cases where volume is smaller than some universal constant v_0. To treat domains with volume $\geq v_0 > 0$, we apply a compactness result due to [8], which provides a uniform lower bound on the isoperimetric profile on $[v_0, Vol(N_1) - v_0]$.

2.3. Lifting domains piece by piece

Imitating [3], we shall cut domains in N_1 using families of parallel planes. Again, the point is to reduce to domains which are null-homotopic and then to lift to the universal covering, where the isoperimetric inequality is known.

Let us explain Bérard and Meyer's idea in the flat torus $T = \mathbf{Z}^3 \setminus \mathbf{R}^3$. For $t = (t_1, t_2, t_3) \in T$, let

$$G_t = \{p \in T\,;\; x(p) = t_1 \text{ or } y(p) = t_2 \text{ or } z(p) = t_3\}.$$

G_t is the projection to the torus of three perpendicular planes. As t moves, these planes stay parallel to themselves. Let Ω be a domain in T. The coarea formula shows that

$$3 \cdot \text{volume}(\Omega) = \int_T \text{area}(\Omega \cap G_t)\, dt.$$

One can pick t such that $\text{area}(\Omega \cap G_t) \leq 3 \cdot \text{volume}(\Omega)$. Then $\Omega \setminus G_t$ lifts to a Euclidean domain Ω' whose boundary is not too much larger than that of Ω,

$$\text{area}(\partial\Omega') \leq \text{area}(\partial\Omega) + 6 \cdot \text{volume}(\Omega).$$

Note that when volume(Ω) is small, it is much smaller than area$(\partial\Omega')$, which is at least volume$(\Omega)^{2/3}$, according to Euclidean isoperimetric inequality. Therefore

Bérard-Meyer's construction provides an isoperimetric inequality valid in T, for domains of small volume.

Due to the anisotropic character of Carnot-Carathéodory geometry, there are two different kinds of planes:

1. Vertical planes, containing lines parallel to the z axis, defined by linear equations in x and y only;
2. Horizontal planes, *i.e.* level sets of the z coordinate.

These families satisfy different coarea formulae, therefore we shall proceed in two steps.

2.4. Reduction to pillars

A first step is to cut domains into pieces called *pillars* that lift to a $\mathbf{Z} \oplus \mathbf{Z}$ covering space Z of N_1.

Definition 2.2. Let ζ denote the center of $\mathbf{H}_{\mathbf{Z}}$ and $[\mathbf{H}, \mathbf{H}]$ the subgroup of commutators of $\mathbf{H}$. Let us call *pillar* a subset of $Z = \zeta \setminus \mathbf{H}$ whose projection to $[\mathbf{H}, \mathbf{H}] \setminus \mathbf{H} = \mathbf{R}^2$ is contained in a unit square. Denote by PI_Z the *pillar profile of* Z, *i.e.*

$$PI_Z(v) = \inf\{P_{\mathbf{H}}(P)\,;\ P \text{ is a pillar, } V_{\mathbf{H}}(P) = v\}.$$

Proposition 2.3 (Reduction to pillars). *The pillar profile of Z bounds the profile of N_1 from below, with an error term:*

$$I_{(N_1,d_c)}(v) \geq PI_Z(v) - 4v.$$

Proof. The coordinate functions x and y on $\mathbf{H}$ pass to the quotient $N_1 \to \mathbf{Z} \setminus \mathbf{R}$. For $u = (s, s') \in (\mathbf{Z} \setminus \mathbf{R})^2$, let

$$G_u = \left\{p \in N_1\,;\ x(p) = s \text{ or } y(p) = s'\right\}.$$

This is the union of two surfaces, each of which is a level set of one of the functions x or y. The complement of G_u has a cyclic fundamental group that maps isomorphically onto ζ.

Let Ω be a domain in N_1. By the coarea formula,

$$V_{\mathbf{H}}(\Omega) = \int_{\mathbf{Z}\setminus\mathbf{R}} P_{\mathbf{H}}(x^{-1}(s) \cap \Omega)\, ds.$$

This coarea formula follows from the fact that the volume element is a 3-form and splits as

$$d\,V_{\mathbf{H}} = dx \wedge dy \wedge \theta = dx \wedge d\,P_{\mathbf{H}},$$

since $dy \wedge \theta = d\,P_{\mathbf{H}}$ along the fibers of x, see Lemma 2.4 below.

The same inequality holds with x replaced with y. This shows that there exists $u = (s, s') \in (\mathbf{Z} \setminus \mathbf{R})^2$ such that

$$P_{\mathbf{H}}\left(x^{-1}(s) \cap \Omega\right) \leq V_{\mathbf{H}}(\Omega), \quad P_{\mathbf{H}}\left(y^{-1}(s') \cap \Omega\right) \leq V_{\mathbf{H}}(\Omega),$$

and thus

$$P_{\mathbf{H}}(G_u \cap \Omega) \leq 2 \cdot V_{\mathbf{H}}(\Omega).$$

The complement $\Omega \setminus G_u$ lifts to the covering space Z. Indeed, it is homotopic to the circle $\{(0, 0, z)\,;\ z \in \mathbf{Z} \setminus \mathbf{R}\}$. Pick some lift. Its closure P is a pillar. Indeed, on P, the real-valued functions x and y take values in intervals of length 1. The boundary of P consists of a part that isometrically and injectively maps to $\partial\Omega$, and of a part that maps two-to-one to $G_u \cap \Omega$. Therefore

$$P_{\mathbf{H}}(\partial P) \leq P_{\mathbf{H}}(\partial\Omega) + 2 \cdot P_{\mathbf{H}}(G_u \cap \Omega) \leq P_{\mathbf{H}}(\partial\Omega) + 4 \cdot V_{\mathbf{H}}(\Omega).$$

If $V_{\mathbf{H}}(\Omega) = v$, this shows that

$$I_{(N_1, d_c)}(v) \geq PI_Z(v) - 4v. \qquad \square$$

Lemma 2.4. *Let F be the vertical plane $\{x = 0\}$ in $\mathbf{H}$. Then the perimeter measure on F is $dy \wedge \theta$.*

Proof. The Riemannian normal is $n_\epsilon = X_1$, it is horizontal and does not depend on ϵ. Its horizontal projection is $n_\epsilon^h = X_1$, whose norm is 1. Since dy and θ are orthogonal, $|dy|_\epsilon = 1$, and $|\theta|_\epsilon = \epsilon$, then the Riemannian area element is $d\text{area}_\epsilon = dy \wedge \frac{1}{\epsilon}\theta$, and the perimeter measure is $d\mathcal{P}_{\mathbf{H}} = \epsilon |n_\epsilon^h|_\epsilon d\text{area}_\epsilon = dy \wedge \theta$. $\square$

2.5. Treatment of pillars

Proposition 2.5 (Treatment of pillars). *The profile of $\mathbf{H}$ bounds the pillar profile of Z from below, with an error term:*

$$PI_Z(v) \geq I_{\mathbf{H}}(v) - 2v.$$

Proof. Let $P \subset Z$ be a pillar. We can assume that its projection to $\mathbf{R}^2$ is contained in $\{0 \leq x \leq 1\}$. Its inverse image $\tilde{P}$ in $\mathbf{H}$ is a ζ-invariant subset with small projection in $\mathbf{R}^2$. Again, we cut $\tilde{P}$ into logs of height 1 using level sets of the z function. This time, we split the volume element as

$$dV_{\mathbf{H}} = dx \wedge dy \wedge dz = dz \wedge (dx \wedge dy) = dz \wedge \frac{1}{|x|} d\,P_{\mathbf{H}} \geq dz \wedge d\,P_{\mathbf{H}}.$$

We have used the expression $d\,P_{\mathbf{H}} = |x|\,dx\,dy$ for the measure induced on horizontal planes $\{z = s\}$, see Lemma 2.6 below.

The coarea formula gives

$$\begin{aligned} V_{\mathbf{H}}(P) &= V_{\mathbf{H}}(\tilde{P} \cap \{0 \le z \le 1\}) \\ &= \int_0^1 \left(\int_{\tilde{P} \cap \{z=s\}} \frac{1}{|x|} d\, P_{\mathbf{H}} \right) ds \\ &\ge \int_0^1 P_{\mathbf{H}}(\tilde{P} \cap \{z = s\})\, ds. \end{aligned}$$

There exists $s \in [0, 1]$ such that

$$P_{\mathbf{H}}(\tilde{P} \cap \{z = s\}) \le V_{\mathbf{H}}(P).$$

Set $\Omega' = \tilde{P} \cap \{s \le z \le s + 1\}$. Then

$$P_{\mathbf{H}}(\partial \Omega') \le P_{\mathbf{H}}(\partial P) + 2 \cdot V_{\mathbf{H}}(P).$$

If P has volume v, this leads to

$$P I_Z(v) \ge I_{\mathbf{H}}(v) - 2v. \qquad \square$$

Lemma 2.6. *Let H be the horizontal plane $\{z = 0\}$ in $\mathbf{H}$. Then the perimeter measure on H is $|x| dx \wedge dy$.*

Proof. Use the parametrization $(x, y) \mapsto (x, y, 0)$. The vectors

$$\frac{\partial}{\partial x} = X_1 \quad \text{and} \quad \frac{\partial}{\partial y} = X_2 - x X_3$$

are tangent to H. Their cross-product

$$X_1 \times (X_2 - x X_3) = \epsilon X_3 + \frac{x}{\epsilon} X_2$$

is normal. Its norm equals

$$\left| \epsilon X_3 + \frac{x}{\epsilon} X_2 \right|_{\epsilon} = \sqrt{1 + \frac{x^2}{\epsilon^2}}.$$

The Riemannian area element is

$$d\text{area}_{\epsilon} = \sqrt{1 + \frac{x^2}{\epsilon^2}} dx \wedge dy.$$

The unit normal is

$$n_{\epsilon} = \frac{1}{\sqrt{1 + \frac{x^2}{\epsilon^2}}} \left(\epsilon X_3 + \frac{x}{\epsilon} X_2 \right).$$

Its horizontal projection is

$$n_\epsilon^h = \frac{1}{\sqrt{1+\frac{x^2}{\epsilon^2}}}\frac{|x|}{\epsilon}X_2.$$

Therefore, the perimeter measure is

$$\begin{aligned} d\mathcal{P}_{\mathbf{H}} &= \epsilon|n_\epsilon^h|_\epsilon d\mathrm{area}_\epsilon \\ &= |x|\frac{1}{\sqrt{1+\frac{x^2}{\epsilon^2}}}\sqrt{1+\frac{x^2}{\epsilon^2}}dx\wedge dy \\ &= |x|dx\wedge dy. \end{aligned}$$

□

2.6. Profile of (N_1, d_c)

Proposition 2.7. (Carnot-Carathéodory isoperimetric inequality for small volumes). *If $v \le v_0 := (12)^{-5}\pi$, then*

$$I_{(N_1,d_c)}(v) \ge \frac{\tilde{c}}{2}v^{3/4},$$

where $\tilde{c} := \left(\frac{\pi}{12}\right)^{\frac{1}{4}}$ is the non-sharp isoperimetric constant for the Carnot-Carathéodory metric appearing in (2.3).

Proof. Combined with Propositions 2.3 and 2.5, the Heisenberg isoperimetric inequality (2.3) yields

$$I_{(N_1,d_c)}(v) \ge \tilde{c}v^{3/4} - 4v - 2v = v^{3/4}(\tilde{c} - 6\,v^{1/4}) \ge \frac{\tilde{c}}{2}v^{3/4},$$

since $v \le v_0 = 12^{-5}\pi$. □

2.7. Proof of Theorem 2.1

There is a notion of Carnot-Carathéodory perimeter, an appropriate topology, namely the L^1_{loc} convergence of the characteristic functions for which $V_{\mathbf{H}}$ is continuous and the perimeter (which coincides with $P_{\mathbf{H}}$ for smooth domains) lower semi-continuous, and a compactness theorem for sets of bounded perimeter in a compact Carnot manifold, due to Garofalo and Nhieu in [8, Theorem 1.28]. This implies that the Carnot-Carathéodory isoperimetric profile $I_{(N_1,d_c)}$ is positive on $(0,1)$ and lower semi-continuous. Therefore, there exists $\eta > 0$ such that $I_{(N_1,d_c)} \ge \eta$ on $[v_0, 1-v_0]$. Set $c = \min\left\{\frac{1}{2}, 2^{3/4}\eta, \frac{\tilde{c}}{2}\right\}$. Then $I_{(N_1,d_c)}(v) \ge \eta = c(\frac{1}{2})^{3/4} \ge c\,v^{3/4}$ for every $v \in \left[v_0, \frac{1}{2}\right]$. On the other hand, Proposition 2.7 shows that $I_{(N_1,d_c)}(v) \ge c\,v^{3/4}$ for all $v \in [0, v_0]$. □

Note that the proof does not provide an effective constant c.

2.8. Riemannian profile

Corollary 2.8. *Let $N_{t,\epsilon}$ denote the quotient $\delta_t(\mathbf{H_Z}) \setminus \mathbf{H}$ equipped with the Riemannian metric induced by g_ϵ. The isoperimetric profile of $N_{t,\epsilon}$ satisfies*

$$I_{N_{t,\epsilon}}(v) \geq \frac{c}{\epsilon^{1/4}} \min\left\{v, \frac{t^4}{\epsilon} - v\right\}^{3/4}.$$

Proof. The homothetic map $N_1 \to N_t$ induced by the automorphism δ_t transports the inequality of Theorem 2.1 to N_t without any change but the fact that $V_{\mathbf{H}}(N_t) = t^4$ replaces 1. The Riemannian volume element of $N_{t,\epsilon}$ is $\frac{1}{\epsilon}V_{\mathbf{H}}$, the Riemannian area induced on surfaces satisfies ϵ area $\geq P_{\mathbf{H}}$ by Equation (2.1). This leads to the indicated dependence on ϵ in the isoperimetric profile of $N_{t,\epsilon}$. □

3. Proof of Theorem 1.2

In this section, complete manifolds are constructed by piecing together compact nilmanifolds like beads. As a warm up, we start with a disjoint union, where the mechanism is more visible. A slight modification will provide a connected example.

3.1. The case of a disjoint union of nilmanifolds

Proposition 3.1. *Let $\tau_n = \frac{1}{n}$, $\epsilon_n = \tau_n^3$ and $t_n = \tau_n^{3/4}(1+\tau_n)^{1/4}$. Let $N = \coprod_n N_{t_n,\epsilon_n}$. Then, for all $v \in [\frac{1}{16}, 1]$, $I_N(v) \geq \frac{c}{8}$, where c is the constant of Theorem* 2.1.

Proof. By construction, $\mathrm{vol}(N_{t_n,\epsilon_n}) = 1 + \tau_n$. Let Ω be a domain in N with $\mathrm{vol}(\Omega) = v$. Write $\Omega = \coprod_n \Omega_n$ where $\Omega_n \subset N_{t_n,\epsilon_n}$ has volume v_n, $\sum_{n=1}^{\infty} v_n = v$.

If some v_n satisfies $v_n \geq \frac{1}{2}(1+\tau_n)$, then

$$\begin{aligned}\mathrm{area}(\partial\Omega_n) &\geq \frac{c}{\epsilon_n^{1/4}}(1+\tau_n - v_n)^{3/4}\\ &\geq \frac{c}{\epsilon_n^{1/4}}\tau_n^{3/4} = c,\end{aligned}$$

the last inequality coming from the fact that $v_n \leq v \in]0,1]$, so that

$$\mathrm{area}(\partial\Omega) \geq c, \tag{3.1}$$

in this case.

Otherwise, for all $n \geq 1$,

$$\mathrm{area}(\partial\Omega_n) \geq \frac{c}{\epsilon_n^{1/4}} v_n^{3/4} \geq c\, v_n^{3/4}.$$

We use the concavity inequality

$$a^\alpha + b^\alpha \geq (a+b)^\alpha,$$

valid for all $0 \leq \alpha \leq 1, a \geq 0$ and $b \geq 0$. This gives

$$\begin{aligned}\operatorname{area}(\partial\Omega) &= \sum_{n=1}^{\infty} \operatorname{area}(\partial\Omega_n) \\ &\geq c \sum_{n=1}^{\infty} v_n^{3/4} \\ &\geq c \left(\sum_{n=1}^{\infty} v_n\right)^{3/4} \geq \left(\frac{1}{16}\right)^{3/4} c = \frac{c}{8}. \qquad \square\end{aligned}$$

3.2. Connecting manifolds

Proof of Theorem 1.2. We construct a noncompact manifold that has the shape of an infinite pearl necklace, adjusting suitable parameters carefully. Let $0 < \tau_n < 1$ be the sequence of positive real numbers chosen in the proof of Proposition 3.1. Pick another sequence of volumes $w_n < 1$, such that

$$\sum_n w_n < \frac{1}{2}, \tag{3.2}$$

and a sequence of areas $a_n > 0$ such that

$$\sum_n a_n < \frac{c}{16}, \tag{3.3}$$

where c is the constant of Theorem 2.1.

The manifolds N_{t_n,ϵ_n} we want to connect to obtain our counterexample M, are like in Proposition 3.1, in particular we retain here that $V(N_{t_n,\epsilon_n}) = 1 + \tau_n$, where V is the Riemannian measure associated to g. Take two small disjoint balls $B_{n,1}$, $B_{n,2}$ inside N_{t_n,ϵ_n} whose boundaries have total area $\leq a_n$, except that for $n = 1$, $B_{1,1} := \emptyset$. Arrange that $B_{n,2}$ and $B_{n+1,1}$ be nearly isometric with the same volume $\tilde{v}_n' = V(B_{n,2}) = V(B_{n+1,1})$. Put $\tilde{N}_n := N_{t_n,\epsilon_n} \setminus \big(B_{n,1} \mathring{\cup} B_{n,2}\big)$, $A \mathring{\cup} B$ denotes set theoretical disjoint union for any pair of sets A, B such that $A \cap B = \emptyset$.

Consider tubes or cylinders T_n of the form $T_n := (S^2(1) \times [0,1], g_n)$, where the metrics g_n are chosen in such a way that $V(g_n) \leq w_n$ and they glue together into a smooth metric on the connected sum $M_n := \tilde{N}_n \# T_n$ where the gluing is done along $i_n(S^2(1) \times \{0\}) \cong \partial B_{n,2}$. Now consider

$$(M, g) := M_1 \# M_2 \# \cdots \# M_n \# M_{n+1} \# \cdots \tag{3.4}$$

where M_n and M_{n+1} are glued together along the boundaries $i_n(S^2(1) \times \{1\}) \cong \partial B_{(n+1),1}$, where $i_n : T_n \to M$ is the isometric embedding associated to our construction.

Consider domains $D_n := \tilde{N}_n$, we get $V(D_n) = 1 + \tau_n - \tilde{v}'_{n-1} - \tilde{v}'_n = 1 + \alpha_n$, with $\alpha_n \to 0$, $\varepsilon'_n := A(\partial D_n) = A_g(\partial B_{n,2} \mathring{\cup} \partial B_{n+1,1}) \to 0$, where A is the 2-dimensional Hausdorff measure with respect to the metric induced by g. This implies readily

$$0 \leq \lim_{n \to +\infty} I_M(1 + \alpha_n) \leq \lim_{n \to +\infty} A(\partial D_n) = 0. \tag{3.5}$$

We show that $I_M(1) > 0$. Let Ω be a domain in M such that $V(\Omega) = 1$. Write $\tilde{\Omega} := \mathring{\bigcup} \tilde{\Omega}_n$, where $\tilde{\Omega}_n := \Omega \cap \tilde{N}_n$. Then

$$V(\tilde{\Omega}) \geq 1 - \sum_n w_n \geq \frac{1}{2}.$$

According to Proposition 3.1,

$$A(\partial \tilde{\Omega}) \geq \frac{c}{8}.$$

We have, for all n,

$$\partial \tilde{\Omega}_n = \left((\partial \Omega) \cap \tilde{N}_n \right) \mathring{\cup} \left(\Omega \cap \partial \tilde{N}_n \right),$$

$$A\left(\partial \tilde{\Omega}_n\right) - A\left((\partial \Omega) \cap \tilde{N}_n\right) \leq A_g\left(\partial B_{n,2} \mathring{\cup} \partial B_{n,1}\right) \leq a_n,$$

thus

$$A(\partial \Omega) \geq A(\partial \tilde{\Omega}) - \sum_n a_n \geq \frac{c}{8} - \frac{c}{16} = \frac{c}{16}.$$

This shows that $I_M(1) \geq \frac{c}{16}$.

This last inequality combined with (3.5) concludes the proof of Theorem 1.2. $\square$

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