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On parametric extensions over number fields

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Abstract. Given a number field F, a finite group G and an indeterminate T, *a G-parametric extension over F* is a finite Galois extension $E/F(T)$ with Galois group G and E/F regular that has all the Galois extensions of F with Galois group G among its specializations. We are mainly interested in producing non-G-parametric extensions, which relates to classical questions in inverse Galois theory like the Beckmann-Black problem. Building on a strategy developed in previous papers, we show that there exists at least one non-G-parametric extension over F for a given non-trivial finite group G and a given number field F under the sole necessary condition that G occurs as the Galois group of a Galois extension $E/F(T)$ with E/F regular.

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1. Introduction

Given a number field F, the *inverse Galois problem* over F asks whether every finite group G occurs as the Galois group of a Galois extension of F. A classical way to obtain such an extension consists in introducing an indeterminate T and in producing a Galois extension $E/F(T)$ with the same Galois group and E/F *regular*[1]: from the *Hilbert irreducibility theorem*, the extension $E/F(T)$ has infinitely many linearly disjoint *specializations* with Galois group G (if G is not trivial). We refer to Subsection 2.1 for basic terminology.

Following recent works [6, Section 4] [5], in the present paper we are interested in finite Galois extensions $E/F(T)$ with E/F regular – from now on, say for short that the extension $E/F(T)$ is an "F-regular Galois extension" – that have all the Galois extensions of F with Galois group G among their specializations. More precisely, let us recall the following definition.

[1] *i.e.*, $E \cap \overline{\mathbb{Q}} = F$.

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Definition 1.1. A finite F-regular Galois extension $E/F(T)$ with Galois group G is *G-parametric over F* if every Galois extension of F with Galois group G occurs as a specialization of $E/F(T)$.

Parametric extensions have been introduced with the aim of a better understanding of the *Beckmann-Black problem* which asks whether the specialization process is optimal to solve the inverse Galois problem. Namely, recall that the Beckmann-Black problem, for the finite group G over the number field F, asks whether every Galois extension L/F with Galois group G is a specialization of some F-regular Galois extension $E_L/F(T)$, possibly depending on L/F, with Galois group G. Although no counter-example is known and only a few positive results have been proved (see *e.g.* [2, Theorem 2.2] for more details), it may be expected that the Beckmann-Black problem fails in general over number fields. However, no line of attack seems to be known and a disproof at the moment is probably out of reach.

Actually, the answer to the following weaker question on parametric extensions seems to be unavailable in the literature. Let us say that a finite group G is a *regular Galois group* over (the given number field) F if G occurs as the Galois group of an F-regular Galois extension of $F(T)$.

Question 1.2. Does there exist a regular Galois group G over F such that no F-regular Galois extension of $F(T)$ with Galois group G is G-parametric over F?

The existence of such a finite group G would be a first step towards a counter example to the Beckmann-Black problem over the number field F. However, although we may expect the answer to be negative almost always, deciding whether a given F-regular Galois extension of $F(T)$ with Galois group G is G-parametric over F or not is a difficult problem in general (even in the easiest case $G = \mathbb{Z}/2\mathbb{Z}$) and only a few non-parametric extensions are available in the literature. In particular, finding a group G as in Question 1.2 seems to be difficult as well.

In [6, Section 4] and [5], we offer a systematic approach to produce F-regular Galois extensions $E/F(T)$ with Galois group G which are not G-parametric over F. It consists in introducing another F-regular Galois extension $E'/F(T)$ with Galois group G and in giving criteria ensuring that some specializations of $E'/F(T)$ with Galois group G are not specializations of $E/F(T)$. Examples with specific finite groups G such as abelian groups, symmetric and alternating groups, non-abelian simple groups, and so on, are then given, under some natural necessary conditions. For example, an obvious obstruction to the existence of at least one non-G-parametric extension over F is that G is not a regular Galois group over F.

Building on this strategy, in this paper we show that the latter obstruction is the only one to the existence of a non-G-parametric extension over F with Galois group G.

Theorem 1.3. *Let G be a non-trivial finite group and F a number field. Assume that G is a regular Galois group over F. Then there exists at least one non-G-parametric extension over F with Galois group G.*

Actually, from any F-regular Galois extension $E/F(T)$ with Galois group G satisfying some mild assumptions on its set of branch points, we derive a sequence $(E_k/F(T))_k$ of F-regular realizations of G such that infinitely many linearly disjoint specializations of $E/F(T)$ with Galois group G are not specializations of $E_k/F(T)$. See Theorem 3.1.

The paper is organized as follows. In Section 2 we recall some material used in the sequel. In Section 3 we prove Theorem 1.3 under an auxiliary result on *prime divisors of polynomials* (Definition 2.1) that has its own interest; see Proposition 3.7. Proposition 3.7 is proved in Section 4. Finally, in Section 5, we make related previous results from [5] more precise thanks to a group theoretic argument communicated to us by Reiter.

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2. Basics

For this section, let F be a number field.

2.1. Specializations of finite Galois extensions of $F(T)$

Given an indeterminate T, let $E/F(T)$ be a finite Galois extension with Galois group G and E/F *regular* (*i.e.*, $E \cap \overline{\mathbb{Q}} = F$). From now on, say for short that $E/F(T)$ is an *F-regular Galois extension*.

Recall that a point $t_0 \in \mathbb{P}^1(\overline{\mathbb{Q}})$ is *a branch point of* $E/F(T)$ if the prime ideal $(T-t_0)\overline{\mathbb{Q}}[T-t_0]$ [2] ramifies in the integral closure of $\overline{\mathbb{Q}}[T-t_0]$ in the *compositum* of E and $\overline{\mathbb{Q}}(T)$ (in a fixed algebraic closure of $F(T)$). The extension $E/F(T)$ has only finitely many branch points.

Given a point $t_0 \in \mathbb{P}^1(F)$ which is not a branch point, the residue extension of $E/F(T)$ at a prime ideal $\mathcal{P}$ lying over $(T-t_0)F[T-t_0]$ is denoted by E_{t_0}/F and is called *the specialization of* $E/F(T)$ *at* t_0. It does not depend on the choice of the prime $\mathcal{P}$ lying over $(T-t_0)F[T-t_0]$, as $E/F(T)$ is Galois. The extension E_{t_0}/F is Galois, with a subgroup of G as Galois group, namely the decomposition group of $E/F(T)$ at $\mathcal{P}$.

2.2. Prime divisors of polynomials

Denote the integral closure of $\mathbb{Z}$ in F by O_F. Let $P(T) \in O_F[T]$ be a non-constant monic polynomial.

[2] Replace $T-t_0$ by $1/T$ if $t_0 = \infty$.

Definition 2.1. Say that a non-zero prime ideal $\mathcal{P}$ of O_F is *a prime divisor of* $P(T)$ if the reduction of $P(T)$ modulo $\mathcal{P}$ has a root in the residue field $O_F/\mathcal{P}$.

The following lemma will be used on several occasions in the sequel. Denote the roots of $P(T)$ by $t_1, \ldots, t_r$. Given an integer $k \geq 1$ and an index $j \in \{1, \ldots, r\}$, let $\sqrt[k]{t_j}$ be a k-th root of t_j. Finally, let L_k be the splitting field of $P(T^k)$ over F and ζ_k a primitive k-th root of unity.

Lemma 2.2. *The following three conditions are equivalent:*

(1) $\bigcup_{j=1}^{r} \bigcup_{l=0}^{k-1} \mathrm{Gal}(L_k/F(\zeta_k^l \sqrt[k]{t_j})) \neq \bigcup_{j=1}^{r} \mathrm{Gal}(L_k/F(t_j))$;

(2) *There exists a set $\mathcal{S}$ of non-zero prime ideals of O_F that has positive density and such that each prime ideal $\mathcal{P}$ in $\mathcal{S}$ is a prime divisor of $P(T)$ but not of $P(T^k)$;*

(3) *There exist infinitely many non-zero prime ideals of O_F each of which is a prime divisor of $P(T)$ but not of $P(T^k)$.*

Proof. We may assume that $P(T)$ is separable. If $P(0) = 0$, then (1), (2) and (3) fail. From now on, we assume that $P(0) \neq 0$. In particular, $P(T^k)$ is separable.

First, assume that (1) holds, *i.e.*, that there exists some σ in

$$\bigcup_{j=1}^{r} \mathrm{Gal}\big(L_k/F(t_j)\big) \setminus \bigcup_{j=1}^{r} \bigcup_{l=0}^{k-1} \mathrm{Gal}\left(L_k/F\left(\zeta_k^l \sqrt[k]{t_j}\right)\right).$$

By the Tchebotarev density theorem, there exists a positive density set $\mathcal{S}$ of primes $\mathcal{P}$ of O_F such that the associated Frobenius element in L_k/F is conjugate to σ. As σ fixes no root of $P(T^k)$, such a $\mathcal{P}$ is not a prime divisor of $P(T^k)$ (up to finitely many). Denote the splitting field of $P(T)$ over F by L_1. Then the Frobenius element associated with $\mathcal{P}$ in L_1/F is the restriction to L_1 of the one in L_k/F. As σ fixes a root of $P(T)$, $\mathcal{P}$ is a prime divisor of $P(T)$ (up to finitely many), as needed for (2).

As implication (2) $\Rightarrow$ (3) is obvious, it remains to prove implication (3) $\Rightarrow$ (1). To do this, assume that (1) does not hold. Let $\mathcal{P}$ be a non-zero prime ideal of O_F that is a prime divisor of $P(T)$ and that is unramified in L_k/F. Denote the associated Frobenius in L_k/F by σ. As $\mathcal{P}$ is a prime divisor of $P(T)$ and $\mathcal{P}$ does not ramify in L_1/F, the associated Frobenius element in L_1/F fixes a root of $P(T)$ (up to finitely many). Since this Frobenius element is the restriction of σ to L_1, we get that σ fixes a root of $P(T)$. As condition (1) fails, σ fixes a root of $P(T^k)$ as well. Hence $\mathcal{P}$ is a prime divisor of $P(T^k)$ (up to finitely many). Then (3) does not hold either, thus ending the proof. □

3. Proof of Theorem 1.3

The aim of this section consists in proving Theorem 3.1 below, of which Theorem 1.3 is a straightforward application.

3.1. Statement of Theorem 3.1

Let F be a number field, O_F the integral closure of $\mathbb{Z}$ in F and G a non-trivial finite group that is a regular Galois group over F (*i.e.*, G occurs as the Galois group of an F-regular Galois extension of $F(T)$).

Given an indeterminate T, let $E/F(T)$ be an F-regular Galois extension with Galois group G, branch points $t_1, \ldots, t_r$ and such that the following two conditions hold[3]:

- (bp-1) $\{0, 1, \infty\} \cap \{t_1, \ldots, t_r\} = \emptyset$;
- (bp-2) $t_1, \ldots, t_r$ all are integral over O_F.

Theorem 3.1. *There exists a sequence of F-regular Galois extensions $E_k/F(T)$, $k \in \mathbb{N} \setminus \{0\}$ (depending on $E/F(T)$), with Galois group G and that satisfies the following property.*

For each finite extension F'/F, there exist infinitely many positive integers k (depending on F') such that the extension $E_kF'/F'(T)$ satisfies the following condition:

(non-G-parametricity) *There exist infinitely many linearly disjoint Galois extensions of F' with Galois group G each of which is not a specialization of $E_kF'/F'(T)$.*

In particular, the extension $E_kF'/F'(T)$ is not G-parametric over F'. Furthermore, these Galois extensions of F' with Galois group G may be produced by specializing the extension $EF'/F'(T)$.

Remark 3.2. As explained in Subsection 3.2.4 below, we are not able to remove the dependence on the number field F' containing F in the set of all suitable positive integers k. In particular, the proof provides no integer k such that the extension $E_kF'/F'(T)$ satisfies the (non-G-parametricity condition) for each finite extension F'/F. See Proposition 5.1 for a result with such a geometric conclusion.

Corollary 3.3. *Let G be a non-trivial finite group. Then there exists some number field F_G that satisfies the following property. For each number field F' containing F_G, there exists an F'-regular Galois extension of $F'(T)$ with Galois group G which satisfies the* (non-G-parametricity condition). *Moreover, one can take F_G equal to a given number field F if and only if G is a regular Galois group over F.*

Remark 3.4. As a classical consequence of the Riemann existence theorem, every finite group G is a regular Galois group over some number field F_G, and then over every number field F' containing F_G. Hence Theorem 3.1 provides the following statement.

[3] These two conditions hold up to applying a suitable change of variable.

3.2. Proof of Theorem 3.1

We break the proof into four parts.

3.2.1. *Notation*

Given a positive integer k and $j \in \{1, \dots, r\}$, let $\sqrt[k]{t_j}$ be a k-th root of t_j. Let F'/F be a finite extension and $O_{F'}$ the integral closure of $\mathbb{Z}$ in F'.

By condition (bp-1), one may consider the polynomial

$$P_E(T) := \prod_{j=1}^{r} (T - t_j).$$

By condition (bp-2), the monic separable polynomial $P_E(T)$ has coefficients in O_F.

3.2.2. *Two lemmas*

Fix a positive integer k.

First, we derive from the extension $E/F(T)$ an F-regular Galois extension of $F(T)$ with group G and specified set of branch points.

Lemma 3.5. *There exists an F-regular Galois extension of $F(T)$ with Galois group G and whose branch points are exactly the k-th roots of those of $E/F(T)$.*

Proof. The proof below follows part of an argument of Dèbes and Zannier given in the proof of [3, Proposition 5.2]. Let $P(T, Y) \in F[T][Y]$ be the irreducible polynomial over $F(T)$ of some primitive element of E over $F(T)$, assumed to be integral over $F[T]$. The polynomial $P(T, Y)$ is absolutely irreducible, as $E/F(T)$ is F-regular, and, as 0 is not a branch point (condition (bp-1)), it has a root in $\overline{\mathbb{Q}}((T))$. By [1, Lemma 0.1], the polynomial $P_k(T, Y) := P(T^k, Y)$ is absolutely irreducible. Denote the field generated by one root of $P_k(T, Y)$ over $F(T)$ by E_k. The extension $E_k/F(T)$ is F-regular, as $P_k(T, Y)$ is absolutely irreducible, and has degree equal to the order of G. Denote the Galois closure of $E_k/F(T)$ by $\widehat{E_k}/F(T)$ and the Galois group of $\widehat{E_k}/F(T)$ by H_k. By the Hilbert irreducibility theorem, there are infinitely many $t_0 \in F$ such that the specialization $(\widehat{E_k})_{t_0}/F$ of $\widehat{E_k}/F(T)$ at t_0 has Galois group H_k. For all but finitely many $t_0 \in F$, the field $(\widehat{E_k})_{t_0}$ is the splitting field over F of the polynomial $P_k(t_0, Y) = P(t_0^k, Y)$, which is in turn the field $E_{t_0^k}$. Hence there is a specialization of $E/F(T)$ with Galois group H_k. In particular, H_k is a subgroup of G. As the order of G divides the order of H_k, we get $G = H_k$. Hence $E_k/F(T)$ is an F-regular Galois extension with Galois group G. By construction, the branch points of $E_k/F(T)$ lying in $\overline{\mathbb{Q}} \setminus \{0\}$ are the k-th roots of those of $E/F(T)$. As neither 0 nor ∞ is a branch point of $E/F(T)$ (condition (bp-1)), the same is true of $E_k/F(T)$, thus ending the proof. □

Let $E_k/F(T)$ be an F-regular Galois extension with Galois group G and whose branch points are exactly the k-th roots of those of $E/F(T)$.

Next, we apply a previous criterion from [6] for the extension $E_k F'/F'(T)$ to satisfy the (non-G-parametricity condition).

Lemma 3.6. *Assume that there exist infinitely many non-zero prime ideals of $O_{F'}$ each of which is a prime divisor of $P_E(T)$ but not of $P_E(T^k)$ (considered as polynomials with coefficients in F'). Then the extension $E_kF'/F'(T)$ satisfies the* (non-G-parametricity condition). *Moreover, the Galois extensions of F' with Galois group G appearing in the* (non-G-parametricity condition) *may be produced by specializing the extension $EF'/F'(T)$.*

Proof. Given an algebraic number $t \neq 0$, denote the irreducible polynomial of t over F' by $m_t(T)$. Consider the following four polynomials:

$$m_{EF'}(T) = \prod_{j=1}^{r} m_{t_j}(T),$$

$$m^*_{EF'}(T) = \prod_{j=1}^{r} m_{1/t_j}(T),$$

$$m_{E_kF'}(T) = \prod_{j=1}^{r}\prod_{l=0}^{k-1} m_{e^{2i\pi l/k}\sqrt[k]{t_j}}(T),$$

$$m^*_{E_kF'}(T) = \prod_{j=1}^{r}\prod_{l=0}^{k-1} m_{1/(e^{2i\pi l/k}\sqrt[k]{t_j})}(T).$$

By [6, Theorem 4.2] and since the branch points of the extension $E_k/F(T)$ are the k-th roots of those of $E/F(T)$, it suffices to prove that there exist infinitely many non-zero prime ideals of $O_{F'}$ each of which is a prime divisor of $m_{EF'}(T)\cdot m^*_{EF'}(T)$ but not of $m_{E_kF'}(T)\cdot m^*_{E_kF'}(T)$.

As ∞ is not a branch point of $EF'/F'(T)$ (condition (bp-1)), one may apply [6, Remark 3.11] to get that $m_{EF'}(T)\cdot m^*_{EF'}(T)$ and $m_{EF'}(T)$ have the same prime divisors (up to finitely many). Since the polynomials $m_{EF'}(T)$ and $P_E(T)$ have the same prime divisors, we get that $m_{EF'}(T)\cdot m^*_{EF'}(T)$ and $P_E(T)$ have the same prime divisors (up to finitely many). By the same argument, every prime divisor of $m_{E_kF'}(T)\cdot m^*_{E_kF'}(T)$ is a prime divisor of $P_E(T^k)$ (up to finitely many). Then, from the assumption in the statement, there exist infinitely many non-zero prime ideals of $O_{F'}$ each of which is a prime divisor of $m_{EF'}(T)\cdot m^*_{EF'}(T)$ but not of $m_{E_kF'}(T)\cdot m^*_{E_kF'}(T)$, as needed. □

3.2.3. *A number theoretical result*

Now, we need the following number theoretical result to ensure that the assumption of Lemma 3.6 holds.

Proposition 3.7. *Given a monic separable polynomial $P(T) \in O_F[T]$ such that $P(0) \neq 0$ and $P(1) \neq 0$, there is an infinite set S of integers $k \geq 1$ such that, for each $k \in S$, there are infinitely many prime ideals of O_F each of which is a prime divisor of $P(T)$ but not of $P(T^k)$.*

Remark 3.8. (1) If either 0 or 1 is a root of $P(T)$, then the conclusion of Proposition 3.7 clearly fails.

(2) The set S depends on the polynomial $P(T)$ and this dependence cannot be removed. Indeed, given an integer $k \geq 1$, all non-zero prime ideals of O_F are prime divisors of $P(T)$ and $P(T^k)$ if $P(T) = T - 2^k$.

(3) Similarly, the set S depends on the number field F and this dependence cannot be removed. Indeed, given a number field F' containing F and an integer $k \geq 1$, all but finitely many prime ideals of $O_{F'}$ are prime divisors of $P(T)$ and $P(T^k)$ if F' contains a root of $P(T^k)$.

Proposition 3.7 is proved in Section 4.

3.2.4. *Conclusion*

As already said, the monic separable polynomial $P_E(T)$ has coefficients in $O_{F'}$. Moreover, by condition (bp-1), one has $P_E(0) \neq 0$ and $P_E(1) \neq 0$. Then, by Proposition 3.7 (applied over F'), there exists an infinite set S of positive integers k (depending on F': see Remark 3.8) such that, for each $k \in S$, there exist infinitely many non-zero prime ideals of $O_{F'}$, each of which is a prime divisor of $P_E(T)$, but not of $P_E(T^k)$. Then it remains to apply Lemma 3.6 to conclude.

4. Proof of Proposition 3.7

This section is organized as follows. In Subsection 4.1, we state Proposition 4.1, which is Proposition 3.7 for polynomials whose roots are all in the base number field. Next, we explain in Subsection 4.2 how to deduce Proposition 3.7 from Proposition 4.1. Finally, Proposition 4.1 is proved in Subsection 4.3.

4.1. Statement of Proposition 4.1

Proposition 4.1. *Given a number field F, let $P(T) \in O_F[T]$* [4] *be a monic separable polynomial whose roots are all in $F \setminus \{0, 1\}$. Then there exist infinitely many positive integers k such that the Galois group of $P(T^k)$ over F has an element fixing no root of $P(T^k)$.*

4.2. Proof of Proposition 3.7 under Proposition 4.1

Let F be a number field and $P(T) \in O_F[T]$ a monic separable polynomial such that $P(0) \neq 0$ and $P(1) \neq 0$. Denote the roots of $P(T)$ by $t_1, \ldots, t_r$ and the splitting field of $P(T)$ over F by L. By Proposition 4.1, there exist infinitely many

[4] As before, O_F denotes the integral closure of $\mathbb{Z}$ in F.

positive integers k such that

$$\mathrm{Gal}(L_k/L) \setminus \bigcup_{j=1}^{r} \bigcup_{l=0}^{k-1} \mathrm{Gal}\left(L_k/L\left(\zeta_k^l \sqrt[k]{t_j}\right)\right)$$

contains some σ_k, where L_k is the splitting field over L of $P(T^k)$, ζ_k is a primitive k-th root of unity and $\sqrt[k]{t_j}$ is a given k-th root of t_j $(j = 1, \ldots, r)$. For each positive integer k, the splitting field of $P(T^k)$ over F is equal to L_k. Then σ_k lies in

$$\bigcup_{j=1}^{r} \mathrm{Gal}(L_k/F(t_j)) \setminus \bigcup_{j=1}^{r} \bigcup_{l=0}^{k-1} \mathrm{Gal}\left(L_k/F\left(\zeta_k^l \sqrt[k]{t_j}\right)\right).$$

It then remains to use implication (1) $\Rightarrow$ (3) of Lemma 2.2 to conclude.

4.3. Proof of Proposition 4.1

We proceed by induction on the degree of the polynomial $P(T)$.

4.3.1. *The case where $P(T)$ has degree* 1

Let F be a number field and $t \in O_F \setminus \{0, 1\}$. The conclusion of Proposition 4.1 for the polynomial $T^k - t$ easily follows from Lemmas 4.2 and 4.3 below[5].

Lemma 4.2. *Assume that t is a root of unity. Then for each number field L containing F, there exist infinitely many integers $k \geq 1$ such that the Galois group of $T^k - t$ over L is not trivial and each non-trivial element of the Galois group of $T^k - t$ over F fixes no root of this polynomial.*

Proof. Assume that t is a primitive N-th root of unity. Let L be a number field containing F and k a positive integer whose prime factors all are prime factors of N. As $t \neq 1$, one has $N \geq 2$ and there exist infinitely many such integers k. Assume that the Galois group of $T^k - t$ over L is trivial. Then L contains a primitive k-th root of unity, which cannot happen if k is sufficiently large (depending on L). One may then assume that the Galois group of $T^k - t$ over L is not trivial. In particular, the Galois group of $T^k - t$ over F is not trivial either. Let σ be a non-trivial element of the latter Galois group. Assume that σ fixes at least one root of $T^k - t$. By the definition of k, each root of $T^k - t$ is a primitive (Nk)-th root of unity. This implies that σ fixes each root of $T^k - t$, which cannot happen. □

Lemma 4.3. *Assume that t is not a root of unity. Then $T^k - t$ is irreducible over F for all but finitely many prime numbers k.*

[5] In the case where t is not a root of unity, one makes use of the following classical lemma: if a given finite group G acts transitively on a given finite set X with cardinality at least 2, then there exists $g \in G$ such that $g.x = x$ for no $x \in X$.

Proof. By Capelli's lemma (see *e.g.* [4, Chapter VI, Section 9, Theorem 9.1]) it suffices to show that, for all but finitely many prime numbers k, t is not a k-th power in F. Denote the absolute logarithmic Weil height on $\overline{\mathbb{Q}}$ by h. Assume that there exist infinitely many integers $k \geq 1$ such that there exists $x_k \in F$ satisfying $t = x_k^k$. One then has

$$h(t) = h(x_k^k) = k \cdot h(x_k).$$

As t is not a root of unity and $t \neq 0$, one has $h(t) \neq 0$. Hence F contains infinitely many elements each of which has height bounded by $h(t)$, which cannot happen [7, Theorem 1]. □

4.3.2. *End of the proof of Proposition* 4.1

Let r be a positive integer. Assume that the following condition holds:

(H) *For each number field F and each monic separable polynomial $P(T) \in O_F[T]$ of degree r whose roots are all in $F \setminus \{0, 1\}$, there exist infinitely many positive integers k such that the Galois group of $P(T^k)$ over F has an element fixing no root of $P(T^k)$.*

Let F be a number field and let $P(T) \in O_F[T]$ be a monic separable polynomial of degree $r + 1$ whose roots are all in $F \setminus \{0, 1\}$. Denote the roots of $P(T)$ by $t_1, \ldots, t_r, t_{r+1}$. By condition (H), there exists an integer $k_0 \geq 1$ such that the Galois group of $(T^{k_0} - t_1) \cdots (T^{k_0} - t_r)$ over F has an element τ fixing no root of this polynomial. Denote the splitting field of $(T^{k_0} - t_1) \cdots (T^{k_0} - t_r)$ over F by L.

(a) Assume that t_{r+1} is a root of unity. By Lemma 4.2, there is an integer $k_1 \geq 1$ such that the Galois group of $T^{k_1} - t_{r+1}$ over L is not trivial and every non-trivial element of the Galois group of $T^{k_1} - t_{r+1}$ over F fixes no root of this polynomial. Let σ be a non-trivial element of the former Galois group. Denote the splitting field of $T^{k_1} - t_{r+1}$ over L by M and let $\hat{\tau} \in \mathrm{Gal}(M/F)$ be a prolongation of τ to M.

First assume that $\hat{\tau}$ fixes no root of $T^{k_1} - t_{r+1}$. Then $\hat{\tau}$ is an element of the Galois group of $(T^{k_0} - t_1) \cdots (T^{k_0} - t_r) \cdot (T^{k_1} - t_{r+1})$ over F fixing no root of this polynomial. Given a positive multiple k of k_0 and k_1, every prolongation of $\hat{\tau}$ to the splitting field M_k over F of $P(T^k)$ is an element of $\mathrm{Gal}(M_k/F)$ fixing no root of this polynomial. Hence the desired conclusion holds.

Now, assume that $\hat{\tau}$ fixes a root of $T^{k_1} - t_{r+1}$. By the definition of k_1, $\hat{\tau}$ fixes each root of $T^{k_1} - t_{r+1}$. Consider the element $\sigma\hat{\tau}$ of $\mathrm{Gal}(M/F)$. If x denotes any k_0-th root of $t_1, \ldots, t_r$, then $\hat{\tau}(x)$ still is a k_0-th root of $t_1, \ldots, t_r$. By the definition of σ, one then has $\sigma\hat{\tau}(x) = \hat{\tau}(x)$, which is not equal to x by the definition of $\hat{\tau}$. If x denotes any k_1-th root of t_{r+1}, then, by the above, one has $\sigma\hat{\tau}(x) = \sigma(x)$, which is not equal to x by the definition of σ. Hence $\sigma\hat{\tau}$ is an element of the Galois group of $(T^{k_0} - t_1) \cdots (T^{k_0} - t_r) \cdot (T^{k_1} - t_{r+1})$ over F fixing no root of this polynomial. As before, the desired conclusion easily follows.

(b) Assume that t_{r+1} is not a root of unity. By Lemma 4.3, $T^{k_1} - t_{r+1}$ is irreducible over L for some prime k_1. As before, denote the splitting field of $T^{k_1} - t_{r+1}$ over L by M and let $\hat{\tau} \in \mathrm{Gal}(M/F)$ be a prolongation of τ to M. Let σ be an element of

$\mathrm{Gal}(M/L)$ fixing no root of $T^{k_1} - t_{r+1}$. If $\hat{\tau}$ fixes no root of $T^{k_1} - t_{r+1}$, then one gets the desired conclusion as in (a) above. We may then assume that $\hat{\tau}(\alpha) = \alpha$ for some root α of $T^{k_1} - t_{r+1}$. Let ζ be a primitive k_1-th root of unity. Up to making the prime number k_1 sufficiently large (depending on L), we may assume that L and $\mathbb{Q}(\zeta)$ are linearly disjoint over $\mathbb{Q}$. Then $L(\zeta)/L$ has degree $k_1 - 1$. Hence $L(\alpha)$ and $L(\zeta)$ are linearly disjoint over L (as $L(\alpha)/L$ has degree k_1). As a consequence, the Galois group $\mathrm{Gal}(M/L(\alpha))$ is generated by some element b satisfying $b(\alpha) = \alpha$ and $b(\zeta) = \zeta^e$ for some $e \in \mathbb{N}$. Consider the restriction w of $\hat{\tau}$ to $F(\zeta)$. One has $w = c^m$ for some integer m, where c is the generator of $\mathrm{Gal}(F(\zeta)/F)$ defined by $c(\zeta) = \zeta^e$. Then $\hat{\tau} b^{-m}$ is a prolongation of τ to M which fixes each root of $T^{k_1} - t_{r+1}$ (as $\hat{\tau}(\alpha) = \alpha$). As in (a) above, one shows that $\sigma \hat{\tau} b^{-m}$ is an element of the Galois group over F of $(T^{k_0} - t_1) \cdots (T^{k_0} - t_r) \cdot (T^{k_1} - t_{r+1})$ fixing no root of this polynomial, thus ending the proof.

5. A geometric variant

The aim of this section is Proposition 5.1 below which makes [5, Corollary 5.2] more precise (this result is recalled as Lemma 5.2 below).

5.1. Statement of Proposition 5.1

Proposition 5.1. *Let G be a non-trivial finite group, not a cyclic p-group. Then there exist a number field F_G and an F_G-regular Galois extension $E/F_G(T)$ with Galois group G such that the following holds:*

(geometric non-G-parametricity). *For every finite extension F'/F_G, there exist infinitely many linearly disjoint Galois extensions of F' with Galois group G each of which is not a specialization of $EF'/F'(T)$*[6].

Unlike Corollary 3.3, it seems unclear whether a number field F_G as in Proposition 5.1 may be specified for a given group G[7]. See [5, Section 7] where this is done in some specific cases.

5.2. Proof of Proposition 5.1

Let G be a non-trivial finite group.

First, recall the following result which is [5, Corollary 5.2].

Lemma 5.2. *There exist a number field F_G and an F_G-regular Galois extension of $F_G(T)$ with Galois group G which satisfies the* (geometric non-G-parametricity condition) *if the following group theoretic condition holds.*

[6] As in the (non-G-parametricity condition), the realizations of G whose existence is claimed may be produced by specialization.

[7] *i.e.*, being a regular Galois group over a given number field F might not be sufficient to take $F_G = F$.

(H2) *There exists a set* $\{C, C_1, \ldots, C_r\}$ *of non-trivial conjugacy classes of* G *such that the elements of* $C_1, \ldots, C_r$ *generate* G *and the remaining conjugacy class* C *is not in the set* $\{C_1^a, \ldots, C_r^a \,/\, a \in \mathbb{N}\}$.

Now, combine Lemmas 5.2 and 5.3 below to get Proposition 5.1.

Lemma 5.3. *Condition* (H2) *fails if and only if* G *is a cyclic* p*-group*.

Proof of Lemma 5.3. It is not hard to see that condition (H2) fails if G is a cyclic p-group. For the converse, we use the following argument due to Reiter. Assume that condition (H2) fails. Let H be a maximal subgroup of G. If H is not a normal subgroup of G, one has

$$G = \Big\langle \bigcup_{g \in G} gHg^{-1} \Big\rangle. \tag{5.1}$$

As condition (H2) fails, (5.1) provides $G = \bigcup_{g \in G} gHg^{-1}$, which cannot happen. Then each maximal subgroup of G is a normal one. Hence G is nilpotent, *i.e.*, G is the product of its Sylow subgroups. Set

$$G = P_1 \times \cdots \times P_s, \tag{5.2}$$

with $P_1, \ldots, P_s$ the Sylow subgroups of G. By the Sylow theorems, and as condition (H2) has been assumed to fail, (5.2) provides

$$G = P_1 \cup \cdots \cup P_s. \tag{5.3}$$

If $s \geq 2$, then, by taking cardinalities in (5.2) and (5.3), we get

$$\prod_{i=1}^{s} |P_i| < \sum_{i=1}^{s} |P_i|,$$

which cannot happen. Hence $s = 1$ and G is a p-group.

Let H_1 and H_2 be two distinct maximal subgroups of G. Then

$$G = \langle H_1 \cup H_2 \rangle. \tag{5.4}$$

As H_1 and H_2 are normal subgroups of G and as condition (H2) fails, (5.4) provides $G = \bigcup_{g \in G} g(H_1 \cup H_2)g^{-1}$. Hence $G = H_1 \cup H_2$. In particular, one has $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$, which cannot happen. Hence G has only one maximal subgroup and is then cyclic, as needed. □

5.3. A conjectural version of Proposition 5.1

Recall that [5] also offers a conjectural version of [5, Corollary 5.2]; see [5, Corollary 5.3]. Below we provide a similar conjectural version of Proposition 5.1, which then makes [5, Corollary 5.3] more precise.

Namely, let G be a non-trivial finite group. Assume that the following conjecture of Fried is satisfied[8].

[8] See Section III.1 of http://www.math.uci.edu/ mfried/deflist-cov/RIGP.html or [5, Section 5] for more details.

Conjecture (Fried). *Each set* $\{C_1, \dots, C_r\}$ *of non-trivial conjugacy classes of* G *that is rational and such that the elements of* $C_1, \dots, C_r$ *generate* G *occurs as the inertia canonical conjugacy class set of some* $\mathbb{Q}$*-regular Galois extension of* $\mathbb{Q}(T)$ *with Galois group* G.

Then, by combining Lemma 5.3 and [5, Corollary 5.3], Proposition 5.1 holds with $F_G = \mathbb{Q}$, *i.e.*, the following holds.

Proposition 5.4. *Assume that* G *is not a cyclic* p*-group. Then there exists a* $\mathbb{Q}$*-regular Galois extension of* $\mathbb{Q}(T)$ *with Galois group* G *that satisfies the geometric* (non-G-parametricity condition).

5.4. Other base fields

We conclude this paper by noticing that similar statements can be given for other base fields. For example, by conjoining Lemma 5.3 and [5, Subsection 5.2], we obtain the following counterpart of Proposition 5.1 for rational function fields.

Proposition 5.5. *Let* G *be a non-trivial finite group, not a cyclic* p*-group,* κ *an algebraically closed field of characteristic zero and* X *an indeterminate such that* T *is transcendental over* $\kappa(X)$. *Then, for some Galois extension* $E/\overline{\mathbb{Q}}(T)$ *with group* G, *the extension* $E\kappa(X)/\kappa(X)(T)$ *satisfies the* (geometric non-G-parametricity condition).

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