Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XVIII (2018), 467-482

Prym varieties of étale covers of hyperelliptic curves

HERBERT LANGE AND ANGELA ORTEGA

Abstract. It is well known that the Prym variety of an étale cyclic covering of a hyperelliptic curve is isogenous to the product of two Jacobians. Moreover, if the degree n of the covering is odd or congruent to 2 mod 4, then the canonical isogeny is an isomorphism. It is a natural question whether this is true for arbitrary degrees. We show that this is not the case by computing the degree of the isogeny for n a power of 2. Furthermore, we compute the degree of a closely related isogeny for arbitrary n.

Mathematics Subject Classification (2010): 14H40 (primary); 14H30 (secondary).

1. Introduction

Let H denote a hyperelliptic curve of genus $g \geq 2$ and $f : X \to H$ an étale cyclic covering of degree $n \geq 2$. Let σ denote the automorphism of X defining f. It is well known that the hyperelliptic involution of H lifts to an involution τ on X. Then σ and τ generate the dihedral group D_n of order $2n$. The Prym variety $P(f)$ of f is defined as the connected component containing 0 of the kernel of the norm map $\mathrm{Nm}\, f : JX \to JH$ of f. For any element $\alpha \in D_n$ we denote by X_α the quotient of X by the subgroup generated by α. The Jacobians JX_τ and $JX_{\tau\sigma}$ are Abelian subvarieties of the Prym variety $P(f)$ (the pullbacks defining these inclusions are omitted along the paper), therefore the addition map

$$a_0 : JX_\tau \times JX_{\tau\sigma} \to P(f)$$

is well defined. Mumford showed in [3] that for $n = 2$ that the map a_0 is an isomorphism. J. Ries proved the same for any odd prime degree n [6]. The second author generalized this statement to show that a_0 is an isomorphism for any odd number and, more importantly, for any even $n \equiv 2 \mod 4$ [4]. Of course, this immediately rises the question: is this map an isomorphism for every positive integer n? We will

The second author was supported by Deutsche Forschungsgemeinschaft, SFB 647.

Received June 5, 2016; accepted in revised form November 17, 2016.
Published online April 2018.

see in this paper that a_0 is not an isomorphism for $n = 2^r$ with $r \geq 3$. However it is an isogeny whose degree we will determine.

For the remaining positive integers n we compute the degree of a similar map. Namely, for $n = s^r m$ with m odd define

$$a : JX_\tau \times JX_{\tau\sigma^m} \to P(f), \qquad (x, y) \mapsto x + y.$$

Our main result is the following combination of Theorems 2.3 and 4.1.

Theorem. *Let n and $f : X \to H$ be as above with $n = 2^r m$, $r \geq 1$ and m odd. Then JX_τ and $JX_{\tau\sigma^m}$ are Abelian subvarieties of the Prym variety $P(f)$ and the addition map*

$$a : JX_\tau \times JX_{\tau\sigma^m} \to P(f)$$

is an isogeny of degree $2^{[(2^r - r - 1)m - (r-1)](g-1)}$.

Corollary. *The map a is an isomorphism for $n \equiv 2 \mod 4$ (*i.e. $r = 1$*). For $n = 2^r$, that is $m = 1$, one has $a_0 = a$, and the degree of a_0 is $2^{(2^r - 2r)(g-1)}$. In particular, it is an isomorphism for $r = 1, 2$ and it is not for $r \geq 3$.*

The proof proceeds by induction on the exponent r, the beginning of the induction ($r = 1$) being Theorem 2.3. For the induction step we consider an intermediate étale double covering $X \to X_{\sigma^{n/2}}$ on the top of the tower of curves (see Diagram (2.4)), which admits the action of the dihedral group $D_{n/2}$.

The proof is somehow intricate because in order to apply the induction hypothesis, one has first to relate the Prym varieties of other double coverings which do not appear in Diagram (2.4). To facilitate the reading, we sketch the steps of the proof, however using the notation introduced in the proof itself (it would take too much space to define it also here). In Section 2 we compute the genera of all the curves appearing in the tower of curves and recall a key result giving the degree of the addition map into the Prym variety of a double covering $Y \to Y_r$, with Y admitting an action of the Kleinian group (Proposition 2.10). In Section 3 we compute the degree of the isogeny $P(b_{\tau\sigma^{n/2}}) \to P(b_{\tau^m})$ (Proposition 3.2) given by the push forward to JX followed by the norm map between the Prym varieties on both sides of the tower (2.4). Finally, in Section 4 we put all the ingredients together. First, by using Proposition 2.10 and the induction hypothesis, we compute the degree of the isogeny

$$\tilde{\phi}_n : f^* JH \times JX_{K_\tau} \times JX_{K_{\tau\sigma^m}} \times P(b_\tau) \times P(b_{\tau\sigma^{n/2}}) \to JX,$$

and then, by means of Proposition 3.2, the degree of

$$\phi_n : f^* JH \times JX_{K_\tau} \times JX_{K_{\tau\sigma^m}} \times P(b_\tau) \times P(b_{\tau\sigma^m}) \to JX.$$

As a corollary we get the degree of the isogeny

$$\psi_n : JX_{K_\tau} \times JX_{K_{\tau\sigma^m}} \times P(b_\tau) \times P(b_{\tau\sigma^m}) \to P(f).$$

All the isogenies are naturally defined by the addition map. To conclude the proof of the main theorem one factorizes the isogeny ψ_n through the product of addition maps and a, and then applies Lemma 2.9.

We thank the referee for pointing out a gap in a previous version of the paper.

2. Preliminaries

Let H be a smooth hyperelliptic curve of genus g with hyperelliptic covering $\pi : H \to \mathbb{P}^1$ and $f : X \to H$ be a cyclic étale covering of degree $n \geq 2$. So X is of genus $g_X = n(g-1)+1$ and the Prym variety $P := P(f)$ of f is an Abelian variety of dimension

$$\dim P = (n-1)(g-1). \tag{2.1}$$

The canonical polarization of JX induces a polarization on P of type

$$(\underbrace{1,\dots,1}_{(n-2)(g-1)}, \underbrace{n,\dots,n}_{g-1})$$

(see [1, Corollary 12.1.5 and Lemma 12.3.1]). The hyperelliptic involution of H lifts to an involution τ on X which together with the automorphism σ defined by the covering f generates the dihedral group

$$D_n := \langle \sigma, \tau \mid \sigma^n = \tau^2 = (\sigma\tau)^2 = 1 \rangle.$$

The automorphism σ induces an automorphism of P of the same order n and compatible with the polarization, which we denote by the same letter. Each eigenvalue ζ_n^i, for $i = 1, \dots, n-1$ (with ζ_n a fixed primitive n-th root of unity) of the induced map on the tangent space T_0P occurs with multiplicity $g-1$.

In the whole paper we write $n = 2^r m$ with $r \geq 0$ and m odd. In any case the group D_n admits n involutions, namely $\tau\sigma^\nu$ for $\nu = 0, \dots n-1$. For odd n, these are all the involutions. For even n, there is one more, namely $\sigma^{\frac{n}{2}}$. For odd n all involutions are conjugate to τ and for even n there are 3 conjugacy classes. They are represented by

$$\tau, \ \tau\sigma^m \quad \text{and} \quad \sigma^{n/2}.$$

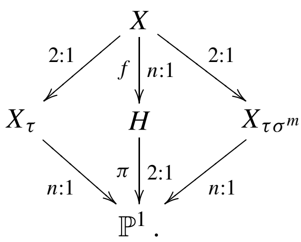
For any subgroup $G \subset D_n$ and for any element $\alpha \in D_n$ we denote by

$$X_G := X/G \quad \text{and} \quad X_\alpha := X/\langle\alpha\rangle$$

the corresponding quotients.

Remark 2.1. The Jacobians $JX_{\tau\sigma}$ and $JX_{\tau\sigma^m}$ are isomorphic since the involutions $\tau\sigma$ and $\tau\sigma^m$ are conjugate (for odd m) in the dihedral group D_n, so the corresponding coverings are isomorphic. However, this isomorphism is not in general compatible with the addition maps, so one can not deduce the degree of a_0 from the degree of a.

Consider the following diagram (for odd n only the left hand side of the diagram, since in this case $m = n$, so both sides are the same):



Let W denote the set of $2g+2$ branch points of the hyperelliptic covering π. Then denote, for arbitrary n,

$$s_0 := \left|\{x \in W \mid (\pi f)^{-1}(x) \text{ contains a fixed point of } \tau\}\right|$$

and

$$s_1 := \left|\{x \in W \mid (\pi f)^{-1}(x) \text{ contains a fixed point of } \tau\sigma^m\}\right|.$$

According to [4, Proposition 2.4] the Jacobians JX_τ and $JX_{\tau\sigma^m}$ are contained in the Prym variety P. With this notation the following theorem is proved in [4].

Theorem 2.2.

(a) *For odd n the map*

$$a_0 : (JX_\tau)^2 \to P, \qquad (x, y) \mapsto x + \sigma(y)$$

is an isomorphism;

(b) *For $n = 2m \equiv 2 \mod 4$ the map*

$$a_0 : JX_\tau \times JX_{\tau\sigma} \to P, \qquad (x, y) \mapsto x + y$$

is an isomorphism. Moreover,

$$g(X_\tau) = m(g-1)+1-\frac{s_0}{2} \quad \textit{and} \quad g(X_{\tau\sigma}) = g(X_{\tau\sigma^m}) = m(g-1)+1-\frac{s_1}{2}.$$

In particular s_0 and s_1 are even.

The following theorem shows that a variation of the map in (b) gives also an isomorphism.

Theorem 2.3. *For $n = 2m$ the map*

$$a : JX_\tau \times JX_{\tau\sigma^m} \to P, \qquad (x, y) \mapsto x + y$$

is an isomorphism.

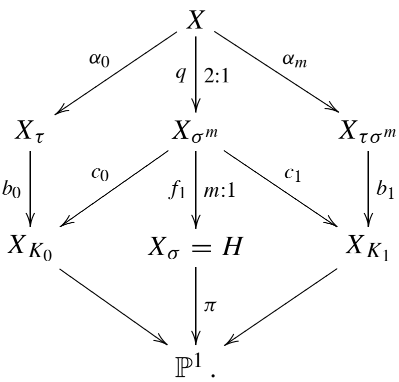
Proof. The proof is very similar to that one of Theorem 2.1 given in [4]. First note that the Kleinian groups

$$K_1 = \langle \tau\sigma, \sigma^m \rangle \quad \text{and} \quad K_m = \langle \tau\sigma^m, \sigma^m \rangle$$

are conjugated. In fact, $\tau\sigma^{\frac{m+1}{2}} K_1 (\tau\sigma^{\frac{m+1}{2}})^{-1} = K_m$; therefore, the curves X_{K_1} are X_{K_m} (and hence their Jacobians) are isomorphic. Let $K_0 = \langle \tau, \sigma^m \rangle$. We will need the following Lemma due to Kempf [2, Theorem 2.3]:

Lemma 2.4. *Let X be an integral algebraic variety with an action of a finite group G. Let F be a G-bundle over X. The bundle F descends to X/G if and only if for every point $x \in X$, the stabilizer of x in G acts trivially on the fiber F_x.*

Consider now the following commutative diagram:



(2.2)

Here the map $b_1 : X_{\tau\sigma^m} \to X_{K_1}$ is the composition of the quotient map $X_{\tau\sigma^m} \to X_{K_m}$ with the isomorphism $X_{K_m} \simeq X_{K_1}$. Since the dimensions on both sides of a are the same it suffices to show that $\operatorname{Ker} a = JX_\tau \cap J_{X_{\tau\sigma^m}} = \{0\}$. Let $F \in JX_\tau \cap J_{X_{\tau\sigma^m}}$. We regard F as a line bundle of degree 0 on X_τ (respectively on $X_{\tau\sigma^m}$). We have $JX_\tau = \operatorname{Im}(1+\tau)$ and $JX_{\tau\sigma^m} = \operatorname{Im}(1+\tau\sigma^m)$ as subvarieties of JX, in particular $F \in \operatorname{Fix}(\tau, \sigma^m)$. Hence there exist line bundles $M_0 \in JX_\tau$ and $M_1 \in JX_{\tau\sigma^m}$ such that

$$\alpha_0^* M_0 \simeq F \simeq \alpha_m^* M_1.$$

We have

$$\alpha_m^* \sigma^{m*} M_1 \simeq \sigma^{m*} \alpha_m^* M_1 \simeq \sigma^{m*} F \simeq F \simeq \alpha_m^* M_1,$$

since the automorphims τ and σ^m commute. The injectivitity of α_m^* implies $\sigma^{m*} M_1 \simeq M_1$.

Notice that the ramification points of the map $b_1 : X_{\tau\sigma^m} \to X_{K_1}$, that is, the fixed points of σ^m, lift to to the fixed points of τ in X: if $p \in \operatorname{Fix}(\sigma^m) \subset JX_{\tau\sigma^m}$ and $\tilde{p} \in X$ is such that $\alpha_m(\tilde{p}) = p$, then

$$\alpha_m(\sigma^m \tilde{p}) = \sigma^m \alpha_m(\tilde{p}) = \sigma^m p = p,$$

so $\sigma^m \tilde{p} \in \alpha_m^{-1}(p) = \{\tilde{p}, \tau\sigma^m \tilde{p}\}$; as q is non-ramified, $\sigma^m \tilde{p} \neq \tilde{p}$, so $\sigma^m \tilde{p} = \tau\sigma^m \tilde{p} = \sigma^m \tau(\tilde{p})$. It follows that $\tau \tilde{p} = \tilde{p}$.

The action of σ^m on the fibers of M_1 over the ramification points is the same as the action of τ on the fibers of F over the fixed points of τ in X since $\alpha_m^* M_1 \simeq F$. Let us set $x \in \mathrm{Fix}(\tau) \subset X$. By Lemma 2.4 $\langle\tau\rangle = \mathrm{Stab}(x)$ acts trivially on the fiber F_x (x being a ramification point of a_0). It follows that σ^m acts trivially on the fiber of $M_{1,\alpha_m(x)}$ and again by Lemma 2.4 there exists a line bundle $N_1 \in JX_{K_1}$ such that $b_1^* N_1 \simeq M_1$. Completely analogous as in [4, Lemma 2.1], one can show that there exists $N_0 \in JX_{K_0}$ such that $b_0^* N_0 \simeq M_0$.

Now, since $q^* c_0^* N_0 \simeq q^* c_1^* N_1 \simeq F$ we have

$$\beta := c_0^* N_0 \otimes (c_1^* N_1)^{-1} \in \mathrm{Ker}\, q^* = \{\mathcal{O}_{X_{\sigma^m}}, \eta\}$$

for some $\eta \in JX_{\sigma^m}[2] \setminus \{0\}$, and

$$\beta \in JX_{K_0} \times JX_{K_1} \simeq P(f_1)$$

by Theorem 2.2 (a). We have

$$q^*\beta \simeq \mathcal{O}_X \simeq \sigma^* \mathcal{O}_X \simeq \sigma^* q^* \beta \simeq q^* \sigma^* \beta,$$

so $\sigma^*\beta \in \mathrm{Ker}\, q^*$, and one checks easily that $\sigma^*\beta \simeq \beta$. As $P(f_1) \subset \mathrm{Ker}(1 + \sigma + \ldots + \sigma^{m-1})$, we have

$$\beta \in JX_{\sigma^m}[m] \cap JX_{\sigma^m}[2] = \{0\}.$$

Hence

$$c_0^* N_0 \simeq c_1^* N_1 \in JX_{K_0} \cap JX_{K_1} = \{0\},$$

where the last equality follows from Lemma [4, Lemma 2.1]. We conclude that $F \simeq q^* \mathcal{O}_{X_{\sigma^m}} \simeq \mathcal{O}_X$. □

It is the aim of this paper to study the map a in the remaining cases, $n = 2^r m$ with $r \geq 2$. So in the sequel we assume $r \geq 2$. We first need some preliminaries.

There are 2 non-conjugate Kleinian subgroups of D_n, namely

$$K_\tau = \left\{1, \sigma^{n/2}, \tau, \tau\sigma^{n/2}\right\} \quad \text{and} \quad K_{\tau\sigma^m} = \left\{1, \sigma^{n/2}, \tau\sigma^m, \tau\sigma^{m+n/2}\right\}.$$

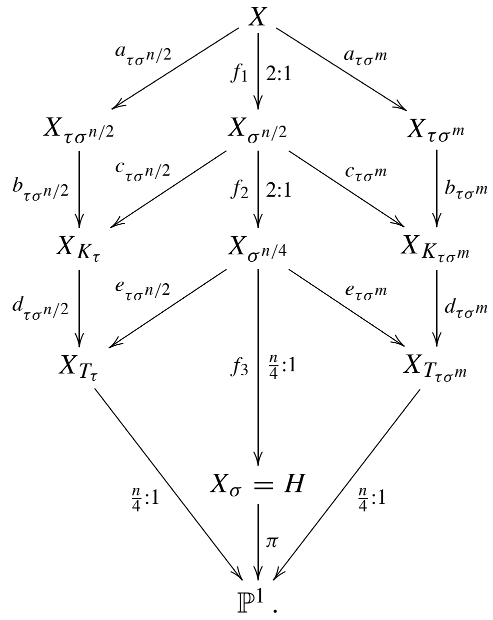
Moreover, consider the dihedral subgroups of order 8,

$$T_\tau = \left\langle \tau, \sigma^{n/4} \right\rangle \quad \text{and} \quad T_{\tau\sigma^m} = \left\langle \tau\sigma^m, \sigma^{n/4} \right\rangle.$$

Note that for $r \geq 3$ the groups T_τ and $T_{\tau\sigma^m}$ are non-conjugate, whereas

$$T_\tau = T_{\tau\sigma^m} \qquad \text{for} \qquad r = 2, \tag{2.3}$$

since then $\frac{n}{4} = m$ and $\langle \tau, \sigma^m \rangle = \langle \tau\sigma^m, \sigma^m \rangle$. In any case we have the following commutative diagram:



(2.4)

In the sequel we use the following notation: if an involution of the group D_n induces an involution on a curve of the diagram, we denote the induced involution by the same letter. In order to compute the genera of the curves in the diagram, we need the following lemma.

Lemma 2.5. *Suppose that the dihedral group $D_n = \langle \sigma, \tau \rangle$ of order $2n$ with $n \geq 3$ acts on a finite set S of n elements such that the subgroup $\langle \sigma \rangle$ acts transitively on S. Then:*

(a) *If n is odd, τ admits exactly one fixed point;*
(b) *For even n, either τ acts fixed-point free or admits exactly 2 fixed points;*
(c) *For even n, exactly one of the involutions τ and $\tau\sigma$ admits at least a fixed point.*

Proof. Let $S = \{x_1, \dots x_n\}$. We may enumerate the x_i in such a way that $\sigma(x_i) = x_{i+1}$ for $i = 1, \dots, n$, where $x_{n+1} = x_1$. If n is odd, then clearly τ admits a fixed point. So in any case we may assume that x_1 is a fixed point of τ. Then we have inductively, for $i = 1, \dots \lfloor \frac{n+3}{2} \rfloor$,

$$\tau(x_i) = x_{n+2-i}. \tag{2.5}$$

In fact, the induction step is $\tau(x_i) = \tau\sigma(x_{i-1}) = \sigma^{-1}\tau(x_{i-1}) = \sigma^{-1}(x_{n-i+3}) = x_{n-i+2}$. Hence for odd n the involution τ admits no further fixed point and for even n, τ admits exactly one additional fixed point, namely $x_{\frac{n+2}{2}}$. This gives (a) and (b).

Now let us prove (c). Suppose that n is even and that τ admits a fixed point, say x_1. Hence we have (2.5) for all i. This implies

$$\tau\sigma(x_i) = \tau(x_{i+1}) = x_{n+1-i},$$

and so $\tau\sigma$ acts without fixed points. Conversely, suppose that τ acts without fixed points and that $\tau(x_1) = x_i$ for some $i \geq 2$. Then $\sigma^{1-i}\tau(x_1) = x_1$. So $\sigma^{1-i}\tau$ admits a fixed point and thus it cannot be conjugated to τ. Hence $\tau\sigma$ is conjugate to $\sigma^{1-i}\tau$ and admits a fixed point. □

Lemma 2.6. *Suppose $n = 2^r m$ with m odd and $r \geq 2$. Then:*

(a) *$s_0 + s_1 = 2g + 2$ with $s_0, s_1 \geq 2$ even;*
(b) *For $r = 2$, $X_{\sigma^{n/4}} \to X_{T_\tau}$ and $X_{\sigma^{n/4}} \to X_{T_{\tau\sigma^m}}$ are ramified exactly at $2g + 2$ points;*
(c) *$X \to X_{\tau\sigma^{n/2}}$ and $X_{\sigma^{n/2}} \to X_{K_\tau}$ as well as $X_{\sigma^{n/4}} \to X_{T_\tau}$, if $r \geq 3$, are ramified exactly at $2s_0$ points. $X \to X_{\tau\sigma^m}$ and $X_{\sigma^{n/2}} \to X_{K_{\tau\sigma^m}}$ as well as $X_{\sigma^{n/4}} \to X_{T_{\tau\sigma^m}}$, if $r \geq 3$, are ramified exactly at $2s_1$ points.*

Proof. The fixed points of τ and $\tau\sigma^m$ lie over the $2g + 2$ Weierstrass points of H. Moreover, according to Lemma 2.5, over each Weierstrass point of H exactly one of τ and $\tau\sigma^m$ admits a fixed point. This gives the first assertion of (i). The evenness of s_0 and s_1 follows from the Hurwitz formula. Now $s_0 = 0$ means that τ acts without fixed-points. Since also σ acts without fixed points, so does $\tau\sigma^m$, which means $s_1 = 0$. But this contradicts the equation $s_0 + s_1 = 2g + 2$. Hence $s_0, s_1 \geq 2$.

If x is a Weierstrass point of H and τ admits a fixed point over x, then D_n acts on the fibre $f^{-1}(x)$. Similarly, the group $D_{n/2} = \langle \sigma^{n/2}, \tau \rangle$ acts on the fibre $(f_3 \circ f_2)^{-1}(x)$ and the group $D_{n/4} = \langle \sigma^{n/4}, \tau \rangle$ acts of the fibre $f_3^{-1}(x)$. Hence Lemma 2.5 implies (ii), since in these cases the order of the fibre is even, and (iii), since in this case the order of the fibre is odd. □

By checking the ramification of the maps in Diagram (2.4) we immediately get from Lemma 2.6 the following corollaries.

Corollary 2.7. *All the vertical left and right hand maps are ramified.*

Corollary 2.8. *If $n = 2^r m$ with m odd and $r \geq 2$, then*

$$g(X) = n(g-1) + 1, \quad g(X_{\sigma^{n/2}}) = \frac{n}{2}(g-1) + 1, \quad g(X_{\sigma^{n/4}}) = \frac{n}{4}(g-1) + 1;$$

$$g(X_{\tau\sigma^{n/2}}) = \frac{n}{2}(g-1) + 1 - \frac{s_0}{2}, \qquad g(X_{\tau\sigma^m}) = \frac{n}{2}(g-1) + 1 - \frac{s_1}{2};$$

$$g(X_{K_\tau}) = \frac{n}{4}(g-1) + 1 - \frac{s_0}{2}, \qquad g(X_{K_{\tau\sigma^m}}) = \frac{n}{4}(g-1) + 1 - \frac{s_1}{2};$$

and for $r \geq 3$,

$$g(X_{T_\tau}) = \frac{n}{8}(g-1) + 1 - \frac{s_0}{2} \qquad g(X_{\tau\sigma^m}) = \frac{n}{8}(g-1) + 1 - \frac{s_1}{2}.$$

For $r = 2$,

$$g(X_{T_\tau}) = g(X_{\tau\sigma^m}) = \frac{1}{2}(m-1)(g-1).$$

Proof. The assertions follow from the Hurwitz formula. For the first line of assertions we use the fact that f is étale. For the other formulas we use Lemma 2.6 (ii) and (iii). □

The following lemma is well known. In fact, it is an easy consequence of [1, Proposition 11.4.3] and [1, Corollary 12.1.4].

Lemma 2.9. *Let* $g : Y \to Z$ *be a covering of smooth projective curves of degree* $d \geq 2$. *The addition map*

$$g^* JZ \times P(g) \to JY$$

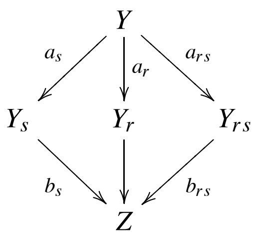
is an isogeny of degree

$$|g^* JZ \cap P(g)| = \frac{|JZ[d]|}{|\ker g^*|^2}.$$

We need a result on curves with an action of the Klein group. Let Y be a curve with an action of the group

$$V_4 = \left\langle r, s \mid r^2 = s^2 = (rs)^2 = 1 \right\rangle.$$

Then we have the following diagram



(2.6)

with $Y_v := Y/\langle v \rangle$ for any $v \in V_4$ and $Z = Y/V_4$. The following theorem is a special case of [5, Theorem 3.2].

Proposition 2.10. *Suppose* a_r *is étale, that* a_s *(respectively* a_{rs}*) is ramified at* $2\alpha_s > 0$ *(respectively* $2\alpha_{rs} > 0$*) points and* Z *is of genus* $g(Z)$. *Then* $P(b_s)$ *and* $P(b_{rs})$ *are subvarieties of* $P(a_r)$ *and the addition map*

$$\phi_r : P(b_s) \times P(b_{rs}) \to P(a_r)$$

is an isogeny of degree $2^{2g(Z)}$.

3. A degree computation

As above, let $n = 2^r m$ with m odd and $r \geq 2$. Again we consider a curve X with action of the dihedral group $D_n := \langle \sigma, \tau \mid \sigma^n = \tau^2 = (\sigma\tau)^2 = 1 \rangle$. With the notation as in Section 2 we have Diagram (2.4) and $s_0, s_1 \geq 2$. Then, apart from f_1, f_2 and f_3, all the maps in Diagram (2.4) are ramified. So the pullbacks of the corresponding Jacobians are embeddings [1, Proposition 11.4.3]. Recall that $P = P(f)$ denotes the Prym variety of the covering f.

We consider the isogenies

$$h := \operatorname{Nm} a_{\tau\sigma^m} \circ a^*_{\tau\sigma^{n/2}} : P(b_{\tau\sigma^{n/2}}) \longrightarrow P(b_{\tau\sigma^m})$$

and

$$h' := \operatorname{Nm} a_{\tau\sigma^{n/2}} \circ a^*_{\tau\sigma^m} : P(b_{\tau\sigma^m}) \longrightarrow P(b_{\tau\sigma^{n/2}}).$$

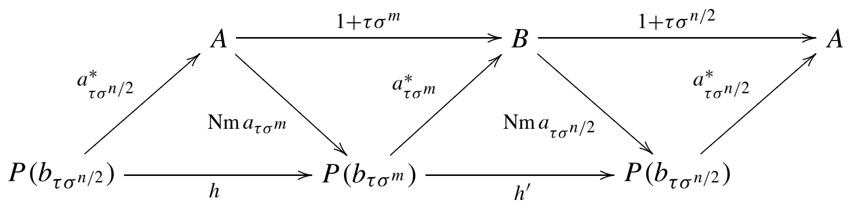
Let

$$A := a^*_{\tau\sigma^{n/2}}(P(b_{\tau\sigma^{n/2}})) \qquad \text{and} \qquad B := a^*_{\tau\sigma^m}(P(b_{\tau\sigma^m}))$$

be subvarieties of JX. Now τ (respectively $\sigma^{n/2}$) induces an involution on $JX_{\tau\sigma^{n/2}}$ (respectively on $JX_{\tau\sigma^m}$), which we denote by the same letter. Thus the Prym variety $P(b_{\tau\sigma^{n/2}})$ is $\operatorname{Ker}(1+\tau)^0$ and $P(b_{\tau\sigma^m}) = \operatorname{Ker}(1+\sigma^{n/2})^0$. In fact, these kernels consist of one connected component since the maps $b^*_{\tau\sigma^{n/2}}$ and $b^*_{\tau\sigma^m}$ are injective [1, Proposition 11.4.3]. Let JX^G denote the set of points in JX invariant under the action of a subgroup $G \subset D_n$. Hence we have (for example by [5, Corollary 2.7]),

$$A = \left\{ z \in JX^{\langle \tau\sigma^{n/2} \rangle} \mid z + \tau z = 0 \right\}, \qquad B = \left\{ w \in JX^{\langle \tau\sigma^m \rangle} \mid w + \sigma^{n/2} w = 0 \right\}.$$

Moreover, as in [5], there is a commutative diagram:



(3.1)

Lemma 3.1. *For any $n = 2^r m$ with m odd and $r \geq 2$ we have*

$$|\operatorname{Ker} h| = |\operatorname{Ker}(1+\tau\sigma^m)_{|A}|$$

and

$$\operatorname{Ker}(1+\tau\sigma^m)_A = (JX[2])^{\langle \tau, \sigma^m \rangle}.$$

Proof. The first assertion follows from Diagram (3.1), since $a^*_{\tau\sigma^{n/2}} : P(b_{\tau\sigma^{n/2}}) \to A$ and $a^*_{\tau\sigma^m} : P(b_{\tau\sigma^m}) \to B$ are isomorphisms. For the last assertion note that $z \in \operatorname{Ker}(1 + \tau\sigma^m)_{|A}$ if and only if

$$\tau\sigma^{n/2}z = z, \qquad \tau z = -z, \qquad \tau\sigma^m(z) = -z,$$

which implies that $\sigma^m z = z$. So

$$z = \tau\sigma^{n/2}z = \tau\sigma^{2^{r-1}m}z = \tau(\sigma^m)^{2^{r-1}}(z) = \tau z = -z,$$

then $z \in A[2]$.

Therefore $z \in \operatorname{Ker}(1 + \tau\sigma^m)_{|A}$ if and only if $z \in JX[2]$ such that $\tau z = z$ and $\sigma^m z = z$, which was to be shown. □

The following proposition is a generalization of a special case of [5, Theorem 4.1, (ii)].

Proposition 3.2. *For every $n = 2^r m$ with m odd and $r \geq 2$, we have*

$$\deg h = 2^{(m-1)(g-1)+s_1-2}.$$

Proof. The proof is by induction on the exponent $r \geq 2$. Suppose first $r = 2$, *i.e.* $n = 4m$. Consider the curve X with the action of the dihedral subgroup

$$D_4 := \langle \sigma^m, \tau \rangle \subset D_n.$$

It has 2 non-conjugate Kleinian subgroups, namely $K_\tau = \langle \sigma^{2m}, \tau \rangle$ and $K_{\tau\sigma^m} = \langle \sigma^{2m}, \tau\sigma^m \rangle$ Note that, by (2.3), $T_\tau = T_{\tau\sigma^m} = \langle \sigma^m, \tau \rangle$. Then according to [5, Theorem 4.1 (ii)] we have

$$|\operatorname{Ker} h| = 2^{2g(X_{T_\tau})-2+s_1}.$$

So Corollary 2.8 gives the proposition in this case.

Suppose now that $r \geq 3$ and the proposition holds for $r - 1$. Let X be a curve with an action of D_n with $X/\langle\sigma\rangle = H$, so that we have the Diagram (2.4). Then the subgroup $D_{\frac{n}{2}} = \langle \sigma^2, \tau \rangle$ of index 2 acts on the curve $X_{n/2}$, so that we can apply the inductive hypothesis. This gives that the map

$$h_{n/2} := \operatorname{Nm} c_{\tau\sigma^m} \circ c^*_{\tau\sigma^{n/2}} : P(d_{\tau\sigma^{n/2}}) \longrightarrow P(d_{\tau\sigma^m})$$

is an isogeny of degree $2^{(m-1)(g-1)-2+s_1}$.

Hence it suffices to show that

$$\operatorname{Ker} h = b^*_{\tau\sigma^{n/2}}(\operatorname{Ker} h_{n/2}).$$

This implies the proposition, since the map $b^*_{\tau\sigma^{n/2}}$ is injective.

By applying Lemma 3.1 to $h_{\frac{n}{2}}$ one obtains

$$\left|\operatorname{Ker} h_{\frac{n}{2}}\right| = \left|(JX_{\tau\sigma^{n/2}}[2])^{\langle\tau,\sigma^m\rangle}\right| = \left|a^*_{\tau\sigma^{n/2}}(JX_{\tau\sigma^{n/2}}[2])^{\langle\tau,\sigma^m\rangle}\right|.$$

But

$$\begin{aligned} a^*_{\tau\sigma^{n/2}}(JX_{\tau\sigma^{n/2}}[2])^{\langle\tau,\sigma^m\rangle} &= \left\{z \in JX[2] \mid \tau\sigma^{n/2}z = z, \tau z = z, \sigma^m z = z\right\} \\ &= \left\{z \in JX[2] \mid \tau z = z, \sigma^m z = z\right\}, \end{aligned}$$

since the equation $\tau\sigma^{n/2}z = z$ is a consequence of the last 2 equations. This gives

$$|\operatorname{Ker} h| = |\operatorname{Ker} h_{\frac{n}{2}}|$$

which completes the proof of Proposition 3.2. □

4. Decomposition for $n = 2^r m$, $r \geq 2$ with m odd

Now let the notation be as in Section 1 with $n = 2^r m$, $r \geq 2$ and m odd. Let $f : X \to H$ be a cyclic étale covering of degree n of a hyperelliptic curve H. The main result of the paper is the following theorem.

Theorem 4.1. *Let n and $f : X \to H$ be as above. Then JX_τ and $JX_{\tau\sigma^m}$ are Abelian subvarieties of the Prym variety $P(f)$ and the addition map*

$$a : JX_\tau \times JX_{\tau\sigma^m} \to P(f)$$

is an isogeny of degree $2^{[(2^r-r-1)m-(r-1)](g-1)}$.

The proof is by induction on r. Since the proofs for $r = 2$ and for the inductive step in case $r \geq 3$ are almost the same, we present them simultaneously. The difference is only that for $r = 2$ we use Theorem 2.3 instead of the induction hypothesis. So in this section we assume that for $r \geq 3$, Theorem 4.1 is true for $r - 1$, *i.e.* for coverings of degree $2^{r-1}m$ for all m. Let $r \geq 2$ and $f : X \to H$ be an étale covering of degree $n = 2^r m$ with odd $m \geq 1$. We use the notation of Diagram (2.4). In addition let $b_\tau : X_\tau \to X_{K_\tau}$ denote the canonical projection.

Proposition 4.2. *The varieties JX_{K_τ}, $JX_{K_{\tau\sigma^m}}$, $P(b_\tau)$ and $P(b_{\tau\sigma^{n/2}})$ are Abelian subvarieties of JX and the addition map*

$$\widetilde{\varphi}_n : f^*JH \times JX_{K_\tau} \times JX_{K_{\tau\sigma^m}} \times P(b_\tau) \times P(b_{\tau\sigma^{n/2}}) \to JX$$

is an isogeny of degree

$$\deg \widetilde{\varphi}_n = m^{2g-2} \cdot 2^{[(2^{r+1}-r)m+r](g-1)+2-s_0}.$$

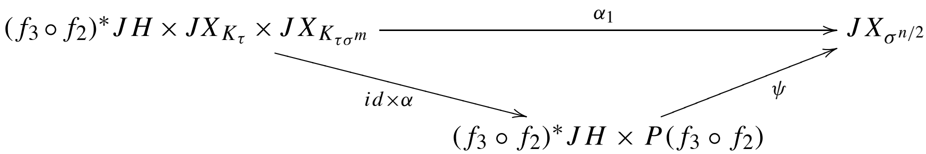
Proof. All the maps in Diagram (2.4) are ramified apart from f_1, f_2 and f_3, which gives the first assertion. The dihedral group $D_{n/2} = \langle \sigma^2, \tau \rangle$ acts on the curve $X_{\sigma^{n/2}}$.

If $r = 2$, we can apply Theorem 2.3 to get that the canonical map

$$\alpha : JX_{K_\tau} \times JX_{K_{\tau\sigma^m}} \to P(f_3 \circ f_2)$$

is an isomorphism. For $r \geq 3$ we can apply the induction hypothesis, which gives that α is an isogeny of degree $2^{[(2^{r-1}-r)m-(r-2)](g-1)}$. Since this number is equal to 1 for $r = 2$, this is valid for all $r \geq 2$.

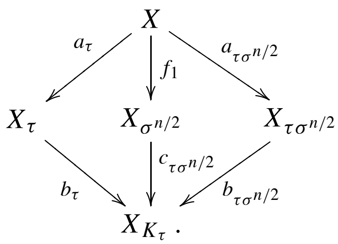
Now the addition map $\alpha_1 : (f_3 \circ f_2)^* JH \times JX_{K_\tau} \times JX_{K_{\tau\sigma^m}} \to JX_{\sigma^{n/2}}$ factorizes as



where ψ is the addition map. Thus Lemma 2.9 implies that

$$\begin{aligned}\deg \alpha_1 &= \deg \alpha \cdot \deg \psi = 2^{[(2^{r-1}-r)m-(r-2)](g-1)} \cdot (2^{r-1}m)^{2g-2} \\ &= m^{2g-2} \cdot 2^{[(2^{r-1}-r)m+r)](g-1)}.\end{aligned}$$

Clearly α_1 and its pullback via f_1^* are of the same degree. Moreover, considering X with the action of the Klein group $\langle \sigma^{n/2}, \tau \rangle$, we have the diagram:

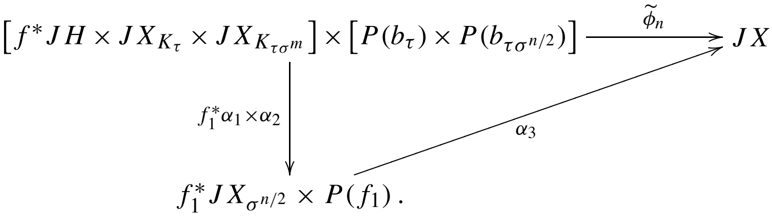


Then by Proposition 2.10 the addition map

$$\alpha_2 : P(b_\tau) \times P(b_{\tau\sigma^{n/2}}) \to P(f_1)$$

is an isogeny of degree $2^{2g(X_{K_\tau})} = 2^{2^{r-1}m(g-1)+2-s_0}$.

Now note that the map $\widetilde{\phi}_n$ factorizes as



By Lemma 2.9 the addition map α_3 is an isogeny of degree $2^{2g(X_{\sigma^{n/2}})-2} = 2^{2^r m(g-1)}$, therefore the map $\widetilde{\phi}_n$ is an isogeny of degree

$$\begin{aligned}\deg \widetilde{\phi}_n &= \deg f_1^* \alpha_1 \cdot \deg \alpha_2 \cdot \deg \alpha_3 \\ &= m^{2g-2} \cdot 2^{[(2^{r-1}-r)m+r)](g-1)} \cdot 2^{2^{r-1}m(g-1)+2-s_0} \cdot 2^{2^r m(g-1)} \\ &= m^{2g-2} \cdot 2^{[(2^{r+1}-r)m+r](g-1)+2-s_0}.\end{aligned}$$

□

Corollary 4.3. *The canonical map*

$$\phi_n : f^* JH \times JX_{K_\tau} \times JX_{K_{\tau\sigma^m}} \times P(b_\tau) \times P(b_{\tau\sigma^m}) \to JX$$

is an isogeny of degree

$$\deg \phi_n = m^{2g-2} 2^{[(2^{r+1}-r-1)m+r-1](g-1)}.$$

Proof. According to Proposition 3.2 the canonical map

$$h : P(b_{\tau\sigma^{n/2}}) \to P(b_{\tau\sigma^m})$$

is an isogeny of degree $2^{(m-1)(g-1)-2+s_1}$.

Now with the definition of the map h one checks that the following diagram commutes

$$\begin{array}{ccc} \left[f^* JH \times JX_{K_\tau} \times JX_{K_{\tau\sigma^m}} \times P(b_\tau)\right] \times P(b_{\tau\sigma^{n/2}}) & \xrightarrow{\widetilde{\phi}_n} & JX \\ \downarrow{\scriptstyle id \times h} & \nearrow{\scriptstyle \phi_n} & \\ \left[f^* JH \times JX_{K_\tau} \times JX_{K_{\tau\sigma^m}} \times P(b_\tau)\right] \times P(b_{\tau\sigma^m}). & & \end{array}$$

So Propositions 4.2 and 3.2 imply that ϕ_n is an isogeny of degree

$$\begin{aligned}\deg \phi_n &= \frac{\deg \widetilde{\phi}_n}{\deg h} \\ &= \frac{m^{2g-2} \cdot 2^{[(2^{r+1}-r)m+r](g-1)+2-s_0}}{2^{(m-1)(g-1)-2+s_1}} = m^{2g-2} \cdot 2^{[(2^{r+1}-r-1)m+r-1](g-1)},\end{aligned}$$

where we used again that $s_0 + s_1 = 2g + 2$. □

Corollary 4.4. *The canonical map*

$$\psi_n : JX_{K_\tau} \times JX_{K_{\tau\sigma^m}} \times P(b_\tau) \times P(b_{\tau\sigma^m}) \to P(f)$$

is an isogeny of degree $2^{[(2^{r+1}-r-1)m-(r+1)](g-1)}$.

Proof. Clearly the addition maps the source of ψ_n into $P(f)$ and the following diagram is commutative

$$\begin{array}{ccc} f^*JH \times \left[JX_{K_\tau} \times JX_{K_{\tau\sigma^m}} \times P(b_\tau) \times P(b_{\tau\sigma^m})\right] & \xrightarrow{\phi_n} & JX \\ \Big\downarrow {\scriptstyle id\times\psi_n} & \nearrow {\scriptstyle \varphi} & \\ f^*JH \times P(f) & & \end{array}$$

where φ denotes the addition map. According to Lemma 2.9, φ is an isogeny of degree $(16m)^{2g-2}$. Hence ψ_n is an isogeny of degree

$$\begin{aligned} \deg \psi_n = \frac{\deg \phi_n}{\deg \varphi} &= \frac{m^{2g-2} \cdot 2^{[(2^{r+1}-r-1)m+r-1](g-1)}}{(2^r m)^{2g-2}} \\ &= 2^{[(2^{r+1}-r-1)m-(r+1)](g-1)}. \end{aligned}$$

□

Proof of Theorem 4.1. The following diagram is commutative

$$\begin{array}{ccc} JX_{K_\tau} \times JX_{K_{\tau\sigma^m}} \times P(b_\tau) \times P(b_{\tau\sigma^m}) & & \\ \Big\downarrow {\scriptstyle \simeq} & \searrow {\scriptstyle \psi_n} & \\ \left[JX_{K_\tau} \times P(b_\tau)\right] \times \left[JX_{K_{\tau\sigma^m}} \times P(b_{\tau\sigma^m})\right] & & P(f) \\ \Big\downarrow {\scriptstyle \varphi_1\times\varphi_2} & \nearrow {\scriptstyle a} & \\ JX_\tau \times JX_{\tau\sigma^m} & & \end{array}$$

where φ_1 and φ_2 denote the addition maps. According to Lemma 2.9, φ_1 and φ_2 are isogenies of degrees $2^{8m(g-1)+2-s_0}$ and $2^{8m(g-1)+2-s_1}$ respectively. This implies that a is an isogeny of degree

$$\deg a = \frac{\deg \psi_n}{\deg \varphi_1 \cdot \deg \varphi_2} = \frac{2^{[(2^{r+1}-r-1)m-(r+1)](g-1)}}{2^{(2^r m-2)(g-1)}} = 2^{[(2^r-r-1)m-(r-1)](g-1)}.$$

which completes the proof of the theorem. □

References

[1] CH. BIRKENHAKE and H. LANGE, "Complex Abelian Varieties", Second ed., Grundlehren der Math. Wiss., Vol. 302, Springer-Verlag, 2004.

[2] J.-M. DREZET and M. S. NARASIMHAN, *Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques*, Invent. Math. **97** (1989), 53–94.

[3] D. MUMFORD, *Prym varieties I*, In: "Contributions to Analysis", L.V. Ahlfors, I. Kra, B. Maskit, and L. Niremberg (eds.), Academic Press, 1974, 325–350.
[4] A. Ortega, *Variétés de Prym associées aux revêtements n-cycliques d'une courbe hyperelliptique*, Math. Z. **245** (2003), 97–103.
[5] S. RECILLAS and R. RODRIGUEZ, *Prym varieties and fourfold covers II, the dihedral case*, Contemp. Math. **397** (2006), 177–191.
[6] J. RIES, *The Prym variety of a cyclic unramified cover of a hyperelliptic surface*, J. Reine Angew. Math. **340** (1983), 59–69.

Department Mathematik
der Universität Erlangen
Cauerstr. 11
91058 Erlangen, Germany
lange@math.fau.de

Institut für Mathematik
Humboldt Universität zu Berlin
Unter den Linden 6
10099 Berlin, Germany
ortega@math.hu-berlin.de