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Cusps and a converse to the Ambrosetti-Prodi theorem

MARTA CALANCHI, CARLOS TOMEI AND ANDRÉ ZACCUR

Abstract. By the Ambrosetti-Prodi theorem, the map $F(u) = -\Delta u - f(u)$ between appropriate functional spaces is a global fold. Among the hypotheses, the convexity of the function f is required. We show in two different ways that convexity is indeed necessary. If f is not convex, there is a point with at least four preimages under F. Even more, F generically admits cusps among its critical points. We present a larger class of nonlinearities f for which the critical set of F has cusps. The results are true for Dirichlet, Neumann and periodic boundary conditions, among others.

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1. Introduction

The celebrated Ambrosetti-Prodi theorem [1], originally a statement about a differential operator acting between Hölder spaces, has been amplified and reformulated by various authors. In particular, Manes and Micheletti [14] weakened the original hypotheses, while Berger, Church and Podolak [3, 5], provided a geometric rephrasing on suitable Sobolev spaces. We present a version for each scenario.

For an open bounded domain $\Omega \subset \mathbb{R}^n$ with piecewise smooth boundary, we consider the Hölder spaces $B_D^2 = C_0^{2,\alpha}(\Omega)$, $B^0 = C^{0,\alpha}(\Omega)$, $\alpha \in (0,1)$, and the Sobolev spaces $H_D^2 = H_0^1(\Omega) \cap H^2(\Omega)$, $H^0 = H^0(\Omega) = L^2(\Omega)$. The Dirichlet Laplacian

$$-\Delta_D : H_D^2 \subset H^0 \to H^0, \quad \sigma(-\Delta_D) = \{0 < \mu_{1,D} < \mu_{2,D} \leq \cdots\}$$

has pure point spectrum $\sigma(-\Delta_D)$ made of eigenvalues $\{\mu_{k,D}\}$ (of finite multiplicity), with an associated set of orthonormal eigenfunctions $\{\psi_{k,D}\}$ in B_D^2 which

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is complete in L^2. Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly convex smooth function with $\overline{f'(\mathbb{R})} = [m, M]$ and

$$m, M \notin \sigma(-\Delta_D), \quad (m, M) \cap \sigma(-\Delta_D) = \{\mu_{1,D}\}, \quad \lim_{|x|\to\infty} f''(x) = 0.$$

Theorem (Ambrosetti-Prodi [1,14]). *The differential operator*

$$F : B^2_D \to B^0 \quad \textit{defined by} \quad F(u) = -\Delta_D u - f(u)$$

is a smooth map with critical set $\mathcal{C} \subset B^2_D$ diffeomorphic to a hyperplane. The complement $B^0 - F(C)$ splits in two connected components $\mathcal{C}_0$ and $\mathcal{C}_2$. Each element of the sets $\mathcal{C}_0$, $F(C)$ and $\mathcal{C}_2$ has respectively zero, one and two preimages under F.

The critical set of F consists of the critical points of F, *i.e.*, the functions $u \in B^2_D$ for which $DF(u)$ is not an isomorphism. Berger and Podolak [5] and Berger and Church [3] introduced additional geometric ingredients. Denote by $\langle u \rangle$ the span of a vector u.

Theorem (Berger-Church-Podolak [3,5]). *For f as above, $F : H^2_D \to H^0$ is a global fold. More precisely, with respect to the orthogonal splitting $H^0 = W^0 \oplus \langle \psi_{1,D} \rangle$, there are global homeomorphisms $\zeta : H^2_D \to W^0 \oplus \mathbb{R}$ and $\xi : H^0 \to W^0 \oplus \mathbb{R}$ for which $\tilde{F}(z,t) = \xi \circ F \circ \zeta^{-1}(z,t) = (z, -t^2)$.*

Said differently, the following diagram commutes:

$$\begin{array}{ccc} H^2_D & \xrightarrow{\quad F \quad} & H^0 \\ \zeta \big\downarrow & & \big\downarrow \xi \\ W^0 \oplus \mathbb{R} & \xrightarrow{(z,t)\mapsto(z,-t^2)} & W^0 \oplus \mathbb{R} \end{array}$$

Such Ambrosetti-Prodi type results for different boundary conditions have also been treated extensively in [8, 15, 16, 21]. A beautiful presentation of both approaches is [20].

Settling a question raised by Dancer [10], we prove a converse result under mild conditions: for a class of boundary conditions, the Ambrosetti-Prodi theorem does not hold if f is not a convex function. We associate to a domain of self-adjointness $H^2_b(\Omega) \subset H^0(\Omega)$ of $-\Delta_b : H^2_b(\Omega) \to H^0(\Omega)$ a *standard boundary condition*, defined in Subsection 2.1, for which $\sigma(-\Delta_b) = \{\mu_{1,b} < \mu_{2,b} \leq \ldots\}$. Dirichlet, Neumann and periodic boundary conditions are standard. For $B^2_b = H^2_b \cap C^{2,\alpha}(\Omega)$, take $F : B^2_b \to B^0$ and let D be a dense subspace of B^2_b in the L^2 norm.

Theorem 1.1. *Consider a standard boundary condition and a smooth function $f : \mathbb{R} \to \mathbb{R}$ satisfying non-resonant strict interaction with $\mu_{1,b}$ and nonconvexity:*

(1) *There are $m, M \in \mathbb{R}$ for which $\overline{f'(\mathbb{R})} = [m, M]$ and $m < \mu_{1,b} < M < \mu_{2,b}$;*

(2) *There is an* $\epsilon > 0$ *such that*

$$f'(x) > \mu_{1,b} + \epsilon \ \textit{for} \ x \to \infty \ \textit{and} \ f'(x) < \mu_{1,b} - \epsilon \ \textit{for} \ x \to -\infty;$$

(3) *For some* $x_*, y_* \in \mathbb{R}$, *one has* $f''(x_*) < 0$ *and* $f''(y_*) > 0$.

Then for some $y \in B^0$, *the equation* $F(u) = y$ *admits (at least) four solutions in* $D \subset B^2_b$.

The result is proved in Section 2 exploiting ideas of Berger and Podolak which were extensively used in [17]. More specifically, in Subsection 2.2 we study *fibers*, that is, inverses under F of certain straight lines. Under hypotheses (1) and (2), for appropriate coordinates, the restriction of F to a fiber is a map from $\mathbb{R}$ to $\mathbb{R}$ which tends to $-\infty$ for $|t| \to \infty$. The existence of a point with four preimages reduces to the search of a fiber on which such a map admits a local minimum.

Section 3 strengthens the previous theorem by asserting that *generically* F admits *cusps*, in the sense presented below. Cusps in infinite dimensions have been characterized by a number of authors [4,13,23,24] and we will follow [4]. For the Jacobian $DF(u) : B^2_b \to B^0$, let $\lambda_1(u)$ be its smallest eigenvalue and $\phi_1(u)$ be the associated L^2-normalized eigenvector. As we shall see in Proposition 3.2, a regular zero u_c of the function

$$\Lambda : B^2_b \to \mathbb{R}^2, \quad \Lambda(u) = \big(\lambda_1(u),\ \delta_1(u) = \ D\lambda_1(u)\, \phi_1(u)\big)\,,$$

i.e., a point for which $\Lambda(u_c) = 0$ and $D\Lambda(u_c)$ is surjective, which additionally satisfies $\tau_1(u_c) = D\delta_1(u_c)\,\phi_1(u_c) \neq 0$ is a (local) cusp of $F : B^2_b \to B^0$.

At a cusp u_c, a function F admits a simple local form [9]: that is, for some Banach space Y, changes of variables near u_c and $F(u_c)$ convert F into

$$\tilde{F}(w, x, y) = \Big(w, x, y^3 - x\,y\Big) \quad \text{for } w \in Y,\ x,\ y \in \mathbb{R}.$$

In particular, points near $F(u_c)$ may have one, two or three preimages near u_c.

Theorem 1.2. *Consider a standard boundary condition and assume hypotheses* (1) *and* (3) *together with the following one:*

(4) *(genericity) The functions* $f' - \mu_{1,b}$, f'' *and* f''' *have no common zero*.

Then either $F : B^2_b \to B^0$ *has a cusp in* D *or there is a family of disjoint arcs each of which is taken by* F *to a single point. In both cases, for some* $g \in B^0$, $F(u) = g$ *has at least three solutions. If* F *is proper, it has a cusp in* D.

An arc is a diffeomorphic image of an open interval. The arcs in the theorem, consist of whole fibers. When they exist, they are abundant, being parameterized by an open set of a codimension 3 subspace in B^2_b. Another simple hypothesis which guarantees the existence of a cusp will be presented in Proposition 3.5.

An ancestor of these results is in [17, Theorem 4.3], which states that, for a strictly convex function $f(x)$ such that $\lim_{|x|\to\infty} f(x) = \infty$, the differential operator

$$G : C^1([0,1]) \to C^0([0,1]), \quad u \mapsto u' + f(u), \quad u(1) = u(0),$$

is a global fold. On the other hand, under generic hypotheses which we do not describe, if f'' is negative at some point then G has points with four preimages.

Cusps are frequently associated with operators $F(u) = -\Delta u - f(u)$ for cubic nonlinearities f, as in Ruf's study [23] of the global geometry of the nonlinearity $f(u) = -u^3 + cu$. Hypothesis (1) excludes such functions, but the following alternative hypothesis is more tolerant:

(H_k) The operator $-\Delta_b : H_b^2 \to H^0$ is self-adjoint, some eigenvalue $\mu_{k,b}$ is isolated and simple and there are points $x_*, y_* \in \mathbb{R}$ for which

$$f'(x_*) = f'(y_*) = \mu_{k,b} \quad \text{and} \quad f''(x_*) < 0\,,\ f''(y_*) > 0\,.$$

Theorem 1.3. *Consider standard boundary conditions and assume hypotheses* (4) *and* (H_k). *Then either $F : B_b^2 \to B^0$ has a cusp in D or there is a family of disjoint arcs each of which is taken by F to a single point. For $k = 1$, a cusp in D necessarily occurs if F is proper or if $f'''(x_*), f'''(y_*) \geq 0$.*

The proof of Theorems 1.2 and 1.3 splits into a few steps. We first show in Subsection 3.1 that the properties of u_c in terms of the function Λ above provide an appropriate description of a cusp. In Subsection 3.2 we find a zero $\overline{u}_{nf}$ of (an extension of) Λ taking only two real values, which is mollified in Subsection 3.3 to obtain zero $u_{nf} \in D$ of Λ. We are left to show in Subsection 3.4 that the remaining hypotheses of Theorem 1.2 are satisfied either by u_{nf} or by some nearby zero $u_c \in D$ of Λ. For Dirichlet conditions, we may take $u_c \in C_0^\infty(\Omega)$.

The technique for mollifying functions respecting nonlinear restrictions used to pass from $\overline{u}_{nf}$ to u_{nf} might be of independent interest: a different version was used in the construction of homotopies in [6].

Under hypotheses (1) and (2), the existence of a cusp u_c implies Theorem 1.1. Indeed, from the local form of F near u_c, there is a point g with three preimages, as for the polynomial $p(x) = x^3 - x$ near zero. A fourth pre-image arises because, by hypothesis (2), F is proper of degree zero (or because along a fiber, for large $|x|$, the function F looks like $x \mapsto -x^2$).

Theorems 1.1 and 1.2 replicate the structure of a pair of papers by Ruf. In [25] he finds points in the image of a semi-linear elliptic boundary value operator with five preimages. In [26] he shows that the operator acting on functions defined on intervals or rectangles (with Neumann boundary conditions) admits a *butterfly* u_b, so that there are points near $F(u_b)$ with five preimages. Since we stop at cusps, our computations are simpler despite of the fact that we handle Laplacians on arbitrary bounded sets. The mollifying arguments allow us to ignore the boundary conditions until the last moment.

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2. Proof of Theorem 1.1

We sketch the proof of Theorem 1.1. Let $\psi_{1,b}$ be the positive (L^2-)normalized eigenvector associated with the free eigenvalue $\mu_{1,b}$. In Subsection 2.2, following [5], we foliate H_b^2 and B_b^2 into *fibers*, the inverses of lines $\{z+s\ \psi_{1,b}\ ,\ s\in\mathbb{R}\}$ which turn out to be connected curves of the form $\{w(z,t)+t\psi_{1,b},\ t\in\mathbb{R},\ \langle w(z,t),\ \psi_{1,b}\rangle = 0\}$ (brackets denote the usual L^2 inner product).

The restriction F^z of F to each fiber $\{u(z,t), t\in\mathbb{R}\}\subset B_b^2$ is of the form $F^z(u(z,t)) = z + h(u(z,t))\psi_{1,b}$ for a real valued *height* h for which, as we shall see in Proposition 2.7,

$$\lim_{|t|\to\infty} h(u(z,t)) = -\infty.$$

Thus, in order for F to be a global fold, the restriction F^z should look (topologically) like $t\mapsto -t^2$. In Subsection 2.4 however we will present fibers $u(z,t)$ on which F^z admits a strict local minimum u_m. The asymptotic behavior of h then implies the existence of points with four preimages, proving Theorem 1.1. The interested reader may find a numerical example in [7, Section 5.3].

By Proposition 2.6, critical points of h along a fiber are exactly the critical points of F. The properties of interest of u_m, namely

- u_m is such that $v\mapsto -\Delta_b v - f'(u_m)v$ has an eigenvalue equal to 0, and
- The height h at u_m has positive second derivative along its fiber,

are verified by checking that $\lambda_1(u_m) = 0,\ \delta_1(u_m) > 0$ for appropriate functionals λ_1 and δ_1, introduced in Theorem 1.2, which extend to bounded functions in H^0. We first find such a point $\overline{u}$ among *two valued potentials*, a class of very simple functions defined in Subsection 2.3. Mollification then yields the required $u_m\in B_b^2$.

2.1. Basic spectral theory and smoothness

We consider boundary conditions associated with domains $H_b^2 = H_b^2(\Omega)\subseteq H^2(\Omega)$ on which $-\Delta_b : H_b^2\to H^0$ is self-adjoint. Set $\sigma(-\Delta_b) = \{\mu_{1,b} < \mu_{2,b} \le \ldots\}$ and let $\psi_{1,b}$ be the positive (L^2-)normalized eigenfunction associated with $\mu_{1,b}$. By the Kato-Rellich theorem, for $q\in L^\infty(\Omega)$, self-adjointness holds for

$$T_q : H_b^2\subset H^0\to H^0,\quad v\mapsto -\Delta_b v - q\, v$$

Definition 2.1. A boundary condition is *standard* if the following conditions hold:

- The smallest eigenvalue λ_1^q of T_q is simple (hence isolated);
- There is a unique L^2-normalized eigenfunction $\phi_1^q > 0$ associated with λ_1^q;
- On bounded sets of potentials q, the sup norm of ϕ_1^q is uniformly bounded.

The boundary condition is implicit in the notation λ_1^q and ϕ_1^q. We recall some basic facts from spectral theory and elliptic regularity [11,19].

Proposition 2.2. *Dirichlet, Neumann and periodic boundary conditions are standard. For standard boundary conditions and $q \leq M < \mu_{2,b}$, one has that T_q is not invertible if and only if $\lambda_1^q = 0$.*

Set $B_b^2 = C^{2,\alpha}(\Omega) \cap H_b^2$. The differentiability of F is also well known [2,9].

Proposition 2.3. *For a smooth function $f : \mathbb{R} \to \mathbb{R}$ the map $F : B_b^2 \to B^0$ is smooth. If f satisfies hypothesis* (1)*, then $F : H_b^2 \to H^0$ is a $\mathcal{C}^1$ map. In both cases, if $f'(u) \leq M < \mu_{2,b}$, the differential $DF(u)v = -\Delta_b v - f'(u)v$ is always a Fredholm operator of index zero, with kernel of dimension at most one.*

In $Z = L^\infty(\Omega)$ consider the L^2-inner product $\langle u, v\rangle$ — notice that Z is not a Banach space with the induced norm. A sequence $\{u_m, m \in \mathbb{N}\} \subset Z$ is *ub-convergent*, $u_m \xrightarrow{\text{ub}} u_\infty$, if $u_m \to u_\infty$ in L^2 and $\{u_m, m \in \mathbb{N}\}$ is bounded in the sup norm (and thus u_∞ satisfies the same L^∞ bound than the u_m's). Given a metric space $\mathbb{M}$, a function $G : X \subset Z \to \mathbb{M}$ is ub-continuous if it takes ub-convergent sequences to convergent sequences in $\mathbb{M}$. In particular, if X is bounded in the sup norm and G is ub-continuous, then it is continuous in the L^2 norm.

For a smooth function $f : \mathbb{R} \to \mathbb{R}$ we are interested in potentials of the form $q = f'(u)$, for $u \in B_b^2$. For a standard boundary condition, denote by $\lambda_1(u) = \lambda_1^q$ the smallest eigenvalue of the Jacobian $DF(u) : H_b^2 \to H^0$ and by $\phi_1(u) = \phi_1^q$ the associated positive L^2-normalized eigenvector, which by standard regularity results is necessarily in B_b^2.

Proposition 2.4. *Assume hypothesis* (1)*. The functions $\lambda_1 : B_b^2 \to \mathbb{R}$ and $\phi_1 : B_b^2 \to H^0$ are smooth. The extensions $\lambda_1 : Z \to \mathbb{R}$ and $\phi_1 : Z \to H^0$ are ub-continuous.*

The proof is given in Subsections 5.1 and 5.2 of the Appendix.

Sobolev spaces are used in Section 2, where we handle folds in a disguised form. To work with cusps in Section 3, however, additional smoothness of λ_1 and ϕ_1 is necessary and we consider F acting between Hölder spaces.

2.2. Fibers and asymptotics on fibers

Locally, the construction of fibers is a Lyapunov-Schmidt decomposition associated with an eigenvector. Hypothesis (1) provides a *global* decomposition. There are analogous results which hold locally under hypothesis (H_k), and this fact will be used in Section 4. The arguments in this section are valid for boundary conditions for which the smallest eigenvalue of $-\Delta_b : H_b^2 \to H^0$ is isolated and simple — the positivity of the ground state is not needed.

Split H_b^2 and H^0 orthogonally into *horizontal* and *vertical* subspaces,

$$H_b^2 = W^2 \oplus V \ , \quad H^0 = W^0 \oplus V, \quad V = \langle \psi_{1,b} \rangle, \quad \|\psi_{1,b}\|_{H^0} = 1 \ .$$

For a fixed $z \in W^0$, the set $\{z + s\,\psi_{1,b}\ ,\ s \in \mathbb{R}\}$ is a vertical line in the image —its inverse under F is a *fiber* F^z. Clearly, the domain H_b^2 is a disjoint union of fibers. Versions of the next result may be found in [5,9,22,27].

Proposition 2.5. *Assume hypothesis* (1). *The fibers F^z are indexed by $z \in W^0$ and are parameterized by $t \in \mathbb{R}$, that is, $u(z,t) = w(z,t) + t\psi_{1,b}$ for a $\mathcal{C}^1$ map $(z,t) \mapsto w(z,t) \in W^2$. The map $F : B_b^2 \to B^0$ admits similar smooth fibers: a fiber of $F : H_b^2 \to H^0$ with a point in B_b^2 is in B_b^2.*

The last statement seems to be new and we sketch a proof. At regular points of $F : B_b^2 \to B^0$, tangent vectors to the fibers are inverses of $\psi_{1,b} \in B^0$ under $DF(u) : B_b^2 \to B^0$. One can extend and normalize it smoothly to the whole space B_b^2 —fibers in B_b^2 are the orbits of this vector field.

The restriction of F to a fiber $F^z = \{u(z,t)\} \subset B_b^2$ is essentially given by the *height* h of its images,

$$F(u(t)) = z + h(u(t))\,\psi_{1,b}, \quad t \in \mathbb{R}, \tag{2.1}$$

clearly a smooth real map. We drop the (fixed) parameter z from the notation.

The critical set of $F : B_b^2 \to B^0$ restricts well to the fibers. Moreover, the critical points of the height are described in terms of spectrum of DF [5,9].

Proposition 2.6. *Let $f : \mathbb{R} \to \mathbb{R}$ be smooth and suppose* (1). *The derivative in t of $h(u(t))$, the height of a fiber $u(t)$, is zero exactly at critical points $u(t_0)$ of $F : B_b^2 \to B^0$. The eigenfunction $\phi_1(u(t_0))$ of $DF(u(t_0))$ is a positive multiple of $u'(t_0)$, the tangent vector to the fiber at $u(t_0)$. Finally, there is a strictly positive smooth function $p : B_b^2 \to \mathbb{R}$ for which*

$$\frac{d}{dt}\ h(u(t)) = Dh(u(t))\,u'(t) = p(u(t))\,\lambda_1(u(t))\,.$$

Proof. Differentiate (2.1) to obtain $DF(u(t))u'(t) = \big(Dh(u(t))\,u'(t)\big)\psi_{1,b}$. Since $u'(t) = w'(t) + \psi_{1,b} \neq 0$, both sides are zero if and only if $u'(t)$ lies in the kernel of $DF(u(t))$. Thus, the critical points of $h(u(t))$ and of F are the same and, at such point $u(t_0)$, $\ker DF(u(t_0)) = \langle u'(t_0) \rangle$ and $u'(t_0) = c\phi_1(u(t_0))$ for some $c \in \mathbb{R}$. Now, $\langle \phi_1(u(t)), \psi_{1,b} \rangle > 0$, as both functions are positive, and $\langle u'(t_0), \psi_{1,b} \rangle = \langle w'(t_0) + \psi_{1,b}, \psi_{1,b} \rangle = \langle \psi_{1,b}, \psi_{1,b} \rangle > 0$, so that $c > 0$.

Since λ_1 and h' have common roots in each fiber, it suffices to show that p is well defined in neighborhoods of these roots. Clearly

$$\langle DF(u(t))\phi_1(u(t)), u'(t) \rangle = \lambda_1(u(t))\langle \phi_1(u(t)), u'(t) \rangle = h'(u(t))\langle \phi_1(u(t)), \psi_{1,b} \rangle.$$

At a critical point $u' = c\,\phi_1(u)$, $c > 0$ and nearby $\langle \phi_1(u(t)), u'(t) \rangle > 0$. Thus p, being a quotient of smooth nonzero inner products, is smooth. □

For the reader's convenience, we transcribe the argument in [7] describing the asymptotic behavior of F along a fiber, already obtained in [5].

Proposition 2.7. *Let f satisfy* (1) *and* (2). *Then on each fiber $u(t) \in B_b^2$ one has*

$$\lim_{t \to \pm\infty} h(u(t)) = -\infty.$$

Proof. Since $F(u(t)) = z + h(u(t))\psi_{1,b}$, for $z \in W^0$ and $W^0 \perp V$, we have $h(u(t)) = \langle F(u(t)), \psi_{1,b} \rangle$. Since $F(u) = -\Delta_b u - f(u)$ and $-\Delta_b \psi_{1,b} = \mu_{1,b}\psi_{1,b}$,

$$h(u(t)) = \mu_{1,b}\, t - \int_\Omega f(u(t))\, \psi_{1,b}.$$

From hypothesis (2), $f(x)$ is bounded below by two lines,

$$(\mu_{1,b} - \epsilon)\, x + c_- \,, \quad (\mu_{1,b} + \epsilon)\, x - c_+ \;<\; f(x).$$

We consider $t \to \infty$. Since $u(t) = w(t) + t\,\psi_{1,b}$, for $w(t) \in W^2 \perp V$,

$$h(u(t)) \le \mu_{1,b} t - \int_\Omega \big((\mu_{1,b} + \epsilon)(w(t) + t\psi_{1,b}) - c_+\big)\, \psi_{1,b} \le -\epsilon t + c_+ \int_\Omega \psi_{1,b}$$

and we are done ($t \to -\infty$ is similar). □

2.3. Two-valued potentials

Denote the usual argument from $\mathbb{R}^2 \setminus \{0\}$ to $[0, 2\pi)$ by arg. The *sector* $S(\theta)$ is

$$S(\theta) = \big\{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n, \quad \arg(x_1, x_2) \le \theta \;\text{ or }\; (x_1, x_2) = 0\big\}.$$

We abuse notation slightly and define $S(0) = \emptyset$ and $S(2\pi) = \mathbb{R}^n$.

For a fixed $p \in \Omega \subset \mathbb{R}^n$, we consider the translated sector $p + S(\theta)$ and split

$$\Omega = \big(\Omega \cap (p + S(\theta))\,\big) \cup \big(\Omega \cap (p + S(\theta))^c\,\big) = \Omega_\theta \cup \Omega_\theta^c$$

in disjoint subsets with characteristic functions χ_θ and χ_θ^c — the point p stays fixed and is omitted. For $\theta \in (0, 2\pi)$, both sets have nonzero measure. For the proof of Theorem 1.1, a family of parallel hyperplanes would suffice, but in Proposition 4.1 of Section 4 an appropriate choice of p is convenient.

The set of *two-valued functions* is

$$\overline{\mathcal{V}} = \{\overline{q}(L, R, \theta) = L\,\chi_\theta + R\,\chi_\theta^c,\; L, R \in \mathbb{R},\; \theta \in [0, 2\pi]\} \;\subset\; L^\infty(\Omega).$$

We simplify notation: for $\overline{q} = \overline{q}(L, R, \theta) = L\,\chi_\theta + R\,\chi_\theta^c \in \overline{\mathcal{V}}$, set

$$T_{\overline{q}} : H^2 \to H^0,\; v \mapsto -\Delta_b v - \overline{q}(L, R, \theta)\, v\,,$$

so that the smallest eigenvalue and positive normalized eigenvector restrict to

$$\lambda_1^{\overline{q}} : \mathbb{R}^2 \times [0, 2\pi] \to \mathbb{R} \quad \text{and} \quad \phi_1^{\overline{q}} : \mathbb{R}^2 \times [0, 2\pi] \to H^0 .$$

A triple $(L, R, \theta) \in \mathbb{R}^2 \times [0, 2\pi]$ is *balanced* if $\lambda_1^{\overline{q}}(L, R, \theta) = 0$. The next lemma shows that balancing is frequently feasible.

Lemma 2.8. *There is a continuous function*

$$\begin{aligned} \Theta : \big((-\infty, \mu_{1,b}] \times (\mu_{1,b}, \infty)\big) \cup \big((\mu_{1,b}, \infty) \times (-\infty, \mu_{1,b}]\big) &\to (0, 2\pi) \\ (L, R) &\mapsto \Theta(L, R) \end{aligned}$$

such that $\lambda_1^{\overline{q}}(L, R, \Theta(L, R)) = 0$. *Also,* $\Theta(\mu_{1,b}, R) = 2\pi$ *and* $\Theta(L, \mu_{1,b}) = 0$.

Proof. Clearly, the map $\iota : (L, R, \theta) \in \mathbb{R}^2 \times [0, 2\pi] \mapsto \overline{q}(L, R, \theta) \in Z$ is continuous, so that $\lambda_1^{\overline{q}} : \mathbb{R}^2 \times [0, 2\pi] \to \mathbb{R}$ and $\phi_1^{\overline{q}} : \mathbb{R}^2 \times [0, 2\pi] \to H^0$ are too: proceed as in the proof of Proposition 2.4 in the Appendix with $\Phi \circ \iota$.

Suppose that $L \le \mu_{1,b} < R$, the other case is similar. We have

$$\lambda_1^{\overline{q}}(L, R, 0) = \mu_{1,b} - R < 0 \quad \text{and} \quad \lambda_1^{\overline{q}}(L, R, 2\pi) = \mu_{1,b} - L > 0\,, \ \text{for}\, L \neq \mu_{1,b}\,,$$

$$\lambda_1^{\overline{q}}(\mu_{1,b}, R, 2\pi) = 0 \quad \text{and} \quad \lambda_1^{\overline{q}}(\mu_{1,b}, R, \theta) \le 0\,, \ \text{for}\, \theta \in [0, 2\pi)\,.$$

Thus, for a given $L \in (-\infty, \mu_{1,b}]$, a balancing Θ exists by continuity in θ of $\lambda_1^{\overline{q}}$. We now show uniqueness and continuity in L.

For $L_1 \le L_2$, $R_1 \le R_2$ and $0 < \theta_1 \le \theta_2 < 2\pi$, consider points (L_1, R_1, θ_1) and (L_2, R_2, θ_2) associated with potentials $\overline{q}_i$, operators $T_{\overline{q}_i}$ and quadratic forms $Q_i(v) = \langle v, T_{\overline{q}_i} v\rangle$, for $i = 1, 2$. Clearly, $\overline{q}_1 \le \overline{q}_2$ pointwise a.e., $\lambda_1^{\overline{q}_1} \ge \lambda_1^{\overline{q}_2}$, $\lambda_1^{\overline{q}_i} = Q_i(\phi_1^{\overline{q}_i})$ and $\phi_1^{\overline{q}_i} > 0$ in Ω. Also, $T_{\overline{q}_1} = T_{\overline{q}_2} + q_+$, for a potential $q_+ \ge 0$. If (L_1, R_1, θ_1) and (L_2, R_2, θ_2) are distinct, $q_+ \not\equiv 0$ and

$$\lambda_1^{\overline{q}_1} = Q_1\left(\phi_1^{\overline{q}_1}\right) = Q_2\left(\phi_1^{\overline{q}_1}\right) + \left\langle q_+ \phi_1^{\overline{q}_1}, \phi_1^{\overline{q}_1}\right\rangle > Q_2\left(\phi_1^{\overline{q}_1}\right) \ge \lambda_1^{\overline{q}_2} .$$

Thus $\lambda_1^{\overline{q}}$ is strictly monotonic on each coordinate and $\Theta(L, R)$ is well defined for $(L, R, \theta) \in (-\infty, \mu_{1,b}) \times (\mu_{1,b}, \infty) \times (0, 2\pi)$. Similarly, Θ is strict by monotonic along the segment $(\mu_{1,b}, R, \theta)$, $\theta \in [0, 2\pi]$, so that $\lambda_1^{\overline{q}}(\mu_{1,b}, R, \theta) < 0$, $\theta \in [0, 2\pi)$, enforcing $\Theta(\mu_{1,b}, R) = 2\pi$. Finally, from the continuity of $\lambda_1^{\overline{q}}$, we see that Θ is continuous. $\square$

2.4. A point with four preimages

As stated in the sketch of proof, we search for a strict local minimum $u(t_0)$ of the height h along a fiber $u(t)$. We use an additional derivative of λ_1.

Proposition 2.9. *Let f satisfy hypothesis* (1). *The derivative of the smooth function $\lambda_1 : B_b^2 \to \mathbb{R}$ along $v \in B_b^2$ is*

$$D\lambda_1(u)\, v = \langle \nabla\lambda_1(u), v\rangle = -\int_\Omega f''(u)\phi_1^2(u)\, v\,, \quad \nabla\lambda_1(u) = -f''(u)\phi_1^2(u) \in Z\,.$$

The extension $\lambda_1 : Z \to \mathbb{R}$ admits Gateaux derivatives along $v \in Z$ given by the same formulas. The map $\nabla\lambda_1 : Z \to Z,\ u \mapsto \nabla\lambda_1(u)$ is ub-continuous.

For the proof see Subsection 5.2. Define the ub-continuous map

$$\delta_1 : Z \to \mathbb{R}\,, \quad u \mapsto D\lambda_1(u)\phi_1(u) = -\Big\langle f''(u)\,\phi_1(u)^2,\ \phi_1(u)\Big\rangle\,.$$

Say $r \sim s$ if both numbers r and s have the same sign. We obtain in Proposition 2.11 a function $u(t_0)$ for which (as in Proposition 2.6),

$$\begin{aligned} \frac{d}{dt} h(u(t))|_{t=t_0} &= p(u(t_0))\,\lambda_1(u(t_0)) = 0\,, \\ \frac{d^2}{dt^2} h(u(t))|_{t=t_0} &= Dp(u(t_0))u'(t_0)\,\lambda_1(u(t_0)) + p(u(t_0))\, D\lambda_1(u(t_0))u'(t_0) \\ &\sim D\lambda_1(u(t_0))u'(t_0) \sim \delta_1(u(t_0)) > 0\,. \end{aligned}$$

The sequences $\{x_m^\pm\}$ in the next lemma play the role of almost critical points. The simple argument in real analysis proving the lemma is left to the reader.

Lemma 2.10. *Suppose that the smooth function $g : \mathbb{R} \to \mathbb{R}$ has a bounded image containing an interior point μ. Then there are sequences $x_m^+, x_m^- \in \mathbb{R}$ such that:*

1. $\lim_{m\to\infty} g'(x_m^+) = \lim_{m\to\infty} g'(x_m^-) = 0\,, \quad g'(x_m^+),\ g'(x_m^-) \neq 0\,;$
2. $\lim_{m\to\infty} g(x_m^-) = R^- < \mu < \lim_{m\to\infty} g(x_m^+) = R^+\,.$

We will use the lemma for the function $g = f'$ and $\mu = \mu_{1,b}$.

Proposition 2.11. *Assume* (1), (2) *and* (3). *Let $D \subset B_b^2$ be a dense subspace of Z. Then there is $u \in D$ with $\lambda_1(u) = 0$, and $\delta_1(u) > 0$.*

Proof. We first show that there is $\overline{u} \in \overline{\mathcal{V}}$ with $\lambda_1(\overline{u}) = 0$, and $\delta_1(\overline{u}) > 0$. From (3), take x_* with $f''(x_*) < 0$. Suppose also $f'(x_*) < \mu_{1,b}$ —the other case is similar.

For $p \in \Omega$, we consider translated sectors $p + S(\theta)$ defined in Subsection 2.3. From (2), $\mu_{1,b}$ is an interior point of $g = f'$. For x_m^+ obtained from Lemma 2.10, set

$$\overline{u}_m^+ = \overline{u}\Big(x_*,\ x_m^+,\ \Theta_m = \Theta\left(f'(x_*),\ f'(x_m^+)\right)\Big) = x_*\,\chi_{\Theta_m} + x_m^+\,\chi_{\Theta_m}^c\,,$$

so that $\lambda_1(\overline{u}_m^+) = 0$ by balancing (Lemma 2.8): notice that $f'(x_m^+) \to R^+ > \mu_{1,b}$. From the definition of a standard boundary condition, the eigenfunctions $\phi_1(\overline{u}_m^+)$ of the potentials $\overline{q}(f'(x_*), f'(x_m^+), \Theta_m) = f'(\overline{u}_m^+)$, are uniformly bounded and converge to the eigenfunction ϕ_1^∞ for the potential $\overline{q}\,(f'(x_*), R^+, \Theta_\infty = \Theta(f'(x_*), R^+))$. By the continuity of δ_1 (Proposition 2.9),

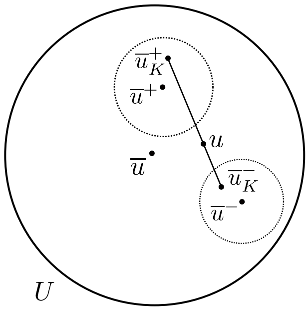
$$\lim_{m\to\infty} \delta_1(\overline{u}_m^+) = -\int_{\Omega_{\theta_\infty}} f''(x_*)(\phi_1^\infty)^3 - \int_{\Omega^c_{\theta_\infty}} \big(\lim_{m\to\infty} f''(x_m^+)\big)(\phi_1^\infty)^3 > 0$$

and thus, for some large K, $\delta_1(\overline{u}_K^+) > 0$: take $\overline{u} = \overline{u}_K^+ = \overline{u}(x_*, x_K^+, \Theta_K) \in Z$.

We now mollify $\overline{u}$. For $\overline{u} \in Z$, we have $\lambda_1(\overline{u}) = 0$, and $\delta_1(\overline{u}) > 0$. Let $U \subset Z$ be an open ball centered at $\overline{u}$ in which δ_1 is positive. Consider the segment $\overline{u}(t) = \overline{u}(x_* + t, x_K^+, \Theta_K) \in H^0$ for t near zero (notice that Θ_K stays fixed). Since $f''(x_*) < 0$, from Proposition 2.9, $\lambda_1(\overline{u}(t))$ at $t = 0$ is strictly monotonic. Indeed,

$$\frac{d}{dt}\bigg|_{t=0} \lambda_1(u(t)) = \left\langle \nabla\lambda_1(\overline{u}(0)), \frac{d}{dt}\bigg|_{t=0} u(t) \right\rangle = \left\langle -f''(\overline{u})\phi_1^2(\overline{u}), u(1,0,\Theta_k) \right\rangle > 0.$$

Thus, there are $\overline{u}^+, \overline{u}^- \in U$ with $\lambda_1(\overline{u}^+) > 0$ and $\lambda_1(\overline{u}^-) < 0$. Now take $u^+, u^- \in U \cap D$ for which $\lambda_1(u^+) > 0$ and $\lambda_1(u^-) < 0$. By continuity, there is $u \in D$ in the segment joining u^+ and u^- for which $\lambda_1(u) = 0$ and $\delta_1(u) > 0$. □



The proof of Theorem 1.1 is complete.

3. Cusps and Theorem 1.2

The proof of Theorem 1.2 requires a few steps. In Subsection 3.1 we validate the characterization of a cusp u_c as a zero u_{nf} of the function Λ, together with the transversality condition $\tau_1(u_{nf}) \neq 0$. The next step is to obtain a zero $\overline{u}_{nf} \in \overline{\mathcal{V}}$ (Subsection 3.2) and then by mollification a smooth zero u_{nf} (Subsection 3.3) for which $\nabla\lambda_1(u_{nf})$ and $\nabla\delta_1(u_{nf})$ are independent. Finally we show that near u_{nf} either there is a cusp u_c or there is a codimension-3 set of points u_i for which each $F(u_i)$ has a full arc of preimages (Subsection 3.4).

3.1. Folds, nonfolds and cusps

From Section 2, there are fibers on which the height h has local maxima and minima. A coalescence of both extrema would yield a critical point of h for which the *second* derivative along a fiber is zero —a *nonfold*— which we proceed to describe in detail. Nonfolds which satisfy additional generic properties are *cusps*.

Nonfolds and cusps are identified with local checks: asymptotic hypotheses like (2) are irrelevant. For maps between spaces of finite dimensions, folds and cusps are special cases of *Morin singularities* [18], generic critical points u for which the differential $DF(u)$ has kernel of dimension equal to one. The original characterization extends for functions between Banach spaces [4,9,13,17,23,24]. We follow closely [9, page 183], which we transcribe.

Theorem (Church-Timourian). *Let X and Y be real Banach spaces, $U \subset X$ be an open set and $G : U \to Y$ be a smooth function. Also, let $\mathcal{C} \subset U$ be the critical set of G and $\mathcal{N} \subset \mathcal{C}$ consist of the points $u \in \mathcal{C}$ for which* $\operatorname{Ker} DG(u) \subset T_u\,\mathcal{C}$. *Then $u_c \in U$ is a cusp of G if and only if:*

(a) *$DG(u_c)$ is a Fredholm map of index* 0 *and* $\dim \operatorname{Ker} DG(u_c) = 1$*;*
(b) *$\mathcal{C} \subset U$ is a manifold of codimension 1 and $u_c \in \mathcal{N}$*, i.e., $\operatorname{Ker} DG(u_c) \subset T_{u_c}\,\mathcal{C}$*;*
(c) *$\mathcal{N} \subset \mathcal{C}$ is a manifold of codimension 1 of $\mathcal{C}$ and* $\operatorname{Ker} DG(u_c) \not\subset T_{u_c}\,\mathcal{N}$.

The smoothness of $F : B_b^2 \to B^0$ in Theorem 1.2 is proved in [23]. For $u \in Z$, by hypothesis (1), the eigenpair $(\lambda_1(u), \phi_1(u))$ is well defined. Let $V = \langle \phi_1(u) \rangle$, and consider the orthogonal split $Z = W \oplus V$, $H_b^2 = W^2 \oplus V$ with associated projection $\Pi_W : Z \to W$. Denote the restriction of an operator T to W by T_W.

Proposition 3.1. *Assume hypothesis* (1). *Let $f : \mathbb{R} \to \mathbb{R}$ be smooth. The function*

$$\Lambda : B_b^2 \to \mathbb{R}^2, \quad \Lambda(u) = \big(\lambda_1(u)\,,\ \delta_1(u)\big)$$

is smooth. The gradient of $\delta_1 : B_b^2 \to \mathbb{R}$ is

$$\nabla\delta_1(u) = -f'''(u)\phi_1^3(u) - 3w(u)f''(u)\phi_1(u) \in Z\,,$$

where $w(u) = D\phi_1(u)\,\phi_1(u) = \big(DF(u) - \lambda_1(u)I\big)_W^{-1}\Pi_W(f''(u)\phi_1^2(u)) \in W^2 \subset H^2$. The extension $\Lambda : Z \to \mathbb{R}^2$, the gradient $\nabla\delta_1 : Z \to Z$ and the functional

$$\tau_1 : Z \to \mathbb{R}\,, \quad \tau_1(u) = D\delta_1(u)\,\phi_1(u) = \langle\, \nabla\delta_1(u)\,,\ \phi_1(u)\,\rangle$$

are ub-continuous. For any 2*-dimensional affine subspace $V_* \subset Z$, the restriction $\Lambda : V_* \to \mathbb{R}^2$ is a $\mathcal{C}^1$ map.*

The proof is given in Subsection 5.3. We now show that the requirements in Theorem 1.2 indeed yield a cusp in the sense of the theorem above.

Proposition 3.2. *Assume standard boundary conditions and hypotheses* (1) *and* (4). *A zero u_c of $\Lambda : B_b^2 \to \mathbb{R}^2$ for which $\tau_1(u_c) \neq 0$ is a bona fide cusp of F.*

Proof. We check the hypotheses of the theorem above for u_c. Given standard boundary conditions and hypothesis (1), λ_1 and ϕ_1 are globally defined, even for $u \in H^0$. In particular, (a) is satisfied. Denote the levels $\lambda_1^{-1}(0)$ and $\delta_1^{-1}(0)$ by $\mathcal{C}$ and $\mathcal{D}$, so that $u_c \in \mathcal{N} = \mathcal{C} \cap \mathcal{D} \subset B_b^2$. Most of the result follows from the linear independence of the gradients of λ_1 and δ_1 at points $u \in \mathcal{N}$: near $\mathcal{N}$ the sets $\mathcal{C}$ and $\mathcal{D}$ are then hypersurfaces and their transversal intersection $\mathcal{N}$ is a manifold.

First, $\nabla\lambda_1(u_c) = -f''(u_c)(\phi_1(u_c))^2 \not\equiv 0$. Indeed, since the zeros of f'' are isolated and $u_c \in B_b^2$ is continuous, if $f''(u_c) \equiv 0$ then u_c must be constant, implying $\lambda_1(u_c) = 0$ and $f'(u_c) = \mu_{1,b}$, contradicting (4).

Thus $\nabla\lambda_1$ and $\nabla\delta_1$ are dependent only if $c\nabla\lambda_1(u_c) = \nabla\delta_1(u_c)$ for some $c \in \mathbb{R}$. From the expression for $\nabla\delta_1$ in Proposition 3.1,

$$cf''(u_c)\phi_1^2(u_c) = -f'''(u_c)\phi_1^3(u_c) - 3wf''(u_c)\phi_1(u_c),$$

and hence $(c\phi_1 + 3w)f'' = -f'''\phi_1^2$. In Ω we have $\phi_1 > 0$, and $u_c \in \mathcal{N}$ implies

$$\delta_1(u_c) = \int_\Omega f''(u_c)\phi_1^3(u_c) = 0\,,$$

so that $f''(u_c)$ changes sign. Thus $f''(u_c)$ and $f'''(u_c)$ have common zeros, again contradicting (4): the gradients are independent. Since $\ker DF(u_0) = \langle\phi_1(u_c)\rangle$, (b) and the first part of (c) are true. The second part of (c) is exactly $\tau_1(u_c) \neq 0$. □

A *fold* u_f is a regular critical point of F for which $\phi_1(u_f) \notin T_{u_f}\mathcal{C}$ (*i.e.* $\delta_1(u_f) \neq 0$). A *nonfold* $u_{nf} \in Z$ is a zero of $\Lambda : Z \to \mathbb{R}^2$ and it is *regular* if $D\Lambda(u_{nf})$ is surjective. Thus, a cusp $u_c \in B_b^2$ is a regular nonfold for which $\tau_1(u_c) \neq 0$.

3.2. A regular nonfold $\overline{u}_{nf}$

The next result plays the role of balancing for the functional δ_1.

Proposition 3.3. *Assume standard boundary conditions and hypotheses* (1), (3) *and* (4). *Then there is a regular nonfold* $\overline{u}_{nf} \in \overline{\mathcal{V}}$.

Proof. Given a continuous function $h : \mathbb{R} \to \mathbb{R}$, we say that two points $x, y \in \mathbb{R}$ are *opposite* if $h(x)h(y) < 0$. To have $\lambda_1(\overline{u}_{nf}) = 0$ for $\overline{u}_{nf} = \ell\,\chi_\theta + r\,\chi_\theta^c$, either ℓ and r are opposite with respect to $f' - \mu_{1,b}$ or $f'(\ell) = f'(r) = \mu_{1,b}$. Conversely, $f'(\ell) = f'(r) = \mu_{1,b}$ implies $\lambda_1(\overline{u}_{nf}) = 0$ for any $\theta \in [0, 2\pi]$, or the unique balancing $\Theta(f'(\ell), f'(r))$ provided by Lemma 2.8 yields $\lambda_1(\overline{u}_{nf}) = 0$.

Similarly, to have $\delta_1(\overline{u}_{nf}) = 0$, either ℓ and r are opposite with respect to f'' or $f''(\ell) = f''(r) = 0$. In what follows, we take ℓ so that $f'''(\ell) = 0$, to be used in the proof that $\overline{u}_{nf}$ is a regular nonfold.

From (3) and (4), the nonempty set $X_* = \{x \mid f''(x) = 0\}$ is closed and consists of isolated points for which $f' - \mu_{1,b}, f''' \neq 0$. The complement $\mathbb{R} - X_* = \cup_i I_i$ is a union of open sets, and again by (3) and (4), there must be an interval I_+ in which

f' increases and $f'(y) = \mu_{1,b}$ for some $y \in I_+$. Suppose without loss that there is $x_* \in X_*$ with $x_* < y$ (the case $x_* > y$ is similar): since X_* consists of isolated points, we may take x_* such that $I_+ = (x_*, x_+)$, where possibly $x_+ = \infty$. Consider also $I_- = (x_-, x_*)$, the interval immediately to the left of I_+, where again we may have $x_- = -\infty$. We now search for ℓ and r.

Suppose first that f' stays below $\mu_{1,b}$ on I_-. There is a point $\ell \in I_-$ for which $f'''(\ell) = 0$. Indeed, if I_- is finite, its endpoints are zeros of f'' and by the mean value theorem there must be a root of f'''. If I_- is infinite, recall that f' is strictly decreasing in I_- with a local minimum at the right endpoint x_*, so that $f'''(x_*) > 0$ by (4). The fact that f' is bounded above by $\mu_{1,b}$ forces a change of curvature, so that again f''' must have a root. Take $r \in (y, x_+)$, the part of I_+ above $\mu_{1,b}$, so that ℓ and r are opposite with respect to both $f' - \mu_{1,b}$ and f''.

We split again in cases: take $x_+ < \infty$, and thus $f''(x_+) = 0$. We compute δ_1 for $r = y$ and $f = x_+$. For $\overline{u} = \overline{u}(\ell, y, \Theta(f'(\ell), f'(y))$, since $f'(y) = \mu_{1,b}$ we have

$$\Theta(f'(\ell), f'(y)) = 0\,, \quad \operatorname{meas}(\Omega_0) = 0\,, \quad f''(y) > 0\,,$$

$$\delta_1\big(\overline{u}(\ell, y, 0)\big) = -\int_{\Omega_0} f''(\ell)\big(\phi_1(\overline{u})\big)^3 - \int_{\Omega_0^c} f''(y)\big(\phi_1(\overline{u})\big)^3 < 0\,.$$

For $\overline{u} = \overline{u}(\ell, x_+, \Theta(f'(\ell), f'(x_+))$, since $f''(\ell) < 0$ and $f''(x_+) = 0$,

$$\delta_1(\overline{u}) = -\int_{\Omega_{\Theta(f'(\ell),f'(x_+))}} f''(\ell)\big(\phi_1(\overline{u})\big)^3 - \int_{\Omega^c_{\Theta(f'(\ell),f'(x_+))}} f''(x_+)\big(\phi_1(\overline{u})\big)^3 > 0\,.$$

By the continuity of δ_1 (Proposition 2.9), we are done: there is $r \in (y, x_+)$ for which $\overline{u}_{nf} = \overline{u}(\ell, r, \Theta(f'(\ell), f'(r))$ is a common zero of λ_1 and δ_1 with $f'''(\ell) = 0$.

Suppose now $x_+ = \infty$. Take $x_m^+ \to \infty, x_m^+ \in I_+$ as in Lemma 2.10 for $g = f'$, so that $f'(x_m^+) > \mu_{1,b}$ and $f''(x_m^+) \to 0$, and follow the proof of Proposition 2.11.

Suppose finally that f' crosses $\mu_{1,b}$ on I_-. Again, there will be $\ell \in I_-$ for which $f'''(\ell) = 0$. If $f'(\ell) < \mu_{1,b}$ we obtain r exactly as before. If instead $f'(\ell) > \mu_{1,b}$ (equality is not permitted, by (4)), search for r in the opposite interval (x_*, y): the signs of δ_1 at the endpoints x_* and y are opposite, and the existence of a nonfold $\overline{u}_{nf}$ for which $f'''(\ell) = 0$ is proved.

We now show that $\overline{u}_{nf}$ with $f'''(\ell) = 0$ is a *regular* nonfold. From hypothesis (4), $f''(\ell) \neq 0$ and $\nabla\lambda_1(\overline{u}_{nf}) = f''(\overline{u}_{nf})\phi_1^2(\overline{u}_{nf}) \neq 0$. Thus the gradients of λ_1 and δ_1 are dependent if and only if $c.\nabla\lambda_1(\overline{u}_{nf}) = \nabla\delta_1(\overline{u}_{nf})$, for $c \in \mathbb{R}$. From the expression for $\nabla\delta_1$ in Proposition 3.1,

$$cf''(\overline{u}_{nf})\phi_1^2(\overline{u}_{nf}) = -f'''(\overline{u}_{nf})\phi_1^3(\overline{u}_{nf}) - 3wf''(\overline{u}_{nf})\phi_1(\overline{u}_{nf}),$$

hence $(c\phi_1 + 3w)f'' = -f'''\phi_1^2$. On $\Omega_{\Theta(f'(\ell),f'(r))}$, where $f'''(\ell) = 0$ (and then $f''(\ell) \neq 0$),

$$w(\overline{u}_{nf}) = c'\phi_1(\overline{u}_{nf}) \quad \text{for some } c' \in \mathbb{R}\,.$$

From the expression for w in Proposition 3.1, taking into account that $\lambda_1(\overline{u}_{nf}) = 0$ and $\delta_1(\overline{u}_{nf}) = 0$, so that $\Pi_W(f''(\overline{u}_{nf})\phi_1^2(\overline{u}_{nf})) = f''(\overline{u}_{nf})\phi_1^2(\overline{u}_{nf})$,

$$w(\overline{u}_{nf}) = \big(DF(\overline{u}_{nf})\big)_W^{-1}\,\big(f''(\overline{u}_{nf})\phi_1^2(\overline{u}_{nf})\big).$$

Since $DF(\overline{u}_{nf})$ is a local operator, one may apply it on both sides of

$$w(\overline{u}_{nf})(x) = c'\phi(\overline{u}_{nf})(x)\,, \quad x \in \Omega_{\Theta(f'(\ell),f'(r))} \neq \emptyset\,,$$

and $f''(\ell)\phi_1^2(\overline{u}_{nf})(x) = 0\,,\ x \in \Omega_{\Theta(f'(\ell),f'(r))}$, clearly a contradiction. □

3.3. Smoothing: from $\overline{u}_{nf}$ to a regular nonfold $u_{nf} \in D$

Let $D \subset B_b^2$ be a dense subspace of Z. We obtain a regular nonfold $u_{nf} \in D$ by mollification of $z_* = \overline{u}_{nf}$ from Proposition 3.3. Consider a ball $B_{z_*}(r) \subset Z$ in the sup norm. By Proposition 2.9, $\Lambda : B_{z_*}(r) \subset Z \to \mathbb{R}^2$ is continuous (ub-continuous functions on L^∞-bounded sets are continuous) and $\Lambda(z_*) = 0$.

The existence of two functions $v_1, v_2 \in D$ with invertible Jacobian

$$\begin{pmatrix} \langle \nabla\lambda_1(z_*)\,,\ v_1 \rangle & \langle \nabla\lambda_1(z_*)\,,\ v_2 \rangle \\ \langle \nabla\delta_1(z_*)\,,\ v_1 \rangle & \langle \nabla\delta_1(z_*)\,,\ v_2 \rangle \end{pmatrix}$$

is clear, since z_* is a regular nonfold. Let $\tilde{V}_* \subset D$ be the span of v_1 and v_2 and set $V_* = z_* + \tilde{V}_*$. Thus $z_* \in Z$ is a regular point of the restriction $\Lambda_* : V_* \to \mathbb{R}^2$, which is a $\mathcal{C}^1$ map from Proposition 3.1. For small balls $B_{z_*}(\epsilon) \subset B_{z_*}(r)$, the topological degree satisfies $\deg(\Lambda_*, B_{z_*}(\epsilon) \cap V_*, 0) = \pm 1$.

Take $z_m \in D \cap B_{z_*}(\epsilon)$ with $z_m \xrightarrow{\text{ub}} z_*$. Define $V_m = z_m + \tilde{V}_* \subset D$. For large m, the restrictions $\Lambda_* : B_{z_*}(\epsilon) \cap V_* \to \mathbb{R}^2$ and $\Lambda_m : B_{z_m}(\epsilon) \cap V_m \to \mathbb{R}^2$ are arbitrarily close in the uniform norm after composing with the obvious translation. For a small ball $B_{z_*}(\epsilon)$,

$$\deg(\Lambda_*, B_{z_*}(\epsilon) \cap V_*, 0) = \deg(\Lambda_m, B_{z_*}(\epsilon) \cap V_m, 0) \neq 0\,.$$

Thus Λ_m has a zero in $B_{z_*}(\epsilon) \cap V_m$ and we obtain a sequence of nonfolds $u_m \in V_m \subset D$ convergent to $z_* = \overline{u}_{nf}$ in Z.

The restrictions $\Lambda_m : V_m \to \mathbb{R}^2$ are smooth and their Jacobians at u_m converge to $D\Lambda_*(z_*)$, by Proposition 3.1. For large m, $D\Lambda(u_m)$ is then surjective and any such u_m is a regular nonfold $u_{nf} \in D$.

3.4. Regular nonfolds imply cusps or worse

The proof of Theorem 1.2 is almost complete. A regular nonfold $u_{nf} \in D \subset B_b^2$ which satisfies the condition $\tau_1(u_{nf}) = D\delta_1(u_{nf})\,\phi_1(u_{nf}) \neq 0$ in Proposition 3.2 is a cusp. If instead $\tau_1(u_{nf}) = 0$, as we shall see, there is an abundance of fibers, each taken by F to a point.

Recall the zero levels $\mathcal{C}, \mathcal{D} \in B_b^2$ of λ_1 and δ_1. From Section 3.3, the set of nonfolds $\mathcal{N} = \mathcal{C} \cap \mathcal{D}$ is nonempty. From the proof of Proposition 3.2, a point $u_{nf} \in \mathcal{N}$ for which $\tau_1(u_{nf}) \neq 0$ is automatically a cusp. Thus,

F has no cusps if and only if τ_1 is identically zero in $\mathcal{N}$.

As in Section 2.2, a fiber is the inverse under F of a line parallel to the free eigenfunction $\psi_{1,b}$.

Proposition 3.4. *Assume $\tau_1 \equiv 0$ in $\mathcal{N} \neq \emptyset$ and the hypotheses of Theorem* 1.2. *Then $\mathcal{N}$ is foliated by fibers, each being sent to a single point by F.*

Proof. As in the proof of Proposition 3.2, $\mathcal{N}$ is the transversal intersection of the sets $\mathcal{C}$ and $\mathcal{D}$, which are manifolds near $\mathcal{N}$, since the gradients $\nabla\lambda_1(u)$ and $\nabla\delta_1(u)$ are linearly independent. The differential equation in B_b^2

$$u' = \tilde{\phi}_1(u) = \frac{\phi_1(u)}{\|\phi_1(u)\|_{B^2}}, \quad u(0) = u_0$$

has a globally defined solution $\gamma = \{u(t), t \in \mathbb{R}\}$. The vector field $u \mapsto \tilde{\phi}_1(u)$ restricts to a vector field tangent to $\mathcal{N}$, since τ_1 is identically zero on $\mathcal{N}$. Indeed, $\delta_1(u) = 0$ and $\tau_1(u) = 0$ imply that $\phi_1(u)$ is orthogonal to $\nabla\lambda_1(u)$ and $\nabla\delta_1(u)$ respectively, so that $\phi_1(u) \in T_u\mathcal{N}$. Thus if $u(0) \in \mathcal{N}$ then $\gamma \subset \mathcal{N}$ and γ consists only of critical points.

From Proposition 2.6, at a critical point $u(t_c)$ the kernel vector $\phi_1(u(t_c))$ of $DF(u(t_c))$ is the tangent vector to the fiber at $u(t_c)$. Thus integration of the vector field above also yields the fibers through points in $\mathcal{N}$: $\mathcal{N}$ is foliated by fibers. Finally, from Proposition 2.6, along fibers $u(t) \in \mathcal{N}$,

$$\frac{d}{dt} h(u(t)) = p(u(t))\, \lambda_1(u(t)) = 0\,.$$

The height $h(u(t))$ does not change: F takes each fiber to a single point. □

The proof of Theorem 1.2 is now complete.

Are there functions as suggested by the alternative in the theorem? At a nonfold u_{nf}, F would take the (local) normal form near the origin:

$$\Psi(w, x, y, z) = (w, x, y, x\alpha(w, x, y, z) + y\beta(w, x, y, z)), \quad w \in W, (x, y, z) \in \mathbb{R}^3,$$

where W is a Banach space. The vertical axes $(w, 0, 0, z)$ are taken to $(w, 0, 0, 0)$: they all collapse under F. The simplicity of the example is misleading: its rarity is due to the fact that the collapse happens for an open set of W. What is not clear is that Ψ indeed is a local form of some function F near some u_{nf}. There are nonlinearities for which a few fibers have locally constant height functions [27], but they are far from being as abundant as in the situation above.

Here are some hypotheses which guarantee the existence of cusps.

Proposition 3.5. *Consider the hypotheses of Theorem* 1.2*. Each of the next two possibilities implies the existence of a cusp of F:*

1. $F : B_b^2 \to B^0$ *is proper;*
2. $\overline{u}_{nf} = \overline{u}(\ell, r, \Theta(f'(\ell), f'(r))) \in \overline{\mathcal{V}}$ *is a zero of* Λ *with* $f'''(\ell), f'''(r) \geq 0$.

Hypothesis (2) implies properness of F [7] and thus forces cusps.

Proof. From the hypotheses, a fiber $u(t)$ through $u(t_0) \in \mathcal{N}$ reaches arbitrary heights t. If there are no cusps, on the other hand, the fibers in $\mathcal{N} \neq \emptyset$ are taken to a single point, as shown above. This cannot happen if F is proper.

For the second possibility, we have $\lambda_1(\overline{u}_{nf}) = 0$ and $\delta_1(\overline{u}_{nf}) = 0$, so that $\Pi_W(f''(\overline{u}_{nf})\phi_1^2(\overline{u}_{nf})) = (f''(\overline{u}_{nf})\phi_1^2(\overline{u}_{nf}))$. We show $\tau_1(\overline{u}_{nf}) \neq 0$. By Proposition 3.1, omitting the dependence in $\overline{u}_{nf}$,

$$\tau_1 = \langle \nabla\delta_1 , \phi_1 \rangle = \langle -f'''\phi_1^3 - 3wf''\phi_1 , \phi_1 \rangle \quad \text{for} \quad w = \big(DF\big)_W^{-1}(f''\phi_1^2)\,.$$

Now, $\big(DF\big)_W^{-1} : \phi_1^\perp \to \phi_1^\perp$ is a positive operator, since from Proposition 2.2 only the smallest eigenvalue λ_1 can be zero. Thus $\langle -3wf''\phi_1 , \phi_1 \rangle < 0$. We also have $\langle -f'''\phi_1^3 , \phi_1 \rangle \leq 0$, since $f'''(\ell), f'''(r) \geq 0$.

A cusp u_c is obtained by smoothing of $\overline{u}_{nf}$ once we know the independence of $\nabla\lambda_1(\overline{u}_{nf})$ and $\nabla\delta_1(\overline{u}_{nf})$. Since $\nabla\lambda_1(\overline{u}_{nf}) \neq 0$ and $\delta_1(\overline{u}_{nf}) = 0$, this follows from

$$\tau_1(\overline{u}_{nf}) = \langle \nabla\delta_1(\overline{u}_{nf}), \phi_1(\overline{u}_{nf}) \rangle \neq 0\,. \qquad \square$$

4. Proof of Theorem 1.3

Most of the argument follows the proof of Theorem 1.2, with simple adaptations. Again, we consider standard boundary conditions. Hypothesis (H_k) guarantees that $\mu_{k,b}$ is simple, so that, from Subsection 5.1 the functional λ_k, the k-th eigenvalue of $DF(u)$, and the corresponding normalized eigenfunction ϕ_k are well defined and appropriately smooth in a neighborhood $U \subset Z$ of u_0 for which $f'(u_0) \equiv \mu_{k,b}$. The characterization of a cusp $u_c \in U$ is analogous: it is a zero of

$$\Lambda : U \cap B_b^2 \to \mathbb{R}^2\,, \quad \Lambda(u) = (\lambda_k(u), \delta_k(u) = D\lambda_k(u)\, \phi_k(u))$$

for which $D\Lambda(u_c) : B_b^2 \to \mathbb{R}^2$ is surjective and $\tau_k(u_c) = D\delta_k(u_c)\, \phi_k(u_c) \neq 0$.

There are small changes to be made in the counterpart to Proposition 3.3.

Proposition 4.1. *Assume* (H_k)*. Then there is a regular nonfold* $\overline{u}_{nf} \in \overline{\mathcal{V}}$.

Proof. For the free eigenfunction $\psi_{k,b}$, take $p \in \Omega$ for which $\psi_{k,b}(p) \neq 0$ (this is automatic for $k = 1$, since the ground state is positive).

The required $\overline{u}_{nf}$ is of the form $\overline{u} = \overline{u}(\ell = x_*, r = y_*, \theta) = x_*\chi_\theta + y_*\chi_\theta^c$, for some $\theta = \theta_0$. For *any* choice of θ, $f'(\overline{u}) \equiv \mu_{k,b}$, so that $\lambda_k(\overline{u}) = 0$ and $f''(\overline{u})$

is nonzero on χ_θ and χ_θ^c. For $\theta = 0$ or 2π, $\bar{u}$ is constant (either x_* or y_*) and $\delta_k(x_*)\delta_k(y_*) \leq 0$. If it is zero, take the $\bar{u}$ for which $\delta_k = 0$ to be $\bar{u}_{nf}$ (in this case, ϕ_k^3 integrates to 0 in Ω). Otherwise, by the ub-continuity of δ_k in Proposition 3.1, an intermediate θ_0 yields $\bar{u}_{nf}$.

We now show regularity. From $f'(\bar{u}_{nf}) = \mu_{k,b}$, we have

$$\lambda_k(\bar{u}_{nf}) = 0\,, \quad \phi_k(\bar{u}_{nf}) = \psi_{k,b}\,, \quad DF(\bar{u}_{nf}) = (-\Delta_b - \mu_{k,b})\,.$$

Since $\delta_k(\bar{u}_{nf}) = 0$, $\langle f''(\bar{u}_{nf})\psi_{k,b}^2\,,\ \psi_{k,b}\rangle = 0$ and $f''(\bar{u}_{nf})\psi_{k,b}^2 \in W$. From (H_k), $\nabla\lambda_k(\bar{u}_{nf}) = f''(\bar{u}_{nf})\,\psi_{k,b}^2 \neq 0$ which combined with the linear dependence of the gradients $\nabla\lambda_k(\bar{u}_{nf})$ and $\nabla\delta_k(\bar{u}_{nf})$ implies collinearity,

$$\nabla\delta_k(\bar{u}_{nf}) \;=\; d\,\nabla\lambda_k(\bar{u}_{nf}) \quad \text{for } d \in \mathbb{R}\,,$$

which, as we shall see, leads to a contradiction. By Proposition 3.1,

$$-f'''(\bar{u}_{nf})\,\psi_{k,b}^3 - 3\,w\,f''(\bar{u}_{nf})\,\psi_{k,b} \;=\; d\,f''(\bar{u}_{nf})\,\psi_{k,b}^2,$$

and

$$\begin{aligned} w \;&=\; (DF(\bar{u}_{nf}) - \lambda_k(\bar{u}_{nf}))|_W^{-1}\,\Pi_W\,(f''(\bar{u}_{nf})\,\psi_{k,b}^2) \\ &=\; (-\Delta_b - \mu_{k,b})|_W^{-1}\,(f''(\bar{u}_{nf})\psi_{k,b}^2) \;\in\; H_b^2\,, \end{aligned}$$

so that (since $f'' \neq 0$ and $\psi_{k,b} = 0$ on a set of measure zero)

$$-\frac{f'''(\bar{u}_{nf})}{f''(\bar{u}_{nf})}\,\psi_{k,b}^2 - 3\,w \;=\; d\,\psi_{k,b}\,.$$

Now, w and $\psi_{k,b}$ belong to H_b^2, and $\psi_{k,b}(p) \neq 0$, so the two-valued fraction (at p, and thus, throughout the jump between Ω_Θ and Ω_Θ^c) is actually a constant, say $a \in \mathbb{R}$. Applying $DF(\bar{u}_{nf}) = (-\Delta_b - \mu_{k,b})$,

$$a\,(-\Delta_b - \mu_{k,b})\psi_{k,b}^2 - 3\,f''(\bar{u}_{nf})\psi_{k,b}^2 \;= d\,DF(\bar{u}_{nf})\psi_{k,b} = 0\,.$$

The first term on the left hand side is smooth, thus $f''(\bar{u}_{nf})$ is also a constant. But f'' has opposite signs at x_* and y_*, a contradiction, so that $\nabla\lambda_k(\bar{u}_{nf})$ and $\nabla\delta_k(\bar{u}_{nf})$ are linearly independent in Z. □

Let $D \subset B_b^2$ be a dense subspace of Z. Mimic Subsection 3.3 to obtain $u_{nf} \in D$ from $\bar{u}_{nf}$ and then Proposition 3.4 to prove the dichotomy in Theorem 1.3.

We finally consider the situations in which F necessarily has a cusp. For the hypothesis $k = 1$ and $f'''(x_*)$, $f'''(y_*) \geq 0$, the argument in the proof of the second case of Proposition 3.5 applies with no change.

Suppose then $k = 1$ and F is proper. We follow the proof of Proposition 3.4, but there are new difficulties. First notice that under hypothesis (H_1), $DF(u)$ may cease to be invertible and still $\lambda_1(u) \neq 0$. For the nonfold $\bar{u}_{nf}$ obtained in the proof of Proposition 4.1, however, $\lambda_1(\bar{u}_{nf}) = 0$. Also, $\lambda_1, \phi_1 > 0$ and δ_1 are still globally defined and smooth. Let $\mathcal{C}_1 = \{u \in B_b^2, \lambda_1(u) = 0\} \subset \mathcal{C}$. Since $\nabla\lambda_1(u) \neq 0$ for $u \in \mathcal{C}_1$ (as in the proof of the proposition above), $\mathcal{C}_1$ is a manifold. It is easy to see that some smoothing u_{nf} obtained from $\overline{u}_{nf}$ belongs to $\mathcal{C}_1$. Define $\mathcal{D}_1$ to be the zero level of the functional δ_1.

Following the proof of Proposition 3.2, every nonfold $u_{nf} \in \mathcal{N}_1 = \mathcal{C}_1 \cap \mathcal{D}_1$ is regular. Again, $\mathcal{N}_1$ is the transversal intersection of the two manifolds $\mathcal{C}_1$ and $\mathcal{D}_1$ near $\mathcal{N}_1$ as in Proposition 3.4. If $u \in \mathcal{N}_1$ and $\tau_1(u) \neq 0$, then u is a cusp of F. Suppose that $\tau_1 \equiv 0$ in $\mathcal{N}_1$ —we derive a contradiction.

As in Proposition 3.4, the vector field $u \mapsto \tilde{\phi}_1(u)$ leaves $\mathcal{N}_1$ invariant, and each integral curve $\gamma = \{u(t), t \in \mathbb{R}\} \subset \mathcal{N}_1$ of $u' = \tilde{\phi}_1(u)$ is sent by F to a single point, $F(\gamma)$. These integral curves may not be periodic. Indeed, the function $t \mapsto \langle \psi_{1,b}, u(t)\rangle$ is a height function along the curve, since it has positive derivative: for $k = 1$, the eigenfunctions $\psi_{1,b}$ and $\phi_1(u(t))$ are positive. Take the integral curve $\gamma_{u_0} = \{u(t), t \in \mathbb{R}, u(0) = u_0\} \subset \mathcal{N}_1$ with $F(\gamma_{u_0}) = z_0$.

The set $F^{-1}(z_0) \cap \mathcal{N}_1$ may possibly disconnect but it still locally an interval: more precisely, for each point $y_0 \in F^{-1}(z_0) \cap \mathcal{N}_1$ there is an open neighborhood $U_{y_0} \subset B_b^2$ of y_0 so that $F^{-1}(z_0) \cap U_{y_0}$ is an arc $\tilde{\gamma}_{y_0}$ (*i.e.*, $\tilde{\gamma}_{y_0}$ is diffeomorphic to an open interval). This follows from a local form of the construction of fibers. Let $\langle \phi_1(y_0)\rangle$ be the subspace spanned by $\phi_1(y_0)$ and split

$$B_b^2 = W \oplus \langle \phi_1(y_0)\rangle, \quad B^0 = \operatorname{Ran} DF(y_0) \oplus \langle \phi_1(y_0)\rangle .$$

For small $w \in W$, $t \in \mathbb{R}$, write

$$F(y_0 + w + t\phi_1(y_0)) = \Pi\, F(y_0 + w + t\phi_1(y_0)) + (I - \Pi)\, F(y_0 + w + t\phi_1(y_0)),$$

where $\Pi : B^0 \to \operatorname{Ran} DF(y_0)$ is the projection with $\ker \Pi = \langle \phi_1(y_0)\rangle$. From simple spectral arguments, the inverse function theorem applies and we learn that, for each fixed t close to 0, the map $w \mapsto \Pi F(y_0 + w + t\phi_1(y_0))$ is a local diffeomorphism near $w = 0$ to $\operatorname{Ran} DF(y_0)$. In particular, the inversion of a small segment $z_0 + t\langle\phi_1(y_0)\rangle$, for t near 0, obtains the small isolated arc $\tilde{\gamma}_{y_0} \in B_b^2 \cap U_{y_0}$ through y_0 —in a sense, this arc is a piece of a fiber.

Take now $y_0 = u_0$ in the construction above and we have, for $F(u_0) = z_0$,

$$\gamma_{u_0} \cap U_{u_0} \subset \tilde{\gamma}_{u_0} = F^{-1}(z_0) \cap U_{u_0} .$$

Since F is proper, the closure $\overline{\gamma}_{u_0} \subset F^{-1}(z_0)$ is compact. Then, at integer times $t = n$, the sequence $u(n)$ accumulates to $u_* \in B_b^2$. By continuity, since $u(n) \in \mathcal{N}_1$, we have that $u_* \in \mathcal{N}_1$ and $F(u_*) = z_0$. Consider the solution γ_{u_*} of $u' = \tilde{\phi}(u)$, $u(t_0) = u_*$. For large n, $u(n) \in \gamma_{u_*} \subset \tilde{\gamma}_{u_*} \subset F^{-1}(z_0) \cap U_{u_*}$.

In particular, the points $u(n)$ lie in the single arc through their accumulation $u_* \in \gamma_{u_*} \cap U_{u_*}$ for some neighborhood U_{u_*}. More, since the tangent vector $\tilde{\phi}(u)$ is of norm one, the points $u(n)$ are far apart along the orbit. Accumulation is not possible then, since the orbit admits a height function: $\overline{\gamma}$ is not compact.

Finally, $F^{-1}(z_0)$ is also not compact (or the closed subset $\overline{\gamma}$ would be too), contradicting the properness of F. The proof of Theorem 1.3 is now complete.

5. Appendix: well-definedness and continuity

We prove that the basic functions in the text satisfy the required continuity and differentiability. We assume standard boundary conditions.

5.1. The eigenpair (λ^q, ϕ^q) for potentials $q \in L^\infty$

The proposition below is a simplified version of [28, Proposition 79.14]. For $X \subset Y$ real Banach spaces, let $\mathcal{B} = \mathcal{B}(X, Y)$ be the Banach space of bounded linear transformations from X to Y, with the operator norm.

Proposition 5.1. *Let $T_0 \in \mathcal{B}$ have eigenvalue $\lambda_0 \in \mathbb{R}$ and eigenvector $\phi_0 \in X$, so that $(T_0 - \lambda_0 I)\phi_0 = 0$. Assume that $T_0 - \lambda_0 I$ is a Fredholm operator of index zero with one-dimensional kernel, and that $\phi_0 \notin \mathrm{Ran}(T_0 - \lambda_0 I)$. Let $\ell \in X^*$ be a linear functional for which $\ell(\phi_0) = 1$ and set $V_2 = \phi_0 + \mathrm{Ker}\,\ell$. Then there is an open neighborhood $U \subset \mathcal{B}$ of T_0 and unique analytic maps $\lambda : U \to \mathbb{R}$ and $\phi : U \to V_2$ for which $(T - \lambda(T)I)\,\phi(T) = 0$ and $\lambda(T_0) = \lambda_0$, $\phi(T_0) = \phi_0$.*

The operators T_q of Proposition 2.2 are smooth functions of $q \in L^\infty(\Omega)$: the linear map taking q to 'multiplication by q'

$$\Phi_q : L^\infty \to \mathcal{B}(H^2_b, H^0)\ , \quad q \mapsto M_q$$

is clearly bounded. For $q \in L^\infty$, Propositions 2.2 and 2.3 imply the hypotheses of the proposition above, and thus $\lambda_1^q : L^\infty \to \mathbb{R}$ and $\phi_1^q : L^\infty \to H^0$ are smooth.

5.2. The eigenpair (λ, ϕ) as a function of $u \in Z$

Let $f : \mathbb{R} \to \mathbb{R}$ be smooth, $B^2_b = H^2_b \cap C^{2,\alpha}(\Omega)$. The function $F : B^2_b \to B^0$ is smooth from Proposition 2.3 and we are interested in $\lambda(u)$ and $\phi(u)$, a simple eigenvalue and corresponding normalized eigenvector of $DF(u) = -\Delta_b - f'(u)$ for the potential $q = f'(u)$. The functions $\lambda(u)$ and $\phi(u)$ can be either $\lambda_1 = \lambda_1^q$ and ϕ_1 or $\lambda_k = \lambda_k^q$ and ϕ_k, depending on the context. The L^2 inner product $\langle u, v\rangle$ makes sense for functions $u, v \in B^0$. Take $\ell(v) = \langle \phi(u), v\rangle$, so that $\ell(\phi(u)) = 1$.

Proof of Proposition 2.4. Proposition 5.1 implies the smoothness of λ and ϕ for $u \in B^2_b$: the smoothness of $u \in B^2_b \mapsto f'(u) \in B^0$ is (easily) proved.

For $u \in Z$, we show that $\Phi : Z \to \mathcal{B}(H^2_b, H^0)$, $u \mapsto M_{f'(u)}$ is ub-continuous. Let $u_m \xrightarrow{\text{ub}} u_\infty$ with $||u_m||_\infty,\ ||u_\infty||_\infty \le C$ and let $||f''||_\infty \le D$ on $[-C, C]$, so that

$$\left\|(f'(u_m) - f'(u_\infty))\, v\right\|_{H^0} \le D\,\|(u_m - u_\infty)v\|_{H^0}\ .$$

Since H^2_b embeds continuously on $L^{\frac{2n}{n-4}}$ for $n \ge 5$ and in every L^p for $n < 5$,

$$\left\|\left(f'(u_m) - f'(u_\infty)\right) v\right\|_{H^0} \le D\,\|u_m - u_\infty\|_{L^{2r}}\,\|v\|_{L^{2s}} \le \tilde{D}\,\|u_m - u_\infty\|_{L^{2r}}\,\|v\|_{H^2_b},$$

where $r = n/4, s = n/(n-4)$ for $n \geq 5$ and are conjugate exponents for $n < 5$. Take $||v||_{H_b^2} \leq 1$: the continuity of Φ follows from $c_m = \|u_m - u_\infty\|_{L^{2r}} \to 0$. But this is the case since

$$c_m = \left(\int_\Omega |u_m - u_\infty|^{(2r-2)+2}\right)^{1/2r} \leq (2C)^{1-1/r} \left(\int_\Omega |u_m - u_\infty|^2\right)^{1/r} \to 0.$$

Finally, compose $u \mapsto M_{f'(u)}$ with the map from bounded potentials to eigenpairs. The normalization yielding L^2-normal eigenvectors is clearly a smooth map. $\square$

The formulas for the derivatives of λ and ϕ at a point u along a direction v are familiar [12]. We confirm their validity for the more unusual scenario $u \in Z$.

Proof of Proposition 2.9. For the directional derivatives $D\lambda(u)v$, subtract

$$DF(u+tv)\phi(u+tv) = \lambda(u+tv)\phi(u+tv)\,, \quad DF(u)\phi(u) = \lambda(u)\phi(u)$$

to obtain, denoting differences $g(u+tv) - g(u)$ by Sg,

$$\begin{aligned} &\big\langle DF(u+tv)S\phi\,,\ \phi(u+tv)\big\rangle + \big\langle SDF\,\phi(u),\ \phi(u+tv)\big\rangle \\ &= \big\langle \lambda(u+tv)S\phi\,,\ \phi(u+tv)\big\rangle + \big\langle S\lambda\,\phi(u),\ \phi(u+tv)\big\rangle. \end{aligned}$$

From the symmetry of $DF(u+tv)$, the first terms on each side cancel each other. We now take limits and use the continuity of $\phi : Z \to H^0$: for $u, v \in Z$,

$$\lim_{t\to 0} \frac{1}{t}\big(\lambda(u+tv) - \lambda(u)\big)\big\langle \phi(u), \phi(u)\big\rangle = \lim_{t\to 0}\left\langle \frac{1}{t}\big(DF(u+tv) - DF(u)\big)\phi(u), \phi(u)\right\rangle$$

and setting $\nabla\lambda(u) = -f''(u)\phi^2(u) \in Z$, by the dominated convergence theorem,

$$D\lambda(u)v = \lim_{t\to 0} -\left\langle \frac{1}{t}\left(f'(u+tv) - f'(u)\right)\phi(u)\,,\ \phi(u)\right\rangle = \langle \nabla\lambda(u), v\rangle\,.$$

For the ub-continuity of $\nabla\lambda(u) : Z \to Z$, take $u_m \xrightarrow{\text{ub}} u_\infty$ with $\|u_m\|_\infty \leq \mathcal{C}$: we show that both terms go to zero in

$$\left\| f''(u_m)\phi^2(u_m) - f''(u_\infty)\phi^2(u_m)\right\| + \left\| f''(u_\infty)\phi^2(u_m) - f''(u_\infty)\phi^2(u_\infty)\right\|\,.$$

The new ingredient is the uniform bound of the sequence $\{\phi(u_m)\}$ from Proposition 2.2. The first term goes to zero because f'' is Lipschitz on $[-C, C]$. $\square$

Let $W^2 \subset H_b^2$, and $W^0 \subset H^0$ be the subspaces of functions orthogonal to $\phi(u)$. Let Π_W be the orthogonal projection from B^0 to W^0.

Lemma 5.2. *$\phi : Z \to \mathbb{R}$ admits Gateaux derivatives along functions $v \in Z$,*

$$D\phi(u)\, v = \big(DF(u) - \lambda(u) I\big)_W^{-1} \Pi_W\big(f''(u)\, \phi(u)\, v\big) \in H^0 .$$

Also, $D\phi : Z \times Z \to H^0$, $D\phi(u, v) = D\phi(u)\, v$, is ub-continuous.

Proof. Let $u, v, \in Z$. Start as in the proof above to obtain

$$\big(SDF(u) - S\lambda(u)\big)\, \phi(u + tv) + \big(DF(u) - \lambda(u)\big)\, S\phi(u) = 0 .$$

After dividing by t and taking $t \to 0$, the first term converges to

$$-f''(u)\, \phi(u)\, v + \big\langle f''(u)\phi^2(u), v\big\rangle\, \phi(u) = -\Pi_W\big(f''(u)\, \phi(u)\, v\big) ,$$

since Π_W projects orthogonally. The restriction $(DF(u) - \lambda(u) I)_W : W^2 \to W^0$ is an isomorphism. The derivative of ϕ in the second term exists and satisfies

$$D\phi(u)\, v = \big(DF(u) - \lambda(u) I\big)_W^{-1}\, \Pi_W\, \big(f''(u)\, \phi(u)\, v\big) \in W^2 .$$

The spaces W^2 and W^0 depend on u, being orthogonal complements of $\phi(u)$: an algebraic argument clarifies continuity. Define $\tilde{T} = \tilde{T}(u) : H_b^2 \to H^0$ as

$$\tilde{T} = DF(u) - \lambda(u) I + \phi(u) \otimes \phi(u) \quad \text{where } \big(\phi(u) \otimes \phi(u)\big) z = \langle \phi(u), z\rangle\, \phi(u) .$$

Notice that $\tilde{T}$ leaves W^2 and $\langle \phi(u) \rangle$ invariant. Also, the restrictions $\hat{T}$ to W^2 of $(DF(u) - \lambda(u) I)$ and $\tilde{T}$ coincide and are invertible. Since $\tilde{T}\phi(u) = \phi(u)$, $\tilde{T}$ is invertible. As in the proof of the ub-continuity of λ, $\tilde{T} \in \mathcal{B}(H_b^2, H^0)$ varies ub-continuously in $u \in Z$. Inversion preserves continuity and thus

$$D\phi(u)\, v \; = \Pi_W\, \hat{T}^{-1}\, \Pi_W\, \big(f''(u)\phi(u)\, v\big) ,$$

is ub-continuous, as well as

$$u \in H_b^2 \mapsto \Pi_W\big(f''(u)\phi^2(u)\big) = f''(u)\phi^2(u) + \big\langle f''(u)\phi^2(u), \phi(u)\big\rangle \phi(u) \in H^0. \quad \square$$

5.3. The functionals δ, τ and the function Λ

Differentiability properties for $\Lambda : Z \to \mathbb{R}^2$ require equivalent statements for λ (Proposition 2.9) and for $\delta : Z \to \mathbb{R}$, which we now prove.

Proof of Proposition 3.1. Since $\delta(u) = \langle \nabla\lambda(u), \phi(u)\rangle$, ub-continuity of δ follows from Propositions 2.4 and 2.9. We now take directional derivatives $D\delta(u)v$:

$$D\delta(u)\, v = -\int_\Omega f'''(u)\, \phi^3(u)\, v - \int_\Omega f''(u)\, 3\, \phi^2(u)\, \big(D\phi(u)\, v\big) .$$

On the second term, using $D\phi(u)\,v \;=\; \Pi_W\,\hat{T}^{-1}\,\Pi_W\,(f''(u)\phi(u)\,v)$, we have

$$
\begin{aligned}
\big\langle f''(u)\,3\,\phi^2(u)\,,\; D\phi(u)\,v\big\rangle &= \Big\langle \Pi_W\,\hat{T}^{-1}\,\Pi_W\,\Big(f''(u)\,3\,\phi^2(u)\Big),\;\Big(f''(u)\,\phi(u)\,v\Big)\Big\rangle \\
&= \Big\langle \Pi_W\,\big(DF(u)-\lambda(u)I\,\big)_W^{-1}\,\Pi_W\,(f''(u)\,3\,\phi^2(u)), \\
&\qquad \Big(f''(u)\,\phi(u)\,v\Big)\Big\rangle \\
&= \big\langle 3\,w(u),\;\big(f''(u)\,\phi(u)\,v\big)\big\rangle
\end{aligned}
$$

and the computation of $D\delta(u)v$ is complete: we are left with showing the continuity of $\nabla\delta(u)$. For $u_m \stackrel{\text{ub}}{\longrightarrow} u_\infty$, we show the L^2 convergences

$$-f'''(u_m)\phi^3(u_m) \to -f'''(u_\infty)\phi^3(u_\infty)\,,$$

$$w(u_m)\,f''(u_m)\phi(u_m) \to w(u_\infty)\,f''(u_\infty)\phi(u_\infty)\,.$$

For the first term, proceed as in the argument for $\nabla\lambda$. For the second, we show

$$
\begin{aligned}
&\big\|w(u_m)\big(\,f''(u_m)\phi(u_m) - f''(u_\infty)\phi(u_\infty)\big)\big\| \\
&\qquad + \big\|\big(w(u_m) - w(u_\infty)\big)f''(u_\infty)\phi(u_\infty)\big)\big\| \to 0\,.
\end{aligned}
$$

Using Lemma 5.2 one can prove that $w(u_m) \to w(u_\infty)$ in L^2. Therefore since $f''(u_\infty)\phi(u_\infty)$ is bounded, $\|\big(w(u_m) - w(u_\infty)\big)f''(u_\infty)\phi(u_\infty)\big)\| \to 0$. For the first term, split again: we show that

$$\big\|\big(w(u_m) - w(u_\infty)\big)\big(\,f''(u_m)\phi(u_m) - f''(u_\infty)\phi(u_\infty)\big)\big\|$$

and

$$\big\|w(u_\infty)\big(\,f''(u_m)\phi(u_m) - f''(u_\infty)\phi(u_\infty)\big)\big\|$$

go to 0. As before, $w(u_m) \to w(u_\infty)$ and $f''(u_m)\phi(u_m) - f''(u_\infty)\phi(u_\infty)$ are uniformly bounded, and the first term is done. The second one follows by the dominated convergence theorem.

The fact that Λ is $\mathcal{C}^1$ on finite-dimensional subspaces V_* follows from the statement just proved: its partial derivatives are continuous. One might use the sup norm in the arguments: on V_*, the L^2 and the L^∞ norms are equivalent. $\square$

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Dipartimento di Matematica
Università di Milano
Via Saldini, 50
20133 Milano, Italia
marta.calanchi@unimi.it

Departamento de Matemática
PUC-Rio
R. Mq. de S. Vicente 225
Rio de Janeiro, RJ 22453-900, Brazil
carlos.tomei@gmail.com
zaccur.andre@gmail.com