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Sign-changing blowing-up solutions for the Brezis–Nirenberg problem in dimensions four and five

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Abstract. We consider the Brezis-Nirenberg problem

$$-\Delta u = \lambda u + |u|^{p-1}u \qquad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 3$, $p = \frac{N+2}{N-2}$ and $\lambda > 0$.

We prove that, if Ω is symmetric and $N = 4, 5$, there exists a sign-changing solution whose positive part concentrates and blowsup at the center of symmetry of the domain, while the negative part vanishes, as $\lambda \to \lambda_1$, where $\lambda_1 = \lambda_1(\Omega)$ denotes the first eigenvalue of $-\Delta$ on Ω, with zero Dirichlet boundary condition.

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1. Introduction and statement of the main results

In this paper we deal with the following problem

$$\begin{cases} -\Delta u = \lambda u + |u|^{p-1}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N = 4, 5$, $\lambda > 0$, and $p + 1 = \frac{2N}{N-2}$ is the critical Sobolev exponent for the embedding of $H_0^1(\Omega)$ into $L^{p+1}(\Omega)$.

Problem (1.1) is known as the Brezis–Nirenberg problem, since the first existence results for positive solutions of (1.1) were given in their celebrated paper [14]. In particular they showed that the dimension N plays a crucial role in the study of problem (1.1). In fact they proved that if $N \geq 4$ there exist positive solutions of (1.1) for every $\lambda \in (0, \lambda_1)$, where $\lambda_1 = \lambda_1(\Omega)$ is the first eigenvalue of $-\Delta$ on Ω with zero Dirichlet boundary condition, while if $N = 3$ there exists

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$\lambda_* = \lambda_*(\Omega) > 0$ such that positive solutions exist if $\lambda \in (\lambda_*, \lambda_1)$. When $\Omega = B$ is a ball they also proved that $\lambda_*(B) = \frac{\lambda_1(B)}{4}$ and a positive solution of (1.1) exists if and only if $\lambda \in \left(\frac{\lambda_1(B)}{4}, \lambda_1(B)\right)$. Moreover, as a consequence of the classical Pohozaev's identity positive solutions do not exist if $\lambda \leq 0$ and Ω is star-shaped.

Since then several results have been obtained for problem (1.1), in particular on the asymptotic analysis of positive solutions, mainly for $N \geq 5$, because also the case $N = 4$ presents more difficulties compared to the higher-dimensional ones.

Concerning the case of sign-changing solutions of (1.1), several existence results have been obtained if $N \geq 4$. In this case one can get sign-changing solutions for every $\lambda \in (0, \lambda_1(\Omega))$, or even $\lambda > \lambda_1(\Omega)$ (see [6,17–21,23,24,42]). In particular, Capozzi, Fortunato and Palmieri in [17] showed that for $N = 4$, $\lambda > 0$ and $\lambda \notin \sigma(-\Delta)$ (the spectrum of $-\Delta$ in $H_0^1(\Omega)$) problem (1.1) has a nontrivial solution. The same holds if $N \geq 5$ for all $\lambda > 0$.

The case $N = 3$ presents the same difficulties enlightened before for positive solutions and even more. In fact, it is not yet known, when $\Omega = B$ is a ball in $\mathbb{R}^3$, if there are nonradial sign-changing solutions of (1.1) when λ is smaller than $\lambda_*(B) = \lambda_1(B)/4$. A partial answer to this question posed by H. Brezis was given in [10].

However, even in the case $N = 4, 5, 6$, some apparently strange phenomenon appears for what concerns radial sign-changing solutions in the ball. Indeed it was first proved by Atkinson, Brezis and Peletier in [5] that for $N = 4, 5, 6$ there exists $\lambda^* = \lambda^*(N)$ such that there are no sign-changing radial solutions of (1.1) for $\lambda \in (0, \lambda^*)$. Later this result was proved in [1] in a different way.

As it will be clear in the sequel, the nonexistence result of Atkinson, Brezis and Peletier is connected to the asymptotic analysis of low-energy sign-changing solutions of (1.1). Ben Ayed, El Mehdi and Pacella investigated the latter question in [10, 11]. More precisely, denoting by $\|\cdot\|$ the $H_0^1(\Omega)$-norm and by S the best Sobolev constant for the embedding $H_0^1(\Omega)$ into $L^{2^*}(\Omega)$, they studied the asymptotic behavior of sign-changing solutions u_λ of (1.1) such that $\|u_\lambda\|^2 \to 2S^{N/2}$, as $\lambda \to 0$ if $N \geq 4$, or $\lambda \to \bar{\lambda}$, if $N = 3$, where $\bar{\lambda}$ is the infimum of the values of λ for which nodal low-energy solutions exist (see [10]). They proved that these solutions blow up at two different points $\bar{a}_1, \bar{a}_2$, which are the limit of the concentration points $a_{\lambda,1}, a_{\lambda,2}$ of the positive and negative part of u_λ. We point out that they need to assume the extra hypothesis that the concentration speeds of the two concentration points are comparable for $N \geq 4$ (see [11]), while in dimension three this was derived without any extra assumption (see [10]).

In view of the results of Ben Ayed, El Mehdi and Pacella we get that, for $N \geq 4$, the question of proving the existence of sign-changing low-energy solutions (*i.e.*, such that $\|u_\lambda\|_\Omega^2$ converges to $2S^{N/2}$ as $\lambda \to 0$) whose positive and negative part concentrate and blow up at the same point, was left open.

In [30], by studying the asymptotic behavior, as $\lambda \to 0$, of low-energy radial sign-changing solutions of (1.1) in the unit ball of $\mathbb{R}^N$, for $N \geq 7$ (for these dimensions they do exist, as proved by Cerami, Solimini and Struwe in [20]), it was proved that the positive and the negative part of such solutions concentrate and blow up at the center of the ball, and their concentration speeds are not comparable. Moreover, in the recent paper [33], it has been proved that for $N \geq 7$ these so called "bubble-tower" solutions for (1.1), exist, as $\lambda \to 0$, in general bounded domains with some symmetry.

We point out that, in the previous result, the assumption $N \geq 7$ on the dimension is not only technically crucial but it also is necessary. In fact, in the recent paper [31], the authors proved that for the low dimensions $N = 4, 5, 6$, and in general bounded domains, there cannot exist sign-changing "bubble-tower" solutions for (1.1), as $\lambda \to 0$. This result is hence the counterpart, in general bounded domains, of the nonexistence theorem of Atkinson, Brezis and Peletier if we think of sign-chaging "bubble-tower" solutions as the functions which play, in general bounded domains, the same role as the radial solutions in the case of the ball.

In view of all these results it is natural to ask what kind of asymptotic profile we can expect for sign-changing solutions in the low dimensions $N = 4, 5, 6$, as λ goes to some strictly positive "limit" value. The case of radial sign-changing solutions in the ball, having two nodal regions, has been investigated in [32]. By studying the associated differential equation, and taking into account the results of [6,7], the authors prove that if (u_λ) is a family of radial sign-changing solutions of (1.1) in the unit ball B_1 of $\mathbb{R}^N$, having two nodal regions, such that $u_\lambda(0) > 0$, and denoting by $\bar{\lambda} = \bar{\lambda}(N)$ the limit value of the parameter λ, which arises from the study of the related ordinary differential equation, then:

(i) if $N = 4, 5$, then $\bar{\lambda} = \lambda_1(B_1)$, where $\lambda_1(B_1)$ is the first eigenvalue of $-\Delta$ in $H_0^1(B_1)$, and u_λ^+ concentrates and blows-up at the center of the ball having the limit profile of a "standard bubble" in $\mathbb{R}^N$ (*i.e.*, a solution of the critical problem in $\mathbb{R}^N$, see (2.1)), while u_λ^- converges to zero uniformly, as $\lambda \to \bar{\lambda}$;

(ii) if $N = 6$, then $\bar{\lambda} \in (0, \lambda_1(B_1))$ and u_λ^+ behaves as in (i) while u_λ^- converges to the unique positive radial solution of (1.1) in B_1, as $\lambda \to \bar{\lambda}$.

The aim of this paper is to show that, in general (symmetric) bounded domains of $\mathbb{R}^N$, when $N = 4, 5$, there exist sign-changing solutions of problem (1.1) having an asymptotic profile, as $\lambda \to \lambda_1(\Omega)$, which is similar to that of radial ones in the ball.

The case $N = 6$ is more delicate and at the moment we can only make some conjecture (see Remark 6.3).

In order to state our results, we denote by e_1 the first (positive, L^2-normalized) eigenfunction of the Laplace operator with Dirichlet boundary condition, namely e_1 solves the problem

$$\begin{cases} -\Delta e_1 = \lambda_1 e_1 & \text{in } \Omega \\ \\ e_1 = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

and $e_1 > 0$ in Ω, $|e_1|_2^2 = \int_\Omega |e_1|^2\, dx = 1$. We construct solutions u_λ of problem (1.1) which, up to a remainder term, are given by the superposition of a "standard bubble" (suitably projected) and the first eigenfunction of the Laplace operator, multiplied by a factor tending to zero, as $\lambda \to \lambda_1$.

More precisely, denoting by $\mathcal{P}$ the projection onto $H_0^1(\Omega)$ (see (2.4)), we get:

Theorem 1.1. *Let $N = 4$. Assume that $0 \in \Omega$ and that Ω is symmetric with respect to $x_1, \dots, x_4$.*
Then, for all $\lambda > \lambda_1$ sufficiently close to λ_1, there exists a sign-changing solution u_λ of problem (1.1) of the form

$$\begin{aligned} u_\lambda(x) = \mathcal{P}\left(\alpha_4 \frac{(\lambda-\lambda_1)e^{-\frac{1}{\lambda-\lambda_1}} s_{1\lambda}}{(\lambda-\lambda_1)^2 e^{-\frac{2}{\lambda-\lambda_1}} s_{1\lambda}^2 + |x|^2}\right) \\ - e^{-\frac{1}{\lambda-\lambda_1}}\left[(s_{2\lambda}-1)^2+1\right] e_1 + \Phi_\lambda \end{aligned} \tag{1.3}$$

where $\alpha_4 = 2\sqrt{2}$, $s_{j\lambda} \to \bar{s}_j > 0$, $j = 1, 2$ as $\lambda \to \lambda_1^+$ and $\Phi_\lambda \to 0$ in $H_0^1(\Omega)$ as $\lambda \to \lambda_1^+$. Moreover u_λ is even with respect to the variables $x_1, \dots, x_4$.

Theorem 1.2. *Let $N = 5$. Assume that $0 \in \Omega$ and that Ω is symmetric with respect to $x_1, \dots, x_5$.*

Then, for all $\lambda < \lambda_1$ sufficiently close to λ_1, there exists a sign-changing solution u_λ of problem (1.1) of the form

$$u_\lambda(x) = \mathcal{P}\left[\alpha_5 \left(\frac{(\lambda_1-\lambda)^{\frac{3}{2}} d_{2\lambda}}{(\lambda_1-\lambda)^2 d_{2\lambda}^2 + |x|^2}\right)^{\frac{3}{2}}\right] - (\lambda_1-\lambda)^{\frac{3}{4}} d_{1\lambda} e_1 + \Phi_\lambda \tag{1.4}$$

where $\alpha_5 = 15\sqrt{15}$, $d_{j\lambda} \to \bar{d}_j > 0$, $j = 1, 2$ as $\lambda \to \lambda_1^-$ and $\Phi_\lambda \to 0$ in $H_0^1(\Omega)$ as $\lambda \to \lambda_1^-$. Moreover u_λ is even with respect to the variables $x_1, \dots, x_5$.

Remark 1.3. We observe that the solutions obtained in Theorem 1.1 and Theorem 1.2 are sign-changing because, in the case $N = 4$ they solve problem 1.1 for $\lambda > \lambda_1$ and it is well known that for these values of the parameter λ there cannot exist solutions of problem (1.1) of constant sign (see [14, Remark 1.1]). In the case $N = 5$, the sign-changingness of the solution is a consequence of the estimates of the L^∞-norm of the remainder term (see the proof of Theorem 1.2 and Proposition 6.1).

We point out that since $\lambda_1(\Omega)$ is reached from above, if $N = 4$, while, it is reached from below, if $N = 5$, our results agree with those of [4, 26] for radial sign-changing solutions in the ball.

Moreover, we observe that, thanks to the estimates of the L^∞-norm of the remainder term in compact subsets of $\overline{\Omega} \setminus \{0\}$ (see the proof of Theorem 1.2, Proposition 6.1 and Remark 6.2), the main contribution to the negative part of the solutions obtained in Theorems 1.1 and 1.2 is given by the first (normalized, positive) eigenfunction of $-\Delta$ in $H_0^1(\Omega)$, multiplied by a factor tending to zero, as $\lambda \to \lambda_1$. Hence, this family of solutions verifies, in the more general setting of bounded (symmetric) domains, a conjecture made by Atkinson, Brezis and Peletier in [6] for nodal radial solutions in the ball, for $N = 4, 5$, which states that the negative part of these nodal radial solutions, converges to zero, in compact subsets of $\overline{B_1} \setminus \{0\}$, as the first eigenfunction of $-\Delta$ in the unit ball multiplied by a vanishing factor, as $\lambda \to \lambda_1$.

We also observe that the energy (see (1.5)) of the solutions obtained in Theorems 1.1 and 1.2 converges, as $\lambda \to \lambda_1(\Omega)$, to the "critical" energy level $\frac{1}{N}S^{N/2}$ for the Palais-Smale condition (as a consequence of (5.17), (5.18) and since the H^1_0-norm of the remainder term goes to zero).

The proof of our results is based on the Lyapunov-Schmidt reduction method which allows us to reduce the problem of finding blowing-up solutions to (1.1) to the problem of finding critical points of a functional (the reduced energy) which depends only on the concentration parameters.

We point out that, since we deal with the critical exponent, there are serious difficulties with the standard procedure when trying to look for critical points for the energy functional associated to (1.1), namely

$$J_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{p+1}\int_\Omega |u|^{p+1}\,dx - \frac{\lambda}{2}\int_\Omega u^2\,dx, \quad u \in H^1_0(\Omega). \tag{1.5}$$

In oder to overcome these difficulties, for the case $N = 5$ we use a new idea, introduced in our paper [33], which is based on the splitting of the remainder term in two parts. Usually the remainder term Φ_λ is found by solving an infinite dimensional problem, called "the auxiliary equation", here, we look for a remainder term which is the sum of two remainder terms, of different orders. Differently from the standard procedure these two functions are found by solving a system of two equations, which is obtained by splitting the auxiliary equation in an appropriate way. We stress that by using the standard procedure it is not possible to prove the existence of a critical point of the reduced energy and consequently to find a solution of problem (1.1) (see [33, Section 1]). We think that this improvement of a very consolidate technique can be used in other contexts for proving existence of solutions. We also note that in order to make the finite dimensional reduction method work, we use some techniques which usually belong to the variational framework. In fact, the standard procedure allows us to get only estimates of the H^1_0-norm of the remainder term, but in our case it is necessary to improve them up to the L^∞-norm (see Lemma 5.2).

For the case $N = 4$ we use the standard procedure, but it requires finer and different estimates, since they are more delicate in this dimension, and it also requires suitable choices of the parameters δ and τ.

We also observe that the symmetry assumption on the domain Ω is only made in order to simplify the computations which however, even in the symmetric context, are long and tough. But there is no reason, a priori, for the previous result not to hold in general domains.

The outline of the paper is the following: in Section 2 we set the notation and recall some preliminary results. In Section 3 we explain the setting of the problem. In Section 4 we look for the remainder term Φ_λ in a suitable space. In Section 5 we study the reduced energy. Finally, in Section 6 we prove Theorems 1.1 and 1.2.

2. Notation and some preliminary results

We introduce the functions

$$\mathcal{U}_\delta(x) = \alpha_N \frac{\delta^{\frac{N-2}{2}}}{\left(\delta^2 + |x|^2\right)^{\frac{N-2}{2}}}, \qquad \delta > 0,\ x \in \mathbb{R}^N \tag{2.1}$$

with $\alpha_N := [N(N-2)]^{\frac{N-2}{4}}$. Is is well known (see [8,16,44]) that (2.1) are the only radial solutions of the equation

$$-\Delta u = u^p \qquad \text{in } \mathbb{R}^N. \tag{2.2}$$

We define φ_δ to be the unique solution to the problem

$$\begin{cases} \Delta\varphi_\delta = 0 & \text{in } \Omega \\ \varphi_\delta = \mathcal{U}_\delta & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

and let

$$\mathcal{P}\mathcal{U}_\delta := \mathcal{U}_\delta - \varphi_\delta \tag{2.4}$$

be the projection of $\mathcal{U}_\delta$ onto $H_0^1(\Omega)$, *i.e.*

$$\begin{cases} -\Delta\mathcal{P}\mathcal{U}_\delta = \mathcal{U}_\delta^p & \text{in } \Omega \\ \mathcal{P}\mathcal{U}_\delta = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

Finally, we introduce the Robin function of a domain Ω, which is defined as $\tau(x) = H(x, x)$.

Here $H(x, y)$, for $x, y \in \Omega$, is given as follows: for all $y \in \Omega$, $H(x, y)$ satisfies

$$-\Delta H(x, y) = 0 \qquad \text{in } \Omega, \qquad H(x, y) = \frac{1}{|x-y|^{N-2}} \qquad \text{for } x \in \partial\Omega.$$

The function H is nothing but the regular part of the Green function. Indeed, if $G(x, y)$ denotes the Green function of the Laplace operator at the boundary $\partial\Omega$, we have:

$$G(x, y) = \gamma_N \left(\frac{1}{|x-y|^{N-2}} - H(x, y) \right)$$

with $\gamma_N := \frac{1}{(N-2)\omega_N}$, where ω_N denotes the surface area of the unit sphere in $\mathbb{R}^N$.

It is well-known that the following expansion holds (see [40])

$$\varphi_\delta(x) = \alpha_N \delta^{\frac{N-2}{2}} H(0,x) + O(\delta^{\frac{N+2}{2}}) \qquad \text{as } \delta \to 0. \tag{2.6}$$

Moreover, from elliptic estimates it follows that

$$0 < \varphi_\delta(x) < c\delta^{\frac{N-2}{2}}, \qquad \text{in } \Omega \tag{2.7}$$

and

$$|\varphi_\delta|_{q,\Omega} \leq C\delta^{\frac{N-2}{2}}, \qquad q \in \left(\frac{p+1}{2}, p+1\right], \tag{2.8}$$

see for instance [45] and references therein.

In what follows we let

$$(u,v) := \int_\Omega \nabla u \cdot \nabla v \, dx, \qquad \|u\| := \left(\int_\Omega |\nabla u|^2 \, dx\right)^{\frac{1}{2}}$$

as the inner product in $H_0^1(\Omega)$ and its corresponding norm while we denote by $(\cdot,\cdot)_{H^1(\mathbb{R}^N)}$ and by $\|\cdot\|_{H^1(\mathbb{R}^N)}$ the scalar product and the standard norm in $H^1(\mathbb{R}^N)$. Moreover we denote by

$$|u|_r := \left(\int_\Omega |u|^r \, dx\right)^{\frac{1}{r}}$$

the $L^r(\Omega)$-standard norm for any $r \in [1,+\infty]$. When $A \neq \Omega$ is any Lebesgue measurable subset of $\mathbb{R}^N$, or, when $A = \Omega$ and we need to specify the domain of integration, we use the alternative notations $\|u\|_A$, $|u|_{r,A}$.

From now on we assume that Ω is a bounded open set with smooth boundary of $\mathbb{R}^N$, symmetric with respect to $x_1, \dots, x_N$ and which contains the origin.

We define then

$$H_{\text{sym}} := \left\{u \in H_0^1(\Omega) \ : \ u \text{ is symmetric with respect to } x_1, \dots, x_N\right\},$$

and for $q \in [1,+\infty]$

$$L^q_{\text{sym}} := \left\{u \in L^q(\Omega) \ : \ u \text{ is symmetric with respect to } x_1, \dots, x_N\right\}.$$

3. Setting of the problem

Let $i^* : L_{\text{sym}}^{\frac{2N}{N+2}} \to H_{\text{sym}}$ be the adjoint operator of the embedding $i : H_{\text{sym}}(\Omega) \to L_{\text{sym}}^{\frac{2N}{N-2}}$, namely if $v \in L_{\text{sym}}^{\frac{2N}{N+2}}$ then $u = i^*(v)$ in H_{sym} is the unique solution of the equation

$$\begin{cases} -\Delta u = v & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

By the continuity of i it follows that

$$\|i^*(v)\| \leq C|v|_{\frac{2N}{N+2}} \qquad \forall v \in L_{\text{sym}}^{\frac{2N}{N+2}} \tag{3.1}$$

for some positive constant C which depends only on N,

$$\begin{cases} u = i^*\left[f(u) + \lambda u\right] \\ u \in H_{\text{sym}} \end{cases} \tag{3.2}$$

where $f(s) = |s|^{p-1}s$, $p = \frac{N+2}{N-2}$.

Let Z_δ the following function:

$$Z_\delta(x) := \partial_\delta \mathcal{U}_\delta(x) = \alpha_N \frac{N-2}{2} \delta^{\frac{N-4}{2}} \frac{|x|^2 - \delta^2}{\left(\delta^2 + |x|^2\right)^{\frac{N}{2}}}. \tag{3.3}$$

We remark that the function Z_δ solves the problem (see [13])

$$-\Delta z = p|\mathcal{U}_\delta|^{p-1}z, \qquad \text{in } \mathbb{R}^N. \tag{3.4}$$

Let $\mathcal{P}Z_\delta$ the projection of Z_δ onto $H_0^1(\Omega)$. Elliptic estimates give

$$\mathcal{P}Z_\delta(x) = Z_\delta(x) - \underbrace{\alpha_N \frac{N-2}{2} \delta^{\frac{N-4}{2}} H(0,x) + O(\delta^{\frac{N}{2}})}_{:=\psi_\delta(x)} \tag{3.5}$$

uniformly in Ω.

We next describe the shape of the solution we are looking for. Let δ, τ be positive parameters defined in the following way: for $N = 4$ we let

$$\begin{aligned} \delta &= \epsilon e^{-\frac{1}{\epsilon}} s_1 \\ \tau &= e^{-\frac{1}{\epsilon}} g(s_2) \\ \text{with} \quad \lambda - \lambda_1 &= \epsilon, \\ g(s_2) &= (s_2 - 1)^2 + 1, \ s_j > 0. \end{aligned} \tag{3.6}$$

Instead, for $N = 5$ we let

$$\begin{aligned} \tau &= \epsilon^{\frac{3}{4}} d_1 \\ \delta &= \epsilon^{\frac{3}{2}} d_2, \\ \text{with} \quad \lambda_1 - \lambda &= \epsilon, \quad d_j > 0. \end{aligned} \tag{3.7}$$

Fix a small $\eta > 0$ and assume that

$$\eta < d_j, s_j < \frac{1}{\eta} \qquad \text{for } j = 1, 2. \tag{3.8}$$

We look for an approximate solution to problem (3.2) which is of the form

$$u_\lambda(x) = \mathcal{P}\mathcal{U}_\delta - \tau e_1 + \Phi_\lambda(x), \tag{3.9}$$

where $e_1 > 0$ is the first eigenfunction of $-\Delta$ corresponding to the first eigenvalue λ_1, and the remainder term Φ_λ is a small function which is even with respect to the variables $x_1, \ldots, x_N$.

Finally let us recall some useful inequalities that we will use in the sequel. Since these are known results, we omit the proof. Recalling that $f(s) = |s|^{p-1}s$, where $p = \frac{N+2}{N-2}$, we have:

Lemma 3.1. *Let $N < 6$. There exists a positive constant c, depending only on p, such that for any $a, b \in \mathbb{R}$*

$$|f(a+b) - f(a) - f'(a)b| \le c\left(|a|^{p-2}|b|^2 + |b|^p\right), \tag{3.10}$$

and

$$|f(a+b) - f(a)| \le c\left(|a|^{p-1}|b| + |b|^p + |a|^{p-2}|b|^2\right). \tag{3.11}$$

Lemma 3.2. *Let $N < 6$. There exists a positive constant c depending only on p such that for any $a, b_1, b_2 \in \mathbb{R}$ we get*

$$\begin{aligned} &\left|f(a+b_1) - f(a+b_2) - f'(a)(b_1 - b_2)\right| \\ &\le c\left(|a|^{p-2}|b_2 - b_1| + |b_1|^{p-1} + |b_2|^{p-1}\right)|b_1 - b_2|. \end{aligned} \tag{3.12}$$

3.1. Scheme of the reduction

Let us consider

$$\mathcal{K}_1 := \operatorname{span}\{e_1\} \subset H_{\mathrm{sym}} \quad \text{and} \quad \mathcal{K} := \operatorname{span}\{\mathcal{P}Z_\delta, e_1\} \subset H_{\mathrm{sym}}$$

and the orthogonal spaces

$$\begin{aligned} \mathcal{K}_1^\perp &:= \left\{\phi \in H_{\mathrm{sym}} \,:\, (\phi, e_1)_{H_0^1(\Omega)} = 0\right\} \\ \mathcal{K}^\perp &:= \left\{\phi \in H_{\mathrm{sym}} \,:\, (\phi, e_1)_{H_0^1(\Omega)} = 0,\ (\phi, \mathcal{P}Z_\delta)_{H_0^1(\Omega)} = 0\right\}. \end{aligned}$$

Let maps $\Pi_1 : H_{\mathrm{sym}} \to \mathcal{K}_1$, $\Pi : H_{\mathrm{sym}} \to \mathcal{K}$ and maps $\Pi_1^\perp : H_{\mathrm{sym}} \to \mathcal{K}_1^\perp$, and $\Pi^\perp : H_{\mathrm{sym}} \to \mathcal{K}^\perp$ be the projections onto $\mathcal{K}_1, \mathcal{K}$ and $\mathcal{K}_1^\perp, \mathcal{K}^\perp$, respectively.

We set

$$V_\lambda(x) := \mathcal{P}\mathcal{U}_\delta(x) - \tau e_1(x). \tag{3.13}$$

We remark that $V_\lambda(x) = V_\lambda(\bar{s}, x)$ for $N = 4$ and $V_\lambda(x) = V_\lambda(\bar{d}, x)$ for $N = 5$ where $\bar{s} := (s_1, s_2) \in \mathbb{R}^2_+$ and $\bar{d} := (d_1, d_2) \in \mathbb{R}^2_+$.

In order to solve problem (1.1) we will solve the pair of equations

$$\Pi^{\perp}\left\{V_\lambda+\Phi_\lambda-i^*\left[f(V_\lambda+\Phi_\lambda)+\lambda(V_\lambda+\Phi_\lambda)\right]\right\}=0, \tag{3.14}$$

$$\Pi\left\{V_\lambda+\Phi_\lambda-i^*\left[f(V_\lambda+\Phi_\lambda)+\lambda(V_\lambda+\Phi_\lambda)\right]\right\}=0. \tag{3.15}$$

Given $\bar{s}$ and $\bar{d}$ satisfying condition (3.8), one has to solve first the equation (3.14) in Φ_λ which is the lower order term in the description of the ansatz and then solve equation (3.15).

We recall now the definition of stable critical point that we will use in the sequel.

Definition 3.3. Let $h:\mathcal{D}\to\mathbb{R}$ be a C^1- function where $\mathcal{D}\subset\mathbb{R}^m$ is an open set. We say that x_0 is a *stable critical point* if

$$\nabla h(x_0)=0$$

and there exists a neighourhood U of x_0 such that

$$\nabla h(x)\neq 0 \qquad \forall\ x\in\partial U$$

$$\nabla h(x)=0,\ x\in U \Longrightarrow h(x)=h(x_0)$$

and

$$\deg(\nabla h,U,0)\neq 0,$$

where deg denotes Brouwer degree.

We remark that any non-degenerate critical point of h is a stable critical point in the sense of Definition 3.3.

Moreover it is easy to see that if x_0 is a minimum or a maximum point of h (not necessarily non-degenerate) then x_0 is a stable critical point in according to Definition 3.3.

4. The auxiliary equation (3.14)

In the sequel we solve (3.14) in both cases $N=4,5$.

4.1. The reduction for $N=4$

We write (3.14) as

$$\mathcal{R}_\lambda+\mathcal{L}(\phi)+\mathcal{N}(\phi)=0, \tag{4.1}$$

where

$$\mathcal{R}_\lambda := \Pi^{\perp}\left\{V_\lambda-i^*\left[f(V_\lambda)+\lambda V_\lambda\right]\right\} \tag{4.2}$$

$$\mathcal{L}(\phi) := \Pi^{\perp}\left\{\phi-i^*\left[f'(\mathcal{U}_\delta)\phi+\lambda\phi\right]\right\} \tag{4.3}$$

and

$$\mathcal{N}(\phi) := \Pi^{\perp}\left\{-i^*\left[f(V_\lambda+\phi)-f(V_\lambda)-f'(\mathcal{U}_\delta)\phi\right]\right\}. \tag{4.4}$$

In what follows we estimate the error term $\mathcal{R}_\lambda$, we analyze the invertibility of the linear operator $\mathcal{L}$ and, at the end, we look for a solution of (4.1) by using a fixed point argument.

4.1.1. *Estimate of the error term*

Proposition 4.1. *For any $\eta > 0$, there exist $\epsilon_0 > 0$ and $c > 0$ such that for all $\epsilon \in (0,\epsilon_0)$, for all $(s_1,s_2) \in \mathbb{R}^2_+$ satisfying (3.8), we have*

$$\|\mathcal{R}_\lambda\| \le c\epsilon e^{-\frac{1}{\epsilon}}.$$

Proof. By continuity of $\Pi^\perp$, by definition of i^* and by using (3.1), we deduce that

$$\begin{aligned}\|\mathcal{R}_\lambda\| \le \underbrace{c|f(\mathcal{P}\mathcal{U}_\delta-\tau e_1)-f(\mathcal{P}\mathcal{U}_\delta)|_{\frac{2N}{N+2}}}_{(I)} + \underbrace{c|f(\mathcal{P}\mathcal{U}_\delta)-f(\mathcal{U}_\delta)|_{\frac{2N}{N+2}}}_{(II)} \\ + \underbrace{c\lambda|\mathcal{P}\mathcal{U}_\delta|_{\frac{2N}{N+2}}}_{(III)} + \underbrace{c\tau|\lambda-\lambda_1||e_1|_{\frac{2N}{N+2}}}_{(IV)}.\end{aligned}$$

Let us fix $\eta > 0$. We begin with estimating (I). By using Lemma 3.1 and recalling the choice of τ and δ (see (3.6)), we deduce that

$$\begin{aligned}(I) &\le c_1\left(\int_\Omega\left[\mathcal{P}\mathcal{U}_\delta^{(p-1)(\frac{2N}{N+2})}(\tau e_1)^{\frac{2N}{N+2}}+(\tau e_1)^{p+1}\right.\right.\\ &\qquad\left.\left.+\tau^{\frac{4N}{N+2}}\mathcal{P}\mathcal{U}_\delta^{\frac{2N(6-N)}{(N-2)(N+2)}}e_1^{\frac{4N}{N+2}}\,dx\right]\right)^{\frac{N+2}{2N}}\\ &\le c_2\left(\tau^{\frac{2N}{N+2}}|e_1|_\infty^{\frac{2N}{N+2}}\int_\Omega\frac{\delta^{-\frac{4N}{N+2}}}{\left(1+\left|\frac{x}{\delta}\right|^2\right)^{\frac{4N}{N+2}}}dx\right.\\ &\qquad\left.+\tau^{p+1}|e_1|_\infty^{p+1}|\Omega|+\tau^{\frac{4N}{N+2}}|e_1|_\infty^{\frac{4N}{N+2}}\delta^{\frac{N(6-N)}{N+2}}\int_\Omega\frac{1}{|x|^{\frac{2N(6-N)}{N+2}}}dx\right)^{\frac{N+2}{2N}}\\ &\le c\left(\epsilon e^{-\frac{2}{\epsilon}}+\epsilon^3 e^{-\frac{3}{\epsilon}}+\epsilon e^{-\frac{3}{\epsilon}}\right)\le c(\eta,\Omega,N)\epsilon e^{-\frac{1}{\epsilon}}.\end{aligned}$$

The estimate of (II) is standard and hence, by making some computations we get

$$(II) \le c_3\delta^{N-2} \le c_4\epsilon e^{-\frac{1}{\epsilon}} \tag{4.5}$$

for all sufficiently small ϵ.

We now estimate (III). Since $\mathcal{P}\mathcal{U}_\delta \leq \mathcal{U}_\delta$ we have:

$$\begin{aligned}(III) &\leq \alpha_N \delta^{\frac{N-2}{2}} \left(\int_\Omega \frac{1}{(\delta^2+|x|^2)^{\frac{N(N-2)}{N+2}}}\, dx \right)^{\frac{N+2}{2N}} \\ &\leq \alpha_N \delta^{\frac{N-2}{2}} \left(\int_\Omega \frac{1}{|x|^{\frac{N(N-2)}{N+2}}}\, dx \right)^{\frac{N+2}{2N}} \leq c\epsilon e^{-\frac{1}{\epsilon}}.\end{aligned}$$

Finally

$$(IV) \leq c\tau\epsilon \leq c\epsilon e^{-\frac{1}{\epsilon}}.$$

Putting together all these estimates the result follows. □

4.1.2. *The linear operator* Let us consider the linear operator $\mathcal{L} : \mathcal{K}^\perp \to \mathcal{K}^\perp$ defined in (4.3). Next results states the invertibility of $\mathcal{L}$ and provides a uniform estimate on the norm of $\mathcal{L}^{-1}$.

Proposition 4.2. *Let $N = 4$ and δ be as in (3.6). Then, for any small $\eta > 0$, there exists $C = C(\eta) > 0$ such that for all λ sufficiently close to λ_1, for any real number $s_1 \in (\eta, \frac{1}{\eta})$ and for any $\phi \in \mathcal{K}^\perp$ it holds that*

$$\|\mathcal{L}(\phi)\| \geq C\|\phi\|.$$

Moreover $\mathcal{L}$ is invertible and $\|\mathcal{L}^{-1}\| \leq \frac{1}{C}$.

Proof. The proof is quite standard and so we limit to sketch the proof of the first part. The invertibility of $\mathcal{L}$ follows by arguing as in Proposition 3.2 of [37].

We argue by contradiction. Assume that there exists a small $\eta > 0$, a sequence $(\lambda_n)_n$ converging to λ_1, a sequence of real numbers $(s_n)_n \subset (\eta, \frac{1}{\eta})$ and a sequence of functions $(\phi_n)_n \subset H_0^1(\Omega)$ such that for all $n \in \mathbb{N}$

$$\begin{aligned}&\phi_n \in \mathcal{K}^\perp \\ &\|\phi_n\| = 1\end{aligned} \tag{4.6}$$

and

$$\mathcal{L}(\phi_n) = h_n \qquad \text{with} \qquad \|h_n\| \to 0, \text{ as } n \to +\infty. \tag{4.7}$$

Since $h_n \in \mathcal{K}^\perp$ we get that there exist some real numbers c_j^n, $j = 0, 1$ such that

$$\phi_n - i^*\left[f'(\mathcal{U}_{\delta_n})\phi_n + \lambda_n \phi_n\right] = h_n + w_n \qquad \text{in } \Omega \tag{4.8}$$

where $w_n = c_0^n \mathcal{P}Z_{\delta_n} + c_1^n e_1$.

First we will show that

$$\lim_{n\to+\infty} \|w_n\| = 0. \tag{4.9}$$

To this end we multiply (4.8) by $\mathcal{P}Z_{\delta_n}$ and by e_1 and we integrate by parts in Ω deducing that

$$(\phi_n, \mathcal{P}Z_{\delta_n})_{H^1_0(\Omega)} - \int_\Omega f'(\mathcal{U}_{\delta_n})\phi_n \mathcal{P}Z_{\delta_n}\,dx - \lambda_n \int_\Omega \phi_n \mathcal{P}Z_{\delta_n}\,dx$$
$$= (h_n, \mathcal{P}Z_{\delta_n})_{H^1_0(\Omega)} + c_0^n(\mathcal{P}Z_{\delta_n}, \mathcal{P}Z_{\delta_n})_{H^1_0(\Omega)}\,dx + c_1^n(e_1, \mathcal{P}Z_{\delta_n})_{H^1_0(\Omega)}$$

and

$$(\phi_n, e_1)_{H^1_0(\Omega)} - \int_\Omega f'(\mathcal{U}_{\delta_n})\phi_n e_1\,dx - \lambda_n \int_\Omega \phi_n e_1\,dx$$
$$= (h_n, e_1)_{H^1_0(\Omega)} + c_0^n(\mathcal{P}Z_{\delta_n}, e_1)_{H^1_0(\Omega)} + c_1^n(e_1, e_1)_{H^1_0(\Omega)}.$$

We remark that since $\mathcal{P}Z_{\delta_n}$ solves (3.4) and $\phi_n \in \mathcal{K}^\perp$ we have

$$0 = (\phi_n, \mathcal{P}Z_{\delta_n})_{H^1_0(\Omega)} = \int_\Omega f'(\mathcal{U}_{\delta_n})\phi_n Z_{\delta_n}\,dx$$

and

$$(\mathcal{P}Z_{\delta_n}, \mathcal{P}Z_{\delta_n})_{H^1_0(\Omega)} = \int_\Omega |\nabla \mathcal{P}Z_{\delta_n}|^2\,dx = \int_\Omega f'(\mathcal{U}_{\delta_n}) Z_{\delta_n} \mathcal{P}Z_{\delta_n}\,dx.$$

Moreover since e_1 solves (1.2)

$$(\mathcal{P}Z_{\delta_n}, e_1)_{H^1_0(\Omega)} = \int_\Omega \nabla e_1 \nabla \mathcal{P}Z_{\delta_n}\,dx = \lambda_1 \int_\Omega e_1 \mathcal{P}Z_{\delta_n}\,dx.$$

and (since $e_1 \in \mathcal{K}^\perp$)

$$0 = (\phi_n, e_1)_{H^1_0(\Omega)} = \lambda_1 \int_\Omega e_1 \phi_n\,dx.$$

Hence the equations become

$$c_0^n \underbrace{\int_\Omega f'(U_{\delta_n}) Z_{\delta_n} \mathcal{P}Z_{\delta_n}\,dx}_{(I)} + c_1^n \lambda_1 \underbrace{\int_\Omega e_1 \mathcal{P}Z_{\delta_n}\,dx}_{(II)}$$
$$= -\int_\Omega f'(\mathcal{U}_{\delta_n})\phi_n(\mathcal{P}Z_{\delta_n} - Z_{\delta_n})\,dx - \lambda_n \int_\Omega \phi_n \mathcal{P}Z_{\delta_n}\,dx - (h_n, \mathcal{P}Z_{\delta_n})_{H^1_0(\Omega)}$$

and $(h_n, \mathcal{P}Z_{\delta_n})_{H^1_0(\Omega)} := 0$ since $h_n \in \mathcal{K}^\perp$, and

$$c_0^n \lambda_1 \underbrace{\int_\Omega e_1 \mathcal{P}Z_{\delta_n}\,dx}_{(II)} + c_1^n \lambda_1 \underbrace{\int_\Omega e_1^2\,dx}_{:=D_0>0} = -\int_\Omega f'(\mathcal{U}_{\delta_n})\phi_n e_1\,dx - (h_n, e_1)_{H^1_0(\Omega)}$$

and $(h_n, e_1)_{H_0^1(\Omega)} := 0$ since $h_n \in \mathcal{K}^\perp$. By definition of projection we have $\mathcal{P}Z_{\delta_n} = Z_{\delta_n} - \psi_{\delta_n}$, where ψ_{δ_n} is an harmonic function and $\psi_{\delta_n} = Z_{\delta_n}$ on $\partial\Omega$. Therefore, by elliptic estimates, it follows that there is a constant $C > 0$ depending only on N and Ω, such that $|\psi_{\delta_n}|_{\infty,\Omega} \le C$ (see also (3.5)).

Hence

$$\int_\Omega f'(\mathcal{U}_{\delta_n})\mathcal{P}Z_{\delta_n} Z_{\delta_n}\, dx = \int_\Omega f'(\mathcal{U}_{\delta_n}) Z_{\delta_n}^2\, dx - \int_\Omega f'(\mathcal{U}_{\delta_n})\psi_{\delta_n} Z_{\delta_n}\, dx.$$

Now

$$\begin{aligned}\int_\Omega f'(\mathcal{U}_{\delta_n}) Z_{\delta_n}^2\, dx &= \alpha_4^{p+1}\delta_n^{-2}\int_{\mathbb{R}^4}\frac{(|y|^2-1)^2}{(1+|y|^2)^6}\, dy + O\left(\delta_n^2\right)\\ &= A\delta_n^{-2} + o(1) \quad \text{as } n \to +\infty\end{aligned}$$

where $A := \alpha_4^{p+1}\int_{\mathbb{R}^4}\frac{(|y|^2-1)^2}{(1+|y|^2)^6}\, dy$. Moreover

$$\int_\Omega f'(\mathcal{U}_{\delta_n})\psi_{\delta_n} Z_{\delta_n}\, dx = \alpha_4^p H(0,0)\int_{\mathbb{R}^4}\frac{1-|y|^2}{(1+|y|^2)^4}\, dy + O(\delta_n) = A_0 + O(\delta_n).$$

Therefore

$$(I) = A\delta_n^{-2} - A_0 + o(1)$$

as $n \to +\infty$. Moreover

$$\int_\Omega e_1\mathcal{P}Z_{\delta_n}\, dx = \int_\Omega e_1 Z_{\delta_n}\, dx - \int_\Omega e_1\psi_{\delta_n}\, dx$$

and now

$$\int_\Omega e_1 Z_{\delta_n}\, dx = B + o(1) \qquad \text{as } n \to +\infty$$

where $B = \int_\Omega e_1\frac{1}{|x|^2}\, dx$. Moreover

$$\int_\Omega e_1\psi_{\delta_n}\, dx = \alpha_4\int_\Omega e_1 H(0,x)\, dx = B_0.$$

We then get

$$(II) = B - B_0 + o(1).$$

Hence the equations become

$$\begin{aligned}&c_0^n\left(A - A_0\delta_n^2 + o(\delta_n^2)\right) + c_1^n\lambda_1\left((B-B_0)\delta_n^2 + o(\delta_n^2)\right)\\ &= \underbrace{-\,\delta_n^2\int_\Omega f'(\mathcal{U}_{\delta_n})\phi_n(\mathcal{P}Z_{\delta_n} - Z_{\delta_n})\, dx}_{(III)} \underbrace{-\,\delta_n^2\lambda_n\int_\Omega \phi_n\mathcal{P}Z_{\delta_n}\, dx}_{(IV)}\end{aligned}$$

and

$$c_0^n\lambda_1\left(B-B_0+o(1)\right)+c_1^n\lambda_1 D_0=-\underbrace{\int_\Omega f'(\mathcal{U}_{\delta_n})\phi_n e_1\,dx}_{(V)}.$$

Now by using (3.5) we get that

$$\begin{aligned}|(III)| &= \left|-\delta_n^2\int_\Omega f'(\mathcal{U}_{\delta_n})\phi_n(\mathcal{P}Z_{\delta_n}-Z_{\delta_n})\,dx\right| \\ &\le \delta_n^2|\mathcal{P}Z_{\delta_n}-Z_{\delta_n}|_{4,\Omega}|\phi_n|_{4,\Omega}|f'(\mathcal{U}_{\delta_n})|_{2,\Omega}=O\left(\delta_n^2\right).\end{aligned}$$

We remark that

$$\|\mathcal{P}Z_{\delta_n}\|^2=\int_\Omega f'(\mathcal{U}_{\delta_n})\mathcal{P}Z_{\delta_n}Z_{\delta_n}\,dx\le\left|\int_\Omega f'(\mathcal{U}_{\delta_n})\mathcal{P}Z_{\delta_n}Z_{\delta_n}\,dx\right|\le C\delta_n^{-2}.$$

Hence we get

$$|(IV)|\le C|\phi_n|_{2,\Omega}\|\mathcal{P}Z_{\delta_n}\|\le C\delta_n.$$

Finally

$$|(V)|\le|e_1|_{\infty,\Omega}\int_\Omega|f'(\mathcal{U}_{\delta_n})\phi_n|\,dx\le C\left|\mathcal{U}_{\delta_n}\right|_{3,\Omega}^{\frac{2}{3}}|\phi_n|_{3,\Omega}\le C\delta_n^{\frac{2}{3}}=o(1).$$

Then

$$\begin{cases} c_0^n\left(A-A_0\delta_n^2+o(\delta_n^2)\right)+c_1^n\lambda_1\left((B-B_0)\delta_n^2+o(\delta_n^2)\right)=o(\delta_n)\\ c_0^n\lambda_1\left(B-B_0+o(1)\right)+c_1^n\lambda_1 D_0=o(1).\end{cases}$$

In both cases the system is definitely non singular and hence it has a solution (c_0^n,c_1^n) such that $c_j^n\to 0$ as $n\to+\infty$.

Moreover $c_0^n=o(\delta_n)$. Now we observe that

$$\begin{aligned}\|w_n\|^2_{H_0^1(\Omega)} &= (\phi_n,w_n)_{H_0^1(\Omega)}-\int_\Omega f'(\mathcal{U}_{\delta_n})\phi_n w_n dx-\lambda_n\int_\Omega\phi_n w_n dx-(h_n,w_n)_{H_0^1(\Omega)}\\ &= -c_0^n\int_\Omega f'(\mathcal{U}_{\delta_n})\phi_n\mathcal{P}Z_{\delta_n}dx-c_1^n\int_\Omega f'(\mathcal{U}_{\delta_n})\phi_n e_1 dx+(\phi_n,w_n)_{H_0^1(\Omega)}\\ &\quad-\lambda_n c_0^n\int_\Omega\phi_n\mathcal{P}Z_{\delta_n}dx-\lambda_n c_1^n\int_\Omega\phi_n e_1 dx-(h_n,w_n)_{H_0^1(\Omega)}\end{aligned}$$

Reasoning as before and using that $c_0^n=o(\delta_n)$, $c_1^n=o(1)$ as $n\to+\infty$ we get that

$$\|w_n\|^2_{H_0^1(\Omega)}=o(1)$$

and the thesis easily follows.

Now let us define

$$\widetilde{\phi}_n(y) := \delta_n \phi_n(\delta_n y).$$

Then $\widetilde{\phi}_n$ solves the problem

$$-\Delta\widetilde{\phi}_n - p\mathcal{U}(y)^{p-1}\widetilde{\phi}_n - \lambda_n\delta_n^2\widetilde{\phi}_n = \delta_n^3\Delta(h_n(\delta_n y) + w_n(\delta_n y)) \qquad \text{in } \frac{\Omega_n}{\delta_n}. \quad (4.10)$$

We point out that since $\|\widetilde{\phi}_n\|_{\frac{\Omega}{\delta_n}}$ is bounded, then, up to a subsequence, $\widetilde{\phi}_n$ converges weakly in $D^{1,2}(\mathbb{R}^4)$ to some ϕ_0. This means that

$$\int_{\frac{\Omega}{\delta_n}} \nabla\tilde{\phi}_n \nabla\varphi\, dx \to \int_{\mathbb{R}^4} \nabla\phi_0\nabla\varphi\, dx \qquad \text{as } n \to +\infty$$

for any $\varphi \in C_0^\infty(\mathbb{R}^4)$.

By multiplying equation (4.10) by $\varphi \in C_0^\infty(\mathbb{R}^4)$ and integrating we get that

$$\begin{aligned}\int_{\frac{\Omega}{\delta_n}} \nabla\tilde{\phi}_n\nabla\varphi\, dx - p\int_{\frac{\Omega}{\delta_n}} \mathcal{U}^{p-1}\tilde{\phi}_n\varphi\, dx - \lambda_n\delta_n^2\int_{\frac{\Omega}{\delta_n}} \tilde{\phi}_n\varphi\, dx \\ = \delta_n^3\int_{\frac{\Omega}{\delta_n}} \nabla\tilde{h}_n\nabla\varphi\, dx + \delta_n^3\int_{\frac{\Omega}{\delta_n}} \nabla\tilde{w}_n\nabla\varphi\, dx,\end{aligned}$$

where $\tilde{h}_n(y) = h_n(\delta_n y)$ and $\tilde{w}_n(y) = w_n(\delta_n y)$. So, as $n \to +\infty$, by using also the results of Step 1, we get that ϕ_0 solves the problem

$$-\Delta\phi_0 = p\,|\mathcal{U}(y)|^{p-1}\,\phi_0 \qquad \text{in } \mathbb{R}^4$$

and satisfies the condition

$$\int_{\mathbb{R}^4} \nabla\phi_0\nabla Z\, dx = 0$$

and hence $\phi_0 \equiv 0$.

Moreover also $\|\phi_n\|_{H_0^1(\Omega)}$ is bounded and so, up to a subsequence, also ϕ_n converges weakly to some ϕ^* in $H_0^1(\Omega)$ and, as before, we get that, as $n \to +\infty$, ϕ^* solves

$$-\Delta\phi^* = \lambda_1\phi^* \qquad \text{in } \Omega$$

with the condition

$$\int_\Omega \nabla e_1\nabla\phi^*\, dx = 0.$$

Hence we get that $\phi^* = 0$.

At the end, in a very standard way, one can prove that $\|\phi_n\| = o(1)$. This immediately gives a contradiction since by assumption $\|\phi_n\|^2 = 1$. □

4.1.3. *Solving equation* (4.1) We are now in position to find a solution $\Phi_\lambda \in \mathcal{K}^\perp$ of the equation (4.1), namely we prove the following result.

Proposition 4.3. *Let $N = 4$, τ and δ as in* (3.6)*. Then, for any $\eta > 0$, there exist $\epsilon_0 > 0$ and $c > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, for all $(s_1, s_2) \in \mathbb{R}^2_+$ satisfying condition* (3.8)*, there exists a unique solution $\bar\Phi_\lambda \in \mathcal{K}^\perp$ of the equation* (4.1)*, such that*

$$\|\bar\Phi_\lambda\| \le c\epsilon e^{-\frac{1}{\epsilon}}. \tag{4.11}$$

Moreover $\bar\Phi_\lambda$ is continuously differentiable with respect to (s_1, s_2).

Proof. The proof is almost standard and hence we sketch it. Let us fix $\eta > 0$ and define the operator $\mathcal{T} : \mathcal{K}^\perp \to \mathcal{K}^\perp$ as

$$\mathcal{T}(\phi) := -\mathcal{L}^{-1}[\mathcal{N}(\phi) + \mathcal{R}_\lambda].$$

We remark that $\mathcal{T}$ is well defined since $\mathcal{L}$ is invertible (see Proposition 4.2).

In order to find a solution of the equation (4.1) we solve the fixed point problem $\mathcal{T}(\phi) = \phi$. Let us define the proper ball

$$B_\epsilon := \left\{\phi \in \mathcal{K}^\perp : \ \|\phi\| \le r\epsilon e^{-\frac{1}{\epsilon}}\right\}$$

for $r > 0$ sufficiently large.

Let us show that $\mathcal{T}$ maps B_ϵ into B_ϵ. From Proposition 4.2, there exists $\epsilon_0 = \epsilon_0(\eta) > 0$ and $c = c(\eta) > 0$ such that:

$$\|\mathcal{T}(\phi)\| \le c(\|\mathcal{N}(\phi)\| + \|\mathcal{R}_\lambda\|), \tag{4.12}$$

for all $\phi \in \mathcal{K}^\perp$, for all $(s_1, s_2) \in \mathbb{R}^2_+$ satisfying (3.8) and for all $\epsilon \in (0, \epsilon_0)$.

In view of Proposition 4.1 we have to estimate only $\|\mathcal{N}_\lambda(\phi)\|$. Indeed:

$$\begin{aligned}
\|\mathcal{N}(\phi)\| \le\ & c|f(\mathcal{P}\mathcal{U}_\delta - \tau e_1 + \phi) - f(\mathcal{P}\mathcal{U}_\delta - \tau e_1) - f'(\mathcal{P}\mathcal{U}_\delta - \tau e_1)\phi|_{\frac{2N}{N+2}}\\
&+ |[f'(\mathcal{P}\mathcal{U}_\delta - \tau e_1) - f'(\mathcal{P}\mathcal{U}_\delta)]\phi|_{\frac{2N}{N+2}}\\
&+ |[f'(\mathcal{P}\mathcal{U}_\delta) - f'(\mathcal{U}_\delta)]\phi|_{\frac{2N}{N+2}}\\
\le\ & c|\mathcal{P}\mathcal{U}_\delta^{p-2}\phi^2|_{\frac{2N}{N+2}} + c|\tau^{p-2}e_1^{p-2}\phi^2|_{\frac{2N}{N+2}}\\
&+ c|\phi^p|_{\frac{2N}{N+2}} + c|\tau^{p-1}e_1^{p-1}\phi|_{\frac{2N}{N+2}}\\
&+ c|(\tau e_1)^{p-2}\mathcal{P}\mathcal{U}_\delta\phi|_{\frac{2N}{N+2}} + c|\varphi_\delta^{p-1}\phi|_{\frac{2N}{N+2}} + c|\varphi_\delta^{p-2}\mathcal{U}_\delta\phi|_{\frac{2N}{N+2}}.
\end{aligned} \tag{4.13}$$

Now since $p - 2 = \frac{6-N}{N-2}$, we have

$$\left(\mathcal{P}\mathcal{U}_\delta^{p-2}\right)^{\frac{2N}{N+2}} = \mathcal{P}\mathcal{U}_\delta^{\frac{2N(6-N)}{(N-2)(N+2)}} \le \mathcal{U}_\delta^{\frac{2N(6-N)}{(N-2)(N+2)}} \le c\delta^{-\frac{N(6-N)}{N+2}}.$$

Hence we get that

$$\left(\int_\Omega \left(\mathcal{P}\mathcal{U}_\delta^{p-2}\phi^2\right)^{\frac{2N}{N+2}} dx\right)^{\frac{N+2}{2N}} \le c\left(\delta^{-N\frac{6-N}{N+2}}\int_\Omega \phi^{\frac{4N}{N+2}} dx\right)^{\frac{N+2}{2N}}$$
$$\le c_1\delta^{-\frac{6-N}{2}}\left(\int_\Omega \phi^{\frac{4N}{N+2}} dx\right)^{\frac{N+2}{2N}} \le c_2\delta^{-\frac{6-N}{2}}\|\phi\|^2.$$

We observe that for $N = 4$, and thanks to the choice of δ we have

$$\delta^{-\frac{6-N}{2}}\|\phi\|^2 \le c\epsilon e^{-\frac{1}{\epsilon}}$$

for all sufficiently small ϵ.

The remaining terms of (4.13) are even simpler and the estimates can be obtained in a similar way. Moreover, with analogous estimates we obtain that $\mathcal{T} : B_\epsilon \to B_\epsilon$ is a contraction. Hence, by the fixed point theorem there exists a unique solution $\bar{\Phi}_\lambda$ of $\mathcal{T}(\phi) = \phi$. Finally, in a standard way one can prove that the map $\bar{\Phi}_\lambda$ is differentiable with respect to (s_1, s_2) (see [2]). The proof is complete. ☐

4.2. The reduction for $N = 5$

As anticipated in the introduction, in the case $N = 5$ we look for a remainder term of the form

$$\Phi_\lambda = \phi_1 + \phi_2,$$

with

$$\|\phi_2\| = o(\|\phi_1\|).$$

To this end we write (3.14) as

$$\mathcal{R}_1 + \mathcal{R}_2 + \mathcal{L}_1(\phi_1) + \mathcal{L}_2(\phi_2) + \mathcal{N}_1(\phi_1) + \mathcal{N}_2(\phi_1, \phi_2) = 0, \tag{4.14}$$

where

$$\mathcal{R}_1 := \Pi_1^\perp\left\{-\tau e_1 - i^*\left[-\lambda\tau e_1\right]\right\}, \tag{4.15}$$

$$\mathcal{R}_2 := \Pi^\perp\left\{\mathcal{P}\mathcal{U}_\delta - i^*\left[\lambda\mathcal{P}\mathcal{U}_\delta + f(\mathcal{P}\mathcal{U}_\delta - \tau e_1)\right]\right\}. \tag{4.16}$$

$$\mathcal{L}_1(\phi_1) := \Pi_1^\perp\left\{\phi_1 - i^*\left[\lambda_1\phi_1\right]\right\} \tag{4.17}$$

$$\mathcal{L}_2(\phi_2) := \Pi^\perp\left\{\phi_2 - i^*\left[f'(\mathcal{U}_\delta)\phi_2 + \lambda\phi_2\right]\right\} \tag{4.18}$$

$$\mathcal{N}_1(\phi_1) := \Pi_1^\perp\{-i^*[f(-\tau e_1 + \phi_1) - (\lambda_1 - \lambda)\phi_1]\}, \tag{4.19}$$

and

$$\begin{aligned}\mathcal{N}_2(\phi_1, \phi_2) := \Pi^\perp\Big\{ & -i^*\big[f(V_\lambda + \phi_1 + \phi_2)\\ & - f'(\mathcal{U}_\delta)\phi_2 - f(-\tau e_1 + \phi_1) - f(\mathcal{P}\mathcal{U}_\delta - \tau e_1)\big]\Big\}.\end{aligned} \tag{4.20}$$

Now, in order to solve equation (4.14) we solve the following system of equations

$$\begin{cases} \mathcal{R}_1 + \mathcal{L}_1(\phi_1) + \mathcal{N}_1(\phi_1) = 0 \\ \mathcal{R}_2 + \mathcal{L}_2(\phi_2) + \mathcal{N}_2(\phi_1, \phi_2) = 0. \end{cases} \tag{4.21}$$

It is clear that a solution of (4.21) gives a solution of (4.14). Moreover we remark that it is not restrictive to consider $\mathcal{R}_1, \mathcal{L}_1(\phi_1), \mathcal{N}_1(\phi_1) \in \mathcal{K}_1^{\perp}$ since only δ_1 appears.

In order to solve (4.21) we apply a fixed point argument twice (see Section 4.2.3). As usual we have to estimate first the error terms $\mathcal{R}_1$ and $\mathcal{R}_2$, then we have to analyze the invertibility of the linear operators $\mathcal{L}_1$ and $\mathcal{L}_2$.

In what follows we let

$$\theta_1 := \frac{5}{4} \quad \text{and} \quad \theta_2 := 3. \tag{4.22}$$

4.2.1. *Estimates of the error terms*

Proposition 4.4. *It holds* $\mathcal{R}_1 = 0$.

Proof. Let us fix $\tau > 0$. By linearity we have $\mathcal{R}_1 = \tau \Pi_1^{\perp}\{-e_1 - i^*[-\lambda e_1]\}$; hence $\mathcal{R}_1 = 0$ if and only if $-e_1 - i^*[-\lambda e_1] = c e_1$ for some $c \in \mathbb{R}$. This is true, since, by definition of i^* and e_1, it holds $-e_1 - i^*[-\lambda e_1] = (-1 + \frac{\lambda}{\lambda_1})e_1$. The proof is complete. □

Proposition 4.5. *For any* $\eta > 0$*, there exist* $\epsilon_0 > 0$ *and* $c > 0$ *such that for all* $\epsilon \in (0, \epsilon_0)$*, for all* $(d_1, d_2) \in \mathbb{R}_+^2$ *satisfying* (3.8)*, we have*

$$\|\mathcal{R}_2\| \le c\, \epsilon^{\frac{\theta_2}{2}+\sigma},$$

for some positive real number σ*, whose choice depends only on* N.

The proof of this result can be obtained by reasoning as in Proposition 4.1.

4.2.2. *The linear operators* Let us first consider the linear operator $\mathcal{L}_1 : \mathcal{K}_1^{\perp} \to \mathcal{K}_1^{\perp}$ defined as in (4.17).

Next result states the invertibility of the operator $\mathcal{L}_1$ and provides a uniform estimate on the norm of $\mathcal{L}_1^{-1}$.

Proposition 4.6. *The linear operator* $\mathcal{L}_1 : \mathcal{K}_1^{\perp} \to \mathcal{K}_1^{\perp}$ *is invertible and* $\|\mathcal{L}_1^{-1}\| \le c$ *for some constant depending only on* N *and* Ω.

Proof. Let us fix $h \in \mathcal{K}_1^{\perp}$. We consider the problem

$$\begin{cases} -\Delta\phi = \lambda_1\phi + h & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.23}$$

Since $h \in \mathcal{K}_1^\perp$ it is well known that (4.23) has a solution $\phi \in H_0^1(\Omega)$ (see [3], Theorem 0.7). Moreover it is elementary to see that the solution is unique in $\mathcal{K}_1^\perp$.

Hence by definition of $\Pi_1^\perp$ and i^* it follows immediately that $\mathcal{L}_1(\phi) = h$ has a unique solution $\bar\phi = \bar\phi(h) \in \mathcal{K}_1^\perp$, and from elliptic estimates we have $\|\bar\phi\| \le c\|h\|$, which implies the boundedness of $\mathcal{L}_1^{-1}$. The proof is complete. □

Let now $\mathcal{L}_2 : \mathcal{K}^\perp \to \mathcal{K}^\perp$ defined in (4.18). Reasoning as in Proposition 4.2 we have the following result.

Proposition 4.7. *Let $N = 5$ and δ as in* (3.7)*. Then, for any small $\eta > 0$, there exists $C = C(\eta) > 0$ such that for all λ sufficiently close to λ_1, for any real number $d_1 \in (\eta, \frac{1}{\eta})$ and for any $\phi \in \mathcal{K}^\perp$ it holds*

$$\|\mathcal{L}_2(\phi)\| \ge C\|\phi\|.$$

Moreover $\mathcal{L}_2$ is invertible and $\|\mathcal{L}_2^{-1}\| \le \frac{1}{C}$.

4.2.3. *The auxiliary equation: solution of the system* (4.21) In this section we solve system (4.21).

The strategy is to solve the first equation of (4.21) by a fixed point argument, finding a unique $\bar\phi_1$ and then, substituting $\bar\phi_1$ in the second equation of (4.21), we obtain an equation depending only on the variable ϕ_2. Hence, using again a fixed point argument, we solve the second equation of (4.21) uniquely. More precisely, by arguing as in the proofs of [33, Propositions 3.1, 3.6], we obtain the following results:

Proposition 4.8. *Let $N = 5$ and τ as in* (3.7)*. Then, for any $\eta > 0$, there exist $\epsilon_0 > 0$ and $c > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, for all $d_1 \in \mathbb{R}_+$ satisfying condition* (3.8) *for $j = 1$, there exists a unique solution $\bar\phi_1 = \bar\phi_1(d_1)$, $\bar\phi_1 \in \mathcal{K}_1^\perp$ of the first equation in* (4.21) *which is continuously differentiable with respect to d_1 and such that*

$$\|\bar\phi_1\| \le c\epsilon^{\frac{\theta_1}{2}+\sigma}, \tag{4.24}$$

where θ_1 is defined in (4.22) *and σ is some positive real number whose choice depends only on N.*

Proposition 4.9. *Let $N = 5$, τ and δ as in* (3.7)*. Then, for any $\eta > 0$, denoting by $\bar\phi_1 \in \mathcal{K}_1^\perp$ the solution of the first equation in* (4.21) *found in Proposition* 4.8*, there exist $\epsilon_0 > 0$ and $c > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, for all $(d_1, d_2) \in \mathbb{R}_+^2$ satisfying condition* (3.8)*, there exists a unique solution $\bar\phi_2 \in \mathcal{K}^\perp$ of the second equation in* (4.21) *with $\phi_1 = \bar\phi_1$, such that*

$$\|\bar\phi_2\| \le c\epsilon^{\frac{\theta_2}{2}+\sigma}, \tag{4.25}$$

where θ_2 is defined in (4.22) *and σ is some positive real number depending only on N. Moreover $\bar\phi_2$ is continuously differentiable with respect to (d_1, d_2).*

5. The reduced problem

We are now left to solve (3.15).

5.1. Estimates for the reduced functional for $N = 4$

Let $\bar{\Phi}_\lambda \in \mathcal{K}^\perp$ be the solution found in Proposition 4.3. Hence $V_\lambda + \bar{\Phi}_\lambda$ is a solution of our original problem (1.1) if we can find $\bar{s}_\lambda = (\bar{s}_{1\lambda}, \bar{s}_{2\lambda})$ which satisfies condition (3.8) and solves equation (3.15).

To this end we consider the reduced functional $\tilde{J}_\lambda : \mathbb{R}^2_+ \to \mathbb{R}$ defined by:

$$\tilde{J}_\lambda(s_1, s_2) := J_\lambda\left(V_\lambda + \bar{\Phi}_\lambda\right),$$

where J_λ is the functional defined in (1.5).

The following result states that solving (3.15) is equivalent to finding critical points $(\bar{s}_{1,\lambda}, \bar{s}_{2,\lambda})$ of the reduced functional $\tilde{J}_\lambda(s_1, s_2)$, moreover it provides a uniform expansion of the reduced functional which will be used in the sequel.

Lemma 5.1. *The following facts hold true:*

(i) *For any small $\eta > 0$ there exists $\epsilon_0 > 0$ such that for all $\lambda \in (\lambda_1, \lambda_1 + \epsilon_0)$ if $(\bar{s}_{1,\lambda}, \bar{s}_{2,\lambda})$ is a critical point of $\tilde{J}_\lambda$ and satisfies* (3.8), *then $V_\lambda + \bar{\Phi}_\lambda$ is a solution of* (1.1);

(ii) *For any $\eta > 0$ there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ it holds*

$$J_\lambda(V_\lambda + \bar{\Phi}_\lambda) = J_\lambda(V_\lambda) + o\left(\epsilon e^{-\frac{2}{\epsilon}}\right).$$

The proof of the above lemma is quite standard (see for instance [37]) and hence we omit it.

5.2. Estimates for the reduced functional for $N = 5$

Let $(\bar{\phi}_1, \bar{\phi}_2) \in \mathcal{K}_1^\perp \times \mathcal{K}^\perp$ be the solution found in Propositions 4.8, 4.9. As in the case $N = 4$, in order to solve (3.15) we consider the reduced functional $\tilde{J}_\lambda : \mathbb{R}^2_+ \to \mathbb{R}$ defined by:

$$\tilde{J}_\lambda(d_1, d_2) := J_\lambda(V_\lambda + \bar{\phi}_1 + \bar{\phi}_2),$$

where J_λ is the functional defined in (1.5).

As before critical points of the reduced functional give rise to solutions of (3.15) (see (i) of Lemma 5.3). Nevertheless, the expansion of the reduced functional is more delicate. In fact, in order to get the estimates of Lemma 5.3 we need informations on the asymptotic behavior of the L^∞-norm of $\bar{\phi}_1$. This is the content of the next lemma.

Lemma 5.2. *Let η be a small positive number and $\bar{\phi}_1 \in \mathcal{K}_1^{\perp}$ be the solution of the first equation in* (4.21)*, found in Proposition* 4.8*. Then, up to a subsequence, as $\epsilon \to 0$, we have*

$$|\bar{\phi}_1|_\infty \to 0,$$

uniformly with respect to d_1 such that $\eta < d_1 < \frac{1}{\eta}$.

Proof. Let us fix a small $\eta > 0$ and remember that $\tau = d_1 \epsilon^{\frac{3}{4}}$, where $d_1 \in]\eta, \frac{1}{\eta}[$. We observe that by definition, since $\bar{\phi}_1 \in \mathcal{K}_1^{\perp}$ solves the first equation of (4.21), then, for all ϵ sufficiently small, for all $d_1 \in]\eta, \frac{1}{\eta}[$, there exists a constant $c_\epsilon = c_\epsilon(d_1)$ such that $\bar{\phi}_1$ weakly solves

$$-\Delta \bar{\phi}_1 = (\lambda_1 - \epsilon)\bar{\phi}_1 + f(-\tau e_1 + \bar{\phi}_1) - \lambda_1 c_\epsilon e_1. \tag{5.1}$$

Testing (5.1) with e_1, and taking into account that $\bar{\phi}_1 \in \mathcal{K}_1^{\perp}$, we deduce that $c_\epsilon \to 0$, as $\epsilon \to 0$, uniformly with respect to $d_1 \in]\eta, \frac{1}{\eta}[$.

We observe that $\bar{\phi}_1$ is a classical solution of (5.1). This comes from standard elliptic regularity theory, the application of a well-known lemma by Brezis and Kato, taking into account that $\bar{\phi}_1 \in H_0^1(\Omega)$ weakly solves (5.1) and the smoothness of e_1, f.

We consider the quantity $\sup_{d_1 \in]\eta, \frac{1}{\eta}[} |\bar{\phi}_1|_\infty$, which is defined for all $\epsilon \in (0, \epsilon_0)$, where $\epsilon_0 > 0$ is given by Proposition 4.8. We want to prove that

$$\lim_{\epsilon \to 0^+} \sup_{d_1 \in]\eta, \frac{1}{\eta}[} |\bar{\phi}_1|_\infty = 0. \tag{5.2}$$

In order to prove (5.2) we argue by contradiction. Assume that (5.2) is false. Then, there exists a positive number $m \in \mathbb{R}^+$, a sequence $(\epsilon_k)_k \subset \mathbb{R}^+$, $\epsilon_k \to 0$ as $k \to +\infty$, such that

$$\sup_{d_1 \in]\eta, \frac{1}{\eta}[} |\bar{\phi}_{1,k}|_\infty > m, \tag{5.3}$$

for any $k \in \mathbb{N}$, where we have set $\bar{\phi}_{1,k} := \bar{\phi}_1(\epsilon_k, d_1) \in B_{1,\epsilon_k}$. We observe that (5.3) contemplates the possibility that $\sup_{d_1 \in]\eta, \frac{1}{\eta}[} |\bar{\phi}_{1,k}|_\infty = +\infty$. From (5.3), for any $k \in \mathbb{N}$, thanks to the definition of sup, we get that there exists $d_{1,k} \in]\eta, \frac{1}{\eta}[$ such that

$$|\bar{\phi}_{1,k}|_\infty(d_{1,k}) > \frac{m}{2}.$$

Hence, if we consider the sequence $\left(|\bar{\phi}_{1,k}|_\infty(d_{1,k})\right)_k$, then, up to a subsequence, as $k \to +\infty$, there are only two possibilities:

(a) $|\bar{\phi}_{1,k}|_\infty(d_{1,k}) \to +\infty$;

(b) $|\bar{\phi}_{1,k}|_\infty(d_{1,k}) \to l$, for some $l \geq \frac{m}{2} > 0$.

We will show that (a) and (b) cannot happen.

Assume (a). We point out that, since $\eta > 0$ is fixed, then, $d_{1,k} \in]\eta, \frac{1}{\eta}[$ for all k, in particular this sequence stays definitely away from 0 and from $+\infty$. Hence, in order to simplify the notation of this proof, we omit the dependence from $d_{1,k}$ in $\bar{\phi}_{1,k}(d_{1,k})$, $c_{\epsilon_k}(d_{1,k})$ and thus we simply write $\bar{\phi}_{1,k}$, c_{ϵ_k}. In particular, we observe that, for any fixed k, $\bar{\phi}_{1,k}$ is a function depending only on the space variable $x \in \Omega$.

Then, for any $k \in \mathbb{N}$, let $a_k \in \Omega$ such that $|\bar{\phi}_{1,k}(a_k)| = |\bar{\phi}_{1,k}|_\infty$ and set $M_k := |\bar{\phi}_{1,k}|_\infty$. We consider the rescaled function

$$\widetilde{\phi}_{1,k}(y) := \frac{1}{M_k}\bar{\phi}_{1,k}\left(a_k + \frac{y}{M_k^\beta}\right), \qquad \beta = \frac{2}{N-2},$$

defined for $y \in \widetilde{\Omega}_k := M_k^{\frac{2}{N-2}}(\Omega - a_k)$.

Moreover let us set

$$\widetilde{e}_{1,k}(y) := \frac{1}{M_k}e_1\left(a_k + \frac{y}{M_k^{\frac{2}{N-2}}}\right) \quad \text{and} \quad \tau_k := d_{1,k}\epsilon_k^{\frac{3}{4}}.$$

It is clear that $\|\widetilde{e}_{1,k}\|_{\infty,\widetilde{\Omega}_k} \to 0$, $\tau_k \to 0$, as $k \to +\infty$. By elementary computations we see that $\widetilde{\phi}_{1,k}$ solves

$$\begin{cases} -\Delta\widetilde{\phi}_{1,k} = \dfrac{\lambda_1 - \epsilon_k}{M_k^{\frac{4}{N-2}}}\widetilde{\phi}_{1,k} + f(-\tau_k\widetilde{e}_{1,k} + \widetilde{\phi}_{1,k}) - \dfrac{\lambda_1 c_{\epsilon_k}}{M_k^{\frac{4}{N-2}}}\widetilde{e}_{1,k} & \text{in } \widetilde{\Omega}_k \\ \widetilde{\phi}_{1,k} = 0, & \text{on } \partial\widetilde{\Omega}_k. \end{cases} \tag{5.4}$$

Let us denote by Π the limit domain of $\widetilde{\Omega}_k$. Since we are assuming (a) we have $M_k \to +\infty$, as $k \to +\infty$, and hence Π is the whole $\mathbb{R}^N$ or an half-space. Moreover, since the family $(\widetilde{\phi}_{1,k})_k$ is uniformly bounded and solves (5.4), then, by the same proof of [11, Lemma 2.2], we get that $0 \in \Pi$ (in particular $0 \notin \partial\Pi$), and, by standard elliptic theory, it follows that, up to a subsequence, as $k \to +\infty$, we have that $\widetilde{\phi}_{1,k}$ converges in $C^2_{\text{loc}}(\Pi)$ to a function w which satisfies

$$\begin{aligned} -\Delta w &= f(w) \text{ in } \Pi \\ w(0) &= 1 \text{ (or } w(0) = -1) \\ |w| &\leq 1 \text{ in } \Pi \\ w &= 0 \text{ on } \partial\Pi. \end{aligned} \tag{5.5}$$

We observe that, thanks to the definition of the chosen rescaling, by elementary computations (see [30, Lemma 2]), it holds $\|\widetilde{\phi}_{1,k}\|^2_{\widetilde{\Omega}_\epsilon} = \|\bar{\phi}_{1,k}\|^2_{\Omega}$. Now, since $\|\bar{\phi}_{1,k}\| \leq c\epsilon_k^{\frac{\theta_1}{2}+\sigma}$, where c depends only on η and σ is some positive number (see Proposition 4.8), we have $\|\widetilde{\phi}_{1,k}\|^2_{\widetilde{\Omega}_k} = \|\bar{\phi}_{1,k}\|^2_{\Omega} \to 0$, as $k \to +\infty$. Hence, since $\widetilde{\phi}_{1,k} \to w$ in $C^2_{\mathrm{loc}}(\Pi)$, by Fatou's lemma, it follows that

$$\|w\|^2_{\Pi} \leq \liminf_{k\to+\infty} \|\widetilde{\phi}_{1,k}\|^2_{\widetilde{\Omega}_k} = 0. \tag{5.6}$$

Therefore, since $\|w\|^2_{\Pi} = 0$ and w is smooth, it follows that w is constant, and from $w(0) = 1$ (or $w(0) = -1$) we get that $w \equiv 1$ (or $w \equiv -1$) in Π. But, since w is constant and solves $-\Delta w = f(w)$ in Π, then necessarily $f(w) \equiv 0$ in Π, and hence w must be the null function, but this contradicts $w \equiv 1$ (or $w \equiv -1$).

Assume (b). We use the same convention on the notation as in previous case. Then $(\bar{\phi}_{1,k})_k$ is uniformly bounded, in particular there exist two positive constants c_1, c_2 such that for all $k \in \mathbb{N}$ it holds

$$c_1 < |\bar{\phi}_{1,k}|_\infty < c_2. \tag{5.7}$$

By definition, $\bar{\phi}_{1,k}$ solves

$$-\Delta\bar{\phi}_{1,k} = (\lambda_1 - \epsilon_k)\bar{\phi}_{1,k} + f(-\tau_k e_1 + \bar{\phi}_{1,k}) - \lambda_1 c_{\epsilon_k} e_1. \tag{5.8}$$

Hence, by standard elliptic theory, it follows that, up to a subsequence, $\bar{\phi}_{1,k}$ converges in $C^2_{\mathrm{loc}}(\Omega)$ to a function w which satisfies

$$\begin{cases} -\Delta w = \lambda_1 w + f(w) & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.9}$$

Now, since $\|\bar{\phi}_{1,k}\| \leq c\epsilon_k^{\frac{\theta_1}{2}+\sigma}$, where $c > 0$ depends only on η and $\bar{\phi}_{1,k} \to w$ in $C^2_{\mathrm{loc}}(\Omega)$, then, by Fatou's Lemma and Sobolev inequality we have that

$$|w|_{p+1} \leq \liminf_{k\to+\infty} |\bar{\phi}_{1,k}|_{p+1} = 0,$$

thus, since w is smooth, it follows that $w \equiv 0$. But, if $a_k \in \Omega$ is such that $|\bar{\phi}_{1,k}|_\infty = \bar{\phi}_{1,k}(a_k)$, by slightly modifications to the proof of [11, Lemma 2.2] we have that $d(a_k, \partial\Omega) \nrightarrow 0$ as $k \to +\infty$. Hence, this fact, $\bar{\phi}_1 \to w$ in $C^2_{\mathrm{loc}}(\Omega)$ and $w \equiv 0$ contradict (5.7).

Alternatively, assuming that $\partial\Omega$ is of class $C^{2,\alpha}$, for some $\alpha \in (0, 1)$, without using the arguments of [11, Lemma 2.2], but using standard elliptic regularity theory and [27, Lemma 6.36], since $\bar{\phi}_{1,k}$ is uniformly bounded, we get that, up to a subsequence $\bar{\phi}_{1,k}$ converges to w in $C^2(\overline{\Omega})$, where w solves (5.9). As before it holds $w \equiv 0$ and hence we contradicts (5.7). The proof is then concluded. □

Lemma 5.3. *The following facts hold true:*

(i) *For any small $\eta > 0$ there exists $\epsilon_0 > 0$ such that for all $\lambda \in (\lambda_1 - \epsilon_0, \lambda_1)$ if $(\bar{d}_{1,\lambda}, \bar{d}_{2,\lambda})$ is a critical point of $\tilde{J}_\lambda$ and satisfies* (3.8)*, then $V_\lambda + \bar{\phi}_1 + \bar{\phi}_2$ is a solution of* (1.1).

(ii) *For any $\eta > 0$ there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ it holds:*

$$J_\lambda(V_\lambda + \bar{\phi}_1) = J_\lambda(V_\lambda) + O(\epsilon^{\theta_1+\sigma}),$$

with

$$O(\epsilon^{\theta_1+\sigma}) = \epsilon^{\theta_1+\sigma} M_1(d_1) + o\left(\epsilon^{\theta_2}\right), \tag{5.10}$$

for some function M_1 depending only on d_1 (and uniformly bounded with respect to ϵ), where θ_1, θ_2 are defined in (4.22)*, σ is some positive real number (depending only on N). These expansion are C^0-uniform with respect to (d_1, d_2) satisfying condition* (3.8).

(iii) *For any $\eta > 0$ there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ it holds:*

$$J_\lambda(V_\lambda + \bar{\phi}_1 + \bar{\phi}_2) = J_\lambda(V_\lambda + \bar{\phi}_1) + O(\epsilon^{\theta_2+\sigma}),$$

C^0-uniformly with respect to (d_1, d_2) satisfying condition (3.8)*, for some positive real number σ depending only on N.*

Proof. The proof of the lemma can be made as in [33, Lemmas 4.3, 4.4]. We limit to sketch the proof of (ii) just to underline where Lemma 5.2 is needed.

Let us fix $\eta > 0$. By direct computation we see that

$$\begin{aligned} J_\lambda(V_\lambda + \bar{\phi}_1) - J_\lambda(V_\lambda) = &\frac{1}{2}\int_\Omega |\nabla\bar{\phi}_1|^2\,dx + \int_\Omega \nabla V_\lambda \cdot \nabla\bar{\phi}_1\,dx \\ &- \frac{\lambda}{2}\int_\Omega |\bar{\phi}_1|^2\,dx - \lambda\int_\Omega V_\lambda\bar{\phi}_1\,dx \\ &- \frac{1}{p+1}\int_\Omega (|V_\lambda + \bar{\phi}_1|^{p+1} - |V_\lambda|^{p+1})\,dx. \end{aligned} \tag{5.11}$$

By definition we have

$$\begin{aligned} \int_\Omega \nabla V_\lambda \cdot \nabla\bar{\phi}_1\,dx &= \int_\Omega \nabla(\mathcal{P}\mathcal{U}_\delta - \tau e_1)\cdot\nabla\bar{\phi}_1\,dx = \int_\Omega (\mathcal{U}_\delta^p - \lambda_1\tau e_1)\bar{\phi}_1\,dx \\ &= \int_\Omega [f(\mathcal{U}_\delta) - \lambda_1\tau e_1]\bar{\phi}_1\,dx, \end{aligned}$$

moreover, since $F(s)=\frac{1}{p+1}|s|^{p+1}$ is a primitive of f, we can write (5.11) as

$$\begin{aligned}
J_\lambda(V_\lambda+\bar\phi_1)-J_\lambda(V_\lambda) &= \frac{1}{2}\|\bar\phi_1\|^2-\frac{\lambda}{2}|\bar\phi_1|_2^2-\lambda\int_\Omega V_\lambda\bar\phi_1\,dx\\
&\quad+\int_\Omega[f(\mathcal{U}_\delta)-\lambda_1\tau e_1]\bar\phi_1\,dx\\
&\quad-\int_\Omega[F(V_\lambda+\bar\phi_1)-F(V_\lambda)]\,dx\\
&= \frac{1}{2}\|\bar\phi_1\|^2-\frac{\lambda}{2}|\bar\phi_1|_2^2-\lambda\int_\Omega \mathcal{P}\mathcal{U}_\delta\bar\phi_1\,dx\\
&\quad+(\lambda-\lambda_1)\int_\Omega\tau e_1\bar\phi_1\,dx\\
&\quad+\int_\Omega[f(\mathcal{U}_\delta)-f(V_\lambda)]\bar\phi_1\,dx\\
&\quad-\int_\Omega[F(V_\lambda+\bar\phi_1)-F(V_\lambda)-f(V_\lambda)\bar\phi_1]\,dx\\
&:= A+B+C+D+E+F.
\end{aligned} \tag{5.12}$$

For the terms A–E, by arguing as in [33, Lemma 4.3] we get that

$$J_\lambda(V_\lambda+\bar\phi_1)-J_\lambda(V_\lambda)=\epsilon^{\theta_1+\sigma}M_1(d_1)+o(\epsilon^{\theta_2}), \tag{5.13}$$

for all sufficiently small ϵ, for some function M_1 depending only on d_1 (and uniformly bounded with respect to ϵ). For the remaining term F, applying elementary inequalities we get that

$$\begin{aligned}
|F| &\le c\int_\Omega\left(|V_\epsilon|^{p-1}\bar\phi_1^2+|\bar\phi_1|^{p+1}\right)\,dx\\
&\le c\int_\Omega(\mathcal{P}\mathcal{U}_\delta^{p-1}\bar\phi_1^2+(\tau e_1)^{p-1}\bar\phi_1^2+|\bar\phi_1|^{p+1})\,dx\\
&= F_1+F_2+F_3.
\end{aligned}$$

For F_1, applying Lemma 5.2, as $\epsilon\to 0$, we have $|\bar\phi_1|_\infty=o(1)$. Hence, taking into account that $\int_\Omega\frac{1}{|x|^4}\,dx$ is finite, we get that

$$\begin{aligned}
F_1 &= \int_\Omega \mathcal{P}\mathcal{U}_\delta^{p-1}\bar\phi_1^2\,dx\le\int_\Omega\mathcal{U}_\delta^{p-1}\bar\phi_1^2\,dx\le c\int_\Omega\frac{\delta^2}{|x|^4}\bar\phi^2\,dx\\
&= o\left(\delta^2\int_\Omega\frac{1}{|x|^4}\,dx\right)=o(\epsilon^{\theta_2}).
\end{aligned}$$

For F_2, thanks to the definition of τ and since $\bar{\phi}_1 \in B_{1,\epsilon}$, we have

$$\int_\Omega (\tau e_1)^{p-1}\bar{\phi}_1^2\,dx \le \tau^{p-1}\|e_1\|_\infty^{p-1}\int_\Omega \bar{\phi}_1^2\,dx \le c\,\tau^{p-1}\int_\Omega |\nabla\bar{\phi}_1|^2\,dx$$
$$\le c_1\epsilon^{\frac{3}{4}\frac{4}{3}}\epsilon^{2(\frac{5}{4}+\sigma)} \le c_1\epsilon^{\theta_2+\sigma}.$$

Finally, for F_3, we have

$$\int_\Omega |\bar{\phi}_1|^{p+1}\,dx \le c\|\bar{\phi}_1\|^{p+1} \le c_1\epsilon^{\frac{10}{3}(\frac{5}{4}+\sigma)} \le c_1\epsilon^{\theta_2+\sigma}.$$

Hence $|F| = o(\epsilon^{\theta_2})$ and combining this with (5.13) we get the desired assertion. □

5.3. Energy expansion of the approximate solution

By the above discussion, in order to prove our main results, we need to find critical points of the reduced functional $\tilde{J}$. To this end we have to analyze the term $J_\lambda(V_\lambda)$, which is the energy of the approximate solution $V_\lambda = \mathcal{P}\mathcal{U}_\delta - \tau e_1$. In the proof of the following lemma we find an expansion for $J_\lambda(V_\lambda)$, and combining it with the expansions obtained in Lemma 5.1, Lemma 5.3 we get:

Proposition 5.4. *The following facts hold:*

(i) *Let $N = 4$. For any $\eta > 0$, as $\lambda \to \lambda_1^+$, the following expansion holds:*

$$\tilde{J}_\lambda(s_1,s_2) = \frac{1}{4}S^2 + \epsilon e^{-\frac{1}{\epsilon}}\left[-b_1 g(s_2)^2 + b_2 g(s_2)s_1 - b_3 s_1^2\right] + o\left(\epsilon e^{-\frac{2}{\epsilon}}\right), \tag{5.14}$$

where $\epsilon = \lambda - \lambda_1$, b_1, b_2, b_3 are positive known constants.

(ii) *Let $N = 5$. For any $\eta > 0$, as $\lambda \to \lambda_1^-$ it holds:*

$$\tilde{J}_\lambda(d_1,d_2) = \frac{1}{5}S^{5/2} + \epsilon^{\frac{5}{2}}\left[a_1 d_1^2 - a_2 d_1^{\frac{10}{3}}\right] + O\left(\epsilon^{\frac{5}{2}+\sigma}\right), \tag{5.15}$$

with

$$O(\epsilon^{\frac{5}{2}+\sigma}) = \epsilon^{\frac{5}{2}+\sigma}M_1(d_1) + \epsilon^3\left[a_3 d_1 d_2^{\frac{3}{2}} - a_4 d_2^2\right] + o\left(\epsilon^3\right), \tag{5.16}$$

for some function M_1 depending only on d_1 (and uniformly bounded with respect to $\epsilon = \lambda_1 - \lambda$), where σ is some positive real number (depending only on N) and a_j, $j = 1,2,3,4$ are some positive and known constants.

The expansions (5.14), (5.15) *and* (5.16) *are C^0-uniform with respect to (s_1,s_2) or (d_1,d_2) satisfying condition* (3.8).

Remark 5.5. We point out that the term M_1 appearing in (5.16) does not depend on d_2 and this will be used in the sequel.

Proof. By making some standard computations we find that

$$\begin{aligned} J_\lambda(\mathcal{P}\mathcal{U}_\delta-\tau e_1)=&\left(\frac{1}{2}-\frac{1}{p+1}\right)\int_\Omega \mathcal{U}_\delta^{p+1}dx+\frac{1}{2}\int_\Omega \mathcal{U}_\delta^p\varphi_\delta dx+\frac{\tau^2}{2}(\lambda_1-\lambda)\int_\Omega e_1^2dx\\ &+\tau(\lambda-\lambda_1)\int_\Omega \mathcal{P}\mathcal{U}_\delta e_1dx-\frac{\lambda}{2}\int_\Omega \mathcal{P}\mathcal{U}_\delta^2dx\\ &-\frac{1}{p+1}\underbrace{\int_\Omega\left[|\mathcal{U}_\delta-\varphi_\delta|^{p+1}-\mathcal{U}_\delta^{p+1}+(p+1)\mathcal{U}_\delta^p\varphi_\delta\right]dx}_{(I)}\\ &-\frac{\tau^{p+1}}{p+1}\int_\Omega e_1^{p+1}dx+\tau\int_\Omega \mathcal{P}\mathcal{U}_\delta^pe_1dx-\tau^p\int_\Omega \mathcal{P}\mathcal{U}_\delta e_1^pdx\\ &-\frac{1}{p+1}\underbrace{\int_\Omega\left[|\mathcal{P}\mathcal{U}_\delta-\tau e_1|^{p+1}-\mathcal{P}\mathcal{U}_\delta^{p+1}-\tau^{p+1}e_1^{p+1}+(p+1)\mathcal{P}\mathcal{U}_\delta^p\tau e_1-(p+1)\mathcal{P}\mathcal{U}_\delta\tau^pe_1^p\right]dx}_{(II)}. \end{aligned}$$

For $N=4,5$ we have that

$$\left(\frac{1}{2}-\frac{1}{p+1}\right)\int_\Omega \mathcal{U}_\delta^{p+1}dx=\frac{1}{N}S_N^{N/2}+O(\delta^N)$$

and

$$\frac{1}{2}\int_\Omega \mathcal{U}_\delta^p\varphi_\delta\,dx=O\left(\delta^{N-2}\right).$$

Now if $N=4$, fixing a small $R>0$ such that $B_R\subset\subset\Omega$, we get

$$\begin{aligned}\int_\Omega \mathcal{U}_\delta^2\,dx&=\delta^2\int_{|x|<R}\frac{\alpha_4^2}{(\delta^2+|x|^2)^2}dx+\delta^2\int_{\Omega\setminus\{|x|<R\}}\frac{\alpha_4^2}{(\delta^2+|x|^2)^2}dx\\ &=\omega_4\alpha_4^2\delta^2\log\frac{1}{\delta}+O\left(\delta^2\right)\end{aligned}$$

where ω_4 denotes the surface area of the unit sphere in $\mathbb{R}^4$. Instead, for $N=5$ we have

$$\int_\Omega \mathcal{U}_\delta^2dx=\delta^{-3}\int_\Omega\frac{\alpha_5^2}{\left(1+|\frac{x}{\delta}|^2\right)^3}dx=\delta^2\int_{\mathbb{R}^5}\mathcal{U}^2dx+O\left(\delta^2\int_{\frac{1}{\delta}}^{+\infty}\frac{r^4}{(1+r^2)^3}dr\right).$$

Hence

$$\begin{aligned}\int_\Omega \mathcal{P}\mathcal{U}_\delta^2\,dx&=\int_\Omega \mathcal{U}_\delta^2\,dx+\int_\Omega\varphi_\delta^2\,dx-2\int_\Omega \mathcal{U}_\delta\varphi_\delta\,dx\\ &=\begin{cases}\omega_4\alpha_4^2\delta^2\log\frac{1}{\delta}+O(\delta^2)+O(|\varphi_\delta|_\infty\int_\Omega \mathcal{U}_\delta\,dx) & \text{for } N=4\\ \delta^2\displaystyle\int_{\mathbb{R}^N}\mathcal{U}^2\,dx+O(\delta^3)+O(|\varphi_\delta|_2|\mathcal{U}_\delta|_2) & \text{for } N=5\end{cases}\end{aligned}$$

and so

$$\int_\Omega \mathcal{P}\mathcal{U}_\delta^2\,dx = \begin{cases} \omega_4\alpha_4^2\delta^2\log\frac{1}{\delta} + O\left(\delta^2\right) & \text{for } N=4 \\ \delta^2\int_{\mathbb{R}^N}\mathcal{U}^2\,dx + O\left(\delta^{\frac{5}{2}}\right) & \text{for } N=5. \end{cases}$$

Moreover

$$\begin{aligned}\int_\Omega \mathcal{P}\mathcal{U}_\delta e_1\,dx &= \int_\Omega e_1\left[\mathcal{U}_\delta - \varphi_\delta\right]dx \\ &= \int_\Omega e_1\left[\alpha_N\frac{\delta^{\frac{N-2}{2}}}{(\delta^2+|x|^2)^{\frac{N-2}{2}}} - \alpha_N\delta^{\frac{N-2}{2}}H(x,0) + O\left(\delta^{\frac{N+2}{2}}\right)\right]dx \\ &= \int_\Omega \alpha_N\delta^{\frac{N-2}{2}}e_1\left[\frac{1}{|x|^{N-2}} - H(x,0)\right]dx + O\left(\delta^{\frac{N+2}{2}}\right) \\ &= \frac{\alpha_N}{\gamma_N}\delta^{\frac{N-2}{2}}\int_\Omega e_1 G(x,0)\,dx + O\left(\delta^{\frac{N+2}{2}}\right) \\ &= \frac{\alpha_N}{\gamma_N\lambda_1}\delta^{\frac{N-2}{2}}e_1(0) + O\left(\delta^{\frac{N+2}{2}}\right),\end{aligned}$$

since $-\Delta e_1 = \lambda_1 e_1$ and hence $e_1(0) = \lambda_1\int_\Omega e_1(x)G(x,0)\,dx$.

Moreover

$$\begin{aligned}\tau\int_\Omega e_1\mathcal{P}\mathcal{U}_\delta^p\,dx &= \tau\int_\Omega e_1\mathcal{U}_\delta^p\,dx + \tau\int_\Omega e_1(\mathcal{P}\mathcal{U}_\delta^p - \mathcal{U}_\delta^p)\,dx \\ &= \tau\delta^{\frac{N-2}{2}}e_1(0)\int_{\mathbb{R}^N}\mathcal{U}^p\,dx + \begin{cases} O\left(\tau\delta^{\frac{N+2}{2}}\log\frac{1}{\delta}\right) & \text{if } N=4 \\ O\left(\tau\delta^{\frac{N+2}{2}}\right) & \text{if } N=5 \end{cases}\end{aligned}$$

and

$$\tau^p\int_\Omega \mathcal{P}\mathcal{U}_\delta e_1^p\,dx = \tau^p\delta^{\frac{N-2}{2}}\frac{\alpha_N}{\gamma_N}\int_\Omega e_1^p G(x,0)\,dx + O\left(\tau^p\delta^{\frac{N+2}{2}}\right).$$

Now

$$\begin{aligned}|(I)| &\le c\left(|\varphi_\delta|_{p+1,\Omega}^{p+1} + \int_\Omega \mathcal{U}_\delta^{p-1}\varphi_\delta^2\,dx\right) \le c_1\left(\delta^N + |\varphi_\delta|_\infty^2\int_\Omega\frac{\delta^2}{(\delta^2+|x|^2)^2}\,dx\right) \\ &\le c_1\delta^N + c_2\delta^{N-2}\begin{cases} C_0\delta^2\log\frac{1}{\delta} + O(1) & \text{for } N=4 \\ \delta^2\int_\Omega\frac{1}{|x|^{N-2}}\,dx & \text{for } N=5 \end{cases} \le c_3\begin{cases} \delta^4\log\frac{1}{\delta} & \text{for } N=4 \\ \delta^5 & \text{for } N=5 \end{cases}\end{aligned}$$

and

$$
\begin{aligned}
|(II)| &\le \left|\int_{B_{\sqrt{\delta}}(0)} \dots\, dx\right| + \left|\int_{\Omega\setminus B_{\sqrt{\delta}}(0)} \dots\, dx\right| \\
&\le \int_{B_{\sqrt{\delta}}(0)} \left| |\mathcal{P}\mathcal{U}_\delta - \tau e_1|^{p+1} - \mathcal{P}\mathcal{U}_\delta^{p+1} + (p+1)\mathcal{P}\mathcal{U}_\delta^{p}\tau e_1 \right| dx \\
&\quad + \tau^{p+1}\int_{B_{\sqrt{\delta}}(0)} e_1^{p+1}\, dx + \tau^p(p+1)\int_{B_{\sqrt{\delta}}(0)} \mathcal{P}\mathcal{U}_\delta e_1^p\, dx \\
&\quad + \int_{\Omega\setminus B_{\sqrt{\delta}}(0)} \mathcal{P}\mathcal{U}_\delta^{p+1}\, dx + \tau(p+1)\int_{\Omega\setminus B_{\sqrt{\delta}}(0)} \mathcal{P}\mathcal{U}_\delta^{p} e_1\, dx \\
&\quad + \int_{\Omega\setminus B_{\sqrt{\delta}}(0)} \left| |\mathcal{P}\mathcal{U}_\delta - \tau e_1|^{p+1} - \tau^{p+1}e_1^{p+1} - (p+1)\tau^p e_1^p \mathcal{P}\mathcal{U}_\delta \right| dx \\
&\le c_1 \left(\tau^2 \int_{B_{\sqrt{\delta}}(0)} \mathcal{P}\mathcal{U}_\delta^{p-1} e_1^2\, dx + \tau^{p+1}\int_{B_{\sqrt{\delta}}(0)} e_1^{p+1}\, dx \right. \\
&\quad + \tau^p \int_{B_{\sqrt{\delta}}(0)} \mathcal{P}\mathcal{U}_\delta e_1^p\, dx + \int_{\Omega\setminus B_{\sqrt{\delta}}(0)} \mathcal{P}\mathcal{U}_\delta^{p+1}\, dx \\
&\quad \left. + \tau^{p-1}\int_{\Omega\setminus B_{\sqrt{\delta}}(0)} \mathcal{P}\mathcal{U}_\delta^{2} e_1^{p-1}\, dx + \tau \int_{\Omega\setminus B_{\sqrt{\delta}}(0)} \mathcal{P}\mathcal{U}_\delta^{p} e_1\, dx \right) \\
&\le c \left(\tau^2\delta^2 \begin{cases} \log\frac{1}{\delta} & \text{for } N=4 \\ \sqrt{\delta} & \text{for } N=5 \end{cases} + \tau^{p+1}\delta^{\frac{N}{2}} \right. \\
&\qquad \left. + \tau^p\delta^{\frac{N}{2}} + \delta^{\frac{N}{2}} + \tau^{p-1} \begin{cases} \delta^2 & \text{for } N=4 \\ \delta^{\frac{5}{2}} & \text{for } N=5 \end{cases} + \tau\delta^{\frac{N}{2}} \right).
\end{aligned}
$$

Putting together all these estimates for $N = 4$ we get that

$$
\begin{aligned}
J_\lambda(\mathcal{P}\mathcal{U}_\delta - \tau e_1) = &\frac{1}{4}S^2 + \epsilon e^{-\frac{2}{\epsilon}}\left[-b_1 g(s_2)^2 + b_2 g(s_2)s_1 - b_3 s_1^2\right] \\
&+ o(\epsilon e^{-\frac{2}{\epsilon}})
\end{aligned}
\tag{5.17}
$$

where

$$
\begin{aligned}
b_1 &:= \frac{1}{2}\int_\Omega e_1^2\, dx \\
b_2 &:= e_1(0)\int_{\mathbb{R}^4} \mathcal{U}^p\, dx \\
b_3 &:= \frac{\lambda_1}{2}\omega_4\alpha_4^2,
\end{aligned}
$$

while for $N = 5$ we get

$$J_\lambda(\mathcal{P}\mathcal{U}_\delta - \tau e_1) = \frac{1}{5}S_5^{\frac{5}{2}} + \epsilon^{\frac{5}{2}}\left[a_1 d_1^2 - a_2 d_1^{\frac{10}{3}}\right] + \epsilon^{\frac{5}{2}+\sigma}M_1(d_1) + \epsilon^3\left[a_3 d_1 d_2^{\frac{3}{2}} - a_4 d_2^2\right] + O(\epsilon^{3+\sigma}) \tag{5.18}$$

where

$$a_1 := \frac{1}{2}\int_\Omega e_1^2\,dx$$
$$a_2 := \frac{1}{p+1}\int_\Omega e_1^{p+1}\,dx$$
$$a_3 := e_1(0)\int_{\mathbb{R}^5}\mathcal{U}^p\,dx$$
$$a_4 := \frac{\lambda_1}{2}\int_{\mathbb{R}^5}\mathcal{U}^2.$$

In the end, combining these expansions with those of Lemma 5.1, Lemma 5.3 the result follows. □

5.4. C^1- estimate of the reduced functional in the case $N = 4$

In the case $N = 4$ we need to be more accurate in order to find a critical point of the reduced functional (see the proof of Theorem 1.1).

Let $\Psi : \mathbb{R}_+^2 \to \mathbb{R}$ the function defined by

$$\Psi(s_1, s_2) := -b_1 g(s_2)^2 + b_2 g(s_2)s_1 - b_3 s_1^2,$$

where b_j, for $j = 1, 2, 3$, are the positive constants appearing in (5.17) and g is the function defined in (3.6). The following result holds.

Lemma 5.6. *For any $\eta > 0$ there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ it holds that*

$$\partial_{s_j} J_\lambda(V_\lambda + \bar{\phi}) = \epsilon e^{-\frac{2}{\epsilon}}\partial_{s_j}\Psi(s_1, s_2) + o\left(\epsilon e^{-\frac{2}{\epsilon}}\right)$$

C^0-uniformly with respect to s_j satisfying (3.8).

The proof can be made as in [34] with some changes and so we omit it.

6. Proof of the main theorems

Proof of Theorem 1.1. Let us fix a small $\eta > 0$. Recalling that $\epsilon = \lambda - \lambda_1$, by (i) of Proposition 5.4, for (s_1, s_2) satisfying (3.8) the reduced functional has the uniform expansion

$$\tilde{J}_\lambda(s_1, s_2) = \frac{1}{4}S^2 + \epsilon e^{-\frac{2}{\epsilon}}\left[\Psi(s_1, s_2)\right] + o\left(\epsilon e^{-\frac{2}{\epsilon}}\right),$$

where

$$\Psi(s_1, s_2) = -b_1 g(s_2)^2 + b_2 g(s_2) s_1 - b_3 s_1^2.$$

It is easy to see that Ψ has a non-trivial critical point in $\left(\frac{b_2}{2b_3}, 1\right)$. Moreover it is a non-degenerate maximum point if $b_2^2 - 4b_1 b_3 < 0$. Hence, since the maximum points are stable under small perturbation, we get that the functional $\tilde{J}_\lambda(s_1, s_2)$ has a critical point in some $(\bar{s}_{1\lambda}, \bar{s}_{2\lambda})$ such that

$$(\bar{s}_{1\lambda}, \bar{s}_{2\lambda}) \to \left(\frac{b_2}{2b_3}, 1\right)$$

as $\lambda \to \lambda_1^+$. If instead $b_2^2 - 4b_1 b_3 = 0$, the point is a degenerate critical point but it is stable according to Definition 3.3 since it is a maximum for Ψ. Indeed

$$\Psi(s_1, s_2) - \Psi\left(\frac{b_2}{2b_3}, 1\right) < 0 \qquad \forall\ (s_1, s_2) \in \mathcal{U},$$

where $\mathcal{U}$ is a neighborhood of the point $\left(\frac{b_2}{2b_3}, 1\right)$, and we get the same conclusion by using also Lemma 5.6.

Furthermore, if $b_2^2 - 4b_1 b_3 > 0$ then $\left(\frac{b_2}{2b_3}, 1\right)$ is a non degenerate critical point but we have a direction in which it is a maximum and a direction in which it is a minimum. However by Lemma 5.6 we get the same conclusion.

In the end the result follows from (i) of Lemma 5.1. □

Proof of Theorem 1.2. Let us set $G_1(d_1) := a_1 d_1^2 - a_2 d_1^{10/3}$, where a_1, a_2 are the positive constants appearing in Proposition 5.4 statement (ii). It is elementary to see that the function $G_1 : \mathbb{R}^+ \to \mathbb{R}$ has a strictly local maximum point at $\bar{d}_1 = \left(\frac{3}{5}\frac{a_1}{a_2}\right)^{\frac{3}{4}}$.

Since $\bar{d}_1$ is a strictly local maximum for G_1, then, for any sufficiently small $\gamma > 0$ there exists an open interval I_{1,σ_1} such that $\overline{I}_{1,\sigma_1} \subset \mathbb{R}^+$, with diameter σ_1, such that $\bar{d}_1 \in I_{1,\sigma_1}$ and for all $d_1 \in \partial I_{1,\sigma_1}$

$$G_1(d_1) \le G_1(\bar{d}_1) - \gamma. \tag{6.1}$$

Clearly as $\gamma \to 0$ we can choose σ_1 so that $\sigma_1 \to 0$.

We set $G_2 : \mathbb{R}^2_+ \to \mathbb{R}$ by $G_2(d_1, d_2) := a_3 d_1 d_2{}^{\frac{3}{2}} - a_4 d_2^2$, where a_3, a_4 are the positive constant appearing in Proposition 5.4, statement (ii). If we fix $d_1 = \bar{d}_1$ then $\hat{G}_2(d_2) := G(\bar{d}_1, d_2)$ has a strictly local maximum point at $\bar{d}_2 := \left(\frac{3}{4}\frac{a_3}{a_4}\bar{d}_1\right)^2$. As in the previous case there exists an open interval I_{2,σ_2} such that $\overline{I}_{2,\sigma_2} \subset \mathbb{R}^+$, with diameter σ_2, such that $\bar{d}_2 \in I_{1,\sigma_1}$ and for all $d_2 \in \partial I_{2,\sigma_2}$

$$\hat{G}_2(d_2) \le \hat{G}_2(\bar{d}_2) - \gamma. \tag{6.2}$$

As $\gamma \to 0$ we can choose σ_2 so that $\sigma_2 \to 0$.

Let us set $K := \overline{I_{1,\sigma_1} \times I_{2,\sigma_2}}$ and let $\eta > 0$ be small enough so that $K \subset]\eta, \frac{1}{\eta}[\times]\eta, \frac{1}{\eta}[$. Thanks to Propositions 4.8 and 4.9, for all sufficiently small ϵ, $\tilde{J}_\lambda : \mathbb{R}^2_+ \to \mathbb{R}$ is defined and it is of class C^1, where we recall that $\epsilon = \lambda_1 - \lambda$. By Weierstrass theorem we know there exists a global maximum point for $\tilde{J}_\lambda$ in K. Let $(d_{1,\lambda}, d_{2,\lambda})$ be that point, it remains to show that there exists ϵ_1 such that, for all $\epsilon < \epsilon_1$, $(d_{1,\lambda}, d_{2,\lambda})$ lies in the interior of K. This can be done as in the proof of [33, Theorem 1.1] and so we skip this part. At the end by (i) of Lemma 5.3 we obtain a solution u_λ of problem 1.1.

It remains to prove that the solution obtained is sign-changing. Let us set $\Phi = \Phi_\lambda := \bar{\phi}_1 + \bar{\phi}_2$. Since $u_\lambda = V_\lambda + \Phi$ is a solution of (1.1) then, by elementary computations, taking into account that by definition $-\Delta V_\lambda = \mathcal{U}_\delta^p - \lambda_1 \tau e_1$ (see (3.13)), we see that Φ solves

$$\begin{cases} -\Delta\Phi = \lambda\Phi + \lambda\mathcal{P}\mathcal{U}_\delta + \epsilon\tau e_1 - \mathcal{U}_\delta^p + f(u_\lambda) & \text{in } \Omega \\ \Phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.3}$$

Since Φ solves (6.3), then, arguing as in the proof in [33, Lemma 3.9] (see also the proofs of Lemma 5.2, Proposition 6.1 in the present paper), we have that $|\Phi|_{\infty,\Omega} = o(\delta^{-\frac{N-2}{2}}) = o(\epsilon^{-9/4})$, [1] for all sufficiently small $\epsilon > 0$. Hence, evaluating u_λ at the origin, we have

$$u_\lambda(0) = c(N)\delta^{-\frac{N-2}{2}} - \tau e_1(0) + o\left(\delta^{-\frac{N-2}{2}}\right) = c(N) d_{2,\lambda}^{-3/2}\epsilon^{-9/4} + o\left(\epsilon^{-9/4}\right) > 0$$

for all sufficiently small $\epsilon > 0$. On the other hand, thanks to Proposition 6.1, if we fix a small ball B_ρ centered at the origin, then, in $\Omega \setminus B_\rho$, we have

$$u_\lambda = O(\delta^{\frac{N-2}{2}}) - \tau e_1 + o(\tau) = -d_{1,\lambda}\epsilon^{3/4} e_1 + o(\epsilon^{3/4}) < 0,$$

for all sufficiently small $\epsilon > 0$. Hence u_λ is sign-changing and the proof is complete. □

Proposition 6.1. *Let Φ_λ be the remainder term appearing in Theorem* 1.2. *Then, for any compact subset K of $\overline{\Omega} \setminus \{0\}$ we have*

$$|\Phi_\lambda|_{\infty,K} = o\left((\lambda_1 - \lambda)^{3/4}\right),$$

as $\lambda \to \lambda_1^-$.

Proof. Let us set $\epsilon := \lambda_1 - \lambda$, and let $\Phi = \Phi_\epsilon := \bar{\phi}_1 + \bar{\phi}_2$ be the remainder term obtained in the proof of Theorem 1.2. We want to show that $|\Phi|_{\infty,K} = o(\epsilon^{3/4})$, as $\epsilon \to 0$. To this end, let us fix a positive number ρ such that $B_\rho = B_\rho(0) \subset\subset \Omega$.

[1] Thanks to the definition of δ and τ (see (3.7)) and since $d_1 = d_{1,\lambda} \to \bar{d}_1 > 0$ and $d_2 = d_{2,\lambda} \to \bar{d}_2 > 0$, as $\epsilon \to 0$, we have $\delta = O(\epsilon^{3/2})$, $\tau = O(\epsilon^{3/4})$, as $\epsilon \to 0$.

As observed in the proof of Theorem 1.2 since $u_\lambda = V_\lambda + \Phi$ is a solution of (1.1), then, Φ solves (6.3). We also point out that Φ is a smooth function since it is the difference between the two smooth functions u_λ and V_λ. Let us set $\Psi = \Psi_\epsilon := \frac{\Phi}{\tau^{1+\gamma}}$, where γ is a small positive number and τ is defined in (3.7) (see also the footnote 1). We want to prove that $|\Psi|_{\infty,\Omega\setminus B_\rho} = O(1)$, for all sufficiently small $\epsilon > 0$. By elementary computations we get that Ψ solves

$$\begin{cases} -\Delta\Psi = \lambda\Psi + \lambda\dfrac{\mathcal{PU}_\delta}{\tau^{1+\gamma}} + \dfrac{\epsilon}{\tau^{1+\gamma}}e_1 - \dfrac{\mathcal{U}_\delta^p}{\tau^{1+\gamma}} + \tau^{p-1-\gamma} f\left(\dfrac{u_\lambda}{\tau}\right) & \text{in } \Omega\setminus B_\rho \\ \Psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.4}$$

We observe that in $\Omega \setminus B_\rho$ it holds $|\mathcal{PU}_\delta|_{\infty,\Omega\setminus B_\rho} \le c(N,\rho)\delta^{\frac{N-2}{2}}$, and hence, taking into account the choice of τ and δ we get that $\frac{|\mathcal{PU}_\delta|_{\infty,\Omega\setminus B_\rho}}{\tau^{1+\gamma}} = o(1)$, as $\epsilon \to 0$. By analogous computations we get that $\frac{|\mathcal{U}_\delta^p|_{\infty,\Omega\setminus B_\rho}}{\tau^{1+\gamma}} = o(1)$ and clearly it also holds

$$\frac{\epsilon}{\tau^{1+\gamma}}\|e_1\|_{\infty,\Omega\setminus B_\rho} \le \frac{\epsilon}{\tau^{1+\gamma}}\|e_1\|_{\infty,\Omega} = o(1), \quad \text{as} \quad \epsilon \to 0.$$

Let us set $M_\epsilon := |\Psi|_{\infty,\Omega\setminus B_\rho}$ and let $a_\epsilon \in \Omega \setminus B_\rho$ such that $|\Psi(a_\epsilon)| = |\Psi|_{\infty,\Omega\setminus B_\rho}$. Assume by contradiction that there exists a subsequence $\epsilon_k \to 0$ (and consequently a sequence of points $a_{\epsilon_k} \in \Omega \setminus B_\rho$) such that

$$M_{\epsilon_k} = |\Psi_{\epsilon_k}|_{\infty,\Omega\setminus B_\rho} = |\Psi_{\epsilon_k}(a_{\epsilon_k})| \to +\infty, \quad \text{as} \quad k \to +\infty.$$

In order to simplify the notation we shall omit the index k and use the notation ϵ to denote that subsequence. We consider the rescaled function

$$\widetilde{\Psi}(y) := \frac{1}{M_\epsilon}\Psi\left(a_\epsilon + \frac{y}{M_\epsilon^\beta}\right) \quad \text{with} \quad \beta = \frac{2}{N-2},$$

defined for $y \in \widetilde{\mathcal{A}}_\epsilon := M_\epsilon^{\frac{2}{N-2}}[(\Omega\setminus B_\rho) - a_\epsilon]$. Let us also set $\widetilde{\Omega}_\epsilon := M_\epsilon^{\frac{2}{N-2}}(\Omega - a_\epsilon)$ By elementary computations we see that $\widetilde{\Psi}$ solves

$$\begin{cases} -\Delta\widetilde{\Psi} = \lambda\dfrac{\widetilde{\Psi}}{M_\epsilon^{2\beta}} + \lambda\dfrac{\mathcal{PU}_\delta\left(a_\epsilon + \frac{y}{M_\epsilon^\beta}\right)}{\tau^{1+\gamma}M_\epsilon^{2\beta+1}} + \dfrac{\epsilon}{\tau^{1+\gamma}M_\epsilon^{2\beta+1}}e_1\left(a_\epsilon + \dfrac{y}{M_\epsilon^\beta}\right) & \\ \qquad\qquad - \dfrac{\mathcal{U}_\delta^p\left(a_\epsilon + \frac{y}{M_\epsilon^\beta}\right)}{\tau^{1+\gamma}M_\epsilon^{2\beta+1}} + \tau^{p-1-\gamma} f\left(\dfrac{u_\lambda\left(a_\epsilon + \frac{y}{M_\epsilon^\beta}\right)}{\tau M_\epsilon}\right) & \text{in } \widetilde{\mathcal{A}}_\epsilon \\ \widetilde{\Psi} = 0 & \text{on } \partial\widetilde{\Omega}_\epsilon, \end{cases} \tag{6.5}$$

As observed before, since we are assuming that $M_\epsilon \to +\infty$, we have

$$\frac{\left|\mathcal{P}\mathcal{U}_\delta\left(a_\epsilon + \frac{y}{M_\epsilon^\beta}\right)\right|_{\infty,\widetilde{\mathcal{A}}_\epsilon}}{\tau^{1+\gamma} M_\epsilon^{2\beta+1}} = o(1)$$

$$\frac{|\mathcal{U}_\delta^p\left(a_\epsilon + \frac{y}{M_\epsilon^\beta}\right)|_{\infty,\widetilde{\mathcal{A}}_\epsilon}}{\tau^{1+\gamma} M_\epsilon^{2\beta+1}} = o(1)$$

$$\frac{\epsilon}{\tau^{1+\gamma} M_\epsilon^{2\beta+1}} e_1\left(a_\epsilon + \frac{y}{M_\epsilon^\beta}\right) = o(1)$$

as $\epsilon \to 0$. In particular, since $\widetilde{\Psi}$ is uniformly bounded we get that $\left|\lambda \frac{\widetilde{\Psi}}{M_\epsilon^{2\beta}}\right|_{\infty,\widetilde{\mathcal{A}}_\epsilon} = o(1)$, and

$$\begin{aligned}&\tau^{p-1-\gamma}\left|f\left(\frac{u_\lambda\left(a_\epsilon + \frac{y}{M_\epsilon^\beta}\right)}{\tau M_\epsilon}\right)\right|_{\infty,\widetilde{\mathcal{A}}_\epsilon}\\ &= \tau^{p-1-\gamma}\left|f\left(\frac{\mathcal{P}\mathcal{U}_\delta\left(a_\epsilon + \frac{y}{M_\epsilon^\beta}\right)}{\tau M_\epsilon} - \frac{\tau e_1\left(a_\epsilon + \frac{y}{M_\epsilon^\beta}\right)}{\tau M_\epsilon} + \tau^\gamma \widetilde{\Psi}\right)\right|_{\infty,\widetilde{\mathcal{A}}_\epsilon} = o(1),\end{aligned}$$

as $\epsilon \to 0$. Now, up to a subequence, by standard elliptic theory $\widetilde{\Psi}$ converges in $C^2_{\mathrm{loc}}(\Pi)$ to some function $\hat{\Psi}$ which satisfies $-\Delta\hat{\Psi} = 0$ in Π, where Π is the limit domain of $\widetilde{\mathcal{A}}_\epsilon$. There are only three possibilities:

(i) $\Pi = \mathbb{R}^N$;
(ii) Π is an half-space and 0 lies in the interior of Π;
(iii) Π is an half-space and $0 \in \partial\Pi$.

We will show that (i), (ii) and (iii) bring to a contradiction.

Assume (i) or (ii). By construction we have that $\|\Psi\|_\Omega \to 0$ as $\epsilon \to 0$, and hence, since $|\widetilde{\Psi}|_{2^*,\widetilde{\mathcal{A}}_\epsilon} = |\Psi|_{2^*,\Omega\setminus B_\rho} \leq |\Psi|_{2^*,\Omega} \leq c\|\Psi\|_\Omega \to 0$, as $\epsilon \to 0$, by Fatou's Lemma we deduce that

$$|\hat{\Psi}|_{2^*,\Pi} \leq \liminf_{\epsilon\to 0} |\widetilde{\Psi}|_{2^*,\widetilde{\mathcal{A}}_\epsilon} = 0.$$

Since $\hat{\Psi}$ is smooth, we deduce that $\hat{\Psi} \equiv 0$, but, since we are assuming (i) or (ii) then 0 lies in the interior of Π, and by definition $\widetilde{\Psi}(0) = 1$ (or $\widetilde{\Psi}(0) = -1$), and hence $\hat{\Psi}(0) = 1$ (or $\hat{\Psi}(0) = -1$), and we get a contradiction.

Assume (iii). Then $\partial\Pi$ is an hyperplane and $0 \in \partial\Pi$. We consider a closed ball $\overline{B}$ such that $\overline{B} \subset \overline{\Pi}$ and ∂B is tangent at Π in 0. Since the limit domain of $\widetilde{\mathcal{A}}_\epsilon$

is Π and thanks to the choice of $\overline{B}$ we get that $\widetilde{\mathcal{A}}_\epsilon \cap \overline{B} = \overline{B}$ for all sufficiently small $\epsilon > 0$. Since $\widetilde{\Psi}$ is smooth and uniformly bounded and thanks to the estimates made before, we deduce that the right-hand side of the equation in (6.5) is smooth (it is sufficient it is of class $C^{0,\alpha}$) and uniformly bounded. Hence, by standard elliptic theory (see [27, Theorem 6.6 and Lemma 6.36]), we get that, up to a subsequence, the restriction of $\widetilde{\Psi}$ to $\overline{B}$ converges in $C^2(\overline{B})$ to a function $\hat{\Psi}$. As before we have that $\hat{\Psi} \equiv 0$ in B, but, since we have the convergence in $C^2(\overline{B})$, we also have $\hat{\Psi}(0) = 1$ (or $\hat{\Psi}(0) = -1$) which contradicts the smoothness of $\hat{\Psi}$. Hence, we have that M_ϵ is uniformly bounded and hence $|\Phi|_{\infty,\Omega\setminus B_\rho} = o(\tau) = o(\epsilon^{3/4})$, as $\epsilon \to 0$. The proof is complete. □

Remark 6.2. We point out that, even for $N = 4$, we can prove that for any compact subset K of $\overline{\Omega} \setminus \{0\}$, the remainder term Φ_λ (appearing in Theorem 1.1) verifies $|\Phi_\lambda|_{\infty,K} = o(e^{-\frac{1}{\lambda-\lambda_1}})$, as $\lambda \to \lambda_1^+$. The key ingredient of the proof is that the remainder term verifies $\|\Phi_\lambda\| = O(\epsilon e^{-\frac{1}{\epsilon}})$, as $\epsilon \to 0$ (see Proposition 4.3), and hence, considering, $\Psi := \frac{\Phi_\lambda}{\epsilon^\alpha e^{-\frac{1}{\epsilon}}}$, where α is any fixed number in (0, 1), then, it still holds $\|\Psi\| \to 0$. Hence, arguing as in the previous proof, we get the same conclusion.

Remark 6.3. We believe that in the case $N = 6$ the limit profile of a sign-changing solution of the problem (2.2) is given by

$$u_\lambda(x) = \mathcal{P}\mathcal{U}_\delta - v_\lambda(x) + \Phi_\lambda$$

as $\lambda \to \bar{\lambda} \in (0, \lambda_1)$, where v_λ is a positive solution of (2.2) whose existence is garanteed by [14] and Φ_λ is a remainder term such that $\|\Phi_\lambda\| \to 0$ as $\lambda \to \bar{\lambda}$. Moreover we have that

$$\bar{\lambda} = 2v_{\bar{\lambda}}(0)$$

and

$$\lambda \to \bar{\lambda}^+.$$

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