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Analytic adjoint ideal sheaves associated to plurisubharmonic functions

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Abstract. In this article, we will present that the analytic adjoint ideal sheaves associated to plurisubharmonic functions are not coherent, in general.

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1. Introduction

The adjoint ideal sheaf on a smooth complex algebraic variety X is a variant of the multiplier ideal sheaf in algebraic geometry (see [1,3,4] for more details).

In [1], Guenancia gave an analytic definition of an adjoint ideal sheaf associated to a quasi-plurisubharmonic function φ along a simple normal crossing (SNC) divisor $D = \sum D_i$ and established the compatibility with the algebraic adjoint ideal whenever φ has analytic singularities.

Let X be a complex manifold, $D = \sum D_i$ an SNC divisor and φ a quasi-plurisubharmonic function on X. Let $Adj^{\alpha}_{D,*}(\varphi) \subset \mathcal{O}_X$ be the ideal sheaf of germs of holomorphic functions $f \in \mathcal{O}_{X,x}$ such that

$$|f|^2 \prod_{k=1}^{p} \frac{1}{|h_k|^2(-\log|h_k|)^{\alpha}} e^{-\varphi}$$

is integrable with respect to the Lebesgue measure in some local coordinates near x, where $h = h_1 \cdots h_p$ is the minimal defining function of D near x and $\alpha > 1$.

In [1] (see also [2]), Guenancia gave the following analytic definition of adjoint ideal sheaf, which generalized the algebraic adjoint ideal sheaf ([1, Proposition 2.11]; see also [2, Proposition 5.1]).

Definition 1.1 ([1,2]). The ideal sheaf $Adj^{\alpha}_{D}(\varphi) := \cup_{\varepsilon>0} Adj^{\alpha}_{D,*}((1+\varepsilon)\varphi)$ is called the *analytic adjoint ideal sheaf* associated to φ along D.

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Note that in [1] Guenancia used $\alpha = 2$ in the definition, and later Kim in [2] extended the definition to $\alpha > 1$ case. When e^{φ} is locally Hölder continuous, Guenancia established the coherence of Adj_D^{α} for smooth divisor D with $\varphi|_D \not\equiv -\infty$ (see [1, Corollary 2.19]).

As mentioned by Guenancia and Kim, it is natural to ask the following

Question 1.2 ([1,2]). For $\alpha > 1$ and φ a general quasi-plurisubharmonic function on X, is the analytic adjoint ideal sheaf $Adj_D^{\alpha}(\varphi)$ coherent?

In this article, we will present the following negative answer to Question 1.2.

Theorem 1.3. *There exists a plurisubharmonic function φ on a neighborhood U of the origin $o \in \mathbb{C}^n$ $(n \geq 3)$ and a smooth divisor D with $\varphi|_D \not\equiv -\infty$ such that for any $\alpha > 1$, the analytic adjoint ideal sheaf $Adj_D^{\alpha}(\varphi)$ is not coherent.*

Specifically, we will construct φ and D with $\varphi|_D \not\equiv -\infty$ near o such that the zero set of $Adj_D^{\alpha}(\varphi)$ is not an analytic set near o.

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2. Proof of main result

We are now in a position to prove Theorem 1.3.

Proof of Theorem 1.3. Let $D = \{z_1 = 0\}$ and

$$\varphi(z) = \max\left\{\sum_{k=2}^{\infty} \alpha_k \log\left(|z_1| + \left|z_2 - \frac{1}{k}\right|^{\beta_k}\right), \lambda \log|z_3|\right\},$$

where $\alpha_k = \frac{1}{2^{k!}}$, $\beta_k = 3 \cdot 2^{k!}$ and $\lambda > 6$. Then D is smooth and $\varphi(z)$ is a plurisubharmonic function near $o \in \mathbb{C}^n$ $(n \geq 3)$. Without loss of generality, we assume $n = 3$ and U contained in the unit polydisk Δ^3 is a neighborhood of o such that $\log(|z_1| + |z_2 - \frac{1}{k}|^{\beta_k}) < 0$ for any $k \geq 2$ on U.

Step 1. Is showing nonintegrability of $\frac{1}{|z_1|^2(-\log|z_1|)^{\alpha}} e^{-\varphi}$ near $(0, \frac{1}{k}, 0)$ for any $(0, \frac{1}{k}, 0) \in U$.

Since

$$\begin{aligned}\varphi(z) &= \max\left\{\sum_{k=2}^{\infty}\alpha_k \log\left(|z_1| + \left|z_2 - \frac{1}{k}\right|^{\beta_k}\right), \lambda \log|z_3|\right\}\\ &\le \max\left\{\alpha_k \log\left(|z_1| + \left|z_2 - \frac{1}{k}\right|^{\beta_k}\right), \lambda \log|z_3|\right\}\\ &\le \log\left(\left(|z_1| + \left|z_2 - \frac{1}{k}\right|^{\beta_k}\right)^{\alpha_k} + |z_3|^{\lambda}\right)\end{aligned}$$

near o, replacing $z_2 - \frac{1}{k}$ by z_2 near $(0, \frac{1}{k}, 0)$, for sufficiently small polydisk Δ_r^3 we have

$$\int_{\Delta_r^3} \frac{1}{|z_1|^2(-\log|z_1|)^{\alpha}} e^{-\varphi(z_1, z_2+\frac{1}{k}, z_3)} dV_3$$

$$\ge \int_{\Delta_r^3} \frac{dV_3}{\big((|z_1| + |z_2|^{\beta_k})^{\alpha_k} + |z_3|^{\lambda}\big)|z_1|^2(-\log|z_1|)^{\alpha}}$$

$$= \int_{\Delta_r^*} \frac{dV_1}{|z_1|^2(-\log|z_1|)^{\alpha}}$$

$$\times \int_{\Delta_r^2} \frac{|z_1|^{\frac{2}{\beta_k}+\frac{2\alpha_k}{\lambda}}}{|z_1|^{\alpha_k}} \frac{\left(\frac{\sqrt{-1}}{2}\right)^2 d\left(\frac{z_2}{|z_1|^{\frac{1}{\beta_k}}}\right) \wedge d\left(\frac{\bar{z}_2}{|z_1|^{\frac{1}{\beta_k}}}\right) \wedge d\left(\frac{z_3}{|z_1|^{\frac{\alpha_k}{\lambda}}}\right) \wedge d\left(\frac{\bar{z}_3}{|z_1|^{\frac{\alpha_k}{\lambda}}}\right)}{\left(1 + \frac{|z_2|^{\beta_k}}{|z_1|}\right)^{\alpha_k} + \frac{|z_3|^{\lambda}}{|z_1|^{\alpha_k}}}$$

$$\ge \int_{\Delta_r^*} \frac{dV_1}{|z_1|^{2+(\alpha_k - \frac{2}{\beta_k} - \frac{2\alpha_k}{\lambda})}(-\log|z_1|)^{\alpha}} \int_{\Delta} \frac{\left(\frac{\sqrt{-1}}{2}\right)^2 dw_2 \wedge d\bar{w}_2 \wedge dw_3 \wedge d\bar{w}_3}{(1 + |w_2|^{\beta_k})^{\alpha_k} + |w_3|^{\lambda}}$$

$$\ge C \cdot \int_{\Delta_r^*} \frac{dV_1}{|z_1|^{2+(\alpha_k - \frac{2}{\beta_k} - \frac{2\alpha_k}{\lambda})}(-\log|z_1|)^{\alpha}} = +\infty,$$

where $C > 0$ is some constant and the last equality is from $\alpha_k - \frac{2}{\beta_k} - \frac{2\alpha_k}{\lambda} > 0$. It follows that $\frac{1}{|z_1|^2(-\log|z_1|)^{\alpha}} e^{-\varphi}$ is not locally integrable near $(0, \frac{1}{k}, 0)$ for any $(0, \frac{1}{k}, 0) \in U$.

Step 2. The second step is the integrability of $\frac{1}{|z_1|^2(-\log|z_1|)^{\alpha}} e^{-(1+\varepsilon)\varphi}$ near $(0, z_2, 0) \in U$ with $z_2 \neq \frac{1}{k}, 0$.

Take $0 < \varepsilon_0 < 1$ such that $\alpha - \varepsilon_0 > 1$ and $|z_2 - \frac{1}{k}| > \varepsilon_0$ for any k. Then, for any $N \geq 1$ with $\varepsilon_0^{\beta_{N+1}} < |z_1| \leq \varepsilon_0^{\beta_N}$, we obtain

$$\begin{aligned}
\varphi(z) &\geq \sum_{k=2}^{\infty} \alpha_k \log\left(|z_1| + \left|z_2 - \frac{1}{k}\right|^{\beta_k}\right) \geq \sum_{k=2}^{\infty} \alpha_k \log\left(|z_1| + \varepsilon_0^{\beta_k}\right) \\
&\geq \sum_{k\leq N} \alpha_k \log\left(|z_1| + \varepsilon_0^{\beta_k}\right) + \sum_{k\geq N+1} \alpha_k \log\left(|z_1| + \varepsilon_0^{\beta_k}\right) \\
&\geq \sum_{k\leq N} \alpha_k\beta_k \log\varepsilon_0 + \sum_{k\geq N+1} \alpha_k \log|z_1| \geq 3N\log\varepsilon_0 + \sum_{k\geq N+1} \alpha_k\beta_{N+1}\log\varepsilon_0 \\
&\geq 3N\log\varepsilon_0 + 2\alpha_{N+1}\beta_{N+1}\log\varepsilon_0 = 3(N+2)\log\varepsilon_0
\end{aligned}$$

and

$$\log(-\log|z_1|) \geq \log(-\beta_N\log\varepsilon_0) = N!\log 2 + \log 3 + \log(-\log\varepsilon_0). \qquad (2.1)$$

Set $C_N := N!\log 2 + \log 3 + \log(-\log\varepsilon_0)$. Since

$$0 < \frac{3(N+2)\log\varepsilon_0}{-C_N} \to 0 \ (N \to \infty),$$

it follows from (2.1) that for large N, we have

$$\frac{3(N+2)\log\varepsilon_0}{-\log(-\log|z_1|)} \leq \frac{3(N+2)\log\varepsilon_0}{-C_N} \leq \frac{\varepsilon_0}{1+\varepsilon}.$$

Hence, for $\varepsilon_0^{\beta_{N+1}} < |z_1| \leq \varepsilon_0^{\beta_N}$, we obtain

$$(1+\varepsilon)\varphi \geq 3(1+\varepsilon)(N+2)\log\varepsilon_0 \geq -\varepsilon_0\log(-\log|z_1|)$$

for large enough N. Thus, when $|z_1|$ is small enough, we have

$$\frac{1}{|z_1|^2(-\log|z_1|)^\alpha} e^{-(1+\varepsilon)\varphi} \leq \frac{1}{|z_1|^2(-\log|z_1|)^{\alpha-\varepsilon_0}},$$

which is locally integrable near $(0, z_2, 0) \in U$ with $z_2 \neq \frac{1}{k}, 0$ by $\alpha - \varepsilon_0 > 1$.

Step 3. The third and last step at the proof is to show the incoherence of the analytic adjoint ideal sheaf $Adj_D^\alpha(\varphi)$ near o.

Suppose that $Adj_D^\alpha(\varphi)$ is a coherent ideal sheaf on U. Then, the zero set

$$N\left(Adj_D^\alpha(\varphi)\right) := \left\{x \in U \middle|\ Adj_D^\alpha(\varphi)_x \neq \mathcal{O}_x\right\}$$

of $Adj_D^\alpha(\varphi)$ is an analytic set in U. However, it follows from Step 1 and Step 2 that on U,

$$\begin{aligned} N\left(Adj_D^\alpha(\varphi)\right) &= \left\{x \in U \middle| \, Adj_D^\alpha(\varphi)_x \neq \mathcal{O}_x\right\} \\ &= \left\{x \in U \middle| \, \frac{1}{|z_1|^2(-\log|z_1|)^\alpha} e^{-(1+\varepsilon)\varphi} \text{ is not integrable near } x\right\} \\ &= \left\{\left(0, \frac{1}{k}, 0\right)\right\} \cup \{o\}, \end{aligned}$$

which is not analytic at o, contradicting to the assumption. □

Remark 2.1. In Theorem 1.3, if the condition $\varphi|_D \not\equiv -\infty$ is not required, then the result also holds for $n = 2$ by a slight modification. Indeed, we can take $D = \{z_1 = 0\}$, and $\varphi(z_1, z_2) = \sum_{k=2}^{\infty} \alpha_k \log(|z_1| + |z_2 - \frac{1}{k}|^{\beta_k})$ with $\alpha_k = \frac{1}{2^{k!}}$, and $\beta_k = 3 \cdot 2^{k!}$ and U contained in the unit polydisk Δ^2 to be a neighborhood of o such that $\log(|z_1| + |z_2 - \frac{1}{k}|^{\beta_k}) < 0$ for any $k \geq 2$ on U. Then, by a similar calculation to the proof of Theorem 1.3, we can establish the nonintegrability of $\frac{1}{|z_1|^2(-\log|z_1|)^\alpha} e^{-\varphi}$ near $(0, \frac{1}{k})$ for any $(0, \frac{1}{k}) \in U$ and the integrability of $\frac{1}{|z_1|^2(-\log|z_1|)^\alpha} e^{-(1+\varepsilon)\varphi}$ near $(0, z_2) \in U$ with $z_2 \neq \frac{1}{k}, 0$. Thus, it follows from the third step of the proof that the analytic adjoint ideal sheaf $Adj_D^\alpha(\varphi)$ is not coherent near o.

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