Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XVIII (2018), 363-390

Symmetric tensors: rank, Strassen's conjecture and *e*-computability

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Al nostro grande amico Tony

Abstract. In this paper we introduce a new method to produce lower bounds for the Waring rank of symmetric tensors. We also introduce the notion of e-computability and we use it to prove that Strassen's conjecture holds in infinitely many new cases.

Mathematics Subject Classification (2010): 14Q20 (primary); 14M10, 15A21 (secondary).

1. Introduction

Let k be a field of characteristic zero and let $F \in k[x_0, x_1, \dots, x_n] = S = \oplus S_i$ ($i \geq 0$ and $n \geq 1$) be a homogeneous polynomial (*form*) of degree d, *i.e.*, $F \in S_d$. It is well known that in this case each S_i has a basis consisting of i^{th} powers of linear forms. Thus we may write

$$F = \sum_{i=1}^{r} \alpha_i L_i^d \qquad \alpha_i \in k,\, L_i \in S_1.$$

If k is algebraically closed (which we now assume for the rest of the paper) then each $\alpha_i = \beta_i^d$ for some $\beta_i \in k$ and so we can write

$$F = \sum_{i=1}^{r} (\beta_i L_i)^d = \sum_{i=1}^{r} \tilde{L}_i^d. \tag{1.1}$$

We call a description of F as in (1.1), a *Waring decomposition of* F. The least integer r such that F has a Waring decomposition with exactly r summands is called the *Waring Rank* (or simply the *rank*) of F.

These authors enjoyed support from NSERC (Canada). The first, second, and third authors were also supported by GNSAGA of INDAM and by MIUR (Italy).

Received February 12, 2016; accepted in revised form November 7, 2016.
Published online March 2018.

There are several variants on this notion in the literature (see, *e.g.*, [1,7,10]). In this paper we will only be interested in the notion of rank described above.

It is easy to see that F has rank one if and only if $[F] \in \mathbb{P}(S_d)$ is a point of the *Veronese variety*, $\mathbb{V} \subset \mathbb{P}(S_d)$. If F has rank r then $[F] \in \mathbb{P}(S_d)$ is on $\sigma_r(\mathbb{V})$, the $(r-1)^{st}$ *secant variety* of $\mathbb{V}$.

Given a Waring decomposition of F

$$F = L_1^d + \ldots + L_\ell^d \text{ with } L_i = a_{i0}x_0 + \ldots + a_{in}x_n,$$

we can associate a set of ℓ points in $\mathbb{P}^n$ to this decomposition, namely

$$\mathbb{X} = \big\{[a_{10} : \ldots : a_{1n}], \ldots, [a_{\ell 0} : \ldots : a_{\ell n}]\big\}.$$

The importance of this set will be explained a bit further on.

Let $T = k[X_0, \ldots, X_n] = \oplus T_i$ $(i \geq 0)$ be another polynomial ring and let T act on S by setting

$$X_i \circ F = (\partial/\partial x_i)(F)$$

and extending linearly (see [5] or [6]). With respect to this action we write

$$F^\perp = \{g \in T \mid g \circ F = 0\}.$$

If F is a form of degree d, then every form in T of degree $\geq d+1$ is in $F^\perp$ and so $F^\perp$ is an Artinian ideal of T. It is a classical theorem of Macaulay that $T/F^\perp$ is also a Gorenstein ring with socle in degree d. Moreover, every Gorenstein Artinian quotient of T with socle in degree d is of the form $T/F^\perp$, with F a form of degree d.

Suppose that $F = L^d$ where $L = a_0x_0 + \ldots + a_nx_n$ and $g \in T_\delta$. Then

$$g \circ L^d = (d!/\delta!)g(a_0, \ldots, a_n)L^{d-\delta}.$$

It follows that if $F \in S_d$ has a Waring decomposition

$$F = L_1^d + \ldots + L_\ell^d \text{ where } L_i \leftrightarrow p_i \in \mathbb{P}^n \text{ and } \mathbb{Y} = \{p_1, \ldots p_\ell\}$$

then for all $g \in T$ such that $g(p_i) = 0, i = 1, \ldots, \ell, g \in F^\perp$, that is

$$I_\mathbb{Y} \subset F^\perp$$

where $I_\mathbb{Y} \subset T$ is the ideal of the set $\mathbb{Y}$.

The opposite implication is also true, namely if $I_\mathbb{Y} \subset F^\perp$, with $\mathbb{Y}$ a finite set of ℓ points in $\mathbb{P}^n$, then $F = L_1^d + \ldots + L_\ell^d$, where the L_i correspond to the points in $\mathbb{Y}$, as described above.

These containments are referred to as the *apolarity lemma* and one can find proofs in [6,9].

Having a particular Waring decomposition of F, or equivalently the ideal of a set of distinct points in $F^\perp$, will thus give us upper bounds for the rank of F. We also need some good lower bounds for the rank of F. The importance of finding such lower bounds was underscored in the papers of [8] and in further work [12]. In [8], generalizing a result of Sylvester, a lower bound was found in terms of ranks of catalecticant matrices and dimensions of the singularity loci in the spaces defined by varieties coming from catalecticant ideals. In [10] the authors found a new lower bound on the rank and using it they computed $\mathrm{rk}((x_0 \cdot \ldots \cdot x_n)^a)$, that is the rank of a monomial with all the exponents equal to each other. In [3] a generalized bound produced the rank of all monomials. Theorem 3.3 uses different invariants of F to further generalizes the ranks of [10] and of [3].

Our new approach to the study of the rank is particularly effective in the direction of Strassen's conjecture. This famous conjecture was stated in the 1973 paper [11] and is still open (for some recent progress see [2]). The symmetric version of Strassen's conjecture can be stated as follows: the rank is additive on the sum of forms in different sets of variables, that is

$$\mathrm{rk}(F_1 + \ldots + F_m) = \mathrm{rk}(F_1) + \ldots + \mathrm{rk}(F_m)$$

if the forms F_i are in distinct sets of variables. In [3] it was proved that the conjecture holds if the forms F_i are monomials. In Theorem 6.1 we find several other families of summands for which Strassens's conjecture is true.

The paper is organized in the following way. In Section 2 we recall some of the basic ideas we will use. In Section 3 we introduce the notion of e-computability and use it to establish our new lower bound for the rank of F. In Section 4 we find several infinite families of forms which are e-computable and thus compute their rank. In Sections 5 and 6 we show how useful the notion of e-computability is in dealing with Strassens's conjecture by giving many new examples of families of forms for which Strassens's conjecture is true. In Section 7 we give an example of an infinite family of forms whose rank is computable by *ad hoc* methods. We show that the first member of this family is not 1-computable.

ACKNOWLEDGEMENTS. The first, second, and third authors wish to thank Queens University, in the person of the fourth author, for their kind hospitality during the preparation of this work.

2. Basic facts

Let

$$S = k[x_0, \ldots, x_n] \qquad \text{and} \qquad T = k[X_0, \ldots, X_n],$$

where k is an algebraically closed field of characteristic zero. We let T act via differentiation on S as above.

Given a homogeneous ideal $I \subseteq T$ we denote by

$$HF(T/I, i) = \dim_k T_i - \dim_k I_i$$

the *Hilbert function* of T/I in degree i. It is well known that the function $HF(T/I, i)$ is eventually a polynomial function with rational coefficients, and this polynomial is called the *Hilbert polynomial* of T/I. We say that an ideal $I \subseteq T$ is *one dimensional* if the Krull dimension of T/I is one, equivalently the Hilbert polynomial of T/I is some integer constant, say ℓ. In the case that $I \subseteq T$ is one dimensional, then this eventually constant value of the Hilbert function of T/I is called the *multiplicity* of T/I. If, in addition, I is a radical ideal, then I is the ideal of a set of ℓ distinct points in $\mathbb{P}^n$. We will use the fact that if I is a saturated ideal and T/I is one dimensional of multiplicity ℓ, then $HF(T/I, i)$ is always at most ℓ.

Our main tool is the apolarity lemma, whose proof can be found in [6, Lemma 1.31].

Lemma 2.1 (Apolarity lemma). *Let* $\mathbb{X} = \{[L_1], \ldots, [L_\ell]\} \subset \mathbb{P}(S_1)$ *be a set of* ℓ *distinct points, corresponding to the linear forms* $L_1, \ldots, L_\ell \in S_1$. *If* $F \in S_d$, *then*

$$F = c_1 L_1^d + \ldots + c_\ell L_\ell^d,$$

for $c_1, \ldots, c_\ell \in k$, *if and only if*

$$I_{\mathbb{X}} \subset F^{\perp}.$$

Note that the coefficients c_i are necessary even if k is algebraically closed since some of them could be zero; this is not a minimal decomposition. With the apolarity lemma in mind, we make the following definition.

Definition 2.2.

a) If F is a form in S and $\mathbb{X} \subset \mathbb{P}^n$ is a set of reduced points for which $I_{\mathbb{X}} \subset F^{\perp}$, then we say that $\mathbb{X}$ is *apolar* to F;
b) If $\mathbb{X}$ is apolar to F and $|\mathbb{X}| \leq |\mathbb{Y}|$ for any other $\mathbb{Y}$ apolar to F, then we say that $\mathbb{X}$ *minimally decomposes* F.

We conclude with the following trivial, but useful, remark (see [3, Remark 2.3]).

Remark 2.3. The computation of the rank of F is independent of the polynomial ring in which we consider F.

More precisely, consider a rank r form $F \in k[x_0, \ldots, x_n]$. Then F has rank r also if we consider F as a form in $k[x_0, \ldots, x_n, x_{n+1}, \ldots, x_{n+t}]$.

3. Lower bound for rank

It is useful to recall the following well known results.

Remark 3.1. Let $J \subset T$ be the ideal of a zero-dimensional scheme and $t \in T_e$ a homogeneous differentiation of degree e. If t is not a zero divisor in T/J, then from the exact sequence

$$0 \longrightarrow (T/J)_{i-e} \xrightarrow{\cdot t} (T/J)_i \longrightarrow (T/(J + (t)))_i \longrightarrow 0, \tag{3.1}$$

we get, for $s \gg 0$,

$$e \cdot HF(T/J, s) = \sum_{i=0}^{s} HF(T/(J + (t)), i). \tag{3.2}$$

Lemma 3.2. *Let $F(x_0, \dots, x_n) \in S_d$, then*

$$F^{\perp} : X_i = (X_i \circ F)^{\perp}.$$

Proof. Let $g \in T$ and suppose that we have $g \in F^{\perp} : X_i$. Now

$$g \in F^{\perp} : X_i \iff (gX_i) \circ F = 0 \iff g \circ (X_i \circ F) = 0 \iff g \in (X_i \circ F)^{\perp},$$

and the conclusion follows. $\square$

We are now ready to state and prove our first theorem.

Theorem 3.3. *Let $F \in S_d$ and let $\mathbb{X} \subset \mathbb{P}(S_1)$ be apolar to F (so $I_{\mathbb{X}} \subset F^{\perp}$). Let $I \subset T$ be any ideal generated in degree $e > 0$ and let $t \in I_e$. If t is not a zero divisor in $T/(I_{\mathbb{X}} : I)$, then for $s \gg 0$ we have*

$$e \cdot |\mathbb{X}| \geq \sum_{i=0}^{s} HF(T/(I_{\mathbb{X}} : I + (t)), i) \geq \sum_{i=0}^{s} HF\left(T/\left(F^{\perp} : I + (t)\right), i\right).$$

Proof. Note that $I_{\mathbb{X}} : I$ is the saturated ideal of $\mathbb{Y} \subseteq \mathbb{X}$ consisting of all points of $\mathbb{X}$ not lying on the zero set of I. Thus, by Remark 3.1, we have

$$\frac{1}{e} \cdot \sum_{i=0}^{s} HF(T/(I_{\mathbb{X}} : I + (t)), i) = |\mathbb{Y}|$$

for $s \gg 0$. Moreover for any s,

$$\sum_{i=0}^{s} HF(T/(I_{\mathbb{X}} : I + (t)), i) \geq \sum_{i=0}^{s} HF(T/(F^{\perp} : I + (t)), i),$$

since $I_{\mathbb{X}}$ is contained in $F^{\perp}$, and so we are done. $\square$

The following corollary gives a useful lower bound for the rank of F.

Corollary 3.4. *Let $F \in S_d$. Let $I \subset T$ be any ideal generated in degree $e > 0$ and let t be a general form in I_e. For $s \gg 0$ we have*

$$\operatorname{rk}(F) \geq \left(\frac{1}{e}\right) \sum_{i=0}^{s} HF\left(T/\left(F^{\perp} : I + (t)\right), i\right).$$

Proof. Let $\mathbb{X}$ minimally decompose F, so $|\mathbb{X}| = \mathrm{rk}(F)$. If $I \subset I_{\mathbb{X}}$ the statement is trivially true. So assume $I \not\subset I_{\mathbb{X}}$. Since $t \in I_e$ is a general form, then t is not a zero divisor in $T/I_{\mathbb{X}} : I$. So by Theorem 3.3 we are done. □

Notice that the summation on the right side cannot decrease as s increases and, indeed, the summands are all zero for s big enough. Hence we often use the corollary above with $s = \infty$.

Definition 3.5. Let $F \in S_d$ and $e > 0$ be an integer. We say that F is *e-computable* if there exists an ideal $I \subset T$ generated in degree e such that for general $t \in I_e$ we have

$$\mathrm{rk}(F) = \left(\frac{1}{e}\right) \sum_{i=0}^{\infty} HF\left(T/\left(F^{\perp} : I + (t)\right), i\right).$$

In this case we say that the *rank of F is computed by I and t*. In case $I = (t)$, we simply say that the *rank of F is computed by t*.

Proposition 3.6. *Let $F \in S_d$ and assume that* $\mathrm{rk}(F)$ *is computed by I and t. If $\mathbb{X}$ minimally decomposes F and if we let $I_{\mathbb{X}'} = I_{\mathbb{X}} : I$, then $\mathbb{X} = \mathbb{X}'$ and $I_{\mathbb{X}} + (t) = F^{\perp} + (t)$.*

Proof. Since $\mathrm{rk}(F) > 0$, then $I_{\mathbb{X}} : I \neq T$ and, since t is general, we may assume that t is a non-zero divisor in $T/I_{\mathbb{X}} : I$. By (3.2) we get

$$|\mathbb{X}'| = \left(\frac{1}{e}\right) \sum_{i=0}^{\infty} HF(T/(I_{\mathbb{X}} : I + (t)), i).$$

Hence we have

$$\begin{aligned} \mathrm{rk}(F) = |\mathbb{X}| \geq |\mathbb{X}'| &= \left(\frac{1}{e}\right) \sum_{i=0}^{\infty} HF(T/(I_{\mathbb{X}} : I + (t)), i) \\ &\geq \left(\frac{1}{e}\right) \sum_{i=0}^{\infty} HF\left(T/\left(F^{\perp} : I + (t)\right), i\right) = \mathrm{rk}(F). \end{aligned}$$

It follows that $\mathbb{X} = \mathbb{X}'$ and $I_{\mathbb{X}} : I + (t) = F^{\perp} : I + (t)$. Hence

$$F^{\perp} + (t) \subseteq F^{\perp} : I + (t) = I_{\mathbb{X}} : I + (t) = I_{\mathbb{X}'} + (t) \subset I_{\mathbb{X}} + (t) \subseteq F^{\perp} + (t),$$

and the conclusion follows. □

4. Forms which are *e*-computable

In this section we give several examples of forms which are e-computable for various values of e.

We start by considering forms in two variables, that is $F \in S = k[x_0, x_1]$, and we recall Sylvester's algorithm to compute the rank of F, see [4]. Since $F^\perp$ is a Gorenstein Artinian ideal and $F^\perp \subset T = k[X_0, X_1]$, we have that

$$F^\perp = (h_1, h_2)$$

where $\deg h_1 = d_1 \leq \deg h_2 = d_2$ and $d_1 + d_2 = \deg F + 2$ with h_1 and h_2 having no common factor. If h_1 is square free then $\mathrm{rk}(F) = d_1$, otherwise $\mathrm{rk}(F) = d_2$.

Proposition 4.1. *If $F \in S = k[x_0, x_1]$ and $F^\perp = (h_1, h_2)$ as above, then*

(i) *if h_1 is not square free and $h_1 = t^2\widetilde{h}_1$, then F is e-computable, where $e = \deg t$;*
(ii) *if h_1 is square free and $d_1 < d_2$, then F is e-computable for any $e \leq \frac{d_2-d_1+1}{2}$;*
(iii) *if $d_1 = d_2$ we can assume we are in case* (i).

Proof.

(i) h_1 is not square free, so $\mathrm{rk}(F) = d_2$;
Since in this case, $h_1 = t^2\widetilde{h}_1$, it is easy to see that $F^\perp : (t) = (t\widetilde{h}_1, h_2)$. It follows that $F^\perp : (t) + (t) = (t, h_2)$. Noting that (t, h_2) is a complete intersection of degree $e \cdot d_2$, we have $\sum_{i=0}^{\infty} HF(T/(F^\perp : (t) + (t)), i) = e \cdot d_2 = e \cdot \mathrm{rk}(F)$, and this completes the proof of (i);
(ii) h_1 is square free and $d_1 < d_2$, so $\mathrm{rk}(F) = d_1$.
Let t be a form of degree $e \leq \frac{d_2-d_1+1}{2}$ such that $t|h_2$. We claim that

$$F^\perp : (t) + (t) = (t, h_1).$$

It is easy to show that $F^\perp : (t) = (h_1, h_2/t)$, hence $F^\perp : (t) + (t) = (t, h_1, h_2/t)$. But (t, h_1) contains all forms of degree at least $e + d_1 - 1$, and $\deg h_2/t = d_2 - e \geq e + d_1 - 1$. Thus $(t, h_1, h_2/t) = (t, h_1)$, and we have proved the claim. Hence,

$$\sum_{i=0}^{\infty} HF(T/(F^\perp : (t) + (t)), i) = e \cdot d_1 = e \cdot \mathrm{rk}(F);$$

(iii) If $d_1 = d_2$ then, using the discriminant of a general combination of h_1 and h_2, we can assume that h_1 is not square free. □

We now consider monomials in $S = k[x_0, \dots, x_n]$. It is shown in [3] that any monomial is 1-computable. In the next proposition we generalize this fact.

Proposition 4.2. *Let $F = x_0^{a_0} x_1^{a_1} \cdots x_n^{a_n}$ where $0 < a_0 \leq a_1 \leq \ldots \leq a_n$. Then F is e-computable for*

$$1 \leq e \leq \frac{a_0 + 1}{2}.$$

Proof. We know that $\mathrm{rk}(F) = \Pi_{i=1}^{n}(a_i + 1)$ (see [3]). Now

$$\begin{aligned} F^{\perp} : \left(X_0^e\right) + \left(X_0^e\right) &= \left(x_0^{a_0-e} x_1^{a_1} \cdots x_n^{a_n}\right)^{\perp} + \left(X_0^e\right) \\ &= \left(X_0^{a_0-e+1}, X_1^{a_1+1}, \dots, X_n^{a_n+1}, X_0^e\right) \\ &= \left(X_1^{a_1+1}, \dots, X_n^{a_n+1}, X_0^e\right). \end{aligned}$$

Hence

$$\sum_{i=0}^{\infty} HF\left(T/\left(F^{\perp} : \left(X_0^e\right) + \left(X_0^e\right)\right), i\right) = e \cdot \Pi_{i=1}^{n}(a_i + 1) = e \cdot \mathrm{rk}(F). \qquad \square$$

Remark 4.3. It would be interesting to know if the forms of Propositions 4.1 and 4.2 are e-computable for e's different from those described in the two propositions.

In the following propositions we exhibit several other families of e-computable forms.

Consider

$$F = x_0^a \left(x_1^b + \ldots + x_n^b\right).$$

Since, both for $n = 1$ and, by a change of coordinates, for $b = 1$, F is a monomial, we skip those known cases (see [3]).

Proposition 4.4. *Let $b \geq 2$, $n \geq 2$ and let*

$$F = x_0^a \left(x_1^b + \ldots + x_n^b\right) \in S = k[x_0, \dots, x_n].$$

If $a+1 \geq b$, then F is 1-computable, the rank of F is computed by $I = (X_1, \dots, X_n)$ and a general linear form $t \in I$, and we have

$$\mathrm{rk}(F) = (a + 1)n.$$

Proof. Consider the ideal $I = (X_1, \dots, X_n) \subset T$. We first calculate $F^{\perp} : I$.

$$F^{\perp} : I = \left(F^{\perp} : (X_1, \dots, X_n)\right) = \left(F^{\perp} : (X_1)\right) \cap \cdots \cap \left(F^{\perp} : (X_n)\right).$$

Thus, by Lemma 3.2,

$$\begin{aligned} F^{\perp} : I &= \left(x_0^a x_1^{b-1}\right)^{\perp} \cap \cdots \cap \left(x_0^a x_n^{b-1}\right)^{\perp} \\ &= \left(X_0^{a+1}, X_1^b, X_2, \dots, X_n\right) \cap \cdots \cap \left(X_0^{a+1}, X_1, \dots, X_{n-1}, X_n^b\right) \\ &= \left(X_0^{a+1}, X_1^b, \dots, X_n^b, X_1X_2, \dots, X_{n-1}X_n\right). \end{aligned}$$

Now consider $\widetilde{I} = F^{\perp} : I + (t)$, where $t = \alpha_1 X_1 + \ldots + \alpha_n X_n \in I_1$ is a general form. We have

$$\begin{aligned}\widetilde{I} &= F^{\perp} : I + \big(\alpha_1 X_1 + \ldots + \alpha_n X_n\big) \\ &= \Big(X_0^{a+1}, X_1^2, \ldots, X_n^2, X_1X_2, \ldots, X_{n-1}X_n, \alpha_1 X_1 + \ldots + \alpha_n X_n\Big).\end{aligned}$$

We want to apply Corollary 3.4, so we compute $\sum_{i=0}^{s} HF(T/\widetilde{I}, i)$ for s large enough.

For $a+1 = 2$ and $b = 2$, we have $F = x_0(x_1^2 + \ldots + x_n^2)$ and

$$\widetilde{I} = \Big(X_0^2, X_1^2, \ldots, X_n^2, X_1X_2, X_1X_3, \ldots, X_{n-1}X_n, \alpha_1 X_1 + \ldots + \alpha_n X_n\Big).$$

So we can easily see that the following table holds true:

i	0	1	2	3
$HF(T/\widetilde{I}, i)$	1	n	$n-1$	0.

From this we get $\sum_{i=0}^{s} HF(T/\widetilde{I}, i) = 2n$.

For $a+1 > 2$ we have

$$\widetilde{I} = \Big(X_0^{a+1}, X_1^2, \ldots, X_n^2, X_1X_2, X_1X_3 \ldots, X_{n-1}X_n, \alpha_1 X_1 + \ldots + \alpha_n X_n\Big).$$

A simple computation shows that:

i	0	1	2...	a	$a+1$	$a+2$
$HF(T/\widetilde{I}, i)$	1	n	$n \ldots$	n	$n-1$	0.

From this we get $\sum_{i=0}^{s} HF(T/\widetilde{I}, i) = (a+1)n$.

Hence, we get $\operatorname{rk}(F) \geq (a+1)n$ in both cases using Corollary 3.4.

Now consider $F^{\perp}$. Since

$$F^{\perp} \supseteq \Big(X_0^{a+1}, X_1^b - X_2^b, \ldots, X_1^b - X_n^b, X_1X_2, X_1X_3, \ldots, X_{n-1}X_n\Big),$$

then the ideal

$$\Big(X_0^{a+1} + \Big(X_1^{a+1-b} + \ldots + X_n^{a+1-b}\Big)\Big((n-1)X_1^b - \ldots - X_n^b\Big), \\ X_1X_2, X_1X_3, \ldots, X_{n-1}X_n\Big)$$

is contained in $F^{\perp}$. This last is the ideal of $(a+1)n$ distinct points lying on the n lines whose defining ideal is $(X_1X_2, X_1X_3, \ldots, X_{n-1}X_n)$.

By the apolarity lemma, it follows that $\operatorname{rk}(F) \leq (a+1)n$, and we are done. $\square$

Remark 4.5. For some special F in Proposition 4.4 the rank of F can be computed by t, instead of by I and t. For instance, if $F = x(y^2 + z^2)$ we have $\mathrm{rk}(F) = 4$. Note that in the proof of Proposition 4.4 we showed that the rank was computed by $I = (Y, Z)$ and $t = \alpha_1 Y + \alpha_2 Z$. However, the rank is also computed by $t = X$, in other words:

$$\sum_{i=0}^{\infty} HF\left(T/\left(F^{\perp} : (X) + (X)\right), i\right) = 4.$$

We do not know if the rank of F can always be computed by t. For instance, if $F = x^2(y^2+z^2+w^2)$ we have $\mathrm{rk}(F) = 9$ (see Proposition 4.9 below). In the proof of Proposition 4.4 we showed that the rank was computed by $I = (Y, Z, W)$ and $t = \alpha_1 Y + \alpha_2 Z + \alpha_3 W$. Note that

$$\sum_{i=0}^{\infty} HF\left(T/\left(F^{\perp} : (Y + Z + W) + (Y + Z + W)\right), i\right) = 3,$$

and that

$$\sum_{i=0}^{\infty} HF\left(T/\left(F^{\perp} : (X) + (X)\right), i\right) = 5,$$

that is, neither $t = Y + Z + W$, nor $t = X$ compute the rank. We do not know if there is a t which computes the rank of this F.

Remark 4.6. Let $M_i = x_0^a x_i^b$, so the polynomial F of the previous proposition, becomes

$$F = x_0^a \left(x_1^b + \ldots + x_n^b\right) = M_1 + \ldots + M_n.$$

In case $a + 1 = b$ we have (see [3] for the rank of the M_i)

$$\mathrm{rk}(F) = (a + 1)n < \mathrm{rk}(M_1) + \cdots + \mathrm{rk}(M_n) = (a + 2)n.$$

Thus, an analogue of Strassen's conjecture is certainly not true if a form is the sum of forms which have a common factor. On the other hand, when $a+1 > b$, we have

$$(a + 1)n = \mathrm{rk}(F) \leq \mathrm{rk}(M_1) + \ldots + \mathrm{rk}(M_n) = (a + 1)n.$$

Thus, in some cases, the rank is additive over summands, even when the summands have a common factor.

Proposition 4.7. *Let $b \geq 2$, $a \geq 1$, and let*

$$F = x_0^a \left(x_1^b + x_2^b\right).$$

(i) *If $a + 1 \geq b$, then the rank of F is computed by $I = (X_1, X_2)$ and t and* $\mathrm{rk}(F) = 2(a + 1)$*;*
(ii) *If $a + 1 \leq b$, then the rank of F is computed by $t = X_0$ and* $\mathrm{rk}(F) = 2b$*.*

Proof.

(i) Follows from Proposition 4.4.
(ii) In this case let $I = (X_0) \subset T$. Obviously t is a general form in I_1. Hence we consider the ideal $\widetilde{I} = F^{\perp} : (X_0) + (X_0)$, and we have

$$\begin{aligned}\widetilde{I} &= (X_0 \circ F)^{\perp} + (X_0) = \left(x_0^{a-1}\left(x_1^b + x_2^b\right)\right)^{\perp} + (X_0) \\ &= \left(X_0, X_1X_2, X_1^b - X_2^b\right).\end{aligned}$$

Since

i	0	1	2	$\dots$	$b-1$	b
$HF(T/\widetilde{I}, i)$	1	2	2	$\dots$	2	1

we have $\sum_{i=0}^{b} HF(T/\widetilde{I}, i) = 2b$. Hence from Corollary 3.4, we get $\operatorname{rk}(F) \geq 2b$. Since

$$\left(X_1X_2, X_0^b + X_1^b - X_2^b\right)$$

is the ideal of $2b$ points apolar to F, by the apolarity lemma we are done. $\square$

Remark 4.8. Note that for $a + 1 \leq b$ and $F = x_0^a(x_1^b + x_2^b)$ we have

$$\operatorname{rk}(F) = 2b < \operatorname{rk}\left(x_0^a x_1^b\right) + \operatorname{rk}\left(x_0^a x_2^b\right) = 2b + 2.$$

Now we study the rank of the forms $G = F + x_0^{a+b}$, where F is as in Propositions 4.4 and 4.7, that is,

$$G = x_0^a\left(x_1^b + \ldots + x_n^b\right) + x_0^{a+b}.$$

We will show that F and G have the same rank.

Proposition 4.9. *Let $b \geq 2$, $n \geq 2$ and let*

$$G = x_0^a\left(x_1^b + \ldots + x_n^b\right) + x_0^{a+b} = x_0^a\left(x_0^b + x_1^b + \ldots + x_n^b\right) \in S.$$

If $a + 1 \geq b$, then the rank of G is computed by $I = (X_1, \ldots, X_n)$ and t and

$$\operatorname{rk}(G) = (a+1)n.$$

Proof. As in the proof of Proposition 4.4, we consider the ideal $I=(X_1,\dots,X_n)\subset T$ and the linear general form $t=\alpha_1X_1+\dots+\alpha_nX_n$. Let $\widetilde{I}=G^{\perp}:I+(t)$. We have

$$\begin{aligned}\widetilde{I}&=G^{\perp}:(X_1,\dots,X_n)+(\alpha_1X_1+\dots+\alpha_nX_n)\\&=\left(G^{\perp}:(X_1)\right)\cap\dots\cap\left(G^{\perp}:(X_n)\right)+(\alpha_1X_1+\dots+\alpha_nX_n).\end{aligned}$$

Hence, by Lemma 3.2,

$$\widetilde{I}=\left(x_0^ax_1^{b-1}\right)^{\perp}\cap\dots\cap\left(x_0^ax_n^{b-1}\right)^{\perp}+(\alpha_1X_1+\dots+\alpha_nX_n).$$

Note that this is exactly the ideal $\widetilde{I}$ that we constructed in the proof of Proposition 4.4, thus we may proceed in the same way and we get $\mathrm{rk}(G)\geq(a+1)n$.

Now consider $G^{\perp}$. It is easy to show that $G^{\perp}$ contains the ideal

$$\left(nX_0^{a+1}-\binom{a+b}{b}\left(X_1^b+\dots+X_n^b\right)X_0^{a+1-b},X_1^{b+1},\dots,X_n^{b+1},\right.$$
$$\left.X_1X_2,X_1X_3,\dots,X_{n-1}X_n\right).$$

If $a+1=b$, then the ideal

$$\left(nX_0^{a+1}-\binom{a+b}{b}\left(X_1^b+\dots+X_n^b\right)X_0^{a+1-b},X_1X_2,X_1X_3,\dots,X_{n-1}X_n\right),$$

is contained in $G^{\perp}$ and defines $(a+1)n$ points apolar to G lying on the n lines whose defining ideal is $(X_1X_2,X_1X_3,\dots,X_{n-1}X_n)$. Hence, we conclude using the apolarity lemma.

If $a+1>b$, then consider the ideal

$$\mathcal{A}=\left(\alpha\left(nX_0^{a+1}-\binom{a+b}{b}X_0^{a+1-b}\left(X_1^b+\dots+X_n^b\right)\right)+\beta X_1^{a+1}+\dots+\beta X_n^{a+1},\right.$$
$$\left.X_1X_2,X_1X_3,\dots,X_{n-1}X_n\right),$$

where $\alpha,\beta\in k$. It is easy to see that $\mathcal{A}$ is contained in $G^{\perp}$. Moreover, for generic values of α and β, $\mathcal{A}$ is the ideal of $(a+1)n$ distinct points lying on the n lines whose defining ideal is $(X_1X_2,X_1X_3,\dots,X_{n-1}X_n)$. In fact, consider the line whose ideal is $(X_2,\dots,X_n)$ (and analogously for the other $n-1$ lines). We have

$$\mathcal{A}+(X_2,\dots,X_n)=\left(\alpha\left(nX_0^{a+1}-\binom{a+b}{b}X_0^{a+1-b}X_1^b\right)+\beta X_1^{a+1},X_2,\dots,X_n\right),$$

hence, in order to find the $a+1$ points, we have to solve the equation

$$\alpha\left(nX_0^{a+1}-\binom{a+b}{b}X_0^{a+1-b}X_1^b\right)+\beta X_1^{a+1}=0,$$

or, in other words, we have to consider the linear series cut out on $\mathbb{P}^1$ by the linear system

$$\Sigma=\left\langle nX_0^{a+1}-\binom{a+b}{b}X_0^{a+1-b}X_1^b,\ X_1^{a+1}\right\rangle,$$

whose general element is reduced by Bertini's theorem.

Thus, using the apolarity lemma, it follows that $\operatorname{rk}(G)\geq(a+1)n$, and we are done. □

Remark 4.10. The lower bound in [8, Proposition 4.7] can only prove the case $a=1$ and $b=2$ of our Proposition 4.9.

Proposition 4.11. *Let $b\geq 2$ and*

$$G=x_0^a\left(x_1^b+x_2^b\right)+x_0^{a+b}=x_0^a\left(x_0^b+x_1^b+x_2^b\right)\in S.$$

(i) *If $a+1\geq b$, then the rank of G is computed by $I=(X_1,X_2)$ and a general $t\in I_1$, and* $\operatorname{rk}(G)=2(a+1)$*;*
(ii) *If $a+1\leq b$, then the rank of G is computed by $t=(X_0)$ and* $\operatorname{rk}(G)=2b$.

Proof.

(i) This is a particular case of Proposition 4.9;
(ii) As in Proposition 4.7, let $I=(X_0)$ and $t=X_0$. Consider the ideal $\widetilde{I}=G^\perp:(X_0)+(X_0)$. We have

$$\begin{aligned}\widetilde{I}&=(X_0\circ G)^\perp+(X_0)=\left(x_0^{a-1}\left(x_1^b+x_2^b\right)\right)^\perp+(X_0)\\&=\left(X_0,X_1X_2,X_1^b-X_2^b\right),\end{aligned}$$

which is the same ideal we found in the proof of Proposition 4.7. So $\operatorname{rk}(G)\geq 2b$ follows in the same way.
Now notice that

$$\left(2X_0^b-\binom{a+b}{b}\left(X_1^b+X_2^b\right),X_1X_2\right)$$

is the ideal of $2b$ points which are apolar to G. Thus, by the apolarity lemma, $\operatorname{rk}(G)\leq 2b$, and we are done. □

Remark 4.12. With a bit more effort one can show the following:

a) In Propositions 4.4, 4.7 (i), 4.9 and 4.11 (i) the forms are e-computable if $2e \leq b$. The rank is computed by $I = (X_1^e, \dots, X_n^e)$ and a general form $t \in I_e$;
b) In Propositions 4.7 (ii) and 4.11 (ii) the forms are e-computable if $2e \leq a+1$ and the rank of F is computed by $I = (X_0^e)$ and $t = X_0^e$.

Now we study forms $F \in S = k[x_0, \dots, x_n]$ for which

$$F^{\perp} = \big(q^a, g_1, \dots, g_n\big) \subset T$$

is a complete intersection such that

$$a \geq 2 \quad \text{and} \quad ae \leq d_1 \leq \dots \leq d_n,$$

where $e = \deg q,\ d_1 = \deg g_1, \dots, d_n = \deg g_n$.

We need the following lemma:

Lemma 4.13. *Let $J = (q^a, g_1, \dots, g_n)$ be a complete intersection as above. Then there exist $f_1, \dots, f_n$ such that*

$$J = (q^a, f_1, \dots, f_n),$$

where $\deg f_i = \deg g_i$ and, for all j, $1 \leq j \leq n$ the ideal $(f_j, f_{j+1}, \dots, f_n)$ defines a smooth complete intersection in $\mathbb{P}^n$ of codimension $n - j + 1$ and having degree $\Pi_{i=j}^{d} d_i$.

Proof. Consider the linear system of forms of degree d_n in J. This system has no base points and so by Bertini's theorem, the general element is smooth. Since the general element is a linear combination of g_n and other forms of degree d_n in J, there is no loss of generality in choosing a generator for J of the type $f_n = g_n +$ (other forms of degree d_n). We call this new generator f_n. Now consider the linear system of codimension two varieties cut out on $V(f_n)$ by all the other hypersurfaces in J of degree d_{n-1}. This linear system is clearly base point free in $V(f_n)$ and so the general element of this system cuts out a smooth variety on $V(f_n)$ of codimension 2 in $\mathbb{P}^n$. We can then replace g_{n-1} by a general element of this system. Continuing in this same way we arrive at hypersurfaces $f_1, \dots, f_n$ where $\deg f_i = \deg g_i$ and $(f_1, \dots, f_n)$ describes a set of $\Pi_{i=1}^{n} d_i$ points. □

We have the following result.

Theorem 4.14. *Let $F \in S$ be a homogeneous polynomial. If*

$$F^{\perp} = (q^a, g_1, \dots, g_n)$$

is a complete intersection such that

$$a \geq 2 \text{ and both } e = \deg q > 0 \text{ and } ae \leq d_1 = \deg g_1 \leq \dots \leq d_n = \deg g_n,$$

then F is e-computable, the rank of F is computed by q and we have

$$\operatorname{rk}(F) = \Pi_{i=1}^{n} d_i = (1/e) \sum_{i=0}^{\infty} HF\left(T/\left(F^{\perp} : (q) + (q)\right), i\right).$$

Proof. Using Lemma 4.13 we know that $\operatorname{rk}(F) \leq \Pi_{i=1}^{n} d_i$.

Since $\{q^a, g_1, \ldots, g_n\}$ is a regular sequence, $F^{\perp} : (q) = (q^{a-1}, g_1, \ldots, g_n)$. Hence

$$F^{\perp} : (q) + (q) = (q, g_1, \ldots, g_n).$$

So by Corollary 3.4 we have

$$\operatorname{rk}(F) \geq \left(\frac{1}{e}\right) \sum_{i=0}^{\infty} HF(T/(q, g_1, \ldots, g_n), i) = \Pi_{i=1}^{n} d_i,$$

and the conclusion follows. □

We now give an example of a form which is 2-computable but not 1-computable.

Example 4.15. If

$$\begin{aligned} F = {} & x^{11} - 22x^9y^2 + 33x^7y^4 - 22x^9z^2 + 396x^7y^2z^2 - 462x^5y^4z^2 \\ & + 33x^7z^4 - 462x^5y^2z^4 + 385x^3y^4z^4, \end{aligned}$$

then F is 2-computable and $\operatorname{rk}(F) = 25$. In fact, using the software CoCoA[1] we get

$$F^{\perp} = \left(\left(X^2 + Y^2 + Z^2\right)^2, G_1, G_2\right),$$

where $G_1 = Y^5 + Z(X^2 + Y^2 + Z^2)^2$ and $G_2 = Z^5 + X(X^2 + Y^2 + Z^2)^2$.

Hence

$$\begin{aligned} \operatorname{rk}(F) & \geq (1/2) \sum_{i=0}^{\infty} HF\left(T/\left(F^{\perp} : \left(X^2 + Y^2 + Z^2\right) + \left(X^2 + Y^2 + Z^2\right)\right), i\right) \\ & = 25, \end{aligned}$$

and the ideal $(G_1, G_2) \subset F^{\perp}$ is the ideal of 25 distinct points.

We will see, in Example 4.23, that this form is not 1-computable.

Proposition 4.16. *Let $F = x_0^a G \in S$ for some a and some form $G \in k[x_1, \ldots, x_n]$. The following hold:*

[1] COCOA TEAM, *A system for doing Computations*, In: "Commutative Algebra", available at http://cocoa.dima.unige.it, 2004.

(i) $F^\perp = (X_0^{a+1}, G^\perp)$, *where* $G^\perp$ *is considered in* $k[X_1, \dots, X_n]$*;*
(ii) *If* $G^\perp$ *is a complete intersection and all generators of* $G^\perp$ *have degree at least* $a+1$*, then* F *is* 1*-computable.*

Proof. First of all, let $g \in F^\perp$. We can write $g = h_0 + X_0 h_1 + \cdots + X_0^a h_a + X_0^{a+1}\widetilde{g}$ where $h_0, \dots, h_a \in k[X_1, \dots, X_n]$ and $\widetilde{g} \in k[X_0, \dots X_n]$. By assumption,

$$\begin{aligned} 0 &= g \cdot F \\ &= \left(h_0 + X_0 h_1 + \cdots + X_0^a h_a + X_0^{a+1}\widetilde{g}\right) \cdot x_0^a G(x_1, \dots, x_n) \\ &= x_0^a (h_0 \cdot G) + a x_0^{a-1}(h_1 \cdot G) + \cdots + (a!)(h_a \cdot G). \end{aligned}$$

Since $h_0 \cdot G, h_1 \cdot G, \dots, h_a \cdot G \in \mathbb{C}[x_1, \dots, x_n]$, we have $h_0 \cdot G = h_1 \cdot G = \dots = h_a \cdot G = 0$ and hence $h_0, \dots, h_a \in G^\perp$. This proves that $F^\perp = (X_0^{a+1}, G^\perp)$.

(ii) Obvious from Theorem 4.14. □

Let $V_n = \prod_{1 \le i < j \le n}(x_i - x_j) \in k[x_1, \dots, x_n]$ be the Vandermonde determinant. Since V_n is the fundamental skew-symmetric invariant of the symmetric group, it is known that the perp ideal $V_n^\perp = (\sigma_1, \sigma_2, \dots, \sigma_n) \subset k[X_1, \dots, X_n]$ where σ_i is the i-th elementary symmetric polynomial in $X_1, \dots, X_n$ for $i = 1, \dots, n$ (see [13] and its bibliography). For later use, let σ_i' be the i-th elementary symmetric polynomial on the variables $X_2, \dots, X_n$ for $i = 1, \dots, (n-1)$. One can see that

- $\sigma_1 = X_1 + \sigma_1'$;
- $\sigma_2 = X_1 \sigma_1' + \sigma_2'$;
- $\cdots$
- $\sigma_{n-1} = X_1 \sigma_{n-2}' + \sigma_{n-1}'$;
- $\sigma_n = X_1 \sigma_{n-1}'$.

Proposition 4.17 ([13]). $\mathrm{rk}(V_n) = (n-1)!$.

Proof. We give a different proof from the one in [13] in order to illustrate the use of e-computable forms. We have $\mathrm{rk}(V_n) \ge (n-1)!$ by the Ranested-Schreyer bound (see [10]). For the upper bound, take $I = (\sigma_1, \dots, \sigma_{n-1}) \subset V_n^\perp$. By the apolarity lemma, it remains to show that I is the homogenous ideal of a set of $(n-1)!$ distinct points. To this end, we will show that on the affine piece $X_1 \neq 0$, the zero locus of the ideal I consists of exactly $(n-1)!$ distinct points. This is enough because I is a complete intersection of forms of degrees $1, 2, \dots, (n-1)$. Now letting $X_1 = 1$, we have

$$\begin{aligned} &\left\{(X_2, \dots, X_n) | \sigma_1(1, X_2, \dots, X_n) = \cdots = \sigma_{n-1}(1, X_2, \dots, X_n) = 0\right\} \\ &= \left\{(X_2, \dots, X_n) | 1 + \sigma_1'(X_2, \dots, X_n) = \cdots = \sigma_{n-2}'(X_2, \dots, X_n) + \sigma_{n-1}' = 0\right\} \\ &= \left\{(X_2, \dots, X_n) | \sigma_1' = -1, \dots, \sigma_i' = (-1)^i, \dots, \sigma_{n-1}' = (-1)^{n-1}\right\} \\ &= \big\{(X_2, \dots, X_n) | X_2, \dots, X_n \text{ are the distinct } (n-1) \text{ solutions of the equation} \\ &\qquad \mathbf{t}^{n-1} + \cdots + \mathbf{t} + 1 = 0\big\}. \end{aligned}$$

This proves that the ideal I defines a set of $(n-1)!$ distinct points. $\square$

Proposition 4.18. *The rank of the Vandermonde determinant V_n is computed by the linear form X_1.*

Proof. Due to Proposition 4.17 it will be enough to show that the length of $T/(V_n^\perp : (X_1) + (X_1))$ is $(n-1)!$. We first observe that since $\sigma_1, \dots, \sigma_n$ form a regular sequence and $\sigma_n = X_1\sigma_n'$ we have that both $\sigma_1, \dots, \sigma_{n-1}, X_1$ and $\sigma_1, \dots, \sigma_{n-1}, \sigma_n'$ form regular sequences. It is also clear that

$$V_n^\perp + (X_1) = (X_1, \sigma_1, \dots, \sigma_{n-1}, \sigma_n) = \left(X_1, \sigma_1', \dots, \sigma_{n-1}'\right).$$

Obviously $X_1, \sigma_1', \dots, \sigma_{n-1}'$ is a regular sequence and so

$$\sum_{i=0}^{\infty} HF\left(T/\left(V_n^\perp + (X_1)\right), i\right) = (n-1)!.$$

Thus from the exact sequence

$$0 \to T/\left(V_n^\perp : (X_1)\right) \to T/V_n^\perp \to T/\left(V_n^\perp + (X_1)\right) \to 0$$

we obtain

$$\sum_{i=0}^{\infty} HF\left(T/\left(V_n^\perp : (X_1)\right), i\right) = n! - (n-1)! = (n-1)! \cdot (n-1).$$

Now notice that

$$V_n^\perp : (X_1) \supseteq (\sigma_1, \dots, \sigma_{n-1}, \sigma_{n-1}').$$

But the length of $T/(V_n^\perp : (X_1))$ is $(n-1)(n-1)!$ and this is exactly the length of $T/(\sigma_1, \dots, \sigma_{n-1}, \sigma_{n-1}')$. It follows that

$$V_n^\perp : (X_1) = \left(\sigma_1, \dots, \sigma_{n-1}, \sigma_{n-1}'\right).$$

Hence $V_n^\perp : (X_1) + (X_1) = (X_1, \sigma_1, \dots, \sigma_{n-1}, \sigma_{n-1}')$ and this is easily seen to be $V_n^\perp + (X_1)$. But we have already shown that $\sum_{i=0}^{\infty} HF(T/(V_n^\perp + (X_1)), i) = (n-1)!$ and thus we are done. $\square$

Note that the Vandermonde determinant is 1-computable and in $V_n^\perp$ there is a form of degree one. A natural question arises: does there exist a change of coordinates such that, after this change, we may consider V_n in a smaller polynomial ring, in which V_n is still 1-computable and $(V_n^\perp)_1 = 0$?

In Proposition 4.21 we give a positive answer to this question, but first we observe the following:

Remark 4.19. Recall that $T = k[X_0, \dots, X_n]$ and suppose that $Y_0, \dots, Y_n$ is another basis for T_1, where

$$Y_i = \sum_{i=0}^{n} \alpha_{i,j} X_j.$$

We can write T as a polynomial ring in the new variables $Y_0, \dots, Y_n$. To avoid confusion we set $\widetilde{T} = k[Y_0, \dots, Y_n]$, even though $\widetilde{T} = T$. The change of coordinates transformation on T can be considered as

$$\psi : T \to \widetilde{T}$$

where

$$X_i = \psi_i(Y_0, \dots, Y_n).$$

It follows that, for a form $G(X_0, \dots, X_n) \in T$,

$$\psi(G) = G(\psi_0(Y_0, \dots, Y_n), \dots, \psi_n(Y_0, \dots, Y_n)) \in \widetilde{T}.$$

Now let $y_0, \dots, y_n \in S_1$ be the dual basis to $Y_0, \dots, Y_n$. As with the discussion above we can consider

$$\varphi : S = k[x_0, \dots, x_n] \to \widetilde{S} = k[y_0, \dots, y_n]$$

the isomorphism which extends the isomorphism induced by ψ from $S_1 \to \widetilde{S}_1$.

Since $X_i \circ x_j = \delta_{i,j}$ and $Y_i \circ y_j = \delta_{i,j}$, we have, for $G \in T$ and $F \in S$,

$$\varphi(G \circ F) = \psi(G) \circ \varphi(F).$$

Lemma 4.20. *Let $Y_0, \dots, Y_n$ be a basis for T_1 and let $y_0, \dots, y_n \in S_1$ be the dual basis. Let $\widetilde{T} = k[Y_0, \dots, Y_n]$, and $\widetilde{S} = k[y_0, \dots, y_n]$, and let $\psi : T \to \widetilde{T}$ and $\varphi : S \to \widetilde{S}$ be the changes of coordinates.*

If $F(x_0, \dots, x_n) \in S$ then

$$\psi\left(F^{\perp}\right) = \varphi(F)^{\perp}.$$

Proof. Let $F^{\perp} = (G_1, \dots, G_s)$, so $\psi(F^{\perp}) = (\psi(G_1), \dots, \psi(G_s))$. Since $\psi(G_i) \circ \varphi(F) = \varphi(G_i \circ F) = 0$, we get $\psi(G_i) \in \varphi(F)^{\perp}$. For the opposite inclusion, let $\widetilde{G} \in \varphi(F)^{\perp}$, and $G = \psi^{-1}(\widetilde{G})$. We have that $\psi(G) \circ \varphi(F) = 0$. But $\psi(G) \circ \varphi(F) = \varphi(G \circ F)$, hence $G \circ F = 0$, that is, $G \in F^{\perp}$, and so $\widetilde{G} \in \psi(F^{\perp})$. $\square$

Proposition 4.21. *Let $F \in S = k[x_0, \dots, x_n]$ and assume that*

$$\left(F^{\perp}\right)_1 = (Y_{n-s+1}, \dots, Y_n) \subset T_1,$$

where the Y_i are linearly independent linear forms in the X_i.

Let $Y_0, \dots, Y_{n-s}, Y_{n-s+1}, \dots, Y_n$ be a basis of T_1 and let $y_0, \dots, y_n \in S_1$ be its dual basis. There exists a change of coordinates φ such that $\varphi(F)$ involves only the variables $y_0, \dots, y_{n-s}$, and considering $\varphi(F)$ in $k[y_0, \dots, y_{n-s}]$, we have $(\varphi(F)^{\perp})_1 = 0$. Moreover, if F is 1-computable, then $\varphi(F)$ also is 1-computable.

Proof. Let φ and ψ be as in Lemma 4.20, then we get

$$\left(\psi\left(F^{\perp}\right)\right)_1 = \left(\varphi(F)^{\perp}\right)_1.$$

Since $(\psi(F^{\perp}))_1 = (Y_{n-s+1}, \dots, Y_n) \subset \widetilde{T}_1$, we have that $Y_i \circ \varphi(F) = 0$ for $n - s + 1 \le i \le n$. It follows that $\varphi(F) \in k[y_0, \dots, y_{n-s}]$. Now assume that F is 1-computable, and that the rank of F is computed by I and t, that is,

$$\operatorname{rk}(F) = \sum_{i=0}^{\infty} HF\left(T/\left(F^{\perp} : I + (t)\right)\right).$$

Since $\psi(F^{\perp} : I + (t)) = \psi(F^{\perp}) : \psi(I) + \psi(t) = \varphi(F)^{\perp} : \psi(I) + \psi(t)$ and $T/(F^{\perp} : I + (t)) \simeq \widetilde{T}/(\psi(F^{\perp} : I + (t))) = \operatorname{rk}\varphi(F)$, then $\varphi(F)$ is 1-computable, and we are done. $\square$

Remark 4.22. By a change of coordinates φ as in Proposition 4.21, we may assume that the form $\varphi(V_n)$, where V_n is the Vandermonde determinant, is 1-computable and $(\varphi(V_n)^{\perp})_1 = 0$.

We close this section by exhibiting a family of forms which are e-computable ($e > 1$) but are not 1-computable.

Example 4.23. Let T be a polynomial ring in three variables. Let $Q \in T$ be an irreducible quadratic form and let $G_1, G_2 \in T$ be two general forms of degree d, $d > 4$. By Macaulay duality, there exists a form F in the dual ring S whose apolar ideal is

$$F^{\perp} = \left(Q^2, G_1, G_2\right).$$

By Theorem 4.14 we know that F is 2-computable and $\operatorname{rk}(F) = d^2$.

We claim that F is not 1-computable.

Note that $(G_1, G_2) \subset F^{\perp}$ is the ideal of a set of d^2 distinct points, say $\mathbb{X}$. By Proposition 3.6, if F were 1-computable by I and t (t general in I), then

$$I_{\mathbb{X}} + (t) = F^{\perp} + (t).$$

Thus, we would have then $(G_1, G_2, t) = (Q^2, G_1, G_2, t)$, which is impossible since t does not divide Q. Hence F is not 1-computable.

Remark 4.24. Following example 4.23, Proposition 3.6 allows us to construct examples of forms which are e-computable but which are not 1-computable. It is enough to take a general form Q of degree e and to increase the degrees of G_1, G_2 accordingly.

Example 4.25. In Section 7 we exhibit a form F whose rank we can compute using *ad hoc* methods. We show it is not 1-computable and wonder if it is e-computable for some $e > 1$.

5. Strassen's conjecture for *e*-computable forms

Fix the following notation:

$$S = k\left[x_{1,0}, \ldots, x_{1,n_1}, \ldots, x_{m,0}, \ldots, x_{m,n_m}\right],$$
$$T = k\left[X_{1,0}, \ldots, X_{1,n_1}, \ldots, X_{m,0}, \ldots, X_{m,n_m}\right].$$

For $i = 1, \ldots, m$, we let

$$S^{[i]} = k[x_{i,0}, \ldots, x_{i,n_i}],$$
$$T^{[i]} = k[X_{i,0}, \ldots, X_{i,n_i}],$$
$$F_i \in S^{[i]}_d,$$

and

$$F = F_1 + \cdots + F_m \in S_d.$$

If we consider $F_i \in S$, then we write

$$F_i^{\perp} = \{g \in T \mid g \circ F_i = 0\}.$$

On the other hand, if we consider $F_i \in S^{[i]}$, then we also write

$$F_i^{\perp} = \left\{g \in T^{[i]} \mid g \circ F_i = 0\right\}.$$

Given this notation, it is important to know precisely in which ring we are considering F_i.

So, for instance, if we consider $F_1 \in S$ then

$$F_1^{\perp} = \left\{g \in T^{[1]} \mid g \circ F_1 = 0\right\} \cup (X_{2,0}, \ldots, X_{2,n_2}, \ldots, X_{m,0}, \ldots, X_{m,n_m}),$$

while if we consider $F_1 \in S^{[1]}$ then

$$F_1^{\perp} = \left\{g \in T^{[1]} \mid g \circ F_1 = 0\right\}.$$

Remark 5.1. We assume that each F_i essentially involves n_i variables, thus $F_i^{\perp}$ does not have linear forms involving the variables of $T^{[i]}$, and in $F^{\perp}$ there are no linear forms.

Moreover, we let $I^{[i]} \subset T^{[i]}$ be ideals with $t_i \in I^{[i]}$ $(i = 1, \cdots, m)$ all of the same degree and we set

$$J_i = \left(F_i^{\perp} : I^{[i]}\right) + (t_i) \subset T;$$

where we consider each F_i as a form in S.

Lemma 5.2. *With the notation above and $a_i \in k$ we have*

$$\left(F^\perp : \left(I^{[1]} + \cdots + I^{[m]}\right)\right) + (a_1 t_1 + \cdots + a_m t_m) \subseteq J_1 \cap \cdots \cap J_m.$$

Proof. Since $F_i \in S^{[i]}$ (although we are considering it in S) we always have that $X_{j,0}, \ldots, X_{j,n_j}$ are in $F_i^\perp$ for all $j \neq i$. Hence $t_j \in F_i^\perp$ for $j \neq i$. So $t_1, \ldots, t_m \in J_1 \cap \cdots \cap J_m$ and it is enough to prove that

$$\left(F^\perp : \left(I^{[1]} + \cdots + I^{[m]}\right)\right) \subseteq J_1 \cap \cdots \cap J_m,$$

that is,

$$\left(F^\perp : I^{[1]}\right) \cap \cdots \cap \left(F^\perp : I^{[m]}\right) \subseteq J_1 \cap \cdots \cap J_m.$$

Let $g \in F^\perp : I^{[i]}$, $(1 \leq i \leq m)$, so $gl \circ F = 0$, for any $l \in I^{[i]}$. Since for $j \neq i$, $l \circ F_j = 0$, then $gl \circ F_i = 0$, that is, $gl \in F_i^\perp$, by considering $F_i \in S$. It follows that $g \in F_i^\perp : I^{[i]} \subseteq J_i$, for $i = 1, \ldots, m$, that is, $g \in J_1 \cap \cdots \cap J_m$. □

Lemma 5.3. *Let $t_i \in I^{[i]}$ be a general form and assume that the rank of F_i is computed by $I^{[i]}$ and t_i. Set $J_i = (F_i^\perp : I^{[i]}) + (t_i) \subset T$. If $s \gg 0$, then*

(i) $$\sum_{i=0}^{s} HF(T/J_1 \cap \ldots \cap J_m, i) = \sum_{i=0}^{s} HF(T/J_1, i) + \ldots + \sum_{i=0}^{s} HF(T/J_m, i) - m + 1,$$

and

(ii) $$\sum_{i=0}^{s} HF(T/J_1 \cap \ldots \cap J_m, i) = e(\mathrm{rk}(F_1) + \cdots + \mathrm{rk}(F_m)) - m + 1.$$

Proof. To prove (i) we proceed by induction on m. If $m = 1$ the equality is obvious. Let $m > 1$ and consider the following short exact sequence:

$$\begin{aligned} 0 \longrightarrow T/(J_1 \cap \ldots \cap J_m) &\longrightarrow T/J_1 \oplus T/(J_2 \cap \ldots \cap J_m) \\ &\longrightarrow T/(J_1 + (J_2 \cap \ldots \cap J_m)) \longrightarrow 0. \end{aligned}$$

Since $J_1 + J_2 \cap \ldots \cap J_m$ is the irrelevant ideal of T we get the conclusion by the inductive hypothesis.

Part (ii) follows from (i) since $T/J_i \simeq T^{(i)}/F_i^\perp : I^{[i]} + (t_i)$, where now F_i is considered as a form in $S^{[i]}$ (so $F_i^\perp = \{g \in T^{[i]} \mid g \circ F_i = 0\}$). Hence, for $s \gg 0$, we have

$$e \cdot \mathrm{rk}(F_i) = \sum_{j=0}^{s} HF\left(T^{[i]}/\left(F_i^\perp : I^{[i]} + (t_i)\right), j\right). \qquad \square$$

Remark 5.4. Recall that in [2, Proposition 3.1], it was shown that Strassen's conjecture holds for forms of the type

$$F(x_0, \ldots, x_n) + y^d,$$

where F is a form of degree d. In other terms, adding the power of a new variable increases the rank by exactly one.

Because of this remark, in the following theorem we may assume that the polynomial rings all have at least two variables.

Theorem 5.5. *Let $F = F_1 + \cdots + F_m \in S$, where $F_i \in S^{[i]}$ with $n_i \geq 1$. If all the forms F_i are e-computable and $(F_i^{\perp})_e = 0$ then*

$$\operatorname{rk}(F) = \operatorname{rk}(F_1) + \cdots + \operatorname{rk}(F_m),$$

that is the Strassen conjecture is true for F.

Proof. Let $I^{[i]} \subset T^{[i]}$ and t_i ($\deg t_i = e$) compute the rank of F_i and let V_i be the zero set of $I^{[i]}$. It is enough to prove that

$$\operatorname{rk}(F) \geq \operatorname{rk}(F_1) + \cdots + \operatorname{rk}(F_m),$$

since the opposite inequality is obvious.

If $\mathbb{X}$ minimally decomposes F, then the ideal $I_{\mathbb{X}} : (I^{[1]} + \cdots + I^{[m]})$ is the homogeneous ideal of the subset $\mathbb{X}'$ of $\mathbb{X}$ not lying on $V_1 \cap \cdots \cap V_m$.

For a general choice of $a_i \in k$, the form $a_1 t_1 + \cdots + a_m t_m$ is a non zero divisor for $T/I_{\mathbb{X}'}$. Now consider $I_{\mathbb{X}'} + (a_1 t_1 + \cdots + a_m t_m)$. We have

$$\begin{aligned} I_{\mathbb{X}'} + (a_1 t_1 + \cdots + a_m t_m) &= \Big(I_{\mathbb{X}} : \Big(I^{[1]} + \cdots + I^{[m]}\Big)\Big) + (a_1 t_1 + \cdots + a_m t_m) \\ &\subseteq \Big(F^{\perp} : \Big(I^{[1]} + \cdots + I^{[m]}\Big)\Big) + (a_1 t_1 + \cdots + a_m t_m). \end{aligned}$$

Hence, by Lemma 5.2,

$$I_{\mathbb{X}'} + (a_1 t_1 + \cdots + a_m t_m) \subseteq J_1 \cap \cdots \cap J_m,$$

where $J_i = (F_i^{\perp} : I^{[i]}) + (t_i) \subset T$, considering $F_i \in S$.

We say that a degree e form $h \in I_{\mathbb{X}'}$ is *uniform* if

$$h = h_1 + \ldots + h_m,$$

and h_i ($i = 1, \ldots, m$) is zero or a degree e form in $T^{[i]}$, that is $h_i \in T_e^{[i]}$.

Claim 1: *If $h \in (I_{\mathbb{X}'})_e$ is uniform, then $h = 0$.*

Assume that $h \in (I_{\mathbb{X}'})_e$ is uniform. Since $I_{\mathbb{X}'} = I_{\mathbb{X}} : (I^{[1]} + \cdots + I^{[m]})$, and $I_{\mathbb{X}} \subset F^{\perp}$, then $hl_i \in F^{\perp}$, for any $l_i \in I^{[i]}$. Hence, for every $i = 1, \dots, m$,

$$hl_i \in F^{\perp} \Rightarrow hl_i \circ F = 0 \Rightarrow hl_i \circ F_i = 0 \Rightarrow h_i l_i \circ F_i = 0.$$

Now, considering $F_i \in S^{[i]}$, the last equality implies $h_i \in F_i^{\perp} : l_i$, so that $h_i \in (F_i^{\perp} : I^{[i]})$ and $h_i \in (F_i^{\perp} : I^{[i]}) + (t_i) \subset T^{[i]}$.

Hence, by Proposition 3.6, $h_i \in I_{\mathbb{X}_i} + (t_i)$, where $\mathbb{X}_i$ minimally decomposes F_i. By hypothesis $(F_i^{\perp})_e = 0$, hence there are no degree e forms in $I_{\mathbb{X}_i}$. Thus we have $h_i = \mu_i t_i$, and

$$h = \mu_1 t_1 + \ldots + \mu_m t_m.$$

Recall that $h \in I_{\mathbb{X}'}$ and hence it vanishes on all the points of $\mathbb{X}'$, that is the points of $\mathbb{X}$ not lying on $V_1 \cap \cdots \cap V_m$. Since $t_i \in I^{[i]}$, we have that h vanishes also on $V_1 \cap \cdots \cap V_m$. It follows that $h \in I_{\mathbb{X}} \subset F^{\perp}$. Thus $h \circ F = 0$. Now

$$\begin{aligned} h \circ F &= h \circ (F_1 + \cdots + F_m) = (\mu_1 t_1 + \ldots + \mu_m t_m) \circ (F_1 + \cdots + F_m) \\ &= \mu_1 t_1 \circ F_1 + \ldots + \mu_m t_m \circ F_m. \end{aligned}$$

Since $n_i \geq 1$ for all $i = 1, \dots, m$, the hypothesis $(F_i)_e^{\perp} = 0$ implies that $\deg F_i > e$, and hence $\deg t_i \circ F_i > 0$. It follows that $\mu_i t_i \circ F_i = 0$ for all $i = 1, \dots, m$, that is $\mu_i t_i \in F_i^{\perp}$ (considering $F_i \in S^{[i]}$). Since $(F_i)_e^{\perp} = 0$, we get that $\mu_i = 0$ for every i, and hence $h = 0$. This completes the proof of Claim 1.

Claim 2: *If B is a basis of $(I_{\mathbb{X}'})_e$, then $B \cup \{t_1, \dots, t_m\}$ is a set of linearly independent forms.*

For $e = 1$ Claim 2 follows immediately from Claim 1, so assume $e > 1$.

Let

$$B = \{\alpha_1 + \widetilde{\alpha}_1, \dots, \alpha_l + \widetilde{\alpha}_l\},$$

where the α_i are uniform and the $\widetilde{\alpha}_i$ are not uniform. Now if t_1 (and analogously for $t_2, \dots, t_m$) satisfies:

$$t_1 = \mu_1(\alpha_1 + \widetilde{\alpha}_1) + \cdots + \mu_l(\alpha_l + \widetilde{\alpha}_l) + \nu_2 t_2 + \cdots + \nu_m t_m,$$

we get $\mu_1 \widetilde{\alpha}_1 + \cdots + \mu_l \widetilde{\alpha}_l = 0$. Hence

$$\mu_1(\alpha_1 + \widetilde{\alpha}_1) + \cdots + \mu_l(\alpha_l + \widetilde{\alpha}_l) = \mu_1 \alpha_1 + \cdots + \mu_l \alpha_l \in (I_{\mathbb{X}'})_e.$$

Claim 1 yields $\mu_1 \alpha_1 + \cdots + \mu_l \alpha_l = 0$. It follows that t_1 is a linear combination of $t_2, \dots, t_m$, thus a contradiction. This finishes the proof of Claim 2.

Recall that, by Lemma 5.2, we have

$$I_{\mathbb{X}'} + (a_1 t_1 + \cdots + a_m t_m) \subseteq J_1 \cap \ldots \cap J_m.$$

Since $B \cup \{a_1t_1 + \cdots + a_mt_m\}$ is a basis of $(I_{\mathbb{X}'} + (a_1t_1 + \cdots + a_mt_m))_e$ and, by Claim 2, $B \cup \{t_1, \ldots, t_m\} \subseteq J_1 \cap \cdots \cap J_m$ is a set of linearly independent forms, then we have

$$HF(T/I_{\mathbb{X}'} + (a_1t_1 + \cdots + a_mt_m), e) - HF(T/J_1 \cap \cdots \cap J_m, e) \geq m - 1.$$

Since $a_1t_1 + \cdots + a_mt_m$ is a non zero divisor for $T/I_{\mathbb{X}'}$, for $s \gg 0$ we have

$$\begin{aligned}
\mathrm{rk}(F) &= |\mathbb{X}| \geq |\mathbb{X}'| = HF(T/I_{\mathbb{X}'}, s) \\
&= \binom{1}{e} \sum_{i=0}^{s} HF(T/(I_{\mathbb{X}'} + (a_1t_1 + \cdots + a_mt_m)), i) \\
&\geq \binom{1}{e} \Bigg(\sum_{i=0}^{e-1} HF(T/J_1 \cap \cdots \cap J_m, i) + (HF(T/J_1 \cap \cdots \cap J_m, e) + m - 1) \\
&\qquad + \sum_{i=e+1}^{s} HF(T/J_1 \cap \cdots \cap J_m, i) \Bigg).
\end{aligned}$$

Hence, for $s \gg 0$, by Lemma 5.3, we get

$$\mathrm{rk}(F) \geq \binom{1}{e} \left(\sum_{i=0}^{s} HF(T/J_1 \cap \ldots \cap J_m, i) + m - 1 \right) = \mathrm{rk}(F_1) + \cdots + \mathrm{rk}(F_m). \qquad \square$$

6. Forms for which the Strassen conjecture holds

Theorem 6.1. *Let $F = F_1 + \cdots + F_m \in S_d$, where $F_i \in S_d^{[i]}$. If, for $i = 1, \ldots, m$, F_i is of one of the following types:*

- *F_i is a monomial;*
- *F_i is a form in one or two variables;*
- *$F_i = x_0^a(x_1^b + \cdots + x_n^b)$ with $a + 1 \geq b$;*
- *$F_i = x_0^a(x_1^b + x_2^b)$;*
- *$F_i = x_0^a(x_0^b + x_1^a + \cdots + x_n^b)$ with $a + 1 \geq b$;*
- *$F_i = x_0^a(x_0^b + x_1^b + x_2^b)$;*
- *$F_i = x_0^a G(x_1, \ldots, x_n)$ where $G^{\perp} = (g_1, \ldots, g_n)$ is a complete intersection and $a < \deg(g_i)$ for $i = 1, \ldots, n$;*
- *F_i is a Vandermonde determinant;*

then the Strassen conjecture holds for F.

Proof. All the forms above are 1-computable, hence the conclusion follows from Proposition 4.21, Remark 5.4, Theorem 5.5 with $e = 1$, and in the case of Vandermonde determinant, Remark 4.22. $\square$

Remark 6.2. If F is a form which is e-computable, but not 1-computable, we can only combine it with other e-computable forms to get a form satisfying Strassen's conjecture.

For example, if F is the form of Example 4.15, then we know that F is 2-computable and $\mathrm{rk}(F) = 25$, but we know F is not 1-computable by Example 4.23.

If $G_1 = x_0x_1^4x_2^5$ then we showed that G_1 is 1-computable and $\mathrm{rk}(G_1) = 30$. But we do not know if G_1 is 2-computable.

Thus we cannot use the theorem to find the rank of $F+G_1$, although Strassen's conjecture says that the rank should be $25 + 30$.

However, if $G_2 = x_0^3x_1^4x_2^5$, by Proposition 4.2, we know that G_2 is 2-computable and $\mathrm{rk}(G_2) = 30$. Hence

$$\mathrm{rk}(F + G_2) = 25 + 30 = 55.$$

Remark 6.3. It would be interesting to have a characterization of those $F \in k[x_0, x_1]$ for which $F^\perp = (q^a, h_2)$ with $a \geq 2$. If we had that, we would have examples which were $\deg q$-computable. This would give us more forms for which Strassen's conjecture is true.

7. Some examples

Lemma 7.1. *Let $F = x_0^a(x_1^b + \cdots + x_n^b)$ with $a + 1 \leq b$, $n \geq 3$. If $\mathbb{X}$ is apolar to F, then $|\mathbb{X} \setminus \{X_i = 0\}| \geq b$ for all $i = 1, \ldots, n$.*

Proof. Since $I_\mathbb{X} : (X_i) \subseteq F^\perp : (X_i) = (x_0^a x_i^{b-1})^\perp$ and $\mathrm{rk}(x_0^a x_i^{b-1}) = b$ (see [3]), the apolarity lemma, yields that the ideal $I_\mathbb{X} : (X_i)$ is the homogeneous ideal of a set of at least b points. That is, $|\mathbb{X} \setminus \{X_i = 0\}| \geq b$ for all $i = 1, \ldots, n$. $\square$

Proposition 7.2. *If $F = x_0^a(x_1^b + \cdots + x_n^b)$ with $2 \leq a + 1 \leq b$ and $n \geq 3$, then*

$$bn - n + 3 \leq \mathrm{rk}(F) \leq bn.$$

In particular, we have $\mathrm{rk}(x_0^a(x_1^b + x_2^b + x_3^b)) = 3b$.

Proof. Note that $F^\perp = (X_0^{a+1}, X_1X_2, X_1X_3, \ldots, X_{n-1}X_n, X_1^b - X_2^b, \ldots, X_1^b - X_n^b)$.

We split the proof into four steps.

Step 1: $\mathrm{rk}(F) \leq bn$.

It is easy to see that

$$I = (X_1X_2, X_1X_3, \ldots, X_{n-1}X_n, (n-1)X_1^b - X_2^b - \cdots - X_n^b - X_0^b) \subseteq F^\perp$$

is the homogenous ideal of a set of bn distinct points. By the apolarity lemma $\mathrm{rk}(F) \leq bn$.

Step 2: $bn - n + 2 \leq \mathrm{rk}(F)$.

Let $\tilde{I} = F^\perp : (X_0) + (X_0) = (X_0, X_1X_2, X_1X_3, \ldots, X_{n-1}X_n, X_1^b - X_2^b, \ldots, X_1^b - X_n^b)$. Thus we have

i	0	1	$\cdots$	$b-1$	b	$b+1$
$HF(T/\tilde{I}, i)$	1	n	$\cdots$	n	1	0.

Hence, by Corollary 3.4, we get $\mathrm{rk}(F) \geq \sum_{i\geq 0} HF(T/\tilde{I}, i) = bn - n + 2$.

Step 3: *Let $\mathbb{X}$ be apolar to F and $a = 1$. If $X_iX_j + c_{ij}X_0^2 \in I_{\mathbb{X}}$ for all $1 \leq i < j \leq n$, then $c_{ij} = 0$ for all i, j.*

Suppose that $c_{ij} \neq 0$ for some $i < j$. Say, $c_{12} \neq 0$, then we have $X_1X_2 + c_{12}X_0^2,\ X_1X_3 + c_{13}X_0^2 \in I_{\mathbb{X}}$. Thus $X_1(c_{13}X_2 - c_{12}X_3) \in I_{\mathbb{X}}$. Thus we have

$$X_1 \in (I_{\mathbb{X}} : (c_{13}X_2 - c_{12}X_3))$$

and hence

$$c_{12}X_0^2 \in (I_{\mathbb{X}} : (c_{13}X_2 - c_{12}X_3)).$$

Since the ideal is radical we get

$$X_0 \in (I_{\mathbb{X}} : (c_{13}X_2 - c_{12}X_3))$$

and thus

$$X_0(c_{13}X_2 - c_{12}X_3) \in I_{\mathbb{X}}$$

and this yields the contradiction $c_{12} = 0$ and $c_{13} = 0$.

Step 4: $bn - n + 2 < \mathrm{rk}(F)$.

Suppose that $\mathrm{rk}(F) = bn - n + 2 = |\mathbb{X}|$, where $\mathbb{X}$ minimally decomposes F.

By the proof of Step 2, the rank of F is computed by X_0, hence by Proposition 3.6 we get $I_{\mathbb{X}} + (X_0) = F^{\perp} + (X_0)$. In particular we have $X_iX_j \in I_{\mathbb{X}} + (X_0)$ for all $1 \leq i < j \leq n$, and so $X_iX_j + L_{ij}X_0 \in I_{\mathbb{X}}$ for some linear form L_{ij}. Since $I_{\mathbb{X}} \subset F^{\perp}$ and $X_iX_j \in F^{\perp}$, then $L_{ij}X_0 \in F^{\perp}$.

If $a > 1$, then $L_{ij} = 0$.

Let $a = 1$. We get $L_{ij} = c_{ij}X_0$ and hence $X_iX_j + c_{ij}X_0^2 \in I_{\mathbb{X}}$. By Step 3, we have $c_{ij} = 0$.

Consequently, $X_iX_j \in I_{\mathbb{X}}$ for all $1 \leq i < j \leq n$ and for any $a \geq 1$. Now, since the ideal $(X_1X_2, X_1X_3, \ldots, X_{n-1n})$ is the homogeneous ideal of n lines $l_1, \ldots, l_n$ where $l_i = \{X_1 = X_2 = \cdots = \hat{X}_i = \cdots = X_n = 0\}$, it follows that all the points of $\mathbb{X}$ lie on the union of the lines l_i. Since $\mathbb{X} \setminus \{X_i = 0\} = \mathbb{X} \cap (l_i \setminus (1, 0, \ldots, 0))$, by Lemma 7.1 we have that

$$bn - n + 2 = |\mathbb{X}| \geq \sum_{i=1}^{n} |\mathbb{X} \setminus \{X_i = 0\}| \geq bn,$$

a contradiction. □

Remark 7.3. The form $F = w(x^3 + y^3 + z^3) \in k[x, y, z, w]$ is not 1-computable.

If F is 1-computable, then there exists an ideal $I \subset T = k[X, Y, Z, W]$ of a linear space L such that

$$\mathrm{rk}(F) = \sum_{i=0}^{\infty} HF\left(T/\left(F^{\perp} : I + (t)\right), i\right),$$

where $t = aX + bY + cZ + dW \in I$ is a general linear form.

If t has at least two of the coefficients a, b, c, d different from zero, since

$$F^{\perp} = \left(W^2, YZ, XZ, XY, Y^3 - Z^3, X^3 - Z^3\right),$$

we get that

$$\begin{aligned} HF(T/(F^{\perp} + (t)), 0) &= 1 \\ HF(T/(F^{\perp} + (t)), 1) &= 3 \\ HF(T/(F^{\perp} + (t)), 2) &\leq 3 \\ HF(T/(F^{\perp} + (t)), 3) &\leq 1 \\ HF(T/(F^{\perp} + (t)), 4) &= 0. \end{aligned}$$

By Proposition 7.2 we know that $\mathrm{rk}(F) = 9$ and since

$$F^{\perp} + (t) \subseteq F^{\perp} : I + (t)$$

we get

$$8 \geq \sum_{i=0}^{\infty} HF\left(T/\left(F^{\perp} + (t)\right), i\right) \geq \sum_{i=0}^{\infty} HF\left(T/\left(F^{\perp} : I + (t)\right), i\right) = \mathrm{rk}(F) = 9,$$

and this is a contradiction.

Now if L is a point or a line, and $\{t = 0\}$ is a general plane through L, then t has at least two of the coefficients a, b, c, d different from zero. If L is a plane, then $(t) = I$, and the only planes with three coefficients zero between a, b, c, d are the coordinate planes. Hence the only possibility for F to be 1-computable, is with $L = \{X = 0\}, \{Y = 0\}, \{Z = 0\}, \{W = 0\}$, but

$$\sum_{i=0}^{\infty} HF\left(T/\left(F^{\perp} : (X) + (X)\right), i\right) = 2,$$

(analogously for Y and Z) and

$$\sum_{i=0}^{\infty} HF\left(T/\left(F^{\perp} : (W) + (W)\right), i\right) = 8.$$

Hence, F is not 1-computable.

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