Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XVIII (2018), 343-362

# Isoperimetric inequality on CR-manifolds with nonnegative $Q'$-curvature

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**Abstract.** In this paper we study contact forms on the three-dimensional Heisenberg manifold with its standard CR structure. We discover that the $Q'$-curvature, introduced by Branson, Fontana and Morpurgo [3] on the CR three-sphere and then generalized to any pseudo-Einstein CR three-manifold by Case and Yang [6], controls the isoperimetric inequality on such a CR-manifold. As the first and important step to show this, we prove that the nonnegative Webster curvature at infinity implies that the metric is normal, which is analogous to the behavior on a Riemannian four-manifold.

**Mathematics Subject Classification (2010):** 32V05 (primary); 32V20, 35H20, 53C21 (secondary).

## 1. Introduction

On a four-dimensional manifold, the Paneitz operator $P_4$ and Branson's $Q$-curvature [2] have many properties analogous to those of the Laplacian operator $\Delta_g$ and the Gaussian curvature $K_g$ on surfaces. The Paneitz operator is defined as

$$P_g = \Delta^2 + \delta\left(\frac{2}{3}Rg - 2\,\mathrm{Ric}\right)d,$$

where $\delta$ is the divergence, $d$ is the differential, $R$ is the scalar curvature of $g$, and Ric is the Ricci curvature tensor. The $Q$-curvature is defined as

$$Q_g = \frac{1}{12}\left\{-\Delta R + \frac{1}{4}R^2 - 3|E|^2\right\},$$

where $E$ is the traceless part of Ric, and $|\cdot|$ is taken with respect to the metric $g$. The two most important properties for the pair $(P_g, Q_g)$ are that under the conformal change $g_w = e^{2w}g_0$,
1. $P_g$ transforms by $P_{g_w}(\cdot) = e^{-4w}P_{g_0}(\cdot)$;
2. $Q_g$ satisfies the fourth-order equation

$$P_{g_0}w + 2Q_{g_0} = 2Q_{g_w}e^{4w}.$$

The research of the first author is partially supported by NSF grant DMS-1547878, and NSF grant DMS-1612015. The research of the second the author is partially supported by NSF grant DMS-1509505.

Received January 22, 2016; accepted in revised form November 07, 2016.
Published online March 2018.

As proved by Beckner [1] and Chang-Yang [4], the pair $(P_g, Q_g)$ also appears in the Moser-Trudinger inequality for higher order operators.

On CR manifolds, it is a fundamental problem to study the existence and properties of CR invariant pairs analogous to $(P_g, Q_g)$. Graham and Lee [14] have studied a fourth-order CR covariant operator with leading term $\Delta_b^2 + T^2$ and Hirachi [16] has identified the $Q$-curvature which is related to $P$ through a change of contact form. However, although the integral of the $Q$-curvature on a compact three-dimensional CR manifold is a CR invariant, it is always equal to zero. And in many interesting cases when the CR three-manifold is the boundary of a strictly pseudoconvex domain, by [11] the $Q$-curvature vanishes everywhere. As a consequence, it is desirable to search for some other invariant operators and curvature invariants on a CR manifold that are more sensitive to the CR geometry. The work of Branson, Fontana and Morpurgo [3] aims to find such a pair $(P', Q')$ on the CR sphere. Later, the definition of $Q'$-curvature is generalized to all pseudo-Einstein CR manifolds by the work of Case-Yang [4] and that of Hirachi [17]. The construction uses the strategy of analytic continuation in dimension by Branson [2], restricted to the subspace of the CR pluriharmonic functions:

$$P_4' := \lim_{n \to 1} \frac{2}{n-1} P_{4,n}|_{\mathcal{P}}. \tag{1.1}$$

Here $P_{4,n}$ is the fourth-order CR covariant operator that exists for every contact form $\theta$ by the work of Gover and Graham [13]. By [14], the space of CR pluriharmonic functions $\mathcal{P}$ is always contained in the kernel of $P_{4,1}$. On the Heisenberg spaces with its standard contact structure, the expression of $P'$ simplifies to be

$$P'u = 2\Delta_b^2 u. \tag{1.2}$$

In this paper, we want explore the geometric meaning of this newly introduced conformal invariant $Q'$-curvature.

In Riemannian geometry, a classical isoperimetric inequality on a complete simply connected surface $M^2$, called Fiala-Huber's [12,18] isoperimetric inequality, states that

$$\mathrm{Vol}(\Omega) \le \frac{1}{2(2\pi - \int_{M^2} K_g^+ dv_g)} \mathrm{Area}(\partial\Omega)^2, \tag{1.3}$$

where $K_g^+$ is the positive part of the Gaussian curvature $K_g$. Also $\int_{M^2} K_g^+ dv_g < 2\pi$ is the sharp bound for the isoperimetric inequality to hold.

In [20], the first author generalizes the Fiala-Huber's isoperimetric inequality to all even dimensions, replacing the role of the Gaussian curvature in dimension two by that of the $Q$-curvature in higher dimensions.

Let $(M^n, g) = (\mathbb{R}^n, e^{2u}|dx|^2)$ be a complete noncompact even dimensional manifold. Let $Q^+$ and $Q^-$ denote the positive and negative part of $Q_g$ respectively, and let $dv_g$ denote the volume form of $M$. Suppose $g = e^{2u}|dx|^2$ is a *normal* metric, *i.e.*

$$u(x) = \frac{1}{c_n} \int_{\mathbb{R}^n} \log \frac{|y|}{|x-y|} Q_g(y) dv_g(y) + C, \tag{1.4}$$

where $c_n = 2^{n-2}(\frac{n-2}{2})!\pi^{\frac{n}{2}}$, and $C$ is some constant. If

$$\beta^+ := \int_{M^n} Q^+ dv_g < c_n, \tag{1.5}$$

and

$$\beta^- := \int_{M^n} Q^- dv_g < \infty, \tag{1.6}$$

then $(M^n, g)$ satisfies the isoperimetric inequality with isoperimetric constant depending only on $n$, $\beta^+$ and $\beta^-$. Namely, for any bounded domain $\Omega \subset M^n$ with smooth boundary,

$$|\Omega|_g \leq C(n, \beta^+, \beta^-)|\partial\Omega|_g^{\frac{n}{n-1}}. \tag{1.7}$$

It is well known that if the scalar curvature is nonnegative at infinity, then one can show that the metric is a normal metric. For interested readers, the proof of such a fact when $n = 4$ was given in [5]. For higher even dimensions, one can prove by a similar manner.

In the main result of this paper, we prove that the $Q'$-curvature and $P'$ operator are the relevant CR scalar invariant and CR covariant operator to study the isoperimetric inequalities in the CR setting. The Webster [21] curvature at infinity imposes important geometric rigidity on the CR manifold. We also notice that the class of pluriharmonic functions $\mathcal{P}$ is the relevant subspace of functions for the conformal factor $u$. We derive the following isoperimetric inequality on any CR three-manifold with $Q'$ curvature assumptions.

**Theorem 1.1.** *Let $(\mathbb{H}^1, e^u\theta)$ be a complete CR manifold, where $\theta$ denotes the standard contact form on the Heisenberg group $\mathbb{H}^1$ and $u$ is a pluriharmonic funcion on $\mathbb{H}^1$. Suppose additionally the $Q'$ curvature is nonnegative, the Webster scalar curvature is nonnegative at infinity and*

$$\int_{\mathbb{H}^1} Q' e^{4u}\theta \wedge d\theta < c_1'. \tag{1.8}$$

*Then the isoperimetric inequality is valid,* i.e. *for any bounded domain $\Omega$,*

$$\mathrm{Vol}(\Omega) \leq C\,\mathrm{Area}(\partial\Omega)^{4/3}. \tag{1.9}$$

*Here $C$ depends only on the integral of the $Q'$-curvature, and $c_1'$ is the constant in the fundamental solution of $P'$ operator. (See Section 2.)*

**Remark 1.2.** It is worth noting that the homogeneous dimension $N$ of $M^3$ is 4. Therefore the power on the right-hand side of the isoperimetric inequality is equal to $\frac{N}{N-1} = 4/3$.

**Remark 1.3.** We also remark that $c_1'$ is the critical constant for the validity of the isoperimetric inequality. In fact, there is a CR contact form $e^u\theta$ with $\int_{\mathbb{H}^1} Q' e^{4u}\theta \wedge d\theta = c_1'$, that does not satisfy the isoperimetric inequality. We give this example in Example 4.6.

In fact, we have proved a stronger result.

**Theorem 1.4.** *Suppose the $Q'$-curvature of $(\mathbb{H}^1, e^u\theta)$ is nonnegative. Suppose additionally the metric is normal and $u$ is a pluriharmonic function on $\mathbb{H}^1$. If*

$$\int_{\mathbb{H}^1} Q' e^{4u} \theta \wedge d\theta < c_1', \tag{1.10}$$

*then $e^{4u}$ is an $A_1$ weight.*

We will introduce the meaning of $A_1$ weight in Section 4.

ACKNOWLEDGEMENTS. We would like to thank the referee for valuable suggestions to improve the presentation of the paper.

## 2. Fundamental solution of $P'$ operator

In this section we compute the fundamental solution of the Paneitz operator $P'$ on the Heisenberg group $\mathbb{H}^1$. Let $p, q$ be two points on $\mathbb{H}^1$. Let $\rho$ denotes the distance function on $\mathbb{H}^1$. We show that $P'(\log \rho(q^{-1}p))$ is equal to the real part of Szegö kernel. Therefore, $P'$ restricted to the space of pluriharmonic functions has the fundamental solution $\log \rho(q^{-1}p)$.

Let us first consider the case for $p = (z, t) \in \mathbb{H}^1$, and $q = (0, 0) \in \mathbb{H}^1$. Note that

$$\begin{aligned} \Delta_b \log \rho(q^{-1}p) &= \Delta_b \log(|z|^4 + t^2)^{\frac{1}{4}} \\ &= \frac{1}{4}(\partial_x + 2y\partial_t)(\partial_x + 2y\partial_t) \log(|z|^4 + t^2) \\ &\quad + \frac{1}{4}(\partial_y - 2x\partial_t)(\partial_y - 2x\partial_t) \log(|z|^4 + t^2). \end{aligned} \tag{2.1}$$

$$\begin{aligned} &(\partial_x + 2y\partial_t)(\partial_x + 2y\partial_t) \log(|z|^4 + t^2) \\ &= (\partial_x + 2y\partial_t)\left[\frac{1}{(|z|^4 + t^2)}(4x|z|^2 + 4yt)\right] \\ &= \frac{-1}{(|z|^4 + t^2)^2}(4x|z|^2 + 4yt)^2 + \frac{1}{|z|^4 + t^2}(4|z|^2 + 8x^2 + 8y^2) \\ &= \frac{1}{(|z|^4 + t^2)^2}\left[-16(x^2|z|^4 + 2xyt|z|^2 + y^2t^2) + 12|z|^2(|z|^4 + t^2)\right]. \end{aligned} \tag{2.2}$$

Similarly, one can see

$$\begin{aligned} &(\partial_y - 2x\partial_t)(\partial_y - 2x\partial_t) \log(|z|^4 + t^2) \\ &= \frac{1}{(|z|^4 + t^2)^2}\left[-16(y^2|z|^4 - 2xyt|z|^2 + x^2t^2) + 12|z|^2(|z|^4 + t^2)\right]. \end{aligned} \tag{2.3}$$

Thus, we obtain

$$\begin{aligned}\Delta_b \log(|z|^4+t^2)^{\frac{1}{4}} &= \frac{1}{4(|z|^4+t^2)^2}\Big[-16(|z|^6+|z|^2t^2)+24|z|^2(|z|^4+t^2)\Big] \\ &= \frac{2|z|^2}{|z|^4+t^2}.\end{aligned} \tag{2.4}$$

We now need to compute $\Delta_b \dfrac{|z|^2}{|z|^4+t^2}$.

$$\begin{aligned}&(\partial_x+2y\partial_t)(\partial_x+2y\partial_t)\frac{|z|^2}{|z|^4+t^2} \\ &= (\partial_x+2y\partial_t)\left[\frac{2x}{|z|^4+t^2}+\frac{-|z|^2}{(|z|^4+t^2)^2}(4x|z|^2+4yt)\right] \\ &= \frac{-2x}{(|z|^4+t^2)^2}(4x|z|^2+4yt)+\frac{2}{|z|^4+t^2} \\ &\quad+\frac{2|z|^2}{(|z|^4+t^2)^3}(4x|z|^2+4yt)^2 \\ &\quad+\frac{-|z|^2}{(|z|^4+t^2)^2}(4|z|^2+8x^2+8y^2)\frac{-2x}{(|z|^4+t^2)^2}(4x|z|^2+4yt) \\ &= \frac{2}{|z|^4+t^2}+\frac{1}{(|z|^4+t^2)^2}\Big[-8x^2|z|^2-16xyt-12|z|^4-8x^2|z|^2)\Big] \\ &\quad+\frac{32|z|^2}{(|z|^4+t^2)^3}(x|z|^2+yt)^2.\end{aligned} \tag{2.5}$$

Similarly,

$$\begin{aligned}&(\partial_y-2x\partial_t)(\partial_y-2x\partial_t)\frac{|z|^2}{|z|^4+t^2} \\ &= (\partial_y-2x\partial_t)\left[\frac{2y}{|z|^4+t^2}+\frac{-|z|^2}{(|z|^4+t^2)^2}(4y|z|^2-4xt)\right] \\ &= \frac{-2y}{(|z|^4+t^2)^2}(4y|z|^2-4xt)+\frac{2}{|z|^4+t^2}+\frac{2|z|^2}{(|z|^4+t^2)^3}(4y|z|^2-4xt)^2 \\ &\quad+\frac{-|z|^2}{(|z|^4+t^2)^2}(4|z|^2+8x^2+8y^2)+\frac{-2y}{(|z|^4+t^2)^2}(4y|z|^2-4xt) \\ &= \frac{2}{|z|^4+t^2}+\frac{1}{(|z|^4+t^2)^2}\Big[-8y^2|z|^2+16xyt-12|z|^4-8y^2|z|^2)\Big] \\ &\quad+\frac{32|z|^2}{(|z|^4+t^2)^3}(y|z|^2-xt)^2.\end{aligned} \tag{2.6}$$

Therefore, by (2.5) and (2.6) we have

$$
\begin{aligned}
\Delta_b \frac{|z|^2}{|z|^4+t^2} &= \frac{4}{|z|^4+t^2} + \frac{1}{(|z|^4+t^2)^2}(-8|z|^4 - 24|z|^4 - 8|z|^4) \\
&\quad + \frac{32|z|^2}{(|z|^4+t^2)^3}(|z|^6 + |z|^2 t^2) \\
&= \frac{4}{|z|^4+t^2} - \frac{8|z|^4}{(|z|^4+t^2)^2} \\
&= 4\frac{t^2 - |z|^4}{(|z|^4+t^2)^2}.
\end{aligned} \tag{2.7}
$$

So we have show that

$$
\begin{aligned}
P'(\log(|z|^4+t^2)^{\frac{1}{4}}) &= 2\Delta_b \frac{|z|^2}{|z|^4+t^2} \\
&= 8\frac{t^2-|z|^4}{(|z|^4+t^2)^2}.
\end{aligned} \tag{2.8}
$$

Note that this is equal to the real part of the Szegö kernel $\mathrm{Re}(S_{\mathbb{H}^1}(p,q))$, up to a multiplicative constant. So we have proved that $\log(|z|^4+t^2)^{\frac{1}{4}}$ is proportional to the fundamental solution of the operator $P'$ on the space of pluriharmonic functions at point $p=(z,t)$ and $q=(0,0)$. Since the norm $\rho$ and $P'$ are both left invariant, this computation is also valid for arbitrary value of $q$. Thus we have proved that $\log(\rho(q^{-1}p))$ is proportional to the fundamental solution of $P'$. We denote $G_{\mathbb{H}^1}(u,v) = c_1' \cdot \log \rho(q^{-1}p)$.

## 3. Nonnegative Webster scalar curvature at $\infty$

In this section we describe the property of CR-manifolds with nonnegative Webster scalar curvature at infinity. We will see this geometric condition has a strong analytic implication. We denote the volume form $\theta \wedge d\theta$ of $\mathbb{H}^1$ by $dv$.

**Proposition 3.1.** *Let $\theta$ be the standard contact form of the Heisenberg group $\mathbb{H}^1$, and $\hat{\theta} = e^u\theta$ be the conformal change of it. Suppose $u \in \mathcal{P}$ is a pluriharmonic function on $\mathbb{H}^1$, $\Delta_b^2 u \in L^1(\mathbb{H}^1)$ and $\hat{\theta}$ has nonnegative Webster scalar curvature near $\infty$,* i.e. $-\Delta_b u \geq |\nabla_b u|^2$. *Then $\hat{\theta}$ is a normal,* i.e.

$$
u(p) = \int_{\mathbb{H}^1} G_{\mathbb{H}^1}(p,q) P'u(q) dv(q) + C, \tag{3.1}
$$

*where $C$ is a constant.*

It is proved by [3] that the Green function for $P'_{\mathbb{S}^3}$ is given by

$$G_{\mathbb{S}^3}(\zeta, \eta) = \log|1 - \zeta \cdot \bar{\eta}|. \tag{3.2}$$

It satisifes the equation

$$P'_{\mathbb{S}^3} G_{\mathbb{S}^3}(u, v) = S_{\mathbb{S}^3}(u, v) - \frac{1}{\mathrm{vol}(\mathbb{S}^3)}, \tag{3.3}$$

where $S_{\mathbb{S}^3}(u, v)$ is the real part of the Szegö kernel. We proved in section 2 that the fundamental solution for $P'_{\mathbb{H}^1}$ is given by $\log \rho(v^{-1}u)$. We recall that the homogeneous norm on $\mathbb{H}^1$ is given by $\rho(z, t) = (|z|^4 + t^2)^{1/4}$.

**Definition 3.2.** Let $u \in \mathcal{P}$ such that $P'u \in L^1(\mathbb{H}^1)$. Define

$$v(p) := \int_{\mathbb{H}^1} G_{\mathbb{H}^1}(p, q) P'u(q) dv(q).$$

This is well-defined when $P'u \in L^1(\mathbb{H}^1)$. We want to prove that $w := u - v$ is a linear function in $t$.

**Lemma 3.3.** *Under the same assumption as Proposition* 3.1, *we have* $\Delta_b w = constant$.

*Proof.* First, we observe that

$$P'w = P'u - P'v = 0.$$

We can then apply the mean value property to the function $\Delta_b w$ which satisfies the equation $\Delta_b(\Delta_b w) = 0$. Let $K_r(x, y)$ denotes the Poisson kernel. We apply the Poisson integral formula to $\Delta_b w$ and derive

$$\Delta_b w(p) = \int_{\partial B(p,r)} \Delta_b w(q) K_r(p, q) dv(q), \tag{3.4}$$

for arbitrary sphere $B(p, r)$ of radius $r$. Here the radius is with respect to the distance given by $\rho(\cdot)$ on $\mathbb{H}^1$. Note that $\Delta_b u \le -|\nabla_b u|^2 \le 0$, and $\Delta_b v$ tends to zero for large spheres $\partial B(p, r)$. Thus by taking $r \to \infty$,

$$\Delta_b w \le 0,$$

at $\infty$. Thus $\Delta_b w$ is bounded from above by (3.4) and the fact that the Poisson kernel is nonnegative.

Now $\Delta_b w$ is bounded from above and $\Delta_b(\Delta_b w) = 0$. Thus, analogously to the harmonic function on the Euclidean spaces, by the Liouville's theorem for $\Delta_b$ operator, we have

$$\Delta_b w = c_1. \tag{3.5}$$

□

Next, besides $\Delta_b w = c_1$, we observe that $Tw$ is also a constant, because $\Delta_b^2 w + T^2 w = 0$. We denote the constant of $Tw$ by $c_2$. This allows us to show that

**Lemma 3.4.** *$w_x(x, y, t)$ is independent of variable the $t$,* i.e.

$$w_x(x, y, t) = w_x(x, y, 0).$$

*Proof.* We recall that

$$X = \partial_x + 2y\partial_t, \quad Y = \partial_y - 2x\partial_t, \quad T = \partial_t.$$

Since $X$ and $T$ commute, we have

$$\begin{aligned} 0 =& XTw = TXw = T(w_x + 2yw_t) \\ =& Tw_x. \end{aligned} \tag{3.6}$$

Thus $w_x$ is independent of $t$ variable. In other words, for any $(x, y, t)$,

$$w_x(x, y, t) = w_x(x, y, 0).$$

□

Similarly since $Y$ and $T$ commute, $w_y$ is independent of $t$ variable.

**Lemma 3.5.** *$w_{xx} + w_{yy}$ is independent of $t$ variable,* i.e.

$$w_{xx}(x, y, t) + w_{yy}(x, y, t) = w_{xx}(x, y, 0) + w_{yy}(x, y, 0).$$

*Proof.* This can be seen from the following computation

$$\begin{aligned} 0 &= T\Delta_b w \\ &= T[(XX + YY)]w \\ &= T\big[(\partial_x + 2yT)(\partial_x + 2yT) + (\partial_y - 2xT)(\partial_y - 2xT)\big]w \\ &= T\big[w_{xx} + 2yT\partial_x w + \partial_x(2yTw) + 2yT(2yTw) \\ &\quad + w_{yy} - \partial_y(2xTw) - 2xT(\partial_y w) + 2xT(2xTw)\big]. \end{aligned} \tag{3.7}$$

By the fact that $Tw$ is a constant, and that $T$ commutes with both $\partial_x$ and $\partial_y$, we obtain the above is equal to

$$T(w_{xx} + w_{yy}).$$

Thus the lemma holds. □

**Lemma 3.6.** *It holds $\Delta_b \partial_x w = 0$ and $\Delta_b \partial_y w = 0$.*

*Proof.* If $\Delta_b$ and $\partial_x$ commute, then since $\Delta_b w = c_1$, we have proved the lemma. In general, $\Delta_b$ and $\partial_x$ might not commute. However, we will use the fact that $Tw$ is a constant to achieve the goal.

$$\begin{aligned}\Delta_b \partial_x w &= \big[(\partial_x + 2yT)(\partial_x + 2yT)\partial_x w + (\partial_y - 2xT)(\partial_y - 2xT)\partial_x w\big] \\ &= w_{xxx} + \partial_x(2yT\partial_x w) + 2yT\partial_x(\partial_x w) + 2yT(2yT\partial_x w) \\ &\quad + w_{xyy} - 2xT\partial_y\partial_x w - \partial_y(2xT\partial_x w) + 2xT(2xT\partial_x w) \\ &= w_{xxx} + w_{xyy}.\end{aligned} \tag{3.8}$$

The last equality uses the fact that $T$ commutes with both $\partial_x$ and $\partial_y$, and the fact that $Tw$ is a constant: thus cross terms

$$\partial_x(2yT\partial_x w); \quad 2yT\partial_x(\partial_x w); \quad 2yT(2yT\partial_x w);$$

$$2xT\partial_y\partial_x w; \quad \partial_y(2xT\partial_x w); \quad 2xT(2xT\partial_x w)$$

vanish. □

**Lemma 3.7.** *The functions $|w_x|$ and $|w_y|$ are at most of linear growth.*

*Proof.* We have

$$|\nabla_b w|^2 = w_x^2 + w_y^2 + 4c_2^2(x^2 + y^2) - 4c_2(xw_y - yw_x). \tag{3.9}$$

The right-hand side is greater than

$$(1-\alpha)(w_x^2 + w_y^2) + 4c_2^2\left(-\frac{1}{\alpha} + 1\right)(x^2 + y^2),$$

for any $\alpha > 0$. Let us fix $\alpha = 1/2$. Note that $|\nabla_b w|^2 \le 2|\nabla_b u|^2 + 2|\nabla_b v|^2$ and

$$|\nabla_b u|^2 \le -\Delta_b u$$

near $\infty$. Also, $|\nabla_b v|$ tends to 0 near $\infty$. Thus $|\nabla_b w|^2 \le -2c_1 + 1$ near $\infty$, where $c_1 \le 0$ is the constant value of function $\Delta_b w$. Thus $|\nabla_b w|$ has an upper bound. It follows that $|\partial_x w|$ and $|\partial_y w|$ are at most of linear growth. □

This together with Lemma 3.6 implies that $\partial_x w$ is a linear function. Similarly, $\partial_y w$ is also a linear function. Suppose both $\partial_x w$ and $\partial_y w$ are not constant, then $w$ is a quadratic function. Since $c_1 \le 0$, we see that $e^u\theta$ gives rise to an incomplete metric. This is a contradiction. Thus both $\partial_x w$ and $\partial_y w$ are constant. So $w$ is linear in both $x$ and $y$. Again, $e^u\theta$ is incomplete unless $w$ is a constant in both $x$ and $y$. In other words, $w$ only depends on $t$. On the other hand, we also have $Tw = c_2$. So $w$ is a linear function of $t$. We now use the assumption that the Webster scalar curvature $R$ is nonnegative to show that $w$ must be a constant.

To do this, we first note that by a simple computation,

$$-\Delta_b(e^{c_2t}) = -4c_2^2(x^2+y^2)e^{c_2t} < 0.$$

Also

$$\begin{aligned} \mathrm{Re}^{2u} &= -\Delta_b(e^u) \\ &= -\Delta_b(e^{c_2t+v}) \\ &= -\Delta_b(e^{c_2t})e^v - 2X(e^{c_2t})X(e^v) - 2Y(e^{c_2t})Y(e^v) - \Delta_b(e^v)e^{c_2t} \\ &= -4c_2^2(x^2+y^2)e^{c_2t}e^v - 4c_2ye^{c_2t}X(e^v) + 4c_2xe^{c_2t}Y(e^v) \\ &\quad -(\Delta_b v + |\nabla_b v|^2)e^v e^{c_2t}. \end{aligned} \tag{3.10}$$

**Lemma 3.8.**

$$\frac{1}{|\partial B_r|}\int_{\partial B_r} |\nabla_b v|(x)d\sigma(x) = O\left(\frac{1}{r}\right) \quad as \quad r\to\infty. \tag{3.11}$$

*Proof.* By direct computation, we have

$$X\left(\log(|z|^4+t^2)^{1/4}\right) = \frac{1}{\rho^4}(|z|^2x+ty) \le \frac{|z|}{\rho^2} \le \frac{1}{\rho},$$
$$Y\left(\log(|z|^4+t^2)^{1/4}\right) = \frac{1}{\rho^4}(|z|^2y-tx),$$

and

$$|\nabla_b\left(\log(|z|^4+t^2)^{1/4}\right)| = \frac{|z|}{\rho^2} \le \frac{1}{\rho}.$$

Therefore

$$\begin{aligned} &\frac{1}{|\partial B_r|}\int_{\partial B_r} |\nabla_b v|(x)d\sigma(x) \\ &\le \int_{\mathbb{H}^1} \frac{1}{|\partial B_r|}\int_{\partial B_r} \frac{1}{\rho(y^{-1}x)}|Q'(y)|e^{4u(y)}dv(y)dv(x). \end{aligned} \tag{3.12}$$

Now we need to show

$$\frac{1}{|\partial B_r|}\int_{\partial B_r} \frac{1}{\rho(y^{-1}x)}d\sigma(x) \le O\left(\frac{1}{r}\right)$$

where $C$ is independent of $y$.

This is true because we can dilate and take the integration over the unit sphere.

$$\frac{1}{|\partial B_r|}\int_{\partial B_r} \frac{1}{\rho(y^{-1}x)}d\sigma(x) = \frac{1}{r}\cdot\frac{1}{|\partial B_1|}\int_{\partial B_1} \frac{1}{\rho((r^{-1}y)^{-1}x)}d\sigma(x).$$

If $|r^{-1}y| \ge 1+\delta$ or $|r^{-1}y| \le 1-\delta$, then it is easy to see that

$$\frac{1}{|\partial B_1|}\int_{\partial B_1} \frac{1}{\rho((r^{-1}y)^{-1}x)}d\sigma(x) \le C$$

for a constant $C$ independent of $x$.

If $1-\delta \leq |r^{-1}y| \leq 1+\delta$, then we need to use spherical coordinates to prove

$$\frac{1}{|\partial B_1|}\int_{\partial B_1}\frac{1}{\rho((r^{-1}y)^{-1}x)}d\sigma(x) \leq C. \tag{3.13}$$

It is obvious that we only need to deal with the limiting case when $r^{-1}y$ is on the unit sphere $\partial B_1$. Let $r^{-1}y=(y_1,y_2,s)$ and $x=(x_1,x_2,t)$. Let $(r',\theta')$ be the polar coordinates centered at $(y_1,y_2)$ in the $xy$-plane (by our notation $x=(x_1,x_2,t)$, it is the $x_1x_2$-plane).

$$\rho((y_1,y_2,s),(x_1,x_2,t)) \geq \sqrt{(x_1-y_1)^2+(x_2-y_2)^2} = r'. \tag{3.14}$$

The area form of the unit sphere is given by

$$d\sigma = \sqrt{(u_{x_1}-x_2)^2+(u_{x_2}+x_1)^2}dx_1dx_2,$$

where $u(x_1,x_2)=t=\pm\sqrt{1-(x_1^2+x_2^2)^2}$. One can directly compute that

$$d\sigma = \sqrt{\frac{r^2(1+3r^4)}{(1-r^2)(1+r^2)}}rdrd\theta.$$

Here $(r,\theta)$ are polar coordinates of $(x_1,x_2)$ centered at $(0,0)$. It is obvious that $rdrd\theta = r'dr'd\theta'$. Therefore,

$$\begin{aligned}&\int_{\partial B_1}\frac{1}{\rho((r^{-1}y)^{-1}x)}d\sigma(x)\\ \leq& 2\int_{x_1^2+x_2^2\leq 1}\frac{1}{r'}\sqrt{\frac{r^2(1+3r^4)}{(1-r^2)(1+r^2)}}r'dr'd\theta'.\end{aligned} \tag{3.15}$$

Case 1: $\sqrt{y_1^2+y_2^2}<1$.
We can denote $\sqrt{y_1^2+y_2^2}=1-\eta$, where $\eta>0$. Then the integral (3.15) is bounded by

$$C+2\int_{1-\frac{\eta}{2}\leq r\leq 1}\sqrt{\frac{r^2(1+3r^4)}{(1-r^2)(1+r^2)}}dr'd\theta'. \tag{3.16}$$

Here $r$ is a function of $(r',\theta')$ by the change of variable formula. The last inequality in (3.15) is because $r=1$ is the only singularity of such an integration.

Now, since $\sqrt{y_1^2+y_2^2} = 1-\eta$ and $1-\frac{\eta}{2} \leq r \leq 1$, we have $r' \geq \frac{\eta}{2}$. Thus $dr'd\theta' = \frac{r}{r'}drd\theta \leq \frac{2}{\eta}rdrd\theta$. Therefore

$$\begin{aligned}&\int_{1-\frac{\eta}{2}\leq r\leq 1}\sqrt{\frac{r^2(1+3r^4)}{(1-r^2)(1+r^2)}}dr'd\theta'.\\ \leq&\frac{2}{\eta}\int_{1-\frac{\eta}{2}\leq r\leq 1}\sqrt{\frac{r^2(1+3r^4)}{(1-r^2)(1+r^2)}}rdrd\theta.\end{aligned} \tag{3.17}$$

The last integral is bounded, because

$$2\int_{r\leq 1}\sqrt{\frac{r^2(1+3r^4)}{(1-r^2)(1+r^2)}}rdrd\theta = 2\int_{r\leq 1}d\sigma = |\partial B_1| < \infty. \tag{3.18}$$

Case 2: $\sqrt{y_1^2+y_2^2} = 1$.
Without loss of generality, we can assume that $(y_1, y_2) = (1, 0)$. We adopt the notation that $\theta'$ is the angle between the ray and the positive $x_2$-axis. Since the unit sphere on the $x_1x_2$-plane is completely on the left-hand side of $(1, 0)$, we have $\theta' \in [0, \pi]$.
Now

$$\begin{aligned}&\int_{x_1^2+x_2^2\leq 1}\frac{1}{r'}\sqrt{\frac{r^2(1+3r^4)}{(1-r^2)(1+r^2)}}r'dr'd\theta'\\ \leq&\int_0^\pi\int_{r'>\epsilon/2}\sqrt{\frac{r^2(1+3r^4)}{(1-r^2)(1+r^2)}}dr'd\theta' + \int_0^\pi\int_{r'\leq\epsilon/2}\sqrt{\frac{r^2(1+3r^4)}{(1-r^2)(1+r^2)}}dr'd\theta'.\end{aligned} \tag{3.19}$$

Note that

$$\int_0^\pi\int_{r'>\epsilon/2}\sqrt{\frac{r^2(1+3r^4)}{(1-r^2)(1+r^2)}}dr'd\theta' \leq C$$

because when $r' > \epsilon/2$, we can apply the argument in Case 1 again, using $dr'd\theta' = \frac{r}{r'}drd\theta \leq \frac{2}{\epsilon}rdrd\theta$.

For $r' \leq \epsilon/2$, by a direct computation, for very small $\epsilon$, $1-r \approx r'\theta'$.

$$\begin{aligned}&\int_0^\pi\int_{r'\leq\epsilon/2}\sqrt{\frac{r^2(1+3r^4)}{(1-r^2)(1+r^2)}}dr'd\theta'\\ \leq&\int_0^\pi\int_{r'\leq\epsilon/2}\sqrt{\frac{r^2(1+3r^4)}{(1-r^2)(1+r^2)}}\frac{1}{\sqrt{r'\theta'}}dr'd\theta'.\end{aligned} \tag{3.20}$$

Since we have

$$\sqrt{\frac{r^2(1+3r^4)}{(1+r)(1+r^2)}} < C,$$

$$\int_{r'<\epsilon/2} \frac{1}{\sqrt{r'}} dr' < \infty,$$

and

$$\int_0^\pi \frac{1}{\sqrt{\theta'}} d\theta' < \infty,$$

the integration in the second line of (3.20) is finite. This completes the proof of (3.13). □

By a similar proof, one can show the average estimate of $|\Delta_b v|$ and $|v|$ as well.

**Lemma 3.9.**

$$\frac{1}{|\partial B_r|} \int_{\partial B_r} |\Delta_b v|(x) d\sigma(x) = O\left(\frac{1}{r^2}\right) \quad as \quad r \to \infty. \tag{3.21}$$

**Lemma 3.10.**

$$\frac{1}{|\partial B_r|} \int_{\partial B_r} |v|(x) d\sigma(x) = O(1) \quad as \quad r \to \infty. \tag{3.22}$$

So there exists a sequence of points $\{p_i\}$, $|p_i| \to \infty$, such that

$$|v|(p_i) \le C, \tag{3.23}$$

$$|\nabla_b v|(p_i) + |\Delta_b v|(p_i) \le \epsilon. \tag{3.24}$$

Moreover, we can choose $p_i$, such that they lie in the half space $c_2 t \ge 0$, and away from the $t$-axis. In other words, we can require that $c_2 t(p_i) \ge 0$, and that $(x(p_i), y(p_i))$ does not tend to $(0,0)$. Here we adopt the notation that $p_i = (x(p_i), y(p_i), t(p_i))$.

When $|x| + |y| \ge L$ for some $L > 0$, we have

$$|4c_2 y e^{c_2 t} X(e^v)| \le |y| e^{c_2 t} e^v |\nabla_b v| \le \epsilon |y| e^{c_2 t} e^v \le \epsilon (x^2 + y^2) e^{c_2 t} e^v; \tag{3.25}$$

$$|4c_2 x e^{c_2 t} Y(e^v)| \le |x| e^{c_2 t} e^v |\nabla_b v| \le \epsilon |x| e^{c_2 t} e^v \le \epsilon (x^2 + y^2) e^{c_2 t} e^v; \tag{3.26}$$

and

$$|\Delta_b(e^v) e^{c_2 t}| = |(\Delta_b v + |\nabla_b v|^2) e^v e^{c_2 t}| \le \epsilon e^v e^{c_2 t}.$$

Thus

$$|2X(e^{c_2 t}) X(e^v) + 2Y(e^{c_2 t}) Y(e^v) + \Delta_b(e^v) e^{c_2 t}| \le 3\epsilon (x^2 + y^2) e^{c_2 t} e^v. \tag{3.27}$$

We want to show $c_2 = 0$. We prove this by contradiction. Suppose $c_2 \neq 0$. Then, by applying (3.27) in (3.10), we obtain that

$$\begin{aligned} \mathrm{Re}^{2u}(p_i) &= -4c_2^2(x^2+y^2)e^{c_2 t}e^v - 4c_2 y e^{c_2 t} X(e^v) \\ &\quad + 4c_2 x e^{c_2 t} Y(e^v) d - (\Delta_b v + |\nabla_b v|^2) e^v e^{c_2 t} \\ &\leq -3c_2^2(x(p_i)^2 + y(p_i)^2) e^{c_2 t(p_i)} e^v(p_i), \end{aligned} \tag{3.28}$$

when $\epsilon$ is small enough.

By our choice of $\{p_i\}$, $|v(p_i)| \leq C$ and $c_2 t(p_i) \geq 0$ for all $i$. Thus $e^v \geq \eta > 0$, and $e^{c_2 t(p_i)} \geq 1$. Since $c_2 \neq 0$, we get

$$-3c_2^2(x(p_i)^2 + y(p_i)^2) e^{c_2 t(p_i)} e^v(p_i) < 0,$$

as $i \to \infty$. In fact, this quantity goes to $-\infty$ unless $(x(p_i), y(p_i))$ tends to $(0, 0)$. Because if $(x(p_i)^2 + y(p_i)^2)$ is bounded, then $c_2 t(p_i) \to +\infty$. This contradicts the assumption on the nonnegativity of Webster scalar curvature $R$. Therefore $c_2 = 0$.

This completes the proof of Proposition 3.1.

## 4. Main results

To begin this section, we recall some preliminary Poincaré inequalities for Heisenberg groups $\mathbb{H}^n$ of arbitrary dimension. Let us denote the homogenous dimension by $N$. For $\mathbb{H}^n$, $N = 2n + 2$.

**Proposition 4.1.** *For any ball $B$ in Heisenberg group,*

$$\int_B \int_B |g(x) - g(y)| dv(x) dv(y) \leq C|B|^{\frac{N+1}{N}} \int_{2B} |\nabla_b g| dv(x). \tag{4.1}$$

*Here $2B$ denotes the concentric ball of $B$ with double radius, and $|\cdot|$ denotes the volume with respect to the Haar measure on $\mathbb{H}^n$.*

In fact, the above inequality is a direct consequence of the following 1-Poincaré inequality.

**Proposition 4.2 ([19]).** *For any ball $B$ in Heisenberg group,*

$$\int_B |g(x) - g_B| dv(x) \leq C|B|^{\frac{1}{N}} \int_{2B} |\nabla_b g| dv(x). \tag{4.2}$$

*Here $2B$ denotes the concentric ball of $B$ with double radius, $g_B$ denotes the average of $g(x)$ on $B$, and $|\cdot|$ denotes the volume with respect to the Haar measure on $\mathbb{H}^n$.*

This implies Proposition 4.1 because

$$\begin{aligned}&\int_B\int_B |g(x)-g(y)|dv(x)dv(y)\\ &\le \int_B\int_B |g(x)-g_B|+|g(y)-g_B|dv(x)dv(y) \qquad (4.3)\\ &\le C|B|^{\frac{N+1}{N}}\int_{2B}|\nabla_b g|dv(x).\end{aligned}$$

David Jerison [19] proved a stronger version of the 2-Poincaré inequality:

$$\int_B |g(x)-g_B|^2 dv(x) \le C|B|^{\frac{2}{N}}\int_B |\nabla_b g|^2 dv(x). \qquad (4.4)$$

The same method also implies a stronger version of 1-Poincaré inequality (see [15]).

$$\int_B |g(x)-g_B|dv(x) \le C|B|^{\frac{1}{N}}\int_B |\nabla_b g|dv(x). \qquad (4.5)$$

For the purpose of this paper, we only need the weaker statement Proposition 4.1, in which the integration is over $2B$ on the right-hand side of the inequality.

Given a bounded domain with smooth boundary, as a special case of the above proposition, one can take $g$ to be (a smooth approximation of) the characteristic function $\chi_\Omega$, and derive

$$|B\cap\Omega|\cdot|B\cap\Omega^c| \le C|\partial\Omega\cap 2B|\cdot|B|^{\frac{N+1}{N}}. \qquad (4.6)$$

This immediately gives rise to the following:

**Corollary 4.3.** *For all balls $B\subset\mathbb{H}^n$, such that,*

$$|B\cap\Omega|\ge\frac{1}{2}|B| \quad \textit{and} \quad |B\cap\Omega^c|\ge\frac{1}{2}|B|,$$

*we have, by* (4.6)*,*

$$|B|^{\frac{N-1}{N}} \le C|\partial\Omega\cap 2B|.$$

**Theorem 4.4.** *Suppose $\omega(x)\ge 0$ is an $A_1$ weight on $\mathbb{H}^n$. Namely, there exists a constant $C_0$ (independent of $B$), so that for any ball $B\subset\mathbb{H}^n$,*

$$\frac{1}{|B|}\int_B \omega(p)dv(p) \le C_0 \inf_{z\in B}\omega(z). \qquad (4.7)$$

*Then the weighted isoperimetric inequality holds for $\omega(x)$: for any domain $\Omega\subset\mathbb{H}^n$ with smooth boundary,*

$$\int_\Omega \omega(x)dv(x) \le C_1\left(\int_{\partial\Omega}\omega(x)^{\frac{N-1}{N}}d\sigma(x)\right)^{\frac{N}{N-1}}, \qquad (4.8)$$

*where $C_1$ only depends on the $A_1$ bound $C_0$ of $\omega(x)$ and the homogeneous dimension $N=2n+2$.*

We now give the proof of this theorem by Proposition 4.1.

*Proof*. Consider a covering $\cup_{\alpha\in\Lambda} B_\alpha$ of the domain $\Omega$ such that each $B_\alpha$ satisfies the properties:

$$\left|\frac{1}{2}B_\alpha \cap \Omega\right| \geq \frac{1}{2}\left|\frac{1}{2}B_\alpha\right|, \quad \left|\frac{1}{2}B_\alpha \cap \Omega^c\right| \geq \frac{1}{2}\left|\frac{1}{2}B_\alpha\right|. \tag{4.9}$$

In other words, $|\frac{1}{2}B_\alpha \cap \Omega|$ and $|\frac{1}{2}B_\alpha \cap \Omega^c|$ are both comparable to $|\frac{1}{2}B_\alpha|$. By Vitali covering theorem, there exists a countable subset $\cup_{i=1}^\infty B_i$ such that $\Omega \subset \cup_{i=1}^\infty B_i$, and $\{\frac{1}{2}B_i\}$ are mutually disjoint. Therefore,

$$\begin{aligned} \omega(\Omega) &= \int_\Omega \omega(x)dv(x) \\ &\leq \sum_{i=1}^\infty \int_{B_i\cap\Omega} \omega(x)dv(x) \\ &\leq \sum_{i=1}^\infty \int_{B_i} \omega(x)dv(x) \\ &\leq \sum_{i=1}^\infty C_0|B_i|\omega(p_i) \\ &\leq C_2(n)\sum_{i=1}^\infty \left|\frac{1}{4}B_i\right|\omega(p_i). \end{aligned} \tag{4.10}$$

Here $\omega(p_i) = \inf_{x\in B_i}\omega(x)$.

By using Corollary 4.3 to $B = \frac{1}{4}B_i$,

$$\begin{aligned} \omega(\Omega) &\leq C_3\sum_{i=1}^\infty \left|\partial\Omega \cap \frac{1}{2}B_i\right|^{\frac{N}{N-1}} \omega(p_i) \\ &\leq C_3\sum_{i=1}^\infty \left(\int_{\partial\Omega\cap\frac{1}{2}B_i} \omega(x)^{\frac{N-1}{N}} d\sigma(x)\right)^{\frac{N}{N-1}} \\ &\leq C_3\left(\sum_{i=1}^\infty \int_{\partial\Omega\cap\frac{1}{2}B_i} \omega(x)^{\frac{N-1}{N}} d\sigma(x)\right)^{\frac{N}{N-1}} \\ &\leq C_3\left(\int_{\partial\Omega} \omega(x)^{\frac{N-1}{N}} d\sigma(x)\right)^{\frac{N}{N-1}}. \end{aligned} \tag{4.11}$$

□

**Lemma 4.5.** *$\frac{1}{\rho(u)^\alpha}$ is an $A_1$ weight for $0 < \alpha < N = 2n+2$ on the Heisenberg group $\mathbb{H}^n$.*

One can directly check this fact by estimating the maximal function of $\frac{1}{\rho(u)^\alpha}$.

In the following, we will give a proof of Theorem 1.4. Theorem 1.1 is then a consequence of Theorem 1.4, because if $e^{4u}$ is an $A_1$ weight, by Theorem 4.4, on such a conformal Heisenberg group, the isoperimetric inequality is valid. Moreover, the isoperimetric constant depends only on the integral of the $Q'$-curvature.

*Proof of Theorem* 1.4. The PDE that the conformal factor $u$ satisfies is

$$P'u = Q'e^{4u}.$$

Since $u$ is a pluriharmonic function, one has $\Delta_b^2 u = T^2 u$. Recall that the fundamental solution of Paneitz operator $P' = 2\Delta_b^2$ is given by $c_1' \log \frac{1}{\rho(y^{-1}x)}$. By section 3, as the Webster scalar curvature at $\infty$ is nonnegative, we have the metric is normal. Namely, $u$ has an integral representation

$$u(x) = \frac{1}{c_1'} \int_{\mathbb{H}^1} \log \frac{\rho(y)}{\rho(y^{-1}x)} Q'(y) e^{4u(y)} dv(y) + C. \tag{4.12}$$

We now want to prove $e^{4u}$ is an $A_1$ weight. In other words, for any ball $B \subset \mathbb{H}^1$,

$$M(e^{4u})(x) \le C(\alpha) e^{4u(x)}, \tag{4.13}$$

for a.e. $x \in \mathbb{H}^1$, where

$$M(f)(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dv(y).$$

Define $\alpha := \int_{\mathbb{H}^1} Q'e^{4u} dv(x)$. By assumption, $\alpha < c_1'$. Note that we can assume $\alpha \neq 0$. As if $\alpha = 0$, then $u$ is a constant. So the conclusion follows directly.

$$\begin{aligned}
&\frac{M(e^{4u})(x)}{e^{4u(x)}} \\
&= \sup_{r>0} \frac{\frac{1}{|B(x,r)|} \int_{B(x,r)} \exp\left( \frac{4}{c_1'} \int_{\mathbb{H}^1} \log \frac{\rho(p)}{\rho(p^{-1}y)} Q'(p) e^{4u(p)} dv(p) \right) dv(y)}{\exp\left( \frac{4}{c_1'} \int_{\mathbb{H}^1} \log \frac{\rho(p)}{\rho(p^{-1}x)} Q'(p) e^{4u(p)} dv(p) \right)} \\
&= \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} \exp\left( \frac{4\alpha}{c_1'} \int_{\mathbb{H}^1} \log \frac{\rho(p^{-1}x)}{\rho(p^{-1}y)} \cdot \frac{Q'(p) e^{4u(p)}}{\alpha} dv(p) \right) dv(y).
\end{aligned} \tag{4.14}$$

This is bounded by

$$\begin{aligned}&\sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}\int_{\mathbb{H}^1}\left(\frac{\rho(p^{-1}x)}{\rho(p^{-1}y)}\right)^{\frac{4\alpha}{c_1'}}\frac{Q'(p)e^{4u(p)}}{\alpha}dv(p)dv(y)\\&=\sup_{r>0}\int_{\mathbb{H}^1}\frac{1}{|B(x,r)|}\int_{B(x,r)}\left(\frac{\rho(p^{-1}x)}{\rho(p^{-1}y)}\right)^{\frac{4\alpha}{c_1'}}dv(y)\frac{Q'(p)e^{4u(p)}}{\alpha}dv(p).\end{aligned}\tag{4.15}$$

We know that by Lemma 4.5 $\rho(x)^{-\frac{4\alpha}{c_1'}}$ is an $A_1$ weight. And so is $(\rho(p^{-1}x))^{-\frac{4\alpha}{c_1'}}$ for each fixed $p$. This means

$$\frac{\frac{1}{|B(x,r)|}\int_{B(x,r)}\frac{1}{\rho(p^{-1}y)^{\frac{4\alpha}{c_1'}}}dv(y)}{\frac{1}{\rho(p^{-1}x)^{\frac{4\alpha}{c_1'}}}}\leq C(\alpha),\tag{4.16}$$

for each fixed $p$. Observe that $C$ is independent of $p$, one can substitute this inequality to the estimate (4.15) and obtain that (4.15) is bounded by

$$\int_{\mathbb{H}^1}C(\alpha)\frac{Q'(p)e^{4u(p)}}{\alpha}dv(p)=C(\alpha).$$

This shows that $e^{4u}$ is an $A_1$ weight. Once we have the $A_1$ property of $e^{4u}$, we can apply Theorem 4.4 to it. It completes the proof of Theorem 1.4. □

Finally, we give the example that shows $c_1'$ is the critical constant for the validity of the isoperimetric inequality.

**Example 4.6.** Let $e^u\theta$ be a contact form on $\mathbb{H}^1$. And suppose $u$ is given by the following integral formula.

$$u(x)=\frac{1}{c_1'}\int_{\mathbb{H}^1}\log\frac{\rho(y)}{\rho(y^{-1}x)}c_1'\delta_0dv(y),\tag{4.17}$$

where $\delta_0$ denotes Dirac delta function. It is obvious that the volume form $e^{4u(x)}=\frac{1}{\rho(x)^4}$ on $\mathbb{H}^1$ is not an $A_1$ weight. Moreover, such a CR manifold does not satisfy the isoperimetric inequality. This is because $e^u\theta=\frac{1}{\rho}\theta$ is the standard contact form on the cylinder $\mathbb{R}\times S^2\cong\mathbb{H}^1\setminus\{(0,0,0)\}$. In particular, one can choose a sequence of rotationally symmetric annular domains $A(r_0,r)$ on $\mathbb{H}^1, r\to\infty$. The area of $\partial A(r_0,r)$ with respect to $e^u\theta$ is bounded in $r$. But the volume of $A(r_0,r)$ with respect to $e^u\theta$ tends to $\infty$ as $r\to\infty$. This gives a counterexample to the isoperimetric inequality. In this construction, $u$ is singular at the origin. But we can use the approximation argument to deal with the issue. By choosing $\phi_\epsilon(y)$ to be a sequence of compactly supported smooth functions approximating $c_1'\delta_0$, and defining

$$u_\epsilon(x) = \frac{1}{c_1'}\int_{\mathbb{H}^1} \log\frac{\rho(y)}{\rho(y^{-1}x)}\phi_\epsilon(y)dv(y), \tag{4.18}$$

we construct a sequence of $u_\epsilon$ that approximates $u(x) = \log\frac{1}{\rho(x)}$ locally uniformly away from the origin. Since $\phi_\epsilon(y)$ are compactly supported, when the annular domains $A(r_0, r), r \to \infty$ are chosen such that $r_0$ is big enough (but fixed), the CR manifold $(\mathbb{H}^1, e^{u_\epsilon}\theta)$ does not satisfy the isoperimetric inequality.

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