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Trace and extension theorems for functions of bounded variation

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Abstract. In this paper we show that every L^1-integrable function on $\partial\Omega$ can be obtained as the trace of a function of bounded variation in Ω whenever Ω is a domain with regular boundary $\partial\Omega$ in a doubling metric measure space. In particular, when Ω supports a 1-Poincaré inequality, the trace class of $BV(\Omega)$ is $L^1(\partial\Omega)$. We also construct a bounded linear extension from a Besov class of functions on $\partial\Omega$ to $BV(\Omega)$.

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1. Overview

In Dirichlet boundary value problems in analysis, one prescribes the trace value of the solution at the boundary of the domain. Given a domain Ω, it is therefore natural to ask what class of functions on the boundary can be realized as the traces of functions of specified regularity on the domain.

The model problem that motivates our study is the problem of finding least gradient functions from the class of functions of bounded variation (BV), with prescribed boundary data, see [4, 19, 32, 38]. Therefore the regularity of the extended function inside the domain is BV regularity.

The paper [4] first studied the trace and extension problem for functions of bounded variation in Euclidean Lipschitz domains. It was shown there that the trace functions of BV functions on the domain lie in the L^1-class of the boundary. In contrast, the work [15] demonstrated that every L^1-function on the boundary of a Euclidean half-space (and hence boundaries of Lipschitz domains) has a BV extension to the half-space. Together, these two results indicate that the trace class of BV functions on a Euclidean Lipschitz domain is the L^1-class of its boundary.

In the metric setting, a version of the Dirichlet problem associated with BV functions was considered in [16–18, 21], but their notion of trace required the BV

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function to be defined on a larger domain. In [30] this requirement was dispensed with for domains whose boundaries are more regular (Euclidean Lipschitz domains satisfy this regularity condition). In [30] it was shown that if in addition the domain supports a 1-Poincaré inequality, then the trace of a BV function on the domain lies in a suitable L^1-class of the boundary, thus providing an analog of the results of [4] in the metric setting. The recent work [39] gave an analog of the extension result of [15] for Lipschitz domains in Carnot–Carathéodory spaces, which indicated that it is possible to identify the trace class of BV functions in more general metric measure spaces. The goal of this paper is to provide such an identification, by adapting the technique of [15] to the metric setting.

In this paper Ω denotes a bounded domain in a metric measure space (X, d, μ). To consider functions of bounded variation on Ω, we need a measure on Ω. The natural measure on Ω is the restriction of μ to Ω. The measure we consider on the boundary $\partial\Omega$ is the co-dimension 1 Hausdorff measure $\mathcal{H} := \mathcal{H}|_{\partial\Omega}$ (see (1.2) below). The function spaces related to $\partial\Omega$ will have norms computed using the measure $\mathcal{H}$, and this being understood, we will not explicitly mention the measure in the notation representing these function spaces.

We now state the two main theorems of this paper. In what follows, the map $T : BV(\Omega) \to \mathcal{F}$, where $\mathcal{F}$ is the collection of all Borel functions on $\partial\Omega$, is the trace operator as constructed in [30], see (2.7). In the event that Ω does not support a 1-Poincaré inequality, the trace need not be defined for each function in $BV(\Omega)$, but it would still be well-defined in the sense of (2.7) for certain functions in $BV(\Omega)$. Thus in the next two theorems, by stating that $T \circ E$ is the identity operator on the corresponding function space, we are also implicitly claiming that for each u in that function space the trace of Eu is well-defined.

Theorem 1.1. *Let Ω be a bounded domain in X satisfying the co-dimension* 1 *Ahlfors regularity* (1.5) *and the local measure density property* (1.6). *Then there is a bounded linear extension operator $E : B^0_{1,1}(\partial\Omega) \to BV(\Omega)$ such that $T \circ E$ is the identity operator on $B^0_{1,1}(\partial\Omega)$.*

Theorem 1.2. *With Ω a bounded domain in X satisfying the co-dimension* 1 *Ahlfors regularity* (1.5) *and the local measure density property* (1.6), *there is a* nonlinear *bounded extension operator* Ext : $L^1(\partial\Omega) \to BV(\Omega)$ *such that* $T \circ$ Ext *is the identity operator on $L^1(\partial\Omega)$.*

The extension from $L^1(\partial\Omega)$ to $BV(\Omega)$ cannot in general be linear; this is not an artifact of our proof, see [34,35], and the discussion in Section 5 below.

Remark 1.3. In proving Theorem 1.1 we actually prove a stronger but less elegant statement. We show that if the boundary of Ω, equipped with the co-dimension 1 Hausdorff measure $\mathcal{H}$, is *lower* Ahlfors regular, that is, if

$$C\mathcal{H}(B(x,r) \cap \partial\Omega) \geq \mu(B(x,r))/r$$

whenever $x \in \partial\Omega$ and $0 < r < 2\operatorname{diam}(\partial\Omega)$, then there is a bounded linear extension operator $E : B^0_{1,1}(\partial\Omega) \to BV(\Omega)$. We then show that in the event that $\partial\Omega$ also satisfies the requirement of *pointwise upper* co-dimension 1 Ahlfors regularity in the sense that for $\mathcal{H}$-almost every $x \in \partial\Omega$ there are constants $C(x) \geq 1$, $R(x) > 0$ such that for $0 < r \leq R(x)$,

$$\mathcal{H}(B(x,r) \cap \partial\Omega) \leq C(x) \frac{\mu(B(x,r))}{r},$$

then $T \circ E$ is the identity operator on $B^0_{1,1}(\partial\Omega)$.

Combining the above two theorems with those of [30] we obtain the following identification of the trace class of $BV(\Omega)$.

Corollary 1.4. *Let X support a 1-Poincaré inequality. With Ω a bounded domain in X that satisfies the density condition,* i.e.,

$$\mu(B(z,r) \cap \Omega) \geq C^{-1}\mu(B(z,r)) \text{ for all } z \in \Omega \text{ and } 0 < r < \operatorname{diam}(\Omega), \tag{1.1}$$

the co-dimension 1 Ahlfors regularity (1.5)*, and 1-Poincaré inequality, we have that the trace class of $BV(\Omega)$ is $L^1(\partial\Omega)$.*

In the above we cannot drop any of the respective conditions we impose on the domain Ω. The requirement of the support of a 1-Poincaré inequality is needed only in order to obtain the trace theorem from [30]. As the example of a slit disc shows, eliminating the support of a Poincaré inequality might result in the failure of the trace theorem, though as the example of the Euclidean (planar) domain

$$\Omega = (-2,2)^2 \setminus \left\{(x,y) \in \mathbb{R}^2 \,:\, -1 \leq x \leq 1, |x| \leq |y| \leq 1\right\}$$

shows, the support of 1-Poincaré inequality is not essential in obtaining the trace theorem of [30]. The measure density condition (1.1) is also needed to obtain the trace theorem. Again, this property might not be a requirement for obtaining the trace theorem of [30] and hence the above corollary, but if the requirement is removed, some other property of the domain needs to be required as the following example shows. This example is also a planar Euclidean domain, obtained by pasting a sequence of thin tubes, with relatively narrower and narrower trunks, to a rectangular base. For each positive integer n let U_n be the domain given by

$$U_n = \left(\frac{1}{n^2} - 4^{-n}, \frac{1}{n^2} + 4^{-n}\right) \times [0, 2^{-n}),$$

and let

$$\Omega = (-1,2) \times (-1,0) \cup \bigcup_{n \in \mathbb{N}} U_n.$$

The trace theorem of [30] fails here because the trace $T(u_n)$ of the function $u_n = \chi_{(\frac{1}{n}-4^{-n},\frac{1}{n}+4^{-n})\times[0,2^{-n}]}$ has $L^1(\partial\Omega)$-norm of the order of 2^{-n}, while the BV-norm of u_n is of the order of 4^{-n}. Note that Ω fails the measure density condition (1.1), but does satisfy the local version (1.6), and hence the extension theorems of this paper apply to this domain as well.

For clarity, we note that the statements of Theorems 1.1 and 1.2 do not require the domains or the ambient metric measure space to support any Poincaré inequality, which allows the domains to have interior cusps or slits. Note that the interior cusps and slits do not violate the measure density condition (1.1). On the other hand, exterior cusp points violate (1.1), but if there are only $\mathcal{H}$-measure zero worth of them, then the local measure density condition (1.6) is not violated and our results apply here as well. Indeed, we can decompose $\partial\Omega$ into three parts, namely the measure-theoretic interior $\partial_I\Omega$ consisting of points $z \in \partial\Omega$ for which

$$\liminf_{r\to 0^+} \frac{\mu(B(z,r)\cap\Omega)}{\mu(B(z,r))} = 1,$$

the measure-theoretic exterior $\partial_E\Omega$ consisting of points $z \in \partial\Omega$ for which

$$\liminf_{r\to 0^+} \frac{\mu(B(z,r)\setminus\Omega)}{\mu(B(z,r))} = 1,$$

and the measure-theoretic boundary $\partial_m\Omega = \partial\Omega \setminus (\partial_I\Omega \cup \partial_E\Omega)$. Thus, we have the validity of (1.6) at each point in $\partial_I\Omega$ and if the ambient metric space X supports a 1-Poincaré inequality, then (1.6) holds true also at $\mathcal{H}$-a.e. point in $\partial_m\Omega$ (this follows from the fact that Ω is of finite perimeter as $\mathcal{H}(\partial\Omega) < \infty$, see [1]). Hence, if $\mathcal{H}(\partial_E\Omega) = 0$ and X admits a 1-Poincaré inequality, then Ω satisfies the local measure density condition. Thus, even in the Euclidean setting our methods give rise to new results, as the results of [15] and [39] are in the setting of Lipschitz domains. Smooth bounded Euclidean domains and bounded smooth domains in a Riemannian manifold with positive Ricci curvature would satisfy the hypotheses listed in the above three results. Indeed, such domains are uniform domains, and as uniform domains in a metric measure space supporting a 1-Poincaré inequality do support a 1-Poincaré inequality (see [10]), the trace class of the class of BV functions on such domains is the L^1-class of the boundary of the domain. In general, balls in the space with center in the smooth domain need not be connected, but there is a scaling factor $\lambda > 0$ such that each ball with center in the domain can be connected in the λ-times enlarged ball (that is, all the points in the original ball belongs to the same connected component of the enlarged ball). The property of connecting a ball inside a fixed scaled concentric ball is called *linear local connectivity* in [27]. Thus the scaling factor λ on the right-hand side of the Poincaré inequality given in Definition 1.8 cannot in general be removed.

A related problem is to investigate the extensions of functions from a domain Ω to the whole space. See [5,9,24,29,31].

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1.1. Notation and definitions

In this section (X, d, μ) denotes a metric measure space with μ a Radon measure. We say that μ is *doubling* if there is a constant C_D such that for each $x \in X$ and $r > 0$,

$$0 < \mu(B(x, 2r)) \le C_D\, \mu(B(x, r)) < \infty.$$

Given a Lipschitz function f on a subset $A \subset X$, we set

$$\mathrm{LIP}(f, A) := \sup_{x,y \in A\,:\, x \ne y} \frac{|f(x) - f(y)|}{d(x, y)}.$$

When x is a point in the interior of $A \subset X$, we set

$$\mathrm{Lip}\, f(x) := \limsup_{y \to x} \frac{|f(y) - f(x)|}{d(y, x)}.$$

We follow [33] to define the function class $BV(X)$. The space $BV(X)$ of functions of bounded variation consists of functions in $L^1(X)$ that also have finite total variation on X. The total variation of a function on a metric measure space is measured using upper gradients; the notion of upper gradients, first formulated in [27] (with the terminology "very weak gradients"), plays the role of $|\nabla u|$ in the metric setting where no natural distributional derivative structure exists. A Borel function $g : X \to [0, \infty]$ is an upper gradient of $u : X \to \mathbb{R} \cup \{\pm\infty\}$ if the following inequality holds for all (rectifiable) curves $\gamma : [a, b] \to X$, (denoting $x = \gamma(a)$ and $y = \gamma(b)$),

$$|u(y) - u(x)| \le \int_\gamma g\, ds$$

whenever $u(x)$ and $u(y)$ are both finite, and $\int_\gamma g\, ds = \infty$ otherwise. For each function u as above, we set $I(u : X)$ to be the infimum of the quantity $\int_X g\, d\mu$ over all upper gradients (in X) g of u.

Remark 1.5. We note here that if u is a (locally) Lipschitz function on X, then $\mathrm{Lip}\, u$ is an upper gradient of u; see for example [26]. We refer the interested reader to [7,25] for more on upper gradients.

The *total variation* of the function $u \in L^1(X)$ is given by

$$\|Du\|(X) := \inf \left\{ \liminf_{i \to \infty} I(u_i : X) \colon u_i \in \mathrm{Lip}_{\mathrm{loc}}(X),\, u_i \to u \text{ in } L^1(X) \right\}.$$

Remark 1.6. From Remark 1.5 we know that if u is a locally Lipschitz continuous function on X, then $\|Du\|(X) \le \int_X \mathrm{Lip}\, u\, d\mu$.

For each open set $U \subset X$ we can set $\|Du\|(U)$ similarly:

$$\|Du\|(U) := \inf \left\{ \liminf_{i \to \infty} I(u_i : U) \colon u_i \in \mathrm{Lip}_{\mathrm{loc}}(U),\, u_i \to u \text{ in } L^1(U) \right\}.$$

It was shown in [33] that if $\|Du\|(X)$ is finite, then $U \mapsto \|Du\|(U)$ is the restriction of a Radon measure to open sets of X. We use $\|Du\|$ to also denote this Radon measure.

Definition 1.7. The space $BV(X)$ of functions of bounded variation is equipped with the norm

$$\|u\|_{BV(X)} := \|u\|_{L^1(X)} + \|Du\|(X).$$

This definition of BV agrees with the standard notion of BV functions in the Euclidean setting, see [2,14,40]. See also [3] for more on the BV class in the metric setting.

We say that a measurable set $E \subset X$ is of *finite perimeter* if $\chi_E \in BV(X)$. It follows from [33] that the superlevel set $E_t := \{z \in X : u(z) > t\}$ has finite perimeter for almost every $t \in \mathbb{R}$ and that the coarea formula

$$\|Du\|(A) = \int_{\mathbb{R}} \|D\chi_{E_t}\|(A)\, dt$$

holds true whenever $A \subset X$ is a Borel set.

Definition 1.8 (*cf.* [1]). A metric space X supports a 1-Poincaré inequality if there exist positive constants λ and C such that for all balls $B \subset X$ and all $u \in L^1_{\mathrm{loc}}(X)$,

$$\fint_B |u - u_B|\, d\mu \le C \operatorname{rad}(B) \frac{\|Du\|(\lambda B)}{\mu(\lambda B)}.$$

Here and in the rest of the paper, f_A denotes the *integral mean* of a function $f \in L^0(X)$ over a measurable set $A \subset X$ of finite positive measure, defined as

$$f_A = \fint_A f\, d\mu = \frac{1}{\mu(A)} \int_A f\, d\mu$$

whenever the integral on the right-hand side exists, not necessarily finite though. Furthermore, given a ball $B = B(x,r) \subset X$ and $\lambda > 0$, the symbol λB denotes the inflated ball $B(x, \lambda r)$.

Given $A \subset X$, we define its *co-dimension* 1 *Hausdorff measure* $\mathcal{H}(A)$ by

$$\mathcal{H}(A) = \lim_{\delta \to 0^+} \inf \left\{ \sum_i \frac{\mu(B_i)}{\operatorname{rad}(B_i)} : B_i \text{ balls in } X, \operatorname{rad}(B_i) < \delta,\ A \subset \bigcup_i B_i \right\}. \tag{1.2}$$

It was shown in [1] that if μ is doubling and supports a 1-Poincaré inequality, then there is a constant $C \ge 1$ such that whenever $E \subset X$ is of finite perimeter,

$$C^{-1}\mathcal{H}(\partial_m E) \le \|D\chi_E\|(X) \le C\, \mathcal{H}(\partial_m E),$$

where $\partial_m E$ is the *measure-theoretic boundary* of E. It consists of those points $z \in X$ for which

$$\limsup_{r \to 0^+} \frac{\mu(B(z,r) \cap E)}{\mu(B(z,r))} > 0 \quad \text{and} \quad \limsup_{r \to 0^+} \frac{\mu(B(z,r) \setminus E)}{\mu(B(z,r))} > 0.$$

We next turn our attention to the definition of other function spaces to be considered in this paper. The Besov classes, much studied in the Euclidean setting, made their first appearance in the metric setting in [11] and were explored further in [20].

Definition 1.9. Let (Z, d) be a metric space equipped with a Radon measure ν. For a fixed $R > 0$, the *Besov space* $B^\theta_{1,1}(Z)$ of smoothness $\theta \in [0, 1]$ consists of functions of finite Besov norm that is given by

$$\|u\|_{B^\theta_{1,1}(Z)} = \|u\|_{L^1(Z)} + \int_0^R \int_Z \fint_{B(x,t)} |u(y) - u(x)|\, d\nu(y)\, d\nu(x) \frac{dt}{t^{1+\theta}}. \quad (1.3)$$

In our application of Besov spaces, the metric space Z will be the boundary of a bounded domain in X, and the measure ν will be the restriction of the co-dimension 1 Hausdorff measure $\mathcal{H}$ to this boundary.

We will show that the function class $B^\theta_{1,1}(Z)$ is in fact independent of the choice of $R \in (0, \infty)$, see Lemma 3.2 below.

The following *fractional John–Nirenberg space* was first generalized to the metric measure space setting in [24]. In the Euclidean setting it was first studied in [12] and [13], but the case $\theta = 0$ in the Euclidean setting appeared in the earlier work of John and Nirenberg [28]. The fractional John–Nirenberg space $A^\theta_{1,\tau}(Z)$, where $\theta \in [0, 1]$ is its smoothness and $\tau \geq 1$ the dilation factor, is defined via its norm

$$\|u\|_{A^\theta_{1,\tau}(Z)} = \|u\|_{L^1(Z)} + \sup_{\mathcal{B}_\tau} \sum_{B \in \mathcal{B}_\tau} \frac{1}{\mathrm{rad}(B)^\theta} \int_{\tau B} |u - u_{\tau B}|\, d\nu, \quad (1.4)$$

where the supremum is taken over all collections $\mathcal{B}_\tau$ of balls in Z of radius at most R/τ such that $\tau B_1 \cap \tau B_2$ is empty whenever $B_1, B_2 \in \mathcal{B}_\tau$ with $B_1 \neq B_2$. The class $A^\theta_{1,\tau}(Z)$ is also independent of the exact choice of $R \in (0, \infty)$.

1.2. Standing assumptions

Throughout this paper (X, d, μ) is a metric measure space, with μ a Borel regular measure. We assume that X is complete and that μ is doubling on X. Furthermore, $\Omega \subset X$ is a bounded domain and there is a constant $C \geq 1$ such that for all $x \in \partial\Omega$ and $0 < r \leq \mathrm{diam}(\Omega)$, we have

$$C^{-1} \frac{\mu(B(x, r))}{r} \leq \mathcal{H}(B(x, r) \cap \partial\Omega) \leq C \frac{\mu(B(x, r))}{r}. \quad (1.5)$$

The property of satisfying (1.5) will be called *Ahlfors codimension* 1 *regularity* of $\partial\Omega$. Finally, we also assume that Ω satisfies a *local measure density condition*, that is, for $\mathcal{H}$-almost every $z \in \partial\Omega$ there exist constants $r_z > 0$ and $C_z \geq 1$ such that whenever $0 < r < r_z$,

$$\mu(B(z, r) \cap \Omega) \geq \frac{\mu(B(z, r))}{C_z}. \quad (1.6)$$

Throughout the paper C represents various constants that depend solely on the doubling constant, constants related to the Poincaré inequality, and the constants related to (1.5). The precise value of C is not of interest to us at this time, and its value may differ in each occurrence. Given expressions a and b, we say that $a \approx b$ if there is a constant $C \geq 1$ such that $C^{-1}a \leq b \leq Ca$.

2. Bounded linear extension from Besov class to BV class: proof of Theorem 1.1

2.1. Whitney cover and partition of unity

The following theorem from [25, Section 4.1] gives the existence of a Whitney covering of an open subset Ω of a doubling metric space X by balls whose radii are comparable to their distance from the boundary, see also [8].

Theorem 2.1. *Let $\Omega \subsetneq X$ be bounded and open. Then there exists a countable collection $\mathcal{W}_\Omega = \{B(p_{j,i}, r_{j,i}) = B_{j,i}\}$ of balls in Ω so that:*

- $\bigcup_{j,i} B_{j,i} = \Omega$;
- $\sum_{j,i} \chi_{B(p_{j,i}, 2r_{j,i})} \leq 2C_D^5$;
- $2^{j-1} < r_{j,i} \leq 2^j$ *for all* i;
- *and so that* $r_{j,i} = \frac{1}{8}\operatorname{dist}(p_{j,i}, X \setminus \Omega)$.

Here the constant C_D is the doubling constant of the measure μ.

The radii of the balls are small enough so that $2B_i \subset \Omega$. Also, since we are only concerned with bounded domains Ω, there is a largest exponent j that occurs in the cover; we denote this exponent by j_0. Hence $-j \in \mathbb{N} \cup \{0, \cdots, -j_0\}$. Note that 2^{j_0} is comparable to $\operatorname{diam}(\Omega)$. One wishing to keep track of the relationships between various constants should therefore keep in mind that the constants that depend on j_0 then depend on $\operatorname{diam}(\Omega)$. We also note that no ball in level j intersects a ball in level $j+2$. This follows by the reverse triangle inequality $d(p_{j,i}, p_{j+2,k}) \geq 2^{j+4} - 2^{j+3} = 2^{j+3}$ and the bounds on the radii: $2^{j-1} < r_{j,i} \leq 2^j$ and $2^{j+1} < r_{j+2,k} \leq 2^{j+2}$. As in [25, Section 4.1], there is a Lipschitz partition of unity $\{\varphi_{j,i}\}$ subordinate to the Whitney decomposition $\mathcal{W}_\Omega$, that is, $\sum_{j,i} \varphi_{j,i} \equiv \chi_\Omega$ and for every ball $B_{j,i} \in \mathcal{W}_\Omega$, we have that $\chi_{1/2B_{j,i}} \leq \varphi_{j,i} \leq \chi_{2B_{j,i}}$ and $\varphi_{j,i}$ is $C/r_{j,i}$-Lipschitz continuous.

2.2. An extension of Besov functions

Suppose that $f : \partial\Omega \to \mathbb{R}$ is a function in $B^0_{1,1}(\partial\Omega)$. We want to define a function $F : \Omega \to \mathbb{R}$ whose trace is the original function f on $\partial\Omega$.

Consider the center of the Whitney ball $p_{j,i} \in \Omega$ and choose a closest point $q_{j,i} \in \partial\Omega$. Define $U_{j,i} := B(q_{j,i}, r_{j,i}) \cap \partial\Omega$. We set $a_{j,i} := \fint_{U_{j,i}} f(y)\, d\mathcal{H}(y)$. Then for $x \in \Omega$ set

$$F(x) := \sum_{j,i} a_{j,i}\varphi_{j,i}.$$

In subsequent results in this section we will show that $F \in BV(\Omega)$. From the following proposition and Remark 1.6 we obtain the desired bound for $\|DF\|(\Omega)$.

Proposition 2.2. *Given $\Omega \subset X$, there exists $C > 0$ such that for all $f \in B^0_{1,1}(\partial\Omega)$,*

$$\int_\Omega \operatorname{Lip} F\, d\mu \le C\|f\|_{B^0_{1,1}(\partial\Omega)}.$$

Proof. Fix a ball $B_{\ell,m} \in \mathcal{W}_\Omega$, and fix a point $x \in B_{\ell,m}$. For all $y \in B_{\ell,m}$,

$$\begin{aligned}
|F(y) - F(x)| &= \Big|\sum_{j,i} a_{j,i}(\varphi_{j,i}(y) - \varphi_{j,i}(x))\Big| \\
&= \Big|\sum_{j,i}(a_{j,i} - a_{\ell,m})(\varphi_{j,i}(y) - \varphi_{j,i}(x))\Big| \\
&\le \sum_{\substack{j,i \text{ s.t.} \\ 2B_{j,i}\cap B_{\ell,m}\neq\emptyset}} |a_{j,i} - a_{\ell,m}|\frac{C}{r_{j,i}} d(y, x).
\end{aligned}$$

The last inequality in the above sequence follows from the Lipschitz constant of $\varphi_{j,i}$. Rearranging and noting that if the balls intersect then $|j - \ell| \le 1$, we see that

$$\frac{|F(y) - F(x)|}{d(y, x)} \le \frac{C}{r_{\ell,m}} \sum_{\substack{j,i \text{ s.t.} \\ 2B_{j,i}\cap B_{\ell,m}\neq\emptyset}} |a_{j,i} - a_{\ell,m}|.$$

Hence, we want to bound terms of the form $|a_{j,i} - a_{\ell,m}|$:

$$\begin{aligned}
|a_{j,i} - a_{\ell,m}| &= \left|\fint_{U_{j,i}} f(z)\, d\mathcal{H}(z) - \fint_{U_{\ell,m}} f(z)\, d\mathcal{H}(z)\right| \\
&= \left|\fint_{U_{j,i}}\fint_{U_{\ell,m}} \big(f(z) - f(w)\big)\, d\mathcal{H}(w)\, d\mathcal{H}(z)\right|.
\end{aligned}$$

Thus,

$$\begin{aligned}
|a_{j,i} - a_{\ell,m}| &\le \fint_{U_{j,i}} \fint_{U_{\ell,m}} |f(z) - f(w)| \, d\mathcal{H}(w) \, d\mathcal{H}(z) \\
&= \frac{1}{\mathcal{H}(U_{j,i})\mathcal{H}(U_{\ell,m})} \int_{U_{j,i}} \int_{U_{\ell,m}} |f(z) - f(w)| \, d\mathcal{H}(w) \, d\mathcal{H}(z) \\
&\le \frac{C}{\mathcal{H}(U^*_{\ell,m})\mathcal{H}(U^*_{\ell,m})} \int_{U_{j,i}} \int_{U_{\ell,m}} |f(z) - f(w)| \, d\mathcal{H}(w) \, d\mathcal{H}(z) \qquad (2.1) \\
&\le \frac{C}{\mathcal{H}(U^*_{\ell,m})\mathcal{H}(U^*_{\ell,m})} \int_{U^*_{\ell,m}} \int_{U^*_{\ell,m}} |f(z) - f(w)| \, d\mathcal{H}(w) \, d\mathcal{H}(z) \\
&= C \fint_{U^*_{\ell,m}} \fint_{U^*_{\ell,m}} |f(z) - f(w)| \, d\mathcal{H}(w) \, d\mathcal{H}(z),
\end{aligned}$$

where $U^*_{\ell,m}$ denotes the expanded subset of the boundary:

$$U^*_{\ell,m} := B(q_{\ell,m}, 2^6 r_{\ell,m}) \cap \partial\Omega. \qquad (2.2)$$

By the doubling property of X, the boundary regularity condition on $\partial\Omega$, and the definition of codimension-1 Hausdorff measure, we have

$$\mathcal{H}(U^*_{\ell,m}) \le C\mathcal{H}(U_{\ell,m}),$$

which gave inequality (2.1). The above estimates together with the bounded overlap of the Whitney balls yield the following inequality:

$$\begin{aligned}
\operatorname{Lip} F(x) &= \limsup_{y \to x} \frac{|F(y) - F(x)|}{d(y, x)} \\
&\le \frac{C}{r_{\ell,m}} \fint_{U^*_{\ell,m}} \fint_{U^*_{\ell,m}} |f(z) - f(w)| \, d\mathcal{H}(w) \, d\mathcal{H}(z) \qquad (2.3)
\end{aligned}$$

for $x \in B_{\ell,m}$. From (2.3) we see that

$$\begin{aligned}
\int_\Omega \operatorname{Lip} F(x) \, d\mu(x) &\le \sum_{\ell,m} \int_{B_{\ell,m}} \operatorname{Lip} F(x) \, d\mu(x) \\
&\le \sum_{\ell,m} \mu(B_{\ell,m}) \frac{C}{r_{\ell,m}} \fint_{U^*_{\ell,m}} \fint_{U^*_{\ell,m}} |f(z) - f(w)| \, d\mathcal{H}(w) \, d\mathcal{H}(z).
\end{aligned}$$

Using (1.5) now, we get

$$\begin{aligned}\int_\Omega \operatorname{Lip} F(x)\,d\mu(x) &\le C\sum_{\ell,m}\mathcal{H}(U_{\ell,m}) \fint_{U^*_{\ell,m}}\fint_{U^*_{\ell,m}} |f(z)-f(w)|\,d\mathcal{H}(w)\,d\mathcal{H}(z)\\ &\le C\sum_{\ell=-\infty}^{j_0}\sum_m \int_{U^*_{\ell,m}}\fint_{U^*_{\ell,m}} |f(z)-f(w)|\,d\mathcal{H}(w)\,d\mathcal{H}(z)\\ &\le C\sum_{\ell=-\infty}^{j_0}\int_{\partial\Omega}\fint_{B(z,2^{7+\ell})} |f(z)-f(w)|\,d\mathcal{H}(w)\,d\mathcal{H}(z)\,.\end{aligned}$$

Here the last inequality follows from the uniformly bounded overlap of the balls $U^*_{\ell,m}$ for each ℓ. Without loss of generality, we may choose $R = 2^{j_0+7}$ in the definition of the Besov norm (1.3), note that $R \approx \operatorname{diam}(\Omega)$ then. The following estimate (*cf.* the proof of [20, Theorem 5.2]) concludes the proof:

$$\begin{aligned}&\sum_{\ell=-\infty}^{j_0}\int_{\partial\Omega}\fint_{B(z,2^{7+\ell})} |f(z)-f(w)|\,d\mathcal{H}(w)\,d\mathcal{H}(z)\\ &\qquad\approx \int_{t=0}^{2^{j_0+7}}\int_{\partial\Omega}\fint_{B(z,t)} |f(z)-f(w)|\,d\mathcal{H}(w)\,d\mathcal{H}(z)\frac{dt}{t} \qquad (2.4)\\ &\qquad\le C\|f\|_{B^0_{1,1}(\partial\Omega)}\,.\end{aligned}$$

□

We will use the extension constructed in this section in formulating a nonlinear bounded extension from $L^1(\partial\Omega,\mathcal{H})$ to $BV(\Omega)$ in the subsequent sections. There we will need the following estimates for the integral of the gradient and the function on layers of Ω.

Lemma 2.3. *For* $0 \le \rho_1 < \rho_2 < \operatorname{diam}(\Omega)/2$*, set*

$$\Omega(\rho_1,\rho_2) := \{x\in\Omega : \rho_1 \le \operatorname{dist}(x, X\setminus\Omega) < \rho_2\}. \qquad (2.5)$$

Let $\mathcal{J}(\rho_1,\rho_2)$ *be the collection of all* $\ell\in\mathbb{Z}$ *such that there is some* $m\in\mathbb{N}$ *with* $B_{\ell,m}\cap\Omega(\rho_1,\rho_2)$ *non-empty. Then*

$$\int_{\Omega(\rho_1,\rho_2)} \operatorname{Lip} F\,d\mu \le C\sum_{\ell\in\mathcal{J}(\rho_1,\rho_2)}\int_{\partial\Omega}\fint_{B(z,2^{7+\ell})} |f(z)-f(w)|\,d\mathcal{H}(w)\,d\mathcal{H}(z).$$

Proof. For each $\ell \in \mathcal{J}(\rho_1, \rho_2)$ let $\mathcal{I}(\ell)$ denote the collection of all $m \in \mathbb{N}$ for which $B_{\ell,m} \cap \Omega(\rho_1, \rho_2)$ is non-empty. Then by (2.3) and (1.5),

$$\begin{aligned}
\int_{\Omega(\rho_1,\rho_2)} \operatorname{Lip} F \, d\mu &\le \sum_{\ell\in\mathcal{J}(\rho_1,\rho_2)} \sum_{m\in\mathcal{I}(\ell)} \int_{B_{\ell,m}} \operatorname{Lip} F \, d\mu \\
&\le C \sum_{\ell\in\mathcal{J}(\rho_1,\rho_2)} \sum_{m\in\mathcal{I}(\ell)} \frac{\mu(B_{\ell,m})}{r_{\ell,m}} \fint_{U^*_{\ell,m}} \fint_{U^*_{\ell,m}} |f(z) - f(w)| \, d\mathcal{H}(w)\, d\mathcal{H}(z) \\
&\le C \sum_{\ell\in\mathcal{J}(\rho_1,\rho_2)} \sum_{m\in\mathcal{I}(\ell)} \mathcal{H}(U_{\ell,m}) \fint_{U^*_{\ell,m}} \fint_{U^*_{\ell,m}} |f(z) - f(w)| \, d\mathcal{H}(w)\, d\mathcal{H}(z) \\
&= C \sum_{\ell\in\mathcal{J}(\rho_1,\rho_2)} \sum_{m\in\mathcal{I}(\ell)} \int_{U^*_{\ell,m}} \fint_{U^*_{\ell,m}} |f(z) - f(w)| \, d\mathcal{H}(w)\, d\mathcal{H}(z) \\
&\le C \sum_{\ell\in\mathcal{J}(\rho_1,\rho_2)} \sum_{m\in\mathcal{I}(\ell)} \int_{U^*_{\ell,m}} \fint_{B(z,2^{7+\ell})} |f(z) - f(w)| \, d\mathcal{H}(w)\, d\mathcal{H}(z) \\
&\le C \sum_{\ell\in\mathcal{J}(\rho_1,\rho_2)} \int_{\partial\Omega} \fint_{B(z,2^{7+\ell})} |f(z) - f(w)| \, d\mathcal{H}(w)\, d\mathcal{H}(z). \qquad \square
\end{aligned}$$

Corollary 2.4. *Using the notation of Lemma* 2.3*, we have that*

$$\int_{\Omega(\rho_1,\rho_2)} \operatorname{Lip} F \, d\mu \le C\rho_2 \mathcal{H}(\partial\Omega) \operatorname{LIP}(f, \partial\Omega)$$

whenever f is Lipschitz on $\partial\Omega$.

Proof. For a fixed $\ell \in \mathbb{Z}$, we can estimate

$$\begin{aligned}
\int_{\partial\Omega} \fint_{B(z,2^{7+\ell})} &|f(z) - f(w)| \, d\mathcal{H}(w)\, d\mathcal{H}(z) \\
&\le \int_{\partial\Omega} \fint_{B(z,2^{7+\ell})} \operatorname{LIP}(f, \partial\Omega) \mathrm{d}(z, w) \, d\mathcal{H}(w)\, d\mathcal{H}(z) \\
&\le C\mathcal{H}(\partial\Omega) \operatorname{LIP}(f, \partial\Omega)\, 2^{7+\ell}.
\end{aligned}$$

Therefore,

$$\int_{\Omega(\rho_1,\rho_2)} \operatorname{Lip} F \, d\mu \le C\mathcal{H}(\partial\Omega) \operatorname{LIP}(f, \partial\Omega) \sum_{\ell\in\mathcal{J}(\rho_1,\rho_2)} 2^\ell.$$

Every ball $B = B(p, r) \in \mathcal{I}(\ell)$ satisfies $2^{\ell-1} < r \le 2^\ell$ and $\operatorname{dist}(p, X \setminus \Omega) = 8r$. There is $C \ge 1$ such that $C^{-1}\rho_1 \le 2^\ell \le C\rho_2$ whenever $\ell \in \mathcal{J}(\rho_1, \rho_2)$. Thus, $\sum_{\ell\in\mathcal{J}(\rho_1,\rho_2)} 2^\ell \le C\rho_2$. $\square$

We next turn our attention to the L^1-estimates for F.

Lemma 2.5. *There exists $C > 0$ such that*

$$\int_\Omega |F|\, d\mu \le C \operatorname{diam}(\Omega) \|f\|_{L^1(\partial\Omega)}.$$

Proof. We first consider a fixed ball $B_{\ell,m}$ from the Whitney cover. Then

$$\begin{aligned}
\int_{B_{\ell,m}} |F(x)|\, d\mu(x) &= \int_{B_{\ell,m}} \left| \sum_{j,i} \fint_{U_{j,i}} f(y)\, d\mathcal{H}(y) \varphi_{j,i}(x) \right| d\mu(x) \\
&\le \int_{B_{\ell,m}} \sum_{j,i} \left| \fint_{U_{j,i}} f(y)\, d\mathcal{H}(y) \right| \varphi_{j,i}(x)\, d\mu(x) \\
&= \int_{B_{\ell,m}} \sum_{\substack{j,i \text{ s.t.} \\ 2B_{j,i} \cap B_{\ell,m} \neq \emptyset}} \left| \fint_{U_{j,i}} f(y)\, d\mathcal{H}(y) \right| \varphi_{j,i}(x)\, d\mu(x).
\end{aligned}$$

Recall that if $2B_{j,i} \cap B_{\ell,m} \neq \emptyset$, then $|j-\ell| \le 1$, so $\mathcal{H}(U_{j,i}) \approx \mathcal{H}(U^*_{\ell,m})$. Also, for $U^*_{\ell,m}$ as defined in equation (2.2), $U_{j,i} \subset U^*_{\ell,m}$. Furthermore, by the construction of the Whitney decomposition, each point is in a fixed number of dilated Whitney balls $2B_{j,i}$. Hence,

$$\begin{aligned}
&\int_{B_{\ell,m}} \sum_{\substack{j,i \text{ s.t.} \\ 2B_{j,i} \cap B_{\ell,m} \neq \emptyset}} \left| \fint_{U_{j,i}} f(y)\, d\mathcal{H}(y) \right| \varphi_{j,i}(x)\, d\mu(x) \\
&\qquad \le C \int_{B_{\ell,m}} \fint_{U^*_{\ell,m}} |f(y)|\, d\mathcal{H}(y)\, d\mu(x) \le C \mu(B_{\ell,m}) \fint_{U^*_{\ell,m}} |f(y)|\, d\mathcal{H}(y).
\end{aligned}$$

In view of (1.5), we obtain that

$$\int_{B_{\ell,m}} |F(x)|\, d\mu(x) \le C\, r_{\ell,m} \int_{U^*_{\ell,m}} |f(y)|\, d\mathcal{H}(y). \tag{2.6}$$

Summing up and noting that $\Omega = \bigcup_{\ell,m} B_{\ell,m}$, we have

$$\begin{aligned}
\int_\Omega |F|\, d\mu &\le C \sum_{\ell=-\infty}^{j_0} \sum_m r_{\ell,m} \int_{U^*_{\ell,m}} |f|\, d\mathcal{H} \le C \sum_{\ell=-\infty}^{j_0} 2^\ell \sum_m \int_{U^*_{\ell,m}} |f|\, d\mathcal{H} \\
&\le C \sum_{\ell=-\infty}^{j_0} 2^\ell \int_{\partial\Omega} |f|\, d\mathcal{H} \le C \operatorname{diam}(\Omega) \|f\|_{L^1(\partial\Omega)}.
\end{aligned}$$

□

We now aim to obtain an analogous of Lemma 2.3 for the L^1-norm of F on the layer $\Omega(\rho_1, \rho_2)$.

Lemma 2.6. *Let $z \in \partial\Omega$ and $r \in (0, \operatorname{diam}(\Omega)/2)$. Then,*

$$\int_{B(z,r)\cap\Omega(\rho_1,\rho_2)} |F|\,d\mu \le C\, \min\{r, \rho_2\} \int_{B(z,2^8 r)\cap\partial\Omega} |f|\,d\mathcal{H}$$

whenever $0 \le \rho_1 < \min\{r, \rho_2\}$ *and* $\rho_2 < \operatorname{diam}(\Omega)/2$.

Proof. Since $B(z, r) \cap \Omega(\rho_1, \rho_2) = B(z, r) \cap \Omega(\rho_1, \min\{r, \rho_2\})$, we do not lose any generality by assuming that $\rho_2 \le r$. Similarly as in the proof of Lemma 2.3, we set $\mathcal{J}'(\rho_1, \rho_2)$ to be the collection of all ℓ for which there is some m such that $B_{\ell,m} \cap \Omega(\rho_1, \rho_2) \cap B(z, r)$ is non-empty, and for each $\ell \in \mathcal{J}'(\rho_1, \rho_2)$ we set $\mathcal{I}'(\ell)$ to be the collection of all $m \in \mathbb{N}$ for which $B_{\ell,m} \cap \Omega(\rho_1, \rho_2) \cap B(z, r)$ is non-empty. Then by (2.6),

$$\begin{aligned}\int_{B(z,r)\cap\Omega(\rho_1,\rho_2)} |F|\,d\mu &\le \sum_{\ell\in\mathcal{J}'(\rho_1,\rho_2)} \sum_{m\in\mathcal{I}'(\ell)} \int_{B_{\ell,m}} |F|\,d\mu \\ &\le C \sum_{\ell\in\mathcal{J}'(\rho_1,\rho_2)} \sum_{m\in\mathcal{I}'(\ell)} r_{\ell,m} \int_{U^*_{\ell,m}} |f|\,d\mathcal{H}.\end{aligned}$$

The triangle inequality yields that

$$d(z, q_{\ell,m}) \le d(z, p_{\ell,m}) + d(p_{\ell,m}, q_{\ell,m}) \le 2d(z, p_{\ell,m}) \le 2(r + r_{\ell,m}),$$

where $B_{\ell,m} = B(p_{\ell,m}, r_{\ell,m})$ and $U_{\ell,m} = B(q_{\ell,m}, r_{\ell,m}) \cap \partial\Omega$ with $q_{\ell,m} \in \partial\Omega$ being a boundary point lying closest to $p_{\ell,m}$. Moreover,

$$8r_{\ell,m} = \operatorname{dist}(p_{\ell,m}, X \setminus \Omega) \le d(p_{\ell,m}, z) \le r + r_{\ell,m}.$$

hence, $r_{\ell,m} \le \frac{1}{7}r$. Consequently, $d(z, q_{\ell,m}) \le \frac{16}{7}r$ and $U_{\ell,m} \subset B(z, (\frac{16}{7} + \frac{1}{7})r)$. Thus, $U^*_{\ell,m} \subset B(z, 2^8 r)$ and

$$\begin{aligned}\int_{B(z,r)\cap\Omega(\rho_1,\rho_2)} |F|\,d\mu &\le C \sum_{\ell\in\mathcal{J}'(\rho_1,\rho_2)} 2^\ell \int_{B(z,2^8 r)\cap\partial\Omega} |f|\,d\mathcal{H} \\ &\le C\rho_2 \int_{B(z,2^8 r)\cap\partial\Omega} |f|\,d\mathcal{H},\end{aligned}$$

where the last inequality can be verified as follows: every ball $B = B(p, r) \in \mathcal{I}'(\ell)$ satisfies $2^{\ell-1} < r \le 2^\ell$ and $\operatorname{dist}(p, X \setminus \Omega) = 8r$. There is $C \ge 1$ such that $C^{-1}\rho_1 \le 2^\ell \le C\rho_2$ whenever $\ell \in \mathcal{J}'(\rho_1, \rho_2)$. Thus, $\sum_{\ell\in\mathcal{J}'(\rho_1,\rho_2)} 2^\ell \le C\rho_2$. □

By covering $\partial\Omega$ by balls of radii r, whose overlap is bounded, we obtain the following corollary.

Corollary 2.7. *With the notation of Lemma* 2.3, *we have*

$$\int_{\Omega(\rho_1,\rho_2)} |F|\, d\mu \le C\, \rho_2 \int_{\partial\Omega} |f|\, d\mathcal{H}.$$

2.3. Trace of extension is the identity mapping

From the above lemma we know that given a function $f \in B^0_{1,1}(\partial\Omega)$ the corresponding function F is in the class $N^{1,1}(\Omega) \subset BV(\Omega)$, where $N^{1,1}(\Omega)$ is a Newtonian class introduced in [36]. The mapping $f \mapsto F$ is denoted by the operator $E : B^0_{1,1}(\partial\Omega) \to BV(\Omega)$. This operator is bounded by Proposition 2.2 and Lemma 2.5, and it is linear by construction.

We now wish to show that the trace of F returns the original function f, *i.e.*, $T \circ E$ is the identity function on $B^0_{1,1}(\partial\Omega)$. It was shown in [30] that if Ω satisfies our standing assumptions, then for each $u \in BV(\Omega)$ and for $\mathcal{H}$-a.e. $z \in \partial\Omega$ there is a number $Tu(z) \in \mathbb{R}$ such that

$$\limsup_{r\to 0^+} \fint_{B(z,r)\cap\Omega} |u(y) - Tu(z)|\, d\mu(y) = 0. \tag{2.7}$$

Strictly speaking, [30, Theorem 3.4] asks for Ω to satisfy the measure density condition (1.1), but the proof of existence of trace given there remains valid even with the weaker condition of local measure density of Ω as assumed by us. The map $u \mapsto Tu$ is called the *trace* of $BV(\Omega)$. Moreover, if Ω supports a 1-Poincaré inequality and satisfies the (global) measure density condition (1.1), then $Tu \in L^1(\partial\Omega)$ for $u \in BV(\Omega)$.

Note also that $B^0_{1,1}(\partial\Omega) \subset L^1(\partial\Omega)$ and the inclusion is strict in general, which is shown in Example 3.9 below. Further properties of the Besov classes are explored in Section 3.

For the sake of clarity, let us explicitly point out that the following lemma shows that the BV extension of a function of the Besov class $B^0_{1,1}(\partial\Omega)$, as constructed above, has a well-defined trace even though no Poincaré inequality for Ω or for X is assumed.

Lemma 2.8. *For f and F as above, and for $\mathcal{H}$-a.e. $z \in \partial\Omega$,*

$$\lim_{r\to 0^+} \fint_{B(z,r)\cap\Omega} |F(x) - f(z)|\, d\mu(x) = 0.$$

That is, $TEf(z)$ exists for $\mathcal{H}$-a.e. $z \in \partial\Omega$.

Proof. Since $f \in B^0_{1,1}(\partial\Omega) \subset L^1(\partial\Omega)$, we know by the doubling property of $\mathcal{H}|_{\partial\Omega}$ that $\mathcal{H}$-a.e. $z \in \partial\Omega$ is a Lebesgue point of f, and by the standard assumption on Ω we know that at $\mathcal{H}$-a.e. $z \in \partial\Omega$ the measure density condition (1.6) holds. Let z be such a point. Since $\sum_{j,i}\varphi_{j,i} = \chi_\Omega$, we have

$$\begin{aligned}
\fint_{B(z,r)\cap\Omega} |F - f(z)|\, d\mu &= \fint_{B(z,r)\cap\Omega} \left|\sum_{j,i}\left(\fint_{U_{j,i}} f\, d\mathcal{H}\right)\varphi_{j,i}(x) - f(z)\right| d\mu(x) \\
&= \fint_{B(z,r)\cap\Omega} \left|\sum_{j,i}\left(\fint_{U_{j,i}} (f - f(z))\, d\mathcal{H}\right)\varphi_{j,i}(x)\right| d\mu(x) \\
&\le \fint_{B(z,r)\cap\Omega} \sum_{j,i}\left(\fint_{U_{j,i}} |f - f(z)|\, d\mathcal{H}\right)\varphi_{j,i}(x) d\mu(x).
\end{aligned}$$

By the properties of the Whitney covering $\mathcal{W}_\Omega$,

$$\begin{aligned}
&\int_{B(z,r)\cap\Omega} \sum_{j,i}\left(\fint_{U_{j,i}} |f - f(z)|\, d\mathcal{H}\right)\varphi_{j,i}(x)\, d\mu(x) \\
&\quad\le \sum_{\substack{\ell,m \text{ s.t.} \\ B_{\ell,m}\cap B(z,r)\neq\emptyset}} \int_{B_{\ell,m}} \sum_{j,i}\left(\fint_{U_{j,i}} |f - f(z)|\, d\mathcal{H}\right)\varphi_{j,i}(x)\, d\mu(x) \\
&\quad\le \sum_{\substack{\ell,m \text{ s.t.} \\ B_{\ell,m}\cap B(z,r)\neq\emptyset}} \int_{B_{\ell,m}} \sum_{\substack{j,i \text{ s.t.} \\ 2B_{j,i}\cap B_{\ell,m}\neq\emptyset}} \left(\fint_{U_{j,i}} |f - f(z)|\, d\mathcal{H}\right) d\mu(x).
\end{aligned}$$

If $2B_{j,i} \cap B_{\ell,m}$ is non-empty, then $U_{j,i} \subset U^*_{\ell,m}$ and $\mathcal{H}(U_{j,i}) \approx \mathcal{H}(U^*_{\ell,m})$. Therefore by (1.5) we have

$$\begin{aligned}
&\int_{B(z,r)\cap\Omega} \sum_{j,i}\left(\fint_{U_{j,i}} |f - f(z)|\, d\mathcal{H}\right)\varphi_{j,i}(x)\, d\mu(x) \\
&\quad\le C \sum_{\substack{\ell,m \text{ s.t.} \\ B_{\ell,m}\cap B(z,r)\neq\emptyset}} \left(\fint_{U^*_{\ell,m}} |f - f(z)|\, d\mathcal{H}\right)\mu(B_{\ell,m}) \\
&\quad\le C \sum_{\substack{\ell,m \text{ s.t.} \\ B_{\ell,m}\cap B(z,r)\neq\emptyset}} r_{\ell,m} \int_{U^*_{\ell,m}} |f - f(z)|\, d\mathcal{H}.
\end{aligned}$$

Let $\mathcal{J}(B(z,r))$ denote the collection of all $\ell \in \mathbb{Z}$ for which there is some $m \in \mathbb{N}$ such that $B_{\ell,m} \cap B(z,r)$ is non-empty. For each $\ell \in \mathcal{J}(B(z,r))$, set $\mathcal{I}(\ell)$ to be the collection of all $m \in \mathbb{N}$ for which $B_{\ell,m} \cap B(z,r)$ is non-empty. Then,

$$\begin{aligned}\int_{B(z,r)\cap\Omega} \sum_{j,i} \left(\fint_{U_{j,i}} |f - f(z)|\, d\mathcal{H} \right) \varphi_{j,i}(x)\, d\mu(x) \\ &\le C \sum_{\ell\in\mathcal{J}(B(z,r))} 2^{\ell} \sum_{m\in\mathcal{I}(\ell)} \int_{U^*_{\ell,m}} |f - f(z)|\, d\mathcal{H} \\ &\le C \sum_{\ell\in\mathcal{J}(B(z,r))} 2^{\ell} \int_{B(z,2^7 r)\cap\partial\Omega} |f - f(z)|\, d\mathcal{H} \\ &\le C\, r \int_{B(z,2^7 r)\cap\partial\Omega} |f - f(z)|\, d\mathcal{H}.\end{aligned}$$

In the above, we used the fact that $\sum_{\ell\in\mathcal{J}(B(z,r))} 2^{\ell} \approx r$, since only the indices $\ell \in \mathbb{Z}$ for which $2^{\ell} \approx \operatorname{dist}(B_{\ell,m}, X \setminus \Omega) \le r$ are allowed to be in $\mathcal{J}(B(z,r))$. From the fact that z is a Lebesgue point of f, we now have

$$\begin{aligned}\fint_{B(z,r)\cap\Omega} |F - f(z)|\, d\mu &\le C \frac{r}{\mu(B(z,r)\cap\Omega)} \int_{B(z,2^7 r)\cap\partial\Omega} |f - f(z)|\, d\mathcal{H} \\ &\le C \fint_{B(z,2^7 r)\cap\partial\Omega} |f - f(z)|\, d\mathcal{H} \to 0 \text{ as } r \to 0^+.\end{aligned}$$

In the last inequality we used both the local measure density property (1.6) and the codimension 1 Ahlfors regularity (1.5). In particular, the constant C in the last inequality depends on C_z from (1.6), but the key here is that it does *not* depend on r as long as r is small enough (which is what we need in order to get estimates as $r \to 0$). This completes the proof of the lemma. □

3. Comparison of $B^0_{1,1}(\partial\Omega)$ and other function spaces

We now wish to show that $B^0_{1,1}(\partial\Omega)$ has more interesting functions than mere constant functions. What functions are in $B^0_{1,1}(\partial\Omega)$? Since the results of this section deal with function spaces based on more general doubling metric measure spaces, we consider the underlying metric measure space (Z, d, ν). The other function spaces include L^1, BV, and the fractional John–Nirenberg spaces as well as the class of Lipschitz functions.

Let $Z = (Z, \mathrm{d}, \nu)$ be a metric space endowed with a doubling measure. In applications in this paper, Z will be $\partial\Omega \subset X$ and ν will be the Hausdorff codimension 1 measure $\mathcal{H}|_{\partial\Omega}$.

3.1. Preliminary results

Lemma 3.1. *For every $\tau > 3$, there is $C = C(C_D, \tau) \ge 1$ such that for every $r > 0$ there is an at most countable set of points $\{x_j\}_j \subset Z$ (alternatively, $\{x_j\}_j \subset \Omega$,* *where $\Omega \subset Z$ is arbitrary) such that*

- $B(x_j, r) \cap B(x_k, r) = \emptyset$ *whenever* $j \neq k$*;*
- $Z = \bigcup_j B(x_j, \tau r)$ *(alternatively,* $\Omega \subset \bigcup_j B(x_j, \tau r)$*);*
- $\sum_j \chi_{B(x_j,\tau r)} \le C$.

The above lemma is widely known to experts in the field, but we were unable to find it in current literature; hence we provide a sketch of its proof.

Proof. An application of Zorn's lemma or [25, Lemma 4.1.12] gives a countable set $A \subset Z$ such that for distinct points $x, y \in A$ we have $\mathrm{d}(x, y) \ge r$, and for each $z \in Z$ there is some $x \in A$ such that $\mathrm{d}(z, x) < r$. The countable collection $\{B(x, r) : x \in A\}$ can be seen to satisfy the requirements set forth in the lemma because of the doubling property of ν. □

Lemma 3.2. *Let $f \in L^1(Z)$. Then,*

$$\int_0^r \int_Z \fint_{B(x,t)} |f(y) - f(x)|\, d\nu(y)\, d\nu(x) \frac{dt}{t^{1+\theta}} < \infty$$

if and only if

$$\int_0^R \int_Z \fint_{B(x,t)} |f(y) - f(x)|\, d\nu(y)\, d\nu(x) \frac{dt}{t^{1+\theta}} < \infty\,,$$

where $0 < r < R < \infty$. If $\theta > 0$, then the equivalence holds true even for $R = \infty$.

Proof. By the triangle inequality, we obtain for $t > 0$ that

$$\begin{aligned}\int_Z \fint_{B(x,t)} |f(y)-f(x)|\, d\nu(y)\, d\nu(x) &\le \int_Z \fint_{B(x,t)} \big(|f(y)| + |f(x)|\big)\, d\nu(y)\, d\nu(x) \\ &= \int_Z \left(|f(x)| + \fint_{B(x,t)} |f(y)|\, d\nu(y)\right) d\nu(x) \\ &= \|f\|_{L^1(Z)} + \int_Z \fint_{B(x,t)} |f(y)|\, d\nu(y)\, d\nu(x).\end{aligned}$$

The Fubini theorem and the doubling condition then yield

$$\begin{aligned}\int_Z \fint_{B(x,t)} |f(y)|\, d\nu(y)\, d\nu(x) &\approx \int_{Z\times Z} \frac{|f(y)|\chi_{(0,t)}(\mathrm{d}(x, y))}{\nu(B(y, t))}\, d(\nu \times \nu)(x, y) \\ &= \int_Z |f(y)| \fint_{B(y,t)} d\nu(x)\, d\nu(y) = \|f\|_{L^1(Z)}\,.\end{aligned}$$

For $r > 0$ set

$$I(r) := \int_0^r \int_Z \fint_{B(x,t)} |f(y) - f(x)|\, d\nu(y)\, d\nu(x) \frac{dt}{t^{1+\theta}}.$$

Then,

$$I(R) = I(r) + \int_r^R \int_Z \fint_{B(x,t)} |f(y) - f(x)|\, d\nu(y)\, d\nu(x) \frac{dt}{t^{1+\theta}}$$
$$\le I(r) + C \int_r^R \|f\|_{L^1(Z)} \frac{dt}{t^{1+\theta}} = I(r) + C\|f\|_{L^1(Z)} \Big(\frac{1}{r^\theta} - \frac{1}{R^\theta} \Big),$$

with obvious modification for $\theta = 0$. □

Lemma 3.3. *Let* $R \le 2\,\mathrm{diam}(Z)$ *and* $\theta \in [0, 1]$. *Then,*

$$I(R) \approx \int_Z \int_{B(x,R)} \frac{|f(y) - f(x)|}{\nu(B(x, \mathrm{d}(x, y)))\mathrm{d}(x, y)^\theta}\, d\nu(y)\, d\nu(x).$$

Proof. The equivalence follows from the Fubini theorem, see also [20, Theorem 5.2]. □

Lemma 3.4. *There is a constant* $C \ge 1$ *and there are collections of balls* $\mathcal{B}^k$, $k = 0, 1, \ldots$, *such that*

$$C^{-1} I(1) \le \sum_{k=0}^\infty \sum_{B \in \mathcal{B}^k} \frac{1}{\mathrm{rad}(B)^\theta} \int_B |f - f_B|\, d\nu \le C I(4).$$

Moreover, $\mathrm{rad}(B) \approx 2^{-k}$ *whenever* $B \in \mathcal{B}^k$, *and the balls within each collection* $\mathcal{B}^k$ *have bounded overlap (also after inflation by a given factor* $\tau \ge 1$*).*

Proof. By Lemma 3.1, there is a constant $C = C(C_D, \tau) \ge 1$ and collections of balls $\widetilde{\mathcal{B}}^k$, $k = 0, 1, \ldots$, such that $\mathrm{rad}(B) = 2^{-k}$ for every $B \in \widetilde{\mathcal{B}}^k$ and, in addition, $1 \le \sum_{B \in \widetilde{\mathcal{B}}^k} \chi_{2\tau B}(x) \le C$ for all $x \in Z$. Then, by (2.4),

$$I(1) \le \sum_{k=0}^\infty 2^{k\theta} \sum_{B \in \widetilde{\mathcal{B}}^k} \int_B \fint_{B(x,2^{-k})} |f(y) - f(x)|\, d\nu(y)\, d\nu(x)$$
$$\le C \sum_{k=0}^\infty 2^{k\theta} \sum_{B \in \widetilde{\mathcal{B}}^k} \int_B \fint_{2B} |f(y) - f(x)|\, d\nu(y)\, d\nu(x)$$
$$\le C \sum_{k=0}^\infty \sum_{B \in \widetilde{\mathcal{B}}^k} \frac{1}{(2^{-k})^\theta} \int_{2B} \fint_{2B} |f(y) - f(x)|\, d\nu(y)\, d\nu(x)$$
$$\le C \sum_{k=0}^\infty \sum_{B \in \widetilde{\mathcal{B}}^k} \frac{1}{\mathrm{rad}(B)^\theta} \int_{2B} |f - f_{2B}|\, d\nu.$$

Thus, we choose $\mathcal{B}^k = \{2B : B \in \widetilde{\mathcal{B}}^k\}$, $k = 0, 1, \ldots$, to conclude the proof of the first inequality.

The proof of the second inequality follows analogous steps backwards. Recall that $\operatorname{rad}(B) = 2^{1-k}$ whenever $B \in \mathcal{B}^k$. Thus,

$$\begin{aligned}\sum_{k=0}^{\infty} \sum_{B\in\mathcal{B}^k} \frac{1}{\operatorname{rad}(B)^\theta} \int_B |f - f_B|\, d\nu \\ &\approx \sum_{k=0}^{\infty} \sum_{B\in\mathcal{B}^k} 2^{k\theta} \int_B \fint_B |f(y) - f(x)|\, d\nu(y)\, d\nu(x) \\ &\le C \sum_{k=0}^{\infty} 2^{k\theta} \sum_{B\in\mathcal{B}^k} \int_B \fint_{B(x,2^{2-k})} |f(y) - f(x)|\, d\nu(y)\, d\nu(x) \\ &\le C \sum_{k=0}^{\infty} 2^{k\theta} \int_Z \fint_{B(x,2^{2-k})} |f(y) - f(x)|\, d\nu(y)\, d\nu(x),\end{aligned}$$

where we used the fact that the balls have uniformly bounded overlap within each collection $\mathcal{B}^k$. □

Remark 3.5. Let $0 \le \theta < \eta \le 1$. Then, $\|u\|_{B^\theta_{1,1}} \le C(1 + R^{\eta-\theta})\|u\|_{B^\eta_{1,1}}$ and $\|u\|_{A^\theta_{1,\tau}} \le C(1 + R^{\eta-\theta})\|u\|_{A^\eta_{1,\tau}}$ for any $\tau \ge 1$.

3.2. Comparison of function spaces with $B^\theta_{1,1}(Z)$

Proposition 3.6. *Let $\theta \in [0, 1]$ and $\tau \ge 1$ be arbitrary. Then there is a constant $C \ge 1$, which depends on θ and τ, such that*

$$\|u\|_{A^\theta_{1,\tau}} \le C\|u\|_{B^\theta_{1,1}}.$$

Proof. Let $\mathcal{B}_\tau$ be a fixed collection of non-overlapping balls in Z of radius at most R/τ. Then,

$$\begin{aligned}\sum_{B\in\mathcal{B}_\tau} \frac{1}{\operatorname{rad}(B)^\theta} \int_{\tau B} |u - u_{\tau B}|\, d\nu \\ &\approx \sum_{B\in\mathcal{B}_\tau} \frac{1}{(\tau \operatorname{rad}(B))^\theta} \int_{\tau B} \fint_{\tau B} |u(x) - u(y)|\, d\nu(y)\, d\nu(x) \\ &= \int_Z \sum_{B\in\mathcal{B}_\tau} \frac{\chi_{\tau B}(x)}{(\tau \operatorname{rad}(B))^\theta} \fint_{\tau B} |u(x) - u(y)|\, d\nu(y)\, d\nu(x).\end{aligned}$$

Therefore

$$\sum_{B\in\mathcal{B}_\tau} \frac{1}{\operatorname{rad}(B)^\theta} \int_{\tau B} |u - u_{\tau B}|\, d\nu$$
$$\leq \int_Z \sum_{B\in\mathcal{B}_\tau} \chi_{\tau B}(x) \int_{\tau B} \frac{|u(x)-u(y)|}{\nu(B(x,\mathrm{d}(x,y)))\mathrm{d}(x,y)^\theta}\, d\nu(y)\, d\nu(x)$$
$$\leq \int_Z \int_{B(x,2R)} \frac{|u(x)-u(y)|}{\nu(B(x,\mathrm{d}(x,y)))\mathrm{d}(x,y)^\theta}\, d\nu(y)\, d\nu(x) \approx I(2R),$$

where we used that ν is doubling and $B(x, \mathrm{d}(x, y)) \subset 3\tau B$ for all $x, y \in \tau B$. Taking supremum over all collections of balls concludes the proof. □

Proposition 3.7. *Assume that Z is bounded. Let $0 \leq \theta < \eta \leq 1$ and $\tau \geq 1$. Then, $A^\eta_{1,\tau}(Z) \subset B^\theta_{1,1}(Z)$.*

Proof. We use the characterization of Besov functions from Lemma 3.4.

$$\sum_{k=0}^\infty \sum_{B\in\mathcal{B}^k} \frac{1}{\operatorname{rad}(B)^\theta} \int_B |f - f_B|\, d\nu \leq C \sum_{k=0}^\infty \sum_{B\in\mathcal{B}^k} \frac{2^{k(\theta-\eta)}}{\operatorname{rad}(B)^\eta} \int_B |f - f_B|\, d\nu$$
$$\leq C \sum_{k=0}^\infty 2^{k(\theta-\eta)} \|f\|_{A^\eta_{1,\tau}(Z)}.$$

□

Lemma 3.8. *For every $\tau \geq 1$, we have $L^1(Z) = A^0_{1,\tau}(Z)$.*

Proof. Let $\mathcal{B}_\tau$ be a fixed collection of balls in Z that remain pairwise disjoint after being inflated τ-times. Then, the triangle inequality yields that

$$\sum_{B\in\mathcal{B}_\tau} \frac{1}{\operatorname{rad}(B)^0} \int_{\tau B} |u - u_{\tau B}|\, d\nu \approx \sum_{B\in\mathcal{B}_\tau} \left(\int_{\tau B} |u| + \left| \fint_{\tau B} u\, d\nu \right| \right) d\nu$$
$$= \sum_{B\in\mathcal{B}_\tau} 2 \int_{\tau B} |u|\, d\nu$$
$$\leq 2 \int_Z |u|\, d\nu\,.$$

Hence, $\|u\|_{A^0_{1,\tau}(Z)} \leq 3\|u\|_{L^1(Z)}$.

Conversely, $\|u\|_{A^0_{1,\tau}(Z)} \geq \|u\|_{L^1(Z)}$ by definition (1.4). □

The following example shows that the inclusion $B^0_{1,1}(Z) \subset A^0_{1,\tau}(Z)$ may, in general, be strict.

Example 3.9. Let

$$f(x) = \sum_{j=1}^{\infty} \chi_{[1/(j+1),1/j)}(x)u(4^j x), \quad x \in (0,1),$$

where u is the 1-periodic extension of $\chi_{[0,1/2)}$. Obviously, $f \in L^\infty(0,1)$. Hence, $f \in L^1(0,1) = A^0_{1,\tau}(0,1)$. On the other hand, $u \notin B^0_{1,1}(0,1)$, which we are about to show.

We will use the characterization of $B^0_{1,1}(0,1)$ from Lemma 3.4. There, we may choose $\mathcal{B}^k = \{(l2^{-k}, (l+2)2^{-k}) : l = 0, 1, \dots, 2^k - 2\}$ to get $\fint_B |f - f_B| \approx 1$ whenever $B \subset (0, 1/j)$ and $B \in \mathcal{B}^k$ for some $k \le j$. Then,

$$\sum_{k=0}^{\infty} \sum_{B \in \mathcal{B}^k} \int_B |f - f_B| \ge C^{-1} \sum_{k=0}^{\infty} \sum_{\substack{B \in \mathcal{B}^k \\ B \subset (0,k^{-1})}} |B| \approx \sum_{k=0}^{\infty} \frac{1}{k} = \infty.$$

Next, we will provide a family of examples that show that $BV(Z)$ is, in general, a strictly smaller space than $B^\theta_{1,1}(Z)$ for every $\theta \in [0,1)$.

Example 3.10. Let $\alpha \in (\theta, 1)$. Then, the Weierstrass function

$$u_\alpha(x) = \sum_{k=1}^{\infty} \frac{\cos(2^k \pi x)}{2^{k\alpha}}, \quad x \in [0,1],$$

is α-Hölder continuous but nowhere differentiable in [0, 1] by Hardy [23]. Hence, $u_\alpha \notin BV[0,1]$ as it would have been differentiable a.e. otherwise. Since $\alpha > \theta$, we have $\mathcal{C}^{0,\alpha}[0,1] \subset B^\theta_{1,1}[0,1]$ by [20, Lemma 6.2].

In conclusion, we have now proved the following theorem.

Theorem 3.11. *Let $\tau \ge 1$ and $\theta \in (0,1]$ be arbitrary. Then,*

$$L^1(Z) = A^0_{1,\tau}(Z) \supset B^0_{1,1}(Z) \supset A^\theta_{1,\tau}(Z) \supset A^1_{1,\tau}(Z) \subset BV(Z),$$

where all but the last of the inclusions are strict in general. Furthermore, Lipschitz functions on Z belong to $B^0_{1,1}(Z)$.

We know from [24, Theorem 1.1] that $A^1_{1,\tau}(Z) \subset BV(Z)$. Note however that $A^1_{1,\tau}(Z) = BV(Z)$ holds by [24, Corollary 1.3] whenever Z supports a 1-Poincaré inequality.

4. Extension theorem for L^1 boundary data: proof of Theorem 1.2

Given an L^1-function on $\partial\Omega$, we will construct its BV extension in Ω using the linear extension operator for $B^0_{1,1}(\partial\Omega)$ boundary data. Observe however that the mapping $f \in L^1(\partial\Omega) \mapsto F \in BV(\Omega)$ will be nonlinear, which is not surprising in view of [34].

Instead of constructing the extension using a Whitney decomposition of Ω, we will set up a sequence of layers inside Ω whose widths depend not only on their distance from $X \setminus \Omega$, but also on the function itself (more accurately, on the choice of the sequence of Lipschitz approximations of the function in L^1-class). Using a partition of unity subordinate to these layers, we will glue together BV extensions (from Theorem 1.1) of Lipschitz functions on $\partial\Omega$ that approximate the boundary data in $L^1(\partial\Omega)$. Roughly speaking, the closer the layer lies to $X \setminus \Omega$, the better we need the approximating Lipschitz data to be. The core idea of such a construction can be traced back to Gagliardo [15] who discussed extending $L^1(\mathbb{R}^{n-1})$ functions to $W^{1,1}(\mathbb{R}^n_+)$.

First, we approximate f in $L^1(\partial\Omega)$ by a sequence of Lipschitz continuous functions $\{f_k\}_{k=1}^\infty$ such that $\|f_{k+1} - f_k\|_{L^1(\partial\Omega)} \le 2^{2-k}\|f\|_{L^1(\partial\Omega)}$. Note that this requirement of rate of convergence of f_k to f also ensures that $f_k \to f$ pointwise $\mathcal{H}$-a.e. in $\partial\Omega$. For technical reasons, we choose $f_1 \equiv 0$.

Next, we choose a decreasing sequence of real numbers $\{\rho_k\}_{k=1}^\infty$ such that:

- $\rho_1 \le \operatorname{diam}(\Omega)/2$;
- $0 < \rho_{k+1} \le \rho_k/2$;
- $\sum_k \rho_k \operatorname{LIP}(f_{k+1}, \partial\Omega) \le C\|f\|_{L^1(\partial\Omega)}$.

These will now be used to define layers in Ω. Let

$$\psi_k(x) = \max\left\{0, \min\left\{1, \frac{\rho_k - \operatorname{dist}(x, X \setminus \Omega)}{\rho_k - \rho_{k+1}}\right\}\right\}, \quad x \in \Omega.$$

Then, the sequence of functions $\{\psi_{k-1} - \psi_k : k = 2, 3, \ldots\}$ serves as a partition of unity in $\Omega(0, \rho_2)$ subordinate to the system of layers given by $\{\Omega(\rho_{k+1}, \rho_{k-1}) : k = 2, 3, \ldots\}$.

Recall that Lipschitz continuous functions lie in the Besov class $B^0_{1,1}$. Thus, we can apply the linear extension operator $E : B^0_{1,1}(\partial\Omega) \to BV(\Omega)$, whose properties were established in Section 2, to define the extension of $f \in L^1(\partial\Omega)$ by extending its Lipschitz approximations in layers, *i.e.*,

$$\begin{aligned} F(x) &:= \sum_{k=2}^\infty \big(\psi_{k-1}(x) - \psi_k(x)\big) Ef_k(x) \\ &= \sum_{k=1}^\infty \psi_k(x)\big(Ef_{k+1}(x) - Ef_k(x)\big), \quad x \in \Omega. \end{aligned} \tag{4.1}$$

The following result shows that the above extension is in the class $BV(\Omega)$ with appropriate norm bounds (see Remark 1.6). Indeed, we will see that the extension given below lies in $N^{1,1}(\Omega) \subset BV(\Omega)$.

Proposition 4.1. *Given $f \in L^1(\partial\Omega)$, the extension defined by* (4.1) *satisfies*

$$\|F\|_{L^1(\Omega)} \le C \operatorname{diam}(\Omega) \|f\|_{L^1(\partial\Omega)}$$

and

$$\|\operatorname{Lip} F\|_{L^1(\Omega)} \le C(1 + \mathcal{H}(\partial\Omega)) \|f\|_{L^1(\partial\Omega)}.$$

Proof. Corollary 2.7 allows us to obtain the desired L^1 estimate for F. Since the extension on $B^0_{1,1}(\partial\Omega)$ is linear, we have that $Ef_{k+1} - Ef_k = E(f_{k+1} - f_k)$. Therefore,

$$\begin{aligned}
\|F\|_{L^1(\Omega)} &\le \sum_{k=1}^{\infty} \|\psi_k E(f_{k+1} - f_k)\|_{L^1(\Omega)} \\
&\le \sum_{k=1}^{\infty} \|E(f_{k+1} - f_k)\|_{L^1(\Omega(0,\rho_k))} \\
&\le C \sum_{k=1}^{\infty} \rho_k \|f_{k+1} - f_k\|_{L^1(\partial\Omega)} \\
&\le C\rho_1 \|f\|_{L^1(\partial\Omega)} \\
&\le C \operatorname{diam}(\Omega) \|f\|_{L^1(\partial\Omega)} .
\end{aligned}$$

In order to obtain the L^1 estimate for $\operatorname{Lip} F$, we first apply the product rule for locally Lipschitz functions, which yields that

$$\begin{aligned}
\operatorname{Lip} F &= \sum_{k=1}^{\infty} \big(|E(f_{k+1} - f_k)| \operatorname{Lip} \psi_k + \psi_k \operatorname{Lip}(E(f_{k+1} - f_k)) \big) \\
&\le \sum_{k=1}^{\infty} \left(\frac{|E(f_{k+1} - f_k)| \chi_{\Omega(\rho_{k+1},\rho_k)}}{\rho_k - \rho_{k+1}} + \chi_{\Omega(0,\rho_k)} \operatorname{Lip}(E(f_{k+1} - f_k)) \right).
\end{aligned}$$

It follows from Corollary 2.7 that

$$\begin{aligned}
\sum_{k=1}^{\infty} \left\| \frac{E(f_{k+1} - f_k)}{\rho_k - \rho_{k+1}} \right\|_{L^1(\Omega(\rho_{k+1},\rho_k))} &\le C \sum_{k=1}^{\infty} \frac{\rho_k}{\rho_k - \rho_{k+1}} \|f_{k+1} - f_k\|_{L^1(\partial\Omega)} \\
&\le C \sum_{k=1}^{\infty} \|f_{k+1} - f_k\|_{L^1(\partial\Omega)} \\
&\le C \|f\|_{L^1(\partial\Omega)} .
\end{aligned}$$

Next, we apply Corollary 2.4 to see that

$$\begin{aligned}\sum_{k=1}^{\infty}\big\|\operatorname{Lip} E(f_{k+1}-f_k)\big\|_{L^1(\Omega(0,\rho_k))}&\\ \le C\sum_{k=1}^{\infty}\rho_k\mathcal{H}(\partial\Omega)\operatorname{LIP}(f_{k+1}-f_k,\partial\Omega)&\\ \le C\mathcal{H}(\partial\Omega)\sum_{k=1}^{\infty}\rho_k\big(\operatorname{LIP}(f_{k+1},\partial\Omega)+\operatorname{LIP}(f_k,\partial\Omega)\big)&\\ \le C\mathcal{H}(\partial\Omega)\|f\|_{L^1(\partial\Omega)},&\end{aligned}$$

where we used the defining properties of $\{\rho_k\}_{k=1}^{\infty}$ to obtain the ultimate inequality. Altogether, $\|\operatorname{Lip} F\|_{L^1(\Omega)} \le C(1+\mathcal{H}(\partial\Omega))\|f\|_{L^1(\partial\Omega)}$. □

4.1. Trace of the extended functions

In this section we complete the proof of Theorem 1.2 by showing that the trace of the extended function yields the original function back.

Proposition 4.2. *Let $F \in BV(\Omega)$ be the extension of $f \in L^1(\partial\Omega)$ as constructed in* (4.1). *Then,*

$$\lim_{r\to 0} \fint_{B(z,r)\cap\Omega} |F - f(z)|\, d\mu = 0$$

for $\mathcal{H}$-a.e. $z \in \partial\Omega$.

Proof. Let E_0 be the collection of all $z \in \partial\Omega$ for which the local measure density condition (1.6) is satisfied and $\lim_k f_k(z) = f(z)$, and for $k \in \mathbb{N}$ let E_k be the collection of all $z \in \partial\Omega$ for which $TEf_k(z) = f_k(z)$ exists. Lemma 2.8 yields that $\mathcal{H}(\partial\Omega \setminus \bigcap_{k=0}^{\infty} E_k) = 0$. We define also an auxiliary sequence $\{F_n\}_{n=1}^{\infty}$ of functions approximating F by

$$F_n = \sum_{k=2}^{n}(\psi_{k-1}-\psi_k)Ef_k + \sum_{k=n+1}^{\infty}(\psi_{k-1}-\psi_k)Ef_n, \quad n \in \mathbb{N}.$$

It can be shown that $F_n \to F$ in $BV(\Omega)$, but we will not need this fact here. Note that $F_n = Ef_n$ in $\Omega(0,\rho_n)$ and hence the trace of F_n exists on $\partial\Omega$ and coincides with the trace of Ef_n, *i.e.*, with f_n.

Fix a point $z \in \bigcap_{k=0}^{\infty} E_k$ and let $\varepsilon > 0$. Then, we can find $j \in \mathbb{N}$ such that $|f_k(z) - f(z)| < \varepsilon$ for every $k \ge j$. Recall that $r_z > 0$ is the restriction on the radius imposed by the local measure density condition (1.6) at z. Next, we choose

$k_0 > j$ such that $R := \rho_{k_0} < r_z$ satisfies:

- $R\,\mathrm{LIP}(f_j, \partial\Omega) < \varepsilon$;
- $\fint_{B(z,r)} |F_j - f_j(z)|\, d\mu < \varepsilon$ for every $r < R$;
- $\sum_{k=k_0}^{\infty} \rho_k\, \mathrm{LIP}(f_{k+1}, \partial\Omega) < \varepsilon$.

For every $r \in (0, \rho_{k_0+1}) \subset (0, R/2)$, we can then estimate

$$\begin{aligned}\fint_{B(z,r)\cap\Omega} &|F - f(z)|\, d\mu \\ &\le \fint_{B(z,r)\cap\Omega} |F - F_j|\, d\mu + \fint_{B(z,r)\cap\Omega} |F_j - f_j(z)|\, d\mu + |f_j(z) - f(z)| \\ &\le \fint_{B(z,r)\cap\Omega} |F - F_j|\, d\mu + 2\varepsilon. \end{aligned} \tag{4.2}$$

For such r, choose $k_r > k_0$ such that $\rho_{k_r+1} \le r < \rho_{k_r}$. Then,

$$\begin{aligned}\int_{B(z,r)\cap\Omega} |F - F_j|\, d\mu &\le \sum_{k=k_r}^{\infty} \int_{B(z,r)\cap\Omega} (\psi_{k-1} - \psi_k)\big|E(f_k - f_j)\big|\, d\mu \\ &\le \sum_{k=k_r}^{\infty} \int_{B(z,r)\cap\Omega(\rho_{k+1},\rho_{k-1})} \big|E(f_k - f_j)\big|\, d\mu \\ &\le C \sum_{k=k_r}^{\infty} \min\{r, \rho_{k-1}\} \int_{B(z,2^8 r)\cap\partial\Omega} |f_k - f_j|\, d\mathcal{H} \end{aligned} \tag{4.3}$$

by Lemma 2.6. In the last inequality above, we used the fact that when $k = k_r$, we must have $B(z,r) \cap \Omega(\rho_{k_r+1}, \rho_{k_r-1}) = B(z,r) \cap \Omega(\rho_{k_r+1}, r)$ by the choice of $r < \rho_{k_r}$.

Let us, for the sake of brevity, write $U_r = B(z, 2^8 r) \cap \partial\Omega$. As $f_k - f_j$ is Lipschitz continuous, we have by the choice of j, and the fact that $k \ge j$,

$$\begin{aligned}\int_{U_r} |f_k - f_j|\, d\mathcal{H} &\le \int_{U_r} \big|f_k - f_j - (f_k(z) - f_j(z))\big|\, d\mathcal{H} + |f_k(z) - f_j(z)|\mathcal{H}(U_r) \\ &\le Cr\mathcal{H}(U_r)\, \mathrm{LIP}(f_k - f_j, U_r) + 2\varepsilon\mathcal{H}(U_r). \end{aligned} \tag{4.4}$$

Observe that $r\mathcal{H}(U_r) \approx \mu(B(z,r))$ by (1.5), and the doubling condition for μ. Note that $\sum_{k=k_r}^{\infty} \rho_{k-1} \le C\rho_{k_r-1} \le CR$. Combining this with (4.3) and (4.4) gives us that

$$
\begin{aligned}
\int_{B(z,r)\cap\Omega} &|F - F_j|\, d\mu \\
&\le \sum_{k=k_r}^{\infty} C\rho_{k-1}\mu(B(z,r))\big(\mathrm{LIP}(f_k,\partial\Omega) + \mathrm{LIP}(f_j,\partial\Omega)\big) \\
&\quad + 2\varepsilon\mu(B(z,r)) \sum_{k=k_r}^{\infty} \frac{\min\{r,\rho_{k-1}\}}{r} \\
&\le C\mu(B(z,r))\bigg(\sum_{k=k_0}^{\infty} \big(\rho_k\,\mathrm{LIP}(f_{k+1},\partial\Omega)\big) + R\,\mathrm{LIP}(f_j,\partial\Omega) + \varepsilon\bigg) \\
&\le C\mu(B(z,r))\varepsilon \le C\, C_z\, \mu(B(z,r)\cap\Omega)\,\varepsilon.
\end{aligned}
$$

In the above, $C_z \ge 1$ is from (1.6). Plugging this estimate into (4.2) completes the proof. □

5. Summary and further discussion

In conclusion, we have shown that every function in $L^1(\partial\Omega)$ has an extension to $BV(\Omega)$ in such a way that the trace of the extension returns the original function. This extension is nonlinear, but it is bounded. In a preceding section we demonstrated that there is a bounded linear extension from the subclass $B^0_{1,1}(\partial\Omega)$ to $BV(\Omega)$. Note that $B^0_{1,1}(\partial\Omega)$, containing all the Lipschitz functions on $\partial\Omega$, must necessarily be dense in $L^1(\partial\Omega)$. It therefore follows that this extension from $L^1(\partial\Omega)$ to $BV(\Omega)$ cannot be continuous on $L^1(\partial\Omega)$ since if it were, then the extension from $L^1(\partial\Omega)$ would be bounded and linear–and this is not possible (see [34] for the fact that in general any extension from $L^1(\partial\Omega)$ to $BV(\Omega)$ cannot be both bounded and linear).

On the other hand, in the setting of Corollary 1.4, using this corollary we see that the trace operator T of [30] is a continuous *surjective* linear mapping of the Banach space $BV(\Omega)$ to the Banach space $L^1(\partial\Omega)$, and hence there exists a continuous (non-linear) right inverse of T by the Bartle–Graves theorem [6, Corollary 7.1]. Should we know that Ω is a domain for which each function $f \in L^1(\partial\Omega)$ has an associated function $u_f \in BV(\Omega)$ such that u_f is of least gradient (that is, 1-harmonic) in Ω and with trace $Tu = f$ $\mathcal{H}$-a.e. in $\partial\Omega$, then the natural continuous inverse map would be the map $f \mapsto u_f$; the stability results of [22] would indicate that this map is continuous. However, even under the best of circumstances, for example Ω the unit disk in $\mathbb{R}^2$, no u_f exists for general $f \in L^1(\partial\Omega)$ (see [37]). It is not clear what the Bartle-Graves inverse map is.

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