Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XVIII (2018), 241-282

Extension property of semipositive invertible sheaves over a non-archimedean field

HUAYI CHEN AND ATSUSHI MORIWAKI

Abstract. In this article, we prove an extension property of semipositively metrized ample invertible sheaves on a projective scheme over a complete non-archimedean valued field. As an application, we establish a Nakai-Moishezon type criterion for adelically normed graded linear series.

Mathematics Subject Classification (2010): 14C20 (primary); 14G40 (secondary).

1. Introduction

Let k be a field and X be a reduced projective scheme over $\operatorname{Spec} k$, equipped with an ample invertible $\mathscr{O}_X$-module L. If Y is a reduced closed subscheme of X, then for any sufficiently positive integer n, any section ℓ of $L|_Y^{\otimes n}$ on Y extends to a global section of $L^{\otimes n}$ on X. In other words, the restriction map $H^0(X, L^{\otimes n}) \to H^0(Y, L|_Y^{\otimes n})$ is surjective. A simple proof of this result relies on Serre's vanishing theorem, which ensures that $H^1(X, \mathcal{I}_Y \otimes L^{\otimes n}) = 0$ for any sufficiently positive integer n, where $\mathcal{I}_Y$ is the ideal sheaf of Y.

The metrized version (with $k = \mathbb{C}$) of this result has been widely studied in the literature and has divers applications in complex analytic geometry and in arithmetic geometry. We assume that the ample invertible sheaf L is equipped with a continuous (with respect to the analytic topology) metric $|\cdot|_h$, which induces a continuous metric $|\cdot|_{h^n}$ on each tensor power sheaf $L^{\otimes n}$, where $n \in \mathbb{N}$, $n \geq 1$. The metric $|\cdot|_{h^n}$ leads to a supremum norm $\|\cdot\|_{h^n}$ on the global section space $H^0(X, L)$ such that

$$\|s\|_{h^n} = \sup_{x \in X(\mathbb{C})} |s|_{h^n}(x) \text{ for all } s \in H^0(X, L).$$

Huayi Chen is partially supported by the research fundings ANR-14-CE25-0015 and NSFC1127 1021. Atsushi Moriwaki is partially supported by the Grant-in-Aid for Scientific Research (S), Grant Number JP16H06335.

Received January 4, 2016; accepted in revised form October 11, 2016.
Published online March 2018.

Similarly, it induces a supremum norm $\|\cdot\|_{Y,h^n}$ on the space $H^0(Y, L|_Y^{\otimes n})$ with

$$\|s\|_{Y,h^n} = \sup_{y\in Y(\mathbb{C})} |s|_{h^n}(y).$$

Note that for any section $s \in H^0(X, L^{\otimes n})$ one has $\|s|_Y\|_{Y,h^n} \le \|s\|_{h^n}$. The metric extension problem consists of studying the extension of global sections of $L|_Y$ to those of L with an estimation on the supremum norms. Note that a positivity condition on the metric h is in general necessary to obtain interesting upper bounds. This problem has been studied by using Hörmander's L^2 estimates (see [9] for example), under smoothness conditions on the metric. More recently, it has been proved (without any regularity condition) that, if the metric $|\cdot|_h$ is semi-positive, then for any $\epsilon > 0$ and any section $l \in H^0(Y, L|_Y)$ there exist an integer $n \ge 1$ and $s \in H^0(X, L^{\otimes n})$ such that $s|_Y = l^{\otimes n}$ and that $\|s\|_{h^n} \le e^{\epsilon n}\|s|_Y\|_{Y,h^n}$. We refer the readers to [17,19] for more details.

The purpose of this article is to study the non-archimedean counterpart of the above problem. We will establish the following result (see Theorem 4.5 and Corollary 2.17).

Theorem 1.1. *Let k be a field equipped with a complete and non-archimedean absolute value $|\cdot|$ (which could be trivial). Let X be a reduced projective scheme over* Spec k *and L be an ample invertible sheaf on X, equipped with a continuous and semi-positive metric $|\cdot|_h$. Let Y be a reduced closed subscheme of X and $l \in H^0(Y, L|_Y)$. For any $\epsilon > 0$ there exists an integer $n_0 \ge 1$ such that, for any integer $n \ge n_0$, the section $l^{\otimes n}$ extends to a section $s \in H^0(X, L^{\otimes n})$ verifying $\|s\|_{h^n} \le e^{\epsilon n}\|l\|_{Y,h}^n$.*

The semi-positivity condition of the metric means that the metric $|\cdot|_h$ can be written as a uniform limit of Fubini-Study metrics. We will show that, if the absolute value $|\cdot|$ is non-trivial, then this condition is equivalent to the classical semi-positivity condition (namely uniform limit of nef model metrics, see Proposition 3.14) of Zhang [21], see also [10, 16], and compare with the complex analytic case [20]. The advantage of the new definition is that it also works in the trivial valuation case, where the model metrics are too restrictive. We use an argument of extension of scalars to the ring of formal Laurent series to obtain the result of the above theorem in the trivial valuation case.

As an application, we establish an adelic version of the arithmetic Nakai-Moishezon criterion as follows, see Theorem 5.6 and Corollary 5.9.

Theorem 1.2. *Let X be a geometrically integral projective scheme over a number field K and L be an invertible sheaf on X. For any place v of K, let h_v be a continuous semipositive metric on the pull-back of L on the analytic space X_v^{an}, such that $(H^0(X, L^{\otimes n}), \{\|\cdot\|_{X_v,h_v^n}\})$ forms an adelically normed vector space over K for any $n \in \mathbb{N}$ (see Definition 5.1). Suppose that for any integral closed subscheme Y of X, the restriction of L on Y is big and there exist a positive integer n and a non-zero section $s \in H^0(Y, L|_Y^{\otimes n})$ such that $\|s\|_{Y_v,h_v^n} \le 1$ for any place v of K,* *and that the inequality is strict when v is an infinite place. Then for any sufficiently positive integer n, the $\mathbb{Q}$-vector space $H^0(X, L^{\otimes n})$ has a basis $(\omega_1, \ldots, \omega_{r_n})$ with $\|\omega_i\|_{X_v, h_v^n} \leq 1$ for any place v, where the inequality is strict if v is an infinite place.*

This result generalizes simultaneously [21, Theorem 4.2] and [15, Theorem 4.2] since here we have a weaker assumption on the adelic metric on L. Indeed, in the paper [21], the following conditions are assumed: h_v is semipositive for all places v of K, $\widehat{\deg}(\overline{L}\big|_Y^{\dim Y+1}) > 0$ for all integral subschemes Y of X, and there exist a non-empty open set U of $\mathrm{Spec}(O_K)$ and a positive integer d such that the metric h_v^d of $L_v^{\otimes d}$ ($\forall v \in U$) is induced by a nef model $(\mathscr{X}_U, \mathscr{L}_U)$ of $(X, L^{\otimes d})$ over U. Obviously these assumptions imply our assumptions in Theorem 1.2. The main idea for the proof is to combine the estimation on normed Noetherian graded linear series developed in [15] and the non-archimedean extension property established in the current paper. In the archimedean case we also use the archimedean extension property proved in [17].

The article is organized as follows. In the first section we introduce the notation of the article and prove some preliminary results, most of which concern finite-dimensional normed vector spaces over a non-archimedean field. In the second section, we study various properties of continuous metrics on an invertible sheaf, where an emphasis is made on the positivity of such metrics. In the third section, we prove the extension theorem. Finally, in the fourth and last section, we apply the extension property to prove a generalized arithmetic Nakai-Moishezon's criterion.

ACKNOWLEDGEMENTS. Huayi Chen has benefited from the visiting support of Beijing International Center for Mathematical Research and would like to thank the center for the hospitalities.

We are grateful to the anonymous referee for the careful reading and for the suggestions.

2. Notation and preliminaries

2.1. Notation

Throughout this paper, we fix the following notation.

2.1.1. Fix a field k with a non-archimedean absolute value $|\cdot|$ on k. Unless otherwise stated, we assume that $(k, |\cdot|)$ is complete. The valuation ring of k and the maximal ideal of the valuation ring are denoted by $\mathfrak{o}_k$ and $\mathfrak{m}_k$, respectively, that is,

$$\mathfrak{o}_k := \{a \in k \mid |a| \leq 1\} \quad \text{and} \quad \mathfrak{m}_k := \{x \in k \mid |x| < 1\}.$$

In the case where $|\cdot|$ is discrete, we fix a uniformizing parameter ϖ of $\mathfrak{m}_k$, that is, $\mathfrak{m}_k = \varpi \mathfrak{o}_k$.

2.1.2. A norm $\|\cdot\|$ of a finite-dimensional vector space V over the non-archimedean field k is always assumed to be ultrametric, that is, $\|x+y\| \leq \max\{\|x\|, \|y\|\}$. The pair $(V, \|\cdot\|)$ is called a *finite-dimensional normed vector space over* k.

2.1.3. In Section 1-Section 4, we fix a reduced algebraic scheme X over $\operatorname{Spec} k$, that is, X is a reduced scheme of finite type over $\operatorname{Spec}(k)$. Let X^{an} be the analytification of X in the sense of Berkovich [2]. For any $x \in X^{\mathrm{an}}$, the residue field of the associated scheme point of x is denoted by $\kappa(x)$. Note that the seminorm $|\cdot|_x$ at x yields an absolute value of $\kappa(x)$. By abuse of notation, it is denoted by $|\cdot|_x$. Let $\hat{\kappa}(x)$ be the completion of $\kappa(x)$ with respect to $|\cdot|_x$. The extension of $|\cdot|_x$ to $\hat{\kappa}(x)$ is also denoted by the same symbol $|\cdot|_x$. The valuation ring of $\hat{\kappa}(x)$ and the maximal ideal of the valuation ring are denoted by $\mathfrak{o}_x$ and $\mathfrak{m}_x$, respectively. Let L be an invertible sheaf on X. For any $x \in X^{\mathrm{an}}$, the sheaf $L \otimes_{\mathscr{O}_X} \hat{\kappa}(x)$ is denoted by $L(x)$.

2.1.4. By *continuous metric* on L, we refer to a family $h = \{|\cdot|_h(x)\}_{x \in X^{\mathrm{an}}}$, where $|\cdot|_h(x)$ is a norm on $L \otimes_{\mathscr{O}_X} \hat{\kappa}(x)$ over $\hat{\kappa}(x)$ for each $x \in X^{\mathrm{an}}$, such that for any local basis ω of L over a Zariski open subset U, $|\omega|_h(\cdot)$ is a continuous function on U^{an}. We assume that X is projective. Given a continuous metric h on L, we define a norm $\|\cdot\|_h$ on $H^0(X, L)$ such that

$$\|s\|_h := \sup_{x \in X^{\mathrm{an}}} |s|_h(x) \text{ for all } s \in H^0(X, L).$$

Similarly, if Y is a reduced closed subscheme of X, we define a norm $\|\cdot\|_{Y,h}$ on $H^0(Y, L)$ such that

$$\|l\|_{Y,h} := \sup_{y \in Y^{\mathrm{an}}} |l|_h(y) \text{ for all } l \in H^0(Y, L).$$

Clearly one has

$$\|s\|_h \geq \|s|_Y\|_{Y,h} \tag{2.1}$$

for any $s \in H^0(X, L)$.

• In the following 2.1.5, 2.1.6 and 2.1.7, X is always assumed to be projective.

2.1.5. Given a continuous metric h on L, for each integer $n \geq 1$ the metric induces a continuous metric on $L^{\otimes n}$ which we denote by h^n: for any point $x \in X^{\mathrm{an}}$ and any local basis ω of L over a Zariski open neighborhood of x one has

$$|\omega^{\otimes n}|_{h^n}(x) = |\omega|_h(x)^n.$$

Note that for any section $s \in H^0(X, L)$ one has $\|s^{\otimes n}\|_{h^n} = \|s\|_h^n$. By convention, h^0 denotes the trivial metric on $L^{\otimes 0} = \mathscr{O}_X$, namely $|\mathbf{1}|_{h^0}(x) = 1$ for any $x \in X^{\mathrm{an}}$, where $\mathbf{1}$ denotes the section of unity of $\mathscr{O}_X$.

Conversely, given a continuous metric $g = \{|\cdot|_g(x)\}_{x\in X^{\mathrm{an}}}$ on $L^{\otimes n}$, there is a unique continuous metric h on L such that $h^n = g$. We denote by $g^{1/n}$ this metric. This observation allows to define continuous metrics on an element in $\mathrm{Pic}(X)\otimes\mathbb{Q}$ as follows. Given $M \in \mathrm{Pic}(X)\otimes\mathbb{Q}$, we denote by $\Gamma(M)$ the subsemigroup of $\mathbb{N}_{\geq 1}$ of all positive integers n such that $M^{\otimes n} \in \mathrm{Pic}(X)$. We call *continuous metric* on M any family $g = (g_n)_{n\in\Gamma(M)}$ with g_n being a continuous metric on $M^{\otimes n}$, such that $g_n^m = g_{mn}$ for any $n \in \Gamma(M)$ and any $m \in \mathbb{N}_{\geq 1}$. Note that the family

$$g = (g_n)_{n\in\Gamma(M)}$$

is uniquely determined by any of its elements. In fact, given an element $n \in \Gamma(M)$, one has $g_m = g_{mn}^{1/n} = (g_n^m)^{1/n}$ for any $m \in \Gamma(M)$. In particular, for any positive rational number p/q, the family $g^{p/q} = (g_{Nnp}^{1/Nq})_{n\in\Gamma(M^{\otimes(p/q)})}$ is a continuous metric on $M^{\otimes(p/q)}$, where N is a positive integer such that $M^{\otimes N} \in \mathrm{Pic}(X)$, and the metric $g^{p/q}$ does not depend on the choice of the positive integer N. If L is an element of $\mathrm{Pic}(X)$, equipped with a continuous metric g, by abuse of notation, we use the expression g to denote the metric family $(g^n)_{n\in\mathbb{N}_{\geq 1}}$, viewed as a continuous metric on the canonical image of L in $\mathrm{Pic}(X)\otimes\mathbb{Q}$.

Let M be an element in $\mathrm{Pic}(X)\otimes\mathbb{Q}$ equipped with a continuous metric $g = (g_n)_{n\in\Gamma(M)}$. By abuse of notation, for any $n \in \Gamma(M)$ we also use the expression g^n to denote the continuous metric g_n on $M^{\otimes n}$.

2.1.6. Let $\mathscr{X} \to \mathrm{Spec}(\mathfrak{o}_k)$ be a projective and flat $\mathfrak{o}_k$-scheme such that the generic fiber of $\mathscr{X} \to \mathrm{Spec}(\mathfrak{o}_k)$ is X. We call it a *model* of X. We denote by $\mathscr{X}_\circ := \mathscr{X}\otimes_{\mathfrak{o}_k}(\mathfrak{o}_k/\mathfrak{m}_k)$ the central fiber of $\mathscr{X} \to \mathrm{Spec}(\mathfrak{o}_k)$. By the valuative criterion of properness, for any point $x \in X^{\mathrm{an}}$, the canonical k-morphism $\mathrm{Spec}\,\hat{\kappa}(x) \to X$ extends in a unique way to an $\mathfrak{o}_k$-morphism of schemes $\mathscr{P}_x : \mathrm{Spec}\,\mathfrak{o}_x \to \mathscr{X}$. We denote by $r_{\mathscr{X}}(x)$ the image of $\mathfrak{m}_x \in \mathrm{Spec}\,\mathfrak{o}_x$ by the map $\mathscr{P}_x$. Thus we obtain a map $r_{\mathscr{X}}$ from X^{an} to $\mathscr{X}_\circ$, called the *reduction map* of $\mathscr{X}$.

Let $\mathscr{L}$ be an element of $\mathrm{Pic}(\mathscr{X})\otimes\mathbb{Q}$ such that $\mathscr{L}\big|_X = L$ in $\mathrm{Pic}(X)\otimes\mathbb{Q}$. The $\mathbb{Q}$-invertible sheaf $\mathscr{L}$ yields a continuous metric $|\cdot|_{\mathscr{L}}$ as follows.

First we assume that $\mathscr{L} \in \mathrm{Pic}(\mathscr{X})$ and $\mathscr{L}\big|_X = L$ in $\mathrm{Pic}(X)$. For any $x \in X^{\mathrm{an}}$, let ω_x be a local basis of $\mathscr{L}$ around $r_{\mathscr{X}}(x)$ and $\bar{\omega}_x$ the class of ω_x in $L(x) := L\otimes_{\mathscr{O}_X}\hat{\kappa}(x)$. For any $l \in L\otimes_{\mathscr{O}_X}\hat{\kappa}(x)$, if we set $l = a_x\bar{\omega}_x$ ($a_x \in \hat{\kappa}(x)$), then $|l|_{\mathscr{L}}(x) := |a_x|_x$. Here we set $h := \{|\cdot|_{\mathscr{L}}(x)\}_{x\in X^{\mathrm{an}}}$. Note that h is continuous because, for a local basis ω of $\mathscr{L}$ over an open set $\mathscr{U}$ of $\mathscr{X}$, $|\omega|_{\mathscr{L}}(x) = 1$ for all $x \in r_{\mathscr{X}}^{-1}(\mathscr{U}_\circ)$, where $\mathscr{U}_\circ = \mathscr{U}\otimes_{\mathfrak{o}_k}(\mathfrak{o}_k/\mathfrak{m}_k)$ is the central fiber of $\mathscr{U}$. Moreover,

$$|\cdot|_{h^n}(x) = |\cdot|_{\mathscr{L}^n}(x) \tag{2.2}$$

for all $n \geq 0$ and $x \in X^{\mathrm{an}}$. Indeed, if we set $l = a_x\bar{\omega}_x$ for $l \in L(x)$, then $l^{\otimes n} = a_x^n\bar{\omega}_x^{\otimes n}$. Thus

$$|l^{\otimes n}|_{h^n}(x) = (|l|_h(x))^n = |a_x|_x^n = |l^{\otimes n}|_{\mathscr{L}^n}(x).$$

In general, there are an $\mathscr{M} \in \mathrm{Pic}(\mathscr{X})$ and a positive integer m such that $\mathscr{L}^{\otimes m} = \mathscr{M}$ in $\mathrm{Pic}(\mathscr{X}) \otimes \mathbb{Q}$ and $\mathscr{M}\big|_X = L^{\otimes m}$ in $\mathrm{Pic}(X)$. Then we set

$$|\cdot|_{\mathscr{L}} := (|\cdot|_{\mathscr{M}})^{1/m}.$$

Note that the above definition does not depend on the choice of $\mathscr{M}$ and m. Indeed, let $\mathscr{M}'$ and m' be another choice. As $\mathscr{M}^{\otimes m'} = \mathscr{M}'^{\otimes m}$ in $\mathrm{Pic}(\mathscr{X}) \otimes \mathbb{Q}$, there is a positive integer N such that $\mathscr{M}^{\otimes Nm'} = \mathscr{M}'^{\otimes Nm}$ in $\mathrm{Pic}(\mathscr{X})$, so that, by using (2.2),

$$(|\cdot|_{\mathscr{M}}(x))^{Nm'} = |\cdot|_{\mathscr{M}^{\otimes Nm'}}(x) = |\cdot|_{\mathscr{M}'^{\otimes Nm}}(x) = (|\cdot|_{\mathscr{M}'}(x))^{Nm},$$

as desired.

2.1.7. Let $\mathscr{X}$ be a model of X. As $\mathscr{X}$ is flat over $\mathfrak{o}_k$, the natural homomorphism $\mathscr{O}_{\mathscr{X}} \to \mathscr{O}_X$ is injective. Let Y be a closed subscheme of X and $I_Y \subseteq \mathscr{O}_X$ the defining ideal sheaf of Y. Let $\mathscr{I}_{\mathscr{Y}}$ be the kernel of $\mathscr{O}_{\mathscr{X}} \to \mathscr{O}_X/I_Y$, that is, $\mathscr{I}_{\mathscr{Y}} := I_Y \cap \mathscr{O}_{\mathscr{X}}$. Obviously $\mathscr{I}_{\mathscr{Y}} \otimes_{\mathfrak{o}_k} k = I_Y$, so that if we set $\mathscr{Y} = \mathrm{Spec}(\mathscr{O}_{\mathscr{X}}/\mathscr{I}_{\mathscr{Y}})$, then $\mathscr{Y} \times_{\mathrm{Spec}(\mathfrak{o}_k)} \mathrm{Spec}(k) = Y$. Moreover, $\mathscr{Y}$ is flat over $\mathfrak{o}_k$ because $\mathscr{O}_{\mathscr{Y}} \to \mathscr{O}_Y$ is injective. Therefore, $\mathscr{Y}$ is a model of Y. We say that $\mathscr{Y}$ is the *Zariski closure of Y in $\mathscr{X}$*.

2.2. Normed vector space over a non-archimedean field

In this subsection, we recall several facts on (ultrametric) norms over a non-archimedean field. Throughout this subsection, a norm on a vector space over a non-archimedean field is always assumed to be ultrametric. We also assume that k is complete except in Subsections 2.2.1-2.2.2.

2.2.1. *Topology.* In this subsubsection, k is not necessarily complete. Let V be a finite-dimensional vector space over k and $\|\cdot\|$ be a norm of V over $(k, |\cdot|)$. Let r be the rank of V. We assume that $\|\cdot\|$ extends by continuity to a norm on $V \otimes_k \widehat{k}$, where $\widehat{k}$ denotes the completion of $(k, |\cdot|)$, on which the absolute value extends in a unique way. In particular, any k-linear isomorphism $k^r \to V$ is a homeomorphism, where we consider the product topology on k^r (see [3, Section I.2, n. 3, Theorem 2 and the remark on the page I.15]), and for any vector subspace W of V, W is closed in V and is dense in $W \otimes_k \widehat{k} \subset V \otimes_k \widehat{k}$.

For a basis $\boldsymbol{e} = (e_1, \ldots, e_r)$ of V, we set

$$\|a_1 e_1 + \cdots + a_r e_r\|_{\boldsymbol{e}} := \max\{|a_1|, \ldots, |a_r|\} \text{ for all } (a_1, \ldots, a_r) \in \widehat{k}^r$$

which yields an ultrametric norm on $V \otimes_k \widehat{k}$. Note that the norms $\|\cdot\|_{\boldsymbol{e}}$ and $\|\cdot\|$ on V are equivalent. In particular, if the valuation $|\cdot|$ is discrete and non-trivial, then there exists an integer n such that the unit ball $(V, \|\cdot\|)_{\leq 1} := \{x \in V \mid \|x\| \leq 1\}$ is contained in the free $\mathfrak{o}_k$-module

$$\varpi^n \mathfrak{o}_k e_1 + \cdots + \varpi^n \mathfrak{o}_k e_r,$$

where ϖ is a uniformizing parameter of $\mathfrak{o}_k$ as in 2.1.1. Since $\mathfrak{o}_k$ is a discrete valuation ring, we obtain that $(V, \|\cdot\|_{\leq 1})$ is an $\mathfrak{o}_k$-module of finite type, and hence a free $\mathfrak{o}_k$-module of rank r.

Let $(x_i)_{i=1}^n$ be a family of vectors in V, where $n \geq 2$. Assume that $(a_i)_{i=1}^n \in k^n$ is such that the numbers $(\|a_i x_i\|)_{i=1}^n$ are distinct, then one has

$$\|a_1 x_1 + \cdots + a_n x_n\| = \max_{i \in \{1,\ldots,n\}} \|a_i x_i\|.$$

In particular, $a_1 x_1 + \cdots + a_n x_n$ is non-zero. Therefore, the image of $V \setminus \{0\}$ by the composed map

$$V \setminus \{0\} \xrightarrow{\|\cdot\|} \mathbb{R}_+^\times \longrightarrow \mathbb{R}_+^\times / |k^\times|$$

is a finite set, whose cardinality does not exceed the rank of V over k. In particular, if the valuation $|\cdot|$ is discrete, then the image of $V \setminus \{0\}$ by $\|\cdot\|$ is a discrete subset of $\mathbb{R}_+^\times$; if the valuation $|\cdot|$ is trivial, then the image of $V \setminus \{0\}$ by $\|\cdot\|$ is a finite set, whose cardinality does not exceed the rank of V over k.

Proposition 2.1. *Assume that $|\cdot|$ is discrete. Let W be a quotient vector space of V and $\pi : V \to W$ be the projection map. We equip $W \otimes_k \widehat{k}$ with the quotient norm $\|\cdot\|_W$. Then for any $y \in W$ there is an $x \in V$ such that $\pi(x) = y$ and $\|y\|_W = \|x\|$.*

Proof. We may assume that $y \neq 0$ (the case where $y = 0$ is trivial). We set $M = \mathrm{Ker}(\pi)$. Since M is dense in $M \otimes_k \hat{k} = \mathrm{Ker}(\pi_{\hat{k}})$, we obtain that $\pi^{-1}(\{y\})$ is dense in $\pi_{\hat{k}}^{-1}(\{y\})$. Hence there exists a sequence $(x_n)_{n \geq 0}$ in V such that $\pi(x_n) = y$ for any n and that $\lim_{n \to +\infty} \|x_n\| = \|y\|_W$. Since the image of $V \setminus \{0\}$ by $\|\cdot\|$ is discrete, we obtain that $\|x_n\| = \|y\|_W$ for sufficiently positive n. The proposition is thus proved. □

2.2.2. *Orthogonality of bases.* For $\alpha \in (0, 1]$, a basis $(e_1, \ldots, e_r)$ of V is called an *α-orthogonal basis of V with respect to $\|\cdot\|$* if

$$\alpha \max\{|a_1|\|e_1\|, \ldots, |a_r|\|e_r\|\} \leq \|a_1 e_1 + \cdots + a_r e_r\| \quad \forall a_1, \ldots, a_r \in k.$$

If $\alpha = 1$ (respectively $\alpha = 1$ and $\|e_1\| = \cdots = \|e_r\| = 1$), then the above basis is called an *orthogonal basis of V* (respectively an *orthonormal basis of V*). We refer the readers to [18, Section 2.3] for more details on the orthogonality in the non-archimedean setting. Let $(e_1', \ldots, e_r')$ be another basis of V. We say that $(e_1, \ldots, e_r)$ *is compatible with* $(e_1', \ldots, e_r')$ if $ke_1 + \cdots + ke_i = ke_1' + \cdots + ke_i'$ for $i = 1, \ldots, r$.

Proposition 2.2. *Fix a basis* $(e'_1, \dots, e'_r)$ *of* V. *For any* $\alpha \in (0, 1)$, *there exists an* α*-orthogonal basis* $(e_1, \dots, e_r)$ *of* V *with respect to* $\|\cdot\|$ *such that* $(e_1, \dots, e_r)$ *is compatible with* $(e'_1, \dots, e'_r)$. *Moreover, if the absolute value* $|\cdot|$ *is discrete, then there exists an orthogonal basis* $(e_1, \dots, e_r)$ *of* V *compatible with* $(e'_1, \dots, e'_r)$ *(see* [7, *Proposition* 2.5]).

Proof. We prove it by induction on $\dim_k V$. If $\dim_k V = 1$, then the assertion is obvious. By the hypothesis of induction, there is a $\sqrt{\alpha}$-orthogonal basis $(e_1, \dots, e_{r-1})$ of $V' := ke'_1 + \cdots + ke'_{r-1}$ with respect to $\|\cdot\|$ such that

$$ke_1 + \cdots + ke_i = ke'_1 + \cdots + ke'_i$$

for $i = 1, \dots, r-1$. Choose $v \in V \setminus V'$. Since V' is a closed subset of V, one has

$$\operatorname{dist}(v, V') := \inf\{\|v - x\| : x \in V'\} > 0.$$

There then exists $y \in V'$ such that $\|v - y\| \le (\sqrt{\alpha})^{-1} \operatorname{dist}(v, V')$. We set $e_r = v - y$. Clearly $(e_1, \dots, e_{r-1}, e_r)$ forms a basis of V, which is compatible with $(e'_1, \dots, e'_r)$. It is sufficient to see that

$$\|a_1e_1 + \cdots + a_{r-1}e_{r-1} + e_r\| \ge \alpha \max\{|a_1|\|e_1\|, \dots, |a_{r-1}|\|e_{r-1}\|, \|e_r\|\}$$

for all $a_1, \dots, a_{r-1} \in k$. Indeed, as $\|e_r\| \le (\sqrt{\alpha})^{-1}\|a_1e_1 + \cdots + a_{r-1}e_{r-1} + e_r\|$, we have

$$\alpha\|e_r\| \le \sqrt{\alpha}\|e_r\| \le \|a_1e_1 + \cdots + a_{r-1}e_{r-1} + e_r\|.$$

If $\|a_1e_1 + \cdots + a_{r-1}e_{r-1}\| \le \|e_r\|$, then

$$\begin{aligned}\|a_1e_1 + \cdots + a_{r-1}e_{r-1} + e_r\| &\ge \sqrt{\alpha}\|e_r\| \ge \sqrt{\alpha}\|a_1e_1 + \cdots + a_{r-1}e_{r-1}\| \\ &\ge \sqrt{\alpha}\left(\sqrt{\alpha} \max\{|a_1|\|e_1\|, \dots, |a_{r-1}|\|e_{r-1}\|\}\right) \\ &= \alpha \max\{|a_1|\|e_1\|, \dots, |a_{r-1}|\|e_{r-1}\|\}.\end{aligned}$$

Otherwise,

$$\begin{aligned}\|a_1e_1 + \cdots + a_{r-1}e_{r-1} + e_r\| &= \|a_1e_1 + \cdots + a_{r-1}e_{r-1}\| \\ &\ge \sqrt{\alpha} \max\{|a_1|\|e_1\|, \dots, |a_{r-1}|\|e_{r-1}\|\} \\ &\ge \alpha \max\{|a_1|\|e_1\|, \dots, |a_{r-1}|\|e_{r-1}\|\},\end{aligned}$$

as required.

For the second assertion, it is sufficient to use the discreteness of the set

$$\{\|v - x\| \mid x \in V'\}$$

to show that it has the minimal value (see Section 2.2.1). □

Remark 2.3. We assume that k is not complete. Let $\gamma \in \widehat{k} \setminus k$, we define a norm $\|\cdot\|_\gamma$ on k^2 by

$$\forall (a,b) \in k^2, \quad \|(a,b)\|_\gamma := |a + b\gamma|.$$

Then there is no positive constant C such that $\|(a,b)\|_\gamma \geq C \max\{|a|, |b|\}$ for all $a, b \in k$. In particular, for any $\alpha \in (0,1]$, there is no α-orthogonal basis of k^2 with respect to $\|\cdot\|_\gamma$. Indeed, let us assume the contrary. We can find a sequence $\{a_n\}$ in k with $\lim_{n\to\infty} |a_n - \gamma| = 0$. On the other hand,

$$|a_n - \gamma| = \|(a_n, -1)\|_\gamma \geq C \max\{|a_n|, 1\} \geq C$$

for all n. This is a contradiction. Note that the norm $\|\cdot\|_\gamma$ extends by continuity to a map $\widehat{k}^2 \to \mathbb{R}_{\geq 0}$ sending $(a,b) \in \widehat{k}^2$ to $|a + b\gamma|$. But this map is a semi-norm instead of a norm. Therefore, the hypothesis that the $\|\cdot\|$ extends to a norm on $V \otimes_k \widehat{k}$ is essential.

2.2.3. *Dual norm.* From now on and until the end of the section, we assume that $(k, |\cdot|)$ is complete. Let $(V, \|\cdot\|)$ and $(V', \|\cdot\|')$ be finite-dimensional normed vector spaces over k, and $\phi : V \to V'$ be a k-linear map. By the topological property of V that we resumed in Subsection 2.2.1 we obtain that the linear map ϕ is continuous, which implies that

$$\|\phi\|_{\mathrm{Hom}_k(V,V')} := \sup_{v \in V \setminus \{0\}} \frac{\|\phi(v)\|'}{\|v\|}$$

is finite (in the case where $|\cdot|$ is trivial, we also use the fact that $\|\cdot\|$ and $\|\cdot\|'$ only takes finitely many values). Note that $\|\cdot\|_{\mathrm{Hom}_k(V,V')}$ yields an ultrametric norm on $\mathrm{Hom}_k(V, V')$, called the *operator norm*. We denote $\|\cdot\|_{\mathrm{Hom}_k(V,k)}$ by $\|\cdot\|^\vee$ (*i.e.*, the case where $V' = k$ and $\|\cdot\|' = |\cdot|$), called the *dual norm* of $\|\cdot\|$. By definition, one has

$$|\phi(x)| \leq \|\phi\|^\vee \|x\|$$

for any $x \in V$ and $\phi \in V^\vee$. In particular, one has

$$\|x\|^{\vee\vee} \leq \|x\| \text{ for all } x \in V, \tag{2.3}$$

where we identify V with $(V^\vee)^\vee$ via the natural isomorphism.

Let $(e_i)_{i=1}^r$ be an α-orthogonal basis of V, $\alpha \in (0,1]$, and $(e_i^\vee)_{i=1}^r$ be its dual basis of $V^\vee$. By definition one has $e_i^\vee(\lambda_1 e_1 + \cdots + \lambda_r e_r) = \lambda_i$ for any $(\lambda_1, \ldots, \lambda_r) \in k^r$. Hence

$$\|e_i^\vee\|^\vee = \sup_{(\lambda_1,\ldots,\lambda_r) \neq (0,\ldots,0)} \frac{|\lambda_i|}{\|\lambda_1 e_1 + \cdots + \lambda_r e_r\|} \leq \alpha^{-1} \|e_i\|^{-1}. \tag{2.4}$$

Therefore, for any $\phi = a_1 e_1^\vee + \cdots + a_r e_r^\vee \in V^\vee$, where $(a_1, \ldots, a_r) \in k^r$, one has

$$\|\phi\| \geqslant \frac{|\phi(e_i)|}{\|e_i\|} = \frac{|a_i|}{\|e_i\|} \geq \alpha |a_i| \|e_i^\vee\|^\vee.$$

Namely the dual basis $(e_i^\vee)_{i=1}^r$ is α-orthogonal with respect to the dual norm $\|\cdot\|^\vee$. By the same reason, the basis $(e_i)_{i=1}^r$ is also α-orthogonal with respect to the double dual norm $\|\cdot\|^{\vee\vee}$, and for any $x = \lambda_1 e_1 + \cdots + \lambda_r e_r \in V$ one has

$$\|x\|^{\vee\vee} \geq \frac{e_i^\vee(x)}{\|e_i^\vee\|^\vee} = \frac{|\lambda_i|}{\|e_i^\vee\|^\vee} \geq \alpha|\lambda_i|\|e_i\| \text{ for all } i \in \{1, \ldots, r\}$$

where the second inequality comes from (2.4). We then deduce that

$$\|x\|^{\vee\vee} \geq \alpha \max_{i\in\{1,\ldots,r\}} |\lambda_i|\|e_i\| \geq \alpha\|x\|.$$

By Proposition 2.2 and (2.3), we obtain that the natural isomorphism $V \to (V^\vee)^\vee$ is actually an isometry, where we consider the double dual norm $\|\cdot\|^{\vee\vee}$ on $(V^\vee)^\vee$.

2.2.4. *Scalar extension of norms.* In this subsubsection, we fix a finite-dimensional normed vector space $(V, \|\cdot\|)$ over k.

Definition 2.4. Let k' be an extension field of k, and let $|\cdot|'$ be a complete absolute value of k' which is an extension of $|\cdot|$ (we call $(k', |\cdot|')$ a *complete valued extension* of $(k, |\cdot|)$). We set $V_{k'} := V \otimes_k k'$. Identifying $V_{k'}$ with

$$\mathrm{Hom}_k(\mathrm{Hom}_k(V, k), k'),$$

we define $\|\cdot\|_{k'}$ as the operator norm on $V_{k'}$, that is,

$$\|v'\|_{k'} := \sup\left\{\frac{|(\phi \otimes 1)(v')|'}{\|\phi\|^\vee} \,\middle|\, \phi \in V^\vee\right\} \text{ for all } v' \in V_{k'}.$$

The norm $\|\cdot\|_{k'}$ is called the *scalar extension of* $\|\cdot\|$.

By definition, if $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on V such that $\|\cdot\|_1 \leq \|\cdot\|_2$, then one has $\|\cdot\|_1^\vee \geq \|\cdot\|_2^\vee$ and hence $\|\cdot\|_{1,k'} \leq \|\cdot\|_{2,k'}$. Moreover,

$$\|v \otimes 1\|_{k'} = \sup\left\{\frac{|\phi(v)|}{\|\phi\|^\vee} \,\middle|\, \phi \in V^\vee\right\} = \|v\|^{\vee\vee} = \|v\| \text{ for all } v \in V,$$

see Subsection 2.2.3 for the last equality. In other words, $\|\cdot\|_{k'}$ extends the norm $\|\cdot\|$ on V. It is actually the largest ultrametric norm on $V_{k'}$ extending $\|\cdot\|$. In fact, by an argument similar to that in Subsection 2.2.3 we can show that, if $(e_i)_{i=1}^r$ is an α-orthogonal basis of $(V, \|\cdot\|)$, where $\alpha \in (0, 1]$, then $(e_i \otimes 1)_{i=1}^r$ is an α-orthogonal basis of $(V, \|\cdot\|_{k'})$. Assume that $\|\cdot\|'$ is another ultrametric norm on $V_{k'}$ extending $\|\cdot\|$. If $(e_1, \ldots, e_r)$ is an α-orthogonal basis of V, where $\alpha \in (0, 1)$, then we have for all $(a_1', \ldots, a_r') \in k'$

$$\begin{aligned}\alpha\|a_1'e_1 + \cdots + a_r'e_r\|' &\leq \alpha \max_{i\in\{1,\ldots,r\}} (|a_i|'\|e_i\|') \\ &= \alpha \max_{i\in\{1,\ldots,r\}} (|a_i|'\|e_i\|) \leq \|a_1'e_1 + \cdots + a_r'e_r\|_{k'}.\end{aligned}$$

This maximality property implies that, if $(k'', |\cdot|'')$ is a complete valued extension of $(k', |\cdot|')$, then one has

$$\|\cdot\|_{k''} = \|\cdot\|_{k',k''} \text{ on } V_{k''} = V \otimes_k k'' \cong V_{k'} \otimes_{k'} k''. \tag{2.5}$$

Lemma 2.5. *Let $f : V \to W$ be a surjective homomorphism of finite-dimensional vector spaces over k. We assume that $\dim_k W = 1$ and let $\|\cdot\|_W$ be the quotient norm of $\|\cdot\|$ induced by the surjective map $f : V \to W$. Then the norm $\|\cdot\|_{W,k'}$ identifies with the quotient norm of $\|\cdot\|_{k'}$ induced by the surjective map $f_{k'} := f \otimes \mathrm{id}_{k'} : V_{k'} \to W_{k'}$.*

Proof. Let $\|\cdot\|'_{W_{k'}}$ be the quotient norm of $\|\cdot\|_{k'}$ induced by the surjective map $f_{k'} : V_{k'} \to W_{k'}$. Let ℓ be an non-zero element of W. As $\|\ell\|_{W,k'} = \|\ell\|_W$, it is sufficient to show that $\|\ell\|'_{W_{k'}} = \|\ell\|_W$. Note that

$$\{v \in V \mid f(v) = \ell\} \subseteq \{v' \in V_{k'} \mid f_{k'}(v') = \ell\},$$

so that we have $\|\ell\|_W \geq \|\ell\|'_{W_{k'}}$. In the following, we prove the inequality $\|\ell\|_W \leq \|\ell\|'_{W_{k'}}$. For $\alpha \in (0, 1)$, let $(e_1, \ldots, e_r)$ be an α-orthogonal basis of V such that $(e_2, \ldots, e_r)$ forms a basis of $\mathrm{Ker}(f)$ and that $f(e_1) = \ell$. Then

$$\begin{aligned}\|\ell\|'_{W_{k'}} &= \inf\{\|e_1 + a'_2 e_2 + \cdots + a'_r e_r\|_{V,k'} \mid a'_2, \ldots, a'_r \in k'\} \\ &\geq \inf\{\alpha \max\{\|e_1\|, |a'_2|'\|e_2\|, \ldots, |a'_r|'\|e_r\|\} \mid a'_2, \ldots, a'_r \in k'\} \\ &= \alpha\|e_1\| \geq \alpha\|\ell\|_W.\end{aligned}$$

Therefore, we have $\|\ell\|'_{W_{k'}} \geq \|\ell\|_W$ by taking $\alpha \to 1$. □

Lemma 2.6. *We assume that the absolute value $|\cdot|$ of k is trivial. Let $(k', |\cdot|')$ be a complete valued extension of $(k, |\cdot|)$ such that $|\cdot|'$ is non-trivial. Let $\mathfrak{o}_{k'}$ be the valuation ring of $(k', |\cdot|')$ and $\mathfrak{m}_{k'}$ the maximal ideal of $\mathfrak{o}_{k'}$. Suppose that*

1. *The natural map $k \to \mathfrak{o}_{k'}$ induces an isomorphism $k \xrightarrow{\sim} \mathfrak{o}_{k'}/\mathfrak{m}_{k'}$;*
2. *For all elements v and v' in $V \setminus \{0\}$ such that $\|v\| \neq \|v'\|$, the quotient $\|v'\|/\|v\|$ does not belong to $|k'^{\times}|'$.*

Then $\|\cdot\|_{k'}$ is the only ultrametric norm on $V_{k'}$ extending $\|\cdot\|$.

Proof. We prove the assertion by induction on the rank r of V over k. The case where $r = 1$ is trivial. In the following, we suppose that the assertion has been proved for normed vector spaces of rank $< r$ over k.

Let $\|\cdot\|'$ be another ultrametric norm on $V_{k'}$ extending $\|\cdot\|$. Since $\|\cdot\|_{k'}$ is the largest ultrametric norm on $V_{k'}$ extending $\|\cdot\|$, we obtain that $\|\cdot\|_{k'} \geqslant \|\cdot\|'$. If the equality $\|\cdot\|' = \|\cdot\|_{k'}$ does not hold, then there exists a vector $x \in V_{k'}$ such that

$\|x\|' < \|x\|_{k'}$. Let $(e_i)_{i=1}^r$ be an orthogonal basis of $(V, \|\cdot\|)$, which is also an orthogonal basis of $(V_{k'}, \|\cdot\|_{k'})$. Suppose that x is written in the form $x = a_1e_1 + \cdots + a_re_r$, with $(a_1, \ldots, a_r) \in k'^r$. We will prove that $|a_i|'\|e_i\|$ are the same for $i \in \{1, \ldots, r\}$ arguing by contradiction. Without loss of generality, we assume on the contrary that

$$|a_1|'\|e_1\| \le \cdots \le |a_j|'\|e_j\| < |a_{j+1}|'\|e_{j+1}\| = \cdots = |a_r|'\|e_r\|$$

with $j \in \{1, \ldots, r-1\}$. Note that

$$\|x\|' < \|x\|_{k'} = \max_{i\in\{1,\ldots,r\}} |a_i|'\|e_i\| = |a_r|'\|e_r\|.$$

Moreover, by the induction hypothesis, the norms $\|\cdot\|'$ and $\|\cdot\|_{k'}$ coincide on $k'e_{j+1} + \cdots + k'e_r$. In particular, one has

$$\|a_{j+1}e_{j+1} + \cdots + a_re_r\|' = |a_r|'\|e_r\|.$$

Therefore, if we let $y = a_1e_1 + \cdots + a_je_j$, then we have

$$\|y\|' = \|x-(a_{j+1}e_{j+1} + \cdots + a_re_r)\|' = |a_r|'\|e_r\| > \max_{i\in\{1,\ldots,j\}} |a_i|'\|e_i\| = \|y\|_{k'},$$

which leads to a contradiction since $\|\cdot\|' \le \|\cdot\|_{k'}$. Hence we should have

$$|a_1|'\|e_1\| = \cdots = |a_r|'\|e_r\|.$$

By the condition (2), we have $\|e_1\| = \cdots = \|e_r\|$ (namely the function $\|\cdot\|$ is constant on $V \setminus \{0\}$) and hence $|a_1|' = \cdots = |a_r|' > 0$. By the assumption (1), we obtain that, for any $i \in \{1, \ldots, r\}$ there exists a $b_i \in k^\times$ such that $|a_i - b_ia_r|' < |a_r|'$. Thus

$$\|x\|' = \left\| a_r \sum_{i=1}^r b_ie_i + \sum_{i=1}^r (a_i - b_ia_r)e_i \right\|' = |a_r|'\|e_r\| = \|x\|_{k'}$$

since

$$\left\| a_r \sum_{i=1}^r b_ie_i \right\|' = |a_r|' \left\| \sum_{i=1}^r b_ie_i \right\|' = |a_r|'\|e_r\| \text{ and } \left\| \sum_{i=1}^r (a_i - b_ia_r)e_i \right\|' < |a_r|'\|e_r\|.$$

This leads to a contradiction. The lemma is thus proved. □

Remark 2.7. We assume that $|\cdot|'$ is discrete and

$$|a'|' = \exp(-\alpha \operatorname{ord}_{\mathfrak{o}_{k'}}(a')) \qquad a' \in k'$$

for $\alpha \in \mathbb{R}_{>0}$. If

$$\alpha \notin \bigcup_{v,v'\in V\setminus\{0\}} \mathbb{Q}(\log\|v\| - \log\|v'\|),$$

then the assumption (2) holds. Indeed, we suppose that $|a'|' = \|v\|/\|v'\|$ for some $a' \in k'^\times$ and $v, v' \in V \setminus \{0\}$. Then

$$-\alpha \operatorname{ord}_{\mathfrak{o}_{k'}}(a') = \log\|v\| - \log\|v'\|,$$

so that $\operatorname{ord}_{\mathfrak{o}_{k'}}(a') = 0$, and hence $\|v\| = \|v'\|$, as required.

2.2.5. *Lattices and norms*. From now on and until the end of the subsection, we assume that $|\cdot|$ is non-trivial. Let $\mathscr{V}$ be an $\mathfrak{o}_k$-submodule of V. We say that $\mathscr{V}$ is a *lattice of* V if $\mathscr{V} \otimes_{\mathfrak{o}_k} k = V$ and

$$\sup\{\|v\|_0 \mid v \in \mathscr{V}\} < \infty$$

for some norm $\|\cdot\|_0$ of V. Note that the condition $\sup\{\|v\|_0 \mid v \in \mathscr{V}\} < \infty$ does not depend on the choice of the norm $\|\cdot\|_0$ since all norms on V are equivalent. For any lattice $\mathscr{V}$ of V, we define $\|\cdot\|_{\mathscr{V}}$ to be

$$\|v\|_{\mathscr{V}} := \inf\{|a|^{-1} \mid a \in k^{\times} \text{ and } av \in \mathscr{V}\}.$$

Note that $\|\cdot\|_{\mathscr{V}}$ forms a norm of V. Moreover, for a norm $\|\cdot\|$ of V,

$$(V, \|\cdot\|)_{\leq 1} := \{v \in V \mid \|v\| \leq 1\}$$

is a lattice of V.

Proposition 2.8. *Let $\mathscr{V}$ be a lattice of V. We assume that, as an $\mathfrak{o}_k$-module, $\mathscr{V}$ admits a free basis $(e_1, \dots, e_r)$. Then $(e_1, \dots, e_r)$ is an orthonormal basis of V with respect to $\|\cdot\|_{\mathscr{V}}$.*

Proof. For $v = a_1e_1 + \cdots + a_re_r \in V$ and $a \in k^{\times}$, it holds

$$\begin{aligned} av \in \mathscr{V} &\iff aa_i \in \mathfrak{o}_k \text{ for all } i = 1, \dots, r \\ &\iff |a_i| \leq |a|^{-1} \text{ for all } i = 1, \dots, r \\ &\iff \max\{|a_1|, \dots, |a_r|\} \leq |a|^{-1}, \end{aligned}$$

so that $\|v\|_{\mathscr{V}} = \max\{|a_1|, \dots, |a_r|\}$. □

Lemma 2.9. *Let $\|\cdot\|$ be a norm of V and $\mathscr{V} := (V, \|\cdot\|)_{\leq 1}$. Then*

$$\|v\|_{\mathscr{V}} = \inf\{|b| \mid b \in k^{\times} \text{ and } \|v\| \leq |b|\}.$$

Moreover, $\|\cdot\| \leq \|\cdot\|_{\mathscr{V}}$ and $\|v\|_{\mathscr{V}} < |\alpha|\|v\|$ for all $\alpha \in k^{\times}$ with $|\alpha| > 1$ and $v \in V \setminus \{0\}$.

Proof. The first assertion is obvious because, for $a \in k^{\times}$, $av \in \mathscr{V}$ if and only if $\|v\| \leq |a|^{-1}$.

For $v \in V$, let $a \in k^{\times}$ with $av \in \mathscr{V}$. Then $\|av\| \leq 1$, that is, $\|v\| \leq |a|^{-1}$, and hence $\|v\| \leq \|v\|_{\mathscr{V}}$.

Finally we consider the second inequality, that is, $\|v\|_{\mathscr{V}} < |\alpha|\|v\|$ for $v \in V \setminus \{0\}$. As $|\alpha|^{-1} < 1$, there is an $\epsilon > 0$ with $|\alpha|^{-1}e^{\epsilon} < 1$. By the first assertion, we can choose $b \in k^{\times}$ such that $\|v\| \leq |b| \leq e^{\epsilon}\|v\|_{\mathscr{V}}$. If $\|v\| \leq |b\alpha^{-1}|$, then

$$\|v\|_{\mathscr{V}} \leq |b||\alpha|^{-1} \leq e^{\epsilon}\|v\|_{\mathscr{V}}|\alpha|^{-1}.$$

Thus $1 \leq e^{\epsilon}|\alpha|^{-1}$. This is a contradiction, so that $\|v\| > |b\alpha^{-1}|$. Therefore,

$$\|v\|_{\mathscr{V}} \leq |b| < |\alpha|\|v\|,$$

as required. □

Proposition 2.10. *We assume that $|\cdot|$ is discrete. Then we have the following:*

(1) *Every lattice $\mathscr{V}$ of V is a finitely generated $\mathfrak{o}_k$-module;*
(2) *If we set $\mathscr{V} := (V, \|\cdot\|)_{\leq 1}$ for a norm of $\|\cdot\|$ of V, then $\|v\| \leq \|v\|_{\mathscr{V}} < |\varpi|^{-1}\|v\|$ for $v \in V \setminus \{0\}$.*

Proof. (1) By Subsection 2.2.1, $(V, \|\cdot\|_{\mathscr{V}})_{\leq 1}$ is a finitely generated $\mathfrak{o}_k$-module. Moreover, note that $\mathscr{V} \subseteq (V, \|\cdot\|_{\mathscr{V}})_{\leq 1}$. Thus we have (1) because $\mathfrak{o}_k$ is Noetherian.

Part (2) follows from Lemma 2.9. □

Proposition 2.11. *We assume that $|\cdot|$ is not discrete. If we set $\mathscr{V} := (V, \|\cdot\|)_{\leq 1}$ for a norm of $\|\cdot\|$ of V, then $\|\cdot\| = \|\cdot\|_{\mathscr{V}}$.*

Proof. Since $|\cdot|$ is not discrete, $|k^\times|$ is a dense subgroup of $\mathbb{R}_+^\times$ (see [5, Chapter V, Section 1, no. 1 and Section 4, no. 1]). We can thus find a sequence $\{\beta_n\}_{n=1}^\infty$ such that $|\beta_n| > 1$ and $\lim_{n\to\infty} |\beta_n| = 1$. On the other hand, by Lemma 2.9, it holds

$$\|\cdot\| \leq \|\cdot\|_{\mathscr{V}} \leq |\beta_n|\|\cdot\|.$$

Therefore the assertion follows. □

Proposition 2.12. *We assume that the absolute value $|\cdot|$ is not discrete. Let $\|\cdot\|$ be a norm of V and $\mathscr{V} := (V, \|\cdot\|)_{\leq 1}$. For any $\epsilon > 0$, there is a sub-lattice $\mathscr{V}'$ of $\mathscr{V}$ such that $\mathscr{V}'$ is finitely generated over $\mathfrak{o}_k$ and $\|\cdot\| \leq \|\cdot\|_{\mathscr{V}'} \leq e^\epsilon \|\cdot\|$.*

Proof. Let $(e_1, \ldots, e_r)$ be an $e^{-\epsilon/2}$-orthogonal basis of V with respect to $\|\cdot\|$ (*cf.* Proposition 2.2). We can find a $\lambda_i \in k^\times$ such that $\|e_i\| \leq |\lambda_i| \leq e^{\epsilon/2}\|e_i\|$ for each i. We set $\omega_i := \lambda_i^{-1} e_i$ $(i = 1, \ldots, r)$ and $\mathscr{V}' := \mathfrak{o}_k\omega_1 + \cdots + \mathfrak{o}_k\omega_r$. Note that $\omega_i \in \mathscr{V}$ for all i, that is, $\mathscr{V}'$ is a sub-lattice of $\mathscr{V}$ and $\mathscr{V}'$ is finitely generated over $\mathfrak{o}_k$. By Proposition 2.11, one has $\|\cdot\| = \|\cdot\|_{\mathscr{V}}$, and hence $\|\cdot\| \leq \|\cdot\|_{\mathscr{V}'}$. Moreover, for $c_1, \ldots, c_r \in k$, by Proposition 2.8,

$$\begin{aligned}\|c_1e_1 + \cdots + c_re_r\|_{\mathscr{V}'} = \|c_1\lambda_1\omega_1 + \cdots + c_r\lambda_r\omega_r\|_{\mathscr{V}'} &= \max\{|c_1\lambda_1|, \ldots, |c_r\lambda_r|\} \\ &\leq e^{\epsilon/2}\max\{|c_1|\|e_1\|, \ldots, |c_r|\|e_r\|\} \\ &\leq e^\epsilon \|c_1e_1 + \cdots + c_re_r\|,\end{aligned}$$

hence we have $\|\cdot\|_{\mathscr{V}'} \leq e^\epsilon \|\cdot\|$. □

2.3. Seminorm and integral extension

Let $\mathscr{A}$ be a finitely generated $\mathfrak{o}_k$-algebra, which contains $\mathfrak{o}_k$ as a subring. We set $A := \mathscr{A} \otimes_{\mathfrak{o}_k} k$. Note that A coincides with the localization of $\mathscr{A}$ with respect to $S := \mathfrak{o}_k \setminus \{0\}$. Let $\operatorname{Spec}(A)^{\mathrm{an}}$ be the analytification of $\operatorname{Spec}(A)$, that is, the set of all seminorms of A over the absolute value of k. For $x \in \operatorname{Spec}(A)^{\mathrm{an}}$, let $\mathfrak{o}_x$ and $\mathfrak{m}_x$ be the valuation ring of $(\hat{\kappa}(x), |\cdot|_x)$ and the maximal ideal of $\mathfrak{o}_x$, respectively (see Subsection 2.1.3 for the definition of $\hat{\kappa}(x)$). We denote the natural homomorphism $A \to \hat{\kappa}(x)$ by φ_x. It is easy to see that the following are equivalent:

(1) $\operatorname{Spec}(\hat{\kappa}(x)) \to \operatorname{Spec}(A)$ extends to $\operatorname{Spec}(\mathfrak{o}_x) \to \operatorname{Spec}(\mathscr{A})$, that is, there is a ring homomorphism $\tilde{\varphi}_x : \mathscr{A} \to \mathfrak{o}_x$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\tilde{\varphi}_x} & \mathfrak{o}_x \\ \downarrow & & \downarrow \\ A & \xrightarrow{\varphi_x} & \hat{\kappa}(x)\,; \end{array}$$

(2) $|a|_x \leq 1$ for all $a \in \mathscr{A}$.

Moreover, under the above conditions, the image of $\mathfrak{m}_x$ of $\operatorname{Spec}(\mathfrak{o}_x)$ is given by $\tilde{\varphi}_x^{-1}(\mathfrak{m}_x) = (\mathscr{A}, |\cdot|_x)_{<1}$, and $(\mathscr{A}, |\cdot|_x)_{<1} \in \operatorname{Spec}(\mathscr{A})_\circ$, where

$$\begin{cases} (\mathscr{A}, |\cdot|_x)_{<1} := \{a \in \mathscr{A} \mid |a|_x < 1\}, \\ \operatorname{Spec}(\mathscr{A})_\circ := \{P \in \operatorname{Spec}(\mathscr{A}) \mid P \cap \mathfrak{o}_k = \mathfrak{m}_k\}. \end{cases}$$

Let $\operatorname{Spec}(A)^{\mathrm{an}}_{\mathscr{A}}$ be the set of all $x \in \operatorname{Spec}(A)^{\mathrm{an}}$ such that the above condition (2) is satisfied. The map $r_{\mathscr{A}} : \operatorname{Spec}(A)^{\mathrm{an}}_{\mathscr{A}} \to \operatorname{Spec}(\mathscr{A})_\circ$ given by

$$x \mapsto (\mathscr{A}, |\cdot|_x)_{<1}$$

is called the reduction map (*cf.* Subsection 2.1.6). Note that the reduction map is surjective (*cf.* [2, Proposition 2.4.4] or [11, 4.13 and Proposition 4.14]).

Theorem 2.13. *If we set* $\mathscr{B} := \{\alpha \in A \mid \alpha$ *is integral over* $\mathscr{A}\}$, *then*

$$\mathscr{B} = \bigcap_{x \in \operatorname{Spec}(A)^{\mathrm{an}}_{\mathscr{A}}} (A, |\cdot|_x)_{\leq 1},$$

where $(A, |\cdot|_x)_{\leq 1} := \{\alpha \in A \mid |\alpha|_x \leq 1\}$.

Proof. Let us first see that $\mathscr{B} \subseteq (A, |\cdot|_x)_{\leq 1}$ for all $x \in \operatorname{Spec}(A)^{\mathrm{an}}_{\mathscr{A}}$. If $a \in \mathscr{B}$, then there are $a_1, \ldots, a_n \in \mathscr{A}$ such that $a^n + a_1 a^{n-1} + \cdots + a_n = 0$. We assume that $|a|_x > 1$. Then

$$\begin{aligned} |a|_x^n = |a^n|_x = |a_1 a^{n-1} + \cdots + a_n|_x &\leq \max_{i=1,\ldots,n} \{|a_i|_x |a|_x^{n-i}\} \\ &\leq \max_{i=1,\ldots,n} \{|a|_x^{n-i}\} = |a|_x^{n-1}, \end{aligned}$$

so that $|a|_x \leq 1$, which is a contradiction.

Let $a \in A$ such that a is not integral over $\mathscr{A}$. We show that there exists a prime ideal $\mathfrak{q}$ of $\mathscr{A}$ such that the canonical image of a in $A/S^{-1}\mathfrak{q}$ is not integral over $\mathscr{A}/\mathfrak{q}$. In fact, since A is a k-algebra of finite type, it is a noetherian ring. In particular, it admits only finitely many minimal prime ideals $S^{-1}\mathfrak{p}_1, \ldots, S^{-1}\mathfrak{p}_n$, where $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ are prime ideals of $\mathscr{A}$ which do not intersect $S = \mathfrak{o}_k \setminus \{0\}$. Assume that, for any $i \in \{1, \dots, n\}$, there is a monic polynomial f_i in $(\mathscr{A}/\mathfrak{p}_i)[T]$ on such that $f_i(\lambda_i) = 0$, where λ_i is the class of a in $A/S^{-1}\mathfrak{p}_i$. Let F_i be a monic polynomial in $\mathscr{A}[T]$ whose reduction modulo $\mathfrak{p}_i[T]$ coincides with f_i. One has $F_i(a) \in S^{-1}\mathfrak{p}_i$ for any $i \in \{1, \dots, n\}$. Let F be the product of the polynomials $F_1, \dots, F_n$. Then $F(a)$ belongs to the intersection $\bigcap_{i=1}^n S^{-1}\mathfrak{p}_i$, hence is nilpotent, which implies that a is integral over $\mathscr{A}$. To show that there exists an $x \in \mathrm{Spec}(A)^{\mathrm{an}}_{\mathscr{A}}$ such that $|a|_x > 1$ we may replace $\mathscr{A}$ (respectively A) by $\mathscr{A}/\mathfrak{q}$ (respectively $A/S^{-1}\mathfrak{q}$) and hence assume that $\mathscr{A}$ is an integral domain without loss of generality.

We set $b = a^{-1}$. Let us see that

$$b\mathscr{A}[b] \cap \mathfrak{o}_k \neq \{0\} \quad \text{and} \quad 1 \notin b\mathscr{A}[b].$$

We set $a = a'/s$ for some $a' \in \mathscr{A}$ and $s \in S$. Then $s = ba' \in b\mathscr{A}[b] \cap \mathfrak{o}_k$, so that $b\mathscr{A}[b] \cap \mathfrak{o}_k \neq \{0\}$. Next we assume that $1 \in b\mathscr{A}[b]$. Then

$$1 = a'_1 b + a'_2 b^2 + \cdots + a'_{n'} b^{n'}$$

for some $a'_1, \dots, a'_{n'} \in \mathscr{A}$, so that $a^{n'} = a'_1 a^{n'-1} + \cdots + a'_{n'}$, which is a contradiction.

Let $\mathfrak{p}$ be the maximal ideal of $\mathscr{A}[b]$ such that $b\mathscr{A}[b] \subseteq \mathfrak{p}$. As $\mathfrak{p} \cap \mathfrak{o}_k \neq \{0\}$ and $\mathfrak{p} \cap \mathfrak{o}_k \subseteq \mathfrak{m}_k$, we have $\mathfrak{p} \cap \mathfrak{o}_k = \mathfrak{m}_k$, and hence $\mathfrak{p} \in \mathrm{Spec}(\mathscr{A}[b])_\circ$. Note that $\mathscr{A}[b]$ is finitely generated over $\mathfrak{o}_k$ and $\mathscr{A}[b] \otimes_{\mathfrak{o}_k} k = A[b]$. Thus, since the reduction map

$$r_{\mathscr{A}[b]} : \mathrm{Spec}(A[b])^{\mathrm{an}}_{\mathscr{A}[b]} \to \mathrm{Spec}(\mathscr{A}[b])_\circ$$

is surjective, there is an $x \in \mathrm{Spec}(A[b])^{\mathrm{an}}_{\mathscr{A}[b]}$ such that $r_{\mathscr{A}[b]}(x) = \mathfrak{p}$. Clearly $x \in \mathrm{Spec}(A)^{\mathrm{an}}_{\mathscr{A}}$. As $b \in \mathfrak{p}$, we have $|b|_x < 1$, so that $|a|_x > 1$ because $ab = 1$. Therefore,

$$a \notin \bigcap_{x \in \mathrm{Spec}(A)^{\mathrm{an}}_{\mathscr{A}}} (A, |\cdot|_x)_{\leq 1},$$

as required. □

We assume that X is projective. Let $\mathscr{X} \to \mathrm{Spec}(\mathfrak{o}_k)$ be a flat and projective scheme over $\mathrm{Spec}\,\mathfrak{o}_k$ such that the generic fiber of $\mathscr{X} \to \mathrm{Spec}(\mathfrak{o}_k)$ is X. Let $\mathscr{L}$ be an invertible sheaf on $\mathscr{X}$ such that $\mathscr{L}\big|_X = L$. We set $h := \{|\cdot|_{\mathscr{L}}(x)\}_{x \in X^{\mathrm{an}}}$. For the definition of the metric $|\cdot|_{\mathscr{L}}(x)$ at x, see Subsection 2.1.6.

Corollary 2.14. *Fix $l \in H^0(X, L)$. If $|l|_{\mathscr{L}}(x) \leq 1$ for all $x \in X^{\mathrm{an}}$, then there is an $s \in \mathfrak{o}_k \setminus \{0\}$ such that $sl^{\otimes n} \in H^0(\mathscr{X}, \mathscr{L}^{\otimes n})$ for all $n \geq 0$.*

Proof. Let $\mathscr{X} = \bigcup_{i=1}^N \mathrm{Spec}(\mathscr{A}_i)$ be an affine open covering of $\mathscr{X}$ with the following properties:

(1) $\mathscr{A}_i$ is a finitely generated algebra over $\mathfrak{o}_k$ for every i;
(2) $\mathrm{Spec}(\mathscr{A}_i)_\circ \neq \emptyset$ for all i;
(3) There is a basis ω_i of $\mathscr{L}$ over $\mathrm{Spec}(\mathscr{A}_i)$ for every i.

We set $l = a_i\omega_i$ for some $a_i \in A_i := \mathscr{A}_i \otimes_{\mathfrak{o}_k} k$. By our assumption, $|a_i|_x \le 1$ for all $x \in \operatorname{Spec}(A_i)^{\mathrm{an}}_{\mathscr{A}_i}$. Therefore, by Theorem 2.13, a_i is integral over $\mathscr{A}_i$, so that, by the following Lemma 2.15, we can find $s_i \in S$ such that $s_i a_i^n \in \mathscr{A}_i$ for all $n \ge 0$. We set $s = s_1 \cdots s_N$. Then, as $sa_i^n \in \mathscr{A}_i$ for all $n \ge 0$ and $i = 1, \ldots, N$, we have the assertion. □

Lemma 2.15. *Let A be a commutative ring and S be a multiplicatively closed subset of A, which consists of regular elements of A. If $t \in S^{-1}A$ and t is integral over A, then there is an $s \in S$ such that $st^n \in A$ for all $n \ge 0$.*

Proof. As t is integral over A, there are $a_1, \ldots, a_{r-1} \in A$ such that

$$t^r = a_1 t^{r-1} + \cdots + a_{r-1}t + a_r.$$

We choose $s \in S$ such that $st^i \in A$ for $i = 0, \ldots, r-1$. By induction on n, we prove that $st^n \in A$ for all $n \ge 0$. Note that

$$t^n = a_1 t^{n-1} + \cdots + a_{r-1}t^{n-r+1} + a_r t^{n-r}.$$

Thus, if $st^i \in A$ for $i = 0, \ldots, n-1$, then $st^n \in A$ because

$$st^n = a_1\big(st^{n-1}\big) + \cdots + a_{r-1}\big(st^{n-r+1}\big) + a_r\big(st^{n-r}\big).$$

□

2.4. Extension obstruction index

In this subsection, we introduce an invariant to describe the obstruction to the extension property. Let X be a reduced projective scheme over $\operatorname{Spec} k$, L be an invertible sheaf on X equipped with a continuous metric h, and Y be a reduced closed subscheme of X. For any non-zero element l of $H^0(Y, L|_Y)$, we denote by $\lambda_h(l)$ the following number (if there does not exist any section $s \in H^0(X, L^{\otimes n})$ extending $l^{\otimes n}$, then the infimum in the formula is defined to be $+\infty$ by convention)

$$\lambda_h(l) = \limsup_{n\to+\infty} \inf_{\substack{s\in H^0(X,L^{\otimes n})\\ s|_Y = l^{\otimes n}}} \left(\frac{\log \|s\|_{h^n}}{n} - \log \|l\|_{Y,h}\right) \in [0, +\infty]. \tag{2.6}$$

This invariant allows to describe in a numerically way the obstruction to the metric extendability of the section l. In fact, the following assertions are equivalent:

(a) $\lambda_h(l) = 0$;
(b) for any $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}_{\ge 1}$ such that, for any integer $n \ge n_0$, the element $l^{\otimes n}$ extends to a section $s \in H^0(X, L^{\otimes n})$ such that $\|s\|_{h^n} \le e^{\epsilon n}\|l\|^n_{Y,h}$.

The following proposition shows that, if $l^{\otimes n}$ extends to a global section of $L^{\otimes n}$ for sufficiently positive n (notably this happens when the line bundle L is ample), then the limsup defining $\lambda_h(l)$ is actually a limit.

Proposition 2.16. *For any integer $n \geq 1$, let*

$$a_n = \inf_{\substack{s\in H^0(X,L^{\otimes n})\\ s|_Y=l^{\otimes n}}} \Big(\log \|s\|_{h^n} - n\log\|l\|_{Y,h} \Big).$$

Then the sequence $(a_n)_{n\geq 1}$ is sub-additive, namely one has $a_{m+n} \leq a_m + a_n$ for any $(m,n) \in \mathbb{N}_{\geq 1}$. In particular, if for sufficiently positive integer n, the section $l^{\otimes n}$ lies in the image of the restriction map $H^0(X, L^{\otimes n}) \to H^0(Y, L|_Y^{\otimes n})$, then the limit superior in (2.6) *is actually a limit.*

Proof. By (2.1), one has $a_n \geq 0$ for any integer $n \geq 1$. Moreover, $a_n < +\infty$ if and only if l^n lies in the image of the restriction map $H^0(X, L^{\otimes n}) \to H^0(Y, L|_Y^{\otimes n})$. To verify the inequality $a_{m+n} \leq a_m + a_n$, it suffices to consider the case where both a_m and a_n are finite. Let s_m and s_n be respectively sections in $H^0(X, L^{\otimes m})$ and $H^0(X, L^{\otimes n})$ such that $s_m|_Y = l^{\otimes m}$ and $s_n|_Y = l^{\otimes n}$, then the section $s = s_m \otimes s_n \in H^0(X, L^{\otimes(m+n)})$ verifies the relation $s|_Y = l^{\otimes(n+m)}$. Moreover, one has

$$\|s\|_{h^{m+n}} = \sup_{x\in X^{\mathrm{an}}} |s|_{h^{m+n}}(x) = \sup_{x\in X^{\mathrm{an}}} (|s_m|_{h^m}(x)\cdot|s_n|_{h^n}(x)) \leq \|s_m\|_{h^m}\cdot\|s_n\|_{h^n}.$$

Since s_m and s_n are arbitrary, one has $a_{m+n} \leq a_m + a_n$. Finally, by Fekete's lemma, if $a_n < +\infty$ for sufficiently positive integer n, then the sequence $(a_n/n)_{n\geq 1}$ actually converges in $\mathbb{R}_+$. The proposition is thus proved. □

Corollary 2.17. *Assume that the invertible sheaf L is ample, then the following conditions are equivalent.*

(a) $\lambda_h(l) = 0$*;*
(b) *for any $\epsilon > 0$, there exists an $n \in \mathbb{N}_{\geq 1}$ and a section $s \in H^0(X, L^{\otimes n})$ such that $s|_Y = l^n$ and that $\|s\|_{h^n} \leq e^{\epsilon n}\|l\|_{Y,h}^n$.*

Proof. We keep the notation of the previous proposition. By definition the second condition is equivalent to

$$\liminf_{n\to+\infty} \frac{a_n}{n} = 0. \tag{2.7}$$

Since L is ample, Proposition 2.16 leads to the convergence of the sequence $(a_n/n)_{n\geq 1}$ in $\mathbb{R}_+$. Hence the condition (2.7) is equivalent to $\lambda_h(l) = 0$. □

3. Continuous metrics of invertible sheaves

In this section, we consider several properties of continuous metrics of invertible sheaves. Throughout this section, let X be a reduced scheme of finite type over $\operatorname{Spec} k$ and L be an invertible $\mathscr{O}_X$-module.

3.1. Quotient metric

Let V be a finite-dimensional vector space over k. We assume that there is a surjective homomorphism

$$\pi : V \otimes_k \mathscr{O}_X \to L.$$

For each $e \in V$, $\pi(e \otimes 1)$ yields a global section of L, that is, $\pi(e \otimes 1) \in H^0(X, L)$. We denote it by $\tilde{e}$. Let $\|\cdot\|$ be a norm of V and $\overline{V} := (V, \|\cdot\|)$. Let $\|\cdot\|_{\hat{\kappa}(x)}$ be a norm of $V \otimes_k \hat{\kappa}(x)$ obtained by the scalar extension of $\|\cdot\|$ (*cf.* Definition 2.4). Let $|\cdot|_{\overline{V}}^{\mathrm{quot}}(x)$ be the quotient norm of $L(x) := L \otimes \hat{\kappa}(x)$ induced by $\|\cdot\|_{\hat{\kappa}(x)}$ and the surjective homomorphism $V \otimes_k \hat{\kappa}(x) \to L(x)$.

Proposition 3.1. *The family* $\left\{|\cdot|_{\overline{V}}^{\mathrm{quot}}(x)\right\}_{x \in X^{\mathrm{an}}}$ *defines a continuous metric on* L^{an}.

Proof. The problem is local for the Zariski topology. Hence we may assume without loss of generality that L is the trivial $\mathscr{O}_X$-module. Denote by s_0 the global section of L which trivializes L on X. It suffices to show that the function $|s_0|_{\overline{V}}^{\mathrm{quot}}$ is continuous on X^{an}.

For any point $x \in X^{\mathrm{an}}$ and any element $s \in V \otimes_k \hat{\kappa}(x)$, there exists a unique element $f_x(s) \in \widehat{\kappa}(x)$ such that $s(x) = f_x(s)s_0(x)$, where $s(x)$ denotes the image of s by the natural (surjective) homomorphism $\pi(x) : V \otimes_k \hat{\kappa}(x) \to L \otimes_{\mathscr{O}_X} \hat{\kappa}(x)$. The map f_x is a linear form on $V \otimes_k \hat{\kappa}(x)$, and one has $|s_0|_{\overline{V}}^{\mathrm{quot}}(x) = \left(\|f_x\|_{\hat{\kappa}(x)}^{\vee}\right)^{-1}$, where $\|\cdot\|_{\hat{\kappa}(x)}^{\vee}$ denotes the dual norm of $\|\cdot\|_{\hat{\kappa}(x)}$ (see Subsection 2.2.3).

It remains to prove that the function $(x \in X^{\mathrm{an}}) \mapsto \|f_x\|_{\hat{\kappa}(x)}^{\vee}$ is continuous. We first treat the case where $(V, \|\cdot\|)$ admits an orthogonal basis $(e_i)_{i=1}^r$ (see Subsection 2.2.2 for the notion of orthogonality). Let $(e_i^{\vee})_{i=1}^r$ be its dual basis. For any $x \in X^{\mathrm{an}}$, $(e_i^{\vee})_{i=1}^r$ is an orthogonal basis of $(V \otimes_k \hat{\kappa}(x))^{\vee}$ (see Subsections 2.2.3-2.2.4). Moreover, by construction there exist regular functions $g_1, \ldots, g_r$ on X such that $f_x = g_1(x)e_1^{\vee} + \cdots + g_r(x)e_r^{\vee}$. Note that

$$\|f_x\|_{\hat{\kappa}(x)}^{\vee} = \max_{i \in \{1,\ldots,r\}} |g_i(x)|_x \cdot \left\|e_i^{\vee}\right\|_{\hat{\kappa}(x)}^{\vee} = \max_{i \in \{1,\ldots,r\}} |g_i(x)|_x \cdot \left\|e_i^{\vee}\right\|^{\vee},$$

where $\|\cdot\|^{\vee}$ denotes the dual norm of $\|\cdot\|$. Therefore the function $x \mapsto \|f_x\|_{\hat{\kappa}(x)}^{\vee}$ is continuous.

We now consider the general case. By Proposition 2.2, for any integer $n \geqslant 2$, there exists a basis $(e_i^{(n)})_{i=1}^r$ of V which is $(1-\frac{1}{n})$-orthogonal. Let $\|\cdot\|_n$ be the norm on V such that

$$\left\|\lambda_1 e_1^{(n)} + \cdots + \lambda_r e^{(n)}\right\|_n = \max_{i \in \{1,\ldots,n\}} |\lambda_i| \cdot \left\|e_i^{(n)}\right\| \text{ for all } (\lambda_1, \ldots, \lambda_r) \in k^r.$$

Since the basis $(e_i^{(n)})_{i=1}^r$ is $(1-\frac{1}{n})$-orthogonal, we obtain that

$$\left(1 - \frac{1}{n}\right)\|\cdot\|_n \leq \|\cdot\| \leq \|\cdot\|_n.$$

Therefore, if we denote by $|\cdot|^{\mathrm{quot}}_{(V,\|\cdot\|_n)}(x)$ the quotient norm on $L(x)$ induced by $\|\cdot\|_{n,\hat{\kappa}(x)}$, where $x \in X^{\mathrm{an}}$, one has

$$\left| \log |s_0|^{\mathrm{quot}}_{(V,\|\cdot\|_n)}(x) - \log |s_0|^{\mathrm{quot}}_{\overline{V}}(x) \right| \leq -\log\left(1 - \frac{1}{n}\right) \text{ for all } x \in X^{\mathrm{an}}.$$

By the particular case we have proved above, each function $\log |s_0|^{\mathrm{quot}}_{(V,\|\cdot\|_n)}$ is continuous. Therefore, the function $\log |s_0|^{\mathrm{quot}}_{\overline{V}}$, which is the uniform limit of a sequence of continuous functions, is also continuous. Thus we obtain the continuity of the function $|s_0|^{\mathrm{quot}}_{(V,\|\cdot\|_n)}$ and the proposition is proved. □

From now on and until the end of the subsection, we assume that X is projective and L is generated by global sections. Let $h = \{|\cdot|_h(x)\}_{x\in X^{\mathrm{an}}}$ be a continuous metric of L^{an}. As $H^0(X, L) \otimes_k \mathscr{O}_X \to L$ is surjective, by Proposition 3.1,

$$h^{\mathrm{quot}} = \left\{ |\cdot|^{\mathrm{quot}}_{(H^0(X,L),\|\cdot\|_h)}(x) \right\}_{x\in X^{\mathrm{an}}}$$

yields a continuous metric of L^{an}. For simplicity, we denote $|\cdot|^{\mathrm{quot}}_{(H^0(X,L),\|\cdot\|_h)}(x)$ by $|\cdot|^{\mathrm{quot}}_h(x)$. Moreover, the supremum norm of $H^0(X, L)$ arising from h^{quot} is denoted by $\|\cdot\|^{\mathrm{quot}}_h$, that is, $\|\cdot\|^{\mathrm{quot}}_h := \|\cdot\|_{h^{\mathrm{quot}}}$.

Lemma 3.2. *The following statements hold:*

(1) *We have* $|\cdot|_h(x) \leq |\cdot|^{\mathrm{quot}}_h(x)$ *for all* $x \in X^{\mathrm{an}}$*;*
(2) *We have* $\|\cdot\|_h = \|\cdot\|^{\mathrm{quot}}_h$*;*
(3) *Let* (L', h') *be a pair of an invertible sheaf* L' *on* X *and a continuous metric* $h' = \{|\cdot|_{h'}(x)\}_{x\in X^{\mathrm{an}}}$ *of* L'^{an} *such that* L' *is generated by global sections. Then*

$$|l \cdot l'|^{\mathrm{quot}}_{h\otimes h'}(x) \leq |l|^{\mathrm{quot}}_h(x)|l'|^{\mathrm{quot}}_{h'}(x)$$

for $l \in L(x)$ *and* $l' \in L'(x)$.

Proof. (1) Fix $l \in L(x) \setminus \{0\}$. For $\epsilon > 0$, let $(e_1, \ldots, e_n)$ be an $e^{-\epsilon}$-orthogonal basis of $H^0(X, L)$ with respect to $\|\cdot\|_h$. There is an $s \in H^0(X, L) \otimes_k \hat{\kappa}(x)$ such that $s(x) = l$ and $\|s\|_{h,\hat{\kappa}(x)} \leq e^{\epsilon}|l|^{\mathrm{quot}}_h(x)$. We set $s = a_1e_1 + \cdots + a_ne_n$ $(a_1, \ldots, a_n \in \hat{\kappa}(x))$. Then, by Subsection 2.2.4,

$$\begin{aligned}\|s\|_{h,\hat{\kappa}(x)} &\geq e^{-\epsilon} \max\{|a_1|_x\|e_1\|_h, \ldots, |a_n|_x\|e_n\|_h\} \\ &\geq e^{-\epsilon} \max\{|a_1|_x|e_1|_h(x), \ldots, |a_n|_x|e_n|_h(x)\} \geq e^{-\epsilon}|l|_h(x),\end{aligned}$$

so that $|l|_h(x) \leq e^{2\epsilon}|l|^{\mathrm{quot}}_h(x)$, and hence the assertion follows because ϵ is an arbitrary positive number.

(2) By (1), we have $\|\cdot\|_h \le \|\cdot\|_h^{\mathrm{quot}}$. On the other hand, as $|s|_h^{\mathrm{quot}}(x) \le \|s\|_h$ for $s \in H^0(X, L)$, we have $\|s\|_h^{\mathrm{quot}} \le \|s\|_h$.

(3) For $\epsilon > 0$, there are an $s \in H^0(X, L) \otimes_k \hat{\kappa}(x)$ and an $s' \in H^0(X, L') \otimes_k \hat{\kappa}(x)$ such that

$$s(x) = l,\ s'(x) = l',\ \|s\|_{h,\hat{\kappa}(x)} \le e^{\epsilon}|l|_h^{\mathrm{quot}}(x) \text{ and } \|s'\|_{h',\hat{\kappa}(x)} \le e^{\epsilon}|l'|_{h'}^{\mathrm{quot}}(x).$$

Let us show $\|s \cdot s'\|_{h\otimes h',\hat{\kappa}(x)} \le e^{2\epsilon}\|s\|_{h,\hat{\kappa}(x)}\|s'\|_{h',\hat{\kappa}(x)}$. Let $(s_1, \dots, s_m)$ and $(s'_1, \dots, s'_{m'})$ be $e^{-\epsilon}$-orthogonal bases of $H^0(X, L)$ and $H^0(X, L')$, respectively. If we set $s = t_1 s_1 + \cdots + t_m s_m$ and $s' = t'_1 s'_1 + \cdots + t'_{m'} s'_{m'}$ (with $t_1, \dots, t_m, t'_1, \dots, t'_{m'} \in \hat{\kappa}(x)$), then

$$s \cdot s' = \sum_{i,j} t_i t'_j s_i \cdot s'_j.$$

Thus,

$$\begin{aligned}\|s \cdot s'\|_{h\otimes h',\hat{\kappa}(x)} &\le \max_{i,j}\left\{|t_i|_x |t'_j|_x \|s_i \cdot s'_j\|_{h\otimes h'}\right\} \le \max_{i,j}\left\{|t_i|_x |t'_j|_x \|s_i\|_h \|s'_j\|_{h'}\right\} \\ &\le \max_i \{|t_i|_x \|s_i\|_h\} \max_j \left\{|t'_j|_x \|s'_j\|_{h'}\right\} \\ &\le e^{2\epsilon}\|s\|_{h,\hat{\kappa}(x)}\|s'\|_{h',\hat{\kappa}(x)}.\end{aligned}$$

Therefore, we have $(s \cdot s')(x) = l \cdot l'$ and

$$|l \cdot l'|_{h\otimes h'}^{\mathrm{quot}}(x) \le \|s \cdot s'\|_{h\otimes h',\hat{\kappa}(x)} \le e^{2\epsilon}\|s\|_{h,\hat{\kappa}(x)}\|s'\|_{h',\hat{\kappa}(x)} \le e^{4\epsilon}|l|_h^{\mathrm{quot}}(x)|l'|_{h'}^{\mathrm{quot}}(x),$$

as required. ☐

Proposition 3.3. *If there are a normed finite-dimensional vector space* $(V, \|\cdot\|)$ *and a surjective homomorphism* $V \otimes_k \mathscr{O}_X \to L$ *such that* h *is given by* $\left\{|\cdot|_{(V,\|\cdot\|)}^{\mathrm{quot}}(x)\right\}_{x\in X^{\mathrm{an}}}$, *then* $|\cdot|_{h^n}(x) = |\cdot|_{h^n}^{\mathrm{quot}}(x)$ *for all* $n \ge 1$.

Proof. First we consider the case $n = 1$. Fix $l \in L(x) \setminus \{0\}$. For $\epsilon > 0$, there is an $s \in V \otimes_k \hat{\kappa}(x)$ such that $s(x) = l$ and $\|s\|_{\hat{\kappa}(x)} \le e^{\epsilon}|l|_h(x)$.

Note that $\|u\|_h \le \|u\|$ for all $u \in V$. Let $(e_1, \dots, e_r)$ be an $e^{-\epsilon}$-orthogonal basis of V with respect to $\|\cdot\|$. If we set $s = a_1 e_1 + \cdots + a_r e_r$ (with $a_1, \dots, a_r \in \hat{\kappa}(x)$), then, by Subsection 2.2.4,

$$\begin{aligned}\|s\|_{h,\hat{\kappa}(x)} &\le \max\{|a_1|_x \|e_1\|_h, \dots, |a_r|_x \|e_r\|_h\} \\ &\le \max\{|a_1|_x \|e_1\|, \dots, |a_r|_x \|e_r\|\} \\ &\le e^{\epsilon}\|s\|_{\hat{\kappa}(x)},\end{aligned}$$

so that

$$|l|_h^{\mathrm{quot}}(x) \le \|s\|_{h,\hat{\kappa}(x)} \le e^{\epsilon}\|s\|_{\hat{\kappa}(x)} \le e^{2\epsilon}|l|_h(x),$$

and hence $|l|_h^{\mathrm{quot}}(x) \leq |l|_h(x)$ by taking $\epsilon \to 0$. Thus the assertion for $n = 1$ follows from (1) in Lemma 3.2.

In general, by using (3) in Lemma 3.2,

$$|l^n|_{h^n}(x) = (|l|_h(x))^n = \left(|l|_h^{\mathrm{quot}}(x)\right)^n \geq |l^n|_{h^n}^{\mathrm{quot}}(x),$$

and hence we have the assertion by (1) in Lemma 3.2. □

Lemma 3.4. *We assume that there are a normed finite-dimensional vector space* $(V, \|\cdot\|)$ *and a surjective homomorphism* $V \otimes_k \mathscr{O}_X \to L$ *such that* h *is given by* $\left\{|\cdot|_{(V,\|\cdot\|)}^{\mathrm{quot}}(x)\right\}_{x \in X^{\mathrm{an}}}$. *Let* k' *be a separable extension field of* k, *and let* $|\cdot|'$ *be a complete absolute value of* k' *as an extension of* $|\cdot|$. *We set*

$$X' := X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k'), \quad L' = L \otimes_k k' \quad \text{and} \quad V' := V \otimes_k k'.$$

Let $\|\cdot\|'$ *be a norm of* V' *obtained by the scalar extension of* $\|\cdot\|$. *Moreover, let* h' *be a continuous metric of* L'^{an} *given by the scalar extension of* h. *Then* h' *coincides with* $\left\{|\cdot|_{(V',\|\cdot\|')}^{\mathrm{quot}}(x')\right\}_{x' \in X'^{\mathrm{an}}}$.

Proof. Let $f : X' \to X$ be the projection. For $x' \in X'^{\mathrm{an}}$, we set $x = f^{\mathrm{an}}(x')$. Then $\hat{\kappa}(x) \subseteq \hat{\kappa}(x')$ and $(L \otimes_k \hat{\kappa}(x)) \otimes_{\hat{\kappa}(x)} \hat{\kappa}(x') = L' \otimes_{k'} \hat{\kappa}(x')$, that is, $L(x) \otimes_{\hat{\kappa}(x)} \hat{\kappa}(x') = L'(x')$. Moreover, $V' \otimes_{k'} \hat{\kappa}(x') = (V \otimes_k \hat{\kappa}(x)) \otimes_{\hat{\kappa}(x)} \hat{\kappa}(x')$, and by (2.5), $\|\cdot\|'_{\hat{\kappa}(x')} = \|\cdot\|_{\hat{\kappa}(x')} = \|\cdot\|_{\hat{\kappa}(x),\hat{\kappa}(x')}$. Thus the assertion follows from Lemma 2.5. □

Proposition 3.5. *We assume that there is a subspace* H *of* $H^0(X, L)$ *such that* $H \otimes_k \mathscr{O}_X \to L$ *is surjective and the morphism* $\phi_H : X \to \mathbb{P}(H)$ *induced by* H *is a closed embedding. We identify* X *with* $\phi_H(X)$, *so that* $L = \mathscr{O}_{\mathbb{P}(H)}(1)\big|_X$. *Let* $\|\cdot\|$ *be a norm of* H *such that* H *has an orthonormal basis* $(e_1, \ldots, e_r)$ *with respect to* $\|\cdot\|$. *We set*

$$h := \left\{|\cdot|_{(H,\|\cdot\|)}^{\mathrm{quot}}(x)\right\}_{x \in X^{\mathrm{an}}} \quad \text{and} \quad \mathscr{H} := \mathfrak{o}_k e_1 + \cdots + \mathfrak{o}_k e_r = (H, \|\cdot\|)_{\leq 1}.$$

Let $\mathscr{X}$ *be the Zariski closure of* X *in* $\mathbb{P}(\mathscr{H})$ (cf. *Subsection* 2.1.7) *and* $\mathscr{L} := \mathscr{O}_{\mathbb{P}(\mathscr{H})}(1)\big|_{\mathscr{X}}$. *Then* $|\cdot|_h(x) = |\cdot|_{\mathscr{L}}(x)$ *for all* $x \in X^{\mathrm{an}}$.

Proof. First let us see that $|s|_h(x) \leq |s|_{\mathscr{L}}(x)$ for $s \in H$. Let ω_ξ be a local basis of $\mathscr{L}$ at $\xi = r_{\mathscr{X}}(x)$. If we set $s = s_\xi \omega_\xi$, then

$$|s|_{\mathscr{L}}(x) = |s_\xi|_x.$$

As $s_\xi^{-1}s \in \mathscr{L}_\xi$ and $\mathscr{H} \otimes_{\mathfrak{o}_k} \mathscr{O}_{\mathscr{X},\xi} \to \mathscr{L}_\xi$ is surjective, there are $l_1, \ldots, l_r \in \mathscr{H}$ and $a_1, \ldots, a_r \in \mathscr{O}_{\mathscr{X},\xi}$ such that $s_\xi^{-1}s = a_1 l_1 + \cdots + a_r l_r$. Therefore,

$$\begin{aligned}\left|s_\xi^{-1}s\right|_h(x) &\leq \max\{|a_1 l_1|_h(x), \ldots, |a_r l_r|_h(x)\} \\ &= \max\{|a_1|_x |l_1|_h(x), \ldots, |a_r|_x |l_r|_h(x)\} \leq 1,\end{aligned}$$

so that $|s|_h(x) \leq |s_\xi|_x = |s|_{\mathscr{L}}(x)$, as required.

Next let us see that $|l|_{\mathscr{L}}(x) \leq \|l\|_{\hat{\kappa}(x)}$ for all $l \in H \otimes \hat{\kappa}(x)$. By Subsection 2.2.4, $(e_1, \ldots, e_r)$ is an orthonormal basis of $H \otimes \hat{\kappa}(x)$ with respect to $\|\cdot\|_{\hat{\kappa}(x)}$. Thus, if we set $l = a_1e_1 + \cdots + a_re_r$ $(a_1, \ldots, a_r \in \hat{\kappa}(x))$, then

$$\begin{aligned}|l|_{\mathscr{L}}(x) &\leq \max\{|a_1|_x|e_1|_{\mathscr{L}}(x), \ldots, |a_r|_x|e_r|_{\mathscr{L}}(x)\}\\ &\leq \max\{|a_1|_x, \ldots, |a_r|_x\} = \|l\|_{\hat{\kappa}(x)}.\end{aligned}$$

Finally let us see that $|s|_{\mathscr{L}}(x) \leq |s|_h(x)$ for $s \in H$. For $\epsilon > 0$, we choose $l \in H \otimes \hat{\kappa}(x)$ such that $l(x) = s(x)$ and $\|l\|_{\hat{\kappa}(x)} \leq e^{\epsilon}|s|_h(x)$. Then, by the previous observation,

$$|s|_{\mathscr{L}}(x) = |l|_{\mathscr{L}}(x) \leq \|l\|_{\hat{\kappa}(x)} \leq e^{\epsilon}|s|_h(x).$$

Thus the assertion follows. □

Remark 3.6. We assume that $|\cdot|$ is non-trivial and $\|\cdot\| = \|\cdot\|_{\mathscr{H}}$ for some finitely generated lattice $\mathscr{H}$ of H. Then a free basis $(e_1, \ldots, e_r)$ of $\mathscr{H}$ yields an orthonormal basis of H with respect to $\|\cdot\|$ (*cf.* Proposition 2.8). Moreover, $\mathscr{H} = (H, \|\cdot\|)_{\leq 1}$.

3.2. Semipositive metric

We assume that L is semiample, namely certain tensor power of L is generated by global sections. We say that a continuous metric $h = \{|\cdot|_h(x)\}_{x \in X^{\mathrm{an}}}$ is *semipositive* if there are a sequence $\{e_n\}$ of positive integers and a sequence $\{(V_n, \|\cdot\|_n)\}$ of normed finite-dimensional vector spaces over k such that there is a surjective homomorphism $V_n \otimes_k \mathscr{O}_X \to L^{\otimes e_n}$ for every n, and that the sequence

$$\left\{\frac{1}{e_n}\log\frac{|\cdot|^{\mathrm{quot}}_{(V_n, \|\cdot\|_n)}(x)}{|\cdot|_{h^{e_n}}(x)}\right\}_{n=1}^{\infty}$$

converges to 0 uniformly on X^{an}.

Proposition 3.7. *If X is projective, L is generated by global sections, and h is semipositive, then the sequence*

$$\left\{\frac{1}{m}\log\frac{|\cdot|^{\mathrm{quot}}_{h^m}(x)}{|\cdot|_{h^m}(x)}\right\}_{m=1}^{\infty}$$

converges to 0 *uniformly on* X^{an}.

Proof. We set

$$a_m = \max_{x \in X^{\mathrm{an}}}\left\{\log\frac{|\cdot|^{\mathrm{quot}}_{h^m}(x)}{|\cdot|_{h^m}(x)}\right\}.$$

Then $a_{m+m'} \leq a_m + a_{m'}$ by (3) in Lemma 3.2, and hence $\lim_{m\to\infty} a_m/m = \inf\{a_m/m\}$ by Fekete's lemma. For $\epsilon > 0$, there is an $e_n > 0$ such that

$$e^{-e_n\epsilon}|\cdot|_{h^{e_n}}(x) \leq |\cdot|_{h_n}(x) \leq e^{e_n\epsilon}|\cdot|_{h^{e_n}}(x)$$

for all $x \in X^{\mathrm{an}}$, where $h_n = \left\{|\cdot|^{\mathrm{quot}}_{(V_n,\|\cdot\|_n)}(x)\right\}_{x\in X^{\mathrm{an}}}$. Thus

$$e^{-e_n\epsilon}\|\cdot\|_{h^{e_n}} \leq \|\cdot\|_{h_n} \leq e^{e_n\epsilon}\|\cdot\|_{h^{e_n}},$$

so that $e^{-e_n\epsilon}|\cdot|^{\mathrm{quot}}_{h^{e_n}}(x) \leq |\cdot|^{\mathrm{quot}}_{h_n}(x) \leq e^{e_n\epsilon}|\cdot|^{\mathrm{quot}}_{h^{e_n}}(x)$. Thus, by Proposition 3.3,

$$e^{-e_n\epsilon}|\cdot|^{\mathrm{quot}}_{h^{e_n}}(x) \leq |\cdot|_{h_n}(x) \leq e^{e_n\epsilon}|\cdot|^{\mathrm{quot}}_{h^{e_n}}(x).$$

Therefore,

$$1 \leq \frac{|\cdot|^{\mathrm{quot}}_{h^{e_n}}(x)}{|\cdot|_{h^{e_n}}(x)} = \frac{|\cdot|_{h_n}(x)}{|\cdot|_{h^{e_n}}(x)}\frac{|\cdot|^{\mathrm{quot}}_{h^{e_n}}(x)}{|\cdot|_{h_n}(x)} \leq e^{2e_n\epsilon},$$

that is, $0 \leq a_{e_n}/e_n \leq 2\epsilon$, and hence $0 \leq \lim_{m\to\infty} a_m/m \leq 2\epsilon$, as required. □

Corollary 3.8. *A continuous metric h is semipositive if and only if, for any $\epsilon > 0$, there is a positive integer n such that, for all $x \in X^{\mathrm{an}}$, we can find $s \in H^0(X, L^{\otimes n})_{\hat{\kappa}(x)} \setminus \{0\}$ with $\|s\|_{h^n,\hat{\kappa}(x)} \leq e^{n\epsilon}|s|_{h^n}(x)$.*

Proof. First we assume that h is semipositive. By using Proposition 3.7, we can find a positive integer n such that $L^{\otimes n}$ is generated by global sections and

$$|\cdot|_{h^n}(x) \leq |\cdot|^{\mathrm{quot}}_{h^n}(x) \leq e^{n\epsilon/2}|\cdot|_{h^n}(x)$$

for all $x \in X^{\mathrm{an}}$. On the other hand, there is an $s \in H^0(X, L^{\otimes n})_{\hat{\kappa}(x)} \setminus \{0\}$ such that $\|s\|_{h^n,\hat{\kappa}(x)} \leq e^{n\epsilon/2}|s|^{\mathrm{quot}}_{h^n}(x)$. Thus,

$$\|s\|_{h^n,\hat{\kappa}(x)} \leq e^{n\epsilon/2}|s|^{\mathrm{quot}}_{h^n}(x) \leq e^{n\epsilon}|s|_{h^n}(x).$$

Next we consider the converse. For a positive integer m, there is a positive integer e_m such that, for any $x \in X^{\mathrm{an}}$, we can find $s \in H^0(X, L^{\otimes e_m})_{\hat{\kappa}(x)} \setminus \{0\}$ with $\|s\|_{h^{e_m},\hat{\kappa}(x)} \leq e^{e_m/m}|s|_{h^{e_m}}(x)$. Clearly $L^{\otimes e_m}$ is generated by global sections. Moreover,

$$|s|_{h^{e_m}}(x) \leq |s|^{\mathrm{quot}}_{(H^0(X,L^{\otimes e_m}),\|\cdot\|_{h^{e_m}})}(x) \leq e^{e_m/m}|s|_{h^{e_m}}(x),$$

that is,

$$0 \leq \frac{1}{e_m}\log\left(\frac{|\cdot|^{\mathrm{quot}}_{(H^0(X,L^{\otimes e_m}),\|\cdot\|_{h^{e_m}})}(x)}{|\cdot|_{h^{e_m}}(x)}\right) \leq \frac{1}{m}.$$

Thus h is semipositive. □

Corollary 3.9. *Let h be a continuous metric of L^{an}. If there are a sequence $\{e_n\}$ of positive integers and a sequence $\{h_n\}$ of metrics such that h_n is a semipositive metric of $(L^{\otimes e_n})^{\mathrm{an}}$ for each n and*

$$\frac{1}{e_n} \log \frac{|\cdot|_{h_n}(x)}{|\cdot|_{h^{e_n}}(x)}$$

converges to 0 *uniformly as $n \to \infty$, then h is semipositive.*

Proof. For a positive number $\epsilon > 0$, choose a positive integer n such that

$$e^{-\epsilon e_n/3} h^{e_n} \leq h_n \leq e^{\epsilon e_n/3} h^{e_n}.$$

As h_n is semipositive, by Corollary 3.8, there is a positive integer m such that, for all $x \in X^{\mathrm{an}}$, we can find $s \in H^0(X, L^{\otimes m e_n})_{\hat{\kappa}(x)} \setminus \{0\}$ with $\|s\|_{h_n^m, \hat{\kappa}(x)} \leq e^{m e_n \epsilon/3} |s|_{h_n^m}(x)$, so that

$$\|s\|_{h^{m e_n}, \hat{\kappa}(x)} \leq e^{\epsilon m e_n/3} \|s\|_{h_n^m, \hat{\kappa}(x)} \leq e^{2 m e_n \epsilon/3} |s|_{h_n^m}(x) \leq e^{m e_n \epsilon} |s|_{h^{m e_n}}(x).$$

Therefore, the assertion follows from Corollary 3.8. □

3.3. The functions σ and μ on X^{an}

Throughout this subsection, we assume that X is projective. Let $\widehat{\mathrm{Pic}}_{C^0}(X)$ denote the group of isomorphism classes of pairs (L, h) consisting of an invertible sheaf L on X and a continuous metric h of L^{an}. Fix $\overline{L} = (L, h) \in \widehat{\mathrm{Pic}}_{C^0}(X)$. We assume that L is generated by global sections. We define $\sigma_{\overline{L}}(x)$ to be

$$\sigma_{\overline{L}}(x) := \log \left(\frac{|\cdot|_h^{\mathrm{quot}}(x)}{|\cdot|_h(x)} \right).$$

Lemma 3.10. *For $\overline{L}$ and $\overline{L}' \in \widehat{\mathrm{Pic}}_{C^0}(X)$ such that both L and L' are generated by global sections, we have the following:*

(1) *$\sigma_{\overline{L}} \geq 0$ on X^{an};*
(2) *$\sigma_{\overline{L} \otimes \overline{L}'}(x) \leq \sigma_{\overline{L}}(x) + \sigma_{\overline{L}'}(x)$ for $x \in X^{\mathrm{an}}$;*
(3) *If $\overline{L} \simeq \overline{L}'$, then $\sigma_{\overline{L}} = \sigma_{\overline{L}'}$ on X^{an}.*

Proof. (1) and (3) are obvious. (2) follows from (3) in Lemma 3.2. □

We assume that L is semiample. We set

$$\mathbb{N}(L) := \left\{ n \in \mathbb{Z}_{\geq 1} \mid L^{\otimes n} \text{ is generated by global sections} \right\}.$$

Note that $\mathbb{N}(L) \neq \emptyset$ and $\mathbb{N}(L)$ forms a subsemigroup of $\mathbb{Z}_{\geq 1}$ with respect to the addition of $\mathbb{Z}_{\geq 1}$. For $x \in X^{\mathrm{an}}$, we define $\mu_{\overline{L}}(x)$ to be

$$\mu_{\overline{L}}(x) := \inf\left\{ \frac{\sigma_{\overline{L}^{\otimes n}}(x)}{n} \;\middle|\; n \in \mathbb{N}(L) \right\}.$$

Note that $\mu_{\overline{L}}$ is upper-semicontinuous on X^{an} because $\sigma_{\overline{L}^{\otimes n}}$ is continuous for all $n \in \mathbb{N}(L)$. We set

$$\widehat{\mathrm{Pic}}^{+}_{C^0}(X) := \{(L, h) \in \widehat{\mathrm{Pic}}_{C^0}(X) \mid L \text{ is semiample}\}.$$

Note that $\widehat{\mathrm{Pic}}^{+}_{C^0}(X)$ forms a semigroup with respect to $\otimes$.

Lemma 3.11. *Let $\overline{L} = (L, h)$ and $\overline{L}' = (L', h')$ be elements of $\widehat{\mathrm{Pic}}^{+}_{C^0}(X)$. Then we have the following:*

(1) *It holds $\mu_{\overline{L}} \geq 0$ on X^{an};*
(2) *It holds $\mu_{\overline{L}}(x) = \lim\limits_{\substack{n\to\infty \\ n\in\mathbb{N}(L)}} \dfrac{\sigma_{\overline{L}^{\otimes n}}(x)}{n}$ for $x \in X^{\mathrm{an}}$;*
(3) *It holds $\mu_{\overline{L}\otimes\overline{L}'}(x) \leq \mu_{\overline{L}}(x) + \mu_{\overline{L}'}(x)$ for $x \in X^{\mathrm{an}}$;*
(4) *If $\overline{L} \simeq \overline{L}'$, then $\mu_{\overline{L}} = \mu_{\overline{L}'}$ on X^{an};*
(5) *For $n \geq 0$, $\mu_{\overline{L}^{\otimes n}} = n\mu_{\overline{L}}$ on X^{an}.*

Proof. (1) follows from (1) in Lemma 3.10.

(2) Since $\sigma_{\overline{L}^{\otimes(n+n')}}(x) \leq \sigma_{\overline{L}^{\otimes n}}(x) + \sigma_{\overline{L}^{\otimes n'}}(x)$ for $n, n' \in \mathbb{N}(L)$ by (2) in Lemma 3.10, the assertion follows from Fekete's lemma.

(3) and (4) follow from (2) and (3) in Lemma 3.10 together with (2), respectively.

(5) If $n = 0$, then the assertion is obvious, so that we may assume that $n \geq 1$. We fix $n_0 \in \mathbb{N}(L)$. Then $n_0 \in \mathbb{N}(L^{\otimes n})$. Thus, by (2),

$$\mu_{\overline{L}^{\otimes n}}(x) = \lim_{m\to\infty} \frac{\sigma_{L^{\otimes mn_0n}}(x)}{mn_0} = n \lim_{m\to\infty} \frac{\sigma_{L^{\otimes mn_0n}}(x)}{mn_0n} = n\mu_{\overline{L}}(x). \qquad \square$$

We let $\widehat{\mathrm{Pic}}_{C^0}(X)_{\mathbb{Q}}$ be the quotient space of $\widehat{\mathrm{Pic}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ by the $\mathbb{Q}$-vector subspace generated by $(\mathscr{O}_X, \{e^{-\lambda}|\cdot|^0_x\}) - \lambda(\mathscr{O}_X, \{|\cdot|^0_x\})$, where $\{|\cdot|^0_x\}$ denotes the trivial continuous metric on $\mathscr{O}_X$. Note that $\widehat{\mathrm{Pic}}_{C^0}(X)_{\mathbb{Q}}$ can be identified with the $\mathbb{Q}$-vector space of all pairs (L, h), where L is an element of $\mathrm{Pic}(X) \otimes \mathbb{Q}$ and h is a continuous metric on L (see Subsection 2.1.5). Moreover, we set

$$\widehat{\mathrm{Pic}}^{+}_{C^0}(X)_{\mathbb{Q}} := \left\{(L, h) \in \widehat{\mathrm{Pic}}_{C^0}(X)_{\mathbb{Q}} \mid L \text{ is semiample}\right\}.$$

Let $\iota : \widehat{\mathrm{Pic}}_{C^0}(X) \to \widehat{\mathrm{Pic}}_{C^0}(X)_{\mathbb{Q}}$ be the canonical homomorphism. For $\overline{L} \in \widehat{\mathrm{Pic}}^{+}_{C^0}(X)_{\mathbb{Q}}$, we choose a positive integer n and $\overline{L}_n \in \widehat{\mathrm{Pic}}^{+}_{C^0}(X)$ with $\iota(\overline{L}_n) = \overline{L}^{\otimes n}$. Then

$\mu_{\overline{L}_n}(x)/n$ does not depend on the choice of n and $\overline{L}_n$. Indeed, let us choose another $n' \in \mathbb{Z}_{\geq 1}$ and $\overline{L}_{n'} \in \widehat{\mathrm{Pic}}^{+}_{C^0}(X)$ with $\iota(\overline{L}_{n'}) = \overline{L}^{\otimes n'}$. As $\iota(\overline{L}_n^{\otimes n'}) = \iota(\overline{L}_{n'}^{\otimes n}) = \overline{L}^{\otimes nn'}$, there is a positive integer m such that $\overline{L}_n^{\otimes mn'} = \overline{L}_{n'}^{\otimes mn}$. By (5) in Lemma 3.11,

$$mn'\mu_{\overline{L}_n}(x) = \mu_{\overline{L}_n^{\otimes mn'}}(x) = \mu_{\overline{L}_{n'}^{\otimes mn}}(x) = mn\mu_{\overline{L}_{n'}}(x),$$

that is, $\mu_{\overline{L}_n}(x)/n = \mu_{\overline{L}_{n'}}(x)/n'$, as required. By abuse of notation, it is also denoted by $\mu_{\overline{L}}(x)$.

Lemma 3.12. *For $\overline{L}, \overline{L}' \in \widehat{\mathrm{Pic}}^{+}_{C^0}(X)_{\mathbb{Q}}$, we have the following:*

(1) *It holds $\mu_{\overline{L}\otimes\overline{L}'}(x) \leq \mu_{\overline{L}}(x) + \mu_{\overline{L}'}(x)$ for $x \in X^{\mathrm{an}}$;*
(2) *For $a \in \mathbb{Q}_{\geq 0}$, $\mu_{\overline{L}^{\otimes a}} = a\mu_{\overline{L}}$ on X^{an};*
(3) *Let $\overline{L}_1, \dots, \overline{L}_r$ be elements of $\widehat{\mathrm{Pic}}_{C_0}(X)_{\mathbb{Q}}$. We assume that there are open intervals $I_1, \dots, I_r$ of $\mathbb{R}$ such that*

$$\overline{L} \otimes \overline{L}_1^{\otimes t_1} \otimes \cdots \otimes \overline{L}_r^{\otimes t_r} \in \widehat{\mathrm{Pic}}^{+}_{C^0}(X)_{\mathbb{Q}}$$

for all $(t_1, \dots, t_r) \in (I_1 \times \cdots \times I_r) \cap \mathbb{Q}^r$. Then, for a fixed $x \in X^{\mathrm{an}}$, there is a continuous function $f : I_1 \times \cdots \times I_r \to \mathbb{R}$ such that

$$f(t_1, \dots, t_r) = \mu_{\overline{L}\otimes\overline{L}_1^{\otimes t_1}\otimes\cdots\otimes\overline{L}_r^{\otimes t_r}}(x)$$

for all $(t_1, \dots, t_r) \in (I_1 \times \cdots \times I_r) \cap \mathbb{Q}^r$.

Proof. (1) and (2) are consequences of (3) and (5) in Lemma 3.11, respectively.

(3) We set

$$f_0(t_1, \dots, t_r) := \mu_{\overline{L}\otimes\overline{L}_1^{\otimes t_1}\otimes\cdots\otimes\overline{L}_r^{\otimes t_r}}(x)$$

for $(t_1, \dots, t_r) \in (I_1 \times \cdots \times I_r) \cap \mathbb{Q}^r$. By (1) and (2), for $\lambda \in [0, 1] \cap \mathbb{Q}$ and $(t_1, \dots, t_r), (t'_1, \dots, t'_r) \in (I_1 \times \cdots \times I_r) \cap \mathbb{Q}^r$, we have

$$\begin{aligned} &f_0(\lambda(t_1, \dots, t_r) + (1-\lambda)(t'_1, \dots, t'_r)) \\ &= \mu_{(\overline{L}\otimes\overline{L}_1^{\otimes t_1}\otimes\cdots\otimes\overline{L}_r^{\otimes t_r})^{\otimes\lambda}\otimes(\overline{L}\otimes\overline{L}_1^{\otimes t'_1}\otimes\cdots\otimes\overline{L}_r^{\otimes t'_r})^{\otimes(1-\lambda)}}(x) \\ &\leq \lambda\mu_{\overline{L}\otimes\overline{L}_1^{\otimes t_1}\otimes\cdots\otimes\overline{L}_r^{\otimes t_r}}(x) + (1-\lambda)\mu_{\overline{L}\otimes\overline{L}_1^{\otimes t'_1}\otimes\cdots\otimes\overline{L}_r^{\otimes t'_r}}(x) \\ &= \lambda f_0(t_1, \dots, t_r) + (1-\lambda) f_0(t'_1, \dots, t'_r), \end{aligned}$$

that is, f_0 is concave on $(I_1 \times \cdots \times I_r) \cap \mathbb{Q}^r$. Therefore, the assertion (3) follows from [14, Corollary 1.3.2]. □

Let (L, h) be an element of $\widehat{\mathrm{Pic}}^{+}_{C^0}(X)_{\mathbb{Q}}$. We say that h is semipositive if there is a positive integer n such that $L^{\otimes n} \in \mathrm{Pic}(X)$ and h^n is semipositive. The following characterization of the semipositivity of h is a consequence of Proposition 3.7.

Proposition 3.13. *For* $\overline{L} = (L, h) \in \widehat{\mathrm{Pic}}^{+}_{C^0}(X)_{\mathbb{Q}}$, *h is semipositive if and only if* $\mu_{\overline{L}} = 0$ *on* X^{an}.

We assume that $|\cdot|$ is non-trivial. Let $\mathscr{X}$ be a model of X over $\mathrm{Spec}(\mathfrak{o}_k)$. Let $L \in \mathrm{Pic}(X) \otimes \mathbb{Q}$ and $\mathscr{L} \in \mathrm{Pic}(\mathscr{X}) \otimes \mathbb{Q}$ with $\mathscr{L}\big|_X = L$. Let m be a positive integer such that $L^{\otimes m} \in \mathrm{Pic}(X)$. Then we define $\overline{L} = (L, h)$ to be

$$(L, h) := \left(L^{\otimes m}, \left\{|\cdot|_{\mathscr{L}^{\otimes m}}(x)\right\}_{x \in X^{\mathrm{an}}}\right)^{\otimes 1/m}.$$

Proposition 3.14. *If L is ample and $\mathscr{L}$ is nef, then h is semipositive.*

Proof. First we assume that $\mathscr{L}$ is ample. We choose a positive integer n such that $\mathscr{L}^{\otimes n} \in \mathrm{Pic}(\mathscr{X})$ and $\mathscr{L}^{\otimes n}$ is very ample. Then we have an embedding $\iota : \mathscr{X} \to \mathbb{P}(H^0(\mathscr{X}, \mathscr{L}^{\otimes n}))$ and $\mathscr{L}^{\otimes n} = \iota^*(\mathscr{O}_{\mathbb{P}(H^0(\mathscr{X}, \mathscr{L}^{\otimes n}))}(1))$. Let $(e_1, \ldots, e_r)$ be a free basis of $H^0(\mathscr{X}, \mathscr{L}^{\otimes n})$. We define a norm $\|\cdot\|$ of $H^0(X, L^{\otimes n})$ to be

$$\|a_1 e_1 + \cdots + a_r e_r\| := \max\{|a_1|, \ldots, |a_r|\}.$$

Note that $(H^0(X, L^{\otimes n}), \|\cdot\|)_{\leq 1} = H^0(\mathscr{X}, \mathscr{L}^{\otimes n})$, so that, by Proposition 3.5, we have $|\cdot|^{\mathrm{quot}}_{(H^0(X, L^{\otimes n}), \|\cdot\|)}(x) = |\cdot|_{\mathscr{L}^{\otimes n}}(x)$ for $x \in X^{\mathrm{an}}$. Thus h is semipositive.

In general, let $\mathscr{A}$ be an ample invertible sheaf on $\mathscr{X}$ and $A := \mathscr{A}\big|_X$. We choose $\delta \in \mathbb{Q}_{>0}$ such that $L \otimes A^{\otimes a}$ is ample for all $a \in (-\delta, \delta) \cap \mathbb{Q}$. Note that

$$\overline{L} \otimes \left(A, |\cdot|_{\mathscr{A}}\right)^{\otimes \epsilon} = \left(L \otimes A^{\otimes \epsilon}, |\cdot|_{\mathscr{L} \otimes \mathscr{A}^{\otimes \epsilon}}\right),$$

so that $\mu_{\overline{L} \otimes (A, |\cdot|_{\mathscr{A}})^{\otimes \epsilon}} = 0$ for $\epsilon \in (0, \delta) \cap \mathbb{Q}$ by the previous observation together with Proposition 3.13. On the other hand, by (3) in Lemma 3.12,

$$\mu_{\overline{L}}(x) = \lim_{\substack{\epsilon \downarrow 0 \\ \epsilon \in \mathbb{Q}}} \mu_{\overline{L} \otimes (A, |\cdot|_{\mathscr{A}})^{\otimes \epsilon}}(x).$$

Therefore, $\mu_{\overline{L}} = 0$, and hence h is semipositive by Proposition 3.13. □

Remark 3.15. Assume that the absolute value $|\cdot|$ is non-trivial. Let L be an ample invertible sheaf on X, equipped with a semipositive continuous metric h. Then there exists a sequence $\{(\mathscr{X}_n, \mathscr{L}_n)\}_{n \geq 1}$, where $\mathscr{X}_n$ is a model of X and $\mathscr{L}_n$ is a nef invertible sheaf on $\mathscr{X}_n$ such that $\mathscr{L}_n|_X = L^{\otimes n}$ and that $h_n = (|\cdot|_{\mathscr{L}_n}(x)^{1/n})_{x \in X^{\mathrm{an}}}$ converges uniformly to h. This follows from Proposition 3.7 and the comparison between quotient metrics and model metrics (via the embedding into the projective spaces of lattices). Combining with Proposition 3.14 and Corollary 3.8, we obtain that, in the non-trivial valuation case, our semipositivity coincides with that of Zhang [21] and Moriwaki [16]. We refer the readers to [12, Section 6] and to [8, Section 6.8] for the descriptions of the semipositivity in terms of plurisubharmonic currents. Note that their semipositivity is also equivalent to our semipositivity.

4. Extension theorem

Throughout this section, we assume that X is projective and reduced. Let us begin with a special case of the extension theorem. The general extension theorem is a consequence of the special case.

4.1. Extension theorem for a metric arising from a model

Let $\mathscr{X} \to \operatorname{Spec} \mathfrak{o}_k$ be a model of X. We let $\mathscr{L}$ be an invertible sheaf on $\mathscr{X}$ such that $\mathscr{L}\big|_X = L$. We have seen in Subsection 2.1.6 that $\mathscr{L}$ induces a continuous metric $h = \{|\cdot|_{\mathscr{L}}(x)\}_{x \in X^{\mathrm{an}}}$ of L^{an}.

Theorem 4.1. *We assume that $|\cdot|$ is non-trivial and $\mathscr{L}$ is an ample invertible sheaf. Fix a reduced closed subscheme Y of X, a section $l \in H^0(Y, L|_Y)$ and a positive number ϵ. Then there are a positive integer n and an $s \in H^0(X, L^{\otimes n})$ such that $s|_Y = l^{\otimes n}$ and*

$$\|s\|_{h^n} \leq e^{n\epsilon} \left(\|l\|_{Y,h}\right)^n .$$

Proof. Clearly, we may assume that $l \neq 0$. Let $\mathscr{Y}$ be the Zariski closure of Y in $\mathscr{X}$ (*cf.* Section 2.1.7).

Claim 4.2. There are a positive integer a and an $\alpha \in k^{\times}$ such that

$$e^{-a\epsilon/2} \leq \|\alpha l^{\otimes a}\|_{Y,h^a} \leq 1.$$

Proof. Since the absolute value $|\cdot|$ is not trivial, there exists a non-zero element γ of k such that $\log|\gamma| < 0$. Hence there exists a rational number b/a (with $a \in \mathbb{Z}_{>0}$ and $b \in \mathbb{Z}$) such that

$$-\frac{\log \|l\|_{Y,h}}{\log|\gamma|} \leq \frac{b}{a} \leq -\frac{\log \|l\|_{Y,h}}{\log|\gamma|} - \frac{\epsilon}{2\log|\gamma|},$$

that is, $e^{-\epsilon a/2} \leq |\gamma|^b \|l\|_{Y,h}^a \leq 1$. By the equality $\|l^{\otimes a}\|_{Y,h^a} = \|l\|_{Y,h}^a$ it suffices to take $\alpha = \gamma^b$ to conclude the claim. □

By Corollary 2.14, there is a $\beta \in \mathfrak{o}_K \setminus \{0\}$ such that

$$\beta \left(\alpha l^{\otimes a}\right)^{\otimes m} \in H^0\left(\mathscr{Y}, \mathscr{L}^{\otimes am}\big|_{\mathscr{Y}}\right)$$

for all $m \geq 0$. We choose a positive integer m such that $|\beta|^{-1} \leq e^{am\epsilon/2}$ and

$$H^0\left(\mathscr{X}, \mathscr{L}^{\otimes am}\right) \to H^0\left(\mathscr{Y}, \mathscr{L}^{\otimes am}\big|_{\mathscr{Y}}\right)$$

is surjective, so that we can find an $l_m \in H^0(\mathscr{X}, \mathscr{L}^{\otimes am})$ such that $l_m|_{\mathscr{Y}} = \beta(\alpha l^{\otimes a})^{\otimes m}$. Note that $\|l_m\|_{h^{am}} \leq 1$. Thus, if we set $s = \beta^{-1}\alpha^{-m} l_m$, then $s|_{\mathscr{Y}} = l^{\otimes am}$ and

$$\begin{aligned}\|s\|_{h^{am}} &= |\beta|^{-1}|\alpha|^{-m}\|l_m\|_{h^{am}} \leq e^{am\epsilon/2}|\alpha|^{-m}\\ &\leq e^{am\epsilon/2}|\alpha|^{-m}\left(e^{a\epsilon/2}\|\alpha l^{\otimes a}\|_{Y,h^a}\right)^m = e^{am\epsilon}\left(\|l\|_{Y,h}\right)^{am},\end{aligned}$$

as required. □

4.2. Extension theorem for quotient metrics

Theorem 4.3. *We assume that L is very ample. Let $\|\cdot\|$ be a norm of $H^0(X, L)$ and h a continuous metric of L^{an} given by $\left\{|\cdot|^{\mathrm{quot}}_{(H^0(X,L),\|\cdot\|)}(x)\right\}_{x\in X^{\mathrm{an}}}$. Let Y be a reduced closed subscheme of X and $l \in H^0(Y, L|_Y)$. Then, for any $\epsilon > 0$, there are a positive integer n and an $s \in H^0(X, L^{\otimes n})$ such that $s|_Y = l^{\otimes n}$ and $\|s\|_{h^n} \leq e^{n\epsilon}(\|l\|_{Y,h})^n$.*

Proof. First we assume that $|\cdot|$ is non-trivial. Let us begin with the following:

Claim 4.4. There are a positive integer a and a finitely generated lattice $\mathscr{H}$ of $H^0(X, L^{\otimes a})$ such that

$$\|\cdot\|_{h^a} \leq \|\cdot\|_{\mathscr{H}} \leq e^{a\epsilon/2}\|\cdot\|_{h^a}.$$

Proof. First we assume that $|\cdot|$ is discrete. We choose a positive integer a such that $|\varpi|^{-1} \leq e^{a\epsilon/2}$. We set $\mathscr{H} := \{s \in H^0(X, L^{\otimes a}) \mid \|s\|_{h^a} \leq 1\}$. Note that $\mathscr{H}$ is a finitely generated lattice of $H^0(X, L^{\otimes a})$ by Proposition 2.10. As $\|\cdot\|_{h^a} \leq \|\cdot\|_{\mathscr{H}} \leq |\varpi|^{-1}\|\cdot\|_{h^a}$ by Proposition 2.10, we have the assertion.

Next we assume that $|\cdot|$ is not discrete. By Proposition 2.11, there is a lattice $\mathscr{V}$ of $H^0(X, L)$ such that $\|\cdot\|_h = \|\cdot\|_{\mathscr{V}}$. By Proposition 2.12, there is a finitely generated lattice $\mathscr{H}$ of $H^0(X, L)$ such that $\mathscr{H} \subseteq \mathscr{V}$ and $\|\cdot\|_h \leq \|\cdot\|_{\mathscr{H}} \leq e^{\epsilon/2}\|\cdot\|_h$, as desired. □

Let $\mathscr{X}$ be the Zariski closure of X in $\mathbb{P}(\mathscr{H})$ (*cf.* Subsection 2.1.7) and $\mathscr{L} = \mathscr{O}_{\mathbb{P}(\mathscr{H})}(1)\big|_{\mathscr{X}}$. Moreover, let h' be a continuous metric of $(L^{\otimes a})^{\mathrm{an}}$ given by

$$\left\{|\cdot|^{\mathrm{quot}}_{(H,\|\cdot\|_{\mathscr{H}})}(x)\right\}_{x\in X^{\mathrm{an}}}.$$

Then, by Proposition 3.5 and Remark 3.6, $|\cdot|_{h'} = |\cdot|_{\mathscr{L}}$. Therefore, by virtue of Theorem 4.1, there are a positive integer m and an $s \in H^0(X, L^{\otimes am})$ such that $s|_Y = l^{\otimes am}$ and

$$\|s\|_{h'^m} \leq e^{am\epsilon/2}(\|l^{\otimes a}\|_{Y,h'})^m. \tag{4.1}$$

As $\|\cdot\|_{h^a} \leq \|\cdot\|_{\mathscr{H}} \leq e^{a\epsilon/2}\|\cdot\|_{h^a}$, we have

$$|\cdot|^{\mathrm{quot}}_{h^a}(x) \leq |\cdot|_{h'}(x) \leq e^{a\epsilon/2}|\cdot|^{\mathrm{quot}}_{h^a}(x)$$

for all $x \in X^{\mathrm{an}}$. Therefore, by Proposition 3.3,

$$|\cdot|_{h^a}(x) \leq |\cdot|_{h'}(x) \leq e^{a\epsilon/2}|\cdot|_{h^a}(x) \tag{4.2}$$

for all $x \in X^{\mathrm{an}}$. In particular, $|\cdot|_{h^{am}}(x) \leq |\cdot|_{h'^m}(x)$. Therefore,

$$\|s\|_{h^{am}} \leq \|s\|_{h'^m}. \tag{4.3}$$

On the other hand, by using (4.2),

$$\|l^{\otimes a}\|_{Y,h'} \leq e^{a\epsilon/2} \sup\{|l^{\otimes a}|_{h^a}(y) \mid y \in Y^{\mathrm{an}}\} \leq e^{a\epsilon/2}(\|l\|_{Y,h})^a. \tag{4.4}$$

Thus the assertion follows from (4.1), (4.3) and (4.4).

Next we assume that $|\cdot|$ is trivial. Clearly we may assume that $l \neq 0$. Let k' be the field $k(\!(T)\!)$ of formal Laurent power series over k, that is, the quotient field of the ring $k[\![T]\!]$ of formal power series over k. Note that k' is separable over k. We set

$$\Sigma := \bigcup_{i=0}^{\infty} \left(\bigcup_{s,s' \in H^0(X, L^{\otimes i}) \setminus \{0\}} \mathbb{Q}\left(\log \|s\|_{h^i} - \log \|s'\|_{h^i}\right) \right).$$

As $\left\{\|s\|_{h^i} \mid s \in H^0(X, L^{\otimes i}) \setminus \{0\}\right\}$ is a finite set by virtue of Subsection 2.2.1, we have $\#(\Sigma) \leq \aleph_0$. Therefore, we can find $\alpha \in \mathbb{R}_{>0} \setminus \Sigma$. Here we consider an absolute value $|\cdot|'$ of k' given by

$$|\phi(T)|' := \exp(-\alpha \operatorname{ord}(\phi(T))) \quad (\phi(T) \in k').$$

We set

$$X' := X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k'), \quad Y' := Y \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k') \quad \text{and} \quad L' = L \otimes_k k'.$$

Note that $H^0(X', L') = H^0(X, L) \otimes_k k'$. Let h' be a continuous metric of L'^{an} given by the scalar extension of h. Then, by Lemma 3.4, h' is given by

$$\left\{|\cdot|^{\mathrm{quot}}_{(H^0(X',L'),\|\cdot\|_{k'})}(x')\right\}_{x' \in X'^{\mathrm{an}}},$$

where $\|\cdot\|_{k'}$ is the scalar extension of $\|\cdot\|$. Moreover, for $s \in H^0(X, L)$, it holds $|s|_{h'}(x') = |s|_h(p^{\mathrm{an}}(x'))$ for $x' \in X'^{\mathrm{an}}$, where $p : X' \to X$ is the projection. Note that $p^{\mathrm{an}} : X'^{\mathrm{an}} \to X^{\mathrm{an}}$ is surjective. Therefore, $\|s\|_{h'} = \|s\|_h$ for all $s \in H^0(X, L)$.

By the previous observation, there are a positive integer n and an $s' \in H^0(X', L'^{\otimes n})$ such that

$$s'\big|_{Y'} = l^{\otimes n} \quad \text{and} \quad \|s'\|_{h'^n} \leq e^{n\epsilon}(\|l\|_{Y',h'})^n = e^{n\epsilon}(\|l\|_{Y,h})^n.$$

Note that, for a positive integer d, we have

$$s'^{\otimes d} \in H^0(X', L'^{\otimes dn}), \quad s'^{\otimes d}\Big|_{Y'} = l^{\otimes dn} \quad \text{and} \quad \|s'^{\otimes d}\|_{h'^{dn}} \leq e^{dn\epsilon}(\|l\|_{Y,h})^{dn}.$$

Thus we may assume that $H^0(X, L^{\otimes n}) \to H^0(Y, L|_Y^{\otimes n})$ is surjective. Let $(e_1, \ldots, e_r)$ be an orthogonal basis of $H^0(X, L^{\otimes n})$ with respect to $\|\cdot\|_{h^n}$ such that $(e_{t+1}, \ldots, e_r)$ forms a basis of $\mathrm{Ker}(H^0(X, L^{\otimes n}) \to H^0(Y, L|_Y^{\otimes n}))$ (*cf.* Proposition 2.2). We set

$$s' = a_1(T)e_1 + \cdots + a_t(T)e_t + a_{t+1}(T)e_{t+1} + \cdots + a_r(T)e_r$$

for some $a_1(T), \ldots, a_r(T) \in k' = k(\!(T)\!)$. As $s'\big|_{Y'} = l^{\otimes n} \in H^0(Y, L|_Y^{\otimes n})$ and $(e_1|_Y, \ldots, e_t|_Y)$ forms a basis of $H^0(Y, L|_Y^{\otimes n})$, we have $a_1(T), \ldots, a_t(T) \in k$. Note that

$$\alpha \notin \bigcup_{s,s' \in H^0(X, L^{\otimes n}) \setminus \{0\}} \mathbb{Q}\left(\log \|s\|_{h^n} - \log \|s'\|_{h^n}\right),$$

so that, by Lemma 2.6 and Remark 2.7 together with Subsection 2.2.4, $(e_1, \ldots, e_r)$ forms an orthogonal basis of $H^0(X', L'^{\otimes n})$ with respect to $\|\cdot\|_{h'^n}$. Therefore, if we set $s = a_1(T)e_1 + \cdots + a_t(T)e_t$, then $s \in H^0(X, L^{\otimes n})$, we have $s|_Y = l^{\otimes n}$ and

$$\begin{aligned} \|s\|_{h^n} &= \max\{|a_1(T)|\|e_1\|_{h^n}, \ldots, |a_t(T)|\|e_t\|_{h^n}\} \\ &\le \max\Big\{|a_1(T)|\|e_1\|_{h^n}, \ldots, |a_t(T)|\|e_t\|_{h^n}, |a_{t+1}(T)|'\|e_{t+1}\|_{h^n}, \ldots, \\ &\qquad |a_r(T)|'\|e_r\|_{h^n}\Big\} \\ &= \|s'\|_{h'^n} \le e^{n\epsilon}(\|l\|_{Y,h})^n, \end{aligned}$$

as required. □

4.3. General case

Theorem 4.5. *We assume that L is ample and h is a semipositive continuous metric of L^{an}. Fix a reduced closed subscheme Y, $l \in H^0(Y, L|_Y)$ and $\epsilon \in \mathbb{R}_{>0}$. Then there is a positive integer n_0 such that, for all $n \ge n_0$, we can find an $s \in H^0(X, L^{\otimes n})$ with*

$$s|_Y = l^{\otimes n} \quad \textit{and} \quad \|s\|_{h^n} \le e^{n\epsilon}(\|l\|_{Y,h})^n.$$

Proof. Clearly we may assume that $l \neq 0$. Let us begin with the following claim:

Claim 4.6. For any $\epsilon' > 0$, there are a positive integer N and an $s_N \in H^0(X, L^{\otimes N})$ such that

$$s_N|_Y = l^{\otimes N} \quad \text{and} \quad \|s_N\|_{h^N} \le e^{N\epsilon'}(\|l\|_{Y,h})^N.$$

Proof. By using Proposition 3.7, we can find a positive integer a such that $L^{\otimes a}$ is very ample and

$$|\cdot|_{h^a}(x) \le |\cdot|_{h^a}^{\mathrm{quot}}(x) \le e^{a\epsilon'/2}|\cdot|_{h^a}(x)$$

for all $x \in X^{\mathrm{an}}$. We set $h' = \{|\cdot|^{\mathrm{quot}}_{h^a}(x)\}$. Then, the above inequalities mean that

$$|\cdot|_{h^a}(x) \leq |\cdot|_{h'}(x) \leq e^{a\epsilon'/2}|\cdot|_{h^a}(x) \tag{4.5}$$

for all $x \in X^{\mathrm{an}}$. Furthermore, by Theorem 4.3, there are a positive integer b and an $s_{ab} \in H^0(X, L^{\otimes ab})$ such that $s_{ab}|_Y = l^{\otimes ab}$ and

$$\|s_{ab}\|_{h'^b} \leq e^{ab\epsilon'/2}(\|l^{\otimes a}\|_{Y,h'})^b.$$

By (4.5), it holds

$$\|l^{\otimes a}\|_{Y,h'} \leq e^{a\epsilon'/2}\|l^{\otimes a}\|_{Y,h^a} = e^{a\epsilon'/2}(\|l\|_{Y,h})^a.$$

Moreover, as $|\cdot|_{h^{ab}}(x) \leq |\cdot|_{h'^b}(x)$ by (4.5), we have $\|s_{ab}\|_{h^{ab}} \leq \|s_{ab}\|_{h'^b}$, so that

$$\begin{aligned}\|s_{ab}\|_{h^{ab}} \leq \|s_{ab}\|_{h'^b} &\leq e^{ab\epsilon'/2}(\|l^{\otimes a}\|_{Y,h'})^b \\ &\leq e^{ab\epsilon'/2}(e^{a\epsilon'/2}(\|l\|_{Y,h})^a)^b = e^{ab\epsilon'}(\|l\|_{Y,h})^{ab}.\end{aligned}$$

Therefore, if we set $N = ab$, then we have the assertion of the claim. □

Since L is ample, by Corollary 2.17, the above claim is actually equivalent to the assertion of the theorem. Thus the theorem is proved. □

5. Arithmetic Nakai-Moishezon criterion over a number field

In this section as an application of the extension property (*cf.* [17] and Theorem 4.5), we consider the arithmetic Nakai-Moishezon criterion over a number field under a weaker assumption (adelically normed vector space) than Zhang's paper [21].

5.1. Adelically normed vector space over a number field

Fix a number field K. Let $\mathcal{O}_K$ be the ring of integers in K. We set

$$\begin{cases} M_K^{\mathrm{fin}} := \operatorname{Spec}(\mathcal{O}_K) \setminus \{(0)\} \\ M_K^{\infty} := K(\mathbb{C}) \ (\textit{i.e.}\text{ the set of all embeddings } K \hookrightarrow \mathbb{C}). \end{cases}$$

Moreover, $M_K := M_K^{\mathrm{fin}} \cup M_K^{\infty}$. For $\mathfrak{p} \in M_K^{\mathrm{fin}}$ and $\sigma \in M_K^{\infty}$, the absolute values $|\cdot|_{\mathfrak{p}}$ and $|\cdot|_{\sigma}$ of K are defined by

$$|x|_{\mathfrak{p}} := \#(\mathcal{O}_K/\mathfrak{p})^{-\operatorname{ord}_{\mathfrak{p}}(x)} \quad \text{and} \quad |x|_{\sigma} := |\sigma(x)| \quad (x \in K),$$

respectively. Furthermore, for $\mathfrak{p} \in M_K^{\mathrm{fin}}$, the completion of K with respect to $|\cdot|_{\mathfrak{p}}$ is denoted by $K_{\mathfrak{p}}$. In addition, K_{σ} and $K \hookrightarrow K_{\sigma}$ ($\sigma \in M_K^{\infty}$) are defined to be $\mathbb{C}$

and σ, respectively. By abuse of notation, for $v \in M_K$, the extension absolute of $|\cdot|_v$ to K_v is also denoted by $|\cdot|_v$. In the case where $v = \sigma \in M_K^\infty$, the absolute value $|\cdot|_\sigma$ on $K_\sigma = \mathbb{C}$ is the usual absolute value. If $\mathfrak{p} \in M_K^{\mathrm{fin}}$, the valuation rings of $(K, |\cdot|_\mathfrak{p})$ and $(K_\mathfrak{p}, |\cdot|_\mathfrak{p})$ are denoted by $\mathcal{O}_\mathfrak{p}$ and $\widehat{\mathcal{O}}_\mathfrak{p}$, respectively. Note that $\mathcal{O}_\mathfrak{p}$ is the localization of $\mathcal{O}_K$ with respect to $\mathcal{O}_K \setminus \mathfrak{p}$, and $\widehat{\mathcal{O}}_\mathfrak{p}$ is the completion of the local ring $\mathcal{O}_\mathfrak{p}$.

Definition 5.1. Let H be a finite-dimensional vector space over K. For $v \in M_K$, $H \otimes_K K_v$ is denoted by H_v. For each $v \in M_K$, let $\|\cdot\|_v$ be a norm of H_v over $(K_v, |\cdot|_v)$. In the case where $v \in M_K^{\mathrm{fin}}$, the norm $\|\cdot\|_v$ is always assumed to be ultrametric. Moreover, we assume that the family $(\|\cdot\|_\sigma)_{\sigma \in M_K^\infty}$ is invariant under complex conjugation, namely for any finite family of vectors $(s_i)_{i=1}^n$ in H and vector $(\lambda_i)_{i=1}^n$ of complex numbers, one has

$$\|\overline{\lambda_1} \otimes s_1 + \cdots + \overline{\lambda_n} \otimes s_n\|_{\overline{\sigma}} = \|\lambda_1 \otimes s_1 + \cdots + \lambda_n \otimes s_n\|_\sigma.$$

The family $\{\|\cdot\|_v\}_{v \in M_K}$ of norms is often denoted by $\|\cdot\|$. We set

$$\begin{cases} (H, \|\cdot\|)_{\le 1}^{\mathrm{fin}} := \left\{ x \in H \mid \|x\|_\mathfrak{p} \le 1 \text{ for all } \mathfrak{p} \in M_K^{\mathrm{fin}} \right\} \\ (H, \|\cdot\|)_{\le 1}^{\mathfrak{p}} := \left\{ x \in H \mid \|x\|_\mathfrak{p} \le 1 \right\}. \end{cases}$$

The pair $(H, \|\cdot\|)$ is called an *adelically normed vector space over K* if, for any $x \in H$, $\|x\|_\mathfrak{p} \le 1$ except finitely many $\mathfrak{p} \in M_K^{\mathrm{fin}}$, and $(H, \|\cdot\|)_{\le 1}^{\mathrm{fin}}$ is a finitely generated $\mathcal{O}_K$-module (*cf.* [6, Definition 2.1] and [7, Definition 2.10]).

Lemma 5.2. *We will assume that $(H, \|\cdot\|)$ is an adelically normed vector space over K. Then, the following hold:*

(1) *For $\mathfrak{p} \in M_K^{\mathrm{fin}}$, we have $(H, \|\cdot\|)_{\le 1}^{\mathfrak{p}} = (H, \|\cdot\|)_{\le 1}^{\mathrm{fin}} \otimes_{\mathcal{O}_K} \mathcal{O}_\mathfrak{p}$;*
(2) *We have $(H, \|\cdot\|)_{\le 1}^{\mathrm{fin}} \otimes_{\mathcal{O}_K} K = H$. Moreover, $(H, \|\cdot\|)_{\le 1}^{\mathrm{fin}} \otimes_{\mathbb{Z}} \mathbb{Q} = H$;*
(3) *Let $f : H \to H'$ be a surjective homomorphism of finite-dimensional vector spaces over K. Let $\|\cdot\|_v^{\mathrm{quot}}$ be the quotient norm of H'_v induced by the surjection $f_v : H_v \to H'_v$ and the norm $\|\cdot\|_v$ on H_v. Then $(H', \|\cdot\|^{\mathrm{quot}})$ is an adelically normed vector space over K and*

$$f\left((H, \|\cdot\|)_{\le 1}^{\mathrm{fin}}\right) = (H', \|\cdot\|^{\mathrm{quot}})_{\le 1}^{\mathrm{fin}},$$

where $\|\cdot\|^{\mathrm{quot}} = \{\|\cdot\|_v^{\mathrm{quot}}\}_{v \in M_K}$.

Proof. (1) Obviously $(H, \|\cdot\|)_{\le 1}^{\mathrm{fin}} \otimes_{\mathcal{O}_K} \mathcal{O}_\mathfrak{p} \subseteq (H, \|\cdot\|)_{\le 1}^{\mathfrak{p}}$. Conversely, we assume that $x \in H$ and $\|x\|_\mathfrak{p} \le 1$. We set

$$\{\mathfrak{q} \in M_K^{\mathrm{fin}} \mid \|x\|_\mathfrak{q} > 1\} = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_r\}.$$

By Lemma 5.3 as below, there is an $\alpha \in K^{\times}$ such that

$$\mathrm{ord}_{\mathfrak{q}_i}(\alpha) > 0 \ (\forall i = 1, \ldots, r) \quad \text{and} \quad \mathrm{ord}_{\mathfrak{q}}(\alpha) = 0 \ (\forall \mathfrak{q} \in M_K^{\mathrm{fin}} \setminus \{\mathfrak{q}_1, \ldots, \mathfrak{q}_r\}).$$

We choose a positive integer n such that $\|\alpha^n x\|_{\mathfrak{q}_i} \leq 1$ for all $i = 1, \ldots, r$. Note that $\alpha^n \in \mathcal{O}_{\mathfrak{p}}^{\times}$ and $\alpha^n x \in (H, \|\cdot\|)^{\mathrm{fin}}_{\leq 1}$, so that $x = \alpha^{-n}\alpha^n x \in (H, \|\cdot\|)^{\mathrm{fin}}_{\leq 1} \otimes_{\mathcal{O}_K} \mathcal{O}_{\mathfrak{p}}$.

(2) For $x \in H$, by using Lemma 5.3, we can find a $\beta \in \mathcal{O}_K \setminus \{0\}$ with $\beta x \in (H, \|\cdot\|)^{\mathrm{fin}}_{\leq 1}$, which means that the first assertion holds.

Let $\gamma \in \mathcal{O}_K \setminus \{0\}$. Then there are $a_1, \ldots, a_n \in \mathbb{Z}$ such that

$$\gamma^n + a_1\gamma^{n-1} + \cdots + a_n = 0.$$

Clearly we may assume that $a_n \neq 0$. Thus, if we set

$$\gamma' = -(\gamma^{n-1} + a_1\gamma^{n-1} + \cdots + a_{n-1}),$$

then $\gamma' \in \mathcal{O}_K$ and $\gamma\gamma' = a_n$. Note that $(H, \|\cdot\|)^{\mathrm{fin}}_{\leq 1} \otimes_{\mathcal{O}_K} K$ and $(H, \|\cdot\|)^{\mathrm{fin}}_{\leq 1} \otimes_{\mathbb{Z}} \mathbb{Q}$ are the localizations of $(H, \|\cdot\|)^{\mathrm{fin}}_{\leq 1}$ with respect to $\mathcal{O}_K \setminus \{0\}$ and $\mathbb{Z} \setminus \{0\}$, respectively. Therefore the last assertion follows.

(3) Let us see that

$$f\left((H, \|\cdot\|)^{\mathfrak{p}}_{\leq 1}\right) = (H', \|\cdot\|^{\mathrm{quot}})^{\mathfrak{p}}_{\leq 1} \tag{5.1}$$

for all $\mathfrak{p} \in M_K^{\mathrm{fin}}$. Clearly one has $f\left((H, \|\cdot\|)^{\mathfrak{p}}_{\leq 1}\right) \subseteq (H', \|\cdot\|^{\mathrm{quot}})^{\mathfrak{p}}_{\leq 1}$. The converse inclusion follows from Proposition 2.1. By using (1) together with the equation (5.1), we obtain

$$f\left((H, \|\cdot\|)^{\mathrm{fin}}_{\leq 1}\right) \otimes_{\mathcal{O}_K} \mathcal{O}_{\mathfrak{p}} = (H', \|\cdot\|^{\mathrm{quot}})^{\mathrm{fin}}_{\leq 1} \otimes_{\mathcal{O}_K} \mathcal{O}_{\mathfrak{p}}.$$

Therefore $f\left((H, \|\cdot\|)^{\mathrm{fin}}_{\leq 1}\right) = (H', \|\cdot\|^{\mathrm{quot}})^{\mathrm{fin}}_{\leq 1}$ by [1, Proposition 3.8], as required. □

Lemma 5.3. *Let Σ be a finite subset of M_K^{fin}. Then there is an $\alpha \in K^{\times}$ such that*

$$\mathrm{ord}_{\mathfrak{p}}(\alpha) \begin{cases} > 0 & \text{if } \mathfrak{p} \in \Sigma \\ = 0 & \text{if } \mathfrak{p} \in M_K^{\mathrm{fin}} \setminus \Sigma. \end{cases}$$

Proof. We set $\Sigma = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_e\}$. As the class group of K is finite, for each i, there are a positive integer n_i and an $\alpha_i \in \mathcal{O}_K \setminus \{0\}$ with $\mathfrak{p}_i^{n_i} = \alpha_i\mathcal{O}_K$. Thus, if we set $\alpha = \alpha_1 \cdots \alpha_e$, then the assertion follows. □

5.2. Estimation of $\lambda_{\mathbb{Q}}$ for a graded algebra

A *normed $\mathbb{Z}$-module* is a pair $(\mathscr{M}, \|\cdot\|)$ of a finitely generated $\mathbb{Z}$-module $\mathscr{M}$ and a norm $\|\cdot\|$ of $\mathscr{M} \otimes_{\mathbb{Z}} \mathbb{R}$. We define $\lambda_{\mathbb{Q}}(\mathscr{M}, \|\cdot\|)$ and $\lambda_{\mathbb{Z}}(\mathscr{M}, \|\cdot\|)$ as follows. If $\mathscr{M}$ is a torsion module, then

$$\lambda_{\mathbb{Q}}(\mathscr{M}, \|\cdot\|) = \lambda_{\mathbb{Z}}(\mathscr{M}, \|\cdot\|) = 0.$$

Otherwise, let $\lambda_{\mathbb{Q}}(\mathscr{M}, \|\cdot\|)$ (respectively $\lambda_{\mathbb{Z}}(\mathscr{M}, \|\cdot\|)$) be the infimum of the set of non-negative real numbers λ such that we can find a $\mathbb{Q}$-basis $e_1, \ldots, e_r$ of $\mathscr{M}_{\mathbb{Q}} := \mathscr{M} \otimes_{\mathbb{Z}} \mathbb{Q}$ which is contained in $\mathscr{M}$ (respectively a free basis of $\mathscr{M}/\mathscr{M}_{\mathrm{tor}}$) with $\|e_i\| \le \lambda$ for all $i = 1, \ldots, r$. Note that

$$\lambda_{\mathbb{Q}}(\mathscr{M}, \|\cdot\|) \le \lambda_{\mathbb{Z}}(\mathscr{M}, \|\cdot\|) \le \mathrm{rk}(\mathscr{M})\lambda_{\mathbb{Q}}(\mathscr{M}, \|\cdot\|) \tag{5.2}$$

(*cf.* [15, Lemma 1.2]).

Let $R = \bigoplus_{n=0}^{\infty} R_n$ be a graded $\mathbb{Q}$-algebra of finite type such that R is an integral noetherian domain and $\dim_{\mathbb{Q}} R_n < \infty$ for all $n \ge 0$. Let $\mathscr{R} = \bigoplus_{n=0}^{\infty} \mathscr{R}_n$ be a graded subalgebra of R such that $\mathscr{R}_n$ is a finitely generated $\mathbb{Z}$-module and $\mathscr{R}_n \otimes_{\mathbb{Z}} \mathbb{Q} = R_n$ for all $n \ge 0$. For each $n \ge 0$, let $\|\cdot\|_n$ be a norm of $R_n \otimes_{\mathbb{Q}} \mathbb{R}(= \mathscr{R}_n \otimes_{\mathbb{Z}} \mathbb{R})$. We assume that

$$\left(\mathscr{R}, \|\cdot\|\right) = \bigoplus_{n=0}^{\infty} \left(\mathscr{R}_n, \|\cdot\|_n\right)$$

is a *normed graded $\mathbb{Z}$-algebra*, that is, for $a \in \mathscr{R}_n$ and $b \in \mathscr{R}_{n'}$, it holds $\|a \cdot b\|_{n+n'} \le \|a\|_n \cdot \|b\|_{n'}$.

Let $X := \mathrm{Proj}(R)$ and Y be a closed subvariety of X over $\mathbb{Q}$, that is, Y is a closed, reduced and irreducible subscheme of X over $\mathbb{Q}$. Let $P = \bigoplus_{n=0}^{\infty} P_n$ be the corresponding homogeneous prime ideal of R to Y. We set

$$R_{Y,n} := R_n/P_n \text{ and } \mathscr{R}_{Y,n} := \mathscr{R}_n/P_n \cap \mathscr{R}_n$$
$$R_Y := \bigoplus_{n=0}^{\infty} R_{Y,n} \text{ and } \mathscr{R}_Y := \bigoplus_{n=0}^{\infty} \mathscr{R}_{Y,n}.$$

Let $\|\cdot\|_{Y,n}^{\mathrm{quot}}$ be the quotient norm of $R_{Y,n} \otimes_{\mathbb{Q}} \mathbb{R}$ induced by the surjective homomorphism $R_n \otimes_{\mathbb{Q}} \mathbb{R} \to R_{Y,n} \otimes_{\mathbb{Q}} \mathbb{R}$ and the norm $\|\cdot\|_n$ on $R_n \otimes_{\mathbb{Q}} \mathbb{R}$. Note that $\mathscr{R}_{Y,n} \otimes_{\mathbb{Z}} \mathbb{Q} = R_{Y,n}$ for all $n \ge 0$ and

$$\left(\mathscr{R}_Y, \|\cdot\|_Y^{\mathrm{quot}}\right) = \bigoplus_{n=0}^{\infty} \left(\mathscr{R}_{Y,n}, \|\cdot\|_{Y,n}^{\mathrm{quot}}\right)$$

is a normed graded $\mathbb{Z}$-algebra. Then we have the following:

Theorem 5.4. *Let $\mathfrak{S}_X$ be the set of all subvarieties of X and let $\upsilon : \mathfrak{S}_X \to \mathbb{R}_{>0}$ be a map. We assume that, for every $Y \in \mathfrak{S}_X$, there are a positive integer $n(Y)$ and an $s_Y \in \mathscr{R}_{Y,n(Y)} \setminus \{0\}$ with $\|s_Y\|^{\mathrm{quot}}_{Y,n(Y)} \le \upsilon(Y)^{n(Y)}$. Then there are a positive number B and a finite subset S of $\mathfrak{S}_X$ such that*

$$\lambda_{\mathbb{Q}}(\mathscr{R}_n, \|\cdot\|_n) \le Bn^{d(d+1)/2} \left(\max\{\upsilon(Y) \mid Y \in S\}\right)^n$$

for all $n \ge 1$, where $d = \dim X$.

Proof. It is a generalization of [15, Theorem 3.1]; however, it can be proved in a similar way. For reader's convenience, we give a sketch of the proof.

Step 1. For a positive integer h, we set

$$R^{(h)}_n := R_{hn}, \quad \mathscr{R}^{(h)}_n := \mathscr{R}_{hn}, \quad R^{(h)} = \bigoplus_{n=0}^{\infty} R^{(h)}_n \quad \text{and} \quad \mathscr{R}^{(h)} = \bigoplus_{n=0}^{\infty} \mathscr{R}^{(h)}_n.$$

By using [15, Lemma 2.2 and Lemma 2.4], we can see that if the theorem holds for $\mathscr{R}^{(h)}$ and υ^h, then it holds for $\mathscr{R}$ and υ. Therefore, by [4, Chapter III, Section 1, Proposition 3], we may assume that R is generated by R_1 over R_0 and $s := s_X \in \mathscr{R}_1$. Let $\mathscr{O}_X(1)$ be the tautological invertible sheaf of X arising from R_1.

We prove this theorem by induction on d.

Step 2. In the case where $d = 0$, $X = \mathrm{Spec}(K)$ for some number field K, so that $R_n \subseteq H^0(X, \mathscr{O}_X(n)) \cong K$. Therefore, $\dim_{\mathbb{Q}} R_n \le [K : \mathbb{Q}]$ for all $n \ge 1$, and hence the assertion can be checked by the same arguments as in [15, Claim 3.1.2].

Step 3. We assume $d > 0$. Let I be the homogeneous ideal generated by $s := s_X$, that is, $I = Rs$. By using the same ideas as in [13, Chapter I, Proposition 7.4], we can find a sequence

$$I = I_0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_r = R$$

of homogeneous ideals of R and non-zero homogeneous prime ideals $P_1, \ldots, P_r$ of R such that $P_i \cdot I_i \subseteq I_{i-1}$ for $i = 1, \ldots, r$.

Step 4. We set $\overline{\mathscr{R}}_n = (\mathscr{R}_n, \|\cdot\|_n)$ and $\overline{\mathscr{I}}_{i,n} = (\mathscr{I}_{i,n}, \|\cdot\|_{i,n})$, where $\mathscr{I}_{i,n} := \mathscr{R}_n \cap I_{i,n}$ and $\|\cdot\|_{i,n}$ is the subnorm induced by $\|\cdot\|_n$ and $I_{i,n} \hookrightarrow R_n$. Here we consider the following sequence:

$$\begin{array}{ccccccc}
\overline{\mathscr{R}}_0 \xrightarrow{\cdot s} & \overline{\mathscr{I}}_{0,1} & \hookrightarrow \cdots \hookrightarrow & \overline{\mathscr{I}}_{i,1} & \hookrightarrow \cdots \hookrightarrow & \overline{\mathscr{I}}_{r,1} = \overline{\mathscr{R}}_1 \\
\vdots \quad \vdots & & \vdots & & \vdots & \vdots \quad \vdots \\
\xrightarrow{\cdot s} & \overline{\mathscr{I}}_{0,j} & \hookrightarrow \cdots \hookrightarrow & \overline{\mathscr{I}}_{i,j} & \hookrightarrow \cdots \hookrightarrow & \overline{\mathscr{I}}_{r,j} = \overline{\mathscr{R}}_j \\
\xrightarrow{\cdot s} & \overline{\mathscr{I}}_{0,j+1} & \hookrightarrow \cdots \hookrightarrow & \overline{\mathscr{I}}_{i,j+1} & \hookrightarrow \cdots \hookrightarrow & \overline{\mathscr{I}}_{r,j+1} = \overline{\mathscr{R}}_{j+1} \\
\vdots \quad \vdots & & \vdots & & \vdots & \vdots \quad \vdots \\
\xrightarrow{\cdot s} & \overline{\mathscr{I}}_{0,n} & \hookrightarrow \cdots \hookrightarrow & \overline{\mathscr{I}}_{i,n} & \hookrightarrow \cdots \hookrightarrow & \overline{\mathscr{I}}_{r,n} = \overline{\mathscr{R}}_n .
\end{array}$$

Let $\|\cdot\|_{i,n}^{\mathrm{quot}}$ be the quotient norm of $I_{i,n}/I_{i-1,n}$ induced by $\|\cdot\|_{i,n}$ and $I_{i,n} \to I_{i,n}/I_{i-1,n}$. Note that $\mathscr{I}_{0,n}/\mathscr{R}_{n-1}s$ is a torsion module for all $n \geq 1$, so that, applying [15, Proposition 1.4] to the above sequence, we have

$$\begin{aligned}\lambda_{\mathbb{Q}}(\overline{\mathscr{R}}_n) \leq \sum_{j=1}^{n}\left(\sum_{i=1}^{r} \|s\|_1^{n-j}\lambda_{\mathbb{Q}}(\mathscr{I}_{i,j}/\mathscr{I}_{i-1,j}, \|\cdot\|_{i,j}^{\mathrm{quot}}) \dim_{\mathbb{Q}}(I_{i,j}/I_{i-1,j})\right) \\ + \|s\|_1^n \lambda_{\mathbb{Q}}(\overline{\mathscr{R}}_0) \dim_{\mathbb{Q}} R_0.\end{aligned} \tag{5.3}$$

Step 5. Here we claim the following:

Claim 5.5. The following facts hold:

(1) If $P_i \in \mathrm{Proj}(R)$, then there are positive constants B_i and C_i, and a finite subset S_i of $\mathfrak{S}_X$ such that

$$\lambda_{\mathbb{Q}}(\mathscr{I}_{i,n}/\mathscr{I}_{i-1,n}, \|\cdot\|_{i,n}^{\mathrm{quot}}) \leq B_i n^{d(d-1)/2} (\max\{\upsilon(Y) \mid Y \in S_i\})^n$$

and $\dim_{\mathbb{Q}}(I_{i,n}/I_{i-1,n}) \leq C_i n^{d-1}$ for all $n \geq 1$.

(2) If $P_i \notin \mathrm{Proj}(R)$, then there is a positive integer n_i such that $I_{i,n}/I_{i-1,n} = 0$ for $n \geq n_i$. In particular, $\lambda_{\mathbb{Q}}(\mathscr{I}_{i,n}/\mathscr{I}_{i-1,n}, \|\cdot\|_{i,n}^{\mathrm{quot}}) = 0$ and $\dim_{\mathbb{Q}}(I_{i,n}/I_{i-1,n}) = 0$ for all $n \geq n_i$.

Proof. (1) follows from [15, Proposition 2.3] and the hypothesis of induction. In the case (2), $P_i = \bigoplus_{n=1}^{\infty} R_n$ because R_0 is a number field. As I_i/I_{i-1} is a finitely generated (R/P_i)-module, we can find a positive integer n_i such that $I_{i,n}/I_{i-1,n} = 0$ for $n \geq n_i$. □

Step 6. The assertion of the theorem follows from (5.3) by using (1) and (2) of Claim 5.5. □

5.3. Nakai-Moishezon's criterion

Let X be a geometrically integral projective variety over a number field K. For a closed subvariety Y of X and $v \in M_K$, we set $Y_v := Y \times_{\mathrm{Spec}(K)} \mathrm{Spec}(K_v)$. Let L be an invertible sheaf on X. For $v \in M_K$, let h_v be a continuous metric of L_v^{an} on X_v^{an}, where $L_v := L \otimes_K K_v$. Note that $X(\mathbb{C})$ is canonically identified with $\coprod_{\sigma \in M_K^\infty} X_\sigma(\mathbb{C})$, so that $h_\infty := \{h_\sigma\}_{\sigma \in M_K^\infty}$ yields a metric on L_∞. We assume that h_∞ is invariant by the complex conjugation map F_∞ on $X(\mathbb{C})$. Moreover, for $s \in H^0(Y, L|_Y)$, we set

$$\|s\|_{Y_v, h_v} := \sup\{|s|_{h_v}(x) \mid x \in Y_v^{\mathrm{an}}\}.$$

Theorem 5.6. *We will assume the following:*

(a) *For any* $n \in \mathbb{Z}_{\geq 0}$, *the space* $\left(H^0(X, L^{\otimes n}), \{\|\cdot\|_{X_v,h_v^n}\}_{v\in M_K}\right)$ *is an adelically normed vector space over* K;
(b) *The sheaf* $\overline{L}\big|_Y$ *is big for all subvarieties* Y *of* X, *that is,* $L|_Y$ *is big on* Y *and there are a positive integer* n *and an* $s \in H^0(Y, L|_Y^{\otimes n}) \setminus \{0\}$ *such that* $\|s\|_{Y_\mathfrak{p},h_\mathfrak{p}^n} \leq 1$ *for all* $\mathfrak{p} \in M_K^{\rm fin}$ *and* $\|s\|_{Y_\sigma,h_\sigma^n} < 1$ *for all* $\sigma \in M_K^\infty$;
(c) *It holds that* h_v *is semipositive*[1] *for all* $v \in M_K$.

Then there are positive numbers B *and* υ *such that* $\upsilon < 1$ *and*

$$\lambda_\mathbb{Q}\left(\left(H^0(X, L^{\otimes n}), \|\cdot\|_{h^n}\right)^{\rm fin}_{\leq 1}, \max_{\sigma\in M_K^\infty}\left\{\|\cdot\|_{X_\sigma,h_\sigma^n}\right\}\right) \leq Bn^{d(d+1)/2}\upsilon^n$$

for all $n \geq 1$.

Proof. First note that L is nef because $L|_C$ is big for all curves C on X. Moreover, as $L|_Y$ is big on Y and L is nef, we have $(L|_Y^{\dim Y}) > 0$. Therefore, L is ample on X by virtue of the Nakai-Moishezon criterion for projective algebraic varieties.

We set

$$R_n := H^0(X, L^{\otimes n}), \quad \mathscr{R}_n := (R_n, \|\cdot\|_{h^n})^{\rm fin}_{\leq 1} \text{ and } \|\cdot\|_n := \max_{\sigma\in M_K^\infty}\{\|\cdot\|_{X_\sigma,h_\sigma^n}\}.$$

Note that $\mathscr{R}_n$ is a finitely generated $\mathbb{Z}$-module. We use the same notation as in Subsection 5.2. Note that $X = \mathrm{Proj}(R)$ because L is ample. Fix a closed subvariety Y. For $v \in M_K$, the norm $\|\cdot\|_{X_v,h_v^n}$ on $H^0(X_v, L_v^{\otimes n})$ (respectively the norm $\|\cdot\|_{Y_v,h_v^n}$ on $H^0(Y_v, L|_{Y_v}^{\otimes n})$) is denoted by $\|\cdot\|_{X_v,n}$ (respectively $\|\cdot\|_{Y_v,n}$). Note that $\|\cdot\|_n = \max_{\sigma\in M_K^\infty}\{\|\cdot\|_{X_\sigma,n}\}$. Let $\|\cdot\|^{\rm quot}_{Y_v,n}$ be the quotient norm of $R_{Y,n} \otimes_K K_v$ induced by $\|\cdot\|_{X_v,n}$ and the surjective homomorphism $R_n \otimes_K K_v \to R_{Y,n} \otimes_K K_v$. We also fix a positive integer n_0 such that, for all $n \geq n_0$, $H^0(X, L^{\otimes n}) \to H^0(Y, L|_Y^{\otimes n})$ is surjective.

By (3) in Lemma 5.2 and Theorem 5.4, it is sufficient to show that there are a positive integer $n(Y) \geq n_0$ and an $s \in H^0(Y, L|_Y^{\otimes n(Y)})\setminus\{0\}$ such that $\|s\|^{\rm quot}_{Y_\mathfrak{p},n(Y)} \leq 1$ for all $\mathfrak{p} \in M_K^{\rm fin}$ and $\|s\|^{\rm quot}_{Y_\sigma,n(Y)} < 1$ for all $\sigma \in M_K^\infty$.

As $\overline{L}\big|_Y$ is big, there are an $n_1 > 0$ and an $s' \in H^0(Y, L|_Y^{\otimes n_1})$ such that $\|s'\|_{Y_\mathfrak{p},n_1} \leq 1$ for all $\mathfrak{p} \in M_K^{\rm fin}$ and $\|s'\|_{Y_\sigma,n_1} < 1$ for all $\sigma \in M_K^\infty$. Since $H^0(X, L^{\otimes n_0 n_1}) \to H^0(Y, L|_Y^{\otimes n_0 n_1})$ is surjective, we can find an $l' \in H^0(X, L^{\otimes n_0 n_1})$ such that $l'\big|_Y = s'^{\otimes n_0}$, so that there are $\mathfrak{p}_1, \ldots, \mathfrak{p}_e \in M_K^{\rm fin}$ such that $\|l'\|_{X_\mathfrak{p},n_0n_1} \leq 1$ for all $\mathfrak{p} \in M_K^{\rm fin} \setminus \{\mathfrak{p}_1, \ldots, \mathfrak{p}_e\}$. In particular, $\|s'^{\otimes n_0}\|^{\rm quot}_{Y_\mathfrak{p},n_0n_1} \leq 1$ for all $\mathfrak{p} \in$

[1] In the case where $v \in M_K^\infty$, the semipositivity of h_v can be defined as the uniform limit of the quotient metrics as described in Subsection 3.2. This semipositivity coincides with the positivity of the first Chern current of (L_v, h_v). For details, see [17].

$M_K^{\mathrm{fin}} \setminus \{\mathfrak{p}_1, \ldots, \mathfrak{p}_e\}$. By Lemma 5.3, we can choose $\beta \in \mathcal{O}_K \setminus \{0\}$ such that

$$\mathrm{ord}_v(\beta) \begin{cases} > 0 & \text{if } v \in \{\mathfrak{p}_1, \ldots, \mathfrak{p}_e\} \\ = 0 & \text{if } v \in M_K \setminus \{\mathfrak{p}_1, \ldots, \mathfrak{p}_e\}. \end{cases}$$

Since $\|s'\|_{Y_\sigma, n_1} < 1$ for all $\sigma \in M_K^\infty$, we can find a positive integer n_2 such that

$$\left(\max_{\sigma \in M_K^\infty} \{\|s'\|_{Y_\sigma, n_1}\} \right)^{n_0 n_2} \max\left\{ |\sigma(\beta)| \mid \sigma \in M_K^\infty \right\} < 1. \tag{5.4}$$

Claim 5.7. If we set $s = \beta s'^{\otimes n_0 n_2}$, then s satisfies the following properties:

(i) It holds $\|s\|^{\mathrm{quot}}_{Y_{\mathfrak{p}}, n_2 n_1 n_0} \leq 1$ for all $\mathfrak{p} \in M_K^{\mathrm{fin}} \setminus \{\mathfrak{p}_1, \ldots, \mathfrak{p}_e\}$;
(ii) It holds $\|s\|_{Y_{\mathfrak{p}_i}, n_2 n_1 n_0} < 1$ for all $i = 1, \ldots, e$;
(iii) It holds $\|s\|_{Y_\sigma, n_2 n_1 n_0} < 1$ for all $\sigma \in M_K^\infty$.

Proof. (i) is obvious. (iii) follows from (5.4). Let us consider (ii). As $\mathrm{ord}_{\mathfrak{p}_i}(\beta) > 0$ and $\|s'\|_{Y_{\mathfrak{p}_i}, n_1} \leq 1$, we have

$$\begin{aligned} \|s\|_{Y_{\mathfrak{p}_i}, n_2 n_1 n_0} &= \#(\mathcal{O}_K/\mathfrak{p}_i)^{-\mathrm{ord}_{\mathfrak{p}_i}(\beta)} \|s'^{\otimes n_0 n_2}\|_{Y_{\mathfrak{p}_i}, n_0 n_1 n_2} \\ &= \#(\mathcal{O}_K/\mathfrak{p}_i)^{-\mathrm{ord}_{\mathfrak{p}_i}(\beta)} \left(\|s'\|_{Y_{\mathfrak{p}_i}, n_1} \right)^{n_0 n_2} < 1, \end{aligned}$$

as required. ☐

Next let us prove the following claim:

Claim 5.8. If $\|t\|_{Y_v, m} < 1$ for $v \in M_K$ and $t \in H^0(Y_v, L_v|_{Y_v}^{\otimes m})$, then there is an m_0 such that, for all $m' \geq m_0$, it holds

$$\|t^{\otimes m'}\|^{\mathrm{quot}}_{Y_v, mm'} < 1.$$

Proof. Choose an $\epsilon > 0$ such that $e^\epsilon \|t\|_{Y_v, m} < 1$. By virtue of the extension property (*cf.* [17] and Theorem 4.5), there is an m_0 such that, for all $m' \geq m_0$, we can find $t' \in H^0(X_v, L_v^{\otimes mm'})$ with $t'\big|_{Y_v} = t^{\otimes m'}$ and $\|t'\|_{X_v, mm'} \leq e^{m'\epsilon} (\|t\|_{Y_v, m})^{m'}$. In particular, $\|t'\|_{X_v, mm'} < 1$, so that the assertion follows. ☐

By the above claim, for each $i = 1, \ldots, e$ and $\sigma \in M_K^\infty$, there is a positive integer n_3 such that

$$\|s^{\otimes n_3}\|^{\mathrm{quot}}_{Y_{\mathfrak{p}_i}, n_3 n_2 n_1 n_0} < 1 \quad \text{and} \quad \|s^{\otimes n_3}\|^{\mathrm{quot}}_{Y_\sigma, n_3 n_2 n_1 n_0} < 1.$$

If we set $n(Y) := n_3 n_2 n_1 n_0$ and $s_Y := s^{\otimes n_3}$, then $\|s_Y\|^{\mathrm{quot}}_{Y_{\mathfrak{p}}, n(Y)} \leq 1$ for all $\mathfrak{p} \in M_K^{\mathrm{fin}}$ and $\|s_Y\|^{\mathrm{quot}}_{Y_\sigma, n(Y)} < 1$ for all $\sigma \in M_K^\infty$. ☐

Corollary 5.9. *We will assume* (a), (b) *and* (c) *as in Theorem* 5.6. *Let* (N, g) *be a pair of an invertible sheaf* N *on* X *and a family* $g = \{g_v\}_{v \in M_K}$ *of continuous metrics* g_v *of* N_v^{an} *on* X_v^{an}. *We will assume that* $g_\infty := \{g_\sigma\}_{\sigma \in M_K^\infty}$ *is compatible with respect to* F_∞ *and*

$$\left(H^0(X, L^{\otimes n} \otimes N), \{\|\cdot\|_{X_v, h_v^n g_v}\}_{v \in M_K}\right)$$

is an adelically normed vector space over K *for all* $n \geq 0$. *Then there is a positive integer* n_0 *such that, for* $n \geq n_0$, $\left(H^0(X, L^{\otimes n} \otimes N), \|\cdot\|_{h^n g}\right)_{\leq 1}^{\mathrm{fin}}$ *has a free basis* $(\omega_1, \ldots, \omega_{r_n})$ *over* $\mathbb{Z}$ *with* $\|\omega_i\|_{h_\sigma^n g_\sigma} < 1$ *for all* $i = 1, \ldots, r_n$ *and* $\sigma \in M_K^\infty$, *where* r_n *is the rank of* $H^0(X, L^{\otimes n} \otimes N)$ *over* $\mathbb{Q}$.

Proof. We use the same notation in the proof of Theorem 5.6. Moreover, we set

$$\begin{cases} A_n := H^0(X, L^{\otimes n} \otimes N) \\ \mathscr{A}_n := \left(H^0(X, L^{\otimes n} \otimes N), \|\cdot\|_{h^n g}\right)_{\leq 1}^{\mathrm{fin}} \text{ and } \|\cdot\|_n' := \max\limits_{\sigma \in M_K^\infty} \{\|\cdot\|_{X_\sigma, h_\sigma^n g_\sigma}\}_{\sigma \in M_K^\infty} \\ A := \bigoplus\limits_{n=0}^{\infty} A_n \\ (\mathscr{A}, \|\cdot\|') := \bigoplus\limits_{n=0}^{\infty} (\mathscr{A}_n, \|\cdot\|_n'). \end{cases}$$

Note that $(\mathscr{A}, \|\cdot\|')$ is a normed graded $(\mathscr{R}, \|\cdot\|)$-module (*cf.* [15, Section 2]), where

$$\mathscr{R} = \bigoplus_{n=0}^{\infty} (H^0(X, L^{\otimes n}), \|\cdot\|_{h^n})_{\leq 1}^{\mathrm{fin}}.$$

Furthermore A is a finitely generated over $\bigoplus_{n=0}^{\infty} H^0(X, L^{\otimes n})$ because L is ample. Therefore, by Theorem 5.6 together with [15, Lemma 2.2], there is a positive number B' such that $\lambda_{\mathbb{Q}}\left(\mathscr{A}_n, \|\cdot\|_n'\right) \leq B' n^{d(d+1)/2} \upsilon^n$ for all $n \geq 1$, so that, by (5.2),

$$\lambda_{\mathbb{Z}}\left(\mathscr{A}_n, \|\cdot\|_n'\right) \leq \dim_{\mathbb{Q}} H^0(X, L^{\otimes n} \otimes N) \cdot B' n^{d(d+1)/2} \upsilon^n$$

for all $n \geq 1$. Thus we can find a positive integer n_0 such that $\lambda_{\mathbb{Z}}\left(\mathscr{A}_n, \|\cdot\|_n'\right) < 1$ for $n \geq n_0$, as required. □

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Institut de Mathématiques de Jussieu
Paris Rive Gauche
Université Paris Diderot, Paris 7
Bâtiment Sophie Germain, Boîte 7012
75205 Paris Cedex 13, France
huayi.chen@imj-prg.fr

Department of Mathematics
Faculty of Science
Kyoto University
Kyoto, 606-8502, Japan
moriwaki@math.kyoto-u.ac.jp