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On the Kodaira-Spencer map of Abelian schemes

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Abstract. Let A be an Abelian scheme over a smooth affine complex variety S, Ω_A the $\mathcal{O}_S$-module of 1-forms of the first kind on A, $\mathcal{D}_S\Omega_A$ the $\mathcal{D}_S$-module spanned by Ω_A in the first algebraic De Rham cohomology module, and $\theta_\partial : \Omega_A \to \mathcal{D}_S\Omega_A/\Omega_A$ the Kodaira-Spencer map attached to a tangent vector field ∂ on S. We compare the rank of $\mathcal{D}_S\Omega_A/\Omega_A$ to the maximal rank of θ_∂ when ∂ varies: we show that both ranks do not change when one passes to the "modular case", *i.e.* when one replaces S by the smallest weakly special subvariety of $\mathcal{A}_g$ containing the image of S (assuming, as one may up to isogeny, that A/S is principally polarized); we then analyse the "modular case" and deduce, for instance, that for any Abelian pencil of relative dimension g with Zariski-dense monodromy in Sp_{2g}, the derivative with respect to a parameter of a non zero Abelian integral of the first kind is never of the first kind.

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This paper deals with Abelian integrals depending algebraically on parameters and their derivatives with respect to the parameters. Since the nineteenth century, it has been known that differentiation with respect to parameters does not preserve Abelian integrals of the first kind in general.

We study this phenomenon in the language of modern algebraic geometry, *i.e.* in terms of the algebraic De Rham cohomology $\mathcal{O}_S$-module $\mathcal{H}^1_{dR}(A/S)$ attached to an Abelian scheme A of relative dimension g over a smooth $\mathbb{C}$-scheme S, its submodule Ω_A of forms of the first kind on A, the Gauss-Manin connection ∇ and the associated Kodaira-Spencer map θ, *i.e.* the $\mathcal{O}_S$-linear map $T_S \otimes \Omega_A \xrightarrow{\theta} \mathcal{H}^1_{dR}(A/S)/\Omega_A$ induced by ∇.

We introduce and compare the following (generic) "ranks":

- $r = r(A/S) = \operatorname{rk} \mathcal{D}_S\Omega_A/\Omega_A$;
- $r' = r'(A/S) = \operatorname{rk} \theta$;
- $r'' = r''(A/S) = \max_\partial \operatorname{rk} \theta_\partial$;

where ∂ runs over local tangent vector fields on S (of course, $r'' = r'$ when S is a curve).

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One has $r'' \leq r' \leq r \leq g$ and these inequalities may be strict, even if there is no isotrivial factor (Paragraph 4.1.2). On the other hand, these ranks are insensitive to dominant base change, and depend only on the isogeny class of A/S (1.6). In particular, one may assume that A/S is principally polarized and (replacing S by an etale covering) admits a level $n \geq 3$ structure.

We prove that r and r'' are unchanged if one passes to the "modular case", *i.e.* if one replaces S by the smallest weakly special (= totally geodesic) subvariety of the moduli space $\mathcal{A}_{g,n}$ containing the image of S, and A by the universal Abelian scheme on S (3.1).

We prove that $r = r'$ in the "modular case", *i.e.* when S is a weakly special subvariety of $\mathcal{A}_{g,n}$ (3.2).

We then study the "PEM case", *i.e.* the case when the connected algebraic monodromy group is maximal with respect to the polarization and the endomorphisms, and emphasize the "restricted PEM case", *i.e.* where we moreover assume that if the center F of End $A \otimes \mathbb{Q}$ is a CM field, then Ω_A is a free $F \otimes_{\mathbb{Q}} \mathcal{O}_S$-module (4.1, 4.3); this includes, of course, the case when the algebraic monodromy group is Sp_{2g}.

Building on the previous results, we show that one has $r'' = r' = r = g$ in the restricted PEM case (4.4). If moreover S is a curve, we show that the derivative (with respect to a parameter) of a non zero Abelian integral of the first kind is never of the first kind (4.6).

Our methods are inspired by B. Moonen's paper [18]; we exploit the "bi-algebraic" properties of the Kodaira-Spencer map in the guise of a theorem of "logarithmic Ax-Schanuel type" for tangent vector bundles (2.2).

Since the problems under study occur in various parts of algebraic geometry and diophantine geometry, we have tried to make the results more accessible by including extended reminders: Section 1 about algebraic De Rham cohomology of Abelian schemes, Gauss-Manin connections and Kodaira-Spencer maps; Subsections 3.1 to 3.4 about weakly special subvarieties of connected Shimura varieties, relative period torsors, and automorphic bundles.

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1. Preliminaries

1.1. Invariant differential forms

Let S be a smooth connected scheme over a field k of characteristic zero.

Let $f : G \to S$ be a smooth commutative group scheme; we denote by $m : G \times_S G \to G$ the group law and by $e : S \to G$ the unit section. The *invariant differential* 1*-forms* on G are those satisfying $m^*\omega = p_1^*\omega + p_2^*\omega$ (where p_1, p_2

denote the projections); they form a locally free $\mathcal{O}_S$-module denoted by Ω_G, naturally isomorphic to $e^*\Omega^1_{G/S}$ and to $f_*\Omega^1_{G/S}$, and $\mathcal{O}_S$-dual to the Lie algebra Lie G. One has $f^*\Omega_G \cong \Omega^1_{G/S}$. Moreover, invariant differential 1-forms are closed [19, 3.5] [9, 1.2.1].

Let us consider the special case when $G = A$ is an Abelian scheme of relative dimension g, or $G = A^\natural$ universal vectorial extension of A (Rosenlicht-Barsotti, *cf. e.g.*, [16, I]). Recall that $\mathrm{Ext}(A, \mathbb{G}_a) \cong R^1 f_*\mathcal{O}_A$ (using the fact that any rigidified $\mathbb{G}_a$-torsor over an S-Abelian scheme has a canonical S-group structure), so that $A^\natural$ is an extension of A by the vector group attached to the dual of $R^1 f_*\mathcal{O}_A$, which is a locally free $\mathcal{O}_S$-module of rank g. The projection $A^\natural \to A$ gives rise to an exact sequence of locally free $\mathcal{O}_S$-modules

$$0 \to \Omega_A \to \Omega_{A^\natural} \to R^1 f_*\mathcal{O}_A \to 0, \tag{1.1}$$

in a way compatible with base change $S' \to S$. On the other hand, if $A^t := Pic^0(A)$ denotes the dual Abelian scheme, Ω_{A^t} is naturally dual to $R^1 f_*\mathcal{O}_A$ (Cartier), and $\Omega_{A^{t\natural}}$ is naturally dual to $\Omega_{A^\natural}$ in such a way that the exact sequence (1.1) is dual to corresponding exact sequence for A^t [9, 1.1.1].

1.2. Algebraic De Rham cohomology

The *first algebraic De Rham cohomology* $\mathcal{O}_S$-module $\mathcal{H}^1_{dR}(G/S)$ is the hypercohomology sheaf $\mathbf{R}^1 f_*(\Omega^*_{G/S}, d)$. Assuming S affine, it can be computed *à la* Čech using an affine open cover $\mathcal{U}$ of G and taking as coboundary map on $C^p(\mathcal{U}, \Omega^q_{G/S})$ the sum of the Čech coboundary and $(-)^{p+1}$ times the exterior derivative d. In particular, since invariant differential forms are closed, there is a canonical $\mathcal{O}_S$-linear map $\Omega_G \to \mathcal{H}^1_{dR}(G/S)$.

If $G = A$ is an Abelian scheme, and $A^\natural$ its universal vectorial extension, it turns out that the canonical morphisms

$$\Omega_{A^\natural} \to \mathcal{H}^1_{dR}\big(A^\natural/S\big) \leftarrow \mathcal{H}^1_{dR}(A/S)$$

are isomorphisms [9, 1.2.2]. The exact sequence (1.1) thus gives rise to an exact sequence of locally free $\mathcal{O}_S$-modules

$$0 \to \Omega_A = f_*\Omega^1_{A/S} \to \mathcal{H}^1_{dR}(A/S) \to R^1 f_*\mathcal{O}_A = \Omega^\vee_{A^t} \to 0, \tag{1.2}$$

in a way compatible with base change $S' \to S$ and with duality $A \mapsto A^t$ (*cf.* also [14, 8.0]; $f_*\Omega^1_{A/S}$ and $R^1 f_*\mathcal{O}_A$ are the graded pieces gr^1 and gr^0 of the Hodge filtration of $\mathcal{H}^1_{dR}(A/S)$ respectively).

Any polarization of A endows the rank $2g$ vector bundle $\mathcal{H}^1_{dR}(A/S)$ with a symplectic form, for which Ω_A is a Lagrangian[1] subbundle, and the exact sequence (1.2) becomes autodual.

[1] *I.e.* isotropic of rank g.

When $S = \operatorname{Spec} k$, $\mathcal{H}^1_{dR}(A/S)$ can also be interpreted as the space of differentials of the second kind (*i.e.* closed rational 1-forms which are Zariski-locally sums of a regular 1-form and an exact rational form) modulo exact rational 1-forms. For any rational section τ of $A^\natural \to A$ and any $\eta \in \Omega_{A^\natural}$, $\tau^*\eta$ is of the second kind and depends on τ only up to the addition of an exact rational 1-form.

In the sequel, we abbreviate $\mathcal{H}^1_{dR}(A/S)$ by $\mathcal{H}$.

1.3. Gauss-Manin connection

Since the nineteenth century, it has been known that differentiating Abelian integrals with respect to parameters leads to linear differential equations, the prototype being the Gauss hypergeometric equation in the variable t satisfied by $\int_1^\infty z^{a-c}(1-z)^{c-b-1}(1-tz)^{-a}dz$. Manin gave an algebraic construction of this differential module (in terms of differentials of the second kind), later generalized by Katz-Oda and others to the construction of the Gauss-Manin connection on algebraic De Rham cohomology of any smooth morphism $X \to S$.

Let as before $A \xrightarrow{f} S$ be an Abelian scheme of relative dimension g over a smooth connected k-scheme S. If $k = \mathbb{C}$, the *Gauss-Manin connection* is determined by its analytification ∇^{an}, whose dual is the unique analytic connection on $(\mathcal{H}^\vee)^{an}$ which kills the period lattice

$$\ker \exp_A \cong \ker \exp_{A^\natural} \subset \left(\operatorname{Lie} A^\natural\right)^{an} = \left(\Omega^\vee_{A^\natural}\right)^{an} = (\mathcal{H}^\vee)^{an}. \tag{1.3}$$

The formation of $(\mathcal{H}, \nabla)$ is compatible with base change $S' \to S$ and with duality $A \mapsto A^t$. It is contravariant in A, and S-isogenies lead to isomorphisms between Gauss-Manin connections.

If S is affine and Ω_A and Ω_{A^t} are free, let us take a basis $\omega_1, \ldots, \omega_g$ of Ω_A and complete it into a basis $\omega_1, \ldots, \omega_g, \eta_1, \ldots, \eta_g$ of $\mathcal{H}$. Pairing with a basis $\gamma_1, \ldots, \gamma_{2g}$ of the period lattice on a universal covering $\tilde{S}$ of S^{an}, one gets a full solution matrix

$$Y = \begin{pmatrix} \Omega_2 & \mathrm{N}_2 \\ \Omega_1 & \mathrm{N}_1 \end{pmatrix} \in M_{2g}(\mathcal{O}(\tilde{S})) \tag{1.4}$$

for ∇ (with $(\Omega_1)_{ij} = \int_{\gamma_i} \omega_j$, etc...). This reflects into a family of differential equations[2]

$$\partial Y = Y \begin{pmatrix} R_\partial & S_\partial \\ T_\partial & U_\partial \end{pmatrix}, \tag{1.5}$$

where $R_\partial, S_\partial, T_\partial, U_\partial \in M_g(\mathcal{O}(S))$[3] depend $\mathcal{O}(S)$-linearly on the derivation $\partial \in \Gamma T_S$.

[2] We write the matrix of ∇_∂ on the right in order to let the monodromy act on the left on Y. This convention has many advantages. In particular, it is independent of the choice of $\gamma_1, \ldots, \gamma_{2g}$. Writing Y with the indices 2 above the indices 1 will be justified in Subsection 2.2.2 below.

[3] The fact that these matrices have entries in $\mathcal{O}(S)$ rather than $\mathcal{O}(S^{an})$ reflects the algebraic nature of the Gauss-Manin connection. Alternatively, it can be deduced from the next sentence.

It is well-known that the Gauss-Manin connection is regular at infinity (*cf.*, *e.g.*, [14, 14.1]), hence its $\mathcal{D}$-module theoretic properties are faithfully reflected by monodromy theoretic properties.

Remark 1.1. The Katz-Oda algebraic construction of ∇, in the case of $\mathcal{H}^1_{dR}(A/S)$, goes as follows [15, 1.4]. From the exact sequence

$$0 \to f^*\Omega^1_{S/k} \to \Omega^1_{A/k} \to \Omega^1_{A/S} \to 0, \tag{1.6}$$

passing to exterior powers, one gets the exact sequence of k-linear complexes of $\mathcal{O}_A$-modules

$$0 \to f^*\Omega^1_{S/k} \otimes \Omega^{*-1}_{A/S} \to \Omega^*_{A/k}/\left(f^*\Omega^2_{S/k} \otimes \Omega^{*-2}_{A/S}\right) \to \Omega^*_{A/S} \to 0. \tag{1.7}$$

Then ∇ is a coboundary map in the long exact sequence for $\mathbf{R}^* f_*$ applied to (1.7), that is

$$\mathbf{R}^1 f_*\Omega^*_{A/S} \stackrel{\nabla}{\to} \mathbf{R}^2 f_* \left(f^*\Omega^1_{S/k} \otimes \Omega^{*-1}_{A/S}\right) = \Omega^1_{S/k} \otimes \mathbf{R}^1 f_*\Omega^*_{A/S}, \tag{1.8}$$

and can be computed explicitly *à la* Čech, *cf.* [14, 3.4]. One checks that this map satisfies the Leibniz rule and the associated map

$$T_S = \left(\Omega^1_{S/k}\right)^\vee \stackrel{\partial \mapsto \nabla_\partial}{\to} \operatorname{End}_k \mathcal{H} \tag{1.9}$$

respects Lie brackets, so that ∇ corresponds to a $\mathcal{D}_S$-module structure on $\mathcal{H}$ (here $\mathcal{D}_S$ denotes the sheaf of rings of differential operators on S, which is generated by the tangent bundle T_S). In fact, it can also be interpreted as the first higher direct image of $\mathcal{O}_A$ in the $\mathcal{D}$-module setting (*cf. e.g.*, [6, 4] for an algebraic proof).

An alternative and more precise construction of ∇, which avoids homological algebra, consists in endowing $A^\natural$ with the structure of a commutative algebraic $\mathcal{D}$-group, which automatically provides a connection on (the dual of) its Lie algebra [3, 3.4, H5] [5, 6].

1.4. Kodaira-Spencer map

The Gauss-Manin connection does not preserve the subbundle $\varOmega_A \subset \mathcal{H}$ in general. The composed map

$$\varOmega_A \hookrightarrow \mathcal{H} \stackrel{\nabla}{\to} \Omega^1_S \otimes \mathcal{H} \twoheadrightarrow \Omega^1_S \otimes (\mathcal{H}/\varOmega_A) = \Omega^1_S \otimes \varOmega^\vee_{A^t} \tag{1.10}$$

is the *Kodaira-Spencer map* (or Higgs field). Like the Gauss-Manin connection, its formation commutes with base-change. Unlike the Gauss-Manin connection, it is an $\mathcal{O}_S$-linear map (also called the Higgs field of A/S [22]).

Remark 1.2. This map can be interpreted as a coboundary map in the long exact sequence for $R^* f_*$ applied to (1.6), and computed explicitly *à la* Čech, *cf.* [14, 3.4] [15, 1.3].

It can be rewritten as the map

$$\theta: \ T_S \otimes_{\mathcal{O}_S} \Omega_A \to \Omega^{\vee}_{A^t} = \text{Lie } A^t. \tag{1.11}$$

If $\mathcal{D}_S^{\leq 1} \subset \mathcal{D}_S$ denotes the subsheaf of differential operators of order ≤ 1 on S, and $\mathcal{D}_S \Omega_A \subset \mathcal{H}$ the sub-$\mathcal{D}_S$-module generated by Ω_A in $\mathcal{H} = \mathcal{H}^1_{dR}(A/S)$. One has

$$\text{Im}\, \theta = \mathcal{D}_S^{\leq 1} \Omega_A / \Omega_A \subset \mathcal{D}_S \Omega_A / \Omega_A \subset \mathcal{H}/\Omega_A = \text{Lie } A^t. \tag{1.12}$$

The Kodaira-Spencer map can also be rewritten as the map

$$T_S \overset{\partial \mapsto \theta_\partial}{\to} \text{Lie } A \otimes \text{Lie } A^t, \tag{1.13}$$

which is invariant by duality $A \mapsto A^t$ [7, 9.1]; if A is polarized, it thus gives rise to a map

$$T_S \overset{\partial \mapsto \theta_\partial}{\to} S^2 \text{Lie } A \cong \text{Hom}_{\text{sym}}\big(\Omega_A, \Omega_A^{\vee}\big). \tag{1.14}$$

In the situation and notation of the end of Subsection 1.3, the matrix of θ_∂ is T_∂ (which is a symmetric matrix if one chooses the basis $\omega_1, \ldots, \eta_g$ to be symplectic).

Remarks 1.3.

i) Here is another interpretation of θ_∂ in terms of the universal vectorial extension $A^\natural$, assuming S affine [7, 9]: for any $\omega \in \Gamma \Omega_A$, pull-back the exact sequence of vector bundles associated to (1.6) by the morphism $\mathcal{O}_A \to \Omega^1_{A/S}$ corresponding to ω and get an extension of A by the vector group attached to Ω^1_S, so that the morphism from $A^\natural$ to this vectorial extension gives rise, at the level of invariant differential forms, to a morphism $\Omega_{A^t} \to \Omega^1_S$; thus to any ω and any $\partial \in \Gamma T_S = \text{Hom}(\Omega^1_S, \mathcal{O}_S)$, one gets an element of $\Omega^{\vee}_{A^t}$, which is nothing but $\theta_\partial \cdot \omega$;

ii) The following equivalences are well-known:

- A/S is isotrivial $\Leftrightarrow$ $\theta = 0$ $\Leftrightarrow$ $\mathcal{D}_S \Omega_A = \Omega_A$ $\Leftrightarrow$ ∇ is isotrivial (*i.e.* has finite monodromy).

Remembering that the Kodaira-Spencer map commutes to base-change, the only non trivial implications are: ∇ isotrivial $\Rightarrow$ $\mathcal{D}_S \Omega_A = \Omega_A$, and $\theta = 0 \Rightarrow$ A/S isotrivial. The first implication comes from Deligne's "théorème de la partie fixe" [10, 4.1.2]. An elementary proof of the second one will be given below (Paragraph 2.1.1);

iii) In contrast to $\mathcal{D}_S \Omega_A$, $\mathcal{D}_S^{\leq 1} \Omega_A$ is not locally a direct factor of $\mathcal{H}$ in general: at some points $s \in S$ the rank of θ_s may drop (see however Theorem 3.2). In fact, the condition that the rank of θ_s is constant is very restrictive: for instance, if S is a proper curve, the condition that θ is everywhere an isomorphism is equivalent to the condition that the Arakelov inequality $\deg \Omega_A \leq \frac{g}{2}\deg \Omega^1_S$ is an equality, and implies that A/S is a modular family, parametrized by a Shimura curve [22].

1.4.1. Since the $\mathcal{O}_S$-module $\mathcal{H}/\mathcal{D}_S\Omega_A$ carries a $\mathcal{D}_S$-module structure, it is locally free [14, 8.8], hence $\mathcal{D}_S\Omega_A$ is locally a direct summand of $\mathcal{H}$. In fact, by Deligne's semisimplicity theorem [10, 4.2.6], $\mathcal{D}_S\Omega_A$ is even a direct factor of $\mathcal{H}$ (as a $\mathcal{D}_S$-module, hence as a vector bundle).

Lemma 1.4. *The formation of $\mathcal{D}_S\Omega_A$ commutes with dominant base change $S' \xrightarrow{\pi} S$ (with S' smooth connected).*

Proof. Since $\mathcal{H}$ commutes with base-change and $\mathcal{D}_S\Omega_A$ is locally a direct summand, it suffices to prove the statement after restricting S to a dense affine open subset. In particular, one may assume that π is a flat submersion, so that $T_{S'} \to \pi^*T_S$ and $\mathcal{D}_{S'} \to \pi^*\mathcal{D}_S$ are epimorphisms, and $\mathcal{D}_{S'}\Omega_{A_{S'}} = \pi^*\mathcal{D}_S\pi^*\Omega_A = \pi^*(\mathcal{D}_S\Omega_A)$. □

1.4.2. As in the introduction, let us define

$$r = r(A/S) := \operatorname{rk} \mathcal{D}_S\Omega_A/\Omega_A, \tag{1.15}$$

$$r' = r'(A/S) := \operatorname{rk}\theta = \operatorname{rk} \mathcal{D}_S^{\leq 1}\Omega_A/\Omega_A, \tag{1.16}$$

$$r'' = r''(A/S) := \max_{\partial} \operatorname{rk}\theta_\partial, \tag{1.17}$$

where ∂ runs over local tangent vector fields on S (and rk denotes a generic rank).

Lemma 1.5. *These are invariant by dominant base change $S' \xrightarrow{\pi} S$ (with S' smooth connected), and depend only on the isogeny class of A/S.*

Proof. For r, this follows from the previous lemma. Its proof also shows that $\mathcal{D}_S^{\leq 1}\Omega_A$ commutes with base change by flat submersions, which settles the case of r'. For r'', we may assume that S and S' are affine, that T_S is free and $T_{S'} = \pi^*T_S$, and pick a basis $\partial_1, \ldots, \partial_d$ of tangent vector fields; the point is that $\max_{\lambda_i} \operatorname{rk} \sum \lambda_i\theta_{\partial_i}$ is the same when the λ_i's run in $\mathcal{O}(S)$ or in $\mathcal{O}(S')$ (consider the θ_{∂_i}'s as matrices and note that each minor determinant is a polynomial in the λ_i's).

The second assertion is clear since any isogeny induces an isomorphism at the level of $(\mathcal{H}, \nabla)$. □

Lemma 1.6.

(1) *$r'' = g$ holds if and only if there exists a local vector field ∂ such that $\theta_\partial.\omega \neq 0$ for every non-zero $\omega \in \Gamma\Omega_A$;*
(2) *$r' = g$ holds if and only if for every non-zero $\omega \in \Gamma\Omega_A$, there exists a local vector field ∂ such that $\theta_\partial.\omega \neq 0$.*

Proof. The first equivalence is immediate, while the second uses the symmetry of (1.13): assuming A polarized, and after restricting S to a dense open affine subset, one has $r' = g \Leftrightarrow \forall\omega \in \Gamma\Omega_A \setminus 0,\ \exists\eta \in \Gamma\Omega_A, \exists\partial \in \Gamma T_S,\ (\theta_\partial.\omega)\cdot\eta \neq 0$. Since $(\theta_\partial.\eta)\cdot\omega = (\theta_\partial.\omega)\cdot\eta$, one gets $\forall\omega \in \Gamma\Omega_A\setminus 0,\ \exists\eta \in \Gamma\Omega_A, \exists\partial \in \Gamma T_S,\ (\theta_\partial.\eta)\cdot\omega \neq 0$. □

2. Automorphic vector bundles and bi-algebraicity

2.1. Bi-algebraicity of the Kodaira-Spencer map

2.1.1. Let $\mathcal{A}_{g,n}$ be the moduli scheme of principally polarized Abelian varieties of dimension g with level n structure ($n \geq 3$), and let $\mathcal{X} \to \mathcal{A}_{g,n}$ be the universal Abelian scheme.

The universal covering of $\mathcal{A}_{g,n}^{an}$ is the Siegel upper half space $\mathfrak{H}_g$. We denote by $j_{g,n} : \mathfrak{H}_g \to \mathcal{A}_{g,n}^{an}$ the uniformizing map (for $g = n = 1$, this is the usual j-function). The pull-back of the dual of the period lattice $\ker\exp_{\mathcal{X}}$ on $\mathfrak{H}_g$ is a constant symplectic lattice Λ. On $\mathfrak{H}_g$, the Gauss-Manin connection of $\mathcal{X}/\mathcal{A}_{g,n}$ becomes a trivial connection with solution space $\Lambda_{\mathbb{C}}$.

On the other hand, $\mathfrak{H}_g$ is an (analytic) open subset of its "compact dual" $\mathfrak{H}_g^\vee$, which is the Grassmannian of Lagrangian subspaces $V \subset \Lambda_{\mathbb{C}}^\vee$ (*i.e.* isotropic subspaces of dimension g): the Lagrangian subspace V_τ corresponding to a point $\tau \in \mathfrak{H}_g$ is $\Omega_{\mathcal{X}_{j_{g,n}(\tau)}} \subset \mathcal{H}^1_{dR}(\mathcal{X}_{j_{g,n}(\tau)}) \cong \Lambda_{\mathbb{C}}^\vee$ (note that the latter isomorphism depends on τ, not only on $j_{g,n}(\tau)$). The Grassmannian $\mathfrak{H}_g^\vee$ is a homogeneous space for $Sp(\Lambda_{\mathbb{C}})$ (in block form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ sends $\tau \in \mathfrak{H}_g$ to $(A\tau + B)(C\tau + D)^{-1}$). The vector bundle $j_{g,n}^*\mathrm{Lie}\mathcal{X}$ is the restriction to $\mathfrak{H}_g$ of the tautological vector bundle $\mathcal{L}$ on the Lagrangian Grassmannian $\mathfrak{H}_g^\vee$.

2.1.2. In this universal situation, the Kodaira-Spencer map (in the form of (1.14)) is an isomorphism

$$T_{\mathcal{A}_{g,n}} \xrightarrow{\sim} S^2\mathrm{Lie}\mathcal{X}, \tag{2.1}$$

and its pull-back to $\mathfrak{H}_g$ is the restriction of the canonical isomorphism

$$T_{\mathfrak{H}_g^\vee} \xrightarrow{\sim} S^2\mathcal{L} \tag{2.2}$$

cf. e.g., [8,12].

Any principally polarized Abelian scheme with level n structure A/S is isomorphic to the pull-back of $\mathcal{X}$ by a morphism $S \xrightarrow{\mu} \mathcal{A}_{g,n}$, and the Kodaira-Spencer map of A/S (in the form of (1.14)) is the pull-back by μ of the isomorphism (2.1) composed with $d\mu : T_S \to \mu^*T_{\mathcal{A}_{g,n}}$. In particular, the Kodaira-Spencer map θ of A/S vanishes if and only if the image of $S \to \mathcal{A}_{g,n}$ is a point, *i.e.* A/S is constant; moreover, if A/S is not constant, μ is generically finite, and ∂ is a non zero section of T_S, then θ_∂ is non zero.

2.2. Relative period torsor

2.2.1. The *bi-algebraicity* mentioned above refers to the pair of algebraic structures $\mathcal{A}_{g,n}$, $\mathfrak{H}_g^\vee$, which are transcendentally related via $\mathfrak{H}_g$ and $j_{g,n}$.

On the other hand, there is a purely algebraic relation between these two algebraic structures, through the *relative period torsor*. This is the $Sp(\Lambda_{\mathbb{C}})_{\mathcal{A}_{g,n}}$-torsor $\Pi_{g,n} \stackrel{\pi}{\to} \mathcal{A}_{g,n}$ of solutions of the Gauss-Manin connection ∇ of $\mathcal{X}$. More formally, this is the torsor of isomorphisms $\mathcal{H} \to \Lambda_{\mathbb{C}}^\vee \otimes \mathcal{O}_{\mathcal{A}_{g,n}}$ which respect the ∇-horizontal tensors[4]. Its generic fiber is the spectrum of the Picard-Vessiot algebra[5] attached to ∇, namely Spec $\mathbb{C}(\mathcal{A}_{g,n})[Y_{ij}]_{i,j=1,\dots,2g}$ (with the notation of Subsection 1.3).

2.2.2. The canonical horizontal isomorphism $\mathcal{H} \otimes_{\mathcal{O}_{\mathcal{A}_{g,n}}} \mathcal{O}_{\mathfrak{H}_g} \stackrel{\sim}{\to} \Lambda_{\mathbb{C}}^\vee \otimes_{\mathbb{C}} \mathcal{O}_{\mathfrak{H}_g}$ gives rise to an analytic map

$$k : \mathfrak{H}_g \to \Pi_{g,n} \tag{2.3}$$

with $\pi \circ k = j_{g,n}$. In local bases and with the notation of Subsection 1.3, k sends $\tau \in \mathfrak{H}_g$ to the point $Y(\tau) = \begin{pmatrix} \Omega_2(\tau) & \mathrm{N}_2(\tau) \\ \Omega_1(\tau) & \mathrm{N}_1(\tau) \end{pmatrix}$ of $\Pi_{j_{g,n}(\tau)}$. In particular, the image of k is Zariski-dense in $\Pi_{g,n}$.

On the other hand there is an algebraic $Sp(\Lambda_{\mathbb{C}})$-equivariant map

$$\rho : \Pi_{g,n} \to \mathfrak{H}_g^\vee, \tag{2.4}$$

which sends a point $p \in \Pi_{g,n}(\mathbb{C})$ viewed as an isomorphism $\mathcal{H}_{\pi(p)} \to \Lambda^\vee$ to the image of $\Omega_{\mathcal{X}_{\pi(p)}}$ in $\Lambda_{\mathbb{C}}^\vee$. In local bases and with the notation of Subsection 1.3, ρ sends $\begin{pmatrix} \Omega_2 & \mathrm{N}_2 \\ \Omega_1 & \mathrm{N}_1 \end{pmatrix}$ to $\tau = \Omega_2 \Omega_1^{-1}$; $\rho \circ k$ is the Borel embedding $\mathfrak{H}_g \hookrightarrow \mathfrak{H}_g^\vee$.

One thus has the following diagram

$$\mathfrak{H}_g \to \Pi_{g,n} \stackrel{(\pi,\rho)}{\twoheadrightarrow} \mathcal{A}_{g,n} \times \mathfrak{H}_g^\vee , \tag{2.5}$$

in which the first map has Zariski-dense image, and the second map (π, ρ) is surjective (of relative dimension $\frac{g(3g+1)}{2}$) since the restriction of ρ to any fiber of π is $Sp(\Lambda_{\mathbb{C}})$-equivariant and $\mathfrak{H}_g^\vee$ is homogeneous. It follows that the graph of $j_{g,n}$ is Zariski-dense[6] in $\mathfrak{H}_g^\vee \times \mathcal{A}_{g,n}$.

The function field of $\Pi_{g,n}$ is studied in detail in [4]: it is a differential field both for the derivations of $\mathcal{A}_{g,n}$ and for the derivations $\partial/\partial\tau_{ij}$ of $\mathfrak{H}_g^\vee$. Over $\mathbb{C}(\mathfrak{H}_g^\vee) = \mathbb{C}(\tau_{ij})_{i \le j \le g}$, it is generated by (iterated) derivatives with respect to the $\partial/\partial\tau_{ij}$'s of the modular functions (the field of modular functions being $\mathbb{C}(\mathcal{A}_{g,n})$).

[4] This is a reduction from GSp to Sp of the standard principal bundle considered in [17, III.3].

[5] In general, one has to adjoin the inverse of the wronskian together with the entries of a full solution matrix in order to build a Picard-Vessiot algebra, but here the wronskian is a rational function, since the monodromy is contained in Sp.

[6] This property does not extend to m-jets for $m \geq 3$.

2.3. Connected Shimura varieties and weakly special subvarieties

2.3.1. Let G be a reductive group over $\mathbb{Q}$, G^{ad} the quotient by the center, and $G^{ad}(\mathbb{R})^+$ the connected component of identity of the Lie group $G^{ad}(\mathbb{R})$.

Let X be a connected component of a conjugacy class $\mathfrak{X}$ of real-algebraic homomorphisms $\mathbb{C}^* \to G_{\mathbb{R}}$. For any rational representation $G \to GL(W)$, one then has a collection of real Hodge structures $(W_{\mathbb{R}}, h_x)_{x\in X}$ on $W_{\mathbb{R}}$ parametrized by X. If the weight is defined over $\mathbb{Q}$ (which is the case if $G = G^{ad}$ since the weight is 0 in this case), one even has a collection of rational Hodge structures $(W, h_x)_{x\in X}$.

In the sequel, we assume that $(G, \mathfrak{X})$ satisfies Deligne's axioms for a *Shimura datum*; these axioms ensure that X has a $G^{ad}(\mathbb{R})^+$-invariant metric, which makes X into a hermitian symmetric domain, and that the $(W_{\mathbb{R}}, h_x)$ (respectively (W, h_x)) come from variations of polarized Hodge structures on the analytic variety X (respectively if the weight is defined over $\mathbb{Q}$, for instance if $G = G^{ad}$); moreover, in the case of the adjoint representation on $\mathfrak{g} = \operatorname{Lie} G$, the variation of Hodge structures is of type $(-1, 1) + (0, 0) + (1, -1)$ (*cf. e.g.*, [17, II]).

2.3.2. Let Γ be a discrete subgroup of $G^{ad}(\mathbb{Q})^+$, quotient of a torsion-free congruence subgroup of $G(\mathbb{Q})$. Then $\Gamma\backslash X$ has a canonical structure of algebraic variety (Baily-Borel): the *connected Shimura variety* attached to (G, X, Γ). The variation of Hodge structures descends to it, with monodromy group Γ. The situation of Subsection 2.1 corresponds to the case $G = GSp_{2g}$, $X = \mathfrak{H}_g$, $\Gamma =$ the congruence subgroup of level $n \geq 3$ (*cf. e.g.*, [17, II]).

2.3.3. Let S be the connected Shimura variety attached to (G, X, Γ), and $j : X \to S$ the uniformizing map. An irreducible subvariety $S_1 \subset S$ is *weakly special* if there is a sub-Shimura datum $(H, \mathfrak{Y}) \to (G, \mathfrak{X})$, a decomposition $(H^{ad}, \mathfrak{Y}^{ad}) = (H_1, \mathfrak{Y}_1) \times (H_2, \mathfrak{Y}_2)$, and a point $y \in \mathfrak{Y}_2$ such that S_1 is the image of $Y_1 \times y$ in S (here Y_1 is a connected component of $\mathfrak{Y}_1$ contained in X) [21][7]; in particular, S_1 is isomorphic to the connected Shimura variety attached to $(H_1, Y_1, \Gamma^{ad} \cap H_1)$.

2.4. Automorphic vector bundles

2.4.1. Given a faithful rational representation W of G, the associated family of Hodge filtrations on $W_{\mathbb{C}}$ is parametrized by a certain flag variety $X^\vee$, the *compact dual* of X, which is a $G^{ad}_{\mathbb{C}}$-homogeneous space.

The isotropy group of a point $x \in X^\vee$ is a parabolic subgroup P_x, $K_x := P_x \cap G^{ad}(\mathbb{R})^+$ is a maximal compact subgroup, and there is a $G^{ad}(\mathbb{R})^+$-equivariant *Borel embedding*

$$X = G^{ad}(\mathbb{R})^+/K_x \overset{i}{\hookrightarrow} X^\vee = G^{ad}_{\mathbb{C}}/P_x. \tag{2.6}$$

[7] This is a *special* subvariety if y is a special point.

2.4.2. Associated to W, there is a variation of polarized Hodge structures on $S = \Gamma\backslash X$, hence an integrable connection ∇ with regular singularities at infinity on the underlying vector bundle $\mathcal{W}$. There is again a relative period torsor in this situation.

Assume for simplicity that $G = G^{ad}$. The monodromy group Γ is then Zariski-dense in G. The *relative period torsor*

$$\Pi \stackrel{\pi}{\to} S \tag{2.7}$$

is the G_S-torsor of isomorphisms $\mathcal{W} \to W_C \otimes \mathcal{O}_S$ which respects the ∇-horizontal tensors[8]. Its generic fiber is the Picard-Vessiot algebra attached to ∇.

The canonical horizontal isomorphism $\mathcal{W} \otimes_{\mathcal{O}_S} \mathcal{O}_X \stackrel{\sim}{\to} W_{\mathbb{C}} \otimes \mathcal{O}_X$ gives rise to an analytic map $k : X \to \Pi$ with $\pi \circ k = j$. There is an algebraic $G_{\mathbb{C}}$-equivariant map $\Pi \stackrel{\rho}{\to} X^\vee$ (which sends a point $p \in \Pi(\mathbb{C})$ viewed as an isomorphism $\mathcal{W}_{\pi(p)} \to W$ to the point of $X^\vee$ which parametrizes the image of the Hodge filtration of $\mathcal{W}_{\pi(p)}$); one has $\rho \circ k = i$.

One thus has the following factorization:

$$(j, i) : \; X \to \Pi \stackrel{(\pi,\rho)}{\twoheadrightarrow} S \times X^\vee \tag{2.8}$$

in which the first map has dense image, the second map (π, ρ) is surjective (since the restriction of ρ to any fiber of π is $G_{\mathbb{C}}$-equivariant with homogeneous target).

Since any faithful rational representation of G lies in the tannakian category generated by W and conversely, neither $X^\vee$ nor Π depend on the auxiliary W. On the other hand, π^* provides an equivalence between the category of vector bundles on S and the category of $G_{\mathbb{C}}$-vector bundles on Π [17, III.3.1].

2.4.3. A $G_{\mathbb{C}}$-equivariant vector bundle $\check{\mathcal{V}}$ on $X^\vee = G_{\mathbb{C}}/P_x$ is completely determined by its fiber at $x \in X$ together with the induced P_x-action (or else, the induced K_x-action). The quotient $\mathcal{V} := \Gamma\backslash i^*\check{\mathcal{V}}$ has a canonical structure of algebraic vector bundle on $S = \Gamma\backslash X$, the *automorphic vector bundle* attached to $\check{\mathcal{V}}$ [17, III.2.1, 3.6]. One has the equality of analytic vector bundles on X:

$$j^*\mathcal{V} = i^*\check{\mathcal{V}}. \tag{2.9}$$

There is also a purely algebraic relation between $\mathcal{V}$ and $\check{\mathcal{V}}$, through the relative period torsor [17, III.3.5]: one has the equality of algebraic $G_{\mathbb{C}}$-vector bundles on Π:

$$\pi^*\mathcal{V} = \rho^*\check{\mathcal{V}}. \tag{2.10}$$

[8] It coincides with the standard principal bundle considered in [17, III.3].

2.4.4. Any representation of $G_{\mathbb{C}}$ gives rise to a $G_{\mathbb{C}}$-equivariant vector bundle on $X^\vee$, hence to an automorphic vector bundle (which carries an integrable connection).

On the other hand, $T_{X^\vee}$ is a $G_{\mathbb{C}}$-equivariant vector bundle on $X^\vee$, and the corresponding automorphic vector bundle is nothing but T_S.

In the situation of Paragraph 2.1.2, the tangent bundle $T_{\mathfrak{H}_g^\vee}$ and its tautological bundle $\mathcal{L}$ are equivariant vector bundles, and the universal Kodaira-Spencer map (2.1) is an isomorphism of automorphic vector bundles on $\mathcal{A}_{g,n}$.

2.5. A theorem of logarithmic Ax-Schanuel type for tangent bundles

2.5.1. The theorem of logarithmic Ax-Schanuel type for connected Shimura varieties is the following [11, 2.3.1] (*cf.* also [21])[9]:

Theorem 2.1. *Let S be a connected Shimura variety ($S^{an} = \Gamma\backslash X$). Let $Z \subset S$ be an irreducible locally closed subset, and let $\tilde{Z}$ be an analytic component of the inverse image of Z in X.*

Then the image in S of the intersection with X of the Zariski closure of $\tilde{Z}$ in the compact dual $X^\vee$ is the smallest weakly special subvariety $S_1 \subset S$ containing Z.

Here is a sketch of proof. One can replace S by the smallest special subvariety containing Z. Fix a point $s \in Z(\mathbb{C})$ and a faithful rational representation of G, and consider the associated vector bundle $\mathcal{W}$ with integrable connection ∇ on S. Let $\hat{G}_1 \subset G$ be the Zariski closure of the monodromy group Γ_Z of $(\mathcal{W}_{|Z}, \nabla_{|Z})$ at s. Up to replacing Γ by a subgroup of finite index, $\hat{G}_1$ is connected and a normal subgroup of G (by [2, 5]). This gives rise to a weakly special subvariety $S_1 \subset S$ associated to a factor $G_1 = \hat{G}_1^{ad}$ of G^{ad}, which is in fact the smallest weakly special subvariety of S containing Z (*cf.* [18, 3.6], [21, 4.1] for details). On the other hand, since $\tilde{Z}$ is stable under Γ_Z, its Zariski closure in the $G_{\mathbb{C}}$-homogeneous space $X_1^\vee$ is stable under G_1, hence equal to $X_1^\vee$. □

Here is the analog for tangent vector bundles, assuming Z smooth:

Theorem 2.2. *In this situation, $T_{\tilde{Z}}$ is Zariski-dense in $T_{X_1^\vee}$.*

Proof. We may replace G by G_1 and S by S_1. Let $\overline{T_{\tilde{Z}}}$ be the Zariski closure of $T_{\tilde{Z}} = \tilde{Z} \times_Z T_Z$ in $T_{X^\vee}$. Let $(\mathcal{W}, \nabla)$ be as above, and let Π be the relative period torsor of S (we take over the notation (2.3) (2.5)). Since $(\mathcal{W}_{|Z}, \nabla_{|Z})$ has the same algebraic monodromy group as $(\mathcal{W}, \nabla)$, namely G, the generic fiber of the projection $\Pi_Z \overset{\pi_Z}{\to} Z$ is the spectrum of the Picard-Vessiot algebra attached

[9] Not to be confused with the (exponential) Ax-Schanuel theorem - a.k.a. hyperbolic Ax-Lindemann - for connected Shimura varieties, which concerns the maximal irreducible algebraic subvarieties of $X^\vee$ whose intersection with X is contained in $\tilde{Z}$, and which is a much deeper result (Pila-Tsimerman, Klingler-Ullmo-Yafaev).

to $(\mathcal{W}_{|Z}, \nabla_{|Z})$, and the image of $k_{|\tilde{Z}}(\tilde{Z})$ is Zariski-dense in Π_Z. It follows that $(k_{|\tilde{Z}} \times 1_{T_Z})(T_{\tilde{Z}}) = k_{|\tilde{Z}}(\tilde{Z}) \times_Z T_Z$ is Zariski-dense in $\Pi_Z \times_Z T_Z$.

In fact, $\nabla_{|Z}$ induces a connection on the torsor Π_Z, which amounts to a splitting of the natural exact sequence of $G_{\mathbb{C}}$-equivariant vector bundles on Π_Z:

$$T_{\Pi_Z/Z} \to T_{\Pi_Z} \overset{\leftharpoondown}{\to} \Pi_Z \times_Z T_Z,$$

and since $k_{|\tilde{Z}}(\tilde{Z})$ is horizontal, the Zariski-closure of $k_{|\tilde{Z}*}(T_{\tilde{Z}})$ in T_{Π_Z} is the $G_{\mathbb{C}}$-equivariant vector subbundle $\Pi_Z \times_Z T_Z$.

On the other hand, $T_{\tilde{Z}} \to T_{X^\vee}$ factors through the map $T_{\Pi_Z} \to T_{X^\vee}$ of $G_{\mathbb{C}}$-equivariant vector bundles induced by ρ, and one concludes that $\overline{T_{\tilde{Z}}}$ is a $G_{\mathbb{C}}$-equivariant vector subbundle of $T_{X^\vee}$. Hence $\overline{T_{\tilde{Z}}} = \check{\mathcal{V}}$ for some automorphic vector subbundle $\mathcal{V} \subset T_S$.

It is known (see [13, VIII, 5]) that for any irreducible factor of X, the (real) representation of the corresponding factor of $\mathfrak{k}$ on the corresponding factor of $T_x X$ is irreducible, from which it follows that the automorphic vector subbundles of T_S are of the form $S \times_{S_1} T_{S_1}$ for some factor S_1 of the locally symmetric domain S. Since $\mathcal{V}$ contains T_Z and Z is not contained in any proper S_1, one concludes that $\mathcal{V} = T_S$ and $\overline{T_{\tilde{Z}}} = T_{X^\vee}$. □

Remark 2.3. In general, given an algebraic vector bundle $\mathcal{M}$ on an algebraic variety Y, the Zariski closure of an analytic subbundle over some Zariski-dense analytic subspace of Y is not necessarily an algebraic subbundle of $\mathcal{M}$: for instance, the Zariski closure in $T_{\mathbb{C}^2}$ of the tangent bundle of the graph in $\mathbb{C}^2$ of a Weierstrass $\wp$ function is a quadric bundle over $\mathbb{C}^2$, not a vector subbundle of $T_{\mathbb{C}^2}$ (a similar counterexample holds for the graph of the usual j-function and its bundle of jets of order ≤ 3, since j satisfies a rational non-linear differential equation of order 3).

On the other hand, Theorem 2.2 does not extend to arbitrary automorphic vector bundles, but one has the following easy consequence of Theorem 2.1:

Porism 2.4. In the same situation, let $\mathcal{V}$ be an automorphic vector bundle on S with corresponding vector bundle $\check{\mathcal{V}}$ on $X^\vee$, and let $\mathcal{F}$ be a vector subbundle of the restriction of $\mathcal{V}$ to Z. Then Z and $\mathcal{F}$ are bi-algebraic if and only if Z is a weakly special subvariety and $\mathcal{F}$ is an automorphic vector bundle.

The assumption "Z is bi-algebraic" means that $\tilde{Z}$ is the intersection of X with an algebraic subvariety of $X^\vee$, and according to Theorem 2.1, this amounts to $Z = S_1$.

The assumption "$\mathcal{F}$ is bi-algebraic" means that its pull-back $\tilde{\mathcal{F}}$ in $\check{\mathcal{V}}$ is an algebraic subvariety. Since $Z = S_1$, this amounts to say that this analytic subbundle of $\mathcal{V} \times_S X_1^\vee = \check{\mathcal{V}} \times_{X^\vee} X_1^\vee$ is algebraic. It is in fact a $G_{1\mathbb{C}}$-vector subbundle, so that $\mathcal{F}$ is an automorphic vector bundle on $X_1^\vee$.

Using the relative period torsor, one can also prove the following stronger version of Theorem 2.1:

Scholium 2.5. in the setting of Theorem 2.1, the graph of $j_{|\tilde{Z}}$ is Zariski-dense in $X_1^\vee \times Z$.

Indeed, it follows from (2.8) (with S_1 in place of S) that the map $\Pi_{1|Z} \stackrel{(\rho,\pi_Z)}{\to} X_1^\vee \times Z$ is surjective. On the other hand, the image of $k_{|\tilde{Z}}(\tilde{Z})$ is Zariski-dense in $\Pi_{1|Z}$.

3. Transition to the modular case

We go back to the study of $r(A/S)$, $r'(A/S)$, $r''(A/S)$ for an Abelian scheme A/S. We may assume that the base field k is $\mathbb{C}$. According to Lemma 1.6, these ranks are invariant by dominant base change of S and by isogeny of A, hence one may assume that A/S admits a principal polarization and a Jacobi level n structure for some $n \geq 3$, and then replace S by the smooth locus $Z \subset \mathcal{A}_{g,n}$ of its image in the moduli space of principally polarized Abelian varieties of dimension g with level n structure, and A by the restriction $\mathcal{X}_Z$ of the universal Abelian scheme $\mathcal{X}$ on $\mathcal{A}_{g,n}$.

3.1. From Z to the smallest weakly special subvariety of $\mathcal{A}_{g,n}$ containing Z

Let us consider again the situation of Paragraph 2.5.1, with $S = \mathcal{A}_{g,n}$. Given a (locally closed) subvariety $Z \subset \mathcal{A}_{g,n}$, one constructs the smallest weakly special subvariety $S_1 \subset \mathcal{A}_{g,n}$ containing Z, taking $(\mathcal{W}, \nabla)$ equal to $\mathcal{H}^1_{dR}(\mathcal{X}/\mathcal{A}_{g,n})$ with its Gauss-Manin connection. By construction, $S_1(\mathbb{C}) = \Gamma_1 \backslash X_1$ where X_1 is a hermitian symmetric domain attached to the adjoint group G_1 of the connected algebraic monodromy group of $\nabla_{|Z}$.

Theorem 3.1. *One has* $r(\mathcal{X}_Z/Z) = r(\mathcal{X}_{S_1}/S_1)$ *and* $r''(\mathcal{X}_Z/Z) = r''(\mathcal{X}_{S_1}/S_1)$.

Proof. Fix $s \in Z(\mathbb{C})$. By construction $\nabla_{|Z}$ and $\nabla_{|S_1}$ have the same connected algebraic monodromy group at s, namely $\hat{G}_1 \subset Sp_{2g}$ (up to replacing n by a multiple). It follows that $\mathcal{D}_Z \Omega_{\mathcal{X}_Z} = (\mathcal{D}_{S_1} \Omega_{\mathcal{X}_{S_1}})_{|Z}$, whence $r(\mathcal{X}_Z/Z) = r(\mathcal{X}_{S_1}/S_1)$.

On the other hand, the inequality $r''(\mathcal{X}_Z/Z) \leq r''(\mathcal{X}_{S_1}/S_1)$ is obvious. For any natural integer $h < g$, let Δ_h be the closed subset of $T_{\mathfrak{H}_g^\vee}$ corresponding to quadratic forms in $S^2\mathrm{Lie}\mathcal{X}$ of rank $\leq h$ (this is in fact a $Sp(\Lambda_{\mathbb{C}})$-subvariety; Δ_0 is the 0-section). Then $r''(\mathcal{X}_Z/Z)$ (respectively $r''(\mathcal{X}_{S_1}/S_1)$) is the greatest integer h such that $d\mu(\partial)$ in not contained in $\mu^*\Delta_{h-1}$. In order to prove the inequality $r''(\mathcal{X}_Z/Z) \geq r''(\mathcal{X}_{S_1}/S_1)$, it thus suffices to show that if $T_{\tilde{Z}}$ is not contained in Δ_h, neither is $T_{X_1^\vee}$, which follows from the fact that $T_{\tilde{Z}}$ is Zariski-dense in $T_{X_1^\vee}$ (2.2). □

3.2. Case of a weakly special subvariety of $\mathcal{A}_{g,n}$

We now assume that S is a weakly special subvariety of $\mathcal{A}_{g,n}$, with associated group $G = G^{ad}$, and that there is a finite covering $\hat{G}$ of G contained in $Sp(\Lambda_{\mathbb{Q}})$.

Theorem 3.2. *One has* $\mathrm{Im}\,\theta = \mathcal{D}_S \Omega_{\mathcal{X}_S}/\Omega_{\mathcal{X}_S}$, *hence* $r(\mathcal{X}_S/S) = r'(\mathcal{X}_S/S)$.

Proof. Fix an arbitrary point $x \in X$ and set $s = j(x) \in S$. Then $X = G(\mathbb{R})/K_x$, and $X^\vee = G_\mathbb{C}/P_x$ can also be written $\hat{G}_\mathbb{C}/\hat{P}_x \subset \mathfrak{H}_g^\vee$; $\hat{P}_x$ stabilizes the Lagrangian subspace $V_x := \Omega_{\mathcal{X}_s} \subset \Lambda_\mathbb{C}^\vee$. We write

$$\mathfrak{g} = \mathrm{Lie}\, G_\mathbb{C} = \mathrm{Lie}\, \hat{G}_\mathbb{C}, \quad \mathfrak{k}_\mathbb{C} = \mathrm{Lie}\, K_{x,\mathbb{C}}.$$

The Hodge decomposition of $\mathfrak{g}$ with respect to $ad \circ h_x$ takes the form $\mathfrak{u}^+ \oplus \mathfrak{k}_\mathbb{C} \oplus \mathfrak{u}^-$, where $\mathfrak{u}^+$, the Lie algebra of the unipotent radical of P_x, is of type $(-1,1)$, $\mathfrak{k}_\mathbb{C}$ is of type $(0,0)$, and $\mathfrak{u}^-$ of type $(1,-1)$. One has $\mathfrak{u}^+ \oplus \mathfrak{k}_\mathbb{C} = \mathrm{Lie}\, \hat{P}_x$, $\mathfrak{k}_\mathbb{C} \oplus \mathfrak{u}^-$ is the Lie algebra of an opposite parabolic group P_x^-, and $\mathfrak{k}_\mathbb{C}$ is the common (reductive) Levi factor (*cf*. also [18, 5]).

Looking at the Hodge type, one finds that

$$[\mathfrak{u}^+, \mathfrak{u}^+] = [\mathfrak{u}^-, \mathfrak{u}^-] = 0,\ [\mathfrak{k}_\mathbb{C}, \mathfrak{u}^+] \subset \mathfrak{u}^+,\ [\mathfrak{k}_\mathbb{C}, \mathfrak{u}^-] \subset \mathfrak{u}^-,\ [\mathfrak{u}^+, \mathfrak{u}^-] \subset \mathfrak{k}_\mathbb{C}. \quad (3.1)$$

By the Jacobi identity, it follows that

$$[\mathfrak{k}_\mathbb{C}, [\mathfrak{u}^+, \mathfrak{u}^-]] \subset [\mathfrak{u}^+, \mathfrak{u}^-], \quad (3.2)$$

i.e. $[\mathfrak{u}^+, \mathfrak{u}^-]$ is a Lie ideal of $\mathfrak{k}_\mathbb{C}$, hence a reductive Lie algebra.

We may identify $T_x X^\vee = T_s S$ with $\mathfrak{u}^-$. Note that $\Lambda_\mathbb{C}^\vee$ is a faithful representation of $\mathfrak{g}$ and that V_x is stable under $\mathrm{Lie}\, P_x = \mathfrak{u}^+ + \mathfrak{k}_\mathbb{C}$. Using the Hodge decomposition $V_x \oplus \bar{V}_x = \Lambda_\mathbb{C}^\vee \cong H^1_{dR}(\mathcal{X}_s)$, we can write the elements of $\mathfrak{g}$ as matrices in block form $\left(\begin{smallmatrix} R & S \\ T & \iota(R)\end{smallmatrix}\right)$, with $R \in \mathfrak{k}_\mathbb{C}$, $S \in \mathfrak{u}^+$, $T \in \mathfrak{u}^-$ and ι is the involution exchanging P_x and P_x^-. Identifying $\mathfrak{u}^+$ with $\left(\begin{smallmatrix} 0 & \mathfrak{u}^+ \\ 0 & 0\end{smallmatrix}\right)$ (respectively $\mathfrak{u}^-$ with $\left(\begin{smallmatrix} 0 & 0 \\ \mathfrak{u}^- & 0\end{smallmatrix}\right)$), one may write $[\mathfrak{u}^+, \mathfrak{u}^-] = \left(\begin{smallmatrix} \mathfrak{u}^+\cdot\mathfrak{u}^- & 0 \\ 0 & \iota(\mathfrak{u}^+\cdot\mathfrak{u}^-)\end{smallmatrix}\right)$. Therefore $\mathfrak{u}^+ \cdot \mathfrak{u}^-$ is a reductive Lie algebra acting on V_x. Accordingly, V_x decomposes as $V_0 \oplus V'$, where V_0 is the kernel of this action, and $(\mathfrak{u}^- \cdot \mathfrak{u})V' = V'$.

The identifications $\mathfrak{u}^- = T_s S$ and $V_x = (\Omega_{\mathcal{X}_S})_s$ lead to $V_x \oplus \mathfrak{u}^- V_x = (\mathcal{D}_S^{\leq 1} \Omega_{\mathcal{X}_S})_s$.

Claim. $V_x \oplus \mathfrak{u}^- V_x$ is the smallest $\mathfrak{g}$-submodule of $\Lambda_\mathbb{C}^\vee$ containing V_x. Therefore, it is the fiber at x of $\mathcal{D}_S \Omega_{\mathcal{X}_S}$.

The point is that $V_x + \mathfrak{u}^- V_x = V_x + \mathfrak{u}^- V'$ is stable under $\mathfrak{u}^-$, $\mathfrak{u}^-$ and $\mathfrak{k}_\mathbb{C}$, which follows from (3.1) and from the fact that V_x is stable under $\mathfrak{u}^+ + \mathfrak{k}_\mathbb{C}$. □

4. The case of maximal monodromy (subject to given polarization and endomorphisms)

4.1. Abelian schemes of PEM type

Definition 4.1. A principally polarized Abelian scheme A/S is *of PE-monodromy type – or PEM type* – if its geometric generic fibre is simple and the connected algebraic monodromy is maximal with respect to the polarization ψ and the endomorphisms.

In other words, the Zariski-closure of the monodromy group at $s \in S$ is the maximal algebraic subgroup of $Sp(H^1(A_s), \psi_s)$ which commutes with the action of End A/S (this condition is independent of $s \in S(\mathbb{C})$).

Let us make this more explicit. The endomorphism $\mathbb{Q}$-algebra $D := (\operatorname{End} A/S) \otimes \mathbb{Q}$ is the same as the one of its generic fiber; since the latter is assumed to be geometrically simple, D is also the endomorphism $\mathbb{Q}$-algebra of the geometric generic fibre. According to Albert's classification, its falls into one of the following types:

I: a totally real field $F = D$;
II: a totally indefinite quaternion algebra D over a totally real field F;
III: a totally indefinite quaternion algebra D over a totally real field F;
IV: a division algebra D over a CM field F.

Let $E \supset F$ be a maximal subfield of D, which we can take to be a CM field except for type I, and let E^+ be a maximal totally real subfield. For any embedding $\lambda : E^+ \hookrightarrow \mathbb{R}$, let us order the embeddings $\lambda_1, \lambda_2 : E \hookrightarrow \mathbb{C}$ above λ if $E \neq E^+$ (and set $\lambda_1 = \lambda$ otherwise). We identify λ (respectively λ_1) with a homomorphism $E^+ \otimes \mathbb{C} \to \mathbb{C}$ (respectively $E \otimes \mathbb{C} \to \mathbb{C}$). Let us set

$$\mathcal{H}_\lambda = \mathcal{H} \otimes_{E^+ \otimes \mathbb{C}, \lambda} \mathbb{C} \ \ \text{(respectively } \mathcal{H}_{\lambda_1} = \mathcal{H} \otimes_{E \otimes \mathbb{C}, \lambda_1} \mathbb{C}\text{)}. \tag{4.1}$$

By functoriality of the Gauss-Manin connection, these are direct factors of $\mathcal{H}$ as $\mathcal{D}_S$-modules, and $\mathcal{H}_\lambda$ only depends (up to isomorphism) on the restriction $[\lambda]$ of λ to F^+.

Then the maximal possible connected complex monodromy group at an arbitrary point $s \in S(\mathbb{C})$ is of the form $\Pi_{[\lambda]} G_{[\lambda]}$ where $G_{[\lambda]}$ and its representation on $\mathcal{H}_{\lambda,s}$ are of the form

I: $Sp(\mathcal{H}_{\lambda,s}),\ St$;
II: $Sp(\mathcal{H}_{\lambda_1,s}),\ St \oplus St$;
III: $SO(\mathcal{H}_{\lambda_1,s}),\ St \oplus St$;
IV: $SL(\mathcal{H}_{\lambda_1,s}),\ St \oplus St^\vee$;

where St denotes the standard representation, and $St^\vee$ its dual. Moreover, for types I, II, III, $\mathcal{H}_{\lambda_1,s}$ is an even-dimensional space, *cf.*, *e.g.*, [1, 5].

Remark 4.2. If A/S is endowed with a level n structure, it is of PEM type if and only if the smallest weakly special subvariety of $\mathcal{A}_{g,n}$ containing the image of S is a special subvariety of PEL type in the sense of Shimura, *i.e.* the image in $\mathcal{A}_{g,n}$ of the moduli space for principally polarized Abelian varieties A such that $D \subset (\operatorname{End} A) \otimes \mathbb{Q}$, equipped with level n structure [20] (for $S = \operatorname{Spec} k$, A is of PEM type if and only if A has complex multiplication).

One could also define the related (but weaker) notion of Abelian scheme A/S of PE Hodge type, on replacing the monodromy group by the Mumford-Tate group, *cf.*, *e.g.*, [1]. If A/S is endowed with a level n structure, it is of PE Hodge type if and only if the smallest *special subvariety* of $\mathcal{A}_{g,n}$ containing the image of S is a special subvariety of PEL type in the sense of Shimura.

4.1.1. Parallel to (4.1), one has a decomposition

$$\Omega_{A,\lambda} = \Omega_A \otimes_{E^+\otimes\mathbb{C},\lambda} \mathbb{C} \ \text{(respectively } \Omega_{A,\lambda_1} = \Omega_A \otimes_{E\otimes\mathbb{C},\lambda_1} \mathbb{C}). \tag{4.2}$$

The sequence (1.2) induces an exact sequence

$$0 \to \Omega_{A,\lambda_1} \to \mathcal{H}_{\lambda_1} \to \mathcal{H}_{\lambda_1}/\Omega_{A,\lambda_1} \to 0. \tag{4.3}$$

It turns out that for types I, II, III, $\mathcal{H}_{\lambda_1}/\Omega_{A,\lambda_1} \cong \Omega^{\vee}_{A,\lambda_1}$. This is not the case for type IV, and the pair

$$\big(r_{[\lambda]} = \dim \Omega_{A,\lambda_1,s},\ s_{[\lambda]} = \dim \mathcal{H}_{\lambda_1,s}/\Omega_{A,\lambda_1,s}\big) \tag{4.4}$$

is an interesting invariant called the *Shimura type* (for type IV, the PEL families depend not only on D, the polarization and the level structure, but also on these pairs, when $[\lambda]$ runs among the real embeddings of F^+). On the other hand, $\Omega_{A,\lambda_2} \cong \Omega_{A,\lambda_1}$ for types I, II, III, while $\Omega_{A,\lambda_2} \cong (\mathcal{H}_{\lambda_1}/\Omega_{A,\lambda_1})^{\vee}$ for type IV.

4.1.2. By functoriality, the Kodaira-Spencer map induces a map

$$\theta_{\partial,\lambda_1} :\ \Omega_{A,\lambda_1} \to \mathcal{H}_{\lambda_1}/\Omega_{A,\lambda_1}. \tag{4.5}$$

Therefore, $\operatorname{rk}\theta_{\partial,\lambda_1} \le \min(r_{[\lambda]}, s_{[\lambda]})$. In particular, if for some $[\lambda]$, $r_{[\lambda]} \ne s_{[\lambda]}$, then $r' < g$.

Let us consider for example the Shimura family of PEL type of Abelian 3-folds with multiplication by an imaginary quadratic field E (type IV) and invariant $(r_{[\lambda]} = 1, s_{[\lambda]} = 2)$ (it is non empty by [20]). The base is a Shimura surface, and for this family one has $r'' = 2$, $r' = r = g = 3$. Let A/S be the restriction of this Abelian scheme to a general curve of this surface; then $r'' = r' = 2$, $r = g = 3$.

One gets examples with $r < g$ when $r_{[\lambda]} \cdot s_{[\lambda]} = 0$ for some $[\lambda]$.

4.2. Abelian schemes of restricted PEM type

Definition 4.3. A principally polarized Abelian scheme A/S is of *restricted PEM type* if it is of PEM type and for any (equivalently, for all) $s \in S(\mathbb{C})$, $(\Omega_A)_s$ is a free $E \otimes \mathbb{C}$-module.

In the latter condition, one could replace E by F. It is automatic for types I, II, III. For type IV, it amounts to the equality $r_\lambda = s_\lambda$ for every λ; in that case, $\Omega_{\lambda_1} \cong \Omega^{\vee}_{\lambda_2}$.

Theorem 4.4. *In the restricted PEM case, one has $r'' = r' = r = g$.*

Proof. Thanks to Lemma 1.6 and Theorem 3.1, we are reduced to prove that $r'' = g$ for a Shimura family of PEL type, provided $r_\lambda = s_\lambda$ for every λ in the type IV case. This amounts in turn to showing that there exists ∂ such that $\theta_{\partial,\lambda_1}$ has maximal rank, equal to the rank of $\mathcal{H}_{\lambda_1}$ which is twice the rank m of Ω_{A,λ_1}. Let $\mathfrak{g}$ be one of the Lie algebras $sp(2m)$, $so(2m)$, $sl(2m)$. In the notation of the proof of Theorem 3.2, The point is to show that $\mathfrak{u}^-$ contains an invertible element. But $\mathfrak{u}^+$ consists of lower left quadrants of elements of $\mathfrak{g}$ viewed as a $2m$-$2m$-matrices; and it is clear that the lower left quadrant of a general element of $\mathfrak{g}$ is an invertible m-m-matrix. □

Remarks 4.5.

i) One can be more precise and give an interpretation of the partial Kodaira-Spencer map at the level of $X^\vee$ as induced by isomorphisms

$$T_{G_{[\lambda]}/P_{[\lambda]}} \xrightarrow{\sim} S^2\mathcal{L}_{\lambda_1} \tag{4.6}$$

for type I and II (the lower left quadrant of an element of $\mathfrak{g}$ is symmetric),

$$T_{G_{[\lambda]}/P_{[\lambda]}} \xrightarrow{\sim} \mathcal{L}_{\lambda_1}^{\otimes 2} \tag{4.7}$$

for type III and IV (the lower left quadrant of an element of $\mathfrak{g}$ can be any m-m-matrix).

ii) Of course one has $r = g$ whenever $\mathcal{H}$ is an irreducible $\mathcal{D}_S$-module.

Claim. If $\mathrm{End}_S A = \mathbb{Z}$ and A/S is not isotrivial, then $\mathcal{H}$ is an irreducible $\mathcal{D}_S$-module.

Indeed, the conclusion can be reformulated as: the local system $R_1 f_*^{an}\mathbb{C}$ is irreducible. Since we know that it is semisimple [10, Section 4.2.6], this is also equivalent, by Schur's lemma, to $\mathrm{End}\, R_1 f_*^{an}\mathbb{C} = \mathbb{C}$ and also to $\mathrm{End}\, R_1 f_*^{an}\mathbb{Z} = \mathbb{Z}$. This equality then follows from the assumptions by the results of [10, Section 4.4]. More precisely, let Z be as in *loc. cit.* the center of $\mathrm{End}\, R_1 f_*^{an}\mathbb{Q}$; then Z is contained in $(\mathrm{End}_S A) \otimes \mathbb{Q}$ (*loc. cit.*, 4.4.7), hence equal to $\mathbb{Q}$, and by *loc. cit.* Proposition 4.4.11 (under conditions (a), (b), (c_1) or (c_2)), one deduces that $\mathrm{End}\, R_1 f_*^{an}\mathbb{Z} = \mathbb{Z}$.

It would be interesting to determine whether $r'' = g$ in this case, beyond the PEM case.

4.3. Differentiating Abelian integrals of the first kind with respect to a parameter

From the above results about differentiating differential forms of the first kind with respect to parameters, it is possible to draw results about differentiating their integrals.

An *Abelian integral of the first kind* on A is a $\mathbb{C}$-linear[10] combination of Abelian periods $\int_\gamma \omega$, with $\omega \in \Gamma\Omega_A$ and γ in the period lattice on a universal covering $\tilde{S}$ of S^{an}.

Theorem 4.6. *Assume that A is an Abelian scheme of restricted PEM type over an affine curve S. Let ∂ be a non-zero derivation of $\mathcal{O}(S)$. Then the derivative of a non zero Abelian integral of the first kind is never an Abelian integral of the first kind (on A).*

[10] Or $\mathcal{O}(S)$-linear, this amounts to the same.

Proof. Let us first treat the case when the monodromy of A/S is Zariski-dense in Sp_{2g} for clarity. We may assume that Ω_A is free. Then an Abelian integral of the first kind is an $\mathcal{O}(S)$-linear combination $\sum_{ij} \lambda_{ij} \int_{\gamma_i} \omega_j$ of entries of $\binom{\Omega_2}{\Omega_1}$. By (1.5), $\sum_{ij} \lambda_{ij} \partial \int_{\gamma_i} \omega_j = \sum_{ijk} \lambda_{ij} (\int_{\gamma_i} \omega_k (R_\partial)_{kj} + \int_{\gamma_i} \omega_k (T_\partial)_{kj})$, *i.e.* an $\mathcal{O}(S)$-linear combination of entries of $\binom{\Omega_2 R_\partial + N_2 T_\partial}{\Omega_1 R_\partial + N_1 T_\partial}$.

Since the monodromy of A/S is Zariski-dense in Sp_{2g}, $Y = \begin{pmatrix} \Omega_2 & N_2 \\ \Omega_1 & N_1 \end{pmatrix}$ is the generic point of a $Sp_{2g,\mathbb{C}(S)}$-torsor, by differential Galois theory in the fuchsian case (Picard-Vessiot-Schlesinger-Kolchin). Since there is no linear relations between the entries of a generic element of Sp_{2g}, there is no $\mathbb{C}(S)$-linear relations between the entries of $\begin{pmatrix} \Omega_2 & N_2 \\ \Omega_1 & N_1 \end{pmatrix}$, or else between the entries of $\begin{pmatrix} \Omega_2 & N_2 T_\partial \\ \Omega_1 & N_1 T_\partial \end{pmatrix}$ since T_∂ is invertible (4.4). One concludes that $\sum_{ij} \lambda_{ij} \partial \int_{\gamma_i} \omega_j = \sum \mu_{ij} \int_{\gamma_i} \omega_j$ with $\lambda_{ij}, \mu_{ij} \in \mathcal{O}(S)$ implies $\lambda_{ij} = \mu_{ij} = 0$.

The other cases are treated similarly, decomposing $\mathcal{H}$ into pieces of rank $2m$ indexed by λ as above, and replacing Sp_{2g} by Sp_{2m}, SO_{2m} or SL_{2m} according to the type. □

References

[1] S. ABDULALI, *Abelian varieties and the general Hodge conjecture*, Compos. Math. **109** (1997), 341–355.

[2] Y. ANDRÉ, *Mumford-Tate groups of mixed Hodge structures and the theorem of the fixed part*, Compos. Math. **82** (1992), 1–24.

[3] D. BERTRAND and A. PILLAY, *A Lindemann-Weierstrass theorem for semi-Abelian varieties over function fields*, J. Amer. Math. Soc. **23** (2010), 491–533.

[4] D. BERTRAND and V. ZUDILIN, *On the transcendence degree of the differential field generated by Siegel modular forms*, J. Reine Angew. Math. **554** (2003), 47–68.

[5] J.-B. BOST, *Algebraization, transcendence, and D-group schemes*, Notre Dame J. Form. Log. **54** (2013), 377–434.

[6] M. CAILOTTO and L. FIOROT, *Algebraic connections vs. algebraic $\mathcal{D}$-modules: inverse and direct images*, Port. Math. **66** (2009), 303–320.

[7] C.-L. CHAI and G. FALTINGS, "Degenerations of Abelian Varieties", Ergeb. Math. Grenzgeb., Vol. 22, Springer, 1990.

[8] C. CILIBERTO and G. VAN DER GEER, *The moduli space of Abelian varieties and the singularities of the theta divisor*, Surv. Differ. Geom. **7** (2000), 61–81.

[9] R. COLEMAN, *Duality for the De Rham cohomology of an Abelian scheme*, Ann. Inst. Fourier Grenoble **48** (1998), 1379–1393.

[10] P. DELIGNE, *Théorie de Hodge II*, Inst. Hautes Études Sc. Publ. Math. **40** (1971), 5–57.

[11] Z. GAO, "The Mixed Ax-Lindemann Theorem and its Application to the Zilber-Pink Conjecture", PhD thesis Orsay-Leiden, 2014.

[12] G. VAN DER GEER, *The cohomology of the moduli space of Abelian varieties*, In: "Handbook of Moduli", G. Farkas and I. Morrison (eds.), Vol. I, Adv. Lectures in Math., Vol. 24, Higher Education Press, Beijing-Boston, 2012, 415–458.

[13] S. HELGASON, "Differential Geometry, Lie Groups, and Symmetric Spaces", Acad. Press, 1978.

[14] N. KATZ, *Nilpotent connections and the monodromy theorem. Application of a result of Turrittin*, Inst. Hautes Études Sc. Publ. Math. **39** (1970), 155–232.

[15] N. KATZ, *Algebraic solutions of differential equations (p-curvature and the Hodge filtration)*, Invent. Math. **18** (1972), 1–118.
[16] B. MAZUR and W. MESSING, "Universal Extensions and one Dimensional Crystalline Cohomology, Notes in Mathematics, Vol. 370, Springer, 1974.
[17] J. MILNE, *Canonical models of (mixed) Shimura varieties and automorphic vector bundles*, In: "Automorphic Forms, Shimura Varieties, and L-functions", Proc. Conf. Univ. Michigan, 1990, 283–414.
[18] B. MOONEN, *Linearity properties of Shimura varieties, I*, J. Algebraic Geom. **7** (1998), 539–567.
[19] B. MOONEN and G. VAN DER GEER, "Abelian Varieties", draft of a book.
[20] G. SHIMURA, *On an analytic family of polarized Abelian varieties and automorphic functions*, Ann. of Math. **78** (1963), 149–192.
[21] E. ULLMO and A. YAFAEV, *Algebraic flows on Shimura varieties*, preprint, 2015.
[22] E. VIEHWEG and K. ZUO, *A characterization of certain Shimura curves in the moduli stack of Abelian varieties*, J. Differential Geom. **66** (2004), 233–287.

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