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Divisorial Zariski decomposition and some properties of full mass currents

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Abstract. Let α be a big class on a compact Kähler manifold. We prove that a decomposition $\alpha = \alpha_1 + \alpha_2$ into the sum of a modified nef class α_1 and a pseudoeffective class α_2 is the divisorial Zariski decomposition of α if and only if $\mathrm{vol}(\alpha) = \mathrm{vol}(\alpha_1)$. We deduce from this result some properties of full mass currents.

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Introduction

The study of the Zariski decomposition started with the work of Zariski [26] who defined it for an effective divisor in a smooth projective surface. Fujita extended the definition to the case of pseudoeffective divisors [13]. Due to the importance of the Zariski decomposition for surfaces, several generalizations to higher dimension exist (see [22] for a survey of these constructions). The divisorial Zariski decomposition for a cohomology class α on a Kähler manifold has been introduced by Boucksom in [7]. If α is the class of a big divisor on a projective manifold, the divisorial Zariski decomposition coincides with the σ-decomposition introduced by Nakayama [20]. The divisorial Zariski decomposition is a decomposition

$$\alpha = Z(\alpha) + \{N(\alpha)\}$$

into a "positive part", the Zariski projection $Z(\alpha)$, whose non-nef locus has codimension at least 2, and a "negative part" $\{N(\alpha)\}$ which is the class of an effective divisor and is rigid. The class $Z(\alpha)$ encodes some important information about α: $Z(\alpha)$ is big if and only if α is and $\mathrm{vol}(\alpha) = \mathrm{vol}(Z(\alpha))$.

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In this note we give a criterion for a sum of two classes to be a divisorial Zariski decomposition. Our main result is:

Main Theorem. *Let X be a compact Kähler manifold of complex dimension n. Let α be a big class on X. Let $\alpha_1 \in H^{1,1}(X, \mathbb{R})$ be a modified nef class and $\alpha_2 \in H^{1,1}(X, \mathbb{R})$ be a pseudoeffective class. Then $\alpha = \alpha_1 + \alpha_2$ is the divisorial Zariski decomposition of α if and only if* $\mathrm{vol}(\alpha) = \mathrm{vol}(\alpha_1)$.

The relations between the Zariski decomposition of numerical classes of cycles on a projective variety and their volume have been largely studied recently in a series of papers [14, 15, 18]. The Main Theorem also goes in this direction: for instance, if X is projective and $\alpha = \{D\}$ is the class of a big divisor, we recover [15, Proposition 5.3] for cycles of codimension 1.

Our proof relies deeply on a result of existence and uniqueness of weak solutions of complex Monge-Ampère equations.

On the other hand the proof in [15] uses the differentiability of the volume function $f(t) = \mathrm{vol}(\alpha + t\{D\})$, which, at the moment, is known to be true only in the algebraic case. In Remark 2.3 we present a proof of the Main Theorem using the differentiability of the volume. As it is proved by Xiao [24, Proposition 1.1] , the differentiability of the volume is equivalent to the following quantitative version of a Demailly's conjecture [8, Conjecture 10.1], that states:

Let X be a compact Kähler manifold of complex dimension n, and let $\alpha, \beta \in H^{1,1}(X, \mathbb{R})$ be two nef classes. Then we have

$$\mathrm{vol}(\alpha - \beta) \geq \alpha^n - n\,\alpha^{n-1} \cdot \beta. \tag{0.1}$$

While this paper was being published, Witt Nyström [21] proved inequality (0.1) for projective manifolds. This, together with Remark 2.3, provides another proof of the Main theorem in the case where X is projective and α_2 is the class of an effective $\mathbb{R}$-divisor.

In the second part of this note we show that the Main Theorem is strictly related to the invariance of finite energy classes under bimeromorphic maps. More precisely, in Theorem 3.6 we show that finite energy classes are inviariant under a bimeromorphic map if and only if the volumes are preserved. This extends to any dimension, [12, Proposition 2.5], where a similar statement is proved in dimension 2 by the first named author using the Hodge index theorem.

We now give a brief outline of this note. Section 1 reviews background material on the divisorial Zariski decomposition and currents with full Monge-Ampère mass. In Section 2 we prove the Main Theorem and in Section 3 we give some applications to full mass currents. In particular we prove Theorem 3.6.

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1. Preliminaries

Let (X, ω) be a compact Kähler manifold of complex dimension n and let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a real $(1, 1)$-cohomology class. Recall that α is said to be *pseudo-effective* if it can be represented by a closed positive $(1, 1)$-current T; α is *nef* if and only if for any $\varepsilon > 0$ there exists a smooth form $\theta_\varepsilon \in \alpha$ such that $\theta_\varepsilon \geq -\varepsilon\omega$; α is *big* if and only if it can be represented by a *Kähler current*, *i.e.*, if and only if there exists a positive closed $(1, 1)$-current $T \in \alpha$ such that $T \geq \varepsilon\,\omega$ for some $\varepsilon > 0$ and α is a Kähler class if and only if it contains a Kähler form.

Given a smooth representative θ of the class α, it follows from $\partial\bar{\partial}$-lemma that any positive $(1, 1)$-current can be written as $T = \theta + dd^c\varphi$ where the global potential $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ is a *θ-plurisubharmonic function* (θ-psh for short), *i.e.*, it is upper semicontinuous and $\theta + dd^c\varphi \geq 0$ in the sense of currents. Here d and d^c are real differential operators defined as

$$d := \partial + \bar{\partial}, \qquad d^c := \frac{i}{2\pi}\left(\bar{\partial} - \partial\right).$$

Let T be a closed positive $(1, 1)$-current. We denote by $\nu(T, x)$ its Lelong number at a point $x \in X$ defined as

$$\nu(T, x) = \nu(\varphi, x) := \sup\{\gamma \geq 0 \; : \; \varphi(z) \;\leq\; \gamma \log d(x, z) + C\},$$

where z is a coordinate in a coordinate neighborhood of x and d is a distance on it. The *Lelong number* of T along a prime divisor D is

$$\nu(T, D) := \inf\{\nu(T, x) \; : \; x \in D\}.$$

We refer the reader to [11] for a more extensive account on Lelong numbers.

There is a unique decomposition of T as a weakly convergent series

$$T = R + \sum_j \lambda_j [D_j],$$

where:

(i) $[D_j]$ is the current of integration over the prime divisor $D_j \subset X$;
(ii) $\lambda_j := \nu(T, D_j) \geq 0$;
(iii) R is a closed positive current with the property that $\operatorname{codim} E_c(R) \geq 2$ for every $c > 0$.

Recall that

$$E_c(R) := \{x \in X \; : \; \nu(R, x) \geq c\},$$

and that this is an analytic subset of X by a famous result due to Siu [23]. Such a decomposition is called the Siu decomposition of T.

Analytic and minimal singularities

A positive current $T = \theta + dd^c\varphi$ is said to have *analytic singularities* if there exists $c > 0$ such that locally on X,

$$\varphi = \frac{c}{2} \log \sum_{j=1}^{N} |f_j|^2 + u,$$

where u is smooth and $f_1, \dots, f_N$ are local holomorphic functions.

If T and T' are two closed positive currents on X, then T' is said to be *less singular* than T if their local potentials satisfy $\varphi \leq \varphi' + O(1)$.

A positive current T is said to have *minimal singularities* (inside its cohomology class α) if it is less singular than any other positive current in α. Its θ-psh potentials φ will correspondingly be said to have minimal singularities.

Such θ-psh functions with minimal singularities always exist, one can consider for example

$$V_\theta := \sup \{\varphi \ \theta\text{-psh}, \varphi \leq 0 \text{ on } X\}.$$

1.1. Big and modified nef classes

Definition 1.1. If α is a big class, we define its *ample locus* $\mathrm{Amp}(\alpha)$ as the set of points $x \in X$ such that there exists a Kähler current $T \in \alpha$ with analytic singularities and smooth in a neighbourhood of x.

The ample locus $\mathrm{Amp}(\alpha)$ is a Zariski open subset, and it is nonempty thanks to Demailly's regularization result (see [7]).

Observe that a current with minimal singularities $T_{\min} \in \alpha$ has locally bounded potential in $\mathrm{Amp}(\alpha)$.

Definition 1.2. Let α be a big class.

(1) Let $T \in \alpha$ be a positive $(1, 1)$-current, then we set

$$E_+(T) := \{x \in X \ : \ \nu(T, x) > 0\};$$

(2) We define the *non-Kähler locus* of α as

$$E_{nk}(\alpha) := \bigcap_T E_+(T)$$

ranging among all the Kähler currents in α.

By [7, Theorem 3.17(iii)] a class α is Kähler if and only if $E_{nk}(\alpha) = \emptyset$. Moreover by [7, Theorem 3.17(ii)] we have $E_{nk}(\alpha) = X \setminus \mathrm{Amp}(\alpha)$.

Definition 1.3. We say that α is *modified-nef* if and only if for every $\varepsilon > 0$ there exists a closed $(1, 1)$-current $T_\varepsilon \in \alpha$ with $T_\varepsilon \geq -\varepsilon\omega$ and $\nu(T_\varepsilon, D) = 0$ for any prime divisor D.

We recall now an alternative and useful definition of modified nef classes.

Proposition 1.4 ([7, Proposition 3.2]). *Let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a pseudoeffective class. Then α is modified nef if and only if $\nu(\alpha, D) = 0$ for every prime divisor D.*

We refer to [7] for the defintion and properties of the *minimal multiplicity* $\nu(\alpha, D)$. We will be only interested in the case where α is big, and in this case the minimal multiplicity coincides with $\nu(T_{\min}, D)$, the Lelong number along D of a current in α with minimal singularities (*cf.* [7, Proposition 3.6(ii)]).

1.2. The divisorial Zariski decomposition

In this subsection we collect some basic results on the divisorial Zariski decomposition defined in [7]. They can all be found in [7] but we recall some statements frequently used in this note.

Let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a pseudo-effective class. The *divisorial Zariski decomposition* of α is defined as follows:

Definition 1.5. The negative part of α is defined as $N(\alpha) := \sum \nu(\alpha, D)[D]$, where D are prime divisors. The Zariski projection of α is $Z(\alpha) := \alpha - \{N(\alpha)\}$. We call the decomposition $\alpha = Z(\alpha) + \{N(\alpha)\}$ the divisorial Zariski decomposition of α.

Properties. Let $\alpha = Z(\alpha) + \{N(\alpha)\}$ be the divisorial Zariski decomposition of α. Then

(1) The class $Z(\alpha)$ is modified nef [7, Proposition 3.8];
(2) $N(\alpha)$ is a divisor, *i.e.* there is a finite number of prime divisors D such that $\nu(\alpha, D) > 0$ [7, Proposition 3.12];
(3) The set of modified nef classes is a closed convex cone and it is the closure of the convex cone generated by the classes $\mu_\star\tilde{\alpha}$ where $\mu : \tilde{X} \to X$ is a modification and $\tilde{\alpha}$ is a Kähler class on $\tilde{X}$ [7, Proposition 2.3];
(4) The negative part $\{N(\alpha)\}$ is a *rigid* class, *i.e.* it contains only one positive current [7, Proposition 3.13];
(5) Let α be a modified nef and big class, $D_1, \dots, D_k$ be prime divisors and $\lambda_1, \dots, \lambda_k \in \mathbb{R}^+$. Then [7, Proposition 3.18]

$$N\left(\alpha + \sum_i \lambda_i \{D_i\}\right) = \sum_i \lambda_i [D_i]$$

if and only if $D_j \subset E_{nk}(\alpha)$ for any j.

Proposition 1.6 ([7, Proposition 3.6(ii)]). *Let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a big class and let $T_{\min} \in \alpha$ be a current with minimal singularities. Consider the Siu decomposition of $T_{\min}$,*

$$T_{\min} = R + \sum_j a_j [D_j]$$

where $a_j = \nu(T_{\min}, D_j)$. Then $\{R\} = Z(\alpha)$ and $\{\sum_j a_j D_j\} = \{N(\alpha)\}$. In particular, $\nu(\alpha, D) = \nu(T_{\min}, D)$ for any prime divisor D.

1.3. Volume of big classes

Fix $\alpha \in H^{1,1}_{big}(X, \mathbb{R})$. We introduce

Definition 1.7. Let $T_{\min}$ be a current with minimal singularities in α and let Ω a Zariski open set on which the potentials of $T_{\min}$ are locally bounded, then

$$\mathrm{vol}(\alpha) := \int_\Omega T^n_{\min} > 0 \tag{1.1}$$

is called the volume of α.

Note that the Monge-Ampère measure of $T_{\min}$ is well defined in Ω by [1] and that the volume is independent of the choice of $T_{\min}$ and Ω [4, Theorem 1.16].

Let $f : X \to Y$ be a birational modification between compact Kähler manifolds and let $\alpha_Y \in H^{1,1}(Y, \mathbb{R})$ be a big class. The volume is preserved by pull-backs,

$$\mathrm{vol}(f^\star \alpha_Y) = \mathrm{vol}(\alpha_Y)$$

(see [6]). On the other hand, it is not preserved by push-forwards. In general we have

$$\mathrm{vol}(f_\star \alpha_X) \geq \mathrm{vol}(\alpha_X)$$

(see Remark 3.4).

1.4. Full mass currents

Fix X a n-dimensional compact Kähler manifold, $\alpha \in H^{1,1}(X, \mathbb{R})$ be a big class and $\theta \in \alpha$ a smooth representative.

The non-pluripolar product

Given $T_1 := \theta_1 + dd^c\varphi_1, ..., T_p := \theta_p + dd^c\varphi_p$ positive $(1, 1)$-currents, where θ_j are closed smooth $(1, 1)$-forms, following the construction of Bedford-Taylor [2] in the local setting, it has been shown in [4, Proposition 1.6] that the sequence of currents

$$\mathbf{1}_{\bigcap_j \{\varphi_j > V_{\theta_j} - k\}} \big(\theta_1 + dd^c \max(\varphi_1, V_{\theta_1} - k)\big) \wedge ... \wedge \big(\theta_p + dd^c \max(\varphi_p, V_{\theta_p} - k)\big)$$

is non-decreasing in k and converges weakly to the so-called *non-pluripolar product*

$$\big\langle T_1 \wedge ... \wedge T_p \big\rangle.$$

The resulting positive (p, p)-current does not charge pluripolar sets and it is *closed*. In the sequel we will focus on the particular case when $T_1 = \cdots = T_p = T$ and $p = n$. We denote by $\langle T^n \rangle$ the *non-pluripolar measure* of T.

Let us stress that since the non-pluripolar product does not charge pluripolar sets,

$$\mathrm{vol}(\alpha) = \int_X \big\langle T^n_{\min} \big\rangle \tag{1.2}$$

whereas by [4, Proposition 1.20] for any positive (1, 1)-current $T \in \alpha$ we have

$$\mathrm{vol}(\alpha) \geq \int_X \langle T^n \rangle. \tag{1.3}$$

Definition 1.8. A closed positive (1, 1)-current T on X with cohomology class α is said to have *full Monge-Ampère mass* if

$$\int_X \langle T^n \rangle = \mathrm{vol}(\alpha).$$

We denote by $\mathcal{E}(X, \alpha)$ the set of such currents. Let φ be a θ-psh function such that $T = \theta + dd^c\varphi$. The *non-pluripolar Monge-Ampère measure* of φ is

$$\mathrm{MA}(\varphi) := \langle (\theta + dd^c\varphi)^n \rangle = \langle T^n \rangle.$$

We will say that φ has *full Monge-Ampère mass* if $\theta + dd^c\varphi$ has full Monge-Ampère mass. We denote by $\mathcal{E}(X, \theta)$ the set of corresponding functions.

2. Proof of the main theorem

Throughout this section X and Y will be compact Kähler manifolds of complex dimension n.

Theorem 2.1. *Let α be a big class on X. Let $\alpha_1 \in H^{1,1}(X, \mathbb{R})$ be a modified nef class and $\alpha_2 \in H^{1,1}(X, \mathbb{R})$ be a pseudoeffective class. Then $\alpha = \alpha_1 + \alpha_2$ is the divisorial Zariski decomposition of α if and only if* $\mathrm{vol}(\alpha) = \mathrm{vol}(\alpha_1)$.

Remark 2.2. In particular, Theorem 2.1 implies that the pseudoeffective class α_2 will be of the form $\alpha_2 = \sum_{j=1}^N \lambda_j \{D_j\}$ where D_j are prime divisors and $\lambda_j = \nu(\alpha, D_j) \geq 0$.

Proof of Theorem 2.1. If $\alpha = \alpha_1 + \alpha_2$ is the divisorial Zariski decomposition then by [7, Proposition 3.20] we have $\mathrm{vol}(\alpha) = \mathrm{vol}(\alpha_1)$.

Vice versa, assume that we have a decomposition as above with $\mathrm{vol}(\alpha) = \mathrm{vol}(\alpha_1) = V$. Let μ be a smooth volume form on X with total mass V and let $T_1 \in \mathcal{E}(X, \alpha_1)$ be the unique solution of the complex Monge-Ampère equation

$$\langle T_1^n \rangle = \mu.$$

Such T_1 exists and is unique by [4, Theorem 3.1]. Furtheremore, T_1 has minimal singularities in its cohomology class [4, Theorem 4.1]. Let τ be any positive closed (1, 1)-current in α_2 and set $T = T_1 + \tau$. By multilinearity of the non-pluripolar product [4, Proposition 1.4], we have $\langle T^n \rangle \geq \langle T_1^n \rangle$. By (1.2) and (1.3) we have

$$\int_X \langle T^n \rangle \leq \mathrm{vol}(\alpha) = \mathrm{vol}(\alpha_1) = \int_X \langle T_1^n \rangle.$$

Therefore $\langle T^n\rangle = \langle T_1^n\rangle = \mu$. Thus T is a solution of the Monge-Ampère equation $\langle T^n\rangle = \mu$ in the class α and by uniqueness, it follows that α_2 is rigid, *i.e.* there exists a unique positive closed (1, 1)-current in α_2. Moreover, T has minimal singularities. Since $\mathrm{vol}(\alpha) = \int_X \langle T^n\rangle = \mu(X)$, by the multilinearity of the non-pluripolar product we get

$$\sum_{j=0}^{n-1}\binom{n}{j}\big\langle T_1^j \wedge \tau^{n-j}\big\rangle = 0.$$

Let $S \in \alpha_1$ be a Kähler current, *i.e.* $S \geq \varepsilon\omega$ for some $\varepsilon > 0$. Let Ω_1 be a non-empty Zariski open subset where S is smooth and let $\Omega = \mathrm{Amp}(\alpha) \neq \emptyset$. Since T has minimal singularities, then $T \in \alpha$ has locally bounded potential on Ω. In particular, the current τ has locally bounded potential in $\Omega_2 = \Omega \cap \Omega_1 = X \setminus \Sigma$. Then we have

$$0 \leq \varepsilon^{n-1}\int_{\Omega_2}\omega^{n-1}\wedge\tau \leq \int_{\Omega_2} S^{n-1}\wedge\tau \leq \int_{\Omega_2} T_1^{n-1}\wedge\tau = 0,$$

where the last inequality follows from [4, Proposition 1.20]. This implies that the current τ is supported on Σ.

By [11, Corollary 2.14], τ is of the form

$$\tau = \sum_{j=1}^{N}\lambda_j\big[D_j\big],$$

where D_j are irreducibile divisors and $\lambda_j \geq 0$. Moreover, observe that, since α_1 is modified nef and T_1 has minimal singularities, we have $\nu(T_1, D_j) = 0$ for any j by Proposition 1.4 hence $\lambda_j = \nu(T, D_j)$. In other words, $T = T_1 + \tau$ is the Siu decomposition of T. Since α is big and T has minimal singularities, by Proposition 1.6 we have $\nu(\alpha, D) = \nu(T, D)$, hence the conclusion. □

We would like to observe that in the algebraic case, for a projective manifold X, Theorem 2.1 can be proved using the differentiability of the volume [9].

We thank Sébastien Boucksom for the following remark:

Remark 2.3. Let $N^1(X)_{\mathbb{R}} \subset H^{1,1}(X, \mathbb{R})$ denote the real Néron-Severi space and $\alpha \in N^1(X)_{\mathbb{R}}$ be a big class. Assume $\alpha = \alpha_1 + \sum_{i=1}^N \lambda_i\{D_i\}$ with

(i) $\alpha_1 \in N^1(X)_{\mathbb{R}}$ a modified nef class such that $\mathrm{vol}(\alpha) = \mathrm{vol}(\alpha_1)$;
(ii) $\lambda_i \geq 0$;
(iii) D_i are prime divisors for any i.

Then $\alpha = \alpha_1 + \sum_{i=1}^N \lambda_i\{D_i\}$ is the divisorial Zariski decomposition of α. We claim that it is enough to prove that for any prime divisor $D \not\subset E_{nk}(\alpha)$,

$$\mathrm{vol}(\alpha_1 + tD) > \mathrm{vol}(\alpha_1) \quad \forall t > 0. \tag{2.1}$$

Indeed, to prove that $\alpha = \alpha_1 + \sum_{i=1}^{N} \lambda_i \{D_i\}$ is the divisorial Zariski decomposition of α, we have to check that $D_i \subset E_{nk}(\alpha_1)$ by Property 1.2(5). If $\lambda_i > 0$ and $D_i \not\subset E_{nk}(\alpha_1)$ then (2.1) yields

$$\mathrm{vol}(\alpha) \geq \mathrm{vol}(\alpha_1 + \lambda_i D_i) > \mathrm{vol}(\alpha_1) = \mathrm{vol}(\alpha),$$

hence a contradiction.

The inequality (2.1) easily follows from the differentiability of the volume. Indeed, by [9, Theorem A] we have

$$\frac{d}{dt}\Big|_{t=0} \mathrm{vol}(\alpha_1 + tD) = n\langle\alpha_1^{n-1}\rangle \cdot D$$

where $\langle\alpha_1^{n-1}\rangle$ denotes *the positive product* of α defined in [4, Definition 1.17]. Thanks to [9, Remark 4.2 and Theorem 4.9], we have $\langle\alpha_1^{n-1}\rangle \cdot D > 0$, hence $\mathrm{vol}(\alpha_1 + tD)$ is a continuous strictly increasing function for small $t > 0$, and so $\mathrm{vol}(\alpha_1 + tD) > \mathrm{vol}(\alpha_1)$.

Using the results in [21] by Witt Nyström and Boucksom, the above proof works when X is projective and $\alpha \in H^{1,1}(X, \mathbb{R})$.

3. Currents with full Monge-Ampère mass

In this section we state a few consequences of Theorem 2.1. The first result states that currents with full Monge-Ampère mass in α compute the coefficients of the divisorial Zariski decomposition of α.

Theorem 3.1. *Let α be a big class on X. If $T \in \mathcal{E}(X, \alpha)$ and $T_{\min} \in \alpha$ is a current with mininal singularities, then the set*

$$\big\{x \in X \,:\, \nu(T, x) > \nu(T_{\min}, x)\big\}$$

is contained in a countable union of analytic subsets of codimension ≥ 2 contained in $E_{nK}(\alpha)$. In particular, $\nu(T, D) = \nu(T_{\min}, D)$ for any irreducible divisor $D \subset X$.

Proof. If $T \in \mathcal{E}(X, \alpha)$ then $E_+(T) \subset E_{nk}(\alpha)$ because of [12, Proposition 1.9]. On the other hand if we write the Siu decomposition of T as

$$T = T_1 + \sum_{j \geq 1} \lambda_j \big[D_j\big],$$

where D_j are prime divisors and $\operatorname{codim} E_c(T_1) \geq 2$ for all $c > 0$, we have $D_j \subset X \setminus \mathrm{Amp}(\alpha)$. Hence there is a finite number of D_j such that $\lambda_j \neq 0$. In particular, $\nu(T_1, D_j) = 0$ for any j.

Set $\alpha_1 := \{T_1\}$ and note that, since α is big, α_1 is big. Moreover, α_1 is modified nef. Indeed, pick $T_{\min,1} \in \alpha_1$ a current with minimal singularities. Since $0 \leq \nu(T_{\min,1}, D_j) \leq \nu(T_1, D_j) = 0$, we have $\nu(T_{\min,1}, D) = 0$ for any D prime divisor. The claim then follows from Propositions 1.4 and 1.6.

Furthermore, the current $S = T_{\min,1} + \sum_{j=1}^{N} \lambda_j [D_j]$ is less singular than T, hence it has full Monge-Ampère mass [4, Corollary 2.3]. Therefore

$$\mathrm{vol}(\alpha) = \int_X \langle T^n \rangle = \int_X \langle S^n \rangle = \int_X \langle T^n_{\min,1} \rangle = \mathrm{vol}(\alpha_1).$$

We are now under the assumptions of Theorem 2.1, thus $\alpha = \alpha_1 + \sum_{j \geq 1} \lambda_j [D_j]$ is the divisorial Zariski decompostion of α and

$$\nu\big(T, D_j\big) = \lambda_j = \nu\big(\alpha, D_j\big) = \nu\big(T_{\min}, D_j\big),$$

where the last identity is Proposition 1.6.

Moreover,

$$B := \big\{x \in X \ : \ \nu(T, x) > \nu(T_{\min}, x)\big\} \subset \bigcup_{c \in \mathbb{Q}^+} E_c(T_1) \cup \bigcup_{j=1}^{N} \Sigma_j,$$

where $\Sigma_j := \{x \in D_j \ : \ \nu(T, x) > \lambda_j\}$. Indeed, if $x \in B$ is such that $x \in X \setminus \bigcup_{j=1}^{N} D_j$ then $\nu(T, x) = \nu(T_1, x) > \nu(T_{\min,1}, x) \geq 0$. If $x \in D_j$ for some j and $x \in B$ then $\nu(T, x) > \nu(T_{\min}, D_j) = \lambda_j$, that is $x \in \Sigma_j$. Finally, observe that by [23] both $E_c(T_1)$ and Σ_j are analytic subsets of codimension ≥ 2 for any $c > 0$ and j, respectively. □

Remark 3.2. In [19, Theorem 1.1 and Lemma 5.4] Lesieutre constructs an example of a big class α on a 4-dimensional manifold X whose non-nef locus $E_{nn}(\alpha)$ is an infinite countable union of irreducible curves and it is Zariski dense in a divisor $E \subset X$. Hence α is modified nef and Theorem 3.1 implies that if $T \in \mathcal{E}(X, \alpha)$ then the set $E_+(T) := \{x \in X \ : \ \nu(T, x) > 0\}$ contains $E_{nn}(\alpha)$ but it does not contain E. Therefore $E_+(T)$ is not a closed analytic subset. In particular, there does not exist a positive current with analytic singularities $T_+ \in \alpha$ that has full Monge-Ampère mass.

In [12], the first named author proved that finite energy classes (and in particular the energy class $\mathcal{E}$ defined in section 3) are in general not preserved by bimeromorphic maps (see [12, Example 1.7 and Proposition 2.3]). In order to circumvent this problem she introduced a natural condition.

Definition 3.3. Let $f : X \dashrightarrow Y$ be a bimeromorphic map and α be a big class on X. Let $\mathcal{T}_\alpha(X)$ denote the set of positive closed $(1,1)$-currents in α. We say that *Condition* (V) is satisfied if

$$f_\star\Big(\mathcal{T}_\alpha(X)\Big) = \mathcal{T}_{f_\star\alpha}(Y),$$

where $\mathcal{T}_{f_\star\alpha}(Y)$ is the set of positive currents in the image class $f_\star\alpha$.

Remark 3.4. Note that in general we have $f_\star\big(\mathcal{T}_\alpha(X)\big) \subseteq \mathcal{T}_{f_\star\alpha}(Y)$. This means in particular that the push-forward of a current with minimal singularities in α_X has not necessarily minimal singularities in $f_\star\alpha_X$, hence $\mathrm{vol}(f_\star\alpha_X) \geq \mathrm{vol}(\alpha_X)$.

The first named author showed in [12, Proposition 2.3] that Condition (V) implies that $f_\star\mathcal{E}(X, \alpha) = \mathcal{E}(Y, f_\star\alpha)$.

In the following we prove that Condition (V) is equivalent to the preservation of volumes.

Lemma 3.5. *Let $f : X \to Y$ be a birational morphism and let α be a big class on X. Let E_i, F_i be distinct prime divisors contained in the exceptional locus* $\mathrm{Exc}(f)$ *of f, then there exist $a_i, b_i \in \mathbb{R}^+$ such that*

$$\alpha = f^\star f_\star \alpha - \left[\sum_i a_i \{E_i\} - \sum_i b_i \{F_i\} \right]. \tag{3.1}$$

Moreover, Condition ($\vee$) *is equivalent to:*

(i) $a_i \leq \nu(f^\star f_\star \alpha, E_i)$ *for any i;*
(ii) $-b_i \leq \nu(f^\star f_\star \alpha, F_i)$ *for any i.*

The statements in Lemma 3.5 are quite standard but we include a proof for the reader's convenience.

Proof. The identity (3.1) follows from the fact that for any $T \in \alpha$ positive $(1,1)$-current, $T - f^\star f_\star T$ is supported on $\mathrm{Exc}(f)$ since f is a biholomorphism on $X \setminus \mathrm{Exc}(f)$. Therefore we conclude by [11, Corollary 2.14].

Assume Condition ($\vee$) holds, that is, that any positive $(1,1)$-current $S \in f_\star \alpha$ can be written as $S = f_\star T$ for some positive $(1,1)$-current $T \in \alpha$. Since the cohomology classes of the excetional divisors of f are linearly independent, by (3.1) we have an identity of currents,

$$T + \sum_i a_i [E_i] = f^\star f_\star T + \sum_i b_i [F_i].$$

Thus, for any i we have $\nu(f^\star f_\star T, E_i) - a_i \geq 0$ and $\nu(f^\star f_\star T, F_i) + b_i \geq 0$. Hence (i) and (ii) since Condition ($\vee$) holds in particular for currents with minimal singularities in $f_\star \alpha$.

Conversely, let $S \in f_\star \alpha$ be a positive $(1,1)$-current. By the Siu decomposition the current

$$f^\star S - \sum_i \nu(f^\star S, E_i)[E_i] - \sum_i \nu(f^\star S, F_i)[F_i]$$

is positive. For any i, set $\lambda_i := \nu(f^\star S, E_i) - a_i$ and $\mu_i := \nu(f^\star S, F_i) + b_i$ and observe $\lambda_i, \mu_i \geq 0$ by (i) and (ii). Then

$$T := f^\star S - \sum_i \nu(f^\star S, E_i)[E_i] - \sum_i \nu(f^\star S, F_i)[F_i] + \sum_i \lambda_i [E_i] + \sum_i \mu_i [F_i]$$

is a positive $(1,1)$-current in α and by construction we have $f_\star T = S$. □

Theorem 3.6. *Let $f : X \dashrightarrow Y$ be a bimeromorphic map and let α be a big class on X. Then Condition* ($\vee$) *holds if and only if* $\mathrm{vol}(\alpha) = \mathrm{vol}(f_\star \alpha)$.

Proof. Condition ($\vee$) insures that there exists a positive current $T \in \alpha$ such that $f_\star T$ is a current with minimal singularities in $f_\star \alpha$. Then

$$\mathrm{vol}(\alpha) \geq \int_X \langle T^n \rangle = \int_Y \langle (f_\star T)^n \rangle = \mathrm{vol}(f_\star \alpha).$$

By Remark 3.4 we get $\mathrm{vol}(\alpha) = \mathrm{vol}(f_\star \alpha)$.

Let us now prove the converse implication. First, observe that, applying a resolution of singularities, a bimeromorphic map $f : X \dashrightarrow Y$ can be decomposed as $f = h^{-1} \circ g$,

$$\begin{array}{ccccc} & & Z & & \\ & \overset{h}{\swarrow} & & \overset{g}{\searrow} & \\ X & & & & Y, \end{array}$$

where h, g are two birational morphisms and Z denotes a resolution of singularities for the graph of f. By the proof of [4, Proposition 1.12], for every birational morphism h we have $h^\star\big(\mathcal{T}_\alpha(X)\big) = \mathcal{T}_{h^\star\alpha}(Z)$, hence it suffices to prove the claim when f is a birational morphism.

Let E_i, F_i and a_i, b_i as in (3.1). By Lemma 3.5, Condition (V) is equivalent to:

(i) $a_i \le \nu(f^\star f_\star\alpha, E_i)$ for any i;
(ii) $-b_i \le \nu(f^\star f_\star\alpha, F_i)$ for any i.

Condition (ii) is satisfied since $\nu(f^\star f_\star\alpha, F_i) \ge 0$. Thus we are left to prove (i).

Consider $\beta := f^\star f_\star\alpha + \sum_i b_i\{F_i\}$. We notice that $f_\star\beta = f_\star\alpha$. Moreover, by Lemma 3.5, β satisfies Condition (V). Indeed, for any i we have $-b_i \le \nu(f^\star f_\star\beta, F_i) = \nu(f^\star f_\star\alpha, F_i)$. By the first implication of this theorem, we get $\mathrm{vol}(\beta) = \mathrm{vol}(f_\star\beta) = \mathrm{vol}(f_\star\alpha)$.

Let $T_{\min} \in \alpha$ and $S_{\min} \in f_\star\alpha$ be currents with minimal singularities. Then $T_{\min} + \sum_i a_i[E_i]$ and $f^\star S_{\min} + \sum_i b_i[F_i]$ are both positive $(1,1)$-currents in β with full Monge-Ampère mass. Indeed,

$$\int_X \left\langle \left(T_{\min} + \sum_i a_i[E_i] \right)^n \right\rangle = \int_X \langle T_{\min}^n \rangle = \mathrm{vol}(\alpha)$$

$$\int_X \left\langle \left(f^\star S_{\min} + \sum_i b_i[F_i] \right)^n \right\rangle = \int_Y \langle S_{\min}^n \rangle = \mathrm{vol}(f_\star\alpha),$$

and $\mathrm{vol}(\alpha) = \mathrm{vol}(f_\star\alpha) = \mathrm{vol}(\beta)$. By Theorem 3.1

$$a_j \le \nu\left(T_{\min} + \sum_i a_i[E_i], E_j\right) = \nu\left(f^\star S_{\min} + \sum_i b_i[F_i], E_j\right) = \nu\big(f^\star S_{\min}, E_j\big)$$

for any prime divisor E_j, since the prime divisors F_i and E_j are distinct. By Proposition 1.6, $a_j \le \nu(f^\star S_{\min}, E_j) = \nu(f^\star f_\star\alpha, E_j)$, hence the conclusion. □

Theorem 3.7. *Let α be a big class and D be an irreducible divisor such that $D \cap \mathrm{Amp}(\alpha) \neq \emptyset$. Then*

$$\mathrm{vol}(\alpha + tD) > \mathrm{vol}(\alpha) \quad \forall t > 0.$$

Vice versa, if $D \cap \mathrm{Amp}(\alpha) = \emptyset$ then

$$\mathrm{vol}(\alpha + tD) = \mathrm{vol}(\alpha) \quad \forall t > 0.$$

Proof. We first reduce to the case α modified nef and big class. Let $\alpha = Z(\alpha) + \{N(\alpha)\}$ be the divisorial Zariski decomposition of α. By Lemma 3.8 $D \cap \mathrm{Amp}(\alpha) \neq \emptyset$ if and only if $D \cap \mathrm{Amp}(Z(\alpha)) \neq \emptyset$.

If the theorem is true for modified nef and big classes, we have

$$\mathrm{vol}(\alpha + tD) \geq \mathrm{vol}(Z(\alpha) + tD) > \mathrm{vol}(Z(\alpha)) = \mathrm{vol}(\alpha).$$

Thus we can assume that α is a modified nef and big class. Assume by contradiction that there exists t_0 such that $\mathrm{vol}(\alpha + t_0 D) = \mathrm{vol}(\alpha)$. It follows by Theorem 2.1 that $\beta = \alpha + t_0 D$ is the divisorial Zariski decomposition of β and so $D \subset E_{nk}(\alpha)$ Property 1.2(5). Since $E_{nk}(\alpha) = X \setminus \mathrm{Amp}(\alpha)$ [7, Proposition 3.17] we get a contradiction.

Vice versa, if $\alpha = Z(\alpha) + \{N(\alpha)\}$ is the divisorial Zariski decomposition of α and $D \cap \mathrm{Amp}(\alpha) = \emptyset$ (or equivalently $D \subset E_{nk}(Z(\alpha))$ by Lemma 3.8 below and [7, Theorem 3.17]) then by Property 1.2(5) we have that, for any $t > 0$, the divisorial Zariski decomposition of $\alpha + tD$ is

$$\alpha + tD = Z(\alpha) + (N(\alpha) + tD),$$

thus $\mathrm{vol}(\alpha + tD) = \mathrm{vol}(Z(\alpha)) = \mathrm{vol}(\alpha)$. □

Lemma 3.8. *Let $\alpha \in H^{1,1}_{\mathrm{big}}(X, \mathbb{R})$ and let $\alpha = Z(\alpha) + \{N(\alpha)\}$ be its divisorial Zariski decomposition. Then we have*

$$\mathrm{Amp}(\alpha) = \mathrm{Amp}(Z(\alpha)).$$

Proof. We first show the inclusion $\mathrm{Amp}(\alpha) \subset \mathrm{Amp}(Z(\alpha))$. Pick $x \in \mathrm{Amp}(\alpha)$. By definition there exists a Kähler current with analytic singularities $T \in \alpha$ which is smooth in a neighbourhood of x. Moreover $\nu(T_{\min}, x) = 0$ since $0 = \nu(T, x) \geq \nu(T_{\min}, x)$. Let $T = R + \sum_j a_j[D_j]$ be the Siu decomposition of T, then $x \notin \mathrm{supp}\, D_j$ for any j. The current $T - N(\alpha) \in Z(\alpha)$ has clearly analytic singularities, is smooth around x and it is also Kähler since $N(\alpha) \leq \sum_j a_j[D_j]$ by Proposition 1.6. Hence $x \in \mathrm{Amp}(Z(\alpha))$. Conversely, pick $x \in \mathrm{Amp}(Z(\alpha))$, then there exists a Kähler current with analytic singularities $T \in Z(\alpha)$ that is smooth in a neighbourhood of x (see Definition 1.1). It follows from Property 1.2(5) that $x \notin \mathrm{supp}\, N(\alpha)$. This implies that $T + N(\alpha) \in \alpha$ is a Kähler current with analytic singularites that is smooth in a neighbourhood of x. Hence $x \in \mathrm{Amp}(\alpha)$. □

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