Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XVII (2017), 1231-1254

Classification of Kähler homogeneous manifolds of non-compact dimension two

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Abstract. Suppose G is a connected complex Lie group and H is a closed complex subgroup such that $X := G/H$ is Kähler and the codimension of the top non-vanishing homology group of X with coefficients in $\mathbb{Z}_2$ is equal to two. We show that such an X has the structure of a holomorphic fiber bundle whose fiber and base are constructed from certain "basic building blocks", *i.e.*, $\mathbb{C}$, $\mathbb{C}^*$, Cousin groups, and flag manifolds.

Mathematics Subject Classification (2010): 32M10 (primary); 32Q15 (secondary).

1. Introduction

In this paper we consider complex homogeneous manifolds of the form G/H, where G is a connected complex Lie group and H is a closed complex subgroup of G. The existence of complex analytic objects on such a G/H, like non-constant holomorphic functions, plurisubharmonic functions and analytic hypersurfaces, is related to when G/H could be Kähler. So the first question one might consider concerns the existence of Kähler structures and we restrict ourselves to that question here. The structure of compact Kähler homogeneous manifolds is now classical [32] and [13] and the structure in the case of G-invariant metrics is also known [16]. Our investigations here concern non-compact complex homogeneous manifolds having a Kähler metric that is not necessarily G-invariant.

Some results are known under restrictions on the type of group G that is acting. The base of the holomorphic reduction of any complex solvmanifold is always Stein [28], where the proof uses some fundamental ideas in [31]. For G a solvable complex Lie group and G/H Kähler the fiber of the holomorphic reduction of G/H is a Cousin group, see [37] and the holomorphic reduction of a finite covering of G/H is a principal Cousin group bundle, see [20]. If G is semisimple, then G/H

This work was partially supported by an NSERC Discovery Grant.

Received October 6, 2015; accepted in revised form June 12, 2016.
Published online December 2017.

admits a Kähler structure if and only if H is algebraic [11]. For G reductive there is the characterization that G/H is Kähler if and only if $S \cdot H$ is closed in G and $S \cap H$ is an algebraic subgroup of S, a maximal semisimple subgroup of G, see [23, Theorem 5.1]. There is also a result if G is the direct product of its radical and a maximal semisimple subgroup under some additional assumptions on the isotropy subgroup and on the structure of G/H [36].

One way to proceed is to impose some topological restraints on $X := G/H$. In [18] we classified Kähler homogeneous manifolds X having more than one end by showing that X is either the product of a Cousin group of hypersurface type and a flag manifold or X admits a homogeneous fibration as a $\mathbb{C}^*$-bundle over the product of a compact complex torus and a flag manifold. Now in the setting of proper actions of Lie groups Abels introduced the notion of non-compact dimension, see [2] and [3, Section 2]. We do not wish to assume that our Lie group actions are necessarily proper ones, so we take a dual approach and define the non-compact dimension d_X of a connected smooth manifold X to be the codimension of the top non-vanishing homology group of X with coefficients in $\mathbb{Z}_2$, see Section 2. Our goal in this paper is to classify Kähler homogeneous manifolds G/H with $d_{G/H} = 2$. All such spaces are holomorphic fiber bundles where the fibers and the bases of the bundles involved consist of Cousin groups, flag manifolds, $\mathbb{C}$, and $\mathbb{C}^*$. We now present the statement of our main result, where T denotes a compact complex torus, C a Cousin group, and Q a flag manifold. Throughout the rest of this paper, if G is a mixed group, *i.e.*, is neither solvable nor semisimple, then S denotes a maximal semisimple subgroup of G. In particular, if G is simply connected, one has its Levi-Malcev decomposition $G = S \ltimes R$, where R is the radical of G.

Theorem 1.1 (Main theorem). *Suppose $X := G/H$ with $d_X = 2$, where G is a connected complex Lie group and H is a closed complex subgroup of G. Then X is Kähler if and only if X is one of the following:*

Case I. H **discrete:** *A finite covering of X is biholomorphic to a product $C \times A$, with C a Cousin group, A a Stein Abelian Lie group and $d_C + d_A = 2$.*

Case II: H **is not discrete:**

(1) *Suppose $\mathcal{O}(X) = \mathbb{C}$ and let $G/H \to G/N$ be its normalizer fibration;*

  (a) *X is a $(\mathbb{C}^*)^k$-bundle over $C \times Q$ with $d_C + k = d_X = 2$;*

  (b) *X is $T \times G/N$ with $\mathcal{O}(G/N) = \mathbb{C}$ and G/N fibers as a $\mathbb{C}$-bundle over a flag manifold; there are two subcases depending on whether S acts transitively on G/N or not;*

(2) *Suppose $\mathcal{O}(X) \neq \mathbb{C}$ and let $G/H \to G/J$ be its holomorphic reduction;*

  (a) *$d_{G/J} = 2$ and G/J is Stein. Then*

    (i) *$G/J = \mathbb{C}$ or*

    (ii) *G/J is the 2-dimensional affine quadric, and in both of these cases $X = T \times Q \times G/J$ or*

    (iii) *G/J is the complement of a quadric curve in $\mathbb{P}_2$, and X or a two-to-one covering of X is a product $T \times Q \times G/J$ or*

    (iv) *$G/J = (\mathbb{C}^*)^2$ and a finite covering of X is $T \times Q \times G/J$;*

(b) $d_{G/J} = 2$ *and* G/J *is not Stein. Then a finite covering of* X *is biholomorphic to* $T \times Y$ *with* Y *a flag manifold bundle over the holomorphic reduction* G/J*, a* $\mathbb{C}^*$*-bundle over an affine cone minus its vertex;*
(c) $d_{G/J} = 1$ *and* G/J *is Stein. Then a finite covering of* X *is biholomorphic to* $C \times Q \times \mathbb{C}^*$*, where* $d_C = 1$*;*
(d) $d_{G/J} = 1$ *and* G/J *is not Stein. Then a finite covering of* X *is a* $\mathbb{C}^*$*-bundle over* $T \times \widetilde{Y}$*, where* $\widetilde{Y}$ *is the universal covering of* Y *which is a flag manifold bundle over the holomorphic reduction* G/J*, an affine cone minus its vertex. Moreover,* $d_{J/H} = 1$ *and* $\mathcal{O}(J/H) = \mathbb{C}$.

The paper is organized as follows. In section two we gather a number of technical tools. In particular, we note that Proposition 2.11 deals with the setting where the fiber of the normalizer fibration is a Cousin group and its base is a flag manifold. It is essential for Case II (1) (a) in the Main theorem and can be used to simplify the proof when $d_X = 1$ given in [18], see Remark 2.12. Section three is devoted to the case when the isotropy subgroup is discrete. Sections four and five deal with general isotropy and contain the proof of the classification when there are no non-constant holomorphic functions and when there are non-constant holomorphic functions, respectively. In section six we note that the manifolds listed in the classification are indeed Kähler. In the last section we present some examples.

ACKNOWLEDGEMENTS. We thank Prof. A. T. Huckleberry and the referees for their comments that led to significant improvements. We also thank the Arbeitsgruppe Transformationsgruppen at the Ruhr-Universität Bochum for their kind hospitality during a sojourn at which time part of this paper was written.

For H discrete this classification was presented in the first author's dissertation [4].

2. Technical tools

The purpose of this section is to collect a number of definitions and basic tools that are needed in the following.

2.1. Basic notions

Definition 2.1. A *Cousin group* is a complex Lie group G with $\mathcal{O}(G) = \mathbb{C}$. The terminology *toroidal group* is also found in the literature. Every Cousin group is Abelian and is the quotient of $\mathbb{C}^n$ by a discrete subgroup having rank $n+k$ for some k with $1 \leq k \leq n$. For details, we refer the reader to [1].

Definition 2.2. A *flag manifold* (the terminology *homogeneous rational manifold* is also in common usage) is a homogeneous manifold of the form S/P, where S is a connected semisimple complex Lie group and P is a parabolic subgroup of S. One source concerning the structure of flag manifolds is [8, Section 3.1].

Definition 2.3. For X a connected (real) smooth manifold we define

$$d_X := \dim_{\mathbb{R}} X - \min\{r \mid H_k(X, \mathbb{Z}_2) = 0 \quad \forall \quad k > r\},$$

i.e., d_X is the codimension of the top non-vanishing homology group of X with coefficients in $\mathbb{Z}_2$. We call d_X the *non-compact dimension* of X.

Remark 2.4. For later purposes we note that $d_X = 0$ if and only if X is compact.

Proposition 2.5. *Suppose X is a connected Stein manifold. Then* $\dim_{\mathbb{C}} X \leq d_X$.

Proof. For X Stein one has $H_k(X, \mathbb{Z}_2) = 0$ for all $k > \dim_{\mathbb{C}} X$ by [39]. □

2.2. Fibration methods

Throughout the paragraph we make use of a number of fibrations that are now classical.

(1) **Normalizer fibration:** Given G/H let $N = N_G(H^0)$ be the normalizer in G of the connected component of the identity H^0 of H. Since H normalizes H^0, we have $H \subset N$ and the *normalizer fibration* is given by $G/H \to G/N$;
(2) **Holomorphic reduction:** Given G/H we set $J := \{g \in G \mid f(gH) = f(eH)$ for all $f \in \mathcal{O}(G/H)\}$. Then J is a closed complex subgroup of G containing H and we call the fibration $p : G/H \to G/J$ the *holomorphic reduction* of G/H. By construction G/J is holomorphically separable and $\mathcal{O}(G/H) \cong p^*(\mathcal{O}(G/J))$.

Suppose a manifold X admits a locally trivial fiber bundle $X \xrightarrow{F} B$ with F and B connected smooth manifolds. One would then like to know how d_F and d_B are related to d_X whenever possible. The following result was proved in [7, Section 2] using spectral sequences.

Lemma 2.6 (The fibration lemma). *Suppose $X \xrightarrow{F} B$ is a locally trivial fiber bundle with X, F, B smooth manifolds. Then*

(1) *if the bundle is orientable* (e.g., *if* $\pi_1(B) = 0$), *then* $d_X = d_F + d_B$;
(2) *if B has the homotopy type of a q-dimensional CW complex, then* $d_X \geq d_F + (\dim B - q)$;
(3) *if B is homotopy equivalent to a compact manifold, then* $d_X \geq d_F + d_B$.

Remark 2.7. If B is homogeneous, then one knows that B is homotopy equivalent to a compact manifold if:

(1) the isotropy subgroup of B has finitely many connected components [35]; *e.g.*, in an algebraic setting;
(2) if B is a solvmanifold [34]; indeed, every solvmanifold is a vector bundle over a compact solvmanifold [9].

2.3. Special case of a question of Akhiezer

Later we will need a result that is based on [7, Lemma 8]. Since that lemma was stated in a way suitable for its particular application in [7], we reformulate it in a form suitable for the present context.

Lemma 2.8 ([7, Lemma 8]). *Let G be a connected, simply connected complex Lie group with Levi-Malcev decomposition $G = S \ltimes R$ with $\dim_{\mathbb{C}} R = 2$ and Γ a discrete subgroup of G such that $X = G/\Gamma$ is Kähler. Then Γ is contained in a subgroup of G of the form $A \ltimes R$, where A is a proper algebraic subgroup of S.*

This has the following consequence which we use later.

Theorem 2.9. *Suppose G is a connected, simply connected, complex Lie group with Levi-Malcev decomposition $G = S \ltimes R$ with $\dim_{\mathbb{C}} R = 2$. Let Γ be a discrete subgroup of G such that $X = G/\Gamma$ is Kähler, Γ is not contained in a proper parabolic subgroup of G and $\mathcal{O}(G/\Gamma) \simeq \mathbb{C}$. Then $S = \{e\}$,* i.e., *G is solvable.*

Proof. By Lemma 2.8 the subgroup Γ is contained in a proper subgroup of G of the form $A \ltimes R$, where A is a proper algebraic subgroup of S. Since $R \cdot \Gamma$ is closed in G, *e.g.*, see [19], there are fibrations

$$G/\Gamma \longrightarrow G/R \cdot \Gamma \longrightarrow S/A,$$

where $G/R \cdot \Gamma = S/\Lambda$ with $\Lambda := S \cap R \cdot \Gamma$. If A is reductive, then S/A is Stein and we get non-constant holomorphic functions on X as pullbacks using the above fibrations. But this contradicts the assumption that $\mathcal{O}(X) \simeq \mathbb{C}$. If A is not reductive then [29, Theorem 30.1] applies and A is contained in a proper parabolic subgroup of S. But this implies Γ is also contained in a proper parabolic subgroup of G, thus contradicting the assumption that this is not the case. □

2.4. The algebraic setting revisited

Throughout this paragraph we repeatedly use two results of Akhiezer concerning the invariant d_X in the setting where $X = G/H$ and G is a connected linear algebraic group over $\mathbb{C}$ and H is an algebraic subgroup of G. For the convenience of the reader we now state these here.

Theorem 2.10 ($d = 1$ in [5]; $d = 2$ in [6]). *Suppose G is a connected linear algebraic group over $\mathbb{C}$, H is an algebraic subgroup of G and $X := G/H$.*

(1) $d_X = 1 \Longrightarrow$ *H is contained in a parabolic subgroup P of G with $P/H = \mathbb{C}^*$;*
(2) $d_X = 2 \Longrightarrow$ *H is contained in a parabolic subgroup P of G with P/H being:*
   (a) $\mathbb{C}$*;*
   (b) *the affine quadric Q_2;*
   (c) *the complement of a quadric curve in $\mathbb{P}_2$;*
   (d) $(\mathbb{C}^*)^2$.

2.5. Cousin group bundles over flag manifolds

In this section we prove a result concerning the structure of Kähler homogeneous manifolds whose normalizer fibrations are Cousin group bundles over flag manifolds, without assumptions on the invariant d. We show that one can reduce the problem to the case where a complex reductive group is acting transitively and employ some now classical details about the structure of parabolic subgroups, see [8] or [17]. A crucial point occurs in diagram (2.1) below, where the right vertical arrow is a holomorphic fiber bundle and the left vertical one is algebraic as a consequence of [23, Theorem 5.1].

Proposition 2.11. *Suppose $X := G/H$ is a Kähler homogeneous manifold whose normalizer fibration $G/H \to G/N$ has fiber N/H a Cousin group and base $Q := G/N$ a flag manifold. Then there exists a closed complex subgroup I of N containing H such that the fibration $G/H \to G/I$ realizes X as a $(\mathbb{C}^*)^k$-bundle over a product $G/I = Q \times C$, where C is a Cousin group with $d_C = d_X - k$.*

Proof. Our first task is to show that there is a reductive complex Lie group acting holomorphically and transitively on X. Write $N/H = \mathbb{C}^q/\Gamma$ and note that there exists a subgroup $\widehat{\Gamma} < \Gamma$ such that $\widehat{N/H} := \mathbb{C}^q/\widehat{\Gamma}$ is isomorphic to $(\mathbb{C}^*)^q$ and is a covering group of N/H, see [1, Section 1.1]. In particular, the reductive complex Lie group $\widehat{G} := S \times \widehat{N/H}$ acts transitively on X. We drop the hats from now on and assume, by considering a finite covering, if necessary, that $G = S \times Z$ is a reductive complex Lie group, where $Z \cong (\mathbb{C}^*)^q$ is the center of G and S is a maximal semisimple subgroup.

Since X is Kähler, the S-orbit $S \cdot H/H = S/S \cap H$ is closed in X and $S \cap H$ is an algebraic subgroup of S [23, Theorem 5.1]. Consider the induced fibration on the left hand side of the following diagram

$$\begin{array}{ccc} S/S\cap H & \hookrightarrow & G/H = X \\ F\big\downarrow & & \big\downarrow N/H \\ S/P & = & G/N \end{array} \tag{2.1}$$

where $F := P/S \cap H$ is the induced fiber. The bundle $G/H \to G/N$ is defined by a representation $\rho : N \longrightarrow \mathrm{Aut}^0(N/H) \cong N/H$ with the group $\rho(N)$ lying in the connected component of the identity of the automorphism group of N/H since N is connected. Since N/H is Abelian, ρ factors through the canonical projection from N to $N/N' \cong P/P' \times Z$. Note that $P/P' \cong (\mathbb{C}^*)^p$, see [8, Proposition 8, Section 3.1]. Since $P/P' \times Z \cong (\mathbb{C}^*)^{p+q}$ is reductive, the factorized homomorphism is algebraic and the image $\rho(N)$ is a closed subgroup of N/H that is isomorphic to an algebraic torus $(\mathbb{C}^*)^k$ given as the quotient of $P/P' \times Z$ by an algebraic subgroup. Let $\sigma : N \to N/H^0 \to (N/H^0)/(H/H^0) = N/H$ be the composition of the quotient homomorphisms. The subgroup $I := \sigma^{-1} \circ \rho(N)$ is a closed, complex subgroup of N, and therefore of G, that contains H. Thus one has the fibration $G/H \to G/I$ whose typical fiber F is biholomorphic to $(\mathbb{C}^*)^k$. We claim that the bundle $G/I \to G/N$ is holomorphically trivial. This follows from the fact that the N-action on the neutral fiber of the bundle $G/I \to G/N$ is trivial. Otherwise, the dimension of the N-orbit in N/H would be bigger than k, as we assumed above, and this would give a contradiction. Finally, since N/H is a Cousin group, $C := N/I$ is also a Cousin group and the statement about the topological invariant follows because $d_{N/I} = d_{G/I}$, since S/P is compact and simply connected, and $d_{G/I} = d_X - k$. □

Remark 2.12. The case $d_X = 1$ is treated in [18, Proposition 5], where X is assumed to have more than one end. For X Kähler this is equivalent to $d_X = 1$.

3. The discrete case

Throughout this section we assume that $X = G/\Gamma$ is Kähler with $d_X = 2$, where G is a connected, simply connected, complex Lie group and Γ is a discrete subgroup of G. We first show that G is solvable. Then we prove that a finite covering of such an X is biholomorphic to a product $C \times A$, where C is a Cousin group and A is a holomorphically separable complex Abelian Lie group.

3.1. The reduction to solvable groups

We first handle the case when the Kähler homogeneous manifold has no nonconstant holomorphic functions.

Lemma 3.1. *Assume Γ is a discrete subgroup of a connected, simply connected complex Lie group G that is not contained in a proper parabolic subgroup of G, with $X := G/\Gamma$ Kähler, $\mathcal{O}(X) = \mathbb{C}$, and $d_X \le 2$. Then G is solvable.*

Proof. Assume $G = S \ltimes R$ is a Levi decomposition. Since the R-orbits are closed, we have a fibration

$$G/\Gamma \longrightarrow G/R \cdot \Gamma = S/\Lambda,$$

where $\Lambda := S \cap R \cdot \Gamma$ is Zariski dense and discrete in S, see [19]. Now if $\mathcal{O}(R \cdot \Gamma/\Gamma) = \mathbb{C}$, then the result was proved in [19]. Otherwise, let

$$R \cdot \Gamma/\Gamma \longrightarrow R \cdot \Gamma/J =: Y$$

be the holomorphic reduction. Then Y is holomorphically separable and since R acts transitively on Y, it follows that Y is Stein [28]. One has $2 = d_X \ge d_Y \ge \dim_{\mathbb{C}} Y$. Further we claim that J^0 is a normal subgroup of G. In order to see this, note first that $\mathcal{O}(G/N_G(J^0)) = \mathbb{C}$ because one has the fibration $G/\Gamma \to G/N_G(J^0)$. If $N_G(J^0) \ne G$, then it follows from [27, Corollary 6] that $N_G(J^0)$ is contained in a proper parabolic subgroup of G. However, this implies that Γ is also contained in the same proper parabolic subgroup, which contradicts our assumptions. As a consequence, the quotient group $\widehat{R} := R/J^0$ has complex dimension one or two. If $\dim \widehat{R} = 1$, then $\widehat{G} := G/J^0$ is a product $S \times \widehat{R}$ and this implies $S = \{e\}$ by [36]. If $\dim \widehat{R} = 2$, then $\widehat{G}$ is either a product, see [36] again, or it is a non-trivial semidirect product. In the latter case the result follows from Theorem 2.9. □

In the next Proposition we reduce ourselves to the case when the maximal semisimple factor is $SL(2,\mathbb{C})$. We first prove a technical lemma in that setting.

Lemma 3.2. *Suppose G/Γ is Kähler and $d_{G/\Gamma} \leq 2$, where Γ is a discrete subgroup of a connected, complex Lie group of the form $G = SL(2,\mathbb{C}) \ltimes R$ with R the radical of G. Then Γ is not contained in a proper parabolic subgroup of G.*

Proof. Assume the contrary, *i.e.*, that Γ is contained in a proper parabolic subgroup and let P be a maximal such subgroup of G. Note that P is isomorphic to $B \ltimes R$, where B is a Borel subgroup of $SL(2,\mathbb{C})$. Let $P/\Gamma \to P/J$ be the holomorphic reduction. Then P/Γ is a Cousin group [37] and P/J is Stein [28]. Note that $J \neq P$, since otherwise P would be Abelian, giving a contradiction. The Fibration lemma and Proposition 2.5 imply $\dim_{\mathbb{C}} P/J = 1$ or 2. So P/J is biholomorphic to $\mathbb{C}$, $\mathbb{C}^*$, $\mathbb{C}^* \times \mathbb{C}^*$, or the complex Klein bottle.

In the first two cases P/J is equivariantly embeddable in $\mathbb{P}_1$ and by [30] it follows that G/J is Kähler. In the latter two cases the fiber J/Γ is compact by the Fibration lemma and we can push down the Kähler metric on X to obtain a Kähler metric on G/J, see [12]. In particular, the S-orbit $S/(S \cap J)$ in G/J is Kähler and so its isotropy $S \cap J$ is algebraic [11]. Now consider the diagram

$$\begin{array}{ccccc} G/\Gamma & \xrightarrow{F} & G/J & \xrightarrow{Y} & G/P \\ & & & & \| \\ \cup & & \cup & & \mathbb{P}_1 \\ & & & & \| \\ S/S\cap\Gamma & \xrightarrow{F_S} & S/S\cap J & \xrightarrow{Z} & S/B. \end{array}$$

Note that since $Y := P/J$ is noncompact and $d_{G/\Gamma} = 2$, it follows from the Fibration lemma that either $d_F = 1$ or F is compact. Since F is an Abelian Lie group, it is clear that $d_{F_S} \leq d_F$.

We list below, up to isomorphism, the algebraic subgroups of B and in each case we derive a contradiction.

(1) $\dim_{\mathbb{C}} S \cap J = 2$. Then $S \cap J = B$. This yields the contradiction $d_{S/S\cap\Gamma} \leq d_{F_S} + d_{S/B} = 1 + 0 = 1 < 3 = d_{S/S\cap\Gamma}$, since $S \cap \Gamma$ is finite;

(2) $\dim_{\mathbb{C}} S \cap J = 1$. There are two possible cases.

   (a) If $S \cap J = \mathbb{C}^*$, then $S/S \cap J$ is an affine quadric or the complement of a quadric curve in $\mathbb{P}_2$ and thus $Z = \mathbb{C}$. So $P/J \neq \mathbb{C}^*$ and it is either $\mathbb{C}$ or $(\mathbb{C}^*)^2$, *i.e.*, $d_{P/J} = 2$. Then the Fibration lemma implies that F is compact and, since the fiber F_S is closed in F, it must also be compact. But this forces $S \cap \Gamma$ to be an infinite subgroup of $S \cap J$ which is a contradiction;

(b) If $S \cap J = \mathbb{C}$, then $S/S \cap J$ is a finite quotient of $\mathbb{C}^2 \setminus \{(0,0)\}$ and so $Z = \mathbb{C}^*$. Now $P/J = \mathbb{C}$, $(\mathbb{C}^*)^2$ or $\mathbb{C}^*$. In the first two instances F would be compact and we get the same contradiction as in (a). In the last case $d_F = 1$ by the Fibration lemma and F_S is either compact or $\mathbb{C}^*$. Again $S \cap \Gamma$ is infinite with the same contradiction as in (a);

(3) $\dim_{\mathbb{C}} S \cap J = 0$. Here $S \cap J$ is finite, since it is an algebraic subgroup of B. Then $\dim S/S \cap J = 3$ and we see that $\dim G/J = 3$, since we know $\dim G/P = 1$ and $\dim P/J \leq 2$. Then $P/J = (\mathbb{C}^*)^2$ and, since the fiber $S/S \cap J$ is both open and closed in G/J, it follows that $S/S \cap J = G/J$ and $d_{S/S \cap J} = 2$. But F is compact and thus so is F_S and we get the contradiction that $d_{S/S\cap\Gamma} = 2 < 3 = d_{S/S\cap\Gamma}$.

As a consequence, Γ is not contained in a proper parabolic subgroup of G. □

Proposition 3.3. *Suppose G/Γ is Kähler with $d_{G/\Gamma} \leq 2$. Then G is solvable.*

Proof. First note that G cannot be semisimple. If that were so, then Γ would be algebraic, hence finite and thus G/Γ would be Stein. But then $2 = d_{G/\Gamma} \geq \dim_{\mathbb{C}} G/\Gamma = \dim_{\mathbb{C}} G$ which is a contradiction, since necessarily $\dim_{\mathbb{C}} G \geq 3$ for any complex semisimple Lie group G.

So assume $G = S \ltimes R$ is mixed. The proof is by induction on the dimension of G. Now if a proper parabolic subgroup of G contains Γ, then a maximal one does too, it is solvable by induction and thus it has the special form $B \ltimes R$, where B is isomorphic to a Borel subgroup of $S = SL(2,\mathbb{C})$. But this is impossible because of Lemma 3.2.

Lemma 3.1 handles the case $\mathcal{O}(G/\Gamma) = \mathbb{C}$. So we assume $\mathcal{O}(G/\Gamma) \neq \mathbb{C}$ with holomorphic reduction $G/\Gamma \to G/J$. The Main Result in [7] gives the following possibilities for the base G/J:

(1) $\mathbb{C}$;
(2) affine quadric Q_2;
(3) $\mathbb{P}_2 \setminus Q$, where Q is quadric curve;
(4) an affine cone minus its vertex;
(5) $\mathbb{C}^*$-bundle over an affine cone minus its vertex.

In case (1) the bundle is holomorphically trivial, its compact fiber being a torus, and the group that is acting effectively is solvable. In cases (2) and (3) we have fibrations $G/\Gamma \to G/J \to G/P = \mathbb{P}_1$ and so Γ is contained in a proper parabolic subgroup of G, contradicting what was shown in the previous paragraph.

In order to handle cases (4) and (5) we recall that an affine cone minus its vertex fibers equivariantly as a $\mathbb{C}^*$-bundle over a flag manifold. Thus we get fibrations

$$G/\Gamma \longrightarrow G/J \longrightarrow G/P.$$

Note that it cannot be the case that $G \neq P$, since then Γ would be contained in a proper parabolic subgroup, a possibility that has been ruled out. So $G = P$ and there is no flag manifold involved in this setting. Thus G/J (or a 2-1 covering) is biholomorphic to $\mathbb{C}^*$ or $(\mathbb{C}^*)^2$. In the second case the fiber J/Γ is compact and thus a torus, so G is solvable. If $G/J = \mathbb{C}^*$, then J/Γ is Kähler with $\dim J < \dim G$ and $d_{J/\Gamma} = 1$ by the Fibration lemma. By induction J is solvable and so G is solvable too, because $G/J = \mathbb{C}^*$. □

3.2. A product decomposition

In order to prove the classification we need the following splitting result.

Proposition 3.4. *Suppose G is a connected, simply connected solvable complex Lie group that contains a discrete subgroup Γ such that G/Γ is Kähler and has holomorphic reduction $G/\Gamma \to G/J$ with base $(\mathbb{C}^*)^2$ and fiber a torus. Then a finite covering of G/Γ is biholomorphic to a product.*

Proof. First assume that J^0 is normal in G and let $\alpha : G \to G/J^0$ be the quotient homomorphism with differential $d\alpha : \mathfrak{g} \to \mathfrak{g}/\mathfrak{j}$. Then G/J^0 is a two dimensional complex Lie group that is either Abelian or solvable. In the Abelian case $G_0 := \alpha^{-1}(S^1 \times S^1)$ is a subgroup of G that has compact orbits in X, since these orbits fiber as torus bundles over $S^1 \times S^1$ in the base. The result now follows from [22, Theorem 6.14].

Next assume that J^0 is normal and G/J^0 is isomorphic to the two dimensional Borel group B with Lie algebra $\mathfrak{b}$. Let $\mathfrak{n}_\mathfrak{b}$ denote the nilradical of $\mathfrak{b}$. Then $\mathfrak{n} := d\alpha^{-1}(\mathfrak{n}_\mathfrak{b})$ is the nilradical of G. Let N denote the corresponding connected Lie subgroup of G. Now choose $\gamma_N \in \Gamma_N := N \cap \Gamma$ such that $\alpha(\gamma_N) \neq eJ^0$. There exists $x \in \mathfrak{n}$ such that $\exp(x) = \gamma_N$. Let U be the connected Lie group corresponding to $\langle \gamma_N \rangle_\mathbb{C}$. Since Γ centralizes J^0 (see [20, Theorem 1]), it follows that $\mathfrak{n} = \mathfrak{u} \oplus \mathfrak{j}$ and $N = U \times J^0$ is Abelian. Set $\Gamma_U := \Gamma \cap U$ and $\Gamma_J := \Gamma \cap J^0$. Then $N/\Gamma_N = U/\Gamma_U \times J^0/\Gamma_J$.

Since $\Gamma/\Gamma_N = \mathbb{Z}$, we may choose $\gamma \in \Gamma$ such that γ projects to a generator of Γ/Γ_N. Also set $A := \exp(\langle w \rangle_\mathbb{C})$ for fixed $w \in \mathfrak{g} \setminus \mathfrak{n}$. Since A is complementary to N, we have $G = A \ltimes N$. Now there exist $\gamma_A \in A$ and $\gamma_N \in N$ such that $\gamma = \gamma_A \cdot \gamma_N$. Both γ and γ_N centralize J^0 and thus γ_A does too. Also $\gamma_A = \exp(h)$ for some $h = sw$ with $s \in \mathbb{C}$. Therefore,

$$[h, \mathfrak{j}] = 0. \tag{3.1}$$

Since $\mathfrak{a} + \mathfrak{u}$ is isomorphic to $\mathfrak{b} = \mathfrak{g}/\mathfrak{j}$ as a vector space, there exists $e \in \mathfrak{u}$ such that

$$[d\alpha(h), d\alpha(e)] = 2d\alpha(e).$$

Let $\{e_1, \ldots, e_{n-2}\}$ be a basis for $\mathfrak{j}$. There exist structure constants a_i such that

$$[h, e] = 2e + \sum_{i=1}^{n-2} a_i e_i \, ,$$

and the remaining structure constants are all 0 by (3.1). Note that, conversely, any choice of the structure constants a_i determines a solvable Lie algebra $\mathfrak{g}$ and the corresponding connected simply-connected complex Lie group $G = A \ltimes N$.

We now compute the action of $\gamma_A \in A$ on N by conjugation. In order to do this note that the restriction $\mathrm{ad}_h : \mathfrak{n} \to \mathfrak{n}$ of ad_h to $\mathfrak{n}$ is expressed by the matrix

$$M \;:=\; [\mathrm{ad}_h] = \begin{pmatrix} 2 & 0 \ldots 0 \\ a_1 & 0 \ldots 0 \\ \vdots & \vdots \ddots \vdots \\ a_{n-2} & 0 \ldots 0 \end{pmatrix}.$$

So the action of A on N is through the one parameter group of linear transformations $t \mapsto e^{tM}$ for $t \in \mathbb{C}$. For $k \geq 1$

$$(tM)^k = \frac{1}{2}(2t)^k M,$$

and it follows that

$$e^{tM} = \frac{1}{2}\left(e^{2t} - 1\right) M + \mathrm{Id}.$$

Since $\gamma_A \in \Gamma$ and Γ is a subgroup of J, the element $\alpha(\gamma_A)$, where $\alpha : G \to G/J^0$ is the quotient homomorphism defined above, acts trivially on the base $Y = G/J$. So $t = \pi i k$ where $k \in \mathbb{Z}$. Hence γ_A acts trivially on U. Also γ_N acts trivially on N, since N is Abelian. Thus γ acts trivially on N and as a consequence, although G is a non-Abelian solvable group the manifold $X = G/\Gamma$ is just the quotient of $\mathbb{C}^n$ by a discrete additive subgroup of rank $2n - 2$. Its holomorphic reduction is the original torus bundle which, since we are now dealing with the Abelian case, is trivial.

Now assume J^0 is not normal in G, set $N := N_G(J^0)$, and let $G/J \xrightarrow{N/J} G/N$ be the normalizer fibration. Since the base G/N of the normalizer fibration is an orbit in some projective space, G/N is holomorphically separable and thus Stein [28]. Since we also have $2 \geq d_{G/N} \geq \dim_{\mathbb{C}} G/N$ we see that $G/N \cong \mathbb{C}, \mathbb{C}^*$ or $\mathbb{C}^* \times \mathbb{C}^*$. We claim that we must have $G/N = \mathbb{C}^*$, *i.e.*, we have to eliminate the other two possiblities. Assume $G/N \cong \mathbb{C}$. Since $d_X \leq 2$ the Fibration lemma implies $d_{N/J} = 0$, *i.e.*, N/J is biholomorphic to a torus T. Thus $G/J = T \times \mathbb{C}$. However, $G/J = \mathbb{C}^* \times \mathbb{C}^*$ giving a contradiction. Now assume $G/N \cong \mathbb{C}^* \times \mathbb{C}^*$. By Chevalley's theorem [15] the commutator group G' acts algebraically. Hence the G'-orbits are closed and one gets the commutator fibration $G/N \xrightarrow{\mathbb{C}} G/G' \cdot N$. Since G is solvable, it follows that G' is unipotent and the G'-orbits are cells, *i.e.*,

$G' \cdot N/N \cong \mathbb{C}$. By the Fibration lemma the base of the commutator fibration is a torus. But it is proved in [26] that the base of a commutator fibration is always Stein which is a contradiction. This proves the claim that $G/N \cong \mathbb{C}^*$ and by the Fibration lemma $d_{N/J} = 1$ and hence $N/J = \mathbb{C}^*$.

Since G is simply connected, G admits a Hochschild-Mostow hull [25], *i.e.*, there exists a solvable linear algebraic group

$$G_a = (\mathbb{C}^*)^k \ltimes G$$

that contains G as a Zariski dense, topologically closed, normal complex subgroup. By passing to a subgroup of finite index we may, without loss of generality, assume the Zariski closure $G_a(\Gamma)$ of Γ in G_a is connected. Then $G_a(\Gamma) \supseteq J^0$ and $G_a(\Gamma)$ is nilpotent [20]. Let $\pi : \widehat{G_a}(\Gamma) \to G_a(\Gamma)$ be the universal covering and set $\widehat{\Gamma} := \pi^{-1}(\Gamma)$. Since $\widehat{G_a}(\Gamma)$ is a simply connected, nilpotent, complex Lie group, the exponential map from the Lie algebra $\mathfrak{g}_a(\Gamma)$ to $\widehat{G_a}(\Gamma)$ is bijective. For any subset of $\widehat{G_a}(\Gamma)$ and, in particular for the subgroup $\widehat{\Gamma}$, the smallest closed, connected, complex (respectively real) subgroup $\widehat{G_1}$ (respectively $\widehat{G_0}$) of $\widehat{G_a}(\Gamma)$ containing $\widehat{\Gamma}$ is well-defined. By construction $\widehat{G_0}/\widehat{\Gamma}$ is compact, see [38, Theorem 2.1]. Set $G_1 := \pi(\widehat{G_1})$ and $G_0 := \pi(\widehat{G_0})$ and consider the CRS manifold (G_1, G_0, Γ). Note that the homogeneous CR-manifold G_0/Γ fibers as a T-bundle over $S^1 \times S^1$. In order to understand the complex structure on the base $S^1 \times S^1$ of this fibration consider the diagram

$$\begin{array}{ccccc}
\widehat{G}_0/\widehat{\Gamma} & \subset & \widehat{G}_1/\widehat{\Gamma} & \subseteq & \widehat{G}_a(\Gamma)/\widehat{\Gamma} \\
\Vert & & \Vert & & \Vert \\
G_0/\Gamma & \subset & G_1/\Gamma & \subseteq & G_a(\Gamma)/\Gamma \\
T\big\downarrow & & T\big\downarrow & & \big\downarrow T \\
S^1 \times S^1 = G_0/G_0 \cap (J^0 \cdot \Gamma) & \subset & G_1/J^0 \cdot \Gamma & \subseteq & G_a/J^0 \cdot \Gamma\,.
\end{array}$$

As observed in [20, Theorem 1], the manifold $G_a/J^0 \cdot \Gamma$ is a holomorphically separable solvmanifold and thus is Stein [28]. So $G_1/J^0 \cdot \Gamma$ is also Stein and thus $G_0/G_0 \cap (J^0 \cdot \Gamma)$ is totally real in $G_1/J^0 \cdot \Gamma$. The corresponding complex orbit $G_1/J^0 \cdot \Gamma$ is then biholomorphic to $\mathbb{C}^* \times \mathbb{C}^*$. It now follows by [20, Theorem 6.14] that a finite covering of G_1/Γ splits as a product of a torus with $\mathbb{C}^* \times \mathbb{C}^*$ and, in particular, that a subgroup of finite index in Γ is Abelian.

Now set $A := \{ \exp t\xi \mid t \in \mathbb{C} \}$, where $\xi \in \mathfrak{g} \setminus \mathfrak{n}$ and $\mathfrak{n}$ is the Lie algebra of N^0. Then $G = A \ltimes N^0$ and any $\gamma \in \Gamma$ can be written as $\gamma = \gamma_A.\gamma_N$ with $\gamma_A \in A$ and $\gamma_N \in N$. The fiber $G/\Gamma \to G/N$ is the N^0-orbit of the neutral point and Γ acts on it by conjugation. Since N/Γ is Kähler and has two ends, it follows by [18, Proposition 1] that a finite covering of N/Γ is biholomorphic to a product of the torus and $\mathbb{C}^*$. (By abuse of language we still call the subgroup having finite index Γ.) Now the bundle $G/\Gamma \to G/N$ is associated to the bundle

$$\mathbb{C} = G/N^0 \longrightarrow G/N = \mathbb{C}^*$$

and thus $G/\Gamma = \mathbb{C} \times_\rho (T \times \mathbb{C}^*)$, where $\rho : N/N^0 \to \mathrm{Aut}(T \times \mathbb{C}^*)$ is the adjoint representation. Since Γ is Abelian, this implies that γ_A acts trivially on $\Gamma_N := \Gamma \cap N^0$. Now suppose J has complex dimension k. Then γ_A is acting as a linear map on $N^0 = \mathbb{C} \ltimes J^0 = \mathbb{C}^{k+1}$ and commutes with the additive subgroup Γ_N that has rank $2k+1$ and spans N^0 as a linear space. This implies γ_A that acts trivially on N^0 and, as a consequence, the triviality of a finite covering of the bundle, as required. □

3.3. The classification for discrete isotropy

In the following we classify Kähler G/Γ when Γ is discrete and $d_X \leq 2$. Note that $d_X = 0$ means X is compact and this is the now classical result of Borel-Remmert [13]; the case $d_X = 1$ corresponds to X having more than one end and was handled in [18].

Theorem 3.5 ([4]). *Let G be a connected simply connected complex Lie group and Γ a discrete subgroup of G such that $X := G/\Gamma$ is Kähler and $d_X \leq 2$. Then G is solvable and a finite covering of X is biholomorphic to a product $C \times A$, where C is a Cousin group and A is $\{e\}, \mathbb{C}^*, \mathbb{C}$, or $(\mathbb{C}^*)^2$. Moreover, $d_X = d_C + d_A$.*

Proof. By Proposition 3.3 G is solvable. If $\mathcal{O}(X) \cong \mathbb{C}$, then X is a Cousin group [37]. Otherwise, $\mathcal{O}(X) \neq \mathbb{C}$ and let

$$G/\Gamma \xrightarrow{J/\Gamma} G/J$$

be the holomorphic reduction. Its base G/J is Stein [28], its fiber J/Γ is biholomorphic to a Cousin group [37], and a finite covering of the bundle is principal [20]. Since G/J is Stein, by Proposition 2.5 one has

$$\dim_{\mathbb{C}} G/J \leq d_{G/J} \leq d_X \leq 2.$$

If $d_X = 1$, then $d_{G/J} = 1$ and G/J is biholomorphic to $\mathbb{C}^*$. A finite covering of this bundle is principal, with the connected Cousin group as structure group, and so is holomorphically trivial [24]. If $d_X = 2$, the Fibration lemma implies $G/J \cong \mathbb{C}$, $\mathbb{C}^*$, $\mathbb{C}^* \times \mathbb{C}^*$ or a complex Klein bottle [7]. The case of $\mathbb{C}^*$ is handled as above and a torus bundle over $\mathbb{C}$ is trivial by Grauert's Oka Principle [24]. Finally, since a Klein bottle is a 2-1 cover of $\mathbb{C}^* \times \mathbb{C}^*$, it suffices to consider the case $\mathbb{C}^* \times \mathbb{C}^*$. That case is handled by Proposition 3.4. □

Remark 3.6. This theorem proves the classification in Case I in the Main theorem. *i.e.*, if the isotropy is discrete. One should note that any complex manifold X that has a finite covering biholomorphic to $C \times A$, where C is a Cousin group and A a Stein Abelian Lie group, is Kähler.

4. The classification when $\mathcal{O}(X) = \mathbb{C}$

Proof. Let $\pi : G/H \to G/N$ be the normalizer fibration and recall that its base G/N is equivariantly embedded in some complex projective space $\mathbb{P}_q$. Let $\overline{G}$ denote the algebraic closure of the image of G in $PGL(q+1, \mathbb{C})$ and G' be the commutator group of G. Chevalley showed that $G' = \overline{G}'$ (see [15, Theorem 13, page 173] or [14, Corollary II.7.9]) and, as a consequence, G' is acting as an algebraic group on G/N. This fact and the fact that G' is normal in G imply the existence of the fibration $G/N \to G/N \cdot G'$. Now the base $G/N \cdot G'$ of the commutator fibration is an Abelian affine algebraic group that is Stein [26] and thus, because of the assumption $\mathcal{O}(G/H) = \mathbb{C}$, we also have $\mathcal{O}(G/N) = \mathbb{C}$ and the base $G/N \cdot G'$ must be a point. Otherwise, one could pullback non-constant holomorphic functions to G/N in order to obtain a contradiction. Since G' acts on G/N as an algebraic group of transformations and $d_{G/N} \leq d_X = 2$, there is a parabolic subgroup P of G' containing $N \cap G'$ (see [6] or Theorem 2.10) and we now consider the fibrations

$$G/H \longrightarrow G/N = G'/N \cap G' \longrightarrow G'/P.$$

Our strategy in the remainder of the proof is to use the Fibration lemma 2.6 applied to each of the above fibrations and the information we know on the fiber $N/H = (N/H^0)/(H/H^0)$ of the normalizer fibration. Note that H/H^0 is a discrete subgroup of the complex Lie group N/H^0. Since $2 = d_{G/H} \geq d_{N/H}$ and N/H is Kähler whenever G/H is, Theorem 3.5 applies and a finite covering of N/H is biholomorphic to a product $C \times A$, where C is a Cousin group and A is a Stein Abelian Lie group with $d_C + d_A = d_{N/H}$. In particular, $A = \mathbb{C}^p \times (\mathbb{C}^*)^q$ by the classification of complex Abelian Lie groups, see [33, Theorem 3.2], and $d_A = 2p + q$. In addition, we have $2 = d_{G/H} \geq d_{G/N}$ and we look at the cases $d_{G/N} = 0$ (*i.e.*, G/N compact, see Remark 2.4), $d_{G/N} = 1$, and $d_{G/N} = 2$.

First we assume G/N is compact and thus a flag manifold, *i.e.*, $N \cap G' = P$ is a parabolic subgroup of G', and suppose $\mathcal{O}(N/H) = \mathbb{C}$. The fact that N/H is a Cousin group follows from the observations in the previous paragraph. The structure in this case is given in Proposition 2.11: X fibers as a $(\mathbb{C}^*)^k$-bundle over a product $Q \times C$, where Q a flag manifold and C is a Cousin group with $d_C + k = 2$.

Next suppose G/N compact and $\mathcal{O}(N/H) \neq \mathbb{C}$ with holomorphic reduction $N/H \to N/I$. Recall that I is a closed complex subgroup of G containing H and thus we get an intermediate fibration $G/H \to G/I$. In each case below we will show that $\mathcal{O}(G/I) \neq \mathbb{C}$ and this will contradict the assumption that $\mathcal{O}(G/H) = \mathbb{C}$. From what we noted above there are three possibilities for a finite covering of N/H:

(i) $N/I = \mathbb{C}^*$ and $I/H =: C$ is a Cousin group of hypersurface type;
(ii) $N/I = (\mathbb{C}^*)^2$ and $I/H = T$ is a torus;
(iii) $N/I = \mathbb{C}$ and $I/H = T$ is a torus.

In case (i) the space G/I fibers as a $\mathbb{C}^*$-bundle over the flag manifold G/N. Either this bundle is non-trivial and G/I is an affine cone minus its vertex or the bundle is trivial, and so a product $\mathbb{C}^* \times G/N$. In either case one has $\mathcal{O}(G/I) \neq \mathbb{C}$, the desired contradiction.

In case (ii) the space G/I fibers over the flag manifold G/N with typical fiber $N/I = \mathbb{C}^* \times \mathbb{C}^*$. We consider the possibilities for the S-orbits in G/I. First suppose that S acts transitively on $G/I = S/S \cap I$. Then $S \cap I$ is algebraic, since G/I is Kähler, [11]. By the main result in [6] the bundle $S/S \cap I \to S/S \cap P$ is principal. Let $J := \mathbb{C}^*$ be a subgroup of the structure group, *e.g.*, $J := \mathbb{C}^* \times \{e\}$, and consider the right J-action on $S/S \cap I$. This action equivariantly fibers $S/S \cap I$ as a $\mathbb{C}^*$-bundle over a $\mathbb{C}^*$-bundle over $S/S \cap P$. We then proceed as in the previous paragraph. If the S-orbits have complex codimension one in G/I, then we have again a $\mathbb{C}^*$-bundle over a $\mathbb{C}^*$-bundle over G/N, where the latter space still has non-constant holomorphic functions.

Finally, if any S-orbit has complex codimension two in G/I, then it is a section of this bundle because the flag manifold G/N is simply connected. Indeed, we claim that all S-orbits are sections in this setting. As a consequence, G/I splits as a product $\mathbb{C}^* \times \mathbb{C}^* \times G/N$ and again $\mathcal{O}(G/I) \neq \mathbb{C}$.

In (iii) the group action on the fiber $\mathbb{C}$ is by affine transformations [6] and the S-action on the space G/I is transitive. This gives us the following diagram

$$\begin{array}{ccccc} G/I & = & S/S \cap I & = & G/I \\ & & \downarrow & & \downarrow \\ G/N & = & S/S \cap P & = & G/N\,. \end{array}$$

By [10, Proposition 1 in Section 5.2] the group $S \cap I$ contains a maximal torus of $S \cap P$. This implies $S \cap I$ is normal in $S \cap P$ if and only if $S \cap I$ coincides with $S \cap P$ and this is not the case in our setting, since $\dim_{\mathbb{C}} N/I = 1 > 0$. Now the S-orbits are transversal to this one dimensional fiber and thus are coverings of the flag manifold G/N. Again, the fact that G/N is simply connected implies that these orbits are sections and the bundle $G/I \to G/N$ is a product. Once more we have $\mathcal{O}(G/I) \neq \mathbb{C}$.

Now suppose $d_{G/N} = 1$. As noted above, G' acts algebraically and transitively on G/N. It then follows that G/N is an affine cone minus its vertex by [5] or as recalled in Theorem 2.10, and consequently $\mathcal{O}(G/N) \neq \mathbb{C}$ contradicting the assumption that $\mathcal{O}(G/H) = \mathbb{C}$.

Suppose $d_{G/N} = 2$ and a finite covering of $P/N \cap G'$ is biholomorphic to $(\mathbb{C}^*)^2$. An argument analogous to the one given above in (ii) now yields a contradiction.

Finally, suppose $d_{G/N} = 2$ and $P/N \cap G' = \mathbb{C}$. There are two possibilities depending on whether S is transitive on G/N or not, and we first suppose S acts transitively on G/N. By the Fibration lemma N/H is compact, and thus a compact complex torus since G/H is Kähler. Since N/H is a Cousin group, as in the proof of Proposition 2.11 we may assume that G is a reductive complex Lie group. Thus the S-orbits are closed in G/H [23, Theorem 5.1] and one has the following diagram

$$\begin{array}{ccc} S/S\cap H & \hookrightarrow & G/H \\ F\Big\downarrow & & \Big\downarrow N/H \\ S/I & = & G/N\,. \end{array}$$

So F is a closed subgroup of N/H and it is compact. However, $F = I/S \cap H$ is the quotient of algebraic groups. This is only possible if $I/S \cap H$ is finite. Since we have the fibration $S/I \to S/P$ with $P/I = \mathbb{C}$ and S/P a flag manifold, we see that S/I is simply connected. Every S-orbit in X is a holomorphic section of the bundle $G/H \to G/N$ and $X = T \times S/I$ is a product. This is Case II (1) (b) in the Main theorem when S is transitive on G/N.

Otherwise, S does not act transitively on G/N. The radical $R_{G'}$ of G' is a unipotent group acting algebraically on G/N yielding a fibration

$$G/N \xrightarrow{F} G/N \cdot R_{G'}$$

where $F = \mathbb{C}^p$ with $p > 0$. The Fibration lemma and the assumption $d_X = 2$ imply that N/H is compact, thus a torus, $F = \mathbb{C}$ and $Z := G/N \cdot R_{G'}$ is compact and thus a flag manifold. Now $G/N \neq \mathbb{C} \times Z$ because one would then have $\mathcal{O}(G/N) \neq \mathbb{C}$, contradicting $\mathcal{O}(X) = \mathbb{C}$. So G/N is a non-trivial line bundle over Z and there are two S-orbits in G/N, a compact one Y_1 which is the zero section of the line bundle and is biholomorphic to Z and an open one Y_2. The latter holds, since the existence of another closed orbit would imply the triviality of the $\mathbb{C}^*$-bundle $G/N \setminus Y_1$ over Z. We write X as a disjoint union $X_1 \cup X_2$ with $X_i := \pi^{-1}(Y_i)$ for $i = 1, 2$. Then X_1 is a Kähler torus bundle over Z and it is trivial by [13]. A finite covering of X_2 is also trivial since X_2 is Kähler and satisfies $d_{X_2} = 1$ [18, Main theorem, case (b)]. Note that the S-orbits in X_1 (respectively X_2) are holomorphic sections of the torus bundle lying over the corresponding S-orbit Y_1 (respectively Y_2).

Let $x_2 \in X_2$ and $M_2 := S.x_2$. Since X is Kähler, the boundary of M_2 consists of S-orbits of strictly smaller dimension [23, Theorem 3.6], and for dimension reasons, these necessarily lie in X_1. Let $M_1 \subset \overline{M_2}$ be such an S-orbit in X_1 and let $p \in M_1$. As observed in the previous paragraph, M_1 is biholomorphic to Y_1 which is a flag manifold. Therefore, $M_1 = K \cdot p = K/L$, where K is a maximal compact subgroup of S and L is the corresponding isotropy subgroup at the point p and is compact. The L-action at the L-fixed point p can be linearized. This means that there exist an L-invariant open neighbourhood U of p in X, an open neighbourhood V of 0 in $T_p(X)$, a linear map $\Phi : T_p(X) \to T_p(X)$, and a biholomorphic map α of U onto V with $\alpha(p) = 0$ such that $\alpha^{-1} \circ \Phi \circ \alpha$ gives the L-action on U. In this setup Φ leaves invariant the complex vector subspaces $T_p(K/L)$ and $T_p(\pi^{-1}(\pi(p)))$ and thus also a complementary complex vector subspace W of $T_p(X)$. So we have the following decomposition:

$$T_p(X) = T_p\left(\pi^{-1}(\pi(p))\right) \oplus W \oplus T_p(K/L).$$

Let (t, w, v) be the corresponding coordinates in $T_p(X)$. Set $\widehat{M}_1 := S \cdot \widehat{x}_1$ for some $\widehat{x}_1 \in U \cap X_1$ with $\alpha(\widehat{M}_1 \cap U) \cap T_p(\pi^{-1}(\pi(p))) = \{(t_0\, w_0, v_0)\}$, where $t_0 \neq 0$. Since L is a subgroup of S and the S-orbits are transversal to the complex torus $\pi^{-1}(\pi(p))$, it follows that Φ acts as the identity on the subspace $T_p(\pi^{-1}(\pi(p)))$. So every point of $\Phi\circ\alpha(\widehat{M}_1 \cap U)$ has t-coordinate equal to $t_0 \neq 0$. As a consequence, $\widehat{M}_1$ does not intersect $\overline{M}_2$ inside the set U. Now for any other point of M_1 its isotropy subgroup for the K-action is a conjugate of the group L and the argument just given applied to that conjugate of the group L at that fixed point shows that M_2 is the unique S-orbit that contains M_1 in its closure and also that $\overline{M_2} = M_2 \cup M_1$ is a complex submanifold of X that is a holomorphic section of the bundle $\pi : X = G/H \to G/N$. This bundle is thus trivial and X is biholomorphic to $T \times G/N$. This is Case II (1) (b) in the Main theorem when S is not transitive on G/N.

This completes the classification if $\mathcal{O}(X) = \mathbb{C}$. □

5. The classification when $\mathcal{O}(X) \neq \mathbb{C}$

We first prove a generalization of Proposition 3.4 for arbitrary isotropy.

Proposition 5.1. *Let G be a connected, simply connected, solvable complex Lie group, H a closed complex subgroup of G with G/H Kähler, $G/H \to G/J$ its holomorphic reduction with fiber $T = J/H$ a compact complex torus and base $G/J = (\mathbb{C}^*)^2$. Then a finite covering of G/H is biholomorphic to $T \times (\mathbb{C}^*)^2$.*

Proof. If H^0 is normal in G, then this is Proposition 3.4. Otherwise, let $N := N_G(H^0)$ and consider $G/H \to G/N$. Since G/N is an orbit in some projective space, G/N is holomorphically separable and the map $G/H \to G/N$ factors through the holomorphic reduction, *i.e.*, $J \subset N$. We first assume that $J = N$ and consider $\widehat{N} := N_G(J^0)$. Then the argument given in the third paragraph of the proof of Proposition 3.4 shows that $G/\widehat{N} = \mathbb{C}^*$ and $\widehat{N}/J = \mathbb{C}^*$. But then (a finite covering of) $\widehat{N}/H$ is isomorphic to $\mathbb{C}^* \times T$, see [18, Proposition1], implying that H^0 is normal in $\widehat{N}$. This gives the contradiction that $\widehat{N} = N$ while $\dim \widehat{N} > \dim J = \dim N$.

So we are reduced to the case where $J \neq N$ and, after going to a finite covering if necessary, $G/N = \mathbb{C}^*$ and $N/H \cong \mathbb{C}^* \times T$ is an Abelian complex Lie group, since N/H is Kähler with two ends, see [18, Proposition 1]. We have the diagram

$$\begin{array}{ccc} X = G/H & \xrightarrow{T} & G/J = \mathbb{C}^* \times \mathbb{C}^* \\ & \searrow \quad \swarrow & \\ & G/N = \mathbb{C}^*. & \end{array}$$

Since the top line is the holomorphic reduction and X is Kähler, a finite covering of this bundle is a T-principal bundle, see [20, Theorem 1]. Choose $\xi \in \mathfrak{g} \setminus \mathfrak{n}$ and set $A := \exp\langle\xi\rangle_{\mathbb{C}}$ and $B := N/H \cong \mathbb{C}^* \times T$. Then the group $\widehat{G} := A \ltimes B$ acts holomorphically and transitively on X, where A acts from the left and B acts from the right on the principal $\mathbb{C}^* \times T$-bundle $G/H \to G/N$. For dimension reasons the isotropy of this action is discrete and the result now follows by Proposition 3.4. □

Proof. Let $G/H \to G/J$ be the holomorphic reduction. By the Fibration lemma one has $d_{G/J} \leq 2$ and we first consider the case when $d_{G/J} = 2$. In [7] there is a list of the possibilities for G/J which was also given above in the proof of Proposition 3.3. We now employ that list to determine the structure of X.

Suppose $G/J = \mathbb{C}$. By the Fibration lemma J/H is compact, Kähler, and so biholomorphic to the product of a torus T and a flag manifold Q. Thus $X = T \times Q \times \mathbb{C}$ by [24]; this is case (2) (a) (i) in the Main theorem.

Suppose G/J is an affine quadric. By the Fibration lemma we again have $J/H = T \times Q$. Then X is biholomorphic to a product, since G/J is Stein and it is simply connected [24]; this is case (2) (a) (ii) in the Main theorem.

If G/J is the complement of the quadric curve in $\mathbb{P}_2$, then a two-to-one covering of G/J is the affine quadric and the pullback of X to that space is again a product, as in the previous case; this is case (2) (a) (iii) in the Main theorem.

Suppose the base of the holomorphic reduction of G/H is $G/J = (\mathbb{C}^*)^2$. Since G/H is Kähler, every fiber of the fibration $G/H \to G/J$ is Kähler, and it is compact by the Fibration lemma and thus biholomorphic to $T \times Q$ [13], where T is a compact, complex torus and Q is a flag manifold. The S-orbits in the base G/N of the normalizer fibration $G/H \to G/N$ are compact and biholomorphic to the S-orbits in X. This follows from the fact that the fibers of the fibration of any flag manifold have to be flag submanifolds. But, since the isotropy of the fiber of the normalizer fibration is discrete and no positive dimensional flag manifold is parallelizable, the fibers of the induced fibration of the S-orbits by the normalizer fibration must be discrete, *i.e.*, the S-orbits in X cover the S-orbits in G/N and the latter is simply connected, so the covering is one-to-one. Next we consider the commutator fibration $G/N \to G/G' \cdot N$ of the base G/N of the normalizer fibration. Since $G/G' \cdot N$ is an Abelian, Stein Lie group [26], it follows that $G/G' \cdot N$ is isomorphic to $\{e\}$, $\mathbb{C}$, $\mathbb{C}^*$ or $(\mathbb{C}^*)^2$ by the Fibration lemma and Proposition 2.5. Note that $G/G' \cdot N \neq \mathbb{C}$, since, otherwise, the space X would be biholomorphic to a product $T \times Q \times \mathbb{C}$ by the Oka principle [24], with the base of its holomorphic reduction being $\mathbb{C}$, and this would contradict the assumption that this base is $(\mathbb{C}^*)^2$. Let $\sigma : G \to G/G' \cdot N$ be the quotient homomorphism and set $G_0 := \sigma^{-1}(K)$, where K is the maximal compact subgroup of $G/G' \cdot N$. Note that G_0 contains G' and thus also S. It then follows that the G_0-orbits in G/N are compact, homogeneous CR-manifolds that are products $Q \times (S^1)^k$ by [21, Proposition 4.4], where $k := \dim_{\mathbb{R}} K$ is equal to 0, 1, or 2. As a consequence, $G/N = Q \times (\mathbb{C}^*)^k$. Since S is acting trivially on the fiber N/H of the normalizer fibration and on the second factor in the last product decomposition, the composition of the projection map of the normalizer fibration and the projection of G/N onto Q is the fibration $G/H \to G/R \cdot H$ by the orbits of the radical R of G, *i.e.*, the R-orbits in G/H are closed. Since the base $G/R \cdot H = Q$ of the radical fibration is simply connected and the S-orbits in G/H cover this base, we see that the radical fibration has holomorphic sections and hence $G/H = Q \times Z$, where $Z := R \cdot H/H = R/R \cap H$ is a complex solvmanifold. Then Z fibers as a Kähler T-bundle over $(\mathbb{C}^*)^2$ and a finite covering of Z is a product by Proposition 5.1. Putting this together one sees that a finite covering of X is a product $T \times Q \times (\mathbb{C}^*)^2$. This gives Case II (2) (a) (iv) in the Main theorem.

Suppose G/J is a $\mathbb{C}^*$-bundle over an affine cone minus its vertex. Here $d_{G/J} = 2$ and G/J is not Stein. By the Fibration lemma J/H is compact and J/H inherits a Kähler structure from X. If we set $N := N_J(H^0)$, then the normalizer fibration $J/H \to J/N$ is a product with $N/H = T$ and $J/N = Q$ [13]. Since the fiber of the fibration $G/H \to G/N$ is compact, G/N is Kähler [12] and $d_{G/N} = 2$ by the Fibration lemma. First assume that $G/N = S/I$, where I is an algebraic subgroup of S [11]. Then the principal T-bundle $G/H \to S/I$ is of the form $G/H = S \times_\rho T$, where the representation $\rho : I \to \mathrm{Aut}^0(T) \cong T$ factors through I/I'. As in the proof of Proposition 2.11, we may assume that G is reductive and that the image $\rho(I)$ is, on the one hand, an algebraic subtorus $(\mathbb{C}^*)^k$ of the algebraic manifold $S/S \cap H$ which is closed in G/H [23, Theorem 5.1] and, on the other hand, a closed subgroup of the compact complex torus T. Hence $\rho(I) = \{e\}$ and, as a consequence, the bundle $G/H \to S/I$ is trivial. Example 7.4 shows that the Q-bundle $S/I \to S/S \cap J$ is not necessarily trivial. The setting where S is not transitive on G/N occurs if G/J is a product of $\mathbb{C}^*$ with an affine cone minus its vertex. As in the last part of the previous paragraph, one again has closed R-orbits and the radical fibration $G/H \to G/R \cdot H$. A finite covering splits as a product with the typical radical orbit $R/R \cap H$ being a Kähler T-bundle over $\mathbb{C}^*$ that has a finite covering that splits as a product [22, Theorem 6.14]. These considerations yield the possibilities in Case II (2) (b) in the Main theorem.

Suppose next that $d_{G/J} = 1$. By the Fibration lemma $d_{J/H} = 1$ and either $\mathcal{O}(J/H) = \mathbb{C}$ or $\mathcal{O}(J/H) \neq \mathbb{C}$. We first assume the former and show below that the latter gives a contradiction and thus it does not occur. Since J/H is Kähler, the classification given in [18, Proposition 5] applies and the normalizer fibration $J/H \to J/N$, where $N := N_J(H^0)$, realizes J/H as a Cousin bundle over a flag manifold.

The first case occurs if G/J is Stein. By Proposition 2.5 we have $\dim_{\mathbb{C}} G/J = 1$ and thus $G/J = \mathbb{C}^*$. Since S acts trivially on the Cousin group J/H and on G/J, the radical orbits are closed and one has the fibration $G/H \to G/R \cdot H$. Its base is biholomorphic to Q and the S-orbits in G/H are holomorphic sections. So this bundle is holomorphically trivial. Thus X is biholomorphic to $Q \times Z$, where Z is a hypersurface Cousin group bundle over $\mathbb{C}^*$. A finite covering of this splits as a product by [22, Theorem 6.14]. This is Case II (2) (c) in the Main theorem.

The other possibility is that G/J is not Stein and then G/J is an affine cone minus its vertex. By Proposition 2.11 there is a closed complex subgroup I of N containing H with $I/H = \mathbb{C}^*$ and $J/I = N/I \times J/N$, where $N/I =: T$ is a torus and $J/N =: Q$ is a flag manifold. Consider the T-bundle $G/I \to G/N$, set $Y := G/N$, and observe that Y is a Q-bundle over G/J. As such, Y has a finite fundamental group. This follows from the exact homotopy sequence of the bundle

$G/N \to G/J$ and the facts that a flag manifold is connected and simply connected and an affine cone minus its vertex has finite fundamental group. Consider the finite covering $\pi : \widetilde{Y} \to Y$, where $\widetilde{Y}$ is the universal covering of Y and let $\pi^*(G/I)$ be the pullback of G/I via the map π. The S-orbits in $\pi^*(G/I)$ are sections and the torus bundle splits as a product $\pi^*(G/I) = T \times \widetilde{Y}$. Thus a finite covering of X fibers as a $\mathbb{C}^*$-bundle over $T \times \widetilde{Y}$; this is Case II (2) (d) in the Main theorem. Example 7.3 shows that Y itself need not be a product.

Finally we assume that $\mathcal{O}(J/H) \neq \mathbb{C}$ when $d_{G/J} = 1$. Let $J/H \to J/I$ be its holomorphic reduction. Since $\mathcal{O}(J/H) \neq \mathbb{C}$, one has $\dim J/I > 0$ and thus $\dim J > \dim I$ and in all cases we will produce the contradiction that I and J have the same dimension. By [18, Proposition 3] a finite covering of J/H is biholomorphic to $I/H \times J/I$, where $I/H = T$ is a torus and $Z := J/I$ is an affine cone minus its vertex. Since the fibration $G/H \to G/I$ has T as fiber and T is compact, there is a Kähler structure on G/I, see [12] and by the Fibration lemma we have $d_{G/I} = 2$. First assume that a solvable subgroup of G acts transitively on G/I. Then $G/I \to G/J$ is a $\mathbb{C}^*$-bundle over $\mathbb{C}^*$ and G/I is Stein, *e.g.*, see [7, page 904]. Thus G/I is the base of the holomorphic reduction of G/H and this gives the contradiction that $I = J$. Next suppose that a maximal semisimple subgroup S of G acts transitively on G/I. Since $S/S \cap I$ is Kähler, $S \cap I$ is algebraic by [11]. Thus there exists a parabolic subgroup P of S containing $S \cap I$ such that $P/I \cap S$ is isomorphic to $(\mathbb{C}^*)^2$, see [6] or Theorem 2.10. Then G/I or a two-to-one covering of G/I is a homogeneous algebraic principal $\mathbb{C}^*$-bundle over an affine cone minus its vertex and is quasi-affine and thus holomorphically separable, see [7, Proposition 2]. So G/I is the base of the holomorphic reduction of G/H. But this again gives the contradiction $I = J$. The remaining case occurs when G is a mixed group, *i.e.*, G is neither solvable nor semisimple. First suppose $G/J = \mathbb{C}^*$ and J/I is an affine cone minus its vertex. Let $N := N_G(I^0)$ and consider the normalizer fibration $G/I \to G/N$ which we first assume to be a covering. As in the case when $G/J = (\mathbb{C}^*)^2$ handled above, by using the commutator fibration of G/N and [21, Proposition 4.4] we see that G/N is a product and so is holomorphically separable. Therefore, $\dim I = \dim N = \dim J$ gives the desired contradiction. In all the other cases we get a $\mathbb{C}^*$-bundle over an affine cone minus its vertex with codimension one S-orbits. Since an affine cone minus its vertex has a finite fundamental group, by passing to a finite covering we find that the S-orbits are holomorphic sections of the bundle $G/I \to G/J$, and G/I is a product and thus holomorphically separable. So we again get the contradiction that $I = J$. Thus the case $\mathcal{O}(J/H) \neq \mathbb{C}$ does not occur if $d_{G/J} = 1$.

This completes the classification when $\mathcal{O}(X) \neq \mathbb{C}$. □

6. Proof of the converse

The only component manifolds in the Main theorem that are not immediately recognizable as Kähler are the Y's that occur in II (2) (b) and II (2) (d). Since these are flag manifold bundles over holomorphically separable bases, these manifolds are Kähler by [30]. The proof of the converse follows from the following observations:

(1) The product of Kähler manifolds is a Kähler manifold;
(2) A connected, complex manifold X that is a finite, unramified covering of a complex maniold Y is Kähler if and only if Y is Kähler.

7. Examples

We now give non-trivial examples that can occur in the classification.

Example 7.1. The manifolds that occur in Proposition 2.11 need not be biholomorphic to a product of an S-orbit times an orbit of the center. For $k = d_X = 1$, let $\chi : B \to \mathbb{C}^*$ be a non-trivial character, where B is a Borel subgroup of $S := SL(2,\mathbb{C})$. Let C be a non-compact 2-dimensional Cousin group. Then C fibers as a $\mathbb{C}^*$-bundle over an elliptic curve T and let B act on C via the character χ. Set $X := S \times_B C$. Then X fibers as a principal C-bundle over S/B and is Kähler, but neither this bundle nor the corresponding $\mathbb{C}^*$-bundle is trivial.

Example 7.2. Let $S := SL(3,\mathbb{C})$ and

$$I := \left\{ \begin{pmatrix} * & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\} \quad \subset \quad B \quad \subset \quad P := \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\},$$

where B is the Borel subgroup of S consisting of upper triangular matrices. Then $S/I \to S/B$ is an affine $\mathbb{C}$-bundle over the flag manifold S/B. Now consider the fibration $S/I \to S/P$. Its fiber is $P/I = \mathbb{P}_2 \setminus \{\text{point}\}$ and all holomorphic functions on S/I are constant along the fibers by Hartogs' principle and so must come from the base $S/P = \mathbb{P}_2$. But the latter is compact and so $\mathcal{O}(S/P) = \mathbb{C}$ and, as a consequence, we see that $\mathcal{O}(S/I) = \mathbb{C}$. Thus S/I is an example of a space that can be the base of the normalizer fibration as in the Main theorem II (1) (b) when S is transitive on that base.

Example 7.3. Consider the subgroups of $S := SL(5,\mathbb{C})$ defined by

$$I := \left\{ \begin{pmatrix} 1 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \end{pmatrix} \right\} \quad \subset \quad P := \left\{ \begin{pmatrix} * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \end{pmatrix} \right\}$$

and

$$J := P' = \left\{ \begin{pmatrix} 1 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \end{pmatrix} \right\}.$$

Then $J/I = \mathbb{P}_1$, $P/J = \mathbb{C}^*$, and $S/P = Q$ is a flag manifold that can be fibered as a Gr(2, 4)-bundle over $\mathbb{P}_4$. We have the fibrations

$$S/I \xrightarrow{\mathbb{P}_1} S/J \xrightarrow{\mathbb{C}^*} S/P = Q.$$

Note that S/J is holomorphically separable due to the fact that it can be equivariantly embedded as an affine cone minus its vertex in some projective space and since J/H is compact, S/J is the base of the holomorphic reduction of S/I. Since the fibration of S/I is not trivial, the spaces Y that occur in the Main theorem II (2) (d) need not split as the products of flag manifolds and affine cones minus their vertices.

Example 7.4. Using the groups defined in the previous example set $\widehat{S} := S \times S$ and $\widehat{I} := I \times I$. Then $Y := \widehat{S}/\widehat{I} = S/I \times S/I$ is an example that can occur in the Main theorem II (2) (b). Such a Y fibers as a non-trivial flag manifold over a $(\mathbb{C}^*)^2$-bundle over a flag manifold.

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