

Compactness criteria for the resolvent of the Fokker-Planck operator

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Abstract. In this paper we study the spectral property of a Fokker-Planck operator with potential. By virtue of a multiplier method inspired by Nicolas Lerner, we obtain new compactness criteria for its resolvent, involving the control of the positive eigenvalues of the Hessian matrix of the potential.

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1. Introduction and main results

The Fokker-Planck operator is written

$$P = y \cdot \partial_x - \partial_x V(x) \cdot \partial_y - \Delta_y + \frac{|y|^2}{4} - \frac{n}{2}, \quad (x, y) \in \mathbb{R}^{2n}, \quad (1.1)$$

where x denotes the space variable and y denotes the velocity variable, and $V(x)$ is a potential defined in the whole spatial space \mathbb{R}_x^n . In this work we are mainly concerned with the compact resolvent property for the non-selfadjoint Fokker-Planck operator, and this is motivated by a conjecture stated by Helffer and Nier (see [7, Conjecture 1.2]), which reveals the close link between the compact resolvent property for the Fokker-Planck operator and the same property for the corresponding Witten Laplacian. Precisely, we have the following statement.

Conjecture 1.1 (Helffer-Nier's conjecture). The Fokker-Planck operator P has a compact resolvent if and only if the Witten Laplacian $\Delta_{V/2}^{(0)}$, defined by

$$\Delta_{V/2}^{(0)} = -\Delta_x + \frac{1}{4} |\partial_x V(x)|^2 - \frac{1}{2} \Delta_x V(x),$$

has a compact resolvent.

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The necessity part, that the Witten Laplacian $\Delta_{V/2}^{(0)}$ has a compact resolvent if the Fokker-Planck operator P is with compact resolvent, has already established by Helffer and Nier (cf. [7, Theorem 1.1]). The reverse implication still remains open up to now for general potential, and it is indeed valid under some conditions on the potential V . For instance, following the analysis in [7, 13] with some improvements, the author ([17]) proved that if V satisfies that

$$\forall |\alpha| = 2, \exists C_\alpha > 0, \text{ such that } |\partial_x^\alpha V(x)| \leq C_\alpha \langle \partial_x V(x) \rangle^s \quad \text{with } s < \frac{4}{3}, \quad (1.2)$$

then Fokker-Planck operator has a compact resolvent provided the Witten-Laplacian has a compact resolvent or $\lim_{|x| \rightarrow +\infty} |\partial_x V(x)| = +\infty$, and moreover a constant C exists such that the following weighted estimate

$$\| |\partial_x V(x)|^{2/3} u \|_{L^2} \leq C (\|Pu\|_{L^2} + \|u\|_{L^2})$$

holds for all $u \in C_0^\infty(\mathbb{R}^{2n})$. Here and throughout the paper we will use the notation

$$\langle \cdot \rangle = \left(1 + |\cdot|^2\right)^{1/2},$$

which is equivalent to the modulus $|\cdot|$, and use $\|\cdot\|_{L^2}$ and $\langle \cdot, \cdot \rangle_{L^2}$ to denote respectively the norm and inner product of the complex Hilbert space $L^2(\mathbb{R}^{2n})$, and denote by $C_0^\infty(\mathbb{R}^{2n})$ the set of smooth compactly supported functions.

We remark the drawback of the condition (1.2) is that it doesn't give any information for the dependence on the sign of V , which plays import role in the analysis of compact resolvent property for Witten Laplacian. For instance it is well-known (see [7, 18]) that the Witten Laplacian $\Delta_{V/2}^{(0)}$ with $V = -x_1^2 x_2^2$ has a compact resolvent, while 0 actually belongs to the essential spectrum of Witten Laplacian $\Delta_{V/2}^{(0)}$ with $V = x_1^2 x_2^2$ and thus its resolvent cannot be compact. By the general criteria for Schrödinger operators we see that if

$$\frac{1}{4} |\partial_x V(x)|^2 - \frac{1}{2} \Delta_x V \rightarrow +\infty, \text{ as } |x| \rightarrow +\infty,$$

or more generally (see [7, Proposition 3.1] for instance), if

$$\frac{t}{4} |\partial_x V(x)|^2 - \frac{1}{2} \Delta_x V \rightarrow +\infty, \text{ as } |x| \rightarrow +\infty \quad (1.3)$$

for some $t \in]0, 2[$, then the Witten Laplacian $\Delta_{V/2}^{(0)}$ has a compact resolvent. We refer the reader to [7] for other criteria presented with detailed discussion. These criteria show the microlocal property, *i.e.*, the dependence on the sign of V , for the compact resolvent of Witten Laplacian. As far as Fokker-Planck operator is concerned, Helffer-Nier's conjecture suggests strongly it should have the similar

microlocal property as the Witten Laplacian. And this kind of dependence property for Fokker-Planck operator is not clear by now. In the present work we will give some sufficient conditions for the compact resolvent of Fokker-Planck operator, mainly based on the sign of the eigenvalues of the Hessian matrix $(\partial_{x_i x_j} V)_{1 \leq i, j \leq n}$. Our results can be stated as follows.

Theorem 1.2. Denote by $\lambda_\ell(x)$, $1 \leq \ell \leq n$, the eigenvalues of the Hessian matrix

$$(\partial_{x_i x_j} V(x))_{1 \leq i, j \leq n}.$$

We associate a set I_x of indices defined by to each $x \in \mathbb{R}^n$

$$I_x = \left\{ 1 \leq \ell \leq n; \lambda_\ell(x) > 0 \right\}.$$

Suppose that there exists a constant C such that

$$\sum_{j \in I_x} \lambda_j(x) \leq C \left(\partial_x V(x) \right)^{4/3} \quad \text{for all } x \in \mathbb{R}^n. \quad (1.4)$$

Then the following conclusions hold.

(i) There exists a constant C_* such that

$$\| |\partial_x V(x)|^{1/16} u \|_{L^2} \leq C_* (\|Pu\|_{L^2} + \|u\|_{L^2}) \quad \text{for all } u \in C_0^\infty(\mathbb{R}^{2n}).$$

As a result, the Fokker-Planck operator P has a compact resolvent if

$$\lim_{|x| \rightarrow +\infty} |\partial_x V(x)| = +\infty;$$

(ii) Suppose there exists a number $\alpha \geq 0$, such that

$$\lim_{|x| \rightarrow +\infty} \left(\alpha |\partial_x V(x)|^2 - \Delta_x V(x) \right) = +\infty. \quad (1.5)$$

Then we can find a constant \tilde{C} , depending on α , such that

$$\begin{aligned} & \| |\alpha |\partial_x V(x)|^2 - \Delta_x V(x)|^{1/80} u \|_{L^2} \\ & \leq \tilde{C} (\|Pu\|_{L^2} + \|u\|_{L^2}) \quad \text{for all } u \in C_0^\infty(\mathbb{R}^{2n}), \end{aligned}$$

and thus the Fokker-Planck operator P has a compact resolvent as a result.

The assumption (1.4) is an improvement of the condition (1.2). We mention that the index $4/3$ in (1.4) is not sharp, and the following Theorem 1.3 and Corollary 1.4 are devoted to showing a better index $14/5$ may be expected.

Theorem 1.3. *Suppose that there exists a number $\tau \geq 0$, such that the matrix*

$$A_\tau(x) = \left(a_{ij}^\tau(x) \right)_{1 \leq i, j \leq n} \text{ with } a_{ij}^\tau = \tau \langle \partial_x V \rangle^{\frac{4}{3}} (\partial_{x_i} V) (\partial_{x_j} V) - \partial_{x_i x_j} V + \tau \delta_{ij}$$

is positive-definite for all $x \in \mathbb{R}^n$, where δ_{ij} is the Kronecker Delta. Then there is a constant C , such that

$$\| |\partial_x V(x)|^{1/20} u \|_{L^2} + \sum_{1 \leq i, j \leq n} \| |a_{ij}^\tau(x)|^{1/80} u \|_{L^2} \leq C \left(\|Pu\|_{L^2} + \|u\|_{L^2} \right) \tag{1.6}$$

for all $u \in C_0^\infty(\mathbb{R}^2)$.

As a result, the Fokker-Planck operator P has a compact resolvent if

$$\lim_{|x| \rightarrow +\infty} \left(|\partial_x V(x)| + \sum_{1 \leq i, j \leq n} |a_{ij}^\tau(x)| \right) = +\infty.$$

As an immediate consequence, when $n = 1$ we have the compactness criteria for Fokker-Planck operator, which is an improvement of the corresponding condition (1.3) for Witten Laplacian. Precisely,

Corollary 1.4. *Let $n = 1$ and let $V(x) \in C^2(\mathbb{R})$. Suppose that there exists $\tau \geq 0$, such that*

$$\lim_{|x| \rightarrow +\infty} \tau |\partial_x V(x)|^{14/5} - \Delta_x V(x) = +\infty.$$

Then the Fokker-Planck operator P has a compact resolvent.

Remark 1.5. In the special case when $n = 1$, using Corollary 1.4 and the necessity part in Helffer-Nier Conjecture (cf. [7, Theorem 1.1]), we can also improve the criteria (1.3) for Witten Laplacian, by allowing t to range over $[0, +\infty[$ instead of $]0, 2[$ and relaxing the index 2 there by $14/5$.

Remark 1.6. The hypotheses in Theorem 1.2 and Theorem 1.3, are related to the sign of the eigenvalues of the Hessian matrix $(\partial_{x_i x_j} V)_{1 \leq i, j \leq n}$. In fact these assumptions are obviously fulfilled when the Hessian matrix is negative-semidefinite. When the Hessian matrix is positive-semidefinite or indefinite, we requires that the positive eigenvalues of Hessian matrix, instead of all the second derivatives in the condition (1.2), are dominated by $\langle \partial_x V \rangle^{4/3}$. Now look back at the aforementioned potential $V = \pm x_1^2 x_2^2$, and it is clear that these hypotheses are fulfilled by $V = -x_1^2 x_2^2$ and violated by $V = x_1^2 x_2^2$.

Remark 1.7. In [7, 8], the authors introduced a compactness criterion for Witten Laplacian with polynomial potential V , based on the group theory. And it is also

natural and interesting to expect the similar group theoretical compactness criteria for Fokker-Planck operator. Now consider a potential, not necessary to be a polynomial, such that the matrix

$$\tilde{A}_\tau(x) = \left(\tilde{a}_{ij}^\tau(x) \right)_{1 \leq i, j \leq n}$$

with $\tilde{a}_{ij}^\tau(x) = \tau |\partial_x V(x)|^{4/5} (\partial_{x_i} V(x)) (\partial_{x_j} V(x)) - \partial_{x_i x_j} V(x)$

is positive-semidefinite for some $\tau \geq 0$. This condition is slightly stronger than the one in Theorem 1.3, and it yields

$$-\Delta_x V(x) + \tau |\partial_x V(x)|^{4/5} \geq 0.$$

Thus repeating the arguments used to prove maximum principle for elliptic equations, we see V doesn't have local minimum in \mathbb{R}_x^n , except the constant-valued potentials. So this kind of microlocal property is imposed directly on the potential rather than on its "limiting polynomials" in the sense of [7, 8].

Due to the lack of estimates on the higher derivatives of V , we can't follow the global symbolic calculus to prove our results, although this method is efficiently explored to investigate the hypoellipticity and the compact resolvent of Fokker-Planck operator (*cf.* [7, 13]). Instead we will use a multiplier method inspired by N. Lerner (see for instance [14, 15] and references therein), which is based on the Poisson bracket analysis for the real and imaginary parts of the Fokker-Planck operator. We hope this method not only applies to analyze the weighted estimate and the compact resolvent, but also may give insights on the sign conditions to investigate the subellipticity (see [4, 5, 7, 13, 17, 19] for instance) of Fokker-Planck operator.

We end up the introduction by mentioning that as a diffusive models, the study of Fokker-Planck equation is of independent interest in kinetic theory and nonequilibrium statistical physics. Here one of the basic problems is to analyze the large time behavior of solutions to the time-dependent Fokker-Planck equation and prove that these solutions converge exponentially towards the equilibrium as t goes to $+\infty$. Various approaches, such as hypoellipticity, hypocoercivity, entropy method and so on, are developed to study this problem, and satisfactory results are achieved. We refer to [6, 7, 9–11, 13, 17, 20] and references therein for more detail and [2] for the spectral analysis on the non-selfadjoint Schrödinger operators with compact resolvent. Finally we remark that in order to study the exponentially trend problem, an efficient method is to investigate the spectral gap, which is usually reduced to analyze the compactness of resolvent. On the other hand, when the Fokker-Planck operator has an essential spectrum, only polynomial convergence rate is expected, see the recent work [21] for the study on short-range potentials.

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2. Proof of the main results

First, we list some notations and facts used throughout the paper. The proofs of the main results, Theorem 1.2 and Theorem 1.3, are presented in Subsection 2.1 and Subsection 2.2, where two multipliers \mathcal{M} and \mathcal{K} (see Lemma 2.1 and Lemma 2.2 below) are respectively introduced. This kind of multiplier method is inspired by N. Lerner [16], and here it means that we have to choose carefully an operator M (*multiplier*) which is bounded and self-adjoint in L^2 space, such that

$$\operatorname{Re} \langle (y \cdot \partial_x - \partial_x V(x) \cdot \partial_y)u, Mu \rangle_{L^2}$$

has a good lower bound (weighted estimate here) on one side, and on the other side,

$$\left| \langle (-\Delta_y + |y|^2/4 - n/2)u, Mu \rangle_{L^2} \right|$$

is bounded from above by $\|Pu\|_{L^2} + \|u\|_{L^2}$. The multipliers chosen here are motivated by the Poisson bracket analysis for the real and imaginary parts of symbol for the Fokker-Planck operator. Precisely, if we denoted by $[Q_1, Q_2]$ the commutator between two operators Q_1 and Q_2 , which is defined by

$$[Q_1, Q_2] = Q_1 Q_2 - Q_2 Q_1,$$

and also use the notation that $X_0 = y \cdot \partial_x - \partial_x V(x) \cdot \partial_y$ and $X_j = \partial_{y_j} + \frac{y_j}{2}$, for $j = 1, \dots, n$, then we can rewrite the Fokker-Planck operator P define in (1.1) as

$$P = X_0 + \sum_{j=1}^n X_j^* X_j,$$

and moreover we compute

$$\left[X_0, \sum_{j=1}^n X_j^* X_j \right] = -\frac{1}{2} \partial_x V \cdot y + 2 \partial_x \cdot \partial_y \tag{2.1}$$

and

$$\begin{aligned} \left[X_0, \left[X_0, \sum_{j=1}^n X_j^* X_j \right] \right] &= -2\Delta_x + \frac{1}{2} |\partial_x V|^2 - \frac{1}{2} \sum_{1 \leq i, j \leq n} (\partial_{x_i x_j} V) y_i y_j \\ &\quad + 2 \sum_{1 \leq i, j \leq n} (\partial_{x_i x_j} V) \partial_{y_i} \partial_{y_j}. \end{aligned}$$

Thus the properties of subelliptic and weighted estimates in x variable can be deduced from the commutator above if some kind of conditions (negative semi-definite for instance) are imposed on the Hessian matrix $(\partial_{x_i x_j} V)_{1 \leq i, j \leq n}$. This suggests that the multipliers \mathcal{M} and \mathcal{K} here (see Lemma 2.1 and Lemma 2.2 below)

should be chosen the first-commutator analysis in (2.1). In this work we will investigate only the weighted estimate and thus the essential part in the multipliers is the term $\partial_x V \cdot y$ in (2.1). Moreover it seems reasonable that the term $\partial_x \cdot \partial_y$ in (2.1) shouldn't be involved in the multipliers, since it corresponds to the hypoellipticity and thus more estimates on the higher derivatives of V are required rather than the ones of second order. We refer to [1, 12] for the multipliers introduced to deduce the hypoellipticity of kinetic operators.

Next we will give some estimates to be used frequently. Observe X_0 is an anti-selfadjoint operator in L^2 and thus it is clear that

$$\begin{aligned} \sum_{j=1}^n \|X_j u\|_{L^2} &\leq \| \langle y \rangle u \|_{L^2} + \sum_{1 \leq j \leq n} \| \partial_{y_j} u \|_{L^2} \\ &\leq C (\|Pu\|_{L^2} + \|u\|_{L^2}) \quad \text{for all } u \in C_0^\infty(\mathbb{R}^{2n}). \end{aligned} \tag{2.2}$$

We will use the following result which is just a consequence of Hörmander's bracket condition (cf. [3] for instance), *i.e.*, a constant C exists such that for any vector-valued function $\theta(x) = (\theta_1(x), \dots, \theta_n(x))$ of x variable and for any $v \in C_0^\infty(\mathbb{R}_y^n)$, we have

$$\| |\theta_k(x)|^{1/2} v \|_{L^2(\mathbb{R}_y^n)} \leq C \left(\| (\theta(x) \cdot y) v \|_{L^2(\mathbb{R}_y^n)} + \| \partial_{y_k} v \|_{L^2(\mathbb{R}_y^n)} \right). \tag{2.3}$$

2.1. Proof of Theorem 1.2

We prove in this subsection Theorem 1.2. To do so we begin with the following estimate which holds for quite general potential.

Lemma 2.1. *Let $V(x) \in C^2(\mathbb{R}^n)$. Then for all $\sigma \in]0, 1[$, there exists a constant $C_\sigma > 0$ such that for any $u \in C_0^\infty(\mathbb{R}^{2n})$ we have*

$$\begin{aligned} &\sigma \left\langle \frac{(1 + |y|^8) \langle \partial_x V(x) \rangle^2}{\rho^3} \frac{y^2}{1 + y^2} u, u \right\rangle_{L^2} + \left\langle \frac{\rho}{\langle y \rangle^4} u, u \right\rangle_{L^2} \\ &- \sum_{1 \leq i, j \leq n} \left\langle \frac{(1 + |y|^8) (\partial_{x_i x_j} V) y_i y_j}{\rho^3} \frac{y^2}{1 + y^2} u, u \right\rangle_{L^2} \\ &\leq C_\sigma \left(\|Pu\|_{L^2}^2 + \|u\|_{L^2}^2 \right), \end{aligned}$$

where the function $\rho \in C^1(\mathbb{R}^{2n})$ is defined by

$$\rho = \rho(x, y) = \left(1 + |y|^8 + |\partial_x V(x) \cdot y|^2 \right)^{\frac{1}{2}}.$$

Proof. To simplify the notations we will use C in the proof to denote different constants, and similarly use C_ε to denote different constants depending on ε . This

lemma is to be proven by the multiplier method. Firstly we introduce a multiplier \mathcal{M} , which is a $C^1(\mathbb{R}^{2n})$ function defined by

$$\mathcal{M} = \mathcal{M}(x, y) = \frac{2\partial_x V(x) \cdot y}{\rho(x, y)} \frac{y^2}{1 + y^2},$$

with ρ given the statement of Lemma 2.1. Recall that $X_0 = y \cdot \partial_x - \partial_x V(x) \cdot \partial_y$. Then using the relation

$$\frac{1}{2} [\mathcal{M}, X_0] = \frac{\partial_x V(x) \cdot y}{\rho(x, y)} \left[\frac{y^2}{1 + y^2}, X_0 \right] + \left[\frac{\partial_x V(x) \cdot y}{\rho(x, y)}, X_0 \right] \frac{y^2}{1 + y^2},$$

we calculate

$$\begin{aligned} \frac{1}{2} [\mathcal{M}, X_0] &= \frac{2|\partial_x V(x) \cdot y|^2}{\rho(x, y)(1 + y^2)^2} + \frac{(1 + |y|^8)|\partial_x V(x)|^2}{\rho^3} \frac{y^2}{1 + y^2} \\ &\quad - \frac{(1 + |y|^8) \sum_{1 \leq i, j \leq n} (\partial_{x_i x_j} V) y_i y_j}{\rho^3} \frac{y^2}{1 + y^2} \\ &\quad - \frac{4|\partial_x V(x) \cdot y|^2 |y|^6}{\rho^3} \frac{y^2}{1 + y^2}. \end{aligned}$$

As a result, observe that

$$\operatorname{Re} \langle X_0 u, \mathcal{M} u \rangle_{L^2} = \frac{1}{2} \langle [\mathcal{M}, X_0] u, u \rangle_{L^2},$$

and thus

$$\begin{aligned} \operatorname{Re} \langle X_0 u, \mathcal{M} u \rangle_{L^2} &= \left\langle \frac{(1 + |y|^8)|\partial_x V(x)|^2}{\rho^3} \frac{y^2}{1 + y^2} u, u \right\rangle_{L^2} + \left\langle \frac{2|\partial_x V(x) \cdot y|^2}{\rho(x, y)(1 + y^2)^2} u, u \right\rangle_{L^2} \\ &\quad - \left\langle \frac{(1 + |y|^8) \sum_{1 \leq i, j \leq n} (\partial_{x_i x_j} V) y_i y_j}{\rho^3} \frac{y^2}{1 + y^2} u, u \right\rangle_{L^2} \\ &\quad - \left\langle \frac{4|\partial_x V(x) \cdot y|^2 |y|^6}{\rho^3} \frac{y^2}{1 + y^2} u, u \right\rangle_{L^2}. \end{aligned}$$

This, along with the following inequalities

$$\begin{aligned} \left\langle \frac{2\rho}{(1 + y^2)^2} u, u \right\rangle_{L^2} &= \left\langle \frac{2|\partial_x V(x) \cdot y|^2}{\rho(x, y)(1 + y^2)^2} u, u \right\rangle_{L^2} + \left\langle \frac{2(1 + |y|^8)}{\rho(x, y)(1 + y^2)^2} u, u \right\rangle_{L^2} \\ &\leq \left\langle \frac{2|\partial_x V(x) \cdot y|^2}{\rho(x, y)(1 + y^2)^2} u, u \right\rangle_{L^2} + C \|u\|_{L^2}^2 \end{aligned}$$

and

$$\left| \left\langle \frac{4 |\partial_x V(x) \cdot y|^2 |y|^6}{\rho^3} \frac{y^2}{1+y^2} u, u \right\rangle_{L^2} \right| \leq C \|\langle y \rangle u\|_{L^2}^2 \leq C \left(\|Pu\|_{L^2}^2 + \|u\|_{L^2}^2 \right),$$

yields

$$\begin{aligned} & \left\langle \frac{(1+|y|^8) |\partial_x V(x)|^2}{\rho^3} \frac{y^2}{1+y^2} u, u \right\rangle_{L^2} + \left\langle \frac{2\rho}{(1+y^2)^2} u, u \right\rangle_{L^2} \\ & - \left\langle \frac{(1+|y|^8) \sum_{1 \leq i, j \leq n} (\partial_{x_i x_j} V) y_i y_j}{\rho^3} \frac{y^2}{1+y^2} u, u \right\rangle_{L^2} \quad (2.4) \\ & \leq \operatorname{Re} \langle X_0 u, \mathcal{M} u \rangle_{L^2} + C \left(\|Pu\|_{L^2}^2 + \|u\|_{L^2}^2 \right). \end{aligned}$$

On the other hand, since $\mathcal{M} \in L^\infty(\mathbb{R}^{2n})$ with $\|\mathcal{M}\|_{L^\infty} \leq 2$ then it is easy to see that

$$|\operatorname{Re} \langle Pu, \mathcal{M} u \rangle_{L^2}| \leq \|Pu\|_{L^2}^2 + \|u\|_{L^2}^2. \quad (2.5)$$

Recall that $X_j = \partial_{y_j} + \frac{y_j}{2}$, with $j = 1, \dots, n$. Then a direct computation gives, for each $1 \leq j \leq n$,

$$\begin{aligned} [\mathcal{M}, X_j] &= -\frac{2(1+|y|^8) (\partial_{x_j} V(x))}{\rho^3} \frac{y^2}{(1+y^2)} + \frac{8(\partial_x V(x) \cdot y) |y|^6 y_j}{\rho^3} \frac{y^2}{(1+y^2)} \\ & \quad - \frac{4(\partial_x V(x) \cdot y) y_j}{\rho(x, y)(1+y^2)^2}. \end{aligned}$$

Thus, for any $\varepsilon > 0$,

$$\begin{aligned} & |\operatorname{Re} \langle X_j u, [\mathcal{M}, X_j] u \rangle_{L^2}| \\ & \leq \varepsilon \|\rho^{-3} (1+y^2)^{-1} (1+|y|^8) y^2 |\partial_{x_j} V(x)| u\|_{L^2}^2 + C_\varepsilon \left(\|X_j u\|_{L^2}^2 + \|u\|_{L^2}^2 \right) \\ & \leq \varepsilon \left\langle \frac{(1+|y|^8) |\partial_{x_j} V(x)|^2}{\rho^3} \frac{y^2}{1+y^2} u, u \right\rangle_{L^2} + C_\varepsilon \left(\|X_j u\|_{L^2}^2 + \|u\|_{L^2}^2 \right). \end{aligned}$$

This gives, using $\|\mathcal{M}\|_{L^\infty} \leq 2$ and (2.2),

$$\begin{aligned} \left| \operatorname{Re} \left\langle X_j^* X_j u, \mathcal{M} u \right\rangle_{L^2} \right| &\leq \left| \operatorname{Re} \left\langle X_j u, \mathcal{M} X_j u \right\rangle_{L^2} \right| + \left| \operatorname{Re} \left\langle X_j u, [\mathcal{M}, X_j] u \right\rangle_{L^2} \right| \\ &\leq \varepsilon \left\langle \frac{(1 + |y|^8) |\partial_{x_j} V(x)|^2}{\rho^3} \frac{y^2}{1 + y^2} u, u \right\rangle_{L^2} \\ &\quad + C_\varepsilon \left(\|X_j u\|_{L^2}^2 + \|u\|_{L^2}^2 \right) \\ &\leq \varepsilon \left\langle \frac{(1 + |y|^8) |\partial_{x_j} V(x)|^2}{\rho^3} \frac{y^2}{1 + y^2} u, u \right\rangle_{L^2} \\ &\quad + C_\varepsilon \left(\|P u\|_{L^2}^2 + \|u\|_{L^2}^2 \right). \end{aligned}$$

Then

$$\begin{aligned} \sum_{j=1}^n \left| \operatorname{Re} \left\langle X_j^* X_j u, \mathcal{M} u \right\rangle_{L^2} \right| &\leq \varepsilon \left\langle \frac{(1 + |y|^8) |\partial_x V(x)|^2}{\rho^3} \frac{y^2}{1 + y^2} u, u \right\rangle_{L^2} \\ &\quad + C_\varepsilon \left(\|P u\|_{L^2}^2 + \|u\|_{L^2}^2 \right). \end{aligned}$$

Consequently, from (2.4), (2.5) and the relationship

$$\operatorname{Re} \langle X_0 u, \mathcal{M} u \rangle_{L^2} = \operatorname{Re} \langle P u, \mathcal{M} u \rangle_{L^2} - \operatorname{Re} \sum_{j=1}^n \left\langle X_j^* X_j u, \mathcal{M} u \right\rangle_{L^2},$$

it follows that

$$\begin{aligned} &\left\langle \frac{(1 + |y|^8) |\partial_x V(x)|^2}{\rho^3} \frac{y^2}{1 + y^2} u, u \right\rangle_{L^2} + \left\langle \frac{2\rho}{(1 + y^2)^2} u, u \right\rangle_{L^2} \\ &\quad - \left\langle \frac{(1 + |y|^8) \sum_{1 \leq i, j \leq n} (\partial_{x_i x_j} V) y_i y_j}{\rho^3} \frac{y^2}{1 + y^2} u, u \right\rangle_{L^2} \\ &\leq \varepsilon \left\langle \frac{(1 + |y|^8) |\partial_x V(x)|^2}{\rho^3} \frac{y^2}{1 + y^2} u, u \right\rangle_{L^2} + C_\varepsilon \left(\|P u\|_{L^2}^2 + \|u\|_{L^2}^2 \right). \end{aligned}$$

As result, for all σ with $0 < \sigma < 1$, letting $\varepsilon = 1 - \sigma$ gives the desired estimate in Lemma 2.1. The proof is thus complete. \square

The remaining part of this subsection is devoted to the proof of Theorem 1.2.

Proof of Theorem 1.2. For the symmetric Hessian matrix $(\partial_{x_i x_j} V)_{1 \leq i, j \leq n}$, we can find a $n \times n$ orthogonal matrix $Q(x) = (q_{ij}(x))_{1 \leq i, j \leq n}$ such that

$$Q^T \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} Q = (\partial_{x_i x_j} V)_{1 \leq i, j \leq n}, \tag{2.6}$$

where $\lambda_j, 1 \leq j \leq n$ are the eigenvalues of the Hessian $(\partial_{x_i x_j} V)_{1 \leq i, j \leq n}$. Then for any $x \in \mathbb{R}^n$ we can write

$$-\sum_{j \notin I_x} \lambda_j(x) [(Q(x)y)_j]^2 = -\sum_{1 \leq i, j \leq n} (\partial_{x_i x_j} V(x)) y_i y_j + \sum_{j \in I_x} \lambda_j(x) [(Q(x)y)_j]^2, \tag{2.7}$$

where $(Q(x)y)_j$ stands for the j -th component of the vector $Q(x)y$, and

$$I_x = \{1 \leq \ell \leq n; \lambda_\ell(x) > 0\}.$$

Thus it follows from (2.7) and the assumption (1.4) that , for any $x \in \mathbb{R}^n$,

$$\sum_{j \notin I_x} (-\lambda_j(x)) [(Q(x)y)_j]^2 \leq -\sum_{1 \leq i, j \leq n} (\partial_{x_i x_j} V(x)) y_i y_j + C \langle \partial_x V(x) \rangle^{4/3} |y|^2.$$

This together with the estimate in Lemma 2.1 yields, for all $0 < \sigma < 1$ and for any $\varepsilon > 0$,

$$\begin{aligned} & \sigma \left\langle \frac{(1 + |y|^8) \langle \partial_x V(x) \rangle^2}{\rho^3} \frac{y^2}{1 + y^2} u, u \right\rangle_{L^2} + \left\langle \frac{\rho}{\langle y \rangle^4} u, u \right\rangle_{L^2} \\ & + \int_{\mathbb{R}^n} \left(\sum_{j \notin I_x} \int_{\mathbb{R}^n} \frac{(1 + |y|^8) (-\lambda_j(x)) [(Q(x)y)_j]^2}{\rho^3} \frac{y^2}{1 + y^2} u^2 dy \right) dx \\ & \leq C \left\langle \frac{(1 + |y|^8) \langle \partial_x V(x) \rangle^{4/3} |y|^2}{\rho^3} \frac{y^2}{1 + y^2} u, u \right\rangle_{L^2} + C_\sigma \left(\|Pu\|_{L^2}^2 + \|u\|_{L^2}^2 \right) \\ & \leq \varepsilon \left\langle \frac{(1 + |y|^8) \langle \partial_x V(x) \rangle^2}{\rho^3} \frac{y^2}{1 + y^2} u, u \right\rangle_{L^2} + C_\varepsilon \left\langle \frac{(1 + |y|^8) \langle y \rangle^6}{\rho^3} \frac{y^2}{1 + y^2} u, u \right\rangle_{L^2} \\ & + C_\sigma \left(\|Pu\|_{L^2}^2 + \|u\|_{L^2}^2 \right) \\ & \leq \varepsilon \left\langle \frac{(1 + |y|^8) \langle \partial_x V(x) \rangle^2}{\rho^3} \frac{y^2}{1 + y^2} u, u \right\rangle_{L^2} + C_{\varepsilon, \sigma} \left(\|Pu\|_{L^2}^2 + \|u\|_{L^2}^2 \right), \end{aligned}$$

the second and last inequalities holding because

$$\langle \partial_x V(x) \rangle^{4/3} |y|^2 \leq \varepsilon \langle \partial_x V(x) \rangle^2 + C_\varepsilon |y|^6$$

and

$$\left\langle \frac{(1 + |y|^8) \langle y \rangle^6}{\rho^3} \frac{y^2}{1 + y^2} u, u \right\rangle_{L^2} \leq C \| \langle y \rangle u \|_{L^2}^2 \leq C \left(\| Pu \|_{L^2}^2 + \| u \|_{L^2}^2 \right).$$

Now letting $\varepsilon = \frac{\sigma}{2}$, we obtain

$$\begin{aligned} & \frac{\sigma}{2} \left\langle \frac{(1 + |y|^8) \langle \partial_x V(x) \rangle^2}{\rho^3} \frac{y^2}{1 + y^2} u, u \right\rangle_{L^2} + \left\langle \frac{\rho}{\langle y \rangle^4} u, u \right\rangle_{L^2} \\ & + \int_{\mathbb{R}^n} \left(\sum_{j \notin I_x} \int_{\mathbb{R}^n} \frac{(1 + |y|^8) (-\lambda_j(x)) [(Q(x)y)_j]^2}{\rho^3} \frac{y^2}{1 + y^2} u^2 dy \right) dx \\ & \leq C_\sigma \left(\| Pu \|_{L^2}^2 + \| u \|_{L^2}^2 \right), \end{aligned}$$

and thus, choosing $\sigma = 1/2$,

$$\begin{aligned} & \left\langle \frac{(1 + |y|^8) \langle \partial_x V(x) \rangle^2}{\rho^3} \frac{y^2}{1 + y^2} u, u \right\rangle_{L^2} + \left\langle \frac{\rho}{\langle y \rangle^4} u, u \right\rangle_{L^2} \\ & + \int_{\mathbb{R}^n} \left(\sum_{j \notin I_x} \int_{\mathbb{R}^n} \frac{(-\lambda_j(x)) [(Q(x)y)_j]^2 y^2}{\langle \partial_x V(x) \rangle^3 \langle y \rangle^6} u^2 dy \right) dx \quad (2.8) \\ & \leq C \left(\| Pu \|_{L^2}^2 + \| u \|_{L^2}^2 \right) \end{aligned}$$

due to the fact that $-\lambda_j(x) \geq 0$ for $j \notin I_x$ and

$$\frac{1}{\langle \partial_x V(x) \rangle^3 \langle y \rangle^6} \leq C \frac{(1 + |y|^8)}{\rho^3} \frac{1}{1 + y^2}.$$

In the following discussion we will provide the lower bound of the summation on the left side of (2.8). To do so, we use the estimates

$$\begin{aligned} \langle \partial_x V(x) \rangle^{1/4} \frac{y^2}{1 + y^2} & \leq \left(\frac{1}{8} \frac{\langle \partial_x V(x) \rangle^2 \langle y \rangle^8}{\rho^3} + \frac{3}{8} \frac{\rho}{\langle y \rangle^4} + \frac{\langle y \rangle}{2} \right) \frac{y^2}{1 + y^2} \\ & \leq C \frac{\langle \partial_x V(x) \rangle^2 (1 + |y|^8)}{\rho^3} \frac{y^2}{1 + y^2} + \frac{\rho}{\langle y \rangle^4} + \langle y \rangle, \end{aligned}$$

together with (2.8), to conclude that

$$\begin{aligned} \sum_{j=1}^n \left\| \langle \partial_x V(x) \rangle^{\frac{1}{8}} y_j \langle y \rangle^{-1} u \right\|_{L^2}^2 &\leq C \left(\|Pu\|_{L^2}^2 + \|\langle y \rangle^{1/2} u\|_{L^2}^2 \right) \\ &\leq C \left(\|Pu\|_{L^2}^2 + \|u\|_{L^2}^2 \right). \end{aligned} \quad (2.9)$$

Moreover applying (2.3) with $v = \langle y \rangle^{-1} u$ and $\theta(x) \cdot y = \langle \partial_x V(x) \rangle^{\frac{1}{8}} y_j$, we get

$$\begin{aligned} \left\| \langle \partial_x V(x) \rangle^{\frac{1}{16}} \langle y \rangle^{-1} u \right\|_{L^2}^2 &\leq C \left(\left\| \langle \partial_x V(x) \rangle^{\frac{1}{8}} y_j \langle y \rangle^{-1} u \right\|_{L^2}^2 + \left\| \partial_{y_j} \langle y \rangle^{-1} u \right\|_{L^2}^2 \right) \\ &\leq C \left(\left\| \langle \partial_x V(x) \rangle^{\frac{1}{8}} y_j \langle y \rangle^{-1} u \right\|_{L^2}^2 + \left\| \partial_{y_j} u \right\|_{L^2}^2 + \|u\|_{L^2}^2 \right) \\ &\leq C \left(\|Pu\|_{L^2}^2 + \|u\|_{L^2}^2 \right), \end{aligned}$$

the last inequality following from (2.9) and (2.2). As a result, observe

$$\begin{aligned} &\left\| \langle \partial_x V(x) \rangle^{\frac{1}{16}} u \right\|_{L^2}^2 \\ &= \left(\langle \partial_x V(x) \rangle^{\frac{1}{16}} \frac{1}{1 + |y|^2} u, \langle \partial_x V(x) \rangle^{\frac{1}{16}} u \right)_{L^2} \\ &\quad + \sum_{j=1}^n \left(\langle \partial_x V(x) \rangle^{\frac{1}{16}} \frac{y_j^2}{1 + |y|^2} u, \langle \partial_x V(x) \rangle^{\frac{1}{16}} u \right)_{L^2} \\ &= \left\| \langle \partial_x V(x) \rangle^{\frac{1}{16}} \langle y \rangle^{-1} u \right\|_{L^2}^2 + \sum_{j=1}^n \left\| \langle \partial_x V(x) \rangle^{\frac{1}{16}} y_j \langle y \rangle^{-1} u \right\|_{L^2}^2 \\ &\leq \left\| \langle \partial_x V(x) \rangle^{\frac{1}{16}} \langle y \rangle^{-1} u \right\|_{L^2}^2 + \sum_{j=1}^n \left\| \langle \partial_x V(x) \rangle^{\frac{1}{8}} y_j \langle y \rangle^{-1} u \right\|_{L^2}^2, \end{aligned}$$

and thus combining the above inequalities and (2.9), we obtain

$$\left\| \langle \partial_x V(x) \rangle^{\frac{1}{16}} u \right\|_{L^2}^2 \leq C \left(\|Pu\|_{L^2}^2 + \|u\|_{L^2}^2 \right). \quad (2.10)$$

Then the conclusion (i) in Theorem 1.2 follows.

Now we prove the statement (ii). Let $x \in \mathbb{R}^n$ be given and let $1 \leq i, \ell \leq n$ and $j \notin I_x$. Recall $Q(x) = (q_{k\ell}(x))_{1 \leq k, \ell \leq n}$. Similarly as above, applying again (2.3) with

$$\begin{aligned} v &= y_\ell \langle y \rangle^{-3} u, \text{ and } \theta(x) \cdot y = (-\lambda_j(x))^{1/2} \langle \partial_x V(x) \rangle^{-\frac{3}{2}} (Q(x)y)_j \\ &= (-\lambda_j(x))^{1/2} \langle \partial_x V(x) \rangle^{-\frac{3}{2}} \sum_{k=1}^n q_{jk} y_k, \end{aligned}$$

we have

$$\begin{aligned}
& \| (-\lambda_j(x))^{1/4} \langle \partial_x V(x) \rangle^{-3/4} |q_{ji}(x)|^{1/2} y_\ell \langle y \rangle^{-3} u \|_{L^2(\mathbb{R}_y^n)}^2 \\
& \leq C \left(\| (-\lambda_j(x))^{1/2} \langle \partial_x V(x) \rangle^{-3/2} [(Q(x)y)_j] y_\ell \langle y \rangle^{-3} u \|_{L^2(\mathbb{R}_y^n)}^2 \right. \\
& \quad \left. + \| \partial_{y_i} y_\ell \langle y \rangle^{-3} u \|_{L^2(\mathbb{R}_y^n)}^2 \right) \\
& \leq C \int_{\mathbb{R}^n} \frac{(-\lambda_j(x)) [(Q(x)y)_j]^2 y^2}{\langle \partial_x V(x) \rangle^3 \langle y \rangle^6} u^2 dy + C \left(\| \partial_{y_i} u \|_{L^2(\mathbb{R}_y^n)}^2 + \| u \|_{L^2(\mathbb{R}_y^n)}^2 \right).
\end{aligned}$$

Thus, combining (2.8) and (2.2),

$$\begin{aligned}
& \int_{\mathbb{R}^n} \left(\sum_{i=1}^n \sum_{j \notin I_x} \| (-\lambda_j(x))^{1/4} \langle \partial_x V(x) \rangle^{-3/4} |q_{ji}(x)|^{1/2} y_\ell \langle y \rangle^{-3} u \|_{L^2(\mathbb{R}_y^n)}^2 \right) dx \\
& \leq C \int_{\mathbb{R}^n} \left(\sum_{j \notin I_x} \int_{\mathbb{R}^n} \frac{(-\lambda_j(x)) [(Q(x)y)_j]^2 y^2}{\langle \partial_x V(x) \rangle^3 \langle y \rangle^6} u^2 dy \right) dx \tag{2.11} \\
& \quad + C \sum_{i=1}^n \left(\| \partial_{y_i} u \|_{L^2}^2 + \| u \|_{L^2}^2 \right) \\
& \leq C \left(\| Pu \|_{L^2}^2 + \| u \|_{L^2}^2 \right).
\end{aligned}$$

Moreover, using again (2.3) with

$$v = \langle y \rangle^{-3} u \text{ and } \theta \cdot y = (-\lambda_j(x))^{1/4} \langle \partial_x V(x) \rangle^{-3/4} |q_{ji}(x)|^{1/2} y_\ell,$$

we obtain

$$\begin{aligned}
& \int_{\mathbb{R}^n} \left(\sum_{i=1}^n \sum_{j \notin I_x} \| (-\lambda_j(x))^{1/8} \langle \partial_x V(x) \rangle^{-3/8} |q_{ji}(x)|^{1/4} \langle y \rangle^{-3} u \|_{L^2(\mathbb{R}_y^n)}^2 \right) dx \\
& \leq \int_{\mathbb{R}^n} \left(\sum_{i=1}^n \sum_{j \notin I_x} \| (-\lambda_j(x))^{1/4} \langle \partial_x V(x) \rangle^{-3/4} |q_{ji}(x)|^{1/2} y_\ell \langle y \rangle^{-3} u \|_{L^2(\mathbb{R}_y^n)}^2 \right) dx \tag{2.12} \\
& \quad + C \| \partial_{y_\ell} \langle y \rangle^{-3} u \|_{L^2}^2 \\
& \leq C \left(\| Pu \|_{L^2}^2 + \| u \|_{L^2}^2 \right),
\end{aligned}$$

the last inequality following from (2.11) and (2.2). On the other hand, in view of (2.6) we see that

$$\sum_{i=1}^n \sum_{j=1}^n \lambda_j(x) (q_{ji}(x))^2 = \sum_{j=1}^n \lambda_j(x) = \Delta_x V(x). \tag{2.13}$$

Then, by the assumption (1.5) in Theorem 1.2, we can find a constant C_α depending on α , such that

$$\begin{aligned} 0 &\leq \alpha |\partial_x V(x)|^2 - \Delta_x V(x) + C_\alpha \\ &\leq C_\alpha \langle \partial_x V(x) \rangle^2 + \sum_{i=1}^n \sum_{j \notin I_x}^n (-\lambda_j(x)) |q_{ji}(x)|^2, \text{ for all } x \in \mathbb{R}^n, \end{aligned}$$

the last inequality following from (2.13). And thus for any $x \in \mathbb{R}^n$,

$$\left| \alpha |\partial_x V(x)|^2 - \Delta_x V(x) \right|^{1/4} \leq C \langle \partial_x V(x) \rangle^{1/2} + C \sum_{i=1}^n \sum_{j \notin I_x}^n (-\lambda_j(x))^{1/4} |q_{ji}(x)|^{1/2},$$

which, together with (2.12), yields

$$\begin{aligned} &\left\langle \frac{|\alpha |\partial_x V(x)|^2 - \Delta_x V(x)|^{1/4}}{\langle \partial_x V(x) \rangle^{3/4} \langle y \rangle^6} u, u \right\rangle_{L^2} \\ &\leq C \left\langle \frac{\langle \partial_x V(x) \rangle^{1/2}}{\langle \partial_x V(x) \rangle^{3/4} \langle y \rangle^6} u, u \right\rangle_{L^2} \\ &\quad + C \int_{\mathbb{R}^n} \left(\sum_{i=1}^n \sum_{j \notin I_x} \left\langle \frac{(-\lambda_j(x))^{1/4} |q_{ji}(x)|^{1/2}}{\langle \partial_x V(x) \rangle^{3/4} \langle y \rangle^6} u, u \right\rangle_{L^2(\mathbb{R}_y^n)} \right) dx \\ &\leq C \left(\|Pu\|_{L^2}^2 + \|u\|_{L^2}^2 \right). \end{aligned}$$

As a result, we conclude, combining (2.10), (2.2) and the above inequality,

$$\| |\alpha |\partial_x V(x)|^2 - \Delta_x V(x) |^{1/80} u \|_{L^2}^2 \leq C \left(\|Pu\|_{L^2}^2 + \|u\|_{L^2}^2 \right),$$

due to the estimate

$$\begin{aligned} |\alpha |\partial_x V(x)|^2 - \Delta_x V(x) |^{1/40} &\leq \frac{1}{10} \frac{|\alpha |\partial_x V(x)|^2 - \Delta_x V(x)|^{1/4}}{\langle \partial_x V(x) \rangle^{3/4} \langle y \rangle^6} \\ &\quad + \frac{3}{5} \langle \partial_x V(x) \rangle^{1/8} + \frac{3}{10} \langle y \rangle^2. \end{aligned}$$

Thus the proof of Theorem 1.2 is complete. \square

2.2. Proof of Theorem 1.3

This subsection is devoted to proving Theorem 1.3. Similarly as Lemma 2.1 we have the following

Lemma 2.2. *Let $V(x) \in C^2(\mathbb{R}^n)$. Then for all $\sigma \in]0, 1[$ there exists a constant $C_\sigma > 0$ such that for any $u \in C_0^\infty(\mathbb{R}^{2n})$ we have*

$$\begin{aligned} & \sigma \left\langle \frac{\langle \partial_x V(x) \rangle^2}{\langle \partial_x V(x) \cdot y \rangle^3} \frac{y^2}{1+y^2} u, u \right\rangle_{L^2} + \left\langle \frac{2 \langle \partial_x V(x) \cdot y \rangle}{\langle y \rangle^4} u, u \right\rangle_{L^2} \\ & \quad - \sum_{1 \leq i, j \leq n} \left\langle \frac{(\partial_{x_i x_j} V) y_i y_j}{\langle \partial_x V(x) \cdot y \rangle^3} \frac{y^2}{1+y^2} u, u \right\rangle_{L^2} \\ & \leq C_\sigma \left(\|Pu\|_{L^2}^2 + \|u\|_{L^2}^2 \right). \end{aligned}$$

Proof. The proof is quite similar to Lemma 2.1. Let $\mathcal{K} \in C^1(\mathbb{R}^{2n})$ be defined by

$$\mathcal{K} = \mathcal{K}(x, y) = \frac{2\partial_x V(x) \cdot y}{\langle \partial_x V(x) \cdot y \rangle} \frac{y^2}{1+y^2}.$$

Then using the relation

$$\frac{1}{2}[\mathcal{K}, X_0] = \frac{\partial_x V(x) \cdot y}{\langle \partial_x V(x) \cdot y \rangle} \left[\frac{y^2}{(1+y^2)}, X_0 \right] + \left[\frac{\partial_x V(x) \cdot y}{\langle \partial_x V(x) \cdot y \rangle}, X_0 \right] \frac{y^2}{(1+y^2)},$$

we obtain

$$\begin{aligned} \frac{1}{2}[\mathcal{K}, X_0] &= \frac{2|\partial_x V(x) \cdot y|^2}{\langle \partial_x V(x) \cdot y \rangle (1+y^2)^2} \\ & \quad + \left(\frac{|\partial_x V(x)|^2}{\langle \partial_x V(x) \cdot y \rangle^3} - \sum_{1 \leq i, j \leq n} \frac{(\partial_{x_i x_j} V) y_i y_j}{\langle \partial_x V(x) \cdot y \rangle^3} \right) \frac{y^2}{1+y^2}. \end{aligned}$$

Thus using the relation

$$\operatorname{Re} \langle X_0 u, \mathcal{K} u \rangle_{L^2} = \frac{1}{2} \langle [\mathcal{K}, X_0] u, u \rangle_{L^2},$$

we conclude

$$\begin{aligned} & \left\langle \frac{\langle \partial_x V(x) \rangle^2}{\langle \partial_x V(x) \cdot y \rangle^3} \frac{y^2}{1+y^2} u, u \right\rangle_{L^2} + \left\langle \frac{2 \langle \partial_x V(x) \cdot y \rangle}{(1+y^2)^2} u, u \right\rangle_{L^2} \\ & \quad - \sum_{1 \leq i, j \leq n} \left\langle \frac{(\partial_{x_i x_j} V) y_i y_j}{\langle \partial_x V(x) \cdot y \rangle^3} \frac{y^2}{1+y^2} u, u \right\rangle_{L^2} \tag{2.14} \\ & \leq \operatorname{Re} \langle X_0 u, \mathcal{K} u \rangle_{L^2} + 3 \|u\|_{L^2}^2. \end{aligned}$$

On the other hand, since $\mathcal{K} \in L^\infty(\mathbb{R}^{2n})$ with $\|\mathcal{K}\|_{L^\infty} \leq 2$ then it is easy to see that

$$|\operatorname{Re} \langle Pu, \mathcal{K} u \rangle_{L^2}| \leq \|Pu\|_{L^2}^2 + \|u\|_{L^2}^2. \tag{2.15}$$

Recall that $X_j = \partial_{y_j} + \frac{y_j}{2}$, for $j = 1, \dots, n$. Then direct computation gives, for each $1 \leq j \leq n$,

$$[\mathcal{K}, X_j] = -\frac{2\partial_{x_j} V(x)}{\langle \partial_x V(x) \cdot y \rangle^3} \frac{y^2}{1+y^2} - \frac{4\langle \partial_x V(x) \cdot y \rangle y_j}{\langle \partial_x V(x) \cdot y \rangle (1+y^2)^2}.$$

As result, for any $\varepsilon > 0$,

$$\begin{aligned} & |\operatorname{Re} \langle X_j u, [\mathcal{K}, X_j] u \rangle_{L^2}| \\ & \leq \varepsilon \|\langle \partial_x V(x) \cdot y \rangle^{-3} (1+y^2)^{-1} y^2 |\partial_{x_j} V(x)| u\|_{L^2}^2 + C_\varepsilon \left(\|X_j u\|_{L^2}^2 + \|u\|_{L^2}^2 \right) \\ & \leq \varepsilon \left\langle \frac{|\partial_{x_j} V(x)|^2}{\langle \partial_x V(x) \cdot y \rangle^3} \frac{y^2}{1+y^2} u, u \right\rangle_{L^2} + C_\varepsilon \left(\|X_j u\|_{L^2}^2 + \|u\|_{L^2}^2 \right). \end{aligned}$$

This yields, using again the facts that $\|\mathcal{K}\|_{L^\infty} \leq 2$ and (2.2), that

$$\begin{aligned} \sum_{j=1}^n \left| \operatorname{Re} \langle X_j^* X_j u, \mathcal{K} u \rangle_{L^2} \right| & \leq \sum_{j=1}^n \left(\left| \operatorname{Re} \langle X_j u, \mathcal{K} X_j u \rangle_{L^2} \right| \right. \\ & \quad \left. + \left| \operatorname{Re} \langle X_j u, [\mathcal{K}, X_j] u \rangle_{L^2} \right| \right) \\ & \leq \varepsilon \left\langle \frac{|\partial_x V(x)|^2}{\langle \partial_x V(x) \cdot y \rangle^3} \frac{y^2}{1+y^2} u, u \right\rangle_{L^2} \\ & \quad + C_\varepsilon \left(\|Pu\|_{L^2}^2 + \|u\|_{L^2}^2 \right), \end{aligned}$$

which, along with (2.14), (2.15) and the relation

$$\operatorname{Re} \langle X_0 u, \mathcal{K} u \rangle_{L^2} = \operatorname{Re} \langle Pu, \mathcal{K} u \rangle_{L^2} - \operatorname{Re} \sum_{j=1}^n \langle X_j^* X_j u, \mathcal{K} u \rangle_{L^2},$$

implies

$$\begin{aligned} & \left\langle \frac{\langle \partial_x V(x) \rangle^2}{\langle \partial_x V(x) \cdot y \rangle^3} \frac{y^2}{1+y^2} u, u \right\rangle_{L^2} + \left\langle \frac{2\langle \partial_x V(x) \cdot y \rangle}{(1+y^2)^2} u, u \right\rangle_{L^2} \\ & - \sum_{1 \leq i, j \leq n} \left\langle \frac{\langle \partial_{x_i x_j} V \rangle y_i y_j}{\langle \partial_x V(x) \cdot y \rangle^3} \frac{y^2}{1+y^2} u, u \right\rangle_{L^2} \\ & \leq \varepsilon \left\langle \frac{|\partial_x V(x)|^2}{\langle \partial_x V(x) \cdot y \rangle^3} \frac{y^2}{1+y^2} u, u \right\rangle_{L^2} + C_\varepsilon \left(\|Pu\|_{L^2}^2 + \|u\|_{L^2}^2 \right). \end{aligned}$$

Given any $\sigma \in]0, 1[$, letting $\varepsilon = 1 - \sigma$ gives the desired estimate in Lemma 2.2. The proof is thus complete. \square

Lemma 2.3. *Let $\tau \geq 0$ be given. Then for any $\varepsilon > 0$,*

$$\begin{aligned} & \sum_{1 \leq i, j \leq n} \tau \left\langle \frac{\langle \partial_x V(x) \rangle^{4/5} (\partial_{x_i} V(x)) (\partial_{x_j} V(x)) y_i y_j}{\langle \partial_x V(x) \cdot y \rangle^3} \frac{y^2}{1 + y^2} u, u \right\rangle_{L^2} \\ & \leq \varepsilon \left\langle \frac{\langle \partial_x V(x) \rangle^2}{\langle \partial_x V(x) \cdot y \rangle^3} \frac{y^2}{1 + y^2} u, u \right\rangle_{L^2} + \varepsilon \left\langle \frac{\langle \partial_x V(x) \cdot y \rangle}{\langle y \rangle^4} u, u \right\rangle_{L^2} \\ & \quad + C_{\varepsilon, \tau} \left(\|Pu\|_{L^2}^2 + \|u\|_{L^2}^2 \right), \end{aligned}$$

where $C_{\varepsilon, \tau}$ is a constant depending only on ε and τ .

Proof. In the proof we use $C_{\varepsilon, \tau}$ to denote the different constants depending on ε and τ . Direct computation gives

$$\begin{aligned} \tau \frac{\langle \partial_x V \rangle^{4/5} |\partial_x V(x) \cdot y|^2}{\langle \partial_x V(x) \cdot y \rangle^3} \frac{|y|^2}{1 + |y|^2} & \leq \frac{\varepsilon \langle \partial_x V(x) \rangle^2 + C_{\varepsilon, \tau} |\partial_x V(x) \cdot y|^{10/3}}{\langle \partial_x V(x) \cdot y \rangle^3} \frac{|y|^2}{1 + |y|^2} \\ & \leq \varepsilon \frac{\langle \partial_x V(x) \rangle^2}{\langle \partial_x V(x) \cdot y \rangle^3} \frac{|y|^2}{1 + |y|^2} + C_{\varepsilon, \tau} \langle \partial_x V(x) \cdot y \rangle^{1/3} \\ & \leq \varepsilon \frac{\langle \partial_x V(x) \rangle^2}{\langle \partial_x V(x) \cdot y \rangle^3} \frac{|y|^2}{1 + |y|^2} + \varepsilon \frac{\langle \partial_x V(x) \cdot y \rangle}{\langle y \rangle^4} + C_{\varepsilon, \tau} \langle y \rangle^2, \end{aligned}$$

and thus, using (2.2),

$$\begin{aligned} & \tau \left\langle \frac{\langle \partial_x V(x) \rangle^{4/5} |\partial_x V(x) \cdot y|^2}{\langle \partial_x V(x) \cdot y \rangle^3} \frac{y^2}{1 + y^2} u, u \right\rangle_{L^2} \\ & \leq \varepsilon \left\langle \frac{\langle \partial_x V(x) \rangle^2}{\langle \partial_x V(x) \cdot y \rangle^3} \frac{y^2}{1 + y^2} u, u \right\rangle_{L^2} + \varepsilon \left\langle \frac{\langle \partial_x V(x) \cdot y \rangle}{\langle y \rangle^4} u, u \right\rangle_{L^2} \\ & \quad + C_{\varepsilon, \tau} \left(\|Pu\|_{L^2}^2 + \|u\|_{L^2}^2 \right). \end{aligned}$$

Then observing

$$\begin{aligned} & \sum_{1 \leq i, j \leq n} \tau \left\langle \frac{\langle \partial_x V \rangle^{4/5} (\partial_{x_i} V) (\partial_{x_j} V) y_i y_j}{\rho^3} \frac{y^2}{1 + y^2} u, u \right\rangle_{L^2} \\ & = \tau \left\langle \frac{\langle \partial_x V \rangle^{4/5} |\partial_x V \cdot y|^2}{\rho^3} \frac{y^2}{1 + y^2} u, u \right\rangle_{L^2}, \end{aligned}$$

the desired estimate in Lemma 2.3 follows. The proof is complete. \square

The remaining part of this subsection is occupied by the proof of Theorem 1.3.

Proof of Theorem 1.3. By virtue of Lemma 2.2 and Lemma 2.3, we obtain, for all $0 < \sigma < 1$ and for any $\varepsilon > 0$,

$$\begin{aligned} & \sigma \left\langle \frac{\langle \partial_x V(x) \rangle^2}{\langle \partial_x V(x) \cdot y \rangle^3} \frac{y^2}{1+y^2} u, u \right\rangle_{L^2} + \left\langle \frac{2 \langle \partial_x V(x) \cdot y \rangle}{\langle y \rangle^4} u, u \right\rangle_{L^2} \\ & + \sum_{1 \leq i, j \leq n} \left\{ \tau \left\langle \frac{\langle \partial_x V \rangle^{4/5} (\partial_{x_i} V) (\partial_{x_j} V) y_i y_j}{\langle \partial_x V \cdot y \rangle^3} \frac{y^2}{1+y^2} u, u \right\rangle_{L^2} \right. \\ & \quad \left. - \left\langle \frac{(\partial_{x_i x_j} V) y_i y_j}{\langle \partial_x V(x) \cdot y \rangle^3} \frac{y^2}{1+y^2} u, u \right\rangle_{L^2} \right\} \\ & \leq \varepsilon \left\langle \frac{\langle \partial_x V(x) \rangle^2}{\langle \partial_x V(x) \cdot y \rangle^3} \frac{y^2}{1+y^2} u, u \right\rangle_{L^2} + \varepsilon \left\langle \frac{\langle \partial_x V(x) \cdot y \rangle}{\langle y \rangle^4} u, u \right\rangle_{L^2} \\ & \quad + C_{\varepsilon, \tau, \sigma} \left(\|Pu\|_{L^2}^2 + \|u\|_{L^2}^2 \right). \end{aligned}$$

Letting $\varepsilon = \sigma/2$, denoting by C the various constants which may depend on τ and σ , we have

$$\begin{aligned} & \frac{\sigma}{2} \left\langle \frac{\langle \partial_x V(x) \rangle^2}{\langle \partial_x V(x) \cdot y \rangle^3} \frac{y^2}{1+y^2} u, u \right\rangle_{L^2} + \left\langle \frac{\langle \partial_x V(x) \cdot y \rangle}{\langle y \rangle^4} u, u \right\rangle_{L^2} \\ & + \sum_{1 \leq i, j \leq n} \left\{ \tau \left\langle \frac{\langle \partial_x V \rangle^{4/5} (\partial_{x_i} V) (\partial_{x_j} V) y_i y_j}{\langle \partial_x V \cdot y \rangle^3} \frac{y^2}{1+y^2} u, u \right\rangle_{L^2} \right. \\ & \quad \left. - \left\langle \frac{(\partial_{x_i x_j} V) y_i y_j}{\langle \partial_x V(x) \cdot y \rangle^3} \frac{y^2}{1+y^2} u, u \right\rangle_{L^2} \right\} \\ & \leq C \left(\|Pu\|_{L^2}^2 + \|u\|_{L^2}^2 \right), \end{aligned}$$

and thus, using (2.2),

$$\begin{aligned} & \frac{\sigma}{2} \left\langle \frac{\langle \partial_x V(x) \rangle^2}{\langle \partial_x V(x) \cdot y \rangle^3} \frac{y^2}{1+y^2} u, u \right\rangle_{L^2} + \left\langle \frac{\langle \partial_x V(x) \cdot y \rangle}{\langle y \rangle^4} u, u \right\rangle_{L^2} \\ & + \left\langle \frac{y^T A_\tau(x) y}{\langle \partial_x V \cdot y \rangle^3} \frac{y^2}{1+y^2} u, u \right\rangle_{L^2} \\ & \leq C \left(\|Pu\|_{L^2}^2 + \|u\|_{L^2}^2 \right), \end{aligned} \tag{2.16}$$

where $A_\tau(x)$ is the matrix defined in the statement of Theorem 1.3, *i.e.*,

$$\begin{aligned} & A_\tau(x) = \left(a_{ij}^\tau(x) \right)_{1 \leq i, j \leq n} \quad \text{with} \\ & a_{ij}^\tau(x) = \tau \langle \partial_x V(x) \rangle^{\frac{4}{5}} (\partial_{x_i} V(x)) (\partial_{x_j} V(x)) - \partial_{x_i x_j} V(x) + \tau \delta_{ij}. \end{aligned}$$

Now under the assumption that $A_\tau(x)$ is positive-definite, we can find its Cholesky decomposition matrix

$$B_\tau(x) = \left(b_{ij}^\tau(x) \right)_{1 \leq i, j \leq n},$$

satisfying the relation

$$A_\tau = B_\tau^T B_\tau.$$

Then using the following estimate

$$\begin{aligned} & \sum_{\ell=1}^n \sum_{j=1}^n \left\| \sum_{1 \leq k \leq n} b_{jk}^\tau(x) \langle \partial_x V(x) \rangle^{-3/2} y_k y_\ell \langle y \rangle^{-5/2} u \right\|_{L^2}^2 \\ &= \left\langle \frac{B_\tau(x)y}{\langle \partial_x V(x) \rangle^3 \langle y \rangle^3} \frac{y^2}{1+y^2} u, B_\tau(x)y u \right\rangle_{L^2} \\ &\leq C \left\langle \frac{B_\tau(x)y}{\langle \partial_x V(x) \cdot y \rangle^3} \frac{y^2}{1+y^2} u, B_\tau(x)y u \right\rangle_{L^2} = C \left\langle \frac{y^T A_\tau(x)y}{\langle \partial_x V(x) \cdot y \rangle^3} \frac{y^2}{1+y^2} u, u \right\rangle_{L^2} \end{aligned}$$

and (2.16), we have, letting $\sigma = 1/2$, that

$$\begin{aligned} & \left\langle \frac{\langle \partial_x V(x) \rangle^2}{\langle \partial_x V(x) \cdot y \rangle^3} \frac{y^2}{1+y^2} u, u \right\rangle_{L^2} + \left\langle \frac{\langle \partial_x V(x) \cdot y \rangle}{\langle y \rangle^4} u, u \right\rangle_{L^2} \\ & \quad + \sum_{\ell=1}^n \sum_{j=1}^n \left\| \sum_{1 \leq k \leq n} b_{jk}^\tau(x) \langle \partial_x V(x) \rangle^{-3/2} y_k y_\ell \langle y \rangle^{-5/2} u \right\|_{L^2}^2 \quad (2.17) \\ & \leq C \left(\|Pu\|_{L^2}^2 + \|u\|_{L^2}^2 \right). \end{aligned}$$

Moreover observe that

$$\begin{aligned} & \sum_{\ell=1}^n \left\| \langle \partial_x V(x) \rangle^{\frac{1}{10}} y_\ell \langle y \rangle^{-1} u \right\|_{L^2}^2 \\ & \leq \left\langle \frac{|\partial_x V(x)|^2}{\langle \partial_x V(x) \cdot y \rangle^3} \frac{y^2}{1+y^2} u, u \right\rangle_{L^2} + \left\langle \frac{\langle \partial_x V(x) \cdot y \rangle}{\langle y \rangle^4} u, u \right\rangle_{L^2} + \|\langle y \rangle u\|_{L^2}^2, \end{aligned}$$

due to the estimate

$$\begin{aligned} \langle \partial_x V(x) \rangle^{\frac{1}{5}} \frac{y^2}{1+y^2} & \leq \left(\frac{1}{10} \frac{\langle \partial_x V(x) \rangle^2}{\langle \partial_x V(x) \cdot y \rangle^3} + \frac{9}{10} \langle \partial_x V(x) \cdot y \rangle^{1/3} \right) \frac{y^2}{1+y^2} \\ & \leq \frac{\langle \partial_x V(x) \rangle^2}{\langle \partial_x V(x) \cdot y \rangle^3} \frac{y^2}{1+y^2} + \frac{\langle \partial_x V(x) \cdot y \rangle}{\langle y \rangle^4} + \langle y \rangle^2. \end{aligned}$$

Then combining the above inequalities, (2.2) and (2.17), we have

$$\begin{aligned}
 & \sum_{\ell=1}^n \left\| \langle \partial_x V(x) \rangle^{\frac{1}{10}} y_\ell \langle y \rangle^{-1} u \right\|_{L^2}^2 \\
 & + \sum_{\ell=1}^n \sum_{j=1}^n \left\| \sum_{1 \leq k \leq n} b_{jk}^\tau(x) \langle \partial_x V(x) \rangle^{-3/2} y_k y_\ell \langle y \rangle^{-5/2} u \right\|_{L^2}^2 \quad (2.18) \\
 & \leq C \left(\|Pu\|_{L^2}^2 + \|u\|_{L^2}^2 \right).
 \end{aligned}$$

In order to obtain a lower bound of the terms on the left hand side of (2.18), we will use (2.3) with

$$\theta \cdot y = \sum_{\ell=1}^n \langle \partial_x V(x) \rangle^{\frac{1}{10}} y_\ell \quad \text{and} \quad v = \langle y \rangle^{-1} u;$$

this implies that

$$\begin{aligned}
 \left\| \langle \partial_x V(x) \rangle^{\frac{1}{20}} \langle y \rangle^{-1} u \right\|_{L^2} & \leq C \left(\left\| \sum_{\ell=1}^n \langle \partial_x V(x) \rangle^{\frac{1}{10}} y_\ell \langle y \rangle^{-1} u \right\|_{L^2} + \left\| \partial_{y_k} \langle y \rangle^{-1} u \right\|_{L^2} \right) \\
 & \leq C \sum_{\ell=1}^n \left\| \langle \partial_x V(x) \rangle^{\frac{1}{10}} y_\ell \langle y \rangle^{-1} u \right\|_{L^2} + C \left(\|Pu\|_{L^2} + \|u\|_{L^2} \right).
 \end{aligned}$$

As a result, it follows from the above inequalities and (2.18) that

$$\begin{aligned}
 & \left\| \langle \partial_x V(x) \rangle^{\frac{1}{20}} \langle y \rangle^{-1} u \right\|_{L^2}^2 + \sum_{\ell=1}^n \sum_{j=1}^n \left\| \sum_{1 \leq k \leq n} b_{jk}^\tau(x) \langle \partial_x V(x) \rangle^{-3/2} y_k y_\ell \langle y \rangle^{-5/2} u \right\|_{L^2}^2 \\
 & \leq C \left(\|Pu\|_{L^2}^2 + \|u\|_{L^2}^2 \right),
 \end{aligned}$$

and thus, using again (2.18) and repeating the arguments used to prove (2.10),

$$\begin{aligned}
 & \left\| \langle \partial_x V(x) \rangle^{\frac{1}{20}} u \right\|_{L^2}^2 + \sum_{\ell=1}^n \sum_{j=1}^n \left\| \sum_{1 \leq k \leq n} b_{jk}^\tau(x) \langle \partial_x V(x) \rangle^{-3/2} y_k y_\ell \langle y \rangle^{-5/2} u \right\|_{L^2}^2 \quad (2.19) \\
 & \leq C \left(\|Pu\|_{L^2}^2 + \|u\|_{L^2}^2 \right).
 \end{aligned}$$

Similarly, for any $1 \leq j, \ell \leq n$ we use (2.3) again with

$$\theta \cdot y = \sum_{k=1}^n b_{jk}^\tau(x) \langle \partial_x V(x) \rangle^{-3/2} y_k \quad \text{and} \quad v = y_\ell \langle y \rangle^{-5/2} u,$$

to obtain

$$\begin{aligned}
& \sum_{1 \leq p \leq n} \left\| \left| b_{jp}^\tau(x) \right|^{1/2} \langle \partial_x V(x) \rangle^{-3/4} y_\ell \langle y \rangle^{-5/2} u \right\|_{L^2} \\
& \leq C \sum_{1 \leq p \leq n} \left(\left\| \sum_{1 \leq k \leq n} b_{jk}^\tau(x) \langle \partial_x V(x) \rangle^{-3/2} y_k y_\ell \langle y \rangle^{-5/2} u \right\|_{L^2} + \left\| \partial_{y_p} y_\ell \langle y \rangle^{-5/2} u \right\|_{L^2} \right) \\
& \leq C \left\| \sum_{1 \leq k \leq n} b_{jk}^\tau(x) \langle \partial_x V(x) \rangle^{-3/2} y_k y_\ell \langle y \rangle^{-5/2} u \right\|_{L^2} + C \sum_{1 \leq p \leq n} \left(\left\| \partial_{y_p} u \right\|_{L^2} + \|u\|_{L^2} \right) \\
& \leq C \left\| \sum_{1 \leq k \leq n} b_{jk}^\tau(x) \langle \partial_x V(x) \rangle^{-3/2} y_k y_\ell \langle y \rangle^{-5/2} u \right\|_{L^2} + C \left(\|Pu\|_{L^2} + \|u\|_{L^2} \right).
\end{aligned}$$

This, along with (2.19) and (2.18), yields

$$\begin{aligned}
& \left\| \langle \partial_x V(x) \rangle^{\frac{1}{20}} u \right\|_{L^2} + \sum_{\ell=1}^n \left\| \langle \partial_x V(x) \rangle^{\frac{1}{10}} y_\ell \langle y \rangle^{-1} u \right\|_{L^2} \\
& \quad + \sum_{\ell=1}^n \sum_{j=1}^n \sum_{p=1}^n \left\| \left| b_{jp}^\tau(x) \right|^{1/2} \langle \partial_x V(x) \rangle^{-3/4} y_\ell \langle y \rangle^{-5/2} u \right\|_{L^2} \quad (2.20) \\
& \leq C \left(\|Pu\|_{L^2} + \|u\|_{L^2} \right).
\end{aligned}$$

Moreover, observing that

$$\left| b_{jp}^\tau(x) \right|^{1/17} \langle y \rangle^{45/34} \leq \frac{2}{17} \left| b_{jp}^\tau(x) \right|^{1/2} \langle \partial_x V(x) \rangle^{-3/4} + \frac{15}{17} \langle \partial_x V(x) \rangle^{1/10} \langle y \rangle^{3/2}$$

due to Young's inequality, we conclude

$$\begin{aligned}
& \left\| \left| b_{jp}^\tau(x) \right|^{1/17} \langle y \rangle^{45/34} y_\ell \langle y \rangle^{-5/2} u \right\|_{L^2} \\
& \leq C \left\| \left| b_{jp}^\tau(x) \right|^{1/2} \langle \partial_x V(x) \rangle^{-3/4} y_\ell \langle y \rangle^{-5/2} u \right\|_{L^2} \\
& \quad + C \left\| \langle \partial_x V(x) \rangle^{\frac{1}{10}} \langle y \rangle^{3/2} y_\ell \langle y \rangle^{-5/2} u \right\|_{L^2},
\end{aligned}$$

that is,

$$\begin{aligned}
& \left\| \left| b_{jp}^\tau(x) \right|^{\frac{1}{17}} y_\ell \langle y \rangle^{-\frac{20}{17}} u \right\|_{L^2} \\
& \leq C \left\| \left| b_{jp}^\tau(x) \right|^{1/2} \langle \partial_x V(x) \rangle^{-3/4} y_\ell \langle y \rangle^{-5/2} u \right\|_{L^2} + C \left\| \langle \partial_x V(x) \rangle^{\frac{1}{10}} y_\ell \langle y \rangle^{-1} u \right\|_{L^2}.
\end{aligned}$$

Combining the above estimate and (2.20), it follows that

$$\begin{aligned}
& \left\| \langle \partial_x V(x) \rangle^{\frac{1}{20}} u \right\|_{L^2} + \sum_{\ell=1}^n \sum_{j=1}^n \sum_{p=1}^n \left\| \left| b_{jp}^\tau(x) \right|^{\frac{1}{17}} y_\ell \langle y \rangle^{-\frac{20}{17}} u \right\|_{L^2} \quad (2.21) \\
& \leq C \left(\|Pu\|_{L^2} + \|u\|_{L^2} \right).
\end{aligned}$$

Now we use (2.3) to obtain

$$\begin{aligned} \left\| \left| b_{jp}^\tau(x) \right|^{\frac{1}{34}} \langle y \rangle^{-\frac{20}{17}} u \right\|_{L^2} &\leq C \left(\left\| \left| b_{jp}^\tau(x) \right|^{\frac{1}{17}} y_\ell \langle y \rangle^{-\frac{20}{17}} u \right\|_{L^2} + \left\| \partial_{y_\ell} \langle y \rangle^{-\frac{20}{17}} u \right\|_{L^2} \right) \\ &\leq C \left\| \left| b_{jp}^\tau(x) \right|^{\frac{1}{17}} y_\ell \langle y \rangle^{-\frac{20}{17}} u \right\|_{L^2} + C (\|Pu\|_{L^2} + \|u\|_{L^2}), \end{aligned}$$

which together with (2.21) gives

$$\begin{aligned} &\left\| \langle \partial_x V(x) \rangle^{\frac{1}{20}} u \right\|_{L^2} + \sum_{\ell=1}^n \sum_{j=1}^n \sum_{p=1}^n \left\| \left| b_{jp}^\tau(x) \right|^{\frac{1}{17}} y_\ell \langle y \rangle^{-\frac{20}{17}} u \right\|_{L^2} \\ &+ \sum_{j=1}^n \sum_{p=1}^n \left\| \left| b_{jp}^\tau(x) \right|^{\frac{1}{34}} \langle y \rangle^{-\frac{20}{17}} u \right\|_{L^2} \\ &\leq C (\|Pu\|_{L^2} + \|u\|_{L^2}), \end{aligned}$$

that is,

$$\left\| \langle \partial_x V(x) \rangle^{\frac{1}{20}} u \right\|_{L^2} + \sum_{j=1}^n \sum_{p=1}^n \left\| \left| b_{jp}^\tau(x) \right|^{\frac{1}{34}} \langle y \rangle^{-\frac{3}{17}} u \right\|_{L^2} \leq C (\|Pu\|_{L^2} + \|u\|_{L^2}).$$

As a result, using the inequality

$$\left| b_{jp}^\tau(x) \right|^{\frac{1}{40}} \leq \frac{17}{20} \left| b_{jp}^\tau(x) \right|^{\frac{1}{34}} \langle y \rangle^{-\frac{3}{17}} + \frac{3}{20} \langle y \rangle,$$

we conclude

$$\begin{aligned} \left\| \langle \partial_x V(x) \rangle^{\frac{1}{20}} u \right\|_{L^2} + \sum_{j=1}^n \sum_{p=1}^n \left\| \left| b_{jp}^\tau(x) \right|^{\frac{1}{40}} u \right\|_{L^2} &\leq C (\|Pu\|_{L^2} + \|\langle y \rangle u\|_{L^2}) \\ &\leq C (\|Pu\|_{L^2} + \|u\|_{L^2}). \end{aligned} \tag{2.22}$$

Recall that $A_\tau = B_\tau^T B_\tau$, that is

$$a_{ij}^\tau(x) = \sum_{p=1}^n b_{pi}^\tau(x) b_{pj}^\tau(x).$$

Then

$$\sum_{i=1}^n \sum_{j=1}^n \left\| \left| a_{ij}^\tau(x) \right|^{1/80} u \right\|_{L^2} \leq C \sum_{p=1}^n \sum_{j=1}^n \left\| \left| b_{pj}^\tau(x) \right|^{1/40} u \right\|_{L^2}.$$

Thus, combining (2.22), we conclude

$$\| \langle \partial_x V(x) \rangle^{\frac{1}{20}} u \|_{L^2} + \sum_{i=1}^n \sum_{j=1}^n \| |a_{ij}^\tau(x)|^{1/80} u \|_{L^2} \leq C \left(\|Pu\|_{L^2} + \|u\|_{L^2} \right).$$

The proof of Theorem 1.3 is thus complete. \square

References

- [1] R. ALEXANDRE, F. HÉRAU and W.-X. LI, *Global hypoelliptic and symbolic estimates for the linearized Boltzmann operator without angular cutoff*, preprint, arXiv: 1212.4632.
- [2] Y. ALMOG and B. HELFFER, *On the spectrum of non-selfadjoint Schrödinger operators with compact resolvent*, Comm. Partial Differential Equations **40** (2015), 1441–1466.
- [3] P. BOLLEY, J. CAMUS and J. NOURRIGAT, *La condition de Hörmander-Kohn pour les opérateurs pseudo-différentiels*, Comm. Partial Differential Equations **7** (1982), 197–221.
- [4] F. BOUCHUT, *Hypoelliptic regularity in kinetic equations*, J. Math. Pures Appl. **81** (2002), 1135–1159.
- [5] H. CHEN, W.-X. LI and C.-J. XU, *Gevrey hypoellipticity for linear and non-linear Fokker-Planck equations*, J. Differential Equations **246** (2009), 320–339.
- [6] L. DESVILLETES and C. VILLANI, *On the trend to global equilibrium in spatially inhomogeneous entropy-dissipating systems: the linear Fokker-Planck equation*, Comm. Pure Appl. Math. **54** (2001), 1–42.
- [7] B. HELFFER and F. NIER, “Hypoelliptic Estimates and Spectral Theory for Fokker-Planck Operators and Witten Laplacians”, Lecture Notes in Mathematics, Vol. 1862, Springer-Verlag, Berlin, 2005.
- [8] B. HELFFER and J. NOURRIGAT, “Hypoellipticité maximale pour des opérateurs polynômes de champs de vecteurs”, Progress in Mathematics, Vol. 58, Birkhäuser Boston Inc., 1985.
- [9] F. HÉRAU and L. THOMANN, *On global existence and trend to the equilibrium for the Vlasov-Poisson-Fokker-Planck system with exterior confining potential*, preprint, arXiv: 1505.01698.
- [10] F. HÉRAU, *Short and long time behavior of the Fokker-Planck equation in a confining potential and applications*, J. Funct. Anal. **244** (2007), 95–118.
- [11] F. HÉRAU, *Hypoocoercivity and exponential time decay for the linear inhomogeneous relaxation Boltzmann equation*, Asymptot. Anal. **46** (2006), 349–359.
- [12] F. HÉRAU and W.-X. LI, *Global hypoelliptic estimates for Landau-type operator with external potential*, Kyoto J. Math. **53** (2013), 533–565.
- [13] F. HÉRAU and F. NIER, *Isotropic hypoellipticity and trend to equilibrium for the Fokker-Planck equation with a high-degree potential*, Arch. Ration. Mech. Anal. **171** (2004), 151–218.
- [14] L. HÖRMANDER, “The Analysis of Linear Partial Differential Operators”, Springer-Verlag, Berlin, 1985.
- [15] N. LERNER, “Metrics on the Phase Space and Non-selfadjoint Pseudo-differential Operators”, Birkhäuser Verlag, Basel, 2010.
- [16] N. LERNER, *Semi-classical estimates for non-selfadjoint operators*, Asian J. Math. **11** (2007), 217–250.
- [17] W.-X. LI, *Global hypoellipticity and compactness of resolvent for Fokker-Planck operator*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **11** (2012), 789–815.
- [18] F. NIER, *Hypoellipticity for Fokker-Planck operators and Witten Laplacians*, In: “Lectures on the analysis of nonlinear partial differential equations”, Morningside Lect. Math., Vol. 1, Int. Press, Somerville, MA, 2012, Part 1, 31–84.

- [19] F. NIER, *Boundary conditions and subelliptic estimates for geometric Kramers-Fokker-Planck operators on manifolds with boundaries*, preprint, arXiv 1309.5070
- [20] C. VILLANI, “Hypocoercivity”, *Mem. Amer. Math. Soc.*, Vol. 202, 2009, iv+141.
- [21] X. P. WANG, *Large-time asymptotics of solutions to the Kramers-Fokker-Planck equation with a short-range potential*, *Comm. Math. Phys.* **336** (2015), 1435–1471.

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