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# Moderate solutions of semilinear elliptic equations with Hardy potential under minimal restrictions on the potential

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**Abstract.** We study semilinear elliptic equations with Hardy potential

$$-\mathscr{L}_{\mu}u + u^q = 0 \qquad (E)$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^N$. Here $q > 1$, $\mathscr{L}_{\mu} = \Delta + \frac{\mu}{\delta_{\Omega}^2}$ and $\delta_{\Omega}(x) = \mathrm{dist}\,(x, \partial\Omega)$. Assuming that $0 \leq \mu < C_H(\Omega)$, boundary value problems with measure data and discrete boundary singularities for positive solutions of $(E)$ have been studied in [10]. In the case of *convex* domains $C_H(\Omega) = 1/4$. In this case similar problems have been studied in [8]. In the present paper we study these problems, in *arbitrary domains*, assuming only $-\infty < \mu < 1/4$, even if $C_H(\Omega) < 1/4$. We recall that $C_H(\Omega) \leq 1/4$ and, in general, strict inequality holds. The key to our study is the fact that, if $\mu < 1/4$ then in smooth domains there exist local $\mathscr{L}_{\mu}$-superharmonic functions in a neighborhood of $\partial\Omega$ (even if $C_H(\Omega) < 1/4$). Using this fact we extend the notion of *normalized boundary trace*, introduced in [10], to arbitrary domains, provided that $\mu < 1/4$. Further we study the b.v.p. with normalized boundary trace $\nu$ in the space of positive finite measures on $\partial\Omega$. We show that existence depends on two critical values of the exponent $q$ and discuss the question of uniqueness. Part of the paper is devoted to the study of the linear operator: properties of local $\mathscr{L}_{\mu}$-subharmonic and superharmonic functions and the related notion of moderate solutions. Here we extend and/or improve results of [5] and [10] which are later used in the study of the nonlinear problem.

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## 1. Introduction and main results

### 1.1. Introduction

On bounded smooth domains $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ we study semilinear elliptic equations with Hardy potential of the form,

$$-\Delta u - \frac{\mu}{\delta_{\Omega}^2}u + |u|^{q-1}u = 0 \quad \text{in } \Omega, \qquad (P_{\mu})$$

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where $q > 1, -\infty < \mu < 1/4$ and

$$\delta_\Omega(x) := \mathrm{dist}\,(x, \partial\Omega).$$

Equations $(P_0)$ had been extensively studied in the past two decades and by now the structure of the set of positive solutions of such equations is well understood, see [11] and further references therein. Equation $(P_\mu)$ with Hardy potential, *i.e.* with $\mu \neq 0$, had been first considered in [5], where a classification of positive solutions had been introduced and conditions for the existence and nonexistence of *large* solutions for $(P_\mu)$ had been derived.

The study and classification of positive solutions of equation $(P_\mu)$ relies on the properties of the associated linear equation

$$-\mathscr{L}_\mu h = 0 \quad \text{in } \Omega, \tag{1.1}$$

where

$$\mathscr{L}_\mu := \Delta + \frac{\mu}{\delta_\Omega^2}.$$

Denote

$$\alpha_\pm := \frac{1}{2} \pm \sqrt{\frac{1}{4} - \mu}$$

and note that $\alpha_+ + \alpha_- = 1$. For $\rho > 0$ and $\varepsilon \in (0, \rho)$ we use the notation

$$\Omega_\rho := \{x \in \Omega : \delta(x) < \rho\},\ \Omega_{\varepsilon,\rho} := \{x \in \Omega : \varepsilon < \delta(x) < \rho\}$$
$$D_\rho := \{x \in \Omega : \delta(x) > \rho\},\ \Sigma_\rho := \{x \in \Omega : \delta(x) = \rho\}.$$

A function $w \in L^1_{\mathrm{loc}}(G)$ is a $\mathscr{L}_\mu$*-subharmonic* in $\Omega$ if $\mathscr{L}_\mu w \leq 0$ in the distribution sense, *i.e.*,

$$\int_G w(-\Delta\varphi)\,dx - \int_G \frac{\mu}{\delta_\Omega^2} w\varphi\,dx \leq 0 \quad \text{for all} \quad 0 \leq \varphi \in C_c^\infty(\Omega).$$

We say that $w$ is a *local* $\mathscr{L}_\mu$-subharmonic function if there exists $\rho > 0$ such that $w \in L^1_{\mathrm{loc}}(\Omega_\rho)$ is subharmonic in $\Omega_\rho$. Similarly, (local) $\mathscr{L}_\mu$-superharmonic functions are defined with "$\geq$" in the above inequality.

## 1.2. The role of the Hardy constant

The existence and properties of positive $\mathscr{L}_\mu$-harmonic and superharmonic functions in $\Omega$ are controlled by the Hardy constant of the domain, defined as

$$C_H(\Omega) := \inf_{C_c^\infty(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2\,dx}{\int_\Omega \frac{u^2}{\delta_\Omega^2}\,dx}. \tag{1.2}$$

For a bounded Lipschitz domain it is known that $C_H(\Omega) \in (0, 1/4]$. If $\Omega$ is convex then $C_H(\Omega) = 1/4$. In general, $C_H(\Omega)$ varies with the domain and could be arbitrary small (see, *e.g*. [9, Theorem I and Section 4]) for a discussion and examples).

Denote the *local* Hardy constant in $\Omega_\rho$ relative to $\partial\Omega$ by

$$C_H^{\partial\Omega}(\Omega_\rho) := \inf_{C_c^\infty(\Omega_\rho)\setminus\{0\}} \frac{\int_{\Omega_\rho} |\nabla u|^2\, dx}{\int_{\Omega_\rho} \frac{u^2}{\delta_\Omega^2}\, dx}. \tag{1.3}$$

Note the difference between $C_H^{\partial\Omega}(\Omega_\rho)$ and $C_H(\Omega_\rho)$: the distance involved in the first one is $\delta_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$ while in the second it is $\delta_{\Omega_\rho}(x) = \operatorname{dist}(x, \partial\Omega_\rho)$. Obviously $C_H^{\partial\Omega}(\Omega_\rho) \geq C_H(\Omega_\rho)$.

The following lemma shows that in contrast to the "global" Hardy constant $C_H(\Omega)$ the value of the "local" Hardy constant $C_H^{\partial\Omega}(\Omega_\rho)$ does not depend on the shape of $\Omega$, provided that $\rho$ is sufficiently small.

**Lemma 1.1 (local Hardy inequality).** *There exists $\bar\rho = \bar\rho(\Omega) > 0$ such that for every $\rho \in (0, \bar\rho]$ one has $C_H^{\partial\Omega}(\Omega_\rho) = C_H(\Omega_\rho) = 1/4$.*

The fact that $C_H^{\partial\Omega}(\Omega_\rho) = 1/4$ is due to [9, page 3246], while $C_H(\Omega_\rho) = 1/4$ follows from [6, Lemma 1.2].

The relation between the Hardy constant and the existence of positive $\mathscr{L}_\mu$-superharmonics is explained by the following classical result, *cf*. [9, page 3246].

**Lemma 1.2.** *Equation* (1.1) *admits a positive $\mathscr{L}_\mu$-superharmonic function in $\Omega$ if and only if $\mu \leq C_H(\Omega)$.*

*Equation* (1.1) *admits a positive $\mathscr{L}_\mu$-superharmonic in $\Omega_\rho$ with $\rho \in (0, \bar\rho)$ if and only if $\mu \leq 1/4$.*

Thus, according to Lemma 1.1, if $C_H(\Omega) < 1/4$ then, for $\mu \in [C_H(\Omega), 1/4)$, there exist local positive $\mathscr{L}_\mu$-superharmonic functions but no "global" positive $\mathscr{L}_\mu$-superharmonic functions in $\Omega$.

### 1.3. Moderate solutions and normalised boundary trace

In this work we study *moderate* positive solutions of the nonlinear equation $(P_\mu)$ in the range $\mu < 1/4$, including negative values of $\mu$. Recall that in the classical theory of equations $(P_\mu)$ with $\mu = 0$, a moderate solution is a solution which is dominated by a positive harmonic function, *cf*. [11, pages 66-69]. This concept had been extended to equations $(P_\mu)$ with $0 \leq \mu < C_H(\Omega)$ in [10], where an $\mathscr{L}_\mu$-moderate solution is defined as a solution dominated by a positive $\mathscr{L}_\mu$-harmonic function. This definition is not applicable in the range $\mu \in [C_H(\Omega), 1/4)$, when the set of positive $\mathscr{L}_\mu$-harmonic function is empty. Therefore we modify it as follows:

**Definition 1.3.** A solution $u \in L^1_{\text{loc}}(\Omega)$ of equation $(P_\mu)$ is $\mathscr{L}_\mu$*–moderate* if there exists a local positive $\mathscr{L}_\mu$-harmonic function $h$ such that $|u| \leq h$ in $\Omega_\rho$ for some $\rho \in (0, \bar\rho]$.

We are going to show that the nonlinear equation $(P_\mu)$ admits $\mathscr{L}_\mu$-moderate solutions, with prescribed (normalized) boundary data, in the *entire* domain $\Omega$ for every $\mu < 1/4$, even when $C_H(\Omega) < 1/4$. The *existence* of a certain class of positive solutions was observed in [5, Lemma 4.15].

More specifically, we study the generalised boundary trace problem

$$\begin{cases} -\mathscr{L}_\mu u + |u|^{q-1}u = 0 & \text{in } \Omega \\ \mathrm{tr}^*_{\partial\Omega}(u) = \nu, \end{cases} \tag{$P_\mu^\nu$}$$

where $\mu < 1/4, q > 1, \nu \in \mathcal{M}^+(\partial\Omega)$ and $\mathrm{tr}^*_{\partial\Omega}(u)$ denotes the *normalized boundary trace* of a positive Borel function $u$ on $\partial\Omega$. A function $u \in L^q_{\mathrm{loc}}(\Omega)$ is a solution of $(P_\mu^\nu)$ if it satisfies the equation in the distribution sense and attains the indicated boundary data.

The concept of normalised boundary trace was introduced in [10] in order to classify positive moderate solutions of $(P_\mu^\nu)$ in terms of their behaviour at the boundary, when $0 < \mu < C_H(\Omega)$.[1] It is defined as follows.

A nonnegative Borel function $u : \Omega \to \mathbb{R}$ possesses a *normalised boundary trace* $\nu \in \mathfrak{M}^+(\partial\Omega)$ if,

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{\alpha_-}} \int_{\Sigma_\varepsilon} \left|u - \mathbb{K}_\mu^\Omega[\nu]\right| dS = 0 \tag{1.4}$$

where $K_\mu^\Omega$ is the Martin kernel of $\mathscr{L}_\mu$ in $\Omega$. If, for a given $u$ there exists a measure $\nu$ as above then it is unique.

By Ancona [2], if $\mu < C_H(\Omega)$ there is a (1-1) correspondence between the set of positive $\mathscr{L}_\mu$-harmonic functions in $\Omega$ and $\mathfrak{M}^+(\partial\Omega)$; the $\mathscr{L}_\mu$-harmonic function $v$ corresponding to a measure $\nu$ has the representation $v = K_\mu^\Omega[\nu]$. (For details and notation see Subsection 2.1 below.)

We point out that, except in the case $\mu = 0, \mathrm{tr}^*_{\partial\Omega}(u)$ is *not* the standard measure boundary trace of $u$. In fact, when $\mu > 0$, the measure boundary trace of any $\mathscr{L}_\mu$-harmonic function is zero.

In order to extend the definition of normalised boundary trace to arbitrary $\mu < 1/4$ we pick $\rho \in (0, \bar\rho]$ (with $\bar\rho$ as in Lemma 1.1) and employ (1.4) with $K_\mu^{\Omega_\rho}$ instead of $K_\mu^\Omega$. Since $C_H(\Omega_\rho) = 1/4$, $K_\mu^{\Omega_\rho}$ is well defined for every $\mu < 1/4$.

We show that if, for some $\rho$ as above, there exists $\nu \in \mathfrak{M}_+(\partial\Omega)$ such that

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{\alpha_-}} \int_{\Sigma_\varepsilon} \left|u - \mathbb{K}_\mu^{\Omega_\rho}[\nu]\right| dS = 0 \tag{1.5}$$

then (1.5) holds for every $\rho \in (0, \bar\rho]$ and the measure $\nu$ is independent of $\rho$.

In addition we show that a positive solution of equation $(P_\mu)$ possesses a normalised boundary trace if and only if it is a moderate solution.

[1] Actually, the assumption $\mu > 0$ was introduced in [10] only for simplicity: the normalised boundary trace is well-defined and the related results remain valid for any $\mu < C_H(\Omega)$.

### 1.4. Main results

We start with a few results about the linear operator.

**Theorem 1.4.** *Let $\mu < 1/4$. Suppose that $u$ is positive and $\mathscr{L}_\mu$-subharmonic in $\Omega_{\bar\rho}$. Then $u$ has a normalized boundary trace on $\partial\Omega$ if and only if $u$ is dominated in $\Omega_\rho$ (for some $\rho \in (0,\bar\rho)$) by an $\mathscr{L}_\mu$-harmonic function.*

**Theorem 1.5.** *Let $\mu < 1/4$. Suppose that $u$ is a non-negative, $\mathscr{L}_\mu$-subharmonic function in $\Omega_{\bar\rho}$. In addition assume that, for some $\rho \in (0,\bar\rho)$ $u$ is dominated in $\Omega_\rho$ by an $\mathscr{L}_\mu$-harmonic function. Then, one of the following holds:*

(i) $\mathrm{tr}^*_{\partial\Omega} u = 0$, *in which case, for every $\beta \in (0,\rho)$ there exists a constant $c_\beta > 0$ such that*
$$u(x) \le c_\beta \delta(x)^{\alpha_+} \quad \text{in } \Omega_\beta; \tag{1.6}$$

(ii) $\mathrm{tr}^*_{\partial\Omega} u > 0$, *in which case, for every $\beta$ as above,*
$$\frac{1}{c_\beta}\beta^{\alpha_-} \le \int_{\Sigma_\beta} u\,dS \le c_\beta \beta^{\alpha_-} \quad \text{in } \Omega_\beta. \tag{1.7}$$

**Theorem 1.6.** *Let $\mu < 1/4$. Suppose that $u$ is positive and $\mathscr{L}_\mu$-superharmonic in $\Omega_{\bar\rho}$. Then $u$ has a normalized boundary trace. If $\mathrm{tr}^*_{\partial\Omega} u \neq 0$ then (1.7) holds.*

**Corollary 1.7.** *Suppose that $u$ is non-negative and $\mathscr{L}_\mu$-subharmonic in $\Omega_{\bar\rho}$. Then either (1.6) holds or*
$$0 < \limsup_{\beta\to 0} \frac{1}{\beta^{\alpha_-}} \int_{\Sigma_\beta} u\,dS. \tag{1.8}$$

**Remark 1.8.** The corollary is an improved version of [5, Theorem 2.9]. Since we do not assume that $u$ is dominated by an $\mathscr{L}_\mu$-harmonic function the alternative to (1.6) is not necessarily (1.7) but only (1.8) which is nothing more than the negation of the statement $\mathrm{tr}^*_{\partial\Omega} u = 0$.

Clearly every positive subsolution of the nonlinear equation $(P_\mu)$ is $\mathscr{L}_\mu$-subharmonic so that the above results apply to it.

We turn to the nonlinear problem.

**Theorem 1.9.** *Let $\mu < 1/4$ and $\nu \in \mathfrak{M}^+(\partial\Omega) \setminus \{0\}$. Assume that $\mathbb{K}_\mu^{\Omega_\rho}[\nu] \in L^q(\Omega_\rho; \delta^{\alpha_+})$ for some $\rho \in (0,\bar\rho]$. Then the boundary value problem $(P_\mu^\nu)$ admits a positive solution $u$.*

We emphasise that if $C_H(\Omega) < 1/4$ then for $\mu \in [C_H(\Omega), 1/4)$ an $\mathscr{L}_\mu$-harmonic extension of $\nu$ exists only locally in a strip $\Omega_\rho$. Nevertheless, problem $(P_\mu^\nu)$ has a positive solution in $\Omega$, for any $\mu < 1/4$ .

When $\mu < C_H(\Omega)$ problem $(P_\mu^\nu)$ admits at most one solution for every $\nu \in \mathfrak{M}_+(\partial\Omega)$ [10]. However, if $C_H(\Omega) < \mu < 1/4$ uniqueness fails. Indeed, it was proved in [5, Theorem 5.3] that in the latter case there exists a positive solution of $(P_\mu^\nu)$ with $\nu = 0$. An alternative, more direct proof, of this result is presented in Appendix A.

**Theorem 1.10.** *Let $u$ be a positive solution of $(P_\mu)$. Then,*

(i) *$u$ has a normalized boundary trace if and only if $u \in L^q(\Omega;\delta^{\alpha_+})$;*
(ii) *If $u$ has normalized boundary trace $\nu$ then*

$$\lim_{x\to y}\frac{u(x)}{\mathbb{K}_\mu^{\Omega_\rho}[\nu](x)} = 1 \quad \textit{non-tangentially, for } \nu\textit{-a.e. } y\in\partial\Omega. \tag{1.9}$$

In general, the existence of a solution of $(P_\mu^\nu)$ does not imply that $\mathbb{K}_\mu^{\Omega_\rho}[\nu] \in L^q(\Omega;\delta^{\alpha_+})$. In fact, for any $\mu > 0$ and $q > 1$, one can construct functions $f \in L^1(\partial\Omega)$ such that $\mathbb{K}_\mu^{\Omega_\rho}[f] \notin L^q(\Omega;\delta^{\alpha_+})$ while $(P_\mu^\nu)$ has a solution whenever $\nu = f \in L^1(\partial\Omega)$.

Let

$$q_{\mu,c} := \frac{N+\alpha_+}{N-1-\alpha_-} \quad \text{for all} \quad \mu < 1/4. \tag{1.10}$$

The next result has been obtained in [10, Theorems E and F] for $\mu \in (0, C_H(\Omega))$. A similar result is presented in [8, Theorems D and E], under the assumption that $\Omega$ is a convex domain, in which case it is known that $C_H(\Omega) = 1/4$.

**Proposition 1.11.** *Let $\mu < 1/4$. If $1 < q < q_{\mu,c}$ then the boundary value problem $(P_\mu^\nu)$ has a solution for every Borel measure $\nu \in \mathfrak{M}^+(\partial\Omega)$. Moreover, if $q \ge q_{\mu,c}$ then problem $(P_\mu^\nu)$ has no solution when $\nu$ is the Dirac measure.*

In the next proposition, the existence statement is a consequence of Theorem 1.9. The non-existence part is more subtle.

**Proposition 1.12.** *The following facts hold true:*

(i) *For every $\mu < 1/4$ put*

$$q_\mu^* = \begin{cases} \infty & \text{if } \mu \ge 0 \\ 1-\frac{2}{\alpha_-} & \text{if } \mu < 0. \end{cases}$$

*If $1 < q < q_\mu^*$ then problem $(P_\mu^\nu)$ has a solution for every measure $\nu = f\,dS$, $f \in L^1(\partial\Omega)$;*
(ii) *If $q \ge q_\mu^*$ then problem $(P_\mu^\nu)$ has no solution for any $\nu \in \mathfrak{M}_+(\partial\Omega)\setminus\{0\}$.*

**Remark 1.13.** If $\mu < 0$ then $\alpha_- < 0$ so that $q_\mu^* > 1$ and $q_{\mu,c} < q_\mu^*$.

The paper is organised as follows. In Section 2 we study the linear problem. We derive estimates of the Green and Martin kernels of $\mathscr{L}_\mu$ in $\Omega_\rho$ and discuss the boundary behavior of local positive $\mathscr{L}_\mu$-sub and superharmonic functions in terms of the normalized trace.

In Section 3 these results are applied to the study of the nonlinear boundary value problem $(P_\mu^\nu)$.

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## 2. Linear equation and normalised boundary trace

### 2.1. The local behavior of Green and Martin kernels

We recall some results concerning Schrödinger equations, that are needed in what follows. The results are due to Ancona [2]. Let $D$ be a bounded Lipschitz domain and consider the Schrödinger operator $\mathscr{L}^V = \Delta + V$ where $V \in C(D)$ is a potential such that, for some constant $a > 0$, it holds $|V(x)| \leq a\,\mathrm{dist}\,(x, \partial D)^{-2}$ and $\mathscr{L}^V$ possesses a positive supersolution. (If $V \leq 0$ there is always a supersolution namely, $u = 1$.) Then $\mathscr{L}^V$ has a Green function $G^V$ and Martin kernel $K^V$ in $D$. The Martin boundary coincides with $\partial D$ and the following holds

**Theorem 2.1 (representation theorem).** *For every $\nu \in \mathfrak{M}^+(\partial D)$ the function*

$$\mathbb{K}^V[\nu](x) := \int_{\partial D} K^V(x, y) d\nu(y), \quad x \in D,$$

*is $\mathscr{L}^V$-harmonic in $D$. Conversely, if $u$ is a positive $\mathscr{L}^V$-harmonic function in $D$ then there exists a unique measure $\nu \in \mathfrak{M}^+(\partial D)$ such that $u = \mathbb{K}^V[\nu]$.*

In order to state the boundary Harnack principle we need additional notation. Let $y \in \partial D$ and let $\xi = \xi^y$ be a local set of coordinates centered at $y$ such that the $\xi_1$-axis is in the direction of an interior pseudo normal $\mathbf{n_y}$. (If $D$ is a $C^1$ domain we may take $\mathbf{n_y}$ to be the interior unit normal.) Denote

$$T_y(r, \rho) = \left\{\xi = (\xi_1, \xi') : |\xi_1| < \rho, \quad |\xi'| < r\right\}.$$

Assume that $r$ and $\rho$ are so chosen that

$$\omega_y := T_y(r, \rho) \cap D = \left\{\xi : F_y(\xi') < \xi_1 < \rho, \ |\xi'| < r\right\}$$

where $F_y$ is a Lipschitz function in $\mathbb{R}^{N-1}$, with Lipschitz constant $\Lambda$, and such that $F_y(0) = 0$ and $12\Lambda < \rho/r$. Since $D$ is a bounded Lipschitz domain $\Lambda, r, \rho$ can be chosen independently of $y \in \partial D$.

Let $A \in T(r, \rho)$ be the point such that $\xi(A) = (\rho/2, 0)$. Then the boundary Harnack principle reads as follows: if $u, v$ are positive $\mathscr{L}_\mu$-harmonic functions in $\omega_y$ vanishing continuously on $\partial\Omega \cap T_y(r, \rho)$ then

$$C^{-1}\frac{u(A)}{v(A)} \leq \frac{u(\xi)}{v(\xi)} \leq C\frac{u(A)}{v(A)} \quad \text{for all} \quad \xi \in T_y(r/2, \rho/2) \cap D, \tag{2.1}$$

where the constant $C$ depends only on $N$, $M$, $\rho/r$ and the Lipschitz constant of $F_y$, say $\Lambda$. ($\Lambda$ may be taken to be independent of $y \in \partial D$.)

We also need the following consequence of the boundary Harnack principle (*cf.* Ancona [1, Lemma 3.5]): there exist positive numbers $c$, $t_0$ such that

$$c^{-1}|x-y|^{2-N} \le K^V(x,y)G^V(x,x_0) \le c|x-y|^{2-N} \tag{2.2}$$

for every $y \in \partial\Omega'$ and $x$ on the interior pseudo normal at $y$ such that $|x-y| \le t_0$.

Recall that if $V(x) = \mu \mathrm{dist}\,(x, \partial D)^{-2}$ and $\mu < C_H(D)$ then $\mathscr{L}^V$ has a positive supersolution. In particular, if $D = \Omega_{\bar\rho}$ then $C_H(D) = 1/4$. Therefore, in this case, the above results apply to the operator $\mathscr{L}_\mu = \Delta + \frac{\mu}{\delta_\Omega^2}$ for every $\mu < 1/4$.

**Notation.** Let $D$ be a subdomain of $\Omega$ and denote

$$\mathscr{L}_{\mu,D} = \Delta + \frac{\mu}{\delta_D^2} \quad \text{where} \quad \delta_D(x) = \mathrm{dist}\,(x, \partial D).$$

Assume that $\mu < C_H(D)$ and let $D'$ be a subdomain of $D$. Obviously $C_H(D') \ge C_H(D)$. Denote the Green kernel (respectively the Martin kernel) of $\mathscr{L}_\mu$ in $D$ by $G_\mu^D$ (respectively $K_\mu^D$). Denote the Green kernel (respectively the Martin kernel) of $\mathscr{L}_{\mu,D}$ in $D'$ by $G_{\mu,D}^{D'}$ (respectively $K_{\mu,D}^{D'}$).

If $f_1$, $f_2$ are two non-negative functions in a domain $D$ the notation $f_1 \sim f_2$ means that there exists a constant $c$ such that

$$c^{-1} f_1 \le f_2 \le c f_1.$$

**Lemma 2.2.** *Assume that* $\mu < 1/4$. *Let* $\bar\rho$ *be as in Lemma* 1.1 *and* $t \in (0, \bar\rho)$. *Put* $U = \Omega_{\bar\rho} = [\delta(x) < \bar\rho]$, $\Omega_t = [\delta(x) < t]$, *and* $U_t = [\bar\rho > \delta(x) > t]$. *Then,*

$$\begin{aligned} &G_\mu^{\Omega_{t/2}}(x,y) \le \\ &C(t)\inf(|x-y|^{2-N}, \delta(x)^{\alpha_+}\delta(y)^{\alpha_+}|x-y|^{2\alpha_- - N}) \quad \textit{for all} \quad x, y \in \Omega_{t/2} \end{aligned} \tag{2.3}$$

*Proof.* Note that $\mathscr{L}_\mu = \mathscr{L}_{\mu,U}$ in $\Omega_{t/2}$. Hence

$$G_\mu^{\Omega_{t/2}} = G_{\mu,U}^{\Omega_{t/2}}.$$

It is well-known that the Green function is monotone with respect to the domain. Therefore $G_{\mu,U}^{\Omega_{t/2}} < G_{\mu,U}^{\Omega_t}$ which implies

$$G_\mu^{\Omega_{t/2}}(x,y) \le c G_{\mu,U}^{\Omega_t}(x,y) \quad \text{for all} \quad x, y \in \Omega_{t/2}. \tag{2.4}$$

By (2.4) and the estimate of the Green function of $\mathscr{L}_{\mu,U}$ (see [7] and [10, (2.6)]), it follows

$$\begin{aligned} G_\mu^{\Omega_{t/2}}(x,y) &\le c G_{\mu,U}^{\Omega_t}(x,y) \le c G_{\mu,U}^{U}(x,y) \\ &\sim \inf(|x-y|^{2-N}, \delta(x)^{\alpha_+}\delta(y)^{\alpha_+}|x-y|^{2\alpha_- - N}) \end{aligned} \tag{2.5}$$

for every $x, y \in \Omega_{t/2}$. This implies (2.3). $\square$

**Theorem 2.3.** *Assume that $\mu < 1/4$, let $\bar\rho$ be as in Lemma 1.1 and let $t \in (0, \bar\rho/2)$. Using the notation of the previous lemma, pick $x_t \in U_t$ and $x_t' \in \Omega_t$ such that $\delta(x_t) = (t+\bar\rho)/2$ and $\delta(x_t') = t/2$. As usual $G_0^U$ denotes the Green function for $-\Delta$ in $U$. A similar notation is employed for the corresponding Martin kernels. Then,*

$$\begin{aligned} c_1(t)^{-1}G^U_{\mu,U}(x,x_t) \le G^U_\mu(x,x_t) &\le c_1(t)G^U_{\mu,U}(x,x_t) \quad \text{for all} \quad x\in\Omega_t \\ c_2(t)^{-1}G^U_0(x,x_t') \le G^U_\mu(x,x_t') &\le c_2(t)G^U_0(x,x_t') \quad \text{for all} \quad x\in U_t, \end{aligned} \tag{2.6}$$

*and*

$$\begin{aligned} c_3(t)^{-1}K^U_{\mu,U}(x,y) \le K^U_\mu(x,y) &\le c_3(t)K^U_{\mu,U}(x,y) \quad \text{for all} \quad (x,y)\in\Omega_t\times\partial\Omega, \\ c_4(t)^{-1}K^U_0(x,y) \le K^U_\mu(x,y) &\le c_4(t)K^U_0(x,y) \quad \text{for all} \quad (x,y)\in U_t\times\Sigma_{\bar\rho}. \end{aligned} \tag{2.7}$$

*Proof.* Note that $\mathscr{L}_\mu = \mathscr{L}_{\mu,U}$ in $\Omega_{\bar\rho/2}$. Hence both $G^U_\mu(\cdot,x_t)$ and $G^U_{\mu,U}(\cdot,x_t)$ are $\mathscr{L}_\mu$-harmonic in $\Omega_t$ and vanish on $\partial\Omega$. Therefore, by the boundary Harnack principle they are equivalent in a strip $S$ along $\partial\Omega$. In addition they are continuous and bounded away from zero in $\Omega_t\setminus S$. This implies the first inequality in (2.6). For the second inequality: $G^U_\mu(\cdot,x_t')$ is $\mathscr{L}_\mu$-harmonic in $U_t$, $G^U_0(\cdot,x_t')$ is $\Delta$ harmonic in $U_t$ and $\mathscr{L}_\mu-\Delta = \mu/\delta(x)^2$ is bounded in $U_t$. Therefore, since they both vanish on $\Sigma_{\bar\rho}$, we can still apply the boundary Harnack principle (*cf.* Ancona [4]) to deduce that they are equivalent in the strip $U_t$. This implies the second inequality in (2.6).

Recall that, $G^U_{\mu,U}(x,x_t) \sim \delta_U(x)^{\alpha_+}$ in $\Omega_t$ for $t\in(0,\rho)$. (Of course the constants involved in this relation depend on $t$.) Since $\delta_\Omega \sim \delta_U$ in $\Omega_t$, this fact and (2.6) imply that

$$G^U_\mu(x,x_t) \sim \delta_\Omega(x)^{\alpha_+} \quad \text{for all} \quad x\in\Omega_t. \tag{2.8}$$

In what follows we use the notation introduced for the statement of the boundary Harnack principle. Let $y\in\partial\Omega$ and let $\xi=\xi_y$ be a local set of coordinates at $y$ relative to $U$. Thus

$$\omega_y = T_y(r,\rho)\cap U = \{\xi : F_y(\xi') < \xi_1 < \rho,\ |\xi'| < r\}.$$

We assume that $\gamma = \rho/r > 12\Lambda$.

Since $K^U_\mu(\cdot,y)$ and $G^U_\mu(\cdot,x_t)$ satisfy the (classical) Harnack inequality (2.2) remains valid in $\mathcal{C}_y(b)\cap T_y(r,\rho)$. Therefore, assuming that $\rho < t < \bar\rho$,

$$K^U_\mu(\xi,y)G^U_\mu(\xi,x_t) \sim K^U_\mu((\xi_1,0),y)G^U_\mu((\xi_1,0),x_t) \sim |\xi|^{2-N} \tag{2.9}$$

for every $\xi\in\mathcal{C}_y(b)\cap T_y(r,\rho)$. By (2.8) and (2.9),

$$K^U_\mu(\xi,y) \sim |\xi|^{2-N}\delta(\xi)^{-\alpha_+} \quad \text{for all} \quad \xi\in\mathcal{C}_y(b)\cap T_y(r,\rho). \tag{2.10}$$

Let $\eta$ be a point in $\mathbb{R}^{N-1}$ such that $0 < |\eta| < r/2$ and denote by $P$ the point $(F_y(\eta),\eta)$ in the local coordinates $\xi_y$. Then $P\in\partial\Omega$ and $\xi_P := \xi_y - P$ is a standard set of local coordinates at $P$. Choose $r_P$ and $\rho_P$ such that $r_P = |\eta|/2$ and $\rho_P/r_P = \gamma$. Then,

$$|x-y| = |\xi_y| \sim |\xi'_y| \sim r_P \quad \text{for all} \quad x \in \Omega \cap T_P(r_P, \rho_P).$$

Let $A_P = (\rho_P/2, 0)$ in $\xi_P$ coordinates, *i.e.*, $A_P = (F_y(\eta) + \gamma r_P/2, \eta)$ in $\xi_y$ coordinates. Pick $b$ such that $\Lambda < b < 2\Lambda$. Then

$$F_y(\eta) + \rho_P/2 \geq -\Lambda|\eta| - \gamma r_P/2 = |\eta|(-\Lambda + \gamma/4) > 2\Lambda|\eta|.$$

Consequently, $F_y(\eta) < b|\eta| < F_y(\eta) + \rho_P/2$, which implies

$$A_P \in \mathcal{C}_y(b) := \left\{\xi_y = (\xi_1, \xi') : \xi_1 > b|\xi'|\right\}.$$

Observe that

$$\delta_\Omega(A_P) \sim \rho_P/2 \quad \text{and} \quad |\xi_y(A_P)| = |A_P - y| \sim (\rho_P^2 + r_P^2)^{1/2} \sim r_p.$$

Therefore, by (2.10),

$$K_\mu^U(A_P, y) \sim r_P^{2-N-\alpha_+}.$$

In fact,

$$|x-y| = |\xi_y| \sim r_P \quad \text{for all} \quad x \in \Omega \cap T_P(r_P, \rho_P).$$

Therefore applying (2.1) in $\Omega \cap T_P(r_P, \rho_P)$ with $u(x) = K_\mu^U(x, y)$ we obtain

$$\begin{aligned} K_\mu^U(x, y) &\sim K_\mu^U(A_P, y)\frac{G_\mu^U(x, x_t)}{G_\mu^U(A_P, x_t)} \sim r_P^{2-N-\alpha_+}(\delta(x)/r_P)^{\alpha_+} \\ &\sim |x-y|^{2-N-2\alpha_+}\delta(x)^{\alpha_+} = \delta(x)^{\alpha_+}|x-y|^{2\alpha_- - N} \end{aligned} \tag{2.11}$$

for every $x \in \Omega \cap T_P(r_P/2, \rho_P/2)$. Combining (2.10) and (2.11), we obtain

$$K_\mu^U(x, y) \sim |x-y|^{2-N-\alpha_+}(\delta(x)/|x-y|)^{\alpha_+} = \delta(x)^{\alpha_+}|x-y|^{2\alpha_- - N} \tag{2.12}$$

for every $x \in T_y(r/2, \rho/2)$. As (2.12) holds uniformly with respect to $y \in \partial\Omega$ we conclude that there exists $r' > 0$ such that this relation holds for every $(x, y) \in \Omega_{r'} \times \partial\Omega$. Consequently, for every $t \in (0, \bar{\rho})$,

$$K_\mu^U(x, y) \sim |x-y|^{2-N-\alpha_+}(\delta(x)/|x-y|)^{\alpha_+} = \delta(x)^{\alpha_+}|x-y|^{2\alpha_- - N} \tag{2.13}$$

for every $(x, y) \in \Omega_t \times \partial\Omega$ with similarity constants depending on $t$. Since $K_{\mu,U}^U$ behaves precisely in the same way (see [10, Section 2.2]) we obtain the first inequality in (2.7). The second inequality is proved in a similar way. □

We state below two key results concerning the operator $\mathscr{L}_\mu$ in $U = \Omega_{\bar\rho}$. These have been recently proved in [10], with respect to the operator $\mathscr{L}_\mu$ in $\Omega$ under the assumption that $0 < \mu < C_H(\Omega)$. (In fact, the condition $\mu > 0$ is redundant and does not affect the proofs.) Since $C_H(\Omega_{\bar\rho}) = 1/4$, the results apply to the operator $\mathscr{L}_{\mu,\Omega_{\bar\rho}}$ for every $\mu < 1/4$. In view of the relation between the Martin kernels and Green functions of $\mathscr{L}_{\mu,\Omega_{\bar\rho}}$ and $\mathscr{L}_\mu$ in $\Omega_{\bar\rho}$, these results also apply to the operator $\mathscr{L}_\mu$ in $\Omega_{\bar\rho}$.

**Theorem 2.4.** *The following facts hold true:*

(i) *If* $\nu_0 \in \mathfrak{M}^+(\partial\Omega) \setminus \{0\}$ *then there exist positive numbers* $c$ *and* $\rho_0 < \bar\rho$ *such that*

$$c^{-1}\,\|\nu_0\| \le \frac{1}{\varepsilon^{\alpha_-}} \int_{\Sigma_\varepsilon} \mathbb{K}_\mu^{\Omega_\rho}[\nu_0]dS \le c\,\|\nu_0\| \quad \text{if} \quad \epsilon \in (0,\rho_0); \tag{2.14}$$

(ii) *Let* $\rho \in (0,\bar\rho)$ *and let* $\tau$ *be a Radon measure in* $\Omega_{\bar\rho}$. *Denote*

$$\mathbb{G}_\mu^{\Omega_\rho}[\tau](x) := \int_{\Omega_\rho} G_\mu^{\Omega_\rho}(x,y)d\tau(y) \quad \text{for} \quad x \in \Omega_\rho.$$

*If* $\tau \in \mathfrak{M}^+_{\delta^{\alpha_+}}(\Omega_\rho)$ *then for every* $0 < \varepsilon < \rho' < \rho$,

$$\frac{1}{\varepsilon^{\alpha_-}} \int_{\Sigma_\varepsilon} \mathbb{G}_\mu^{\Omega_\rho}[\tau]dS_x \le c \int_{\Omega_\rho} \delta^{\alpha_+} d\tau, \tag{2.15}$$

*where* $c$ *is a constant depending on* $\mu$, $\rho'$, *but not on* $\varepsilon$. *Moreover,*

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{\alpha_-}} \int_{\Sigma_\varepsilon} \mathbb{G}_\mu^{\Omega_\rho}[\tau]\,dS = 0. \tag{2.16}$$

**Remark 2.5.** If $\mathbb{G}_\mu^{\Omega_\rho}[\tau](x') < \infty$ for some point $x' \in \Omega_\rho$ then $\tau \in \mathfrak{M}^+_{\delta^{\alpha_+}}(\Omega_\rho)$ and $\mathbb{G}_\mu^{\Omega_\rho}[\tau](x) < \infty$ for every $x \in \Omega_\rho$. This follows from the fact that there exists $c > 0$ such that for every fixed $x \in \Omega_\rho$, it holds

$$\frac{1}{c}\delta(y)^{\alpha_+} \le G_\mu^{\Omega_\rho}(x,y) \le c\delta(y)^{\alpha_+} \quad \text{for all} \quad y \in \Omega_{\delta(x)/2}.$$

*Proof.* In view of (2.13), inequality (2.14) follows from [10, Corollary 2.11].

The proof of (2.15) and (2.16) is similar to that of [10, Proposition 2.12]. However several modifications are needed; therefore we provide the proof of these statements in detail.

We may assume that $\tau > 0$. Denote $v := \mathbb{G}_\mu^{\Omega_\rho}[\tau]$. We start with the proof of (2.15).

By Fubini's theorem and (2.6)

$$\int_{\Sigma_\beta} v dS_x \le c\Big( \int_\Omega \int_{\Sigma_\beta \cap B_{\frac{\beta}{2}}(y)} |x-y|^{2-N} dS_x\, d\tau(y) \\ + \beta^{\alpha_+} \int_\Omega \int_{\Sigma_\beta \setminus B_{\frac{\beta}{2}}(y)} |x-y|^{2\alpha_- - N} dS_x\, \delta^{\alpha_+}(y) d\tau(y)\Big) \\ = I_1(\beta) + I_2(\beta).$$

Note that, if $x \in \Sigma_\beta$ and $|x-y| \le \beta/2$ then $\beta/2 \le \delta(y) \le 3\beta/2$. Therefore

$$I_1(\beta) \le c_1 \beta^{-\alpha_+} \int_{\Sigma_\beta \cap B_{\frac{\beta}{4}}(y)} |x-y|^{2-N} dS_x \int_{\Omega_\rho} \delta(y)^{\alpha_+}\, d\tau(y) \\ \le c_1' \beta^{1-\alpha_+} \int_{\Omega_\rho} \delta(y)^{\alpha_+}\, d\tau(y) = c_1' \beta^{\alpha_-} \int_{\Omega_\rho} \delta(y)^{\alpha_+}\, d\tau(y)$$

and

$$I_2(\beta) \le c_2 \beta^{\alpha_+} \int_{\beta/4}^{\infty} r^{2\alpha_- - N} r^{N-2} dr \int_{\Omega_\rho} \delta(y)^{\alpha_+}\, d\tau \le c_2' \beta^{\alpha_-} \int_{\Omega_\rho} \delta(y)^{\alpha_+}\, d\tau.$$

This implies (2.15).

Given $\ell \in (0, \|\tau\|_{\mathfrak{M}_{\delta_+^\alpha}(\Omega)})$ and $\beta_1 \in (0, \beta_0)$ put $\tau_1 = \tau \chi_{\bar{D}_{\beta_1}}$ and $\tau_2 = \tau - \tau_1$. Pick $\beta_1 = \beta_1(\ell)$ such that

$$\int_{\Omega_{\beta_1}} \delta(y)^{\alpha_+}\, d\tau \le \ell. \tag{2.17}$$

Thus the choice of $\beta_1$ depends on the rate at which $\int_{\Omega_\beta} \delta_+^\alpha\, d\tau$ tends to zero as $\beta \to 0$.

Put $v_i = \mathbb{G}_\mu^\Omega[\tau_i]$. Then, for $0 < \beta < \beta_1/2$,

$$\int_{\Sigma_\beta} v_1\, dS_x \le c_3 \beta^{\alpha_+} \beta_1^{2\alpha_- - N} \int_{\Omega_\rho} \delta^{\alpha_+}(y) d\tau_1(y).$$

Thus,

$$\lim_{\beta \to 0} \frac{1}{\beta^{\alpha_-}} \int_{\Sigma_\beta} v_1\, dS_x = 0. \tag{2.18}$$

On the other hand, by (2.15) (replacing $\Omega_\rho$ by $\Omega_{\beta_1}$) and (2.17),

$$\frac{1}{\beta^{\alpha_-}} \int_{\Sigma_\beta} v_2\, dS_x \le c\ell \quad \text{for all} \quad \beta < \beta_1. \tag{2.19}$$

This proves (2.16). □

**Corollary 2.6.** *Let $\rho \in (0, \bar{\rho}]$ and assume that $h$ is a nonnegative $\mathscr{L}_\mu$-harmonic function in $\Omega_\rho$ such that*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{\alpha_-}} \int_{\Sigma_\varepsilon} h \, dS = 0. \tag{2.20}$$

*Then:*

(i) $h = \mathbb{K}_\mu^{\Omega_\rho}[\nu_\rho]$ *for some measure* $\nu_\rho \in \mathfrak{M}^+(\Sigma_\rho)$*;*
(ii) *For* $t \in (0, \bar{\rho})$*,*

$$h \sim \delta_\Omega^{\alpha_+} \quad \text{in } \Omega_t, \tag{2.21}$$

*with the similarity constant depending on $t$.*

*Proof.* (i) By the representation theorem, $h = \mathbb{K}_\mu^{\Omega_\rho}[\nu]$ for some $\nu \in \mathfrak{M}(\partial\Omega_\rho)$. By (2.14) and (2.20), $\nu_0 := \nu \mathbf{1}_{\partial\Omega} = 0$. Thus $\nu = \nu_\rho := \nu \mathbf{1}_{\Sigma_\rho}$.
(ii) This is a consequence of (i) and (2.13). □

**Corollary 2.7.** *If $\tau \in \mathfrak{M}^+_{\delta^{\alpha_+}}(\Omega_\rho)\backslash\{0\}$ then there exists a positive constant $c = c(\tau)$ such that*

$$\mathbb{G}_\mu^{\Omega_\rho}[\tau](x) \geq c\delta(x)^{\alpha_+} \quad \forall x \in \Omega_\rho, \tag{2.22}$$

*and*

$$\liminf_{x \to \partial\Omega} \frac{\mathbb{G}_\mu^{\Omega_\rho}[\tau](x)}{\delta(x)^{\alpha_-}} < \infty. \tag{2.23}$$

*Proof.* Let $t \in (0, \rho)$ be a number such that $\tau(\Omega_\rho \setminus \Omega_t) > 0$. Let $\tau' \in \mathfrak{M}_+(\Omega_\rho)$ be defined by $\tau' = \tau$ in $\Omega_\rho \setminus \Omega_t$ and $\tau' = 0$ in $\Omega_t$. Then

$$\mathbb{G}_\mu^{\Omega_\rho}[\tau] \geq G_\mu^{\Omega_\rho}[\tau'] := h.$$

Since $h$ is $\mathscr{L}_\mu$-harmonic in $\Omega_t$, (2.22) is a consequence of (2.21).
Inequality (2.23) follows from (2.15). □

The next result was proved in [10] for $\mathscr{L}_\mu$ in a domain $\Omega$ such that $\mu < C_H(\Omega)$.

**Theorem 2.8.** *Let $w$ be a nonnegative $\mathscr{L}_\mu$-subharmonic function in $\Omega_\rho$. If $w$ is dominated by an $\mathscr{L}_\mu$-superharmonic function in $\Omega_\rho$ then $\mathscr{L}_\mu w = \lambda \in \mathfrak{M}^+_{\delta^{\alpha_+}}(\Omega_\rho)$ and there exists $\nu \in \mathfrak{M}^+(\partial\Omega_\rho)$ such that*

$$w = \mathbb{K}_\mu^{\Omega_\rho}[\nu] - \mathbb{G}_\mu^{\Omega_\rho}[\lambda]. \tag{2.24}$$

*Proof.* There exists a nonnegative Radon measure $\lambda$ in $\Omega_\rho$, such that $-\mathscr{L}_\mu w = -\lambda$ in $\Omega_\rho$. Since $w$ is dominated by an $\mathscr{L}_\mu$-superharmonic function in $\Omega_\rho$ one shows, as in the proof of [10, Proposition 2.14], that $\lambda \in \mathfrak{M}_{\delta^{\alpha_+}}(\Omega_\rho)$. Then $v := w + \mathbb{G}_\mu^{\Omega_\rho}[\lambda]$ is a nonnegative $\mathscr{L}_\mu$-harmonic function in $\Omega_\rho$. By the representation theorem, $v = \mathbb{K}_\mu^{\Omega_\rho}[\nu]$ for some $\nu \in \mathfrak{M}^+(\partial\Omega_\rho)$. □

**Definition 2.9.** A Borel function $u : \Omega \to \mathbb{R}$ possesses a *normalised boundary trace* $\nu_0 \in \mathfrak{M}^+(\partial\Omega)$ if, for some $\rho \in (0, \bar\rho]$,

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{\alpha_-}} \int_{\Sigma_\varepsilon} \left|u - \mathbb{K}_\mu^{\Omega_\rho}[\nu_0]\right| dS = 0. \tag{2.25}$$

The normalised boundary trace on $\partial\Omega$ will be denoted by $\mathrm{tr}^*_{\partial\Omega}(u)$.

**Remark.** Since $u$ is a Borel function $u\lfloor_{\Sigma_\rho}$ is well defined and (2.25) implies that this function is in $L^1(\Sigma_\epsilon)$ for all sufficiently small $\epsilon$.

We say that $u$ has a *measure boundary trace* on $\Sigma_\rho$ if there exists $\nu_1 \in \mathfrak{M}^+(\Sigma_\rho)$ such that

$$\lim_{a\to\rho-0} \int_{\Sigma_a} u\phi\, dS \to \int_{\Sigma_\rho} \phi\, d\nu_1 \quad \text{for all} \quad \phi \in C_0(\bar\Omega_\rho).$$

This trace is denoted by $\mathrm{tr}_{\Sigma_\rho}(u)$. If both $\mathrm{tr}_{\Sigma_\rho}(u)$ and $\mathrm{tr}^*_{\partial\Omega}(u)$ exist then the measure $\nu \in \mathfrak{M}_+(\partial\Omega_\rho)$ given by $\nu\mathbf{1}_{\partial\Omega} = \mathrm{tr}^*_{\partial\Omega}(u)$ and $\nu\mathbf{1}_{\Sigma_\rho} = \mathrm{tr}_{\Sigma_\rho}(u)$ is denoted by $\mathrm{tr}^\mu_{\partial\Omega_\rho}(u)$.

**Lemma 2.10.** *The normalised boundary trace $\nu_0$ is uniquely defined, independently of $\rho$.*

*Proof.* First we note that (2.25) remains valid if $\nu_0$ is replaced by any measure $\nu \in \mathfrak{M}_+(\partial\Omega_\rho)$ such that $\nu_0 = \nu\mathbf{1}_{\partial\Omega}$. This follows from the fact that, for every measure $\nu_\rho \in \mathfrak{M}_+(\Sigma_\rho)$,

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{\alpha_-}} \int_{\Sigma_\varepsilon} \mathbb{K}_\mu^{\Omega_\rho}[\nu_\rho] dS = 0.$$

This implies that if (2.25) holds with respect to some $\rho \in (0, \bar\rho)$ then it is valid for any $\rho'$ in this range. Suppose for instance that $\rho < \rho' < \bar\rho$ and put $v = \mathbb{K}_\mu^{\Omega_{\rho'}}[\nu_0]$. Let $\nu \in \mathfrak{M}_+(\partial\Omega_\rho)$ be the measure equal to $\nu_0$ on $\partial\Omega$ and to $h = v\lfloor_{\Sigma_\rho} d\omega_\rho$ on $\Sigma_\rho$. (Here $\omega_\rho$ is the $\mathscr{L}_\mu$-harmonic measure on $\Sigma_\rho$ relative to $\Omega_{\rho'}$. Since $\Sigma_\rho$ is "smooth" $\omega_\rho$ is absolutely continuous with respect to surface measure.) Then $v = \mathbb{K}_\mu^{\Omega_\rho}[\nu]$ in $\Omega_\rho$ and

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{\alpha_-}} \int_{\Sigma_\varepsilon} |\mathbb{K}_\mu^{\Omega_\rho}[\nu] - \mathbb{K}_\mu^{\Omega_\rho}[\nu_0]| dS = 0.$$

It remains to verify that, if (2.25) holds, then $\nu_0$ is uniquely determined by $u$ in a fixed domain $\Omega_\rho$.

Suppose, by negation, that there exist $\nu_1, \nu_2 \in \mathfrak{M}_+(\partial\Omega)$ such that (2.25) holds for both $v_1 = K_\mu^{\Omega_\rho}[\nu_1]$ and $v_2 = K_\mu^{\Omega_\rho}[\nu_2]$. Then $w := |v_1 - v_2|$ is $\mathscr{L}_\mu$-subharmonic and $\mathrm{tr}^*_{\partial\Omega}(w) = 0$.

Clearly $w$ is dominated by the $\mathscr{L}_\mu$-superharmonic function $v_1+v_2$. Therefore, by Theorem 2.8 there exist $\lambda \in \mathfrak{M}^+_{\delta^{\alpha_+}}(\Omega_\rho)$ and $\chi \in \mathfrak{M}^+(\partial\Omega_\rho)$ such that,

$$w = \mathbb{K}_\mu^{\Omega_\rho}[\chi] - \mathbb{G}_\mu^{\Omega_\rho}[\lambda].$$

Thus $w + \mathbb{G}_\mu^{\Omega_\rho}[\lambda]$ is $\mathscr{L}_\mu$-harmonic. By (2.16) and the fact that $\mathrm{tr}^*_{\partial\Omega} w = 0$ we have $\mathrm{tr}^*_{\partial\Omega}(w + \mathbb{G}_\mu^{\Omega_\rho}[\lambda]) = 0$. Hence $w = 0$ and therefore $\nu_1 = \nu_2$. □

**Theorem 2.11.** *Let $w$ be a nonnegative $\mathscr{L}_\mu$-subharmonic function in $\Omega_\rho$ dominated by an $\mathscr{L}_\mu$-superharmonic function in this domain. Then the boundary trace $\nu = \mathrm{tr}^\mu_{\partial\Omega_\rho}(w)$ is well-defined and*

$$w \le \mathbb{K}_\mu^{\Omega_\rho}[\nu]. \tag{2.26}$$

*If $\nu_0 := \nu \mathbf{1}_{\partial\Omega}$ then*

$$\lim_{x\to\partial\Omega} \frac{w(x)}{\mathbb{K}_\mu^{\Omega_\rho}[\nu_0](x)} = 1 \quad \textit{non-tangentially, } \nu_0\textit{-a.e. on } \partial\Omega. \tag{2.27}$$

*If $\nu_0 = 0$ then*

$$\limsup_{x\to\partial\Omega} \frac{w(x)}{\delta^{\alpha_+}(x)} < \infty. \tag{2.28}$$

*Proof.* The first statement (2.26) follows from (2.24) and Theorem 2.4 (ii).

The second statement (2.27) follows from (2.24) and the fact that $\mathbb{G}_\mu^{\Omega_\rho}[\lambda]$ is an $\mathscr{L}_\mu$-potential (*i.e.*, a positive superharmonic function that does not dominate any positive $\mathscr{L}_\mu$-harmonic function). This fact implies (see, *e.g.*, [3]):

$$\lim_{x\to\partial\Omega} \frac{\mathbb{G}_\mu^{\Omega_\rho}[\lambda](x)}{\mathbb{K}_\mu^{\Omega_\rho}[\nu](x)} \to 0 \quad \nu\text{-a.e. on } \partial\Omega.$$

By Fatou's limit theorem

$$\lim_{x\to\partial\Omega} \frac{\mathbb{K}_\mu^{\Omega_\rho}[\nu_0](x)}{\mathbb{K}_\mu^{\Omega_\rho}[\nu](x)} = 1 \quad \nu\text{-a.e. on } \partial\Omega.$$

Therefore (2.24) implies (2.27).

The third statement (2.28) follows from (2.26) and Corollary 2.6. □

**Corollary 2.12.** *Let $w$ be a nonnegative $\mathscr{L}_\mu$-subharmonic function in $\Omega_\rho$ for some $\rho \in (0, \bar\rho)$. Then $w$ possesses a normalised boundary trace in $\mathfrak{M}^+(\partial\Omega)$ if and only if $w$ is dominated by a positive $\mathscr{L}_\mu$-superharmonic function $v$ in a strip around $\partial\Omega$.*

*Proof.* If $w$ is dominated by a positive $\mathscr{L}_\mu$-superharmonic function in $\Omega_\rho$ then the existence of $\mathrm{tr}^*_{\partial\Omega}(w)$ follows from (2.16) and Theorem 2.8.

Next suppose that $w$ has a normalized boundary trace $\nu_0 \in \mathfrak{M}^+(\partial\Omega)$. Without loss of generality we may assume that it also has a measure boundary trace $\nu_\rho$ on $\Sigma_\rho$. Since $u$ is $\mathscr{L}_\mu$-subharmonic, there exists a positive Radon measure $\tau$ in $\Omega$ such that

$$-\mathscr{L}_\mu u = -\tau.$$

Let $\tau_\beta := \tau \mathbf{1}_{D_\beta \setminus \bar{D}_\rho}$, with $w = \mathbb{K}_\mu^{\Omega_\rho}[\nu_0 + \nu_\rho]$ and $\nu_\beta = w\lfloor_{\Sigma_\beta}$.

Let $u_\beta$ be the solution of the boundary value problem,

$$\begin{aligned} -\mathscr{L}_\mu v &= -\tau_\beta \text{ in } D_\beta \setminus \bar{D}_\rho, \\ v &= \nu_\rho \text{ on } \Sigma_\rho \quad \text{and} \quad v = \nu_\beta \text{ on } \Sigma_\beta. \end{aligned}$$

Then

$$u_\beta + \mathbb{G}_\mu^{D_\beta \setminus \bar{D}_\rho}[\tau_\beta] = w.$$

It follows that

$$G_\mu^{\Omega_\rho}[\tau] = \lim_{\beta \to 0} \mathbb{G}_\mu^{D_\beta \setminus \bar{D}_\rho}[\tau_\beta] < \infty,$$

which in turn implies that $\tau \in \mathfrak{M}_+(\Omega; \delta^{\alpha_+})$ and finally

$$u + G_\mu^{\Omega_\rho}[\tau] = w.$$

In particular,

$$u \leq w = \mathbb{K}_\mu^{\Omega_\rho}[\nu_0 + \nu_\rho]. \tag{2.29}$$

□

**Corollary 2.13.** *The following facts hold true:*

(i) *Suppose that u is positive and $\mathscr{L}_\mu$-subharmonic in $\Omega_{\bar{\rho}}$. Then* $\mathrm{tr}^*_{\partial\Omega} = 0$ *if and only if, for every $\rho \in (0, \bar{\rho})$, there exists a constant $c_\rho$ such that*

$$u(x) \leq c_\rho \delta(x)^{\alpha_+} \quad \textit{for all} \quad x \in \Omega_\rho; \tag{2.30}$$

(ii) *Suppose that u is positive and $\mathscr{L}_\mu$-superharmonic in $\Omega_{\bar{\rho}}$. Then u has a normalized boundary trace $\nu \in \mathfrak{M}_+(\partial\Omega)$ and consequently there exists $c_\rho$ such that*

$$\int_{\Sigma_\beta} u\,dS \leq c_\rho \beta^{\alpha_-} \quad \textit{for all} \quad \beta \in (0, \rho). \tag{2.31}$$

*Proof.* (i). Obviously (2.30) implies that $\mathrm{tr}^*_{\partial\Omega}(u) = 0$. Conversely assume that $\mathrm{tr}^*_{\partial\Omega}(u) = 0$.

By the previous corollary $u$ is dominated by an $\mathscr{L}_\mu$-harmonic function. Therefore, by Theorem 2.8, there exist $\lambda \in \mathfrak{M}^+_{\delta^{\alpha_+}}(\Omega_\rho)$ and $\nu \in \mathfrak{M}^+(\partial\Omega_\rho)$ such that $u = \mathbb{K}^{\Omega_\rho}_\mu[\nu] - \mathbb{G}^{\Omega_\rho}_\mu[\lambda]$. Since $\mathrm{tr}^*_{\partial\Omega}(u) = 0$, $\nu_0 = \nu\mathbf{1}_{\partial\Omega} = 0$. Hence $u < \mathbb{K}^{\Omega_\rho}_\mu[\nu_\rho]$ where $\nu_\rho = \nu\mathbf{1}_{\Sigma_\rho}$. Therefore the result follows from Corollary 2.6.

(ii). By the Riesz decomposition theorem (see [3]), $u = u_p + u_h$ where $u_p$ is an $\mathscr{L}_\mu$-potential and $u_h$ is a nonnegative $\mathscr{L}_\mu$-harmonic function in $\Omega_\rho$. It is known that every $\mathscr{L}_\mu$-potential is the Green potential of a positive measure. Thus there exists $\tau \in \mathfrak{M}_+(\Omega; \delta^{\alpha_+})$ such that $u_p = \mathbb{G}^{\Omega_\rho}_\mu[\tau]$. By the representation theorem $u_h = \mathbb{K}^{\Omega_\rho}_\mu[\nu]$ for some $\nu \in \mathfrak{M}_+(\partial\Omega_\rho)$. Thus

$$u = \mathbb{G}^{\Omega_\rho}_\mu[\tau] + \mathbb{K}^{\Omega_\rho}_\mu[\nu]. \tag{2.32}$$

The required result follows from Theorem 2.4. □

## 3. $\mathscr{L}_\mu$-moderate solutions of nonlinear equation

In this section we study the nonlinear equation

$$-\mathscr{L}_\mu u + |u|^{q-1}u = 0 \quad \text{in } \Omega, \tag{$P_\mu$}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $\mu < 1/4$ and $q > 1$.

### 3.1. Preliminaries

Suppose that $u \in L^q_{\mathrm{loc}}(\Omega)$ is either a subsolution or a supersolution of $(P_\mu)$, in the distribution sense. Then, $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ for $1 \le p < N/(N-1)$. If, in addition, $u$ is a distributional *solution* of $(P_\mu)$ then it is also a classical solution.

Consequently, if $u \in L^q_{\mathrm{loc}}(\Omega)$ is a distributional subsolution in $\Omega$ then

$$\int_\Omega \nabla u \cdot \nabla\varphi \, dx - \int_\Omega \frac{\mu}{\delta^2} u\varphi \, dx + \int_\Omega |u|^{q-1}u\varphi \, dx \le 0 \quad \forall 0 \le \varphi \in C^\infty_c(\Omega). \tag{3.1}$$

If, in addition, $u \in H^1_{\mathrm{loc}}(\Omega)$ then (3.1) holds for every $\varphi \in H^1_c(\Omega)$.

A similar statement holds for supersolutions, in which case the inequality sign in (3.1) is inversed. Of course these statements remain valid for local subsolutions and supersolutions (in a subdomain $G \subset \Omega$).

We state below two results from [5] that will be used in the sequel.

**Lemma 3.1 (Comparison principle [5, Lemma 3.2]).**

(i) *Let $G$ be open with $G \subset \Omega$. Let $0 \le \underline{u}, \overline{u} \in H^1_{\mathrm{loc}}(G) \cap C(G)$ be a pair of sub and supersolutions to $(P_\mu)$ in $G$ such that*

$$\limsup_{x \to \partial G} [\underline{u}(x) - \overline{u}(x)] < 0.$$

*Then $\underline{u} \le \overline{u}$ in $G$;*

(ii) *Let $G$ be open with $\overline{G} \subset \Omega$. Let $\underline{u}, \overline{u} \in H^1(G) \cap C(\overline{G})$ be a pair of sub and supersolutions to $(P_\mu)$ in $G$ and $\underline{u} \le \overline{u}$ on $\partial G$. Then $\underline{u} \le \overline{u}$ in $G$.*

**Lemma 3.2 ([5, Lemma 4.10]).** *Assume that $(P_\mu)$ admits a subsolution $\underline{u}$ and a supersolution $\overline{u}$ in $\Omega$ so that $0 \le \underline{u} \le \overline{u}$ in $\Omega$. Then $(P_\mu)$ has a solution $U$ in $\Omega$ such that $\underline{u} \le U \le \overline{u}$ in $\Omega$.*

In [5, Proposition 3.5] the Keller–Osserman estimate has been extended to equation $(P_\mu)$. Specifically it was proved that every subsolution $u$ of $(P_\mu)$ in $\Omega$ satisfies,

$$u(x) \le \gamma_* \delta^{-\frac{2}{q-1}}(x) \quad \text{in } \Omega, \tag{3.2}$$

where $\gamma_*$ is a constant independent of $u$. In addition it was shown that, if $u$ is a local subsolution in $\Omega_\rho$, continuous at $\Sigma_\rho$, then $u$ satisfies (3.2) in $\Omega_\rho$, but $\gamma_*$ may depend on $u$. We prove below a stronger version that is needed later on.

**Lemma 3.3 (Keller–Osserman estimate).** *If $u$ is a subsolution of $(P_\mu)$ in $\Omega$ then it satisfies* (3.2) *with a constant depending only on $q$, $N$, $\mu$. If $u$ is a subsolution of $(P_\mu)$ in $\Omega_\rho$ then* (3.2) *holds with a constant depending only on $q$, $N$, $\mu$, $\rho$ and $\delta(x)$ replaced by $\delta_\rho(x) := \operatorname{dist}(x, \partial\Omega_\rho)$.*

*Proof.* Without loss of generality we may assume that $u \ge 0$ because $u_+$ is a subsolution. If $\mu \le 0$ then $u$ is also a subsolution of the equation $-\Delta u + u^q = 0$. Therefore in this case (3.2) is a direct consequence of the classical Keller–Osserman inequality.

Now assume that $\mu > 0$. Let $y \in \Omega$ and $R = \delta(y)/2$. Then,

$$-\Delta u - \frac{\mu}{R^2} u + u^q \le 0 \quad \text{in } B_R(y).$$

Therefore in $B_R(y)$ either $u \le (8\mu/R^2)^{\frac{1}{q-1}}$ or $-\Delta u + u^q/2 \le 0$. Hence, by Kato's inequality, the function $v := (u - (8\mu/R^2)^{\frac{1}{q-1}})_+$ satisfies

$$-\Delta v + v^q/2 \le 0 \quad \text{in } B_R(y).$$

By the classical Keller–Osserman inequality,

$$v(y) \le c(q, N) R^{-\frac{2}{q-1}}.$$

Since $u(y) \leq v(y) + (8\mu/R^2)^{\frac{1}{q-1}}$ we conclude that

$$u(y) \leq c(\mu, q, N)\delta_\Omega(y)^{-\frac{2}{q-1}} \quad \text{for all} \quad y \in \Omega. \tag{3.3}$$

Next, let $u$ be a subsolution in $\Omega_\rho$. As before we may assume that $u \geq 0$ and that $\mu > 0$. By the first part of the proof, (3.3) holds in $\Omega_{3\rho/4}$. Further,

$$-\Delta u - (4\mu/\rho^2)u + u^q \leq 0 \quad \text{in } \Omega'_\rho = \{x \text{ s.t. } \rho/2 \leq \delta(x) < \rho\}.$$

Therefore, either $u \leq (8\mu/\rho^2)^{\frac{1}{q-1}}$ or $-\Delta u + u^q/2 \leq 0$. By the same argument as before, the function $v := (u - (8\mu/\rho^2)^{\frac{1}{q-1}})_+$ satisfies

$$v(x) \leq c(q, N)\mathrm{dist}\,(x, \Sigma_\rho)^{-\frac{2}{q-1}} \quad \text{for all} \quad x \text{ s.t. } 3\rho/4 \leq \delta(x) < \rho.$$

Consequently,

$$u(x) \leq c(\mu, q, N, \rho)\mathrm{dist}\,(x, \partial\Omega_\rho)^{-\frac{2}{q-1}} \quad \text{for all} \quad x \in \Omega_\rho. \tag{3.4}$$

□

### 3.2. Moderate solutions

We study the generalised boundary trace problem $(P^\nu_\mu)$ where $\mu < 1/4, q > 1$ and $\nu \in \mathcal{M}^+(\partial\Omega)$. First we prove,

**Lemma 3.4.** *Let $D$ be a $C^2$ domain such that $D \Subset \Omega$. If $0 \leq f \in C(\partial D)$ then there exists a unique solution of the problem*

$$\begin{cases} -\mathscr{L}_\mu u + u^q = 0 & \text{in } D \\ u = f & \text{on } \partial D. \end{cases} \tag{3.5}$$

*Proof.* For $u \in H^1(D)$, let

$$J_D(u) = \int_D \left( \frac{1}{2}|\nabla u|^2 - \frac{\mu}{2\delta_\Omega^2}u^2 + \frac{1}{q+1}|u|^{q+1} \right) dx.$$

Since $\mu\delta_\Omega^{-2} \in L^\infty(D)$, it is standard to see that $J_D$ is coercive and weakly lower semicontinuous on

$$H^1_f(D) = \left\{ u \in H^1(D) : u = f \text{ on } \partial D \right\}.$$

Therefore there exists a minimizer $u_f \in H^1_f(D)$. We may assume that $u_f > 0$ because $|u_f|$ too is a minimizer. The minimizer is a solution of (3.5). The uniqueness is a consequence of the comparison principle. □

Next consider the problem,

$$\begin{cases} -\mathscr{L}_\mu u + u^q = 0 & \text{in } \Omega_\rho \\ \operatorname{tr}^*_{\partial\Omega}(u) = \nu \mathbf{1}_{\partial\Omega} =: \nu_0 \\ \operatorname{tr}_{\Sigma_\rho}(u) = \nu \mathbf{1}_{\Sigma_\rho} =: \nu_\rho. \end{cases} \tag{$P^\nu_\mu(\rho)$}$$

where $\mu < 1/4$ and $q > 1$ whit $\nu \in \mathcal{M}^+(\partial\Omega_\rho)$ and $\rho \in (0, \bar\rho]$.

The following result is an adaptation of [10, Theorem C] to problem $(P^\nu_\mu(\rho))$. Since $C_H(\Omega_{\bar\rho}) = 1/4$ the result applies to every $\mu < 1/4$. The proof follows the argument in [10]; for the convenience of the reader it is presented below.

**Proposition 3.5.** *Let $\nu \in \mathfrak{M}^+(\partial\Omega_\rho)$ and assume that $\mathbb{K}^{\Omega_\rho}_\mu[\nu] \in L^q_{\delta^{\alpha_+}}(\Omega_\rho)$ for some $\rho \in (0, \bar\rho]$. Then $(P^\nu_\mu(\rho))$ admits a unique solution $U_\nu$.*

*Proof.* Let $\{D_n\}$ be a sequence of $C^2$ domains such that $\bar D_n \subset D_{n+1}$ and $D_n \uparrow \Omega_\rho$. Let $u_n$ be the solution of (3.5) with $D = D_n$ and $f = f_n := \mathbb{K}^{\Omega_\rho}_\mu[\nu]\lfloor_{\partial D_n}$. Since $\mathbb{K}^{\Omega_\rho}_\mu[\nu]$ is a supersolution of the equation $\mathscr{L}_\mu v + v^q = 0$ in $\Omega_\rho$ it follows that $u_n$ decreases and $u = \lim u_n$ is a solution of this equation. We claim that $u$ is a solution of $(P^\nu_\mu(\rho))$. Indeed,

$$u_n + \mathbb{G}^{D_n}_\mu[u_n^q] = \mathbb{P}^{D_n}_\mu[f_n] = \mathbb{K}^{\Omega_\rho}_\mu[\nu] \quad \text{in } D_n, \tag{3.6}$$

where $\mathbb{P}^{D_n}_\mu$ denotes the Poisson kernel of $\mathscr{L}_\mu$ in $D_n$.

Since $u_n \le \mathbb{K}^{\Omega_\rho}_\mu[\nu] \in L^q_{\delta^{\alpha_+}}(\Omega)$ it follows that

$$\mathbb{G}^{D_n}_\mu[u_n^q] \to \mathbb{G}^{\Omega_\rho}_\mu[u^q].$$

Hence, by (3.6),

$$u + \mathbb{G}^{\Omega_\rho}_\mu[u^q] = \mathbb{K}^{\Omega_\rho}_\mu[\nu] \quad \text{in } \Omega_\rho.$$

By Theorem 2.4, $\operatorname{tr}^*_{\partial\Omega}(u) = \nu \mathbf{1}_{\partial\Omega}$ and (by (2.7)) $\operatorname{tr}_{\Sigma_\rho}(u) = \nu \mathbf{1}_{\Sigma_\rho}$. □

The next result is an adaptation of [10, Theorem D]. We omit the proof which except for obvious modifications is the same as in [10].

**Proposition 3.6.** *Assume that $u$ is a positive solution of $(P^\nu_\mu(\rho))$. Then*

$$\lim_{x\to\partial\Omega} \frac{u(x)}{\mathbb{K}^{\Omega_\rho}_\mu[\nu_0](x)} = 1 \quad \textit{non-tangentially, } \nu\textit{-a.e. on } \partial\Omega, \tag{3.7}$$

*where $\nu_0 = \nu \mathbf{1}_{\partial\Omega}$.*

**Theorem 3.7.** *Let $\nu \in \mathfrak{M}^+(\partial\Omega)$ and $\rho \in (0, \bar{\rho})$. Let $\nu' \in \mathfrak{M}^+(\partial\Omega_\rho)$ be defined by $\nu' = \nu$ on $\partial\Omega$ and $\nu' = 0$ on $\Sigma_\rho$. Assume that, for some $\rho$ as above, $\mathbb{K}_\mu^{\Omega_\rho}[\nu'] \in L^q_{\delta^{\alpha_+}}(\Omega_\rho)$. Then the boundary value problem $(P_\mu^\nu)$ admits a solution in $\Omega$.*

*Proof.* By Proposition 3.5 there exists a (unique) solution $U_{\nu,0}$ of problem $(P_\mu^{\nu'}(\rho))$. For every $k \geq 0$, let $\nu_k \in \mathfrak{M}^+(\partial\Omega_\rho)$ be the measure given by, $\nu_k \mathbf{1}_{\partial\Omega} = \nu$ and $\nu_k \mathbf{1}_{\Sigma_\rho} = k dS_{\Sigma_\rho}$. By the same proposition there exists a (unique) solution $U_{\nu,k}$ of $(P_\mu^{\nu_k}(\rho))$. Put

$$U_{\nu,\infty} = \lim_{k\to\infty} U_{\nu,k}.$$

Let $R \in (0, \rho)$. By Lemma 3.4 there exists a unique solution $v_R$ of (3.5) in $D_R$ with $f = U_{\nu,0}\lfloor_{\Sigma_R}$. By the comparison principle,

$$U_{\nu,0} \leq v_R \leq U_{\nu,\infty} \quad \text{in } \Omega_\rho \cap D_R.$$

By Proposition 3.3 the family $\{v_R : 0 < R < \rho\}$ is bounded in compact subsets of $\Omega$. Therefore there exists a sequence $\{R_j\}$ converging to zero such that $v_{R_j}$ converges to a solution $v$ of the nonlinear equation in $\Omega$. By construction,

$$U_{\nu,0} \leq v \leq U_{\nu,\infty} \quad \text{in } \Omega_\rho.$$

Therefore $\operatorname{tr}^*_{\partial\Omega}(v) = \nu$. □

**Remark 3.8.** If $\mu < C_H(\Omega)$ then the problem $(P_\mu^\nu)$ has at most one solution, [10, Theorem B]. However uniqueness fails when $C_H(\Omega) < \mu < 1/4$. It was proved in [5, Theorem 5.3] that in this case there exists a positive solution of $(P_\mu^\nu)$ with $\nu = 0$. An alternative, more direct proof, is presented in Appendix A.

**Proposition 3.9.** *Assume that $u \in L^q_{\mathrm{loc}}(\Omega)$ is a positive solution of $(P_\mu)$. Then the following assertions are equivalent:*

(i) *$u$ has a normalized boundary trace;*
(ii) *$u$ is a moderate solution in the sense of Definition 1.3;*
(iii) *$u \in L^q(\Omega; \delta^{\alpha_+})$.*

*Proof.* The assumption implies that $\mathscr{L}_\mu u \leq 0$ in $\Omega$. If $\rho \in (0, \bar{\rho}]$ then, by Lemma 2.12, (i) holds if and only if $u$ is dominated by an $\mathscr{L}_\mu$-superharmonic function in $\Omega_\rho$. Consequently, by Lemma 3.2, (i) holds if and only if $u$ is dominated by an $\mathscr{L}_\mu$-harmonic function in $\Omega_\rho$. Thus (i) and (ii) are equivalent.

If (iii) holds then $v := u + \mathbb{G}_\mu^{\Omega_\rho}[u^q]$ is $\mathscr{L}_\mu$-harmonic. By the representation theorem there exists $\nu \in \mathfrak{M}(\partial\Omega_\rho)$ such that $v = \mathbb{K}_\mu^{\Omega_\rho}[\nu]$. Since $\operatorname{tr}^*_{\partial\Omega}\mathbb{G}_\mu^{\Omega_\rho}[u^q] = 0$ it follows that $\nu\mathbf{1}_{\partial\Omega}$ is the normalized boundary trace of $u$. Conversely if (ii) holds then by Theorem 2.8 we have $\mathscr{L}_\mu u = u^q \in \mathfrak{M}^+_{\delta^{\alpha_+}}(\Omega_\rho)$ which is the same as (iii). □

### 3.3. Critical exponents

The next result provides necessary and sufficient conditions in order that a positive measures $\nu \in \mathfrak{M}^+(\partial\Omega)$ satisfies

$$\mathbb{K}_\mu^{\Omega_\rho}[\nu] \in L^q(\Omega_\rho; \delta_+^\alpha) \tag{3.8}$$

for some $\rho > 0$. Let $\Gamma_a(x-y) = |x-y|^{-(N-a)}$ denote the Riesz kernel of order $0 < a < N$ in $\mathbb{R}^N$.

**Proposition 3.10.** *Let $\nu \in \mathcal{M}^+(\partial\Omega)$.*

(i) *If $\Gamma_1 * \nu \in L^q_{\delta^{1+(q-1)\alpha_-}}(\Omega)$ then $\nu$ satisfies* (3.8)*;*

(ii) *Assume $\mu \geq 0$. If $\nu$ satisfies* (3.8) *then $\mathbb{P}_0^\Omega[\nu] \in L^q(\Omega; \delta^{1+(q-1)\alpha_-})$. Here $P_0^\Omega$ is the Poisson kernel of $-\Delta$ in $\Omega$: $P_0^\Omega(x,y) = \delta(x)|x-y|^{-N}$.*

*Proof.* By (2.13),

$$\begin{aligned} K_\mu^{\Omega_\rho}(x,y) &\sim \frac{\delta(x)^{\alpha_+}}{|x-y|^{N-2\alpha_-}} \sim \delta(x)^{\alpha_-} P_0^\Omega(x,y)\big(|x-y|/\delta(x)\big)^{2\alpha_-} \\ &\sim \delta(x)^{\alpha_-}\Gamma_1(x-y)\big(|x-y|/\delta(x)\big)^{-1+2\alpha_-}, \end{aligned} \tag{3.9}$$

for every $(x,y) \in \Omega_{\rho/2} \times \partial\Omega$.

For every $\mu < 1/4$ we have $-1 + 2\alpha_- < 0$. Consequently,

$$K_\mu^{\Omega_\rho}(x,y) \leq c\delta(x)^{\alpha_-}\Gamma_1(x-y) \quad \text{for all} \quad (x,y) \in \Omega_{\rho/2} \times \partial\Omega. \tag{3.10}$$

Hence,

$$\|\mathbb{K}_\mu^{\Omega_\rho}\nu\|^q_{L^q_{\delta^{\alpha_+}}(\Omega_{\rho/2})} \leq c\int_{\Omega_{\rho/2}} \left(\int_{\partial\Omega} \Gamma_1(x-y)d\nu(y)\right)^q \delta(x)^{q\alpha_- + \alpha_+}dx.$$

This proves (i).

If $\mu \geq 0$, so that $\alpha_- \geq 0$ then, by (3.9),

$$K_\mu^{\Omega_\rho}(x,y) \geq c\delta(x)^{\alpha_-} P_0^\Omega(x,y) \quad \text{for all} \quad (x,y) \in \Omega_{\rho/2} \times \partial\Omega. \tag{3.11}$$

Therefore

$$\|\mathbb{K}_\mu^{\Omega_\rho}[\nu]\|^q_{L^q_{\delta^{\alpha_+}}(\Omega_{\rho/2})} \geq c\int_{\Omega_{\rho/2}} \left(\int_{\partial\Omega} P_0^\Omega(x,y)d\nu(y)\right)^q \delta(x)^{q\alpha_- + \alpha_+}dx.$$

This proves (ii). □

Using this result we provide a necessary and sufficient condition for the existence of positive moderate solutions of $(P_\mu)$.

**Proposition 3.11.** *Let $\nu \in \mathcal{M}^+(\partial\Omega)$.*

(i) *If $\alpha_- > -\frac{2}{q-1}$ then the boundary value problem $(P^\nu_\mu)$ has a solution for every measure $\nu = f\,dS_{\partial\Omega}$ such that $f \in L^1(\partial\Omega)$;*
(ii) *If $\alpha_- \le -\frac{2}{q-1}$ then, for every $\nu \gneq 0$, $(P^\nu_\mu)$ has no solution.*

**Remark.** When $\mu > 0$ and consequently $\alpha_- > 0$, the condition in (i) holds for every $q > 1$.

*Proof.* Let $\nu = f\,dS_{\partial\Omega}$ and $f \in L^\infty(\partial\Omega)^+$. Let $x \in \Omega_{\beta_0}$ and pick $x' \in \partial\Omega$ such that $|x' - x| = \delta(x)$. Then,

$$\begin{aligned}\int_{\partial\Omega} |x-y|^{1-N} f(y)dS(y) &\le c\|f\|_{L^\infty}\Bigg(\int_{\substack{y\in\partial\Omega\\ |x'-y|\ge\delta(x)}} |x'-y|^{1-N}dS(y)+1\Bigg)\\ &\le c\|f\|_{L^\infty}(1+|\ln\delta(x)|) \le c'\|f\|_{L^\infty}|\ln\delta(x)|,\end{aligned} \tag{3.12}$$

where $c'$ is independent of $x$. Therefore, if $(q-1)\alpha_- + 1 > -1$ then $\Gamma_1 * \nu \in L^q_{\delta^{1+(q-1)\alpha_-}}(\Omega)$. Consequently, by Proposition 3.10 (i) and Theorem 3.7, problem $(P^\nu_\mu)$ has a solution.

Next, let $f \in L^1(\partial\Omega)^+$ and $\nu = f\,dS_{\partial\Omega}$. If $\nu_n = \min(f, n)dS_{\partial\Omega}$ then problem $(P^{\nu_n}_\mu)$ has a solution $u_n$ and the sequence $\{u_n\}$ is non-decreasing. In view of the Keller–Osserman estimate (3.2), $\{u_n\}$ converges to a solution $u$ of $(P^\nu_\mu)$. This proves (i).

We turn to part (ii). Suppose that $\alpha_- \le -\frac{2}{q-1}$ and that there exits $\nu \in \mathcal{M}^+(\partial\Omega) \setminus \{0\}$ such that problem $(P^\nu_\mu)$ has a solution $u$. Then, there exists $c > 0$ such that

$$c\beta^{-\frac{2}{q-1}} \le c\beta^{\alpha_-} \le \int_{\Sigma_\beta} \mathbb{K}^{\Omega_\rho}_\mu[\nu]dS \quad \text{for all} \quad \beta \in (0, \beta_0).$$

Since $u = -\mathbb{G}_\mu[u^q] + \mathbb{K}_\mu[\nu]$ and $\mathrm{tr}^*_{\partial\Omega}(\mathbb{G}_\mu[u^q]) = 0$ it follows that, for sufficiently small $\beta_1$,

$$c\beta^{\alpha_-} \le \int_{\Sigma_\beta} u\,dS \quad \text{for all} \quad \beta \in (0, \beta_1). \tag{3.13}$$

But, by the Keller-Osserman estimate, $u(x) \le c_1\delta(x)^{-\frac{2}{q-1}}$ so that

$$c\beta^{\alpha_-} \le \int_{\Sigma_\beta} u\,dS \le c_2\beta^{-\frac{2}{q-1}} \quad \text{for all} \quad \beta \in (0, \beta_1). \tag{3.14}$$

If $\alpha_- < -2/(q-1)$ we reached a contradiction. If $\alpha_- = -2/(q-1)$ then, in view of the Keller-Osserman estimate (3.2) we conclude that $u(x) \sim \delta(x)^{-\frac{2}{q-1}}$. This implies that $u \sim U_{\max}$ (which is the maximal solution of $-\mathscr{L}_\mu v + v^q = 0$). Thus $\sup U_{\max}/u := c < \infty$. Now $cu$ is a supersolution and, if $v$ is the largest solution dominated by $cu$ then $\mathrm{tr}^*(v) = c\,\mathrm{tr}^*(u) = c\nu$. It follows that $U_{\max} \le v$ which is impossible. □

**Remark 3.12.** When $\mu > 0$, and consequently $\alpha_- > 0$, the condition in (i) holds trivially for every $q > 1$. However, if $\mu < 0$ and

$$q \geq q^*_\mu := 1 - \frac{2}{\alpha_-}$$

then equation $(P_\mu)$ has no moderate solution except for the trivial solution.

**Lemma 3.13.** *Let $\mu < C_H(\Omega)$ and put*

$$q_{\mu,c} = \frac{N+1-\alpha_-}{N-1-\alpha_-}.$$

*Then, for $y \in \partial\Omega$,*

$$K^\Omega_\mu(\cdot, y) \in L^q(\Omega, \delta^{\alpha_+}) \Longleftrightarrow q < q_{\mu,c}.$$

*For every $q \in (1, q_{\mu,c})$ there exists a number $c = c(q, N, \mu)$ such that*

$$\|K^\Omega_\mu[\nu]\|_{L^{\frac{N+\alpha_+}{N-1-\alpha_-}}(\Omega,\delta^{\alpha_+})} \leq c\|\nu\| \quad \text{for all} \quad \nu \in \mathfrak{M}(\partial\Omega). \tag{3.15}$$

*Proof.* Recall that

$$K^\Omega_\mu(x, y) \sim |x-y|^{2-N-\alpha_+} \left(\frac{\delta(x)}{|x-y|}\right)^{\alpha_+} = \delta(x)^{\alpha_+}|x-y|^{2\alpha_- - N}, \tag{3.16}$$

(see [10, Section 2.2]). Therefore,

$$c'(\frac{\delta(x)}{|x-y|})^{\alpha_+}|x-y|^{1+\alpha_- -N} \leq K_\mu(x, y) \leq c|x-y|^{1+\alpha_- -N}.$$

It follows that $K_\mu(\cdot, y) \in L^q(\Omega, \delta^{\alpha_+})$ if and only if

$$I := \int_0^1 t^{q(1+\alpha_- -N)} t^{\alpha_+} t^{N-1} dt < \infty$$

and

$$\|K_\mu(\cdot, y)\|_{L^q(\Omega,\delta^{\alpha_+})} \sim I.$$

A simple computation shows that $I < \infty$ if and only if

$$q < q_{\mu,c} = \frac{N+1-\alpha_-}{N-1-\alpha_-}.$$

Finally,

$$\|K^\Omega_\mu[\nu]\|_{L^q(\Omega,\delta^{\alpha_+})} \leq \int_{\partial\Omega} \|K_\mu(\cdot, y)\|_{L^q(\Omega,\delta^{\alpha_+})} d|\nu|(y) \leq c\|\nu\|.$$ □

**Corollary 3.14.** *Let $\mu < 1/4$. If $1 < q < q_{\mu,c}$ then the boundary value problem $(P^\nu_\mu)$ has a solution for every Borel measure $\nu$. Moreover, if $q \geq q_{\mu,c}$ then problem $(P^\nu_\mu)$ has no solution when $\nu$ is the Dirac measure.*

*Proof.* In view of Lemma 3.13, the first assertion follows from Theorem 3.7. The second assertion follows from Proposition 3.6. □

## Appendix

### A. Non-uniqueness for $C_H(\Omega) < \mu < 1/4$

We are going to show that for $C_H(\Omega) < \mu < 1/4$ the problem

$$\begin{cases} -\mathscr{L}_\mu u + u^q = 0 & \text{in } \Omega \\ \operatorname{tr}^*_\mu(u) = 0 \end{cases} \tag{$P^0_\mu$}$$

admits a nontrivial solution. This was proved in [5, Theorem 5.3]. Here we provide a more direct argument.

Recall that if $C_H(\Omega) < 1/4$ then the operator $-\mathscr{L}_{C_H(\Omega)}$ admits a positive ground state solution $\phi_H \in H^1_0(\Omega)$ such that $-\mathscr{L}_{C_H(\Omega)}\phi_H = 0$ in $\Omega$, see [9].

**Proposition A.1.** *Assume that $C_H(\Omega) < \mu < 1/4$ and $q > 1$. Then $(P^0_\mu)$ admits a positive solution $U_0$ such that*

$$\liminf_{x\to\partial\Omega} \frac{U_0(x)}{\phi_H(x)} > 0.$$

*Proof.* Since $-\mathscr{L}_{C_H(\Omega)}\phi_H = 0$ in $\Omega$, for a small $\tau > 0$ we obtain

$$-\mathscr{L}_\mu(\tau\phi_H) + (\tau\phi_H)^q = -\frac{\mu - C_H(\Omega)}{\delta^2}(\tau\phi_H) + (\tau\phi_H)^q \le 0 \quad \text{in } \Omega,$$

so that $\tau\phi_H$ is a subsolution for $(P^0_\mu)$ in $\Omega$.

Fix $\rho \in (0, \bar\rho]$. Similarly to the proof of Theorem 3.7, for every $k \ge 0$ denote $\nu_{\rho,k} = k dS_{\Sigma_\rho}$ and let $\nu \in \mathfrak{M}^+(\partial\Omega_\rho)$ be the measure such that $\nu \mathbf{1}_{\partial\Omega} = 0$ and $\nu\mathbf{1}_{\Sigma_\rho} = \nu_{\rho,k}$. By Proposition 3.5 there exists a (unique) solution of $(P^\nu_\mu(\rho))$ with this boundary data. Denote this solution by $U_{0,k}$ and put

$$U_{0,\infty} = \lim_{k\to\infty} U_{0,k}.$$

Let $R \in (0, \rho)$. By Lemma 3.4 there exists a unique solution $v_R$ of (3.5) in $D_R$ with $f = 2U_{0,\infty}$ on $\Sigma_R$. We define,

$$\overline{u} := \min\{U_{0,\infty}, u_R\} \quad \text{in } D_R \cap \Omega_\rho.$$

Then $\overline{u}$ is a supersolution of $(P_\mu)$ in $D_R \cap \Omega_\rho$ with $\overline{u} = U_{0,\infty}$ in $D_R \cap \Omega_{\rho'}$ for some $\rho' \in (R, \rho)$ and $\overline{u} = u_R$ in $D_{R'} \cap \Omega_\rho$ for some $R' \in (R, \rho')$. Therefore setting $\overline{u} = u_R$ in $\Omega \setminus \Omega_\rho$ and $\overline{u} = U_{0,\infty}$ in $\Omega \setminus D_R$ provides an extension (still denoted by $\overline{u}$) that is a supersolution of $(P_\mu)$ in $\Omega$. As $\overline{u} = U_{0,\infty}$ in a neighborhood of $\partial\Omega$ it follows that $\overline{u} \sim \delta^{\alpha_+}$ in such a neighborhood. On the other hand $\phi_H \sim \delta^{a_+}$ where $a_+ := \frac{1}{2} + \sqrt{\frac{1}{4} - C_H(\Omega)}$. As $C_H(\Omega) < \mu$ it follows that $\alpha_+ < a_+$ so that $\delta^{\alpha_+} > \delta^{a_+}$. Therefore $\tau\phi_H < \overline{u}$ near $\partial\Omega$ and therefore, by Lemma 3.1, everywhere in $\Omega$. Finally by Lemma 3.2 we conclude that there exists a solution $U_0$ of $(P_\mu)$ in $\Omega$ such that $\tau\phi_H < U_0 < \overline{u}$. Thus $U_0$ is a positive solution such that $\operatorname{tr}^*(U_0) = 0$. $\square$

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